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An Auction with Approximated Bidder Preferences - When an Auction has to be Quick

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An auction with approximated bidder preferences.
- When an auction has to be quick*

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Abstract

This paper presents a combinatorial auction which is of particular interest when short completion times are of importance. It is based on a method for approximating the bidders' preferences over two types of items when complementarity between the two may exist. The resulting approximated preference relation is shown to be complete and transitive at any given price vector. It is shown that an approximated Walrasian equilibrium always exists if the approximated preferences of the bidders comply with the gross substitutes condition. This condition also ensures that the set of approximated equilibrium prices forms a complete lattice. A process is proposed which is shown to always reach the smallest approximated Walrasian price vector.

Keywords: Approximate auction; approximated preferences; non-quasi-linear preferences.
JEL classification: D44.

1 Introduction

Auctions are extensively used as a way to determine who gets to buy what good and at which price. It is not uncommon for a seller to simultaneously auction multiple items. Spectrum licenses are often divided into smaller geographical areas rather than one countrywide license and a company can be sold as several divisions rather than one entity. In recent years, the literature on multi-item auctions and in particular combinatorial auctions has grown substantially. In a unit-demand setting, Demange et al (1986) propose a multi-item auction, which is Pareto efficient and strategy-proof. Key to their result is to find the unique minimal Walrasian equilibrium price vector, its existence being guaranteed by the lattice structure of equilibrium prices (Demange and Gale, 1985; Shapley and Shubik, 1971), and to allocate the items in accordance with this price. Allowing bidders to

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demand multiple units of items, the problem becomes more complex. For homogeneous items, Ausubel (2004) presents an ascending-bid auction, which is efficient and where the outcome of the auction coincides with the outcome of the Vickrey auction. Extending to heterogeneous items, Gul and Stacchetti (2000) design a generalized version of Demange et al (1986)'s auction, which also terminates at the unique minimal Walrasian equilibrium price vector[1]. In their setting, the existence of a Walrasian equilibrium is guaranteed when bidders have gross substitute preferences. The gross substitutes condition was introduced by Kelso and Crawford (1982) and is utilized by Ausubel (2006), who suggests a multi-item auction that reaches the Vickrey-Clarkes-Groves outcome and therefore is incentive compatible. Sun and Yang (2006, 2009) introduce the gross substitutes and complements condition, which allows for some complementarity in the bidders' preferences. The authors show that this condition is sufficient for the existence of competitive equilibrium and propose two auction processes that always finds an equilibrium price vector. Ausubel and Milgrom (2002) suggest an ascending-bid proxy auction: Each bidder reports a valuation for each package and then commits to bid straightforwardly according to these reports. When bidders have quasi-linear preferences in money and goods are substitutes, the outcome of the proxy auction coincides with the Vickrey auction and sincere bidding is a Nash equilibrium. By allowing prices to differ across packages and bidders, authors such as de Vries et al (2007) and Mishra and Parkes (2007) have proposed auction processes that reach the VCG outcome for general valuations.

A possible problem with many auction formats is that they may take a long time to carry out. The auction for British telecom licenses, conducted in the year 2000, is one example of this as it took two months to complete (Binmore and Klemperer, 2002). One reason for long completion times is that many auctions are dynamic processes where the prices of the items are either only increased or only decreased[2]. This may result in a time-consuming process as the starting prices have to be set far below or far above the expected final prices to make sure that the process converges to a desired equilibrium. In some cases, however, short completion times of auctions are very important. One such example is the Product-mix auction, which was designed to help the Bank of England during the bank run in the autumn of 2007. Due to the outbreak of the financial crisis, the Bank of England wished to allocate loans to commercial banks in a very rapid fashion. Klemperer (2010) proposed a quick auction procedure for allocating two different types of loans to the banks. The idea was that bidders submitted a number of bids consisting of two prices (interest rates), one for each type of loan, and a quantity (same for both loans), which served as an approximation of the bidders' demand. Based on the supplied quantities of the two loans, prices were determined and the bidders were awarded the loans which gave them the highest, non-negative, profit. In this way, the central bank allocated the loans in a quick fashion.

[1]Auction processes converging to the unique minimal equilibrium price vector is common in the literature, see e.g. Andersson et al (2013); Andersson and Erlanson (2013); Mishra and Talman (2010); Sankaran (1994).

[2]For auction processes that may be both ascending and descending see e.g. Andersson and Erlanson (2013); Ausubel (2006); Erlanson (2014); Grigorieva et al (2007).

Quick auctions are not uncommon in the auction literature. Sealed-bid auctions, such as the famous Vickrey auction, are well studied examples. However, such auction formats, and many more, are usually analyzed under the assumption that bidders have quasi-linear preferences in money. This may be restrictive as it implies that bidders neither exhibit risk-aversion, experience wealth effects, nor face financing- or budget constraints. If bidders' preferences are in fact non-linear in money, this should be taken into account. Optimal auctions, where bidders exhibit risk-aversion, have been studied by Maskin and Riley (1984) and Matthews (1987). Morimoto and Serizawa (2014) analyze allocation rules for multiple indivisible items, allowing bidders to have non-linear preferences in money and unit-demand. Ausubel and Milgrom (2002) also propose a generalized proxy auction, where the seller and the bidders have non-linear but strict preferences over all offers made in the bidding process. This auction is embedded in the matching with contracts model by Hatfield and Milgrom (2005).

Thus far, two problems have been identified: Auctions may take a long time to conduct and bidders may not have quasi-linear preferences in money. This paper proposes a combinatorial auction which is both quick and allows for bidders to have non-linear preferences in money. In order for the auction to be quick, the bidders report all required information prior to the execution of the auction. Consequently and similar to sealed bid auctions, the bidders do not participate in a dynamic auction process. Due to the possible high complexity of the bidders' non-linear preferences in money, requiring a bidder to report her preferences over money seems highly infeasible. Therefore, the bidder will report a fraction of her preferences which will be used to approximate her preferences. More specifically, a bidder reports prices which makes her indifferent between the packages which are available in the auction. Using these indifference prices, linear approximations of the bidder's indifference curves between any two distinct packages will be made. In this context, an indifference curve contains all combinations of prices for the two packages which makes the bidder indifferent between the packages. By combining the linearly approximated indifference curves, a bidder's approximated preferences can be constructed.

As suggested in the literature review, linear approximations of bidders preferences are not uncommon. Importantly, the quasi-linear preferences are contained in the class of preferences corresponding to the approximation procedure of this paper. In particular, if a bidder has quasi-linear preferences in money and reports truthfully, the approximated preferences will coincide with the bidder's true preferences.

It is shown that the approximated preference relation of each bidder is complete and transitive at any price vector. Given the approximated preference relations of the bidders, it is of interest to know whether it is always possible to find an equilibrium assignment. In addition to theoretical interest, equilibrium assignments are particularly important in e.g. spectrum auctions as governments typically want all regions of the country to have coverage. As a bidder's approximated preferences do not necessarily coincide with her true preferences, the equilibrium concept analyzed in this paper is denoted an approximated Walrasian equilibrium. It is shown that imposing the gross substitutes condition on the bidders' approximated preference relations is sufficient for the set of approximated Walrasian equilibrium prices to be non-empty. Moreover, the gross substitutes condition also ensures that the set of approximated Walrasian equilibrium prices forms a complete lattice and hence contains a unique minimal element.

Finally, a process is described which can be used to find the unique minimal approximated Walrasian equilibrium price vector. However, the bidders do not actively participate in any intermediate step of this process. Using the bidders' approximated preferences as input, the process is a structured method for finding the unique minimal approximated Walrasian equilibrium price vector. This price vector may be of particular importance when the auctioneer is concerned with consumer welfare. A government selling spectrum licenses may be interested in assuring low consumer prices. Selling the licenses for the smallest equilibrium prices may aid in achieving this as the resulting producer costs are relatively low.

To sum up, the auction procedure is as follows:

1. Each bidder reports prices which makes her indifferent between the available packages.

2. These prices are used to construct linear approximations of the bidder's indifference curves.

3. Combining a bidder's linearly approximated indifference curves, her approximated preferences are constructed.

4. Using the approximated preferences as input, a process is used to find the unique minimal approximated Walrasian equilibrium price vector.

5. The items are allocated to the bidders in accordance with this price vector.

The paper is outlined as follows: Section 2 introduces the basic model and some definitions. The approximation procedure is described in Section 3. In Section 4, results concerning the existence of approximated Walrasian equilibrium are presented. Section 5 contains a description of the process and related results. Section 6 concludes the paper. All proofs are collected in the appendix.

2 The model

A finite number of *bidders*, collected in the set $N = \{1, 2, \ldots, n\}$, participate in the auction. A seller wishes to auction two types of indivisible *items*, called a and b,[3] of which there may exist multiple *copies*. Let $q_a \geq 1$ and $q_b \geq 1$ denote the finite integer number of copies of each type of item. Copies of the same type are to be sold for some uniform *price* p_a or p_b depending on the type. In order to sell the items, the seller requires at least some prices $r_a \geq 0$ and $r_b \geq 0$ for each type of item. Such prices are referred to as the seller's *reservation prices* and imply that $p_a \geq r_a$ and $p_b \geq r_b$. Each bidder has the outside option

[3]To simplify the notation we let a and b denote both the item and a set containing the item, i.e., $a \equiv \{a\}$ and $b \equiv \{b\}$.

of not acquire anything in the auction. The outside option is represented by a *null-item*, which is denoted 0 and is equal to the empty set. The price of the null-item is normalized to 0 so $p_0 = r_0 = 0$. Each bidder is interested in acquiring at most one copy of item a and b respectively. Let $ab = \{a, b\}$ be the combination of one item of each type and let p_{ab} denote its price. The sets of items which the bidders are interested in purchasing are collected in $\mathcal{I} = \{0, a, b, ab\}$ and any element $x \in \mathcal{I}$ is referred to as a *package*. A bidder's preferences over the packages are determined by the utility generated from consuming the packages and their prices. A *consumption bundle* is therefore defined to be a pair consisting of a package and a price. For any given prices of the packages, the bidders are hence interested in consuming at least one of the consumption bundles $(0, 0)$, (a, p_a), (b, p_b), or (ab, p_{ab}). Each bidder $i \in N$ has a *preference relation*, denoted R_i, over all possible consumption bundles. R_i is complete, transitive, continuous, and finite. Let P_i be the strict relation and I_i the indifference relation associated with R_i. The preferences of the bidders satisfy *price monotonicity*, that is, for any package $x \in \mathcal{I}$ and any two prices $p'_x, p''_x \in \mathbb{R}_+$, if $p'_x > p''_x$, then $(x, p''_x)P_i(x, p'_x)$. Finally, any bidder is indifferent between any two identical consumption bundles. An objective of the auction is to find an assignment of the items to the bidders such that any bidder is assigned either 0, a, b, or ab. While any number of bidders can be assigned the null-item, an assignment needs to be such that the number of assigned items of any type, a or b, does not exceed the available number of copies of the type. Formally, let $\mu : N \to \mathcal{I}$ be an *assignment* such that $\#N_a \leq q_a$ and $\#N_b \leq q_b$, where $N_a = \{i \in N \mid \mu(i) \in \{a, ab\}\}$ and $N_b = \{i \in N \mid \mu(i) \in \{b, ab\}\}$, and where $\mu(i)$ denotes the assignment of bidder $i \in N$.

3 Approximation of the bidders' preferences

In order to approximate the true preference relation, R_i, of any bidder $i \in N$, the bidder makes two reports. The first report, denoted v, consists of one price $v_j \in \mathbb{R}$ for each package $j \in \{a, b, ab\}$. Recalling that the price of the null-item is normalized to 0, these reported prices are interpreted as the bidder being indifferent between the consumption bundles $(0, 0)$, (a, v_a), (b, v_b), and, (ab, v_{ab}). The second report, z, consists of some other prices $z_j < v_j$ for each $j \in \{a, b, ab\}$. The prices in z are interpreted as making the bidder indifferent between the consumption bundles (a, z_a), (b, z_b), and (ab, z_{ab}). Note that any price reported for ab need not necessarily equal the sum of the prices reported for the individual items. Moreover, the assumptions on R_i guarantee the existence of prices which fulfill the requirements of the reports. Assuming that the bidders report truthfully, the two reports will be used to make linear approximations of the bidder's indifference curves between any two distinct packages. The approximations will be referred to as the bidder's *approximated indifference curves*. The approximated indifference curves will be constructed under the restriction that $p_{ab} = p_a + p_b$. Consequently, the package ab will be sold for $p_{ab} = p_a + p_b$ and no price discrimination is hence allowed. In line with this, four constants, which are based on the two reports, are defined: $\alpha_v = v_{ab} - v_b$, $\alpha_z = z_{ab} - z_b$, $\beta_v = v_{ab} - v_a$, and $\beta_z = z_{ab} - z_a$. A constant α_j, where $j \in \{v, z\}$, is interpreted as a price for item a, which would make the bidder indifferent between the consumption bundles $(ab, \alpha_j + j_b)$ and (b, j_b). β_j has the corresponding interpretation for a price of item b. In this way, six pairs of prices, (p_a, p_b), are extracted with the help of which the approximated indifference curves between any two packages, except 0, are constructed.

In the following, a number of formal concepts will be introduced. In order to ease the understanding of the approximation procedure, an example will accompany these concepts. The example is depicted in Figures 1 - 4 and is based on that a bidder i makes the following reports of v and z:

	a	b	ab	α_j	β_j
v	10	8	14	6	4
z	6	5	10	5	4

From the reported prices it follows that $\alpha_v = 6$, $\beta_v = 4$, $\alpha_z = 5$, and $\beta_z = 4$. Assuming truthful reports, two pairs of prices $(10, 8)$ and $(6, 5)$ are obtained such that $(a, p_a)I_i(b, p_b)$ for bidder i. In addition, $(10, 4)$ and $(6, 4)$ are prices for which $(a, p_a)I_i(ab, p_a + p_b)$ and for $(6, 8)$ and $(5, 5)$ it follows that $(b, p_b)I_i(ab, p_a + p_b)$. These six pairs of prices are shown in Figure 1 and will be the basis for the linear approximation of the bidder's indifference curves. In order to construct the approximated indifference curve between the packages a and b in general, the two pairs of prices (v_a, v_b) and (z_a, z_b) are used in constructing the following linear function:

$$f_1(p_a) = z_b + (p_a - z_a)\left(\frac{v_b - z_b}{v_a - z_a}\right) \tag{1}$$

$(v_a, v_b) = (10, 8)$ and $(z_a, z_b) = (6, 5)$ in our example, and f_1 is depicted in Figure 2. By combining an approximated indifference curve with price monotonicity, prices which make the bidder strictly prefer one consumption bundle over another consumption bundle can be approximated. For example, as a bidder reports that she is indifferent between (a, v_a) and (b, v_b), it follows by price monotonicity that the bidder strictly prefers (a, p_a) to (b, p_b) if $p_a \leq v_a$ and $p_b > v_b$ or if $p_a < v_a$ and $p_b \geq v_b$. Similarly, prices p_a and p_b for which the bidder would strictly prefer (b, p_b) to (a, p_a) are found by reversing the inequality signs. By applying this reasoning to any pair of prices (p_a, p_b) for which $f_1(p_a) = p_b$ is true, all pairs of prices that generate strict preferences between (a, p_a) and (b, p_b) are approximated.

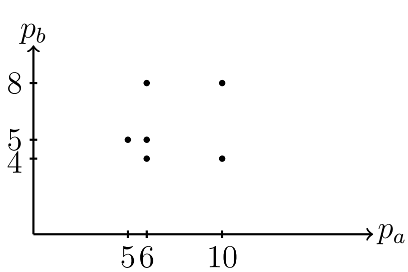

Figure 1

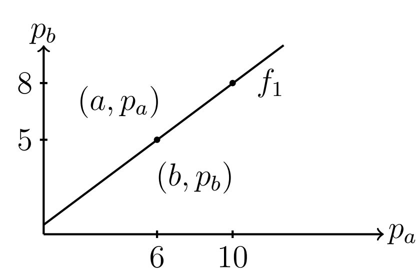

Figure 2

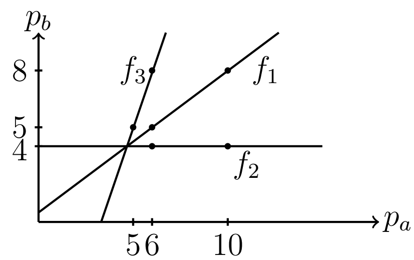

Figure 3

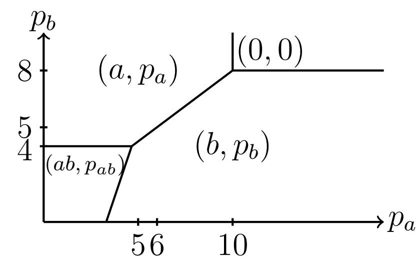

Figure 4

Returning to the example, Figure 2 depicts strict preferences between the consumption bundles (a, p_a) and (b, p_b). (a, p_a) is strictly preferred to (b, p_b) for any pair of prices above and to the left of f_1 whereas (b, p_b) is strictly preferred to (a, p_a) for any pair of prices below and to the right of f_1.

Similarly as for f_1, the pairs of prices (v_a, β_v) and (z_a, β_z) are used to construct the approximated indifference curve between the packages a and ab, while (α_v, v_b) and (α_z, z_b) are used for b and ab, in the following way:

$$f_2(p_a) = \beta_z + (p_a - z_a)\left(\frac{\beta_v - \beta_z}{v_a - z_a}\right) \tag{2}$$

$$f_3(p_b) = \alpha_z + (p_b - z_b)\left(\frac{\alpha_v - \alpha_z}{v_b - z_b}\right) \tag{3}$$

The three approximated indifference curves corresponding to the bidder of our example are displayed in Figure 3. Finally, the approximated indifference curves between 0 and any other package x is given by v_x. As before, by combining an approximated indifference curve and price monotonicity, strict preferences between any two consumption bundles are approximated. In this way, the approximated indifference curves and price monotonicity approximate the true preferences of a bidder. Let $\succsim_i$ denote the *approximated preference relation* of any bidder $i \in N$. Furthermore, $\succ_i$ and $\sim_i$ are the strict and indifference relations associated with $\succsim_i$.

In order for the approximated preference relation of a bidder to be meaningful, it is important that, at any given prices of the items, a consistent ranking of the consumption bundles can be constructed. Proposition 1 ensures that this is the case.

Proposition 1. *For any given prices of the items, the approximated preference relation of each bidder $i \in N$ is complete and transitive.*

Figure 4 shows the combination of prices for which a certain consumption bundle is uniquely most preferred for the bidder in our example.

For a bidder whose preferences are quasi-linear in money, her indifference curves are linear. If prices are reported truthfully, the resulting approximated indifference curves will coincide with the true indifference curves of the bidder. The bidder's approximated- and true preferences will therefore coincide and the quasi-linear preferences are thus contained in the class of preferences corresponding to the approximation procedure described in this section.

4 Existence

Given the approximated preference relations of the bidders, it is of interest to know whether it is always possible to find an equilibrium assignment. A commonly analyzed equilibrium concept is Walrasian equilibrium. However, as the approximated preferences do not necessarily coincide with the true preferences of the bidders, the equilibrium concept of this paper is denoted an approximated Walrasian equilibrium. In order to define this formally, let a price vector be denoted by $p = (0, p_a, p_b) \in \mathbb{R}^3$, which contains a price for the null-item and one price for each type of item. Furthermore, the *approximated demand correspondence* of a bidder $i \in N$ is defined as $D_i(p) = \{x \in \mathcal{I} \mid (x, p_x) \succsim_i (y, p_y) \text{ for all } y \in \mathcal{I}\}$ at any p. If $x \in D_i(p)$, then package x is said to be demanded by bidder $i \in N$.

Definition 1. *The pair $\langle p, \mu \rangle$ constitutes an approximated Walrasian equilibrium if: (i) $\mu(i) \in D_i(p)$ for all $i \in N$ and (ii) if $\#N_x < q_x$ for some $x \in ab$, then $p_x = r_x$.*

Thus, a price vector p and an assignment μ constitute an approximated Walrasian equilibrium if each bidder is assigned a package which she demands and if a copy of an item remains unassigned, then the price of this type of item has to equal the seller's reservation price for the item.

An approximated Walrasian equilibrium does not always exist. For an excellent example, see Milgrom (2000) and recall that the quasi-linear preferences are a special case of the approximated preferences of this paper. However, requiring substitutability in the bidders' preferences has been shown to guarantee the existence of equilibrium in the standard model. Kelso and Crawford (1982) required firms' preferences over workers to comply with the gross substitutes condition to show the existence of a core allocation. This in turn implies that a Walrasian equilibrium exists in Gul and Stacchetti (1999, 2000). Sun and Yang (2006) showed that the more general gross substitutes and complements condition guarantees the existence of competitive equilibrium. Analyzing the simultaneous ascending auction, Milgrom (2000) showed that if objects are mutual substitutes for the bidders, then the objects can be allocated in accordance with a competitive equilibrium. Similarly in the matching with contracts model, a stable allocation exists if hospitals view contracts as substitutes (Hatfield and Milgrom, 2005).

Following Kelso and Crawford (1982), the gross substitutes condition is defined as:

Definition 2. *The approximated preference relation, $\succsim_i$, of any bidder $i \in N$, fulfills the gross substitutes condition if for any two price vectors $p' \geq p$ and any $x \in D_i(p)$, there exists $y \in D_i(p')$ such that $\{w \in x \mid p_w = p'_w\} \subseteq y$.*

The gross substitutes condition implies that a bidder's demand for an item does not decrease as the prices of any other items are raised. Let $\mathcal{P} = \{p \in \mathbb{R}^3_+ \mid \exists \mu \text{ s.t. } \langle p, \mu \rangle \text{ is an approximated Walrasian equilibrium}\}$ be the set of approximated equilibrium prices. Proposition 2 asserts that if the approximated preference relations of each bidder comply with the gross substitutes condition, then there exists an approximated Walrasian equilibrium.

Proposition 2. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$, then the set of approximated equilibrium prices, $\mathcal{P}$, is non-empty.*

It turns out that the gross substitutes condition also guarantees that $\mathcal{P}$ forms a complete lattice. For any two price vectors $p', p'' \in \mathbb{R}^3$, let the meet $p' \wedge p''$ be defined as a vector $s \in \mathbb{R}^3$ with elements $s_j = \min\{p'_j, p''_j\}$. Similarly, let the join $p' \vee p''$ be a vector $h \in \mathbb{R}^3$ with elements $h_j = \max\{p'_j, p''_j\}$. Any $S \subseteq \mathbb{R}^3$ forms a complete lattice if for each $p', p'' \in S$, $s, h \in S$.

Proposition 3. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$, then $\mathcal{P}$ forms a complete lattice.*

Proposition 3 implies that $\mathcal{P}$ contains a unique minimal element. Let this unique minimal approximated Walrasian equilibrium price vector be denoted p^{min}.

5 Process

The proposed process can be used to find p^{min}. It is designed as an English auction; starting at some low prices, prices are increased until p^{min} is reached. As mentioned in Section 1, the bidders do not actively participate in any intermediate step of the process. The process uses the approximated preference relations of each bidder as input in order to find p^{min}. As the approximated preferences are constructed prior to running the process, the process can be executed quickly.

Following Gul and Stacchetti (2000), the process will use the bidders' requirement of the different packages in order to, at least partly, determine how prices should be increased.

Definition 3. *The requirement function $K_i : \mathcal{I} \times \mathbb{R}^3 \to \mathbb{N}_0$ for each $i \in N$ is defined by:*

$$K_i(x, p) = \min_{y \in D_i(p)} \#(x \cap y).$$

Let $K_N(x, p) = \sum_{i \in N} K_i(x, p)$ be the bidders' aggregate requirement of any $x \in \mathcal{I}$ at some p. Proposition 4, below, justifies the interest in the requirement function. Most importantly, it asserts that when, at some p, the bidders' aggregate requirement for each package is weakly less than the number of existing copies of the items contained in the package, it is possible to assign each bidder a package that she demands. Hence, the first condition for an approximated Walrasian equilibrium is fulfilled at p. As any bidder's requirement of the null-object always equals zero, let $q_0 = 0$ and naturally $q_{ab} = q_a + q_b$.

Proposition 4. *For a given price vector p, there exists an assignment μ such that $\mu(i) \in D_i(p)$ for all bidders $i \in N$ if and only if $K_N(x, p) \leq q_x$ for all $x \in \mathcal{I}$.*

Hence, if $K_N(x,p) > q_x$ for some package $x \in \mathcal{I}$, then there is more demand for the items contained in x, at p, than the number of available copies of x. To determine the net demand, in terms of aggregate requirement, for any package at some price vector p, the function $g : \mathcal{I} \times \mathbb{R}^3 \to \mathbb{Z} : g(x,p) = K_N(x,p) - q_x$ is defined. Packages with the greatest net demand at p are collected in $O(p) = \{x \in \mathcal{I} \mid g(x,p) \geq g(y,p) \text{ for all } y \in \mathcal{I}\}$.

Lemma 1. *$O(p)$ has a unique minimal element with respect to cardinality denoted $O_*(p)$.*

Lemma 1 is important for describing the process as whenever $O_*(p)$ contain any of a, b, or ab, in any step of the process, the prices of the items contained in $O_*(p)$ will be the main focus of the price increase.

A price increase consists of one part determining how much the prices are increased relative to each other and a second part deciding the magnitude. For the first part, $\delta(p) \in \mathbb{R}^3_+$ is introduced, which has elements $\delta_x(p)$ for each $x \in \{0, a, b\}$ and p. Let $p^t \in \mathbb{R}^3_+$ denote the price vector at step t of the process. The magnitude of a price increase at any step t is then given by $\varepsilon(t) = \sup\{e \mid O_*(p^t + e\delta(p^t)) = O_*(p^t)\}$. In Step 2 of the process, prices of the items contained in $O_*(p)$ are raised by equal amounts. However, as the approximated preferences of the bidders are not necessarily quasi-linear, such a price increase may not always be possible. To solve this problem, let $x \neq y$ for $x, y \in ab$, and $l_x(t) = \inf\{\delta_x(p^t) \in \mathbb{R}_+ \mid \delta_0(p^t) = 0,\ \delta_y(p^t) = 1, \text{ and } \varepsilon(t) > 0\}$ is defined. $l_x(t)$ and $\delta(p)$ are used to determine the relative price increase of the items.

Process 1. *Set $t = 0$ and let $p^0 = r$*
Step 1: If $O_*(p^t) = 0$ set $p^t = p^T$ and stop. Otherwise, go to step 2.
Step 2: Let $\delta_x(p^t) = 1$ if $x \in O_*(p^t)$ and 0 otherwise.

$$\textit{If} = \begin{cases} \varepsilon(t) \neq 0, & \textit{let } p^{t+1} = p^t + \varepsilon(t)\delta(p^t) \textit{ and set } t := t+1 \textit{ and go to step 1.} \\ \varepsilon(t) = 0, & \textit{go to step 3.} \end{cases}$$

Step 3: Let $\delta_0 = 0$ and

$$\textit{if} = \begin{cases} a, ab \in O_*(p^t), & \textit{then } \delta_a(p^t) = 1 \textit{ and } \delta_b(p^t) = l_b(t). \\ b \in O_*(p^t), & \textit{then } \delta_a(p^t) = l_a(t) \textit{ and } \delta_b(p^t) = 1. \end{cases}$$

Let $p^{t+1} = p^t + \varepsilon(t)\delta(p^t)$ and set $t := t+1$ and go to step 1.

Assuming that the bidders' approximated preferences fulfill the gross substitutes condition, Lemma 2 asserts that the auction process does not get stuck at any step $t < T$.

Lemma 2. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$ and $\varepsilon(t) = 0$ in step 2 of process 1, then $\varepsilon(t) > 0$ in step 3 of process 1.*

As $O_*(p^T) = 0$, Proposition 4 ensures that the first condition for p^T to yield an approximated Walrasian equilibrium is fulfilled. Assuming that each bidder's approximated preference relation complies with the gross substitutes condition, Theorem 1 states that the process always converges to the unique minimal approximated Walrasian equilibrium price vector.

Theorem 1. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$, then Process 1 always terminates at $p^T = p^{min}$.*

Finally we consider an example of Process 1. One item of type a and one item of type b are to be sold and two bidders, i and j, participate in the auction. By reporting v and z, the bidders' preferences have been approximated. The parts of the bidders' approximated indifference curves which are relevant to determine their demand at any price vector are shown in Figure 5. Note that bidder i is the bidder of our example in Section 3. Bidder j has reported $v_a = v_b = 7$, and $v_{ab} = 13$ as well as $z_a = z_b = 5$, and $z_{ab} = 11$. It is left to the reader to confirm that bidder j's reports generate the approximated indifference curves shown in Figure 5. The seller has reservation prices $r_a = 2$ and $r_b = 0$ and the price trajectory of Process 1 is shown by the dashed line in Figure 5. $O_*(p^t)$ and the packages demanded by each bidder at the price vectors corresponding to the different stages of Process 1 are shown in the table below.

p^t	$D_i(p^t)$	$D_j(p^t)$	$O_*(p^t)$
p^0	ab	ab	ab
p^1	b, ab	ab	b
p^2	a, b, ab	ab	ab
p^3	a, b	b, ab	0

- $t = 0$: As $O_*(p^0) = \{ab\}$, Process 1 moves to Step 2 where $\delta_a(p^0) = \delta_b(p^0) = 1$ and $\delta_0(p^0) = 0$. Given this $\delta(p^0)$ it is possible to increase prices and maintain $O_*(p) = \{ab\}$. Consequently, $\varepsilon(0) \neq 0$ and prices are raised from p^0 to p^1 in Figure 5. At p^1, $O_*(p^1) = \{b\}$ due to the change in bidder i's demand. Therefore, p^1 is the upper bound for the price increase at this step. Consequently, $p^1 = p^0 + \varepsilon(0)\delta(p^0)$ and $t = 1$.

- $t = 1$: Since $O_*(p^1) = \{b\}$, we set $\delta_b(p^1)$=1 and $\delta_0(p^1) = \delta_a(p^1) = 0$ in Step 2. For this $\delta(p^1)$, $\varepsilon(1) = 0$ since an increase in p_b would change $O_*(p)$ to contain ab as i would change to only demand ab. Therefore, Process 1 proceeds to Step 3. In this step, we find the smallest relative price increase of p_a to p_b, which makes $\varepsilon(1) \neq 0$. In Figure 5, this is given by the slope of the indifference curve of bidder i. $\delta_a(p^1)$ is therefore adjusted such that $\delta_a(p^1) = l_a(1)$, which makes $\varepsilon(1) \neq 0$. The magnitude of the price increase is bounded by the intersection of bidder i's indifference curves. This is where the demand of bidder i changes. Finally, $p^2 = p^1 + \varepsilon(1)\delta(p^1)$ and $t = 2$.

- $t = 2$: Now $O_*(p^2) = \{ab\}$ and the only price increase which is possible while maintaining $O_*(p) = \{ab\}$ is to follow bidder i's indifference curve. $\delta(p^2)$ is adjusted accordingly and p_a and p_b are increased until the packages demanded by bidder j change. Let $p^3 = p^2 + \varepsilon(2)\delta(p^2)$ and $t = 3$.

- $t = 3$: $O_*(p^3) = \{0\}$ and item a is sold to i for a price of 6 and b is sold to j for a price of 5.

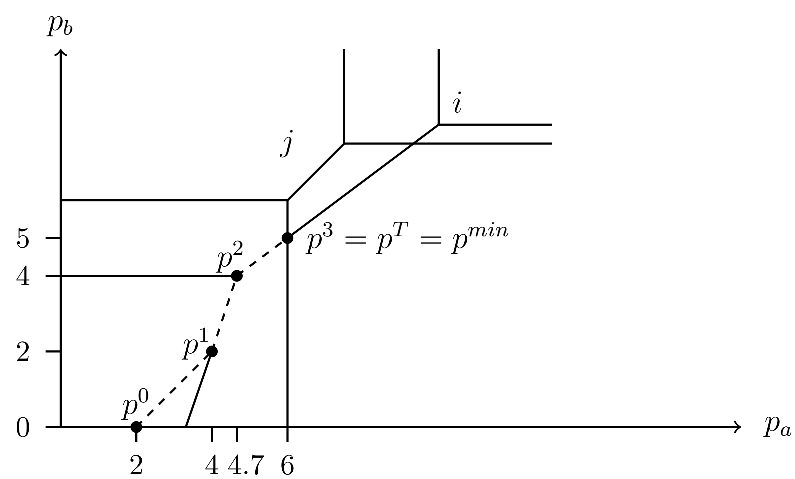

Figure 5

6 Concluding remarks

This paper has provided a procedure for approximating a bidder's preferences over two types of items when complementarity between the two may exist. A quick auction procedure is proposed which is shown to always converge to the unique minimal approximated Walrasian equilibrium price vector. The auction procedure is efficient with respect to the approximated preferences of the bidders. It would therefore be of interest to evaluate the performance of the auction procedure in relation to the bidders' true preferences. Another more complicated question is whether a perhaps similar approximation procedure can be applied to a more general setting, where bidders are interested in more than two items. Finally, the approximation procedure described in this paper assumes that bidders report truthfully and the auction procedure is not strategy-proof. Finding a strategy-proof way of conducting a quick auction, when bidders preferences are not necessarily quasi-linear, would be of great interest and importance.

7 Appendix A: Proofs Related to the Approximation

For proving Proposition 1, completeness of $\succsim_i$ for any $i \in N$ will be shown in Lemma 3. Then Lemma 4, which is of technical nature, will be proven to aid in the proof of the transitivity of $\succsim_i$. Transitivity of $\succsim_i$ will be shown in Lemma 5.

Let the consumption set of a bidder be $Z = \mathcal{I} \times \mathbb{R}_+$ and any consumption bundle is a pair $(x, p_x) \in Z$. Let $Z(p)$ denote the consumption set at any $p = (p_0, p_a, p_b) \in \mathbb{R}^3$. For any bidder $i \in N$, $\succsim_i$ is *complete* if for any given p and for all $(x, p_x), (y, p_y) \in Z(p)$, we have that $(x, p_y) \succsim_i (y, p_y)$ or $(y, p_y) \succsim_i (x, p_x)$ (or both). Let $\mathcal{I}_+ = \{a, b, ab\}$.

Lemma 3. *For any given prices of the items, the approximated preference relation of each bidder $i \in N$ is complete.*

Proof of lemma 3. Fix $p = (p_0, p_a, p_b)$. Then as any bidder is assumed to be indifferent between two identical consumption bundles, we need to show that any pair of the four distinct consumption bundles available at p are related by $\succsim_i$. By the requirements on the bids we know that $(x, v_x) \sim_i (0, 0)$ for any $x \in \mathcal{I}_+$. Assume that $p_x \leq v_x$. Then it follows by price monotonicity that $(x, p_x) \succsim (x, v_x) \sim_i (0, 0)$. By construction, $f_i(p_j) = p_k^i$, for $i = 1, 2, 3$, are some prices of $j, k \in ab$, which would make the bidder indifferent between any two packages $x \neq y$ where $x, y \in \mathcal{I}_+$. Assume that $p_j^i \leq p_j$ for $i = 1, 2, 3$, which by price monotonicity implies that $(x, p_x) \succsim (x, p_x^i) \sim_i (y, p_y^i) \sim_i (y, p_y)$, where the identity of the two packages depend on the identity of i. By replacing $\leq$ with $\geq$ in the arguments above, the same conclusion is derived by symmetry. □

While completeness of the approximated preference relations could be established by only considering one indifference curve at a time, transitivity depends on the construction of different indifference curves. Therefore, it is important to know the relationship of the approximated indifference curves. Let c_i be the *intercept*, m_i the *slope* of f_i for $i = 1, 2, 3$, $c_4 = z_b - \frac{\alpha_z}{m_3}$, and $m_4 = \frac{1}{m_3}$. We start by noting that since $v_j > z_j$ for $j \in ab$, it is always the case that $m_1 = \frac{v_b - z_b}{v_a - z_a} > 0$.

Lemma 4. *The linearly approximated indifference curves have the following relationship:*

i. *If $m_j \neq m_k$ for some $j, k = 1, 2, 4$, then $m_1 \neq m_2 \neq m_4$*

ii. *If $m_1 \neq m_2 \neq m_4$, then there exist unique $p_a^* \in \mathbb{R}$ and $p_b^* \in \mathbb{R}$ such that $f_1(p_a^*) = f_2(p_a^*) = p_b^*$ and $f_3(p_b^*) = p_a^*$.*

iii. *If $m_3 > 0$ and $m_1 \neq m_2 \neq m_4$, then $l > m_1 > k$ for $l, k \in \{m_2, m_4\} \subset \mathbb{R}^2$ where $l \neq k$.*

iv. *$m_j > -1$ for $j = 2, 3$.*

v. *If $m_2 > m_1$, then $m_2 > m_1 > m_4 > 0$.*

vi. *If $m_1 = m_2 = m_4$, then $l \leq c_1 \leq k$ for $l, k \in \{c_2, c_4\} \subset \mathbb{R}^2$ where $l \neq k$.*

vii. *If $c_j \neq c_k$ for some $j, k = 1, 2, 4$, then $c_1 \neq c_2 \neq c_4$*

Proof. i. By symmetry it is enough to consider one case. Let $m_1 \neq m_4$ and to derive a contradiction we assume that $m_2 = m_1 \neq m_4$, which is equivalent to $\frac{\beta_v - \beta_z}{v_a - z_a} = \frac{v_b - z_b}{v_a - z_a} \neq \frac{v_b - z_b}{\alpha_v - \alpha_z}$. Therefore, $\beta_v - \beta_z = v_b - z_b$ and $v_a - z_a \neq \alpha_v - \alpha_z$. By the definition of the four constants β_v, α_v, β_z, and α_z we know that

$$\beta_v + v_a = \alpha_v + v_b \tag{4}$$

and

$$\beta_z + z_a = \alpha_z + z_b \tag{5}$$

Using equations (4) and (5) to replace α_v and α_z we get that $\beta_v - \beta_z \neq v_b - z_b$, which is a contradiction.

ii. As any f_i is a linear function for $i = 1, 2, 3$ and $m_1 \neq m_2$, there must exist a unique p_a^* where $f_1 = f_2$. f_1 and f_2 are defined by equation (1) and (2) respectively. This gives:

$$p_a^* = \frac{z_a(v_b - \beta_v) + v_a(\beta_z - z_b)}{v_b - z_b - \beta_v + \beta_z} \tag{6}$$

Naturally since $m_1 \neq m_2$ we have $v_b - z_b \neq \beta_v - \beta_z$ and $v_b - z_b - \beta_v + \beta_z \neq 0$. Replacing p_a in equation (1) by (6) gives:

$$p_b^* = \frac{v_b \beta_z - z_b \beta_v}{v_b - z_b - \beta_v + \beta_z} \tag{7}$$

We proceed by showing that p_a^* and p_b^* can be found for f_1 and f_3 as well. Replacing p_b in (3) by (1) gives:

$$p_a' = \frac{z_a \alpha_v - \alpha_z v_a}{\alpha_v - \alpha_z - v_a + z_a} \tag{8}$$

As $m_1 \neq m_4$ it is ensured that $\alpha_v - \alpha_z - v_a + z_a \neq 0$. Replacing p_a' in equation (1) by (8) gives:

$$p_b' = \frac{z_b(\alpha_v - v_a) + v_b(z_a - \alpha_z)}{\alpha_v - \alpha_z - v_a + z_a} \tag{9}$$

By using equation (4) in (8) as well as (5) in (9) we get $p_a' = p_a^*$ and $p_b' = p_b^*$.

iii. First note that if $m_3 > 0$, then $m_4 > 0$. As $m_1 \neq m_2 \neq m_4$ we either have $m_1 > m_j$ or $m_1 < m_j$ for some $j = 2, 4$. By symmetry it is enough to consider one case. Let $m_1 > m_4$, then $m_1 = \frac{v_b - z_b}{v_a - z_a} > \frac{v_b - z_b}{\alpha_v - \alpha_z} = m_4 > 0$. As $v_b > z_b$ by construction we have $\alpha_v - \alpha_z > v_a - z_a$. Using equation (4) and (5) to replace α_v and α_z we get $\beta_v - \beta_z > v_b - z_b$ and thus $m_2 = \frac{\beta_v - \beta_z}{v_a - z_a} > m_1 = \frac{v_b - z_b}{v_a - z_a}$.

iv. As we have a requirement on the reports that $v_{ab} > z_{ab}$ we get $v_{ab} = v_a + \beta_v = v_b + \alpha_v > z_a + \beta_z = z_b + \alpha_z = z_{ab}$ or $v_a - z_a > \beta_z - \beta_v$ and $v_b - z_b > \alpha_z - \alpha_v$. Therefore, $1 > \frac{\beta_z - \beta_v}{v_a - z_a}$ and $1 > \frac{\alpha_z - \alpha_v}{v_b - z_b}$ or equivalently, $-1 < m_2 = \frac{\beta_v - \beta_z}{v_a - z_a}$ and $-1 < m_3 = \frac{\alpha_v - \alpha_z}{v_b - z_b}$.

v. $m_2 > m_1$ gives that $\frac{\beta_v - \beta_z}{v_a - z_a} > \frac{v_b - z_b}{v_a - z_a} > 0$ or $\beta_v - \beta_z > v_b - z_b$. Moreover, $m_2 > m_1$ implies that $m_2 \neq m_1 \neq m_4$. Applying (4) and (5) to α and β_z gives that $\alpha_v - \alpha_z > v_a - z_a > 0$ and thus $m_3 = \frac{\alpha_v - \alpha_z}{v_b - z_b} > 0$. The rest follows from point iii of this lemma.

vi. Let $m_1 = m_2 = \frac{1}{m_3} = m$ and then either $c_1 \leq l$ or $c_1 \geq l$ for $l = c_2, c_4$. By symmetry it is enough to consider when $c_1 \geq c_2$, which implies $c_1 = z_b - z_a * m \geq \beta_z - z_a * m = c_2$ or $z_b \geq \beta_z$. Using (5) to replace β_z gives $z_a \geq \alpha_z$ and thus $c_4 = z_b - \alpha_z * m \geq z_b - z_a * m = c_1$.

vii. If $l \neq c_1$ for $l = c_2, c_4$, then by symmetry it is enough to consider one case: Let $c_2 \neq c_1$, which implies $z_a \neq \alpha_z$. Using (5) to replace α_z gives $\beta_z \neq z_b$ and hence $c_4 \neq c_1$. By point *vi*. of this lemma we must have $c_2 \neq c_1 \neq c_4$. If $c_2 \neq c_4$, then by point *vi*. of this lemma we have $l \geq c_1 \geq k$ with at least one weak inequality being a strict inequality and we can use the same argument as before. □

For any bidder $i \in N$, $\succsim_i$ is *transitive* if for any given p and for all $(x, p_x), (y, p_y), (w, p_w) \in Z(p)$, $(x, p_x) \succsim_i (y, p_y)$ and $(y, p_y) \succsim_i (w, p_w)$ imply that $(x, p_x) \succsim_i (w, p_w)$.

Lemma 5. *For any given prices of the items, the approximated preference relation of each bidder $i \in N$ is transitive.*

Proof. As $(x, p_x) \sim_i (x, p_x)$ at any p for any $(x, p_x) \in Z(p)$ it is assumed that $x \neq y \neq w$. Transitivity in any other case follows by completeness. Fix some $p = (p_0, p_a, p_b)$. We start by considering the case when $x, y, w \in \mathcal{I}_+$ and then proceed to where one of x, y, or w is equal to the null-item 0. By point *i*. of Lemma 4 it follows that either $m_1 = m_2 = m_4$ or $m_1 \neq m_2 \neq m_4$. These will have to be treated separately. Assume $m_1 \neq m_2 \neq m_4$ and by point *ii*. of Lemma 4 there exist p_a^* and p_b^* such that $(a, p_a^*) \sim_i (b, p_b^*) \sim_i (ab, p_a^* + p_b^*)$. Let $x \neq y$ for $x, y \in \{b, ab\}$, then we will show the following:

If for any $i \in N$ $(a, p_a) \succsim_i (x, p_x)$ and either (i) $(x, p_x) \succsim_i (y, p_y)$ or (ii) $(y, p_y) \succsim_i (a, p_a)$ at some p, then (i) $(y, p_y) \not\succ_i (a, p_a)$ or (ii) $(x, p_x) \not\succ_i (y, p_y)$.

By symmetry, the following arguments apply when $\succsim_i$ and $\not\succ$ are replaced by $\preceq_i$ and $\not\prec_i$ respectively. Let f_X be the indifference curve between a and x and f_Y be the indifference curve between y and a. Note that $X, Y \in \{1, 2\}$ and $X \neq Y$ as $x \neq y$. Moreover, let $f_X(p_a) = p_b^X$, $f_Y(p_a) = p_b^Y$ and $f_3(p_b) = p_a^3$.

Let $V \neq W$ for $V, W \in \{\succsim_i, \succ_i\}$. In order to derive a contradiction, assume that $(a, p_a) \succsim_i (x, p_x)$, $(x, p_x) W (y, p_y)$, and $(y, p_y) V (a, p_a)$ for any $i \in N$ at some p. By price monotonicity it follows that $p_b^X \leq p_b \leq p_b^Y$ and, depending on the identity of the packages, either $p_a^3 \geq p_a$ or $p_a^3 \leq p_a$, with some weak inequality being a strict inequality.

It will now be shown that $p_a^* \neq p_a$. If $p_a = p_a^*$, then $p_b^* \neq p_b$ since otherwise $(a, p_a) \sim_i (b, p_b) \sim_i (ab, p_a + p_b)$, which contradicts the assumption that bidder $i \in N$ is not indifferent between the three consumption bundles. Combining $p_b^X \leq p_b \leq p_b^Y$ with $p_b^* \neq p_b$ we get that either $p_b^X \neq p_b^*$ and/or $p_b^Y \neq p_b^*$. This together with $p_a = p_a^*$ imply that the slopes $m_X = \frac{p_b^X - p_b^*}{p_a - p_a^*}$ and/or $m_Y = \frac{p_b^Y - p_b^*}{p_a - p_a^*}$ would be undefined. This contradicts the requirement on the bids that $v_a > z_a$. Hence, $p_a \neq p_a^*$.

Assume that $p_a > p_a^*$. Symmetric arguments, to the ones presented below, can be used when $p_a < p_a^*$. As $m_1 \neq m_2$ by assumption, it follows that $m_Y = \frac{p_b^Y - p_b^*}{p_a - p_a^*} > m_X = \frac{p_b^X - p_b^*}{p_a - p_a^*}$.

Case 1: $y = b$. Then $m_1 > m_2$ and either $m_3 = \frac{p_a^3 - p_a^*}{p_b - p_b^*}$ or $m_3 = \frac{p_a^* - p_a^3}{p_b^* - p_b}$. By price monotonicity $y = b$ requires that $p_a^3 \geq p_a > p_a^*$, which implies that we must have $p_b^* \neq p_b$ as m_3 would otherwise be undefined, contradicting that $v_b > z_b$. If $p_b > p_b^*$, then $m_1 = \frac{p_b^Y - p_b^*}{p_a - p_a^*} > m_4 = \frac{p_b - p_b^*}{p_a^3 - p_a^*} > 0$, which contradicts point *iii.* of Lemma 4. If $p_b^* > p_b$, then we must have that $m_1 = \frac{p_b^Y - p_b^*}{p_a - p_a^*} > 0 > \frac{p_b^* - p_b}{p_a^* - p_a^3} = m_4 = \frac{p_b - p_b^*}{p_a^3 - p_a^*} \geq m_2 = \frac{p_b^X - p_b^*}{p_a - p_a^*}$. By point *iv.* of Lemma 4 $m_3 > -1$ and we have $-1 > m_4 \geq m_2$. This is a contradiction of point *iv.* of Lemma 4.

Case 2: $y = ab$. Now $p_a^3 \leq p_a$ and $m_2 > m_1 = \frac{p_b^X - p_b^*}{p_a - p_a^*} > 0$, which requires $p_b \geq p_b^X > p_b^*$. Then it follows by point *v.* of Lemma 4 that $m_1 = \frac{p_b^X - p_b^*}{p_a - p_a^*} > m_4 = \frac{p_b - p_b^*}{p_a^3 - p_a^*} > 0$. This in turn requires $p_b^* < p_b \leq p_b^X$ and $p_a^* < p_a \leq p_a^3$ with some weak inequality being a strict inequality, which is a contradiction.

Next the case when $m_1 = m_2 = m_4 = m$ is considered, which implies that we can rewrite $f_3(p_b) = p_A^3 = c_3 + p_b * m_3$ as $p_b = -\frac{c_3}{m_3} + \frac{p_a^3}{m_3}$. Note that $c_4 = -\frac{c_3}{m_3}$ and thus $p_b = c_4 + p_a^3 * m$. Let $x \neq y \neq w$ for $x, y, w \in \mathcal{I}_+$, then the following will be shown:

If $(x, p_x) \succsim_i (y, p_y)$ and $(y, p_y) \succsim_i (w, p_w)$ for any $i \in N$ at some p, then $(w, p_w) \not\succ_i (x, p_x)$.

To derive a contradiction assume that $(x, p_x) \succsim_i (y, p_y)$, $(y, p_y) \succsim_i (w, p_w)$, and $(w, p_w) \succ_i (x, p_x)$ for some $i \in N$ at some p. Note that by price monotonicity we either have: (i) $f_1(p_a) = p_b^1 \leq p_b \leq p_b^2 = f_2(p_a)$ and $f_3(p_b) = p_a^3 \leq p_a$ or (ii) $f_1(p_a) = p_b^1 \geq p_b \geq p_b^2 = f_2(p_a)$ and $f_3(p_b) = p_a^3 \geq p_a$, with at least one weak inequality being a strict inequality. By symmetry it is enough to consider one case. Assume that the three consumption bundles are related such that $f_1(p_a) = p_b^1 \leq p_b \leq p_b^2 = f_2(p_a)$ and $f_3(p_b) = p_a^3 \leq p_a$, with at least one weak inequality being a strict inequality. From this it follows that $p_b^1 = c_1 + p_a * m \leq p_b = c_4 + p_a^3 * m \leq c_4 + p_a * m$ and $p_b^1 = c_1 + p_a * m \leq p_b^2 = c_2 + p_a * m$. Thus, $c_1 \leq c_4$ and $c_1 \leq c_2$. However, as at least one of the three previous mentioned weak inequalities is a strict inequality we must have that $c_j \neq c_k$ for some $j \neq k$ where $j, k \in \{1, 2, 4\}$. Therefore, $c_1 \neq c_2 \neq c_4$ by point *vii.* of Lemma 4. Hence, $c_1 < c_4$ and $c_1 < c_2$, which is a contradiction of point *vi.* of Lemma 4.

Finally, the case when $x, y, w \in \mathcal{I}$ and where one of x, y, or w is equal to the null-item 0 is considered. By the requirements of the reports we know that $(0, 0) \sim_i (a, v_a) \sim_i (b, v_b) \sim_i (ab, v_{ab})$ for any $i \in N$. Let $x \neq y$ for $x, y \in ab$ and $l \neq k \neq w$ for $l, k, w \in \{0, x, ab\}$, then we will show the following:

1. If $(x, p_x) \succsim_i (0, 0)$ and either (i) $(y, p_y) \succsim_i (x, p_x)$ or (ii) $(0, 0) \succsim_i (y, p_y)$ for any $i \in N$ at some p, then (i) $(0, 0) \not\succ_i (y, p_y)$ or (ii) $(y, p_y) \not\succ_i (x, p_x)$.
2. If $(l, p_l) \succsim_i (k, p_k)$ and $(k, p_k) \succsim_i (w, p_w)$ for any $i \in N$ at some p, then $(w, p_w) \not\succ_i (l, p_l)$

Once again, let $V \neq W$ for $V, W \in \{\succsim_i, \succ_i\}$.

1. To derive a contradiction we assume that $(x, p_x) \succsim_i (0, 0)$, $(y, p_y) V (x, p_x)$, and $(0, 0) W (y, p_y)$. Combining we have: $(y, p_y) V (x, p_x) \succsim_i (0, 0) \sim_i (y, v_y) W (y, p_y)$. By price

monotonicity we have $p_y \leq v_y \leq p_y$, with at least one of the weak inequalities being a strict inequality.

2. Note that $p_{ab} = p_x + p_y$. Let f_X denote the indifference curve between x and ab and let m_X denote its slope. Moreover, let $f_X(p_x) = p_y^X$ for some p_x. Assume that $(l, p_l) \succsim_i (k, p_k)$, $(k, p_k) \succsim_i (w, p_w)$, and $(w, p_w) \succ_i (l, p_l)$ at some p. By price monotonicity we either have: $p_y^X \geq p_y$, $p_x \leq v_x$, and $p_x + p_y \geq v_{ab}$, or $p_y^X \leq p_y$, $p_x \geq v_x$, and $p_x + p_y \leq x_{ab}$, with at least one weak inequality being a strict inequality as $(w, p_w) \succ_i (l, p_l)$. By symmetry it is enough to consider one case. So assume the consumption bundles are related such that $p_y^X \geq p_y$, $p_x \leq v_x$, and $p_x + p_y \geq v_{ab}$, with at least one weak inequality being a strict inequality. By the requirements of the bids we know that $v_{ab} = v_x + \eta$, where η is equal to either α_v or β_v depending on the identity of x. Hence, $p_x + p_y \geq v_x + \eta$. Therefore, $p_y - \eta \geq v_x - p_x$ and $p_y^X - \eta \geq v_x - p_x \geq 0$. If $v_x = p_x$, then $p_y^X = \eta$ as $f_X(v_x) = \eta$ by construction. From this it follows that $p_y = \eta$ as $0 = p_y^X - \eta \geq p_y - \eta \geq 0$. Therefore, $p_y^X = p_y$ and $p_x + p_y = v_{ab}$. Since some of the three weak inequalities above must be a strict inequality, it must be that $p_x < v_x$, which is a contradiction. Hence, $v_x > p_x$ and as $f_X(v_x) = \eta$ we must have $m_X = \frac{\eta - p_x^Y}{v_x - p_x}$. Since $p_y - \eta \geq v_x - p_x$ and $p_y^X \geq p_y$ by assumption, we have $m_X \leq -1$, which is a contradiction. □

Proposition 1. *For any given prices of the items, the approximated preference relation of each bidder $i \in N$ is complete and transitive.*

Proof. Lemma 3 and Lemma 5 together imply Proposition 1 □

8 Appendix B: Proofs Related to Existence

In the following sections, it is assumed that the gross substitutes condition is fulfilled for $\succsim_i$ for any $i \in N$ and if $x \subset y$, then x is a proper subset of y. An item is said to be in excess demand if there are more bidders demanding a package containing the item than the number of copies of the item. Similarly, an item is said to be in under demand if there are less bidders demanding a package containing the item than the existing number of copies of the item.

Proposition 2. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$, then the set of approximated equilibrium prices, $\mathcal{P}$, is non-empty.*

Proof. We start by noting that it is always possible to set p_a, p_b, and thus p, sufficiently high such that it is possible to construct an assignment μ where $\mu(i) \in D_i(p)$ for all $x \in ab$. Let $\mathcal{C} = \{p \in \mathbb{R}^3 \mid \exists \mu \text{ s.t. } \mu(i) \in D_i(p) \text{ for all } i \in N\}$, which we know is non-empty. Moreover, $\mathcal{P} \subset \mathcal{C}$. To derive a contradiction it is assumed that $\mathcal{P} = \emptyset$. From this it follows that for each $p \in \mathcal{C}$ there exists some assignment μ associated with p such that $\#N_x < q_x$ and $p_x > r_x$ for at least some $x \in ab$ and where $\mu(i) \in D_i(p)$ for all $i \in N$. Let μ_p denote an assignment at some price vector p and $\mathcal{A}(p) = \{\mu \mid \mu(i) \in D_i(p) \text{ for all } i \in N\}$ be the set of assignments such that each bidder is assigned a package she demands at price vector

p. Let $r = (r_0, r_a, r_b)$. As $p \geq r$, it follows that $\mathcal{C}$ contains some minimal element. Denote such a minimal element by s. The idea of the proof is to show that if $\mathcal{P} = \emptyset$, then s cannot be a minimal element of $\mathcal{C}$.

If $p_b = r_b$ for some $p \in \mathcal{C}$, then $s_b = r_b$ for some s and it must be that $\#N_a < q_a$ and $s_a > r_a$ for any $\mu_s \in \mathcal{A}(j)$. By symmetry, the following arguments hold when b and a are interchanged. For this part of the proof, price monotonicity and the continuity of the approximated indifference curves will imply that s cannot be a minimal element of $\mathcal{C}$. Let $p' \leq s$ be such that $p'_b = s_b = r_b$, and $r_a \leq p'_a < s_a$. By price monotonicity, the demand for item b has weakly decreased at p' as compared to at s. Moreover, as $p'_b = r_b = s_b$ we know that there does not exist excess demand for item b at p'. Since $p' \notin \mathcal{C}$, it is required that there exist at least some bidder $k \in N$ for whom $\mu_{p'} \notin D_k(p')$ at any $\mu_{p'}$. Since the demand for item a has weakly increased at any p', in comparison to s, it must always be possible to find some p' and $\mu_{p'}$ where either $\#N_a = q_a$, if $p'_a > r_a$, or $\#N_a \leq q_a$, if $p'_a = r_a$, and where $\mu_{p'}(i) \in D_i(p')$ for all $i \in N$. Because if there exists excess demand for item a at any $p' \leq s$ and under demand at s, then there exist at least two bidders who did not demand any package containing a at s and who only demand packages containing a at p'. Collect these bidders in the set F. By price monotonicity and since the approximated indifference curves are continuous, there must exist some price vector p'' such that $p' < p'' < s$ for each bidder $i \in F$ where the bidder is indifferent between a package containing a and another package not containing a. As item a is in under demand at s, there must exist some p'' where it is possible to assign $\mu_s(j)$ to each $j \in N \setminus \{i\}$, and in particular to each $j \in F \setminus \{i\}$, and $w \supset a$ to some $i \in F$. Therefore, $\mu_{p''}(i) \in D_i(p'')$ for all $i \in N$ and $p'' \in \mathcal{C}$, which contradicts the minimality of s.

Now assume that $p_x > r_x$ for all $x \in ab$ and $p \in \mathcal{C}$, which implies that there exists at least some minimal element $s \in \mathcal{C}$ such that $p' \notin \mathcal{C}$ for any $p' \leq s$ where $p'_x < s_x$ for some $x \in ab$. Once again, at s we know that $\#N_x < q_x$ for at least some $x \in ab$ at any $\mu_s \in \mathcal{A}(s)$. Assume that $\#N_a < q_a$ and $\#N_b \leq q_b$ for some $\mu_s \in \mathcal{A}(s)$. By symmetry, the following arguments can be used if a and b are interchanged. Let p' be a price vector such that $r_a < p'_a < s_a$ and $p'_b = s_b$. As $p' \notin \mathcal{C}$ we know that $\mu_{p'}(i) \notin D_i(p')$ for some $i \in N$ and there exists excess demand for item a and/or b.

Assume that item b is in excess demand at p'. Since the demand for item a is weakly lower at s, by price monotonicity, and b must belong to at least some demanded package at s for any bidder who demands any package $w \supseteq b$ at p' by gross substitutes, it follows that b must be in excess demand at s as well. This contradicts that $s \in \mathcal{C}$.

So, it must be that a is the item in excess demand at p'. If $\#N_a < q_a$ for all $\mu_s \in \mathcal{A}(s)$, then the same argument as for the case when $s_b = r_b = p'_b$ can be used to generate a contradiction. Therefore, $\#N_a < q_a$ for some assignment $\mu'_s \in \mathcal{A}(s)$ and $\#N_a = q_a$, $\#N_b < q_b$ for some other assignment $\mu''_s \in \mathcal{A}(s)$ as $s \notin \mathcal{P}$. If $\#N_b < q_b$ for all $\mu_s \in \mathcal{A}$, then we can use symmetric arguments to case when $s_b = r_b = p'_b$ in order to derive a contradiction. It must therefore be that $\#N_a < q_a$ and $\#N_b = q_b$ at μ'_s.

In this part it will be shown that it must be possible to find some $p' \leq s$ such that $p' \in \mathcal{C}$. More specifically, it will be shown that an assignment $\mu_{p'}$ can be constructed such that $\mu_{p'}(i) \in D_i(p')$ for all $i \in N$. To see this, note that for any bidder $i \in N$ who only demands one package, the price decrease can always be made sufficiently small such that $D_i(p') = D_i(s)$. For any bidder $i \in N$ for whom $0, x \in D_i(s)$, where $x \in \{a, b, ab\}$, then either the gross substitutes condition is violated in the case when $x = ab$ as $D_i(p) = 0$ for any $p \geq s$ where $p_x > s_x$ for some $x \in ab$, or it is possible to make the price decrease sufficiently small such that $x \in D_i(p')$ for any such bidder. Note that $\mu_i(s) = x$ at any $\mu_s \in \mathcal{A}(s)$ for any such bidder $i \in N$ as $s \in \mathcal{P}$ otherwise. Therefore, it is possible to construct $\mu_{p'}$ such that $\mu_s(i) = \mu_{p'}(i) = x$ for any bidder $i \in N$ discussed above. Moreover, any bidder who is indifferent between $x \in ab$ and ab at s must have $\mu_s(i) = ab$ at any $\mu_s \in \mathcal{A}(s)$ as $p \in \mathcal{P}$ otherwise. For any price decrease sufficiently small it follows that $D_i(p') \subseteq D_i(s')$. Hence, it is possible to let $\mu_{p'}(i) \subseteq \mu_s(i)$ for any such bidder $i \in N$. The only bidders left to consider are the ones who are indifferent between a and b. Note that some such bidder must exist as $\#N_a < q_a$ and $\#N_b = q_b$ for μ'_s and $\#N_a = q_a$ and $\#N_b < q_b$ for μ''_s. Collect each such bidder in the set S. As $\mu_{p'}(i) \subseteq \mu_s(i)$ for all $i \in N \setminus S$ and $\#N_x < q_x$ for some $x \in ab$ at s, it follows that, at p', there are more copies of item a and b to assign to the bidders in S than number of bidders contained in S. As each bidder $i \in S$ wishes to be assigned only one item at s and prices can always be lowered sufficiently little such that $D_i(p') \subseteq D_i(s)$ for any $i \in S$, there must exist some p' where $\mu_{p'}(i) \in D_i(p')$ for all $i \in S$.

More specifically, let f_1^i be the approximated indifference curve between item a and b for any bidder $i \in S$ and m_1^i its slope. let $T = \{m_1^i \mid i \in S\}$ and as any $m_1^i \in \mathbb{R}_+$, the elements in T can be ordered from smallest to greatest. Let $k = \#\{i \in S \mid \mu'_s(i) = b\}$. As $\#N_a < q_a$ and $\#N_b = q_b$ for μ'_s and $\#N_a = q_a$ and $\#N_b < q_b$, it must be that $k \geq 1$. Pick the kth element from T and denote the corresponding approximated indifference curve by f_1^k. As $\mu_i(p') = \mu_i(s)$ for all $i \in N \setminus S$ it follows that k is the number of copies of b which are possible to assign to any bidder $i \in S$ at p'. Furthermore, $\#S - k + 1$ is the number of copies of a which can be assigned at p'. By lowering prices along f_1^k sufficiently little, it must by price monotonicity be that $(b, p_b) \succ_i (a, p_a)$ for a maximum of $k - 1$ bidders $i \in S$, $(a, p_a) \succ_i (b, p_b)$ for a maximum of $\#S - k$ bidders $i \in S$, and $(a, p_a) \sim_i (b, p_b)$ for at least 1 bidder $i \in S$. As there are more copies of item a and b to assign to the bidders in S than number of bidders contained in S at p' and no bidder requires ab, it is possible to let $\mu_{p'}(i) \in D_i(p')$ for all $i \in S$. Therefore, $\mu_{p'}(i) \in D_i(p')$ for all $i \in N$, which contradicts the minimality of s. □

Lemma 6 will be used in the proof of Proposition 3.

Lemma 6. *For any two price vectors p and p' where $p_x > p'_x$ and $p'_y \geq p_y$ for $x, y \in ab$ and $x \neq y$, if for some $i \in N$, $x \subseteq w$ for some $w \in D_i(p)$, then $x \subseteq w'$ for all $w' \in D_i(p')$.*

Proof. Let the price vector p'' be defined as $p''_j = \max\{p_j, p'_j\}$ for all $j \in \{0, a, b\}$. Since $p''_x = p_x$ we know by gross substitutes that there exists some $w \in D_i(p'')$ such that $x \subseteq w$. By price monotonicity $(w, p'_w) \succ_i (w, p''_w) \succsim_i (o, p''_o) \sim_i (o, p'_o)$ for any $o \in \mathcal{I}$ for which $x \nsubseteq o$. Therefore, $x \in w'$ for all $w' \in D_i(p')$. □

Proposition 3. *If the gross substitutes condition is fulfilled for the approximated preference relation of each bidder $i \in N$, then $\mathcal{P}$ forms a complete lattice.*

Proof. It will first be shown that if $p', p'' \in \mathcal{P}$, then $s \in \mathcal{P}$ and then that $h \in \mathcal{P}$ as well. Combining this with the fact that $\mathcal{P}$ is bounded from below by the seller's reservation prices and from above by some bidder's report v, we can conclude that $\mathcal{P}$ forms a complete lattice.

By definition $p_0 = 0$ for any p, so p_a and p_b are the prices of interest. If $\#N_x < q_x$ for some $x \in ab$ at some $p' \in \mathcal{P}$, then we must have $p_x = r_x$ for all $p \in \mathcal{P}$. Therefore, for any $p', p'' \in \mathcal{P}$, $s \in \mathcal{P}$. Now let $\langle p', \mu' \rangle$ and $\langle p'', \mu'' \rangle$ be two distinct approximated Walrasian equilibria where p' and p'' are such that $p'_a > p''_a > r_a$ and $p''_b > p'_b > r_b$. Hence, $\#N_a = q_a$ and $\#N_b = q_b$ for both μ' and μ''. Let μ_p be an assignment associated with the price vector p. It will first be shown that $\mu'(i) = \mu''(i)$ for all $i \in N$ and secondly that it is possible to let $\mu'(i) = \mu''(i) = \mu_s(i) = \mu_h(i)$ for all $i \in N$. Therefore, $\langle s, \mu_s \rangle$ and $\langle h, \mu_h \rangle$ are two approximated Walrasian equilibria.

If $\mu'(i) = a$ for any $i \in N$, then $a \subseteq \mu''(i)$ by Lemma 6. In order to derive a contradiction, assume $ab \in D_i(p'')$, which by Lemma 6 implies that $b \subseteq w$ for all $w \in D_i(p')$, which is a contradiction. Hence, $\mu'(i) = a$ implies that $\mu''(i) = a$. Now assume $\mu''(i) = a$ and $\mu'(i) \neq a$. Since $\#N_a = q_a$ and $\#N_b = q_b$ under both μ' and μ'', there has to exist some $j \in N \setminus \{i\}$ such that either $a \subseteq \mu'(j)$ and $a \nsubseteq \mu''(j)$, or $b \subseteq \mu''(j)$ and $b \nsubseteq \mu'(j)$, which we know by Lemma 6 does not exist. Therefore, $\mu''(i) = a$ implies that $\mu'(i) = a$. If $\mu'(i) = ab$, then $a \subseteq \mu''(i)$ by Lemma 6, which, by using the same arguments as before, implies that $\mu''(i) = ab$. By symmetry the above arguments apply for the case when a and b, together with the assignments, are interchanged. The previous arguments together imply that if $\mu'(i) = 0$ then $\mu''(i) = 0$.

Now to the second part. For any $i \in N$ for whom $\mu'(i) = \mu''(i) = y$ for any $y \in \{0, a, b\}$ we know by price monotonicity that $(y, s_y) \succsim_i (x, s_x)$ for any $x \in \{0, a, b\}$. In order to derive a contradiction assume that $(ab, s_a + s_b) \succ_i (y, s_y)$. By gross substitutes $a \subseteq w$ for some $w \in D_i(p'')$ and $b \subseteq w$ for some $w \in D_i(p')$. From Lemma 6 it follows that $ab = D_i(p'') = D_i(p')$, which is a contradiction. Finally, for any $i \in N$ for whom $\mu'(i) = \mu''(i) = ab$, it follows by price monotonicity that $(ab, s_a + s_b) \succ_i (0, 0)$, $(ab, s_a + s_b) \succ_i (ab, p'_a + p'_b) \succsim_i (b, p'_b) \sim_i (b, s_b)$, and $(ab, s_a + s_b) \succ_i (ab, p''_a + p''_b) \succsim_i (a, p''_a) \sim_i (a, s_a)$. It is therefore possible to let $\mu(i) = ab$. Therefore, $s \in \mathcal{P}$.

Lastly it will be shown that $h \in \mathcal{P}$ as well. For any $i \in N$ for whom $\mu'(i) = \mu''(i) = y$ for any $y \in \{0, a, b\}$ we know by price monotonicity that $(y, h_y) \succsim_i (x, h_x)$ for any $x \in \mathcal{I}$. If $\mu'(i) = \mu''(i) = ab$, then $a \in w$ and $b \in w'$ for some $w, w' \in D_i(h)$ by gross substitutes. Assume $ab \notin D_i(h)$ and $a, b \in D_i(h)$. However, for any price vector p such that $p_a < h_a$ and $p_b = h_b$ it follows by price monotonicity that for a price decrease sufficiently small, $b \notin D_i(p)$, which contradicts the gross substitutes condition. Thus, $h \in \mathcal{P}$. □

9 Appendix C: Proofs Related to the Process

For many of the proofs in this section, the following sets of packages are introduced: Let $C_a = \{a, ab, \{a, ab\}\}$, $C_b = \{b, ab, \{b, ab\}\}$ and $C_{a,b} = \{\{a, b\}, \{a, b, ab\}\}$. The reason for this is that the approximated demand correspondence of any bidder who demands some package $x \neq 0$, at some p, is a subset of at least one of C_a, C_b, and $C_{a,b}$. Therefore, at any price vector p, it is possible to collect any bidder who demands at least some package $x \neq 0$ into at least one of the following sets: Let $\mathcal{D}_a(p) = \{i \in N \mid D_i(p) \in C_a\}$, $\mathcal{D}_b(p) = \{i \in N \mid D_i(p) \in C_b\}$, $\mathcal{D}_{a,b}(p) = \{i \in N \mid D_i(p) \in C_{a,b}\}$, and $\mathcal{D}_{ab}(p) = \{i \in N \mid D_i(p) = \{ab\}\}$. These sets will be very useful in many of the proofs in this section.

Proposition 4. *For a given price vector p, there exists an assignment μ such that $\mu(i) \in D_i(p)$ for all bidders $i \in N$ if and only if $K_N(x,p) \leq q_x$ for all $x \in \mathcal{I}$.*

Proof. We start by showing the if part of Proposition 4: If there exists an assignment μ for some price vector p such that $\mu(i) \in D_i(p)$ for all $i \in N$, then $K_N(x,p) \leq q_x$ for all $x \in \mathcal{I}$.

We know that $K_N(0,p) \leq q_0$ for all p. Note that if $K_i(a,p) = 1$ for some $i \in N$, then $i \in \mathcal{D}_a(p)$. Thus, $K_N(a,p) = \#\mathcal{D}_a(p)$. Since $\mu(i) \in D_i(p)$ $\forall i \in N$, it is implied that $\mathcal{D}_a(p) \subseteq N_a$. As $\#N_a \leq q_a$ by assumption, it therefore follows that $K_N(a,p) = \#\mathcal{D}_a(p) \leq \#N_a \leq q_a$. $K_N(b,p) \leq q_b$ by symmetrical arguments.

We can also note that $K_N(ab,p) = \#\mathcal{D}_a(p) + \#\mathcal{D}_b(p) + \#\mathcal{D}_{a,b}(p)$ since $K_i(ab,p) = 1$ for any $i \in N$ whenever $D_i(p) \in C_a \cup C_b \cup C_{a,b} \setminus ab$, $K_i(ab,p) = 2$ whenever $D_i(p) = ab$, and $\mathcal{D}_a(p) \cap \mathcal{D}_b(p) \cap \mathcal{D}_{a,b}(p) = \mathcal{D}_{ab}(p)$. Since μ is such that $\mu(i) \in D_i(p)$ for all $i \in N$ by assumption, it follows that $\mathcal{D}_a(p) \cup \mathcal{D}_b(p) \cup \mathcal{D}_{a,b}(p) = N_a \cup N_b$ and $\mathcal{D}_{ab}(p) \subseteq N_a \cap N_b$. Therefore, $K_N(ab,p) = \#\mathcal{D}_a(p) + \#\mathcal{D}_b(p) + \#\mathcal{D}_{a,b}(p) \leq \#N_a + \#N_b \leq q_a + q_b = q_{ab}$.

We continue by showing the only if part of Proposition 4: If $K_N(x,p) \leq q_x$ for all $x \in \mathcal{I}$ at some p, then there exists an assignment μ such that $\mu(i) \in D_i(p)$ for all $i \in N$.

As $K_N(x,p) \leq q_x$ for all $x \in \mathcal{I}$, we know from before that $\#\mathcal{D}_a(p) \leq q_a$, $\#\mathcal{D}_b(p) \leq q_b$ and $\#\mathcal{D}_a(p) + \#\mathcal{D}_b(p) + \#\mathcal{D}_{a,b}(p) \leq q_a + q_b$. Assume that at some price vector p there does not exist a μ such that $\mu(i) \in D_i(p)$ for all $i \in N$, which implies that for all assignments there exists at least one bidder $i \in N$ such that $\mu(i) \notin D_i(p)$. Denote this bidder by k. Note that we can always let $\mu(k) = 0$ so $k \in \mathcal{D}_a(p) \cup \mathcal{D}_b(p) \cup \mathcal{D}_{a,b}(p)$. Moreover, if $\mu(k) = ab$, then it is possible to remove items in order for $\mu(k) \in D_k(p)$. If there would exist a group of bidders $S \subseteq N$ for which $\mu(i) \notin D_i(p)$ for all $i \in S$, then the following arguments would apply to each bidder $i \in S$ individually.

We will focus our attention on an assignment, denoted μ, for which $\#N_x \leq q_x$ for all $x \in ab$, and where each bidder $j \in N \setminus \{k\}$ is matched to a minimal element, w.r.t cardinality, of her demand correspondence. We will show, by way of contradiction, that it is always possible to construct μ such that each bidder is assigned something which she demands. As $\mu(k) \neq ab$, and $\mu(j) = ab$ if and only if $j \in \mathcal{D}_{ab}(p)$ for all $j \in N \setminus \{k\}$ we know that $\mathcal{D}_{ab}(p) \supseteq N_a \cap N_b$.

Obviously, it cannot be that $\#N_x < q_x$ for all $x \in ab$. Let $x \neq y$ for $x, y \in ab$. There are two cases to consider:

Case 1: $\#N_l = q_l$ for all $l \in \{a,b\}$. We cannot have $\mu(k) = 0$ because then $\mathcal{D}_a(p) \cup \mathcal{D}_b(p) \cup \mathcal{D}_{a,b}(p) \supset N_a \cup N_b$ and $K_N(ab,p) = \#\mathcal{D}_a(p) + \#\mathcal{D}_b(p) + \#\mathcal{D}_{a,b}(p) > \#N_a + \#N_b = q_a + q_b = q_{ab}$. Therefore, $\mu(k) = x$ and hence $y \subseteq w$ for all $w \in D_k(p)$, as we otherwise would have $\mu(k) \in D_k(p)$. From this it follows that $k \in \mathcal{D}_y$ and as $y \nsubseteq \mu(k)$ it must either be that $k \in \mathcal{D}_{ab}(p) \supset N_a \cap N_b$, in which case $K_N(ab, p) = \#\mathcal{D}_a(p) + \#\mathcal{D}_b(p) + \#\mathcal{D}_{a,b}(p) > \#N_a + \#N_b = q_a + q_b = q_{ab}$, or $k \in \mathcal{D}_y(p) \setminus \mathcal{D}_{ab}(p)$, which implies that there does not exist a bidder $j \in \mathcal{D}_{a,b}(p)$ such that $y \subseteq \mu(j)$. If this was true, it would be possible to switch the assignment between bidder k and bidder j yielding $\mu(i) \in D_i(p)$ for all $i \in N$. As $y \nsubseteq \mu(j)$ for all $j \in \mathcal{D}_{a,b}(p)$, and $k \in \mathcal{D}_y$, it follows that $N_y \subset \mathcal{D}_y$, and thus $K_N(y, p) = \#\mathcal{D}_y(p) > \#N_y = q_y$, which is a contradiction.

Case 2: $\#N_x < q_x$ and $\#N_y = q_y$. Now we can always let $\mu(i) = x$ and if $N_x = q_x$ in consequence of this, we are back in case 1. As $\mu(k) = x \notin D_k(p)$ we know that $y \in w$ for all $w \in D_k(p)$, and $k \in \mathcal{D}_y$. As $\#N_x < q_x$ it is implied that there does not exist a bidder $j \in \mathcal{D}_{a,b}(p)$ such that $y \in \mu(j)$ because then it would be possible to switch the assignment between bidder k and bidder j. Therefore, $N_y \subset \mathcal{D}_y$, and $K_N(y, p) = \#\mathcal{D}_y > \#N_y = q_y$.
□

Lemma 1. *$O(p)$ has a unique minimal element with respect to cardinality denoted $O_*(p)$.*

Proof. By the construction of $O(p)$ we know that $g(x, p) = g(y, p)$ for all $x, y \in O(p)$. Since $\#0 < \#a = \#b < \#ab$, we need to show that $a, b \in O_*(p)$ can never be true.

We will start by showing that if $x \subseteq y$ for any $x, y \in \mathcal{I}$, then $K_i(x) \leq K_i(y)$ for each $i \in N$. To derive a contradiction, assume that $x \subseteq y$ and $K_i(x) > K_i(y)$ for some $i \in N$, which is equivalent to

$$\min_{w \in D_i(p)} \#(x \cap w) > \min_{w \in D_i(p)} \#(y \cap w)$$

Let $w_1 \in \arg\min_{w \in D_i(p)} \#(x \cap w)$ and $w_2 \in \arg\min_{w \in D_i(p)} \#(y \cap w)$. If $w_1 = w_2 = w$, then $\#(x \cap w) > \#(y \cap w)$ implies that $x \nsubseteq y$. If, on the other hand, $w_1 \neq w_2$, then it must be that $\#(x \cap w_2) \geq \#(x \cap w_1) > \#(y \cap w_2)$, which in turn implies that $x \nsubseteq y$.

We will now show that $K_i(ab, p) \geq K_i(a, p) + K_i(b, p)$ for each $i \in N$. Since $a \subseteq ab$ and $b \subseteq ab$ it follows, by the above, that $K_i(ab, p) \geq \max\{K_i(a, p), K_i(b, p)\}$. Assume that $K_i(ab, p) < K_i(a, p) + K_i(b, p)$ for some $i \in N$ at some p. As $K_i(a, p), K_i(b, p) \in \{0, 1\}$ we must have that $K_i(a, p) = K_i(b, p) = 1$. However, $K_i(a, p) = K_i(b, p) = 1$ implies that $D_i(p) = ab$ and thus that $K_i(ab, p) = K_i(a, p) + K_i(b, p)$ for each $i \in N$.

$K_i(ab, p) \geq K_i(a, p) + K_i(b, p)$ for each $i \in N$ implies that $K_N(ab, p) \geq K_N(a, p) + K_N(b, p)$ as well as $g(ab, p) \geq g(a, p) + g(b, p)$. Since $g(0, p) = 0$ for all p we have that if $O_*(p) = 0$, then $g(x, p) \leq 0$ for all $x \in \mathcal{I}$. So, if $a, b \in O_*(p)$, then $g(a, p) = g(b, p) = s$ for some $s > 0$ and $g(ab, p) \geq 2s$ by the arguments above. This implies that $O(p) = O_*(p) = ab$, which is a contradiction. □

Lemma 2. *If $\varepsilon(t) = 0$ in step 2 of process 1, then $\varepsilon(t) > 0$ in step 3 of process 1.*

Proof. By construction of Process 1, we know that $0 = O_*(p^t)$ if and only if $t = T$. So assume that $t < T$, $O_*(p^t) = x$ for some $x \in \mathcal{I} \setminus 0$ and that $\varepsilon(p^t) = 0$ in step 2. It will be shown that at any p^t there always exist some $e > 0$ and $\delta(p^t)$ such that $O_*(p^t + e\delta(p^t)) = O_*(p^t)$, and hence $\varepsilon(p^t) > 0$.

If $x = O_*(p^t) \in ab$, then by gross substitutes and price monotonicity it must be that by only raising the price of item y, the demand for x is weakly increased and the demand for the other packages contained in $\mathcal{I} \setminus 0$ are weakly decreased. As a consequence, the aggregate requirement of x weakly increases as well. Therefore, if $\delta_0(p^t) = 0$, $\delta_x(p^t) = 1$, and $\delta_y(p^t) = \infty$, then $O_*(p^t + e\delta(p^t)) = O_*(p^t)$ for some $e > 0$ sufficiently small in step 3 of the process and there exists $\varepsilon(t) > 0$.

Assume $O_*(p^t) = ab$. The idea of this part of the proof is to construct a particular price vector $p' \geq p^t$ and to show that the requirement for $ab = O_*(p^t)$ is greater than for any other package at p'. To simplify notation, let $S = \mathcal{D}_{a,b}(p^t) = \{i \in N \mid D_i(p) \in C_{a,b}\}$. Furthermore, let $q_x^S(p) = q_x - K_{N\setminus S}(x, p)$ for any $x \in ab$ at some p. Let p' be a price vector such that $p'_x > p^t_x$ for at least some $x \in ab$. Note that $K_i(ab, p^t) = K_i(a, p^t) + K_i(b, p^t)$, for any $i \in N \setminus S$ at any p^t and that for any $p' \geq p^t$ it is possible to make the price increase sufficiently small such that $K_i(ab, p') = K_i(a, p') + K_i(b, p')$ and $K_i(x, p') \geq K_i(x, p^t)$ for any $x \in \mathcal{I}$. Therefore, at any such p' it must be that $q_a^S(p') \leq q_a^S(p^t)$ and $q_b^S(p') \leq q_b^S(p^t)$. Moreover, for any $i \in S$ we have $K_i(ab, p^t) = 1$, $K_i(x, p^t) = 0$ for any $x \in \mathcal{I} \setminus ab$. Therefore, $g(ab, p^t) = \#S - q_a^S(p^t) - q_b^S(p^t)$.

It will now be shown that for any $p' \geq p^t$, where the price increase is sufficiently small, $D_i(p') \neq \{ab\}$ for any $i \in S$. If $ab \notin D_i(p^t)$ for any $i \in S$, then any such $p' \geq p^t$ can be found by making the price increase sufficiently small. If $D_i(p^t) = \{a, b, ab\}$ however, then $D_i(p') = \{ab\}$ would violate the gross substitutes condition. It can be noted that p^t is the price vector where the three approximated indifference curves, f_1, f_2, and f_3, intersect for bidder $i \in S$. If $D_i(p') = ab$ for some $p' \geq p^t$, then $p'_x > p^t_x$ for all $x \in ab$ and we must by price monotonicity have that $f_2(p'_a) = p''_b > p'_b$, and $f_3(p'_b) = p''_a > p'_a$. Therefore, $m_2 = \frac{p''_b - p^t_b}{p'_a - p^t_a} > m_4 = \frac{p'_b - p^t_b}{p''_a - p^t_a} > 0$. Let c be a price vector such that $f_2(c_a) = c_b$ and $c_a + c_b = v_{ab}$. Since $m_2 > m_4$ it must be that $f_3(c_b) = c'_a > c_a$ and $D_i(c) = \{a, ab, 0\}$. Let c'' be a price vector such that $c''_a = c_a$ and $c''_b = c_b + \gamma$ for some $\gamma > 0$. Then we must have $D_i(c'') = 0$ for some $\gamma > 0$ sufficiently small as it is always possible to find c'' such that $f_3(c''_b) = c'''_a > c''_a$, $c''_a + c''_b > v_{ab}$, and $f_2(c''_a) = c'''_b > c''_b$, which by price monotonicity implies that $(0, 0) \sim_i (ab, v_{ab}) \succ_i (ab, c''_a + c''_b) \succ_i (x, c''_x)$ for $x \in ab$. However, this contradicts the gross substitutes condition as $a \nsubseteq w$ for any $w \in D_i(c'')$.

As $D_i(p') \neq ab$ for any $i \in S$ and $p' \geq p$, where the price increase is sufficiently small, it must be possible to construct p' such that $K_i(ab, p') = 1$, for any $i \in S$. Therefore,

$$0 < g(ab, p^t) = \#S - q_a^S(p^t) - q_b^S(p^t) \leq \#S - q_a^S(p') - q_b^S(p') = g(ab, p').$$

The strict inequality follows from $O_*(p^t) = ab$ and the weak inequality from the fact that $q_x^S(p') \leq q_x^S(p^t)$ for $x \in ab$ and some $p' \geq p^t$. So, if $q_x^S(p^t) < 0$ for all $x \in ab$, then $g(ab, p') = \#S - q_a^S(p') - q_b^S(p') > \#S - q_x^S(p') \geq g(x, p')$ and $x \in ab$. The weak inequality follows from that $K_i(x, p') \in \{0, 1\}$ for any $i \in S$. There are two cases two consider:

Case 1: $q_a^S(p^t) \geq 0$ and $q_b^S(p^t) \geq 0$. For $g(ab, p^t) > 0$ it has to be that $\#S > q_a^S(p^t) + q_b^S(p^t)$. As before, we have $0 < g(ab, p^t) = \#S - q_a^S(p^t) - q_b^S(p^t) \leq g(ab, p')$. Let m_1^i be the slope of f_1^i for bidder $i \in S$, and note that $f_1(p^t_a) = p^t_b$ for all $i \in S$. Define $T = \{m_1^i \in \mathbb{R} \mid i \in S\}$ and let $n = q_a^S(p^t) + 1$. Pick the nth element from T, which we denote m_1^n. Let $\delta_0(p^t) = 0$, $\delta_b(p^t) = m_1^n$, and $\delta_a(p^t) = 1$. By increasing the prices by $p' = p^t + e\delta(p^t)$ for some $e > 0$ sufficiently small, we must by price monotonicity have that

$(a, p'_a) \succ_i (b, p'_b)$ for a maximum of $q_a^S(p^t)$ bidders who belong to S, $(b, p'_b) \succ_i (a, p'_a)$ for a maximum of $\#S - q_a^S(p^t) - 1$ bidders who belong to S, and $(a, p'_a) \sim_i (b, p'_b)$ for at least one bidder $i \in S$. Therefore,

$$\begin{aligned} g(a, p') \leq & q_a^S(p^t) - q_a^S(p') \\ < & \#S - q_b^S(p^t) - q_a^S(p') \\ \leq & \#S - q_b^S(p') - q_a^S(p') \\ = & g(ab, p') \end{aligned}$$

The first weak inequality follows from the fact that $D_i(p') \neq ab$ for any $i \in S$. The strict inequality follows from $\#S - q_b^S(p^t) > q_a^S(p^t)$. Moreover, $g(b, p') \leq \#S - q_a^S(p^t) - 1 - q_b^S(p') < \#S - q_a^S(p') - q_b^S(p') = g(ab, p')$. Hence, $O_*(p') = ab$, and there exist $e, \delta(p^t)$ such that $\varepsilon(t) > 0$ in step 3 of the process.

Case 2: $q_a^S(p^t) \geq 0$ and $q_b^S(p^t) < 0$. For $g(ab, p^t) > 0$ we need $g(ab, p^t) = \#S - q_a^S(p^t) - q_b^S(p^t) > -q_b^S(p^t) = g(b, p^t)$, or $\#S > q_a^S(p^t)$. Moreover, $\#S > q_a^S(p^t) \geq q_a^S(p')$ from before. Let p' be such that $p'_a = p_a^t$ and $p'_b = p_b^t + \gamma$. Then for some $\gamma > 0$ sufficiently small it must by price monotonicity be that $(a, p'_a) \succ_i (b, p'_b)$ for all $i \in S$. Combining this with $D_i(p') \neq ab$ for any $i \in S$ we have, $g(a, p') = \#S - q_a^S(p') < \#S - q_a^S(p') - q_b^S(p') = g(ab, p')$, and $g(b, p') = -q_b^S(p') < \#S - q_a^S(p') - q_b^S(p') = g(ab, p')$ since $\#S > q_a^S(p')$. Hence, $O_*(p') = ab$, and there exist $e, \delta(p^t)$ such that $\varepsilon(t) > 0$ in step 3 of the process. Symmetric arguments can be used if $q_b^S(p^t) \geq 0$ and $q_a^S(p^t) < 0$. □

The proof of Theorem 1 will be decomposed into Lemma 7 and Lemma 9. Lemma 8 will aid in the proof of Lemma 9.

Lemma 7. $p^{min} \leq p^T$

Proof. It will be shown that for any $p \leq p^{min}$, for which $p_x < p_x^{min}$ for some $x \in ab$, it must be that $O_*(p) \neq 0$. As the prices are bounded from below by the seller's reservation prices it is assumed that $p_x^{min} > r_x$ for at least some $x \in ab$. p is constructed such that $p_x < p_x^{min}$ for at least some $x \in ab$. Thus, $p \notin \mathcal{P}$.

If it is possible to construct some assignment μ_p at price vector p such that $\#N_x = q_x$ for any $x \in ab$, or alternatively $\#N_x < q_x$ for any $x \in ab$ for which $p_x = r_x$, then there must exist $i \in N$ for whom $\mu_p(i) \notin D_i(p)$ as $p \in \mathcal{P}$ otherwise. $p \in \mathcal{P}$ would contradict the minimality of p^{min}. By Proposition 4 it follows that $K_N(x, p) > q_x$ for some package $x \in \mathcal{I}$ and since $K_N(0, p) \leq q_0$ for all p it must be that $O_*(p) \neq 0$.

Now assume, in order to derive a contradiction, that μ_p can only be constructed such that $\#N_x < q_x$ and $p_x > r_x$ for at least some $x \in ab$ and that $\mu(i) \in D_i(p)$ for all $i \in N$. Then it must be possible to find a price vector $p' \leq p$ where an assignment can be constructed such that $\mu(i) \in D_i(p')$ for all $i \in N$ and $\#N_w = q_w$ for any $w \in ab$ for which $p'_w > r_w$ and $\#N_w \leq q_w$ for any $w \in ab$ for which $p'_w = r_w$. To see this it can be noted that, by price monotonicity, the demand for any $w \in ab$ weakly increases as p_w is decreased. Therefore, by decreasing p_x to p'_x it must be possible to find a price vector p' and an assignment such that either $p'_x > r_x$ and $\#N_x = q_x$ or $p'_x = r_x$ and

$\#N_x \leq q_x$. Furthermore, the demand for the other item $y \in ab$, for which $y \neq x$, has weakly decreased. Therefore, $\#N_y \leq q_y$, $p'_y \geq r_y$. Moreover, $\mu(i) \in D_i(p')$ for all $i \in N$ as there would otherwise exist excess demand for item x, which could be eliminated by raising its price, as there was no excess demand at p. If $\#N_y < q_y$ and $p'_y > r_y$, then the price of item y can be decreased in the same manner. By repeating this process, it must be possible to find some $p' \leq p$, where an assignment can be constructed, such that $\mu(i) \in D_i(p')$ for all $i \in N$ and $\#N_x = q_x$ for any $x \in ab$ for which $p'_x > r_x$ and $\#N_x \leq q_x$ for any $x \in ab$ for which $p'_x = r_x$. This implies however that $p' \in \mathcal{P}$, contradicting the minimality of p^{min}. There therefore exists $i \in N$ such that $\mu(i) \notin D_i(p)$ and by Proposition 4 it follows that $K_N(x,p) > q_x$ for some package $x \in \mathcal{I}$ and since $K_N(0,p) \leq q_0$ for all p it must be that $O_*(p) \neq 0$. □

For Lemma 8 let $x \neq y$ for $x, y \in ab$.

Lemma 8. *If for any two price vectors p and p' where $p'_x > p_x$, $p'_y = p_y$, and $y \subseteq w$ for all $w \in D_i(p)$ and some $i \in N$, then $y \subseteq w$ for all $w \in D_i(p')$*

Proof. By symmetry it is enough to consider when $x = a$ and $y = b$. If $b \in D_i(p)$ for any $i \in N$, then $(b, p'_b) \succ_i (k, p'_k)$ for all $k \in \mathcal{I} \setminus b$ by price monotonicity. If $ab = D_i(p)$, then $f_2(p_a) = p_b^2 > p_b$ by price monotonicity. If, to derive a contradiction, $a \in D_i(p')$, then $f_2(p'_a) = p_b'^2 \leq p'_b = p_b$ and $m_2 = \frac{p_b'^2 - p_b^2}{p'_a - p_a} < 0$. Let p'' be a price vector where $p''_a = p_a$ and $p''_b = p_b + \gamma$ for some $\gamma > 0$ sufficiently small such that $D_i(p'') = ab$ as well. As $m_2 < 0$ there exists a price vector k, for which $k_a < p'_a$ and $k_b = p''_b$, where $f_2(k_a) = k_b^2 < k_b$ and hence $(a, k_a) \succ_i (ab, k_{ab})$. Moreