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An Algorithm for Identifying Least Manipulable Envy-Free and Budget-Balanced Allocations in Economies with Indivisibilities*

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Abstract

We analyze the problem of allocating indivisible objects and monetary compensations to a set of agents. In particular, we consider envy-free and budget-balanced rules that are least manipulable with respect to agents counting or with respect to utility gains. A key observation is that, for any profile of quasi-linear preferences, the outcome of any such least manipulable envy-free rule can be obtained via so-called agent-k-linked allocations. Given this observation, we provide an algorithm for identifying agent-k-linked allocations.

JEL Classification: C71, C78, D63, D71, D78.
Keywords: Envy-freeness, Budget-balance, Least manipulable, Algorithm.

1 Introduction

People encounter fair division problems in their everyday lives. For example, how to fairly split restaurant bills, taxi fares, and rents. Nowadays, there are even online tools, such as the website `spliddit.org` (Goldman and Procaccia, 2014), that help people to fairly divide costs among themselves. This paper considers the rent division problem where a set of roommates share a house and need to decide who gets which room and at what (room-specific) rent. This problem

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has received considerable attention by economists and computer scientists, see, e.g., Gal et al. (2017), Procaccia et al. (2018), or Velez (2018,2020) for an overview.

In division problems, the concept of envy-freeness (Foley, 1967) is often adopted as fairness notion because it is compelling and since an envy-free allocation always exists when monetary compensations are allowed (Svensson, 1983).[1] An allocation is envy-free if no agent (strictly) prefers the consumption bundle assigned to some other agent over the consumption bundle assigned to herself. In the rent division problem, this means that no agent prefers a room assigned to some of her roommates to the room assigned to herself at the given (room-specific) rents.

Rent division problems are naturally restricted by a balanced budget condition meaning that the sum of the individual rents paid by the roommates must equal the total cost for renting the house. However, if the roommates insist on budget-balance and adopt envy-freeness as a fairness notion, a famous result by Green and Laffont (1979) states that it generally is impossible to prevent roommates from manipulating the outcome of rent division problem in their advantage. This type of friction between desirable properties of social choice rules and matching mechanisms is not unique for the fair rent division problem. In fact, policy makers often adopt mechanisms that are vulnerable to manipulation by strategic misrepresentation, e.g., voting rules, school choice mechanisms, and auction procedures. This has motivated researchers to identify rules and mechanisms that are "least manipulable" according to some predetermined measure.

Two prominent measures of "the degree of manipulability" are (a) to count the number of profiles at which a rule is manipulable (Maus et al., 2007a,b) or (b) to compare (via set inclusion) the preference domains where different rules are manipulable (Pathak and Sönmez, 2013). Even though those measures are natural, Andersson et al. (2014a) demonstrated that, in the context of the fair rent division problem, they do not distinguish envy-free and budget-balanced rules. Consequently, a "finer" measure is needed to identify least manipulable rules among the envy-free and budget-balanced rules.

In Andersson et al. (2014a), rule φ is judged to be more manipulable with respect to *agents counting* than rule ψ if, for each preference profile, the number of agents that can manipulate φ is larger than or equal to the number of agents that can manipulable ψ. Andersson et al. (2014b) and Fujinaka and Wakayama (2015) considered a different approach and calculated the maximal amount by which an agent can gain from manipulating a given rule. In this case, rule φ is defined to be more manipulable with respect to *utility gains* than rule ψ if, for each preference profile, the maximal gain that any agent can obtain by manipulating φ is weakly larger than the maximal gain that any agent can obtain by manipulating ψ. Even though these two finer measures appear to be quite different, they share one important feature. Namely, as observed in this paper, for any given profile of quasi-linear preferences, the outcome of least manipulable envy-free and budget-balanced rules can be identified via agent-k-linked allocations. Here, an allocation is agent-k-linked if for each agent i, there is a sequence of agents from i to k such that any agent in

[1] See, for example, Alkan et al. (1991), Aragones (1995), Nicoló and Velez (2017), Su (1999), Tadenuma and Thomson (1991,1993,1995a,b), or Velez (2018,2020).

the sequence is indifferent between his consumption bundle and the consumption bundle of the next agent in the sequence.

Even if agent-k-linked allocations have played a central role in other contexts than the above mentioned, see, e.g., Alkan et al. (1991), Velez (2011), Fujinaka and Wakayama (2015), or Tadenuma and Thomson (1995a,b), an algorithm for identifying such allocations is lacking in the literature. The main contribution of this paper is to provide an algorithm for identifying envy-free, budget-balanced and agent-k-linked allocations under quasi-linear preferences.

The remaining part of the paper is organized as follows. Section 2 presents the model and some basic definitions. In Section 3, the two least manipulable envy-free and budget-balanced rules from Andersson et al. (2014a,b) are introduced. The section also carefully explains why agent-k-linked allocations are important for identifying the outcome of these rules. Section 4 provides the algorithms and the main convergence theorem.

2 The Model and Basic Definitions

Let $N = \{1, \ldots, n\}$ and $M = \{1, \ldots, m\}$ denote the sets of agents and objects, respectively, with $|N| = |M|$. Each agent $i \in N$ consumes one bundle $(j, x_j) \in M \times \mathbb{R}$ containing one object $j \in M$ and some amount of money $x_j \in \mathbb{R}$. One can think of the agents as roommates and the objects as rooms in the house that they rent jointly. In this interpretation, each agent "consumes" exactly one room j and pays the corresponding rent x_j.

For each agent $i \in N$, i's preferences over bundles (j, x_j) are represented by a quasi-linear utility function u_i where:

$$u_i(j, x_j) = v_{ij} + x_j \text{ for some } v_{ij} \in \mathbb{R}. \tag{1}$$

A list of utility functions $u = (u_1, \ldots, u_n)$ is a (preference) profile. Let $\mathcal{U}$ denote the set of profiles.

An allocation (a, x) is a list of $|N|$ bundles where $a : N \to M$ assigns object a_i to agent $i \in N$ and $x : M \to \mathbb{R}$ assigns monetary compensation x_j to $j \in M$. An allocation (a, x) is *feasible* if $a_i \neq a_j$ whenever $i \neq j$ for $i, j \in N$, and $\sum_{j \in M} x_j \leq \alpha$ for some $\alpha \in \mathbb{R}_+$. If $\sum_{j \in M} x_j = \alpha$, allocation (a, x) is *budget-balanced*. Let $\mathcal{A}$ denote the set of feasible and budget-balanced allocations. For convenience, we write "allocation" instead of "feasible allocation satisfying budget-balance". At profile $u \in \mathcal{U}$, allocation (a, x) is *envy-free* if $u_i(a_i, x_{a_i}) \geq u_i(a_j, x_{a_j})$ for all $i, j \in N$. Let $\mathcal{F}(u)$ denote the set of envy-free allocations at profile $u \in \mathcal{U}$.

A *rule* is a non-empty correspondence φ choosing for each profile $u \in \mathcal{U}$ a non-empty set of allocations $\varphi(u)$ such that $u_i(a_i, x_{a_i}) = u_i(b_i, y_{b_i})$ for all $i \in N$ and all $(a, x), (b, y) \in \varphi(u)$. A rule φ is *envy-free* if $\varphi(u) \subseteq \mathcal{F}(u)$ for each profile $u \in \mathcal{U}$. Given a profile $u \in \mathcal{U}$, a rule φ is *manipulable* at profile u by agent $i \in N$ if there exists $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_i(b_i, y_{b_i}) > u_i(a_i, x_{a_i})$. If rule φ is not manipulable by any agent at profile u, then φ is *non-manipulable* at profile u.

We use the following concepts from Andersson et al. (2014a) to describe indifference relations at any allocation:

Definition 1. Let $(a, x) \in \mathcal{A}$ and $u \in \mathcal{U}$.

(i) For any $i, j \in N$, we write $i \rightarrow_{(a,x)} j$ if $u_i(a_i, x_{a_i}) = u_i(a_j, x_{a_j})$.

(ii) An *indifference chain* at (a, x) consists of a tuple of distinct agents $g = (i_0, \ldots, i_k)$ such that $i_0 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k$.

(iii) Agent $i \in N$ is *linked* to agent $k \in N$ at (a, x) if there exists an indifference chain $(i_0, \ldots, i_t)$ at (a, x) with $i = i_0$ and $i_t = k$.

(iv) Allocation (a, x) is *agent-k-linked* if each agent $i \in N$ is linked to agent $k \in N$.

Definition 2. Let $(a, x) \in \mathcal{A}$. An *indifference component* at (a, x) is a non-empty set $G \subseteq N$ such that for all $i, k \in G$ there exists an indifference chain at (a, x) in G, say $g = (i_0, ..., i_k)$ with $\{i_0, \ldots, i_k\} \subseteq G$, such that $i = i_0$ and $i_k = k$, and there exists no $G' \supsetneq G$ satisfying the previous property at allocation (a, x).

The following lemma from Svensson (2009, Proposition 2) will be important in the analysis.

Lemma 1. Let $u \in \mathcal{U}$. If $(a, x) \in \mathcal{F}(u)$ and $(b, y) \in \mathcal{F}(u)$, then $(a, y) \in \mathcal{F}(u)$ and $(b, x) \in \mathcal{F}(u)$.

3 Least Manipulable Envy-Free and Budget-Balanced Rules

We will next restate three previously established facts that hold for any agent $k \in N$ at any profile $u \in \mathcal{U}$. The first of these facts are from Alkan et al. (1991) whereas the last two are from Andersson et al. (2014a).

Fact 1. There exist an allocation in $\mathcal{F}(u)$ that maximizes agent k's utility in $\mathcal{F}(u)$. Such allocations will be called agent-k-preferred.

Fact 2. An allocation $(a^*, x^*) \in \mathcal{F}(u)$ is agent-k-linked if and only if (a^*, x^*) maximizes agent k's utility in $\mathcal{F}(u)$.

Fact 3. For any envy-free rule φ, there exists a profile $(\hat{u}_k, u_{-k}) \in \mathcal{U}$ such that some $(a^*, x^*) \in \varphi(\hat{u}_k, u_{-k})$ is agent-k-linked (under profile u).

Given a rule φ and a profile $u \in \mathcal{U}$, let $P^\varphi(u)$ denote the set of agents who can manipulate φ at profile u. Rule φ is non-manipulable at u if:

$$|P^\varphi(u)| = 0. \tag{2}$$

Because condition (2) is never satisfied for all profiles when insisting on envy-freeness and budget-balance (Green and Laffont, 1979), an alternative approach is to search for rules where $|P^{\varphi}(u)|$ is minimized for each profile u.

Definition 3. Envy-free rule φ is least manipulable with respect to agents counting if for any envy-free rule ψ, we have $|P^{\varphi}(u)| \leq |P^{\psi}(u)|$ for all profiles $u \in \mathcal{U}$.

Andersson et al. (2014a, Lemma 4) showed that the set of indifference components is invariant for any two envy-free allocations, and second, that agent k cannot manipulate an envy-free rule if and only if all allocations chosen by the rule are agent-k-linked (or, equivalently, agent-k-preferred). An immediate consequence from Andersson et al. (2014a, Theorem 3) is now that the least manipulable envy-free rules with respect to agents counting are exactly "maximally preferred" envy-free rules: for each profile u, we choose some agent k belonging to an indifference component with maximal cardinality and then a non-empty subset of agent-k-linked allocations. Note that such allocations are agent-i-linked for any agent i belonging to the same indifference component as agent k. Hence, to identify the outcome of a least manipulable envy-free rule with respect to agents counting, envy-free agent-k-linked allocations must be identified (and then indifference components with maximal cardinality may be found). Here, it suffices to identify one agent-k-linked allocation for each $k \in N$.

Andersson et al. (2014b) and Fujinaka and Wakayama (2015) determine the maximal utility gain which each agent can obtain by manipulating an envy-free rule. For any profile $u \in \mathcal{U}$ and any allocation $(a, x) \in \varphi(u)$, let:

$$f_k(\varphi, u) = \sup_{(\hat{u}_k, u_{-k}) \in \mathcal{U}} \max_{(b,y) \in \varphi(\hat{u}_k, u_{-k})} u_k(b_k, y_{b_k}) - u_k(a_k, x_{a_k}),$$

denote agent k's maximal gain from manipulating φ at profile u.

Let φ be an envy-free rule, $u \in \mathcal{U}$, $k \in N$, and $(a, x) \in \varphi(u)$. By Fact 1 there exist agent-k-linked $(a^*, x^*) \in \mathcal{F}(u)$. By Lemma 1, now $(a, x^*) \in \mathcal{F}(u)$ and, by envy-freeness, $u_k(a_k, x^*_{a_k}) = u_k(a^*_k, x^*_{a^*_k})$ implying that (a, x^*) is agent-k-linked. Now, the above facts and quasi-linearity imply:

$$f_k(\varphi, u) = v_{ka_k} + x^*_{a_k} - (v_{ka_k} + x_{a_k}) = x^*_{a_k} - x_{a_k}. \tag{3}$$

Hence, $f_k(\varphi, u)$ represents the maximal amount of money that agent k can obtain by manipulating the rule φ at profile u.

Definition 4. An envy-free rule φ is least manipulable with respect to utility gains if for any envy-free rule ψ, we have $\max_{i \in N} f_i(\varphi, u) \leq \max_{i \in N} f_i(\psi, u)$ for all profiles $u \in \mathcal{U}$.

Andersson et al. (2014b, Theorem 5) show that (a) there exist least manipulable envy-free rules φ with respect to utility gains, (b) that any such rule φ satisfies $f_i(\varphi, u) = f_j(\varphi, u)$ for all agents

$i, j \in N$ and all profiles $u \in \mathcal{U}$, and (c) that the allocations chosen by any such rule φ can be identified via agent-k-linked allocations for any profile u.

More explicitly, for a given profile $u \in \mathcal{U}$, start by identifying one agent-k-linked allocation in $\mathcal{F}(u)$, say (a^k, x^k), for any $k \in N$. Using Lemma 1 and the above argument, we may suppose that $a^1 = \cdots = a^n \equiv a$, i.e., that allocation $(a, x^k) \in \mathcal{F}(u)$ is agent-k-linked. By condition (3), for all $k \in N$, $x^k_{a_k} \geq x_{a_k}$ where $(a, x) \in \mathcal{F}(u)$, and $\sum_{k \in N} x^k_{a_k} \geq \alpha$. Thus, the compensations $(x^1_{a_1}, \ldots, x^n_{a_n})$ need to be reduced by $\beta \geq 0$ in order to satisfy budget-balance, i.e., we choose $\beta \geq 0$ such that $\sum_{k \in N}(x^k_{a_k} - \beta) = \alpha$. Andersson et al. (2014b, Theorem 5) show that the allocation $(a, (x^k_{a_k} - \beta)_{k \in N})$ is envy-free. Now, for profile u, any envy-free rule φ choosing $(a, (x^k_{a_k} - \beta)_{k \in N})$ satisfies, by condition (3), $f_i(\varphi, u) = \beta = f_j(\varphi, u)$ for all $i, j \in N$. Hence, the outcome of a least manipulable rule with respect to utility gains may be found via identifying envy-free agent-k-linked allocations. Again, it suffices to identify one agent-k-linked allocation for each $k \in N$.

4 Identification of Agent-k-linked Allocations

For the remaining part of this section, fix a profile $u \in \mathcal{U}$ and an agent $k \in N$. Similarly to Aragones (1995), our algorithm starts with an arbitrary envy-free allocation $(a, x) \in \mathcal{F}(u)$. This assumption is not restrictive since such allocations can be easily found in polynomial time, see, e.g., Klijn (2000) or Haake et al. (2003). Note that, in every step of the algorithm, we keep the assignment a fixed. This will, by Lemma 1, not cause any problems as long as the allocation is envy-free.

Definition 5. A group of agents $C \subsetneq N$ is *isolated* at (a, x) if $i \not\to_{(a,x)} j$ for all $i \in N \setminus C$ and all $j \in C$.

An allocation cannot be agent-k-linked if agent k belongs to an isolated group $C \subsetneq N$ because then at least one agent is not linked to agent k. The termination criterion for our algorithm will be the non-existence of an isolated group containing agent k.

Algorithm 1. Let $(a, x) \in \mathcal{F}(u)$ and set $K^0 = \{k\}$. For each iteration $t = 1, \ldots$:

Step t. Define $K^t \equiv K^{t-1} \cup \{i \in N \setminus K^{t-1} \mid i \to_{(a,x)} j \text{ for some } j \in K^{t-1}\}$. If $K^t = K^{t-1}$, then stop. Otherwise, continue with Step $t + 1$.

Lemma 2. Algorithm 1 identifies an isolated group containing agent k in at most $|N|$ iterations.

Proof. Let Algorithm 1 terminate at Step T. If $K^T \neq N$, then $i \not\to_{(a,x)} j$ for all $i \in N \setminus K^T$ and all $j \in K^T$ by construction. Thus, K^T is isolated and $k \in K^T$ since $\{k\} = K^0 \subseteq K^T$. Furthermore, note that $|K^t| - |K^{t-1}| \geq 1$ for all $t \in \{1, \ldots, T-1\}$, and Algorithm 1 terminates in at most $|N|$ iterations. □

Example 1. Let $N = \{1, 2, 3, 4, 5\}$, $M = \{1, 2, 3, 4, 5\}$ and $\alpha = 0$. Let also and the valuations v_{ij} be given by the matrix:

$$\begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ v_{21} & v_{22} & v_{23} & v_{24} & v_{25} \\ v_{31} & v_{32} & v_{33} & v_{34} & v_{35} \\ v_{41} & v_{42} & v_{43} & v_{44} & v_{45} \\ v_{51} & v_{52} & v_{53} & v_{54} & v_{55} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \tag{4}$$

For these valuations, the allocation (a, x) where $a_i = i$ and $x_{a_i} = 0$ for all $i \in N$ is envy-free and budget-balanced. To identify an isolated group containing agent 1, let $K^0 = \{1\}$. In this case, Algorithm 1 terminates in two steps.

Step 1. From matrix (4), it is clear that $i \to_{(a,x)} 1$ only for $i = 2$. Hence, $K^1 = \{1\} \cup \{2\} = \{1, 2\}$.

Step 2. From matrix (4), it is clear that $i \not\to_{(a,x)} j$ for all $i \in N - K^1$ and all $j \in K^1$. Hence, $K^2 = K^1$ and Algorithm 1 terminates. □

Note that both the distribution x and the assignment a are fixed in Algorithm 1. We next provide an algorithm for identifying an agent-k-linked envy-free allocation given that the distribution x is allowed to change.

Algorithm 2. Let $(a, x) \in \mathcal{F}(u)$ and set $K^0 = \{k\}$ and $x^0 = x$. Let x^t denote the compensations determined in iteration t. For each iteration $t = 1, \ldots$:

Step t. Run Algorithm 1 for (a, x^{t-1}) and let N^t denote the output of Algorithm 1. If $N \setminus N^t = \emptyset$, then stop with output (a, x^{t-1}). Otherwise, let $\lambda^t_{ij} \equiv u_i(a_i, x^{t-1}_{a_i}) - u_i(a_j, x^{t-1}_{a_j})$ for each $i \in N \setminus N^t$ and each $j \in N^t$. Set $\lambda^t \equiv \min_{i \in N \setminus N^t, j \in N^t} \lambda^t_{ij}$, and define x^t by:

$$\begin{aligned} x^t_{a_i} &\equiv x^{t-1}_{a_i} - \frac{|N^t|}{|N|} \cdot \lambda^t \text{ for each } i \in N \setminus N^t, \\ x^t_{a_j} &\equiv x^{t-1}_{a_j} + \frac{|N \setminus N^t|}{|N|} \cdot \lambda^t \text{ for each } j \in N^t, \end{aligned}$$

and continue to Step $t + 1$.

Theorem 1. Algorithm 2 identifies an agent-k-linked envy-free allocation in at most $|N|$ iterations.

Proof. Note that the adjustment of compensations in Step t from x^{t-1} to x^t respects budget-balance because (a, x^0) is budget-balanced, and by induction, if (a, x^{t-1}) is budget-balanced, then:

$$\sum_{i\in N} x^t_{a_i} = \sum_{i\in N} x^{t-1}_{a_i} - \frac{|N^t|}{|N|}\cdot\lambda^t\cdot|N\setminus N^t| + \frac{|N\setminus N^t|}{|N|}\cdot\lambda^t\cdot|N^t| = \sum_{i\in N} x^{t-1}_{a_i} = \alpha.$$

Note that $(a, x^0) \in \mathcal{F}(u)$. By induction, we show that if $(a, x^{t-1}) \in \mathcal{F}(u)$, then $(a, x^t) \in \mathcal{F}(u)$. Equivalently, we show for all $i, j \in N$,

$$\text{if } u_i(a_i, x^{t-1}_{a_i}) \geq u_i(a_j, x^{t-1}_{a_j}), \text{ then } u_i(a_i, x^t_{a_i}) \geq u_i(a_j, x^t_{a_j}). \tag{5}$$

If $i, j \in N^t$ or $i, j \in N \setminus N^t$, then condition (5) holds because $(a, x^{t-1}) \in \mathcal{F}(u)$ and the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical. If $i \in N^t$ and $j \in N \setminus N^t$, then condition (5) holds because $(a, x^{t-1}) \in \mathcal{F}(u)$ and $x^{t-1}_{a_i}$ is increased and $x^{t-1}_{a_j}$ is decreased. If $i \in N \setminus N^t$ and $j \in N^t$, then condition (5) holds because $(a, x^{t-1}) \in \mathcal{F}(u)$ and by definition of λ^t, $\lambda^t \leq \lambda^t_{ij} = u_i(a_i, x^{t-1}_{a_i}) - u_i(a_j, x^{t-1}_{a_j})$, i.e.:

$$\begin{aligned}
u_i(a_i, x^t_{a_i}) &= v_{ia_i} + x^t_{a_i},\\
&= v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|}\cdot\lambda^t,\\
&\geq v_{ia_i} + x^{t-1}_{a_i} - \frac{|N^t|}{|N|}\cdot\lambda^t_{ij},\\
&= u_i(a_i, x^{t-1}_{a_i}) - \lambda^t_{ij} + \frac{|N\setminus N^t|}{|N|}\cdot\lambda^t_{ij},\\
&= u_i(a_j, x^{t-1}_{a_j}) + \frac{|N\setminus N^t|}{|N|}\cdot\lambda^t_{ij},\\
&\geq v_{ia_j} + x^{t-1}_{a_j} + \frac{|N\setminus N^t|}{|N|}\cdot\lambda^t,\\
&= v_{ia_j} + x^t_{a_j},\\
&= u_i(a_j, x^t_{a_j}).
\end{aligned}$$

Because $(a, x^0) = (a, x) \in \mathcal{F}(u)$, now condition (5) yields $(a, x^t) \in \mathcal{F}(u)$.

Finally, we show that Algorithm 2 terminates in at most $|N|$ iterations. By construction of N^t, each agent $i \in N^t$ must belong to an indifference chain $g = (i, \ldots, k)$ at allocation (a, x^{t-1}). Note that at Step t, for $i \in N \setminus N^t$ and $j \in N^t$ such that $\lambda^t_{ij} = \lambda^t$, all the above inequalities become equalities and we obtain $u_i(a_i, x^t_{a_i}) = u_i(a_j, x^t_{a_j})$, $i \rightarrow_{(a,x^t)} j$ and $i \in N^{t+1}$. Note also that $N^t \subseteq N^{t+1}$ because for any $i, j \in N^t$ such that $i \rightarrow_{(a,x^{t-1})} j$, the adjustments of $x^{t-1}_{a_i}$ and $x^{t-1}_{a_j}$ are identical and we also have $i \rightarrow_{(a,x^t)} j$. Thus, $|N^{t+1}| - |N^t| \geq 1$ as long as $N \setminus N^t \neq \emptyset$. Hence, Algorithm 2 terminates in at most $|N|$ iterations. □

Example 2. From Example 1, we know that $K^0 = \{1\}$, $a_i = i$ and $x^0_{a_i} = 0$ for all $i \in N$. We next run Algorithm 2.

Step 1. From Example 1 we know that $N^1 = \{1,2\}$ and $N - N^1 = \{3,4,5\}$. From matrix (4), it is also easy to see that $\lambda^1_{3j} = 1$, $\lambda^1_{4j} = 2$ and $\lambda^1_{5j} = 3$ for all $j \in N^1$. Thus, $\lambda^1 = 1$, so $x^1 = (x^1_1, x^1_2, x^1_3, x^1_4, x^1_5) = (\frac{3}{5}, \frac{3}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5})$.

Step 2. Given the distribution x^1 identified in Step 1, the following holds:

$$(v_{ij} + x^1_j)_{i,j\in N} = \begin{pmatrix} \frac{8}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{8}{5} & \frac{8}{5} & -\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & \frac{8}{5} & \frac{8}{5} & -\frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{13}{5} \end{pmatrix}$$

Thus, when we run Algorithm 1, agent 3 is first included in N^2 (because agent 3 is indifferent between objects 1, 2 and 3) and then agent 4 is included in N^2 (because agent 4 is indifferent between objects 3 and 4). Hence, $N^2 = \{1,2,3,4\}$. Now, $\lambda^2_{51} = \lambda^2_{52} = 2$ and $\lambda^2_{53} = \lambda^2_{53} = 3$. Thus, $\lambda^2 = 2$ and, as a consequence, $x^2 = (x^2_1, x^2_2, x^2_3, x^2_4, x^2_5) = (1, 1, 0, 0, -2)$.

Step 3. Given the distribution x^2 identified in Step 2, the following holds:

$$(v_{ij} + x^2_j)_{i,j\in N} = \begin{pmatrix} 2 & 1 & 0 & 0 & -2 \\ 2 & 2 & 0 & 0 & -2 \\ 1 & 1 & 1 & 0 & -2 \\ 1 & 1 & 2 & 2 & -2 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

By construction of x^2, agent 5 is indifferent between objects 1, 2 and 5 at allocation (a, x^2). Thus, $N^3 = N$ and Algorithm 2 terminates at Step 3. □

We end this paper by stating a few remarks related to the quasi-linearity assumption. A first observation is that the existence of an envy-free allocation is not dependent on this assumption (Svensson, 1983). A second observation is that quasi-linearity often is assumed in real-world applications (see, e.g., the website `spliddit.org`) and in theoretical studies because it is easy for agents to report their preferences, i.e., agents simply have to report a vector of object valuations as illustrated in the matrix (4). For weaker assumptions on preferences, preference elicitation is more cumbersome and it will typically require repeated interactions with the agents and, normally, also some type of approximation. See, e.g., Andersson and Svensson (2018), Arunachaleswaran et al. (2019), or Su (1999).

Furthermore, quasi-linearity implicitly assumes that agents are not budget constrained. For example, if agent i has quasi-linear preferences and is indifferent between objects 1 and 2 at compensations $(x_1, x_2) = (100, 300)$, then the agent is also indifferent between these objects at compensations $(x'_1, x'_2) = (x'_1, x'_1 + 200)$ for any $x'_1 \in \mathbb{R}$. In most real-world applications, agents are budget constrained[2] meaning that agent i will not be able to afford object 1 or 2 for a "sufficiently low" compensation x'_1, or, equivalently, for a "sufficiently high" price $-x'_1$. This well-known limitation of the quasi-linearity assumption has motivated researchers to study the fair rent division problem under more general circumstances.

Budget constraints in rent division were first studied by Nicoló and Velez (2017) in the context of partnership dissolution. Procaccia et al. (2018) extended the fair rent division problem by allowing agents to report their budget restrictions. Given such reports, they provided a computationally feasible algorithm for identifying when there exists an envy-free allocation that respects the individual budget restrictions. However, their solution does not offer any recommendation in the case when the intersection between the set of envy-free and budget-balanced allocations and the set of allocations that respect the (individual) budget restrictions is empty. In a recent contribution, Velez (2020) proposed an intuitive mechanism that allows agents to inform the mechanism designer about their budget constraints on a preference domain that includes the quasi-linear domain as a special case. On this domain, the agents report their valuations of the rooms (exactly as on the quasi-linear domain) together with two additional parameters: one that represents her "housing earmark" and an index that penalizes the utility the agent gets from paying above her reported earmark amount. Velez (2020) convincingly argues that his proposed mechanism retains its practicality also on his proposed preference domain.

The mechanism proposed in this paper is defined on the quasi-linear domain, which implicitly implies that the agents in our model not are budget constrained. Algorithm 2 does not stretch beyond this domain since the adjustment of the compensations in Step t of the algorithm depends on the quasi-linearity assumption. It would, however, be interesting to investigate if these compensation adjustments can be modified to fit into Velez (2020) more realistic framework. This question is beyond the scope of this paper and it is left for future research.

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[2]Financially constrained bidders has been studied in the auction literature since the 1970s, e.g., in the pioneering contribution by Rothkopf (1977). See van der Laan and Yang (2016) for an overview.

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