Singular potentials, rigidity and recurrence in low dimensional dynamics

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2024

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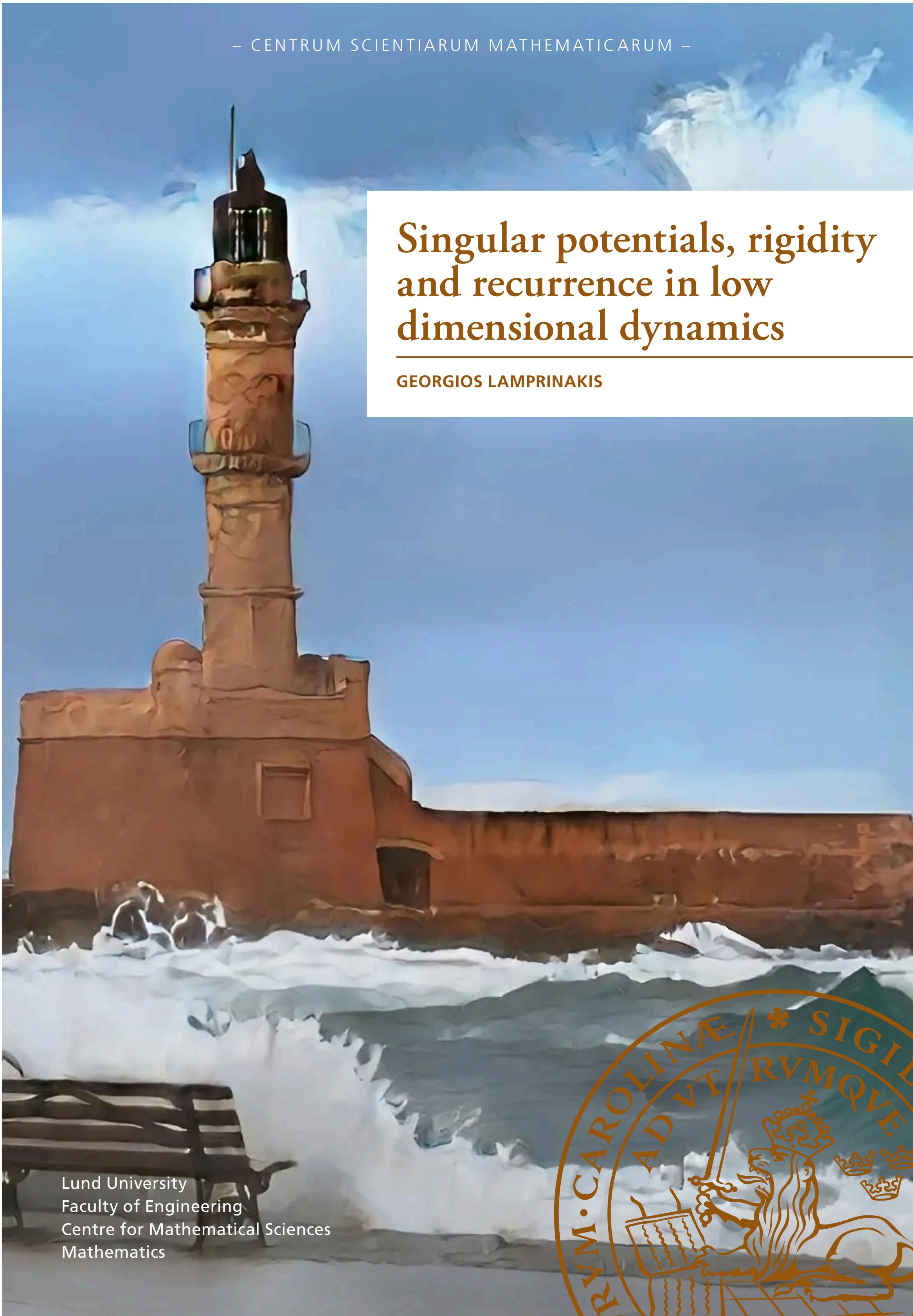

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LUND
UNIVERSITY

Doctoral Thesis
Thesis advisors: Professor Jörg Schmeling, Docent Tomas Persson
Faculty opponent: Professor Marc Kesseböhmer

To be publicly defended, with due permission of the Faculty of Engineering of Lund University, for the Degree of Doctor of Philosophy on Tuesday, 28th of May, 2024, at 10:00 in the Hörmander lecture hall at the Centre of Mathematical Sciences, Sölvegatan 18A, Lund.

DOKUMENTDATABLAD enl SIS 61 41 21

Organization **LUND UNIVERSITY** Centre of Mathematical Sciences Box 118 SE–221 00 LUND Sweden	Document name **DOCTORAL DISSERTATION**
	Date of disputation 2024-05-28
Author(s) Georgios Lamprinakis	Sponsoring organization

Title and subtitle

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Abstract

In Papers I and II we study the behaviour of the equilibrium measure and the Birkhoff sums for a singular potential over the doubling map. A complete multifractal analysis for the the Birkhoff sums and the equilibrium measure (for the 'appropriate' scaling) is given in Paper I. In Paper II we prove a parameter continuity property of the pressure function for a family of singular potentials. In the third paper we study some properties of the invariant sets of a general expanding Markov map of the circle and investigate a rigidity related question, proving that for a fixed such map there are not many (in the topological sense) other maps so that they share common compact invariant sets of 'small' Hausdorff dimension. The last project deviates from the previous ones and focuses on recurrence properties of hyperbolic systems and specifically, hyperbolic automorphisms of the two-dimensional torus. For such a system, we provide a formula for the Hausdorff dimension of the so-called uniform recurrence set.

Key words

Multifractal analysis, thermodynamic formalism, singular potentials, expanding Markov maps, rigidity, hyperbolic automorphisms, recurrence

Classification system and/or index terms (if any)

Supplementary bibliographical information		Language English
ISSN and key title 1404-0034 Doctoral Thesis in Mathematical Sciences 2024:2		ISBN 978-91-8104-042-5 (print) 978-91-8104-043-2 (pdf)
Recipient's notes	Number of pages 168	Price
	Security classification	

Signature ______________________ Date 2024-04-10

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Funding information: The thesis work was supported by the Swedish Research Council (VR).

Faculty of Engineering, Centre of Mathematical Sciences

Doctoral Thesis in Mathematical Sciences 2024:2
ISSN: 1404-0034

ISBN: 978-91-8104-042-5 (print)
ISBN: 978-91-8104-043-2 (pdf)
LUTFMA-1086-2024

Printed in Sweden by Media-Tryck, Lund University, Lund 2024

Media-Tryck is a Nordic Swan Ecolabel certified provider of printed material. Read more about our environmental work at www.mediatryck.lu.se

MADE IN SWEDEN

Dedicated to my parents
and to Danai

Στους γονείς μου
και στη Δανάη

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Paper IV: On uniform recurrence for hyperbolic automorphisms of the 2-dimensional torus 117

List of publications

This thesis is based on the following four papers.

Paper I **Fast dimension spectrum for potential with logarithmic singularity**
P. Gohlke, G. Lamprinakis, J. Schmeling
Journal of Statitstical Physics, **191**, 2024

Paper II **On a family of singular potentials: Parameter dependence of the pressure function**
P. Gohlke, G. Lamprinakis, J. Schmeling
To be submitted

Paper III **Structure and dimension of invariant subsets of expanding Markov maps and joint invariance**
G. Lamprinakis
Dynamical Systems, **38**:3, p.405–426, 2023

Paper IV **On uniform recurrence for hyperbolic automorphisms of the 2-dimensional torus**
G. Lamprinakis, T. Persson and A. Rodriguez Sponheimer
Submitted

Papers I, II and IV are the result of joint effort from the respective authors.

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Acknowledgements

Firstly, I would like to sincerely thank my two supervisors Jörg Schmeling and Tomas Persson. It is hard to express my gratitude towards the both of them in a mere few lines. They truly have been role models for me and undoubtedly, I would not have been able to complete this journey without their help. I specifically thank Jörg Schmeling for his constant support and for his help. His passion and his vast and deep knowledge in mathematics, that he so generously shared with me for the past years, have been a real inspiration and truly helped me discover new depths in mathematics but also in my love of them. I also thank Tomas Persson for his support and belief in me and for always finding the time for me. I truly enjoyed all of our discussions and thank you for all the advice and everything you taught me. Thank you also for reminding me that mathematics, above all, is fun!

I would also like to express my gratitude to Philipp Gohlke for being kind and patient enough to share his knowledge and discuss mathematics with me. I truly enjoyed our collaboration so far and I hope for more in the future. I also thank you for all of our discussions beyond mathematics and for the great times we shared.

I would like to express my sincere thanks to the rest of the dynamics group, Magnus, Mats, Weiwei, Alejandro, Charis for creating such a great working environment. In particular, I thank Mats for all the help with everything since even day one and Alejandro for our excellent collaboration.

I thank of course all my colleagues for the wonderful and warm environment. Especially I thank Julio, Olof, Alexia, Alexandros, Germán, Magnus, Jonathan, Alex, Adem, Joakim, Mårten, Måns, Eskil and Jaime. Special thanks to Jacob Stordal Christiansen, Magnus Goffeng and Carl Olsson for their efforts to create the best possible working and studying environment. Many thanks also to Mikael Abrahamsson, Jessica Kareseit, Hanne Nordkvist and Sara Ingelstrand for always being so kind and helpful.

Many thanks also to Nikos Korkakakis for letting me use his picture for my cover.

I would also like to mention my friends and colleagues from Athens, Thanasis and Panagiotis, with whom we shared this journey, even though we were in different places. Special thanks also to Odysseas Bakas who, apart from being a great mathematician, is a wonderful person who supported me a lot, especially during the beginning of my studies and devoted many hours of his time to discuss with me.

Many thanks to my dear friends Giannis, Panos and Petros for all the support and the great times we have shared throughout all these years. You made this trip a little bit easier for me.

Of course, I owe many thanks to my family and especially to my father, Fotios, my mother, Aikaterini and my sister, Eleni, for always being there for me and for loving me unconditionally. I am grateful beyond words for the freedom you provided in order for me to discover and pursue what I truly love and for all the immense support.

Lastly but certainly not least, I would like to thank my partner Danai, who was the one that had to put up with me on a daily basis all this time and sacrificed a lot to be able to be here on my side. Thank you for all the support and love. I would not have been able to do it without you.

Popular summary

The core objective in the theory of dynamical systems is to study the long-term behaviour of a system as it changes over time under some specific laws. Dynamical systems have application in various fields such as physics, economics, biology and so on, for which the one needs mathematical models in order to obtain a more rigorous study of these systems and have a better understanding of their behaviour. Of course, these mathematical models may be simplifications or approximations of the actual real-life systems. This creates the need of studying the 'stability' of a system in the sense that we would like to know how the dynamics change after a (small) perturbation.

This leads naturally to the notion of *rigidity*, which is the study of sufficient conditions under which two systems share the same dynamical properties. It is of interest to be able to determine whether two systems coincide, given a limited amount of information. In other words, we would like to know whether is is sufficient to consider just a 'part' of a system—a 'subsystem' in other words—in order to uniquely determine the dynamics of the whole system. Of course, from a mathematical point of view, changing the dynamical setting may require a very different approach and often carries some unique, to the specific setting, difficulties. Furthermore, requesting results for a very general family of dynamical systems often imposes some non-trivial problems.

One key property that is often studied in dynamical systems is that of *recurrence*, which describes the property of a system to return back to the same state repeatedly, in finite steps (time). The importance of recurrence in dynamical systems is that it provides predictability, which is the core objective in dynamics, as mentioned before. Clearly, the recurrence properties depend on the dynamics of the system. Many other parameters can be put in that question, as of the 'speed' of return, the first return-time or how often it returns etc.

Typically, even these often simplified, mathematical models appear to be challenging to completely analyse their behaviour. This often creates the need to develop methods and measurements that provide a good understanding of the *average* behaviour of the system at least for *'most'* points. There are many—often 'conflicting'—notions for determining the size of a set. For example we may be able to describe a large proportion in a probabilistic sense of our system but the set containing the excluded points may still be geometrically rich and very informative for the (local) behaviour of the system. This creates the need to further study the properties and structure of the set of these points, which may be neglected by our previous measurements, as it provides a better understanding of the system overall. This represents the basic motivation for the so-called *multifractal analysis*.

Multifractal analysis is very closed related to the *thermodynamic formalism*, which provides some powerful tools to analyse the behaviour of the dynamical system which is subject to some *potential function* (energy). Historically, in thermodynamics and statistical mechanics, the aim was to study the microscopic behaviour, say, of a single particle, and how it leads to macroscopic conclusions for the whole system. As it turned out, thermodynamic formalism found application in a much wider family of mathematical models, which motivated its mathematical generalization. Consequently, it is of great interest to study the potential function, as it informs us about the underlying dynamics of the system. The regularity of the potential is of great importance, in order to have full access to the tools of the (classic) thermodynamic formalism. Often though, *singular potentials* are of interest, as they appear naturally in both physical and mathematical contexts. That being said, the absence of sufficient regularity, makes them much more challenging mathematically to handle.

This thesis is concerned with questions of this type and it contributes to establishing some new results, relying on some previous well-known ones, but also by developing new ideas and methods for each specific problem.

Part I

Introduction and Results

Chapter 1

Introduction

The goal of this chapter is to give a brief exposition of the theory of dynamical systems and to provide an overview of the results that motivated our work. Firstly some well-known notions and results are provided, such as the powerful and very useful tool of coding a system, which is widely applied in dynamical systems and which we used extensively in all of our papers. In Papers I and II we study the behaviour of the equilibrium measure and the Birkhoff sums for a singular potential over the doubling map. In the third paper we study some properties of the invariant sets of a general expanding Markov map of the circle and investigate a rigidity related question. The last project deals with the recurrence properties of hyperbolic systems and specifically, hyperbolic automorphisms of the two-dimensional torus.

1 Dynamical systems

A *discrete dynamical system* is considered to be a function f acting on a set (*phase space*) X to itself, often written as (X, f). The main objective in the theory of dynamical systems is to study its long-term behaviour. In the case of discrete dynamical systems, i.e. for discrete time, this question can be restated as follows: for a point $x \in X$, can we accurately know the behaviour of its orbit,

$$Orb_f(x) := \{x, f(x), f^2(x), f^3(x), \dots\}$$

and make precise predictions of how the system will develop overtime. In general, this question is very hard to answer, even after imposing restrictions and endowing our system with good properties (compactness, metrizability, continuity, smoothness, linearity etc).

Nonetheless, interesting results are acquired, given some appropriate properties on our system.

We assume that X is a compact metric space; usually the torus $\mathbb{T}$—which we also denote as $[0,1)$, to be understood as the unit interval where 0 and 1 are identified—or $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$. Assuming also that the function f is continuous, the pair (X, f) is called a *topological dynamical system*. If further regularity properties are assumed for f, then (X, f) denotes a *smooth system*.

The concept of invariance is important as it gives a better understanding of the dynamics. In general it is important to study *fixed points*, i.e. the points so that $f(x) = x$ and, more broadly, the *periodic points*, i.e. the points so that $f^n(x) = x$, for some $n \in \mathbb{N}$. More generally, it is of interest to study the *invariant sets* of a map, i.e. the sets of $Y \subset X$ so that $f(Y) = Y$. In many cases the system (X, f) is equipped with a measure ν, which we assume to be defined on the Borel σ-algebra, $\mathcal{X}$, of the metric space X. We say that ν is f-invariant, if $\nu = f_*\nu$, where $f_*\nu(A) := \nu(f^{-1}A)$, for all $A \in \mathcal{X}$. We denote by $\mathcal{M}_f(X)$ (or simply $\mathcal{M}_f$ when the ambient space is clear) the space of f-invariant Borel probability measures on X. For a $\nu \in \mathcal{M}$, we call $(X, \mathcal{X}, f, \nu)$, or simply (X, f, ν), a *measure-preserving dynamical system*.

We say that two topological systems, $f\colon X \to X$ and $g\colon Y \to Y$, are *topologically conjugate* if there exists a homeomorphism $h\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes, i.e. $f \circ h = h \circ g$ and we write $f \overset{h}{\sim} g$. If h is surjective but not injective, then we say that f *topologically semiconjugates* to g and then (Y, g) is a *factor* ('subsystem') of the system (X, f). Observe that if two systems that topologically conjugate then they share the same dynamical properties from a topological point of view and it is a natural approach for the classification of dynamical systems.

A function $f\colon [0,1) \to [0,1)$ is called an *expanding Markov map* if, f is a local homeomorphism and there exist finitely many points $x_i \in [0,1)$ such that $f([x_{i-1}, x_i]) = [0,1)$, $f \in C^\alpha(x_{i-1}, x_i)$ and a $\gamma > 1$ so that $|f'(x)| > \gamma$ for all $x \in [0,1)$, on which f is (one-sided) differentiable. We call the intervals $I_i = [x_i, x_{i+1}]$, fundamental intervals. A paradigm of such maps is the doubling map T_2 which is defined by $T_2(x) = 2x \pmod 1$. In a similar fashion, for an $m \in \{2, 3, \ldots\}$ we can define the *linear* map of the circle of m full branches, $T_m\colon x \mapsto mx \pmod 1$. These systems are in general of great interest, one reason being that they share an intimate relation to discrete-time Markov processes. For a more thorough exposition see [Pes97].

The *hyperbolic systems* is another interesting case of dynamical systems inside the realm of smooth dynamics. They have the distinct property of exponential speed of divergence of nearby orbits and they are characterized by the fact that there is a continuous splitting of the tangent space into two parts, the *expanding (unstable)* subbundle and the *contracting (stable)* subbundle. More precisely, consider now X to be a *smooth Riemannian manifold* and $f\colon X \to X$ a diffeomorphism. Then f is called *(uniformly) hyperbolic* or *Anosov diffeomorphism* if for every $x \in X$ there exist $E^u(x)$, $E^s(x) \subset T_xX$, such that $T_xX = E^s(x) \oplus E^u(x)$ and there are constants $C > 0$ and $\lambda > 1$ such that for every $n \in \mathbb{N}$ one has

$$\|D_x f^n(u)\| \leq C\lambda^{-n}\|u\| , \qquad \text{for } u \in E^s(x)$$

and

$$\|D_x f^n(u)\| \geq C\lambda^{n}\|u\| , \qquad \text{for } u \in E^u(x).$$

$E^s(x)$ and $E^u(x)$ are called *stable manifold* and *unstable manifold* respectively. A classic example that illustrates the hyperbolic behaviour is Arnold's cat map $T\colon \mathbb{T}^2 \to \mathbb{T}^2$, which is a linear diffeomorphism defined by a matrix A of the form $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The matrix A admits the two positive eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda_2 = \lambda_1^{-1} = \frac{3-\sqrt{5}}{2}$. Here, the unstable and stable manifold are the eigenlines corresponding to λ_1 and λ_2 respectively. In general the linear maps on $\mathbb{T}^2$ defined by a matrix A, with integer entries and $\det(A) = \pm 1$ admitting eigenvalues λ_1 and λ_2 so that $0 < |\lambda_2| < 1 < |\lambda_1|$ are denoted as *hyperbolic automorphisms*. We refer to [KH95] for more details.

1.1 Shift spaces and coding

Let $\mathbb{S} = \mathbb{S}_d$ denote the *shift space* $\Sigma^+ = \Sigma_d^+ = \mathcal{A}^{\mathbb{N}}$ or $\Sigma = \Sigma_d = \mathcal{A}^{\mathbb{Z}}$ corresponding to the *alphabet* or *symbols*, $\mathcal{A} = \{0, 1, \ldots, d-1\}$. The topology on $\mathbb{S}$ is the product topology and it is a compact, metrizable topological space. A compatible metric is

$$d_{\mathbb{S}}(\underline{x}, \underline{y}) = \varrho^{-k(\underline{x},\underline{y})}$$

where, $\varrho > 1$ and $k(\underline{x}, \underline{y}) = \max\{n \in \mathbb{N} :\ x_i = y_i, \text{ for all } |i| \leq n\}$. Usually, but not exclusively, for the shift space $\mathbb{S}_d$, we set $\varrho = d$. Let σ be the regular shift operator on $\mathbb{S}$, such that $\big(\sigma(\underline{x})\big)_i = x_{i+1}$, for all $\underline{x} \in \mathbb{S}$. The shift operator acts continuously on $\mathbb{S}$.

We call *block* or *word* a finite string of symbols chosen from the alphabet $\mathcal{A}$ which we denote as

$$[a_m \dots a_n]$$

$m, n \in \mathbb{N}$ or $\mathbb{Z}$, $m \leq n$. We call the set

$$C_{[a_m \dots a_n]} = \{x \in \Sigma : \; x_i = a_i \text{ for all } m \leq i \leq n\}$$

a *cylinder*, where $m, n \in \mathbb{N}$ or $\mathbb{Z}$, $m \leq n$ and $a_i \in \{0, 1, \dots, d-1\}$ for $m \leq i \leq n$. For a $\underline{w} \in \mathcal{A}^n$, $n \in \mathbb{N}$, we denote $[\underline{w}]$ the set of all $\underline{x}$ so that $x_i = w_i$, $i \in \{1, 2, \dots n\}$. For a $\underline{y} \in \mathbb{S}$ we denote as $C_n(\underline{y})$ the set of $\underline{x} \in \mathbb{S}$ so that $x_i = y_i$, $|i| \leq n$.

We also endow Σ^+ with the lexicographic order, i.e. if $\underline{a}$, $\underline{b} \in \Sigma_m$, then $\underline{a} < \underline{b}$ if there exists $i_0 \in \mathbb{N}$ such that $a_i = b_i$ for all $1 \leq i < i_0$ and $a_{i_0} < b_{i_0}$.

Let S be a closed subset of $\mathbb{S}$ that is invariant under σ. Any closed invariant subset of $\mathbb{S}$ is determined by a countable collection of forbidden words. Such an S is a *subshift of finite type (SFT)* if there exists a finite list of *forbidden words* $\mathcal{F}$ such that a point $\underline{x} \in \mathbb{S}$ is in S if and only if $\underline{x}$ contains no blocks from $\mathcal{F}$. Of course the whole shift space is an SFT. A forbidden block $w = [w_1 \dots w_\ell]$ can also be described as the collection of larger blocks; for example,

$$\{[w_1 \dots w_\ell 0], [w_1 \dots w_\ell 1], \dots, [w_1 \dots w_\ell (d-1)]\}.$$

Thus we can assume if needed, that all the forbidden words are of the same length, equal to that of the longest of the initial forbidden blocks.

Any subshift of finite type can also be represented by a $d^{\ell-1} \times d^{\ell-1}$ matrix, $\Gamma = (\gamma_{ij})$, with entries in $\{0, 1\}$, where ℓ is the length of the longest forbidden word and $\gamma_{ij} = 1$ when it corresponds to an allowed block and $\gamma_{ij} = 0$ otherwise. The matrix Γ is called *transfer matrix*. Therefore, we sometimes denote a subshift of finite type with transfer matrix Γ as $\Sigma_\Gamma^+ \subset \Sigma^+$ or $\Sigma_\Gamma \subset \Sigma$. For a more thorough exposition we refer to [DGS76] or [LM95].

A *Markov partition* for an expanding Markov map of the circle f is a finite cover $\mathcal{P} = \{P_1, \dots, P_d\}$ of $[0, 1)$ such that

1. each P_i is the closure of its interior, intP_i
2. int$P_i \cap$ int$P_j = \emptyset$
3. each $f(P_i)$ is a union of elements in $\mathcal{P}$.

An expanding Markov map has a Markov partition of arbitrary small diameter. If $\mathcal{P} = \{P_1, \ldots, P_d\}$ is a Markov partition then $([0,1), f)$ can be represented, in a natural way, by a subshift of finite type, Σ_Γ in Σ^+ corresponding to the transfer matrix $\Gamma = (\gamma_{ij})$, where

$$\gamma_{ij} = \begin{cases} 1, & \mathrm{int} P_i \cap f^{-1}(\mathrm{int} P_j) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

This gives a coding map $\pi : \Sigma_\Gamma \to [0,1)$,

$$\pi(\underline{x}) = \bigcap_{j \in \mathbb{N}} f^j(P_{x_j}) , \qquad \text{for } \underline{x} = (x_1 x_2 x_3 \ldots) \in \Sigma_\Gamma \subset \Sigma^+$$

so that $\pi \circ \sigma = f \circ \pi$. For details see for example [Pes97].

Let A be a hyperbolic, area preserving matrix with integer entries and $T : \mathbb{T}^2 \to \mathbb{T}^2$ be defined by $T(x) = Ax \pmod 1, \forall x \in \mathbb{T}^2$. A Markov partition for the system $(\mathbb{T}^2, T)$ is a finite cover $\mathcal{P} = \{P_0, \ldots, P_{d-1}\}$ of $\mathbb{T}^2$ such that,

1. each P_i is the closure of its interior, $\mathrm{int}P_i$ and P_i is convex
2. $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$
3. whenever $x \in \mathrm{int}P_i$ and $T(x) \in \mathrm{int}P_j$, then $W^u_{P_j}\big(T(x)\big) \subset T\big(W^u_{P_i}(x)\big)$ and $T\big(W^s_{P_i}(x)\big) \subset W^s_{P_j}\big(T(x)\big)$.

where, $W^u_{P_i}(y)$ denotes the intersection of the local unstable manifold of y with the element P_i and $W^s_{P_i}(y)$ is defined analogously. For such a system (hyperbolic automorphisms) the elements of the partition can be constructed so that they are parallelograms with sides parallel to the eigendirections. Let $\mathcal{P} = \{P_0, \ldots, P_{d-1}\}$ be a Markov partition of the system $(\mathbb{T}^2, T)$. Then $(\mathbb{T}^2, T)$ can be represented symbolically by a subshift of finite type, Σ_Γ in Σ_d corresponding to the transfer matrix $\Gamma = (\gamma_{ij})$, $i, j \in \{0, 1, \ldots, d-1\}$ where

$$\gamma_{ij} = \begin{cases} 1, & \mathrm{int} P_i \cap T^{-1}(\mathrm{int} P_j) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

This gives rise to the coding map $\pi \colon \Sigma_\Gamma \to \mathbb{T}^2$,

$$\pi(\underline{x}) = \bigcap_{j \in \mathbb{Z}} T^j(P_{x_j}) , \qquad \text{for } \underline{x} = (\ldots x_{-n} \ldots x_{-1} x_0 x_1 \ldots x_n \ldots) \in \Sigma_\Gamma \subset \Sigma$$

so that $\pi \circ \sigma = T \circ \pi$.

In general coding is very useful in dynamics, as it generates a symbolic model of the dynamical system. The Markov partition allows one to transfer the dynamics in a setting which is much more convenient for computations. In particular, for linear systems, the symbolic representation constitutes of the correct metrical correspondent, allowing us to compute 'metric quantities' such as the topological entropy, Hausdorff dimension etc. This is showcased in [Fur67] in the following result.

Theorem 1. *Let A denote a compact invariant set of (Σ_m, σ) and A^* the corresponding invariant set on $([0,1], T_m)$. Then,*

$$\dim_H(A^*) = \dim_{box}(A^*) = \frac{\mathrm{h_{top}}(A)}{\log m}.$$

Adler [Adl98] proved that for a hyperbolic automorphism, we can find a Markov partition so that the spectral radius of the transfer matrix is equal to that of the original matrix A. For the two dimensional torus the results in [Sna91] are also sufficient. In particular we will use the following result.

Theorem 2. *Let A be a hyperbolic 2×2 integer matrix acting on $\mathbb{T}^2$ and let λ be its largest, in absolute value, eigenvalue. Then there is always a Markov partition for which the transfer matrix Γ has the same largest, in absolute value, eigenvalue.*

Corollary 1. *Consider the system $T\colon x \mapsto Ax \pmod 1 \colon \mathbb{T}^2 \to \mathbb{T}^2$. Let Σ_Γ be the corresponding coding space and λ be the eigenvalue of A, of modulus larger than 1. Then, the entropy of Σ_Γ is equal to $\log|\lambda|$.*

2 Thue–Morse measure and g-measures

Firstly we introduce the *Thue–Morse measure* which can be written as an infinite Riesz product on $\mathbb{T}$

$$\mu = \mu_{\mathrm{TM}} = \prod_{m=0}^{\infty} (1 - \cos(2\pi 2^m x))$$

to be understood as the limit of absolutely continuous probability measures with respect to the weak topology. The Thue–Morse measure arises as the diffraction measure of the Thue–Morse substitution [BG13] and it is the *unique equilibrium* measure of a *singular potential*. This will become more precise, after we introduce the notion of *g-measures*; for a more thorough exposition on g-measures see [Kea72] and [Led74].

Consider T_m, $m \in \{2, 3, \ldots\}$, defined by $x \mapsto mx \pmod 1$. Let ν be a Borel probability measure on $\mathbb{T}$. Assume also that ν is invariant under the action of T_m.

Define the measure ν' by taking the local lift of ν, i.e. via the inverse branches of T_m, $x \mapsto \frac{x+i}{m}$, for $i = 0, 1, \ldots, m-1$. The measure ν' is absolutely continuous with respect to ν and thus we can consider the Radon–Nikodym derivative $g = \frac{\mathrm{d}\nu}{\mathrm{d}\nu'}$. One can easily verify that,

$$\sum_{y \in T_m^{-1} x} g(y) = 1. \tag{1.1}$$

for ν-almost every x. With this in mind, consider a $g \colon \mathbb{T} \to [0, 1]$ satisfying the relation (1.1) for all $x \in \mathbb{T}$. We call such a function a *g-function*. A probability measure ν is called a g-measure if its Radon-Nikodym derivative with respect to ν' is equal to g, ν-almost everywhere. Now for such a g and ν we consider the map Φ_g which sends the measure ν to $\Phi_g \nu$ so that

$$\frac{\mathrm{d}\Phi_g \nu}{\mathrm{d}\nu'} = g.$$

The map Φ_g is well defined and send probability measure to probability measures. The map Φ_g can be also understood as follows. We also define the map ϕ_g acting on the space of continuous functions, by

$$(\phi_g f)(x) = \sum_{y \in T_m^{-1} x} g(y) f(y).$$

If g is continuous then $\phi_g f$ is also continuous. For a probability measure ν, we have that

$$\int_{\mathbb{T}} \phi_g f \, \mathrm{d}\nu = \int_{\mathbb{T}} f \, \mathrm{d}\Phi_g \nu.$$

In other words, the map Φ_g is the dual operator of ϕ_g and $\Phi_g \nu$ is nothing else than $(\phi_g)_* \nu$. The g-measures are generated (via a fixed point theorem) as eigenmeasures of the continuous operator ϕ_g.

There is a strong connection between g-measures and invariant measures as it is showcased by the results of Keane [Kea72] below.

Theorem 3. *A probability measure ν is T_m-invariant if and only if it is a g-measure for some $g \colon \mathbb{T} \to [0, 1]$ that satisfies* (1.1) *for all $x \in \mathbb{T}$. For each continuous such g there exists at least one g-measure.*

This gives rise to questions about the uniqueness of g-measures. This requires some stronger regularity conditions for the function g and the number of zeros of g, as showcased in the following result [Kea72].

Theorem 4. *Let $g \colon \mathbb{T} \to [0, 1]$ that satisfies* (1.1) *for all $x \in \mathbb{T}$. Assume also that g is of class C^1 and that it takes the value zero in no more than one point in $\mathbb{T}$. Then there exists exactly one g-measure ν. Furthermore, ν is weakly mixing.*

It is worth noting that one can interpret this procedure as how likely it is to have taken x from each of its preimages. More concretely, it constructs a stochastic process by considering an $x \in \mathbb{T}$ and mapping it to one $y \in T_m^{-1}(x)$ according to the probability vector $(g(y))_{y \in T_m^{-1}(x)}$.

The notion of g-measures appears to be closely related to the dynamical systems as it produces invariant measures. In fact the nice properties of the g-measures and its relation with the g-function are very useful in many areas of dynamics. A particular case that is of interest in this thesis, is their application in the so-called *multifractal analysis* [BGKS19, Fan97, Oli99].

3 Multifractal analysis

We firstly introduce the general concept of *multifractal analysis*. Let X be a nonempty set, $Y \subset X$ a nonempty subsest and consider a function $g\colon Y \to [-\infty, +\infty]$. Consider the *level sets* $X_{g,\alpha} = \{x \in Y : \ g(x) = \alpha\}$. In this way, we obtain the respective decomposition of the phase space

$$X = X \setminus Y \cup \bigcup_{\alpha \in \overline{\mathbb{R}}} X_{g,\alpha}.$$

It is often that the structure of the level sets is complex which makes the analysis of the decomposition impractical. Various notions of "size" of the *level sets* $X_{g,\alpha}$ are useful in order to attain more numerically accessible description that gives information about their structure. A very common and natural quantification of this type comes from considering the Hausdorff dimension of the level sets which enlighten us about their geometric structure.

In dynamics we are interested in creating such a decomposition considering quantities that are compatible with the dynamical system under study and give information about the local or global behaviour. Some common examples are invariant measures, Birkhoff averages, Lyapunov exponents etc.

Consider the so called *dimension spectrum* corresponding to an invariant measure. Given a point $x \in X$ and a Borel probability measure ν on X, we define the *upper* and *lower pointwise dimensions* of ν at x by,

$$\overline{d_\nu}(x) = \limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \qquad \underline{d_\nu}(x) = \liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

If the upper and lower dimensions coincide, we call the common value *pointwise dimension* of ν at x and we write just $d_\nu(x)$. Roughly speaking, the pointwise dimension denotes the (local) scaling exponents of the measure ν, in the sense that, the measure ν behaves locally as $r^{d_\nu(x)}$. It is worth noting that, in general, it is not true that the lower and upper pointwise dimensions coincide [PW96, BS00]. In this setting, by considering the level sets,

$$X_\alpha = \{x \in X : \ d_\nu(x) = \alpha\}$$

where $\alpha \in \overline{\mathbb{R}}$, the multifractal analysis of X offers a more refined description of the scaling properties of the measure ν, which yields the decomposition

$$X = \bigcup_{\alpha \in \overline{\mathbb{R}}} X_\alpha \cup \{x \in X : \ \text{the pointwise dimension } d_\nu(x) \text{ does not exist}\}.$$

For the maps under consideration, i.e. expanding maps, this decomposition can consist of various large sets in the topological sense. More precisely, the level sets X_α can be dense for a plethora of α's which makes difficult to draw any conclusions from this decomposition. This gives motivation for the definition of the *dimension spectrum*, given by considering the Hausdorff dimension of each of the level set X_α,

$$f(\alpha) = f_\nu(\alpha) = \dim_H \{x \in X : \ d_\nu(x) = \alpha\}$$

which gives information about the size of the level sets.

Of course, numerous of interesting and natural questions arise from such a decomposition, like, how large is the set $\mathcal{A} = \{\alpha \in \overline{\mathbb{R}} : \ \exists x \in X \text{ s.t. } d_\nu(x) = \alpha\}$, giving the maximal and minimal scaling of the measure ν (wherever the pointwise dimension exists). Another fundamental question is to determine the size of the *exceptional set*

$$\{x \in X : \ \text{the pointwise dimension } d_\nu(x) \text{ does not exist}\}.$$

The multifractal analysis and especially the analysis of the dimension spectrum is strongly connected to the so-called *thermodynamical formalism* and in particular, to the *pressure function* which in turn motivates the study of *equilibrium measures*.

4 Thermodynamic formalism

In this section we give a brief overview of the field of *thermodynamic formalism* which provides us with some useful tools that allow us to study the behaviour of chaotic dynamics. One central objective in thermodynamic formalism is to establish variational principles, i.e. maximizing specific quantities. The methods have roots in classical statistical mechanics and were extended in the infinite one-dimensional classical lattice by

Ruelle [Rue68]. One tries to maximize the *topological pressure function* (free energy) $P = P(\psi)$ associated to a *potential* (energy) ψ, generating equilibrium measures, each of which corresponds to an observable state of the system. We follow the classical definition of topological pressure given in the pioneering works of Bowen [Bow70], Sinai [Sin72] and Ruelle [Rue78]; see also [Pes97]. We omit here the general definition of the topological pressure and conveniently present its definition for the specific case of a subshift of finite type. Consider $S \subset \Sigma^+$ to be a subshift of finite type and $\psi\colon S \to \mathbb{R}$. The topological pressure is defined by

$$\mathcal{P}_{top}(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\underline{w}\in\mathcal{A}^n} \sup_{\underline{x}\in[\underline{w}]\cap S} \exp\left(\sum_{i=0}^{m-1} \psi(\sigma^i \underline{x})\right)$$

when the limit exists. This specific setting is not restrictive by any means for the (general) definition of the topological pressure. On the other hand, it is worth noting that the specific choice was not only for presentational convenience. The symbolic model of the doubling map (which will be our main focus) via the 2-expansion of a real number, allows us to translate the problem in the symbolic case. In the same spirit we proceed with defining and stating results for the specific case of subshifts of finite type, without that meaning that they cannot be proven and presented in a far more general context. That being said, we sometimes jump back and forth between the initial system $(\mathbb{T}, T_2)$ and its symbolic counterpart (Σ_2^+, σ) without any special mention.

Let us consider a continuous function $\psi\colon S \to [-\infty, +\infty]$ which we denote as the potential. We define the variational pressure of the potential ψ by

$$\mathcal{P}_{var}(\psi) = \sup_{\nu\in\mathcal{M}_\sigma} \left\{ h_\nu(S) + \int_S \psi(\underline{x})\, \mathrm{d}\nu(\underline{x}) \right\}.$$

A measure $\nu_0 \in \mathcal{M}_\sigma$ is called an *equilibrium measure*, if

$$\mathcal{P}_{var}(\psi) = h_{\nu_0}(S) + \int_S \psi(\underline{x})\, \mathrm{d}\nu_0(\underline{x}).$$

The *Gibbs measures* are of particular interest, since they provide an invariant measure which can be determined approximately by the respective potential function. Bowen [Bow70] proved the following classic result.

Theorem 5. *Let $S \subset \Sigma^+$ a subshift of finite type. Let $\psi : S \to \mathbb{R}$ be a Hölder continuous function. Then there exists a unique probability measure ν that is invariant under the shift operator σ for which one can find constants $c_1, c_2 > 0$ and $P \in \mathbb{R}$, such that*

$$c_1 \le \frac{\nu\left(C_n(\underline{x})\right)}{\exp\left(-Pm + \sum_{i=0}^{m-1} \psi(\sigma^i \underline{x})\right)} \le c_2.$$

In particular, $P = \mathcal{P}_{top} = \mathcal{P}_{var} = h_\nu(S) + \int_S \psi(\underline{x})\, \mathrm{d}\nu(\underline{x})$ *and* ν *is the unique invariant probability measure with this property. Also* ν *is mixing.*

In comparison to the g-measures, the Gibbs measures also sprout as eigenmeasures of an appropriate operator and in particular, of the transfer operator

$$(\mathcal{L}_\psi f)(\underline{x}) = \sum_{y \in \sigma^{-1}\underline{x}} e^{\psi(\underline{y})} f(y)$$

via Ruelle's Perron–Frobenius Theorem; see [Bow70] and [Rue68].

A characterization of equilibrium measures was given by Ledrappier [Led74] connecting the notion of equilibrium measures to g-measures. In particular, for good g-functions, the g-measure also carries the Gibbs property.

Theorem 6. *Let* g *be a continuous* g*-function on the shift space* (Σ^+, σ)*. Then,* $\mathcal{P}_{var}(\log g) = 0$ *and* $\nu \in \mathcal{M}_\mathcal{T}$ *is a* g*-measure if and only if it is an equilibrium measure for the potential* $\psi = \log g$*. In particular, if* g *is also strictly positive, it has a unique* g*-measure and it is also a Gibbs measure.*

To this end, we emphasize that the Thue–Morse measure

$$\mu = \mu_{\mathrm{TM}} = \prod_{m=0}^{\infty} (1 - \cos(2\pi 2^m x))$$

can be interpreted as a g-measure under the doubling map $(\mathbb{T}, T_2)$, for the g-function

$$g(x) = \frac{1}{2}\big(1 - \cos(2\pi x)\big).$$

Observe that this function satisfies the conditions of both Theorem 4 and Theorem 6, which confirms our claim in the beginning of the section, that the Thue–Morse measure is the unique equilibrium measure of the potential $\psi(x) = \log(1 - \cos(2\pi x)) - \log 2$, which has a (logarithmic) singularity at 0. Furthermore, consider the natural coding of the doubling map via the 2-expansion. By taking the pull-back of the measure via the coding map, we can assume that $\mu = \mu_{\mathrm{TM}}$ is defined in the whole shift space $\Sigma^+ = \{0,1\}^{\mathbb{N}}$. Furthermore observe that, in the same spirit, if we consider the pull-back of the g-function g described above, we have that the measure μ (now considered on Σ^+) is the unique equilibrium measure for (the pull-back of) the potential $\log(g)$ (now considered acting on Σ^+), which has a singularity at the preimages (through the coding map) of the origin, 0^∞ and 1^∞.

Furthermore, in this context, we consider the upper and lower pointwise dimensions of a measure ν at $\underline{x}$ by,

$$\overline{d_\nu}(\underline{x}) = \limsup_{n\to\infty} \frac{\log \nu(C_n(\underline{x}))}{-n\log 2} \qquad \underline{d_\nu}(\underline{x}) = \liminf_{n\to\infty} \frac{\log \nu(C_n(\underline{x}))}{-n\log 2}$$

since we can consider the balls of radius 2^{-n} which are exactly the cylinders of length n and the dimension spectrum

$$f(\alpha) = f_\nu(\alpha) = \dim_H \left\{\underline{x} \in \Sigma^+ : \ d_\nu(\underline{x}) = \alpha\right\}.$$

As mentioned earlier, the existence of the pointwise dimension may be more rare than one might have hoped. This is true even for good measures, as, for example, the case of Gibbs measures. For example, in [She91] it was showed that there are C^2 Axiom A diffeomorphisms for which any Gibbs measure produces large exceptional sets. More precisely, the exceptional set is dense and of full Hausdorff dimension. In fact, Barreira, Pesin and Schmeling [BPS96] showed, among other things, that for most conformal expanding maps and most Gibbs measures the exceptional set is dense and of full Hausdorff dimension. This gives further motivation to study the size of the exceptional set, something that it will concern us later.

5 Birkhoff and dimension spectrum

We set

$$b(\alpha) = b_\psi(\alpha) = \dim_H \left\{ \underline{x} \in \Sigma : \ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ \sigma^i(\underline{x}) = \alpha \right\}.$$

If ν is a Gibbs measure then we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi \circ \sigma^i(\underline{x}) = P(\psi) - \log 2 d_\nu(\underline{x})$$

if the limits exist. This provides the following multifractal identity

$$b_\psi(\alpha) = f_\nu\left(\frac{P(\psi)-\alpha}{\log 2}\right)$$

which suggest that the multifractal analysis of the Birkhoff sum provides the dimension spectrum (for Gibbs measures) and vice versa.

The (more numerically accessible) pressure function is intimately related to the Hausdorff dimension of the level sets coming from the multifractal analysis of the Birkhoff average of a potential ψ and the dimension spectrum firstly observed in [HJK$^+$86]; see also [CLP87]. In particular the Birkhoff spectrum and the dimension spectrum can be related to the *Legendre transform*[1] of the pressure function. This relation is summarised in the following result.

Theorem 7. *Let $S \subset \Sigma^+$ be a topologically mixing SFT, ψ a Hölder continuous potential and ν the corresponding equilibrium measure. Let also $\nu_{\max}$ denote the measure of maximal entropy. Then,*

1. *If $\nu \neq \nu_{\max}$, then the function $b(\alpha)$ is real analytic and strictly convex on an open interval $(\alpha_-, \alpha_+) \subset \mathbb{R}$. Furthermore,*

$$b(\alpha) = -\frac{p^*(\alpha)}{\log 2}$$

where $p\colon t \to \mathcal{P}_{top}(t\psi)$ and it is strictly convex, real analytic function on $\mathbb{R}$.

2. *For $\alpha_- \leq \alpha \leq \alpha_+$, each of the α-level sets is an uncountable dense subset of S.*

3. *The interval $[\alpha_-, \alpha_+]$ is maximal in the sense that the limit of the Birkhoff averages of ψ does not attain any value outside this interval, i.e. the level sets are empty for $\alpha \in \mathbb{R} \setminus [\alpha_-, \alpha_+]$.*

4. *If $\nu \neq \nu_{\max}$, the set of points for which the limit of the Birkhoff averages of ψ does not exist has maximal Hausdorff dimension, i.e. the Hausdorff dimension of this set equals the Hausdorff dimension of S.*

The proof of this so-called *multifractal miracle* is a combination of the works in [BS00], [PW97] and [Sch99]; see also [PW01].

It is worth noting that this connection between the Hausdorff dimension of the level sets and the pressure function also persists in different settings such as conformal iterated function systems [JK11], continued fractions [JK10, IJ15] etc.

[1] We remind here that the Legendre transform of a real, convex function f, is given by

$$f^*(a) = \sup_{t\in\mathbb{R}} \big(at - f(t)\big).$$

6 Singular potentials

The regularity of the potential is crucial in order for one to have access to the full strength of thermodynamic formalism. For a Hölder continuous potential, the tools provided, allow us to obtain very thoroughly the dimension spectrum. Furthermore, for the case of g-measures, the analysis of the dimension spectrum provides along the way a multifractal analysis for the Birkhoff averages of the potential, essentially for free. In contrast, for non-Hölder potentials, the absence of some of the strong and particularly useful notions described in the previous section, makes the multifractal analysis more challenging.

That being said there is growing interest in the study of singular potentials. Singularities may contribute to the enlargement of the set of a point of "abnormal" behaviour and may create a different multifractal picture. For example, for the dimension spectrum, the points with infinite Birkhoff averages may have full Hausdorff dimension; in contrast with the case of Hölder continuous potentials, where the dimension spectrum $f_\nu(\alpha)$ is defined on an interval $[\alpha_-, \alpha_+]$, $0 < \alpha_- < \alpha_+ < +\infty$.

This is showcased by Kim, Liao, Rams and Wang for example, for the specific case of the Saint-Petersburg potential [KLRW18], which is an unbounded piecewise constant potential defined in terms of continued fractions. A multifractal analysis of the Birkhoff averages of a singular potential is conducted, by transferring the problem to the setting of shift dynamics over an infinite alphabet and for the *renormalized* Birkhoff averages with a different scaling function than the usual n, showcasing also that the relation between the dimension of the level sets and the pressure function persists here as well.

We are interested, in our case, in the classic Thue–Morse measure μ, corresponding to the potential ψ which has a pole at zero; see Section 4. A complete multifractal analysis is given by Baake, Gohlke, Kesseböhmer and Schindler [BGKS19] for the Birkhoff averages. At the same time, exploiting the characterization of the measure μ as g-measure, one gets along the way the dimension spectrum as well. Of course, the main difficulty of this setting is that the potential has a pole and thus the classic thermodynamic formalism is not applicable.

In [GL90], it was observed that

$$\lim_{n\to\infty} \frac{\log\left(\mu(C_n(\underline{x}))\right)}{n^2 \log(2)} = 1$$

for $\underline{x} = 0^\infty$ or $\underline{x} = 1^\infty$. This is essentially a renormalised (or rescaled) local pointwise dimension of μ. In fact, the same holds for any $\underline{x}$ that is a (representation of a) dyadic point, i.e. a preimage of $\underline{x} = 0^\infty$ or $\underline{x} = 1^\infty$ [Goh21]. In general, for singular potentials, one of course expects that an equilibrium measure should 'avoid' the singularities. This can be restated as that the equilibrium measure is expected to have its fastest decay rate at the singularities and their preimages; in our case, zero and the dyadic rationals. With that in mind, the scaling property for dyadic points raises two main questions. Firstly, whether there are more points, apart form the dyadic rationals which share a similar scaling property. Secondly, if there are intermediate (slower) scaling functions, other than the n^2 and especially, giving even sets of non-vanishing Hausdorff dimension. This leads naturally to the study of the scaling properties of the Thue–Morse measure μ (Paper I).

Let now $c \in \mathbb{T}$ and consider the potential

$$\psi_c(x) = \log\big(g_c(x)\big)\,, \quad \text{where} \quad \frac{1}{2} g_c(x) = \big(1 - \cos(2\pi(x-c))\big)$$

related to the so-called generalized Thue–Morse sequences. Considering the variational potential, one expects that the the equilibrium measures should avoid the singularity c in a sense. This leads to the following modified variational pressure function

$$p_c(t) = \sup_{\nu \in \mathcal{M}_{T,c}} \left\{ h(\nu) + \int_{\mathbb{T}} t\psi_c \,\mathrm{d}\nu(t) \right\}$$

where $\mathcal{M}_{T,c}$ denotes the set of all invariant probability Borel measures ν on $\mathbb{T}$ so that $c \notin \operatorname{supp}(\nu)$.

Fan, Schmeling and Shen [FSS21] studied there the scaling properties of the L^∞-norm of the Birkhoff sums for the potential ψ_c, $c \in \mathbb{T}$. In [FSS22] they studied the multifractal properties of the Birkhoff averages for the generalized Thue–Morse sequences. In fact they attained a complete multifractal analysis for the (unbounded) potential in this setting and showcased yet again the relation of the Birkhoff spectrum and the Legendre transform of the respective (modified) variational pressure.

Theorem 8. *Let $0 < c < 1$. There exists an $\alpha_- = \alpha_-(c) \geq -\infty$ and an $\alpha_+ = \alpha_+(c) \in \mathbb{R}$, with $\alpha_- \leq \alpha_+$, such that*

1. *For $\alpha \in (\alpha_-, \alpha_+)$,*
$$b(\alpha) = -\frac{p_c^*(\alpha)}{\log 2}$$
2. *If $\alpha_- = -\infty$, then $b(\alpha) = 1$ for all $\alpha \leq \log 2$.*
3. *For $\alpha > \alpha+$, the level sets are empty and $b(\alpha_+) = 0$. For $\alpha < \alpha_-$, $b(\alpha) = 0$*

As we have already mentioned, for the case of Hölder continuous potentials, the pressure functions showcases some remarkable regularity properties. The (topological) pressure showcases some continuity properties even for less regular potentials. In particular, for a continuous potential over a compact metric space one has the the function $\psi \mapsto \mathcal{P}_{\text{top}}(\psi)$ is Lipschitz continuous [Pes97, Cli14]. This motivates to examine the regularity properties of the pressure function beyond these classic cases.

Observe that $\psi_c(x) = \psi_0(x - c)$, which allows us to interpret the potentials ψ_c as "perturbed" versions of ψ_0 (or any other $\psi_{c'}$ as a matter of fact) corresponding to a perturbation of the singularity c. With that in mind and in view of the regularity results in the classic cases—and because of the key role of the pressure function in the multifractal analysis of the Birkhoff averages proved in [FSS22]—we are interested in studying the dependence of the pressure function on the singularity $c \in \mathbb{T}$. More precisely, we investigate (Paper II) the continuity properties of the map

$$c \mapsto p_c(t) \colon \mathbb{T} \to (-\infty, +\infty].$$

7 Rigidity

One fundamental question in dynamics is when are two systems considered to be the "same", leading to classification questions and notions of equivalence. For example, the notion of conjugacy is a very natural starting point of comparing the dynamical properties of two systems. More precisely, if two topological systems *topologically conjugate* the orbits of one system system are mapped homeomorphically to the orbits of the other system through the conjugation. In other words the two systems share the same dynamics from the topological point of view. A respective notion of equivalence can be defined for different systems, depending on their regularity properties, via a *smooth conjugation*. Even though conjugacy is a very natural starting point of classifying dynamical systems, by creating equivalence classes of dynamical systems that share the same dynamical properties, it does not provide a sufficiently powerful notion of whether two dynamical systems are the same in the strong sense. In fact there is a whole region $\mathcal{U}_{T_2}$ in the C^0-topology such that, every continuous expanding Markov map $f \colon \mathbb{T} \to \mathbb{T}$ in $\mathcal{U}_{T_2}$ conjugates topologically with the doubling map. This type of phenomenon is what we usually call *local rigidity*.

In general the notion of *rigidity* in mathematics suggest that a mathematical object can be uniquely described by less information that one might a priori expect. Typically, rigidity is not common in mathematics. There are a lot of notions of rigidity in dynamics, depending on the structure and regularity of the systems involved, as well as the algebraic properties in between them; for instance, commutativity. Of course, we can study rigidity from different point of view such as, locally (local rigidity), globally or measure rigidity etc. In this thesis we study the a particular rigidity flavoured question revolving around the concept of *joint invariance*. There are various results of joint invariance in different settings [BH17, KS96].

In this thesis we are interested to whether one is able to determine when two systems are the same (in the strict sense), given a limited amount of information. More precisely, we would like to know whether is is sufficient to consider just a factor of a topological dynamical system, in order to uniquely determine the dynamics of the whole system. This leads to the following question: when two maps, acting on the same space, share common (compact) invariant sets?

A particular interesting case is the expanding Markov maps of the circle. The first result towards this direction is due to Furstenberg [Fur67]. In his pioneering work, Furstenberg showed that for the case of linear expanding Markov maps of the circle, T_m and T_n, depending on the multiplicative dependence between m and n, they share only trivial compact joint invariant sets. In particular he showed the following.

Theorem 9 (Furstenberg). *Consider the maps* $T_m = \times m \pmod 1$ *and* $T_n = \times n \pmod 1$*,* $m, n \geq 2$ *integers. Then the only compact joint invariant sets for* T_m *and* T_n *are either finite ones or the whole space* $\mathbb{T}$*, if* $\frac{\log m}{\log n} \notin \mathbb{Q}$*.*

In other words, despite each of these maps individually admits an abundance of compact invariant sets when m and n are *multiplicatively independent*, in the sense of the previous theorem, the only *jointly invariant* ones are the trivial ones. The techniques depend heavily on the fact that the maps under consideration are endomorphisms of the circle and they commute and cannot be generalized for non-linear or non-commuting maps.

Furstenberg's theorem initiated a a particular interest in joint invariance leading to many generalizations, towards different directions. For example regarding the existence and properties of joint invariant measures [EKL06, Rud90] or joint invariance between commuting automorphisms of compact abelian groups [Ber83, Ber84].

On the other hand, Johnson and Rudolph showed that any two commuting Markov maps can be linearized simultaneously [JR92].

Theorem 10. *Let* f_1 *and* f_2 *be two* C^α*,* $\alpha > 1$ *expanding, commuting and orientation preserving maps so that* f_1 *has* m*-branches and* f_2 *has* n*-branches. Assume also that* $\frac{\log m}{\log n} \notin \mathbb{Q}$*. Then, there exists a* C^α *diffeomorphism* g *such that* $g f_1 g^{-1} = T_m$ *and* $g f_2 g^{-1} = T_n$*.*

This reduces the question to jointly invariant sets for two linear Markov maps that was addressed by Furstenberg. In that sense, if the two involved maps are commuting the answer is almost complete. However very little is known in the non-commutative case. A result of this type comes from Hochman [Hoc18], where he managed to extend Furstenberg's result, without assuming any commutative properties. He proved an analogous result for any $f \in C^1$ and a linear map, under some similar assumptions on the relation between $|f'|$ and m, as in Furstenberg's theorem.

Theorem 11 (Hochman). *For every T_n-ergodic measure μ of dimension $0 < s < 1$, there exists an $\epsilon > 0$ such that, for every C^1 expanding Markov map f, every $weak$ accumulation point of the sequence*

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(T_n^i\right)_*(f_*\mu)$$

has dimension at least $s+\epsilon$. In particular, every joint invariant set is either the whole interval or has dimension zero.

A natural approach for the general case is whether we can determine in some sense the 'size' of the set comprised of all the functions that preserve the non-trivial compact invariant set for one map. Moreira's result [Mor11] provided such a characterization in the topological sense. He showed that for a C^1-function, the functions of the same class, that have even one joint invariant set form a meagre set in the C^1-topology.

Theorem 12. *Let f be a C^1 expanding Markov map. Then for $\|\cdot\|_{C^1}$-generic C^1 expanding Markov map g and every compact f-invariant set $K \neq [0,1)$,*

$$g(K) \cap K = \emptyset.$$

Moreira's results utilizes the fact that C^1 perturbations lack of the bounded distortion property, which allows for one to perturb the system in such a way so that greatly enlarges the gaps of the Cantor structure of the compact invariant set, so that generically, intersections of the initial system and the perturbed one are avoided. Of course, this cannot be the case for more regular perturbations.

In this thesis we investigate this type of rigidity question for general expanding Markov maps, i.e. without assuming linearity or any commutative properties (Paper III). The main difficulty is that the methods of the previous results mentioned above cannot be generalized and applied in this case.

8 Recurrence

Recurrence denotes the property that after a sufficiently long but finite time, return to a state, exactly the same as (or close to) their initial state. Of course, as expected, this time may vary greatly depending on the exact initial state and required degree of closeness. As one expects these type of questions appear very naturally in dynamical systems as the recurrence property provides predictability. It also provides deeper understanding of the dynamics overall as it indicates different types of dynamics, such as periodic, chaotic etc.

One of the most important results in dynamical systems is Poincaré's recurrence theorem proved by Carathéodory [Car65] revealing a recurrence property with minimal requirements. Later on Boshernitzan [Bos93] quantified this question by giving information about the *speed* of the *asymptotic recurrence*. An interesting interpretation of Boshernitzan's theorem was given by Barreira and Sausol [BS01], where they unveiled an interesting connection between the speed with which one point returns to itself and the its (lower/upper) pointwise dimension with respect to an arbitrary probability measure μ. This initiated a more systematic study of recurrence properties of a system, as well as its approximation properties and the (more general) shrinking target problems, dynamical Borel–Cantelli lemmas, return/hitting time etc.

In view of Dirichlet's theorem on Diophantine approximations, it is of interest to study the so-called *uniform approximation* and *uniform recurrence* properties of a dynamical system. Dirichlet's theorem states that for any real number ξ and for all integers $N \geq 1$, there exists an integer $1 \leq n \leq N$, such

$$\|n\xi\| \leq N^{-1}$$

where $\|\cdot\|$ denotes the distance to the nearest integer. This can be stated in a dynamical context. Consider the map $T_\xi \colon \mathbb{T} \to \mathbb{T} \colon x \mapsto x+\xi \pmod 1$ and observe that $\|n\xi\| = \|T_\xi^n(x) - x\|$. That way Dirichlet's theorem can be interpreted as a uniform recurrence problem.

Consider the set

$$\begin{aligned} \mathcal{U}_\alpha(y) := \{\xi \in \mathbb{T} :\ \exists M = & M(\xi) \geq 1 \text{ such that} \\ & \forall N \geq M,\ \exists 1 \leq n \leq N \text{ such that } \|n\xi - y\| < N^{-\alpha}\} \end{aligned}$$

where $\alpha > 0$ and $\xi \in \mathbb{T}$. From Dirichlet Theorem we have that $\mathcal{U}_1(0) = \mathbb{R}$. $\mathcal{U}_\alpha(y)$ is the so-called *inhomogeneous analogous* of the classic Dirichlet set $\mathcal{U}_1(0)$. Khintchine [Khi26] showed, in contrast, that for $\alpha > 1, \mathcal{U}_\alpha(0) = \mathbb{Q}$. In fact it is not true in general $\mathcal{U}_1(y) = \mathbb{R}$, for all $y \in \mathbb{T}$. Therefore, there is no inhomogeneous analogous of the Dirichlet Theorem. This rises questions regarding the 'size' of the set $\mathcal{U}_\alpha(y)$. A natural approach of this question is to calculate the Hausdorff dimension[2] of this set.

[2]Observe that the box dimension is 1 for all $\alpha > 0$ and $y \in \mathbb{T}$, since the set $\mathcal{U}_\alpha(y)$ contains $\mathbb{Q}$, i.e. the periodic points of T_ξ, suggesting that the Hausdorff dimension cannot be replaced with the box dimension.

Kim and Liao [KL19] studied the inhomogeneous version of Dirichlet's theorem and in particular they calculated its Hausdorff dimension. Cheung [Che11] studied the corresponding Dirichlet Theorem in higher dimensions by considering simultaneous Diophantine approximations. In particular, he considered all the points (ξ_1, ξ_2) in the two dimensional torus, such that for all 'large' $N \in \mathbb{N}$, there exists an $1 \leq n \leq N$ such that

$$\max\{\|n\xi_1\|, \|n\xi_2\|\} \leq N^{-\alpha}.$$

and he showed that the Hausdorff dimension of this set is $3/4$. In [CC16], Cheung and Chevallier, generalized this result for all arbitrary dimensions.

Bugeaud and Liao [BL16] transferred the question in a different setting by studying the corresponding set for a β-transformation

$$\begin{aligned}\mathcal{U}(\alpha, y) := \{x \in \mathbb{T} : \exists M = M(x) \geq 1 \text{ such that} \\ \forall N \geq M,\ \exists 1 \leq n \leq N \text{ such that } d(T_\beta^n x, y) \leq \beta^{-\alpha N}\}.\end{aligned}$$

They managed to explicitly calculate the Hausdorff dimension of the set $\mathcal{U}(\alpha, y)$, showing that it is $\frac{(1-\alpha)^2}{(1+\alpha)^2}$ for $\alpha \in [0, 1]$ and zero otherwise. Their proof utilizes the natural connection of β-transformations with an appropriate shift space, corresponding to the β-expansion of a real number, which is by far more advantageous computationally. The same set was studied by Kirsebom, Kunde and Persson [KKP20], even for more general maps, such as piecewise expanding maps and some quadratic maps. Specifically for β-transformations, they showed a jump between its Hausdorff and Packing dimension. Zheng and Wu [ZW19] investigated the respective uniform recurrence set for a β-transformation

$$\begin{aligned}\mathcal{U}(\alpha) := \{x \in \mathbb{T} : \exists M = M(x) \geq 1 \text{ such that} \\ \forall N \geq M,\ \exists n \leq N \text{ such that } d(T_\beta^n x, x) \leq \beta^{-\alpha N}\}\end{aligned}$$

and they proved that the Hausdorff dimension is again equal to $\frac{(1-\alpha)^2}{(1+\alpha)^2}$. It is worth noting at this point, that there is (as expected) some dimensional connection between certain recurrence and approximation questions, as it is showcased in the above mentioned results.

Not unexpectedly, similar questions were investigated in higher dimensions. For the expanding case, for example, He and Liao [HL23] gave a formula for the dimension for the asymptotic recurrence when A is a diagonal matrix, not necessarily integer, and with all diagonal elements of modulus larger than 1. Hu and Persson [HP23] studied the asymptotic recurrence for hyperbolic automorphisms on the two dimensional torus. Namely, they explicitly calculated the Hausdorff dimension of the set

$$\mathcal{L}(\alpha) := \left\{x \in \mathbb{T}^2 :\ d_{\mathbb{T}^2}(T^n x, x) \leq |\lambda|^{-\alpha n}, \text{for infinitely many } n\right\}$$

where $T(x) = Ax \pmod 1$, for all $x \in \mathbb{T}^2$ where A is an integer, hyperbolic, area preserving, 2×2 matrix and λ is the eigenvalue of the matrix A of modulus larger than 1. The proof heavily relies on the fact that $\mathcal{L}(\alpha)$ is a $\limsup$ set which often has a *large intersection property* (see [Fal85]).

Theorem 13. *Let A be a hyperbolic 2×2 integer matrix with $\det A = \pm 1$ and let $\lambda \in \mathbb{R}$ be its eigenvalue so that $|\lambda| > 1$. Then*

$$\dim_H \big(\mathcal{L}(\alpha)\big) = \begin{cases} \frac{2}{\alpha+1}\,, & 0 \le \alpha \le 1 \\ \\ \frac{1}{\alpha}\,, & \alpha \ge 1 \end{cases}$$

This result also reveals a phase transition, related to the different choice of optimal covers, depending on α. This phenomenon, of course, does not appear in the one dimensional case where the geometry is much simpler.

In this thesis we are interested in the hyperbolic setting as well and we investigate analogously the uniform recurrence properties of a hyperbolic automorphism on $\mathbb{T}^2$, by studying a similar type of question concerning its Hausdorff dimension (Paper IV).

Chapter 2

Summary of results

Paper I

In this paper we study the Thue–Morse measure

$$\mu = \mu_{\mathrm{TM}} = \prod_{m=0}^{\infty} \big(1 - \cos(2\pi 2^m x)\big)$$

which is in fact the unique equilibrium measure of the singular potential

$$\psi(x) = \log\big(1 - \cos(2\pi x)\big) - \log 2$$

over the doubling $T = \times 2 \pmod 1$ and we give a complete multifractal analysis of the (rescaled) Birkhoff averages and (rescaled) local dimensions of the measure μ, mainly concentrating on the critical scaling n^2. We also study the intermediate scalings n^γ, $\gamma \in (1,2)$.

We firstly transfer the problem in the full shift of two symbols through the usual 2-expansion of real numbers. We manage to obtain a complete multifractal analysis of the fast dimension spectrum and the fast increasing Birkhoff averages

$$\frac{\log \mu(C_n(x))}{-n^2 \log 2} \quad \text{and} \quad \frac{1}{n^2} S_n\psi(x) = \frac{1}{n^2}\sum_{i=0}^{n-1} \psi \circ T^i(x).$$

This scaling property is motivated by the observation that

$$\lim_{n\to\infty} \frac{\log\left(\mu(C_n(x))\right)}{n^2 \log(2)} = 1 \tag{2.1}$$

if x is a dyadic point, firstly observed for the singularity itself. This arises questions about the rescaled pointwise dimension (and Birkhoff sums) in different points. Furthermore, since the dyadic points are of Hausdorff dimension zero, we can conveniently exclude them in our analysis.

Firstly, even though the ψ is not a Hölder continuous potential, we acquire an *asymptotic Gibbs-like* property that relates $\mu(C_n(x))$ with $S_n\psi(x)$. This is achieved from the characterization of the measure as a g-measure for the g-function $g\colon\, x \mapsto \frac{1}{2}\big(1-\cos(2\pi x)\big)$.

Lemma 1. For any two words $w \in \{0,1\}^n$ and $v \in \{0,1\}^m$, we have

$$\mu([wv]) = \int_{[v]} g_n(wx)\, \mathrm{d}\mu(x),$$

where

$$g_n(x) = \prod_{k=0}^{n-1} g(\sigma^k x).$$

In particular,

$$\inf_{x\in[wv]} S_n\psi(x) + \log(\mu[v]) \leqslant \log(\mu[wv]) \leqslant \log(\mu[w]) \leqslant \sup_{x\in[w]} S_n\psi(x).$$

With this in hand, we can now gain estimates for both $\mu(C_n(x))$ with $S_n\psi(x)$ at the same time. Therefore we proceed to acquire estimates about the Birkhoff sums of the potential $S_n\psi(x)$. Exploiting the estimates of the potential ψ with respect to the distance of a point form the singularity, we obtain some estimates about the rate growth of the its Birkhoff sums. By Lemma 1 this yields estimates about $\mu(C_n(x))$ as well.

The growth rate of the Birkhoff sum clearly depends on the distance of one point from the singularity or its preimages. This leads to a very natural representation of each point depending on the size of blocks of consecutive 0's or 1's. In other words, we transfer the problem in the shift of infinite symbols, $\mathbb{N}^{\mathbb{N}}$. More concretely, for a non-dyadic point x, its binary expansion can be uniquely written in an alternating form as $x = a^{n_1}b^{n_2}a^{n_3}b^{n_4}\ldots$, where $a, b \in \{0,1\}$ with $a \neq b$ and $n_i \in \mathbb{N}$ for all $i \in \mathbb{N}$. With this notation, the *alternation coding* is a map τ defined for every non-dyadic point, to $\mathbb{N}^{\mathbb{N}}$, given by

$$\tau\colon\, a^{n_1}b^{n_2}a^{n_3}b^{n_4}\ldots \mapsto n_1n_2n_3n_4\ldots.$$

Observe that this map is 2–1 and the points in any fiber of τ are related via a reflection around the middle point of the torus. Since this is a symmetry of the potential ψ which commutes with the doubling map, it turns out that $S_n\psi(x)$ and $\mu(C_n(x))$ are in fact constant on every fiber of τ.

We can now collectively express the bounds for both the Birkhoff sums and $\mu(C_n(x))$. In the end we get the following key result.

Proposition 2. Let $\tau(x) = (n_1 n_2 n_3 n_4 \ldots)$ and set for every $m \in \mathbb{N}$, $N_m = N_m(x) = \sum_{i=1}^{m} n_i$ and $f_m(x) = \sum_{i=1}^{m} n_i^2$. Let $N_m \leqslant n < N_{m+1}$ for some $m \in \mathbb{N}$, with $r_{m+1} = N_{m+1} - n$ and $s_{m+1} = n - N_m$. Then,

$$\begin{aligned} S_n\psi(x) &= -(f_{m+1}(x) - r_{m+1}^2)\log 2 + O(n), \\ \log \mu(C_n(x)) &= -(f_m(x) + s_{m+1}^2)\log 2 + O(n). \end{aligned}$$

This reduced the problem into studying the averages

$$F_m(x) = \frac{1}{N_m^2}\sum_{i=1}^{m} n_i^2$$

where the n_i's and N_m's are defined as above.

The first result revolves around the scaling properties of $\mu(C_n(x))$ with $S_n\psi(x)$ in the sense of Hausdorff dimension. We show that for any $\gamma \in (1,2)$, the intermediate scaling n^γ gives a trivial Birkhoff and dimension spectrum.

Theorem 3. For each $\gamma \in (1,2)$ and $\alpha \geqslant 0$, the level sets

$$\left\{x \in \mathbb{X} : \lim_{n\to\infty} \frac{\log \mu(C_n(x))}{-n^\gamma \log 2} = \alpha\right\}, \quad \left\{x \in \mathbb{X} : \lim_{n\to\infty} \frac{-S_n\psi(x)}{n^\gamma \log 2} = \alpha\right\}$$

have Hausdorff dimension 1.

The function F_m was introduced as a natural counterpart of the rescaled Birkhoff sums and local dimensions. The estimates established above yield the following result.

Proposition 4. Let $\alpha, \beta \in [0,1]$ $\alpha \leq \beta$. Given a non-dyadic point x, let $\liminf_{m\to\infty} F_m(x) = \alpha$ and $\limsup_{m\to\infty} F_m(x) = \beta$. Then,

$$\liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{-n^2 \log 2} = \frac{\alpha}{1+\alpha}, \quad \limsup_{n\to\infty} \frac{\log \mu(C_n(x))}{-n^2 \log 2} = \beta,$$

and

$$\liminf_{n\to\infty} \frac{-S_n\psi(x)}{n^2 \log 2} = \alpha, \quad \limsup_{n\to\infty} \frac{-S_n\psi(x)}{n^2 \log 2} = \frac{\beta}{1-\beta}.$$

It is worth note that the 'jump' of the accumulation points comes from the strict convexivity of the function $s \mapsto s^2$ which forces the $\liminf \frac{\log \mu(C_n(x))}{-n^2 \log 2}$ to drop compared to $\liminf_{m\to\infty} F_m(x)$. This result indicates that, unlike the case of dyadic points, the limits do not exist. Therefore, it is of particular interest to study the exceptional set. This motivates us to study the joint level level sets. Moreover, in this sense, Proposition 4 suggests that the critical scaling is n^2. Indeed this scaling gives a non-trivial multifractal analysis.

Observe that since the dyadic points have the fastest possible growth of the Birkhoff sum one expects that the lim sup of the rescaled Birkhoff sums should not exceed 1. On the other hand obviously the lim inf of F_m is clearly positive. Therefore, we only look for accumulation points within the closed interval $[0, 1]$.

Theorem 5. Set $\Delta := \{(a, b) : \ a, b \in [0, 1], \ a \le b\}$ and consider the function $f\colon \Delta \to [0, 1]$ defined by

$$f(\alpha, \beta) := \frac{\sqrt{\alpha\beta + \beta - \alpha} - \beta}{\sqrt{\alpha\beta + \beta - \alpha} + \sqrt{\alpha\beta}}.$$

We have

$$f\Big(\frac{\alpha}{1-\alpha}, \beta\Big) = \dim_H \left\{x \in \mathbb{X} : \ \liminf_{n\to\infty} \frac{\log \mu(C_n(x))}{-n^2 \log 2} = \alpha, \ \limsup_{n\to\infty} \frac{\log \mu(C_n(x))}{-n^2 \log 2} = \beta\right\}$$

$$f\Big(\alpha, \frac{\beta}{1+\beta}\Big) = \dim_H \left\{x \in \mathbb{X} : \ \liminf_{n\to\infty} \frac{-S_n\psi(x)}{n^2 \log 2} = \alpha, \ \limsup_{n\to\infty} \frac{-S_n\psi(x)}{n^2 \log 2} = \beta\right\}$$

if $\big(\frac{\alpha}{1-\alpha}, \beta\big)$ and $(\alpha, \frac{\beta}{1+\beta})$ are in the set Δ. Otherwise, the level set is empty.

Paper II

Motivated by the key role of the pressure function for the Birkhoff spectrum, showcased in [FSS22], we study a continuity property of the pressure function. More precisely, consider the (modified) pressure function

$$p_c(t) = \mathcal{P}(t\psi_c) = \sup_{\nu \in \mathcal{M}_{T,c}} \left\{h(\nu) + \int_{\mathbb{T}} t\psi_c \, \mathrm{d}\nu(t)\right\}$$

where $c \in [0, 1]$ and $\mathcal{M}_{T,c}$ denotes the set of all invariant Borel probability measures ν on $\mathbb{T}$ so that $c \notin \operatorname{supp}(\nu)$, corresponding to the potential

$$\psi_c(x) = \log\big(g_c(x)\big), \quad \text{where} \quad g_c(x) = \frac{1}{2}\big(1 - \cos(2\pi(x - c))\big)$$

considered over the doubling map $T = \times 2 \pmod 1$. This definition of the pressure function is justified from the results in [FSS22] of course; but also, intuitively, since ψ_c has a singularity at c, one expects that this will also manifest in the pressure function. Interestingly, one has in fact one the non-trivial result, proved in [GKS], that, for $t \geq 0$

$$p_c(t) = \sup_{\nu \in \mathcal{M}_T} \left\{ h(\nu) + \int_{\mathbb{T}} t\psi_c \, \mathrm{d}\nu(t) \right\} \tag{2.2}$$

where $\mathcal{M}_T$ denotes the set of all invariant Borel probability measures ν on $\mathbb{T}$.

We examine the dependence of the pressure function on the parameter c. The results in [FSS22], indicate that this the function $c \mapsto p_c(t)$ is strongly discontinuous, in the sense that there exist a dense subset of the torus where $p_c(t) = +\infty$ and another dense subset where $p_c(t)$ is finite. That being said, we prove that the the map $c \mapsto p_c(t)$ is lower semicontinuous for all $t \in \mathbb{R}$. When only non-negative t's are considered one can show even more. In particular, for $t \geq 0$ we prove that the pressure function depends continuously on the singularity c.

Theorem 1. Let $t \in \mathbb{R}$. The map

$$c \mapsto p_c(t) \colon \mathbb{T} \to (-\infty, +\infty]$$

lower semicontinuous. For $t \geq 0$, the map

$$c \mapsto p_c(t) \colon \mathbb{T} \to (-\infty, +\infty)$$

is a continuous function on the torus.

In the same spirit as Paper I, the proof revolves around the idea that an equilibrium measure of such a potential should give very little mass in areas of the singularity. With this in mind, we utilize the fact that the (modified) pressure function can be approximated by an increasing sequence of subshifts of finite type, so that they 'avoid' the singularity. The restriction of the potential function $\psi_c = \log g_c$ on each such SFT is a Hölder continuous functions which allows us to use all the tools of the classic thermodynamic formalism, providing good control of the potential which leads to good estimates for the respective measures on each SFT. The relation 2.2 is crucial to show the upper semicontinuity for $t \geq 0$.

Paper III

In this paper we give a partial rigidity answer for the case of expanding Markov maps of the circle, without assuming the commutative and/or linearity.

For a mapping of the circle of class C^α, $\alpha > 1$, we firstly study the topological structure of the set consisting of all compact invariant subsets. Furthermore for a fixed such mapping we examine locally, in the category sense, how big is the set of all maps that have at least one non trivial joint invariant compact subset. Lastly we show the strong dimensional relation between the maximal invariant subset of a given Markov map contained in a subinterval $I \subset [0,1)$ and the set of all right endpoints of its invariant subsets that are contained in the same subinterval, I, as well as the continuous dependence of the dimension on the endpoints of the subinterval I. In what follows we expand a little bit more on our results.

For such a map set $\mathcal{K}_f$ to be the set of all compact and nonempty subsets of $[0,1)$ that are also invariant under the action of f. We endow the set $\mathcal{K}_f$ with the Hausdorff metric d_H. Urbanski [Urb87] and Conley in [Con72] revealed some topological properties of this metric space for C^2 expanding Markov maps and for flows respectively. Motivated by that, we further study the topological structure of the metric space $(\mathcal{K}_f, d_H)$. More precisely, we show that $(\mathcal{K}_f, d_H)$ is a compact and totally disconnected metric space.

Consider the set $\mathcal{E}^\alpha$, $\alpha \in (1, +\infty]$, of all C^α expanding Markov maps of the circle. $\mathcal{E}^\alpha$ is endowed with the $\|\cdot\|_{C^\alpha}$ norm (the usual norm that is considered in C^α). In the same spirit as the result in [Mor11], for a given map f in $\mathcal{E}^\alpha$, we show that for a generic function g in $\mathcal{E}^\alpha$ and for all $K \in \mathcal{K}_f$, with sufficiently small Hausdorff dimension, $K \notin \mathcal{K}_g$. As a matter of fact, we show something even stronger, namely, for generic g and for all K with sufficiently small dimension, the intersection between $g(K)$ and K is empty. The proof relies on a key continuity property of the Hausdorff dimension on compact invariant sets.

Theorem 1. Let $f \in \mathcal{E}^\alpha$. For $\|\cdot\|_{C^\alpha}$-generic $g \in \mathcal{E}^\alpha$ and for every $K \in \mathcal{K}_f$ with $\dim_H(K) < \frac{1}{2}$,

$$g(K) \cap K = \varnothing \, .$$

A similar result without restriction on the dimension can be shown locally. More specifically, there is a open neighbourhood where f is contained in its closure so that for any g in that neighbourhood, there is no joint invariant set.

In the last part of this paper, we further study the invariant sets by their endpoints, motivated by the observation that if K is an invariant set for the orientation preserving maps f, g that are topologically conjugated $h \circ f = g \circ h$ via a homeomorphism h, then h has to map an endpoint to an endpoint since h is monotone as a homeomorphism. We give some dimensional results concerning the relation between the Hausdorff dimension of the largest f-invariant set contained in $[0, c] \subset [0, 1)$ and of the right endpoints of its invariants sets contained in the same subinterval. More precisely, if f is in $\mathcal{E}^\alpha$ then for $c \in [0,1)$, we consider the sets M_c' to be the set of all points for which their orbit remains below c and M_c to be the set of all points x that their orbit stays in $[0,c)$ and $f^n(x) < x$ for all $n > 1$. Observe that the condition for x to be a right-endpoint leads naturally to involvement of β-shifts, by considering the lexicographic order in Σ^+. In [Nil09], Nilsson showed that that for $f = T_2$, the sets M_c and M_c' have in fact the same Hausdorff dimension. Following a similar approach as in [Nil09], we prove that this is true, not only for the doubling map. More generally, we show a similar results for the case in which we place upper and lower bounds on the orbits. Namely, let $M_{c,d}'$ denote the set of all points so that their orbit stays in an interval $[c,d] \subset [0,1)$. Respectively, $M_{c,d}$ is the set of all points x so that not only their orbit remains in $[c,d]$ but also $f^n(x) < x$ for every $n > 1$. Again we can prove that their respective Hausdorff dimensions in fact coincide, i.e. $\dim_H(M_{c,d}) = \dim_H(M_{c,d}')$. Furthermore, we show that the map $(c,d) \mapsto \dim_H(M_{c,d})$ (and thus the map $c \mapsto \dim_H(M_c)$) is continuous.

Paper IV

Motivated by the result in [HP23] we study, in analogy, the uniform recurrence properties of hyperbolic automorphisms in $\mathbb{T}^2$. We consider the set

$$\begin{aligned}\mathcal{U}(\alpha) := \big\{x \in \mathbb{T}^2 :\ &\exists M = M(x) \geq 1 \text{ such that} \\ &\forall N \geq M,\ \exists 1 \leq n \leq N \text{ such that } d_{\mathbb{T}^2}(T^n x, x) \leq |\lambda|^{-\alpha N}\big\}\end{aligned}$$

where $T\colon \mathbb{T}^2 \to \mathbb{T}^2$ is given by $T(x) = Ax \pmod 1$, where A is a hyperbolic, area preserving, 2×2 matrix, with integer entries and λ denotes its eigenvalue of modulus larger than 1. While the dynamical setting is the same as in [HP23], the proofs are closer in spirit to the preceding pieces of work and mostly with [BL16], where they essentially work in an appropriate shift space.

In a similar fashion, we consider the set

$$\begin{aligned}\mathcal{U}'(\alpha) = \big\{\underline{x} \in \Sigma_\Gamma :\ &\exists M = M(\underline{x}) \geq 1 \text{ such that} \\ &\forall N \geq M,\ \exists 1 \leq n \leq N \text{ such that } d_\Sigma(\sigma^n \underline{x}, \underline{x}) \leq |\lambda|^{-\alpha N}\big\}\end{aligned}$$

for an appropriate shift space (SFT) Σ_Γ, coming from the encoding of the original system. We show that this set carries the whole dimension of the original set in the manifold, which allows us to calculate the dimension symbolically, avoiding the complicated geometry inherited by the higher dimensional setting and the intricate structure of $\mathcal{U}(\alpha)$ itself.

Theorem 1. Let A be a hyperbolic 2×2 integer matrix with $\det A = \pm 1$ and let $\lambda \in \mathbb{R}$ be its eigenvalue so that $|\lambda| > 1$. Then

$$\dim_H \left(\mathcal{U}(\alpha)\right) = \begin{cases} 2\frac{(1-\alpha)^2}{(1+\alpha)^2}\,, & 0 \leq \ \alpha \ \leq 3 - 2\sqrt{2} \\[2ex] \frac{(1-\sqrt{2\alpha})^2}{\alpha}\,, & 3-2\sqrt{2} \leq \ \alpha \ \leq 2-\sqrt{3} \\[2ex] \frac{1-3\alpha}{1-\alpha}\,, & 2-\sqrt{3} \leq \ \alpha \ \leq 1/3 \\[2ex] 0\,, & \alpha \geq 1/3 \end{cases}$$

Furthermore, the cardinality of $\mathcal{U}\left(\frac{1}{3}\right)$ is the continuum.

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Part II

Scientific papers

Printed by Media-Tryck, Lund 2024 NORDIC SWAN ECOLABEL 3041 0903

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Doctoral Theses in Mathematical Sciences 2024:2

ISBN 978-91-8104-042-5
ISSN 1404-0034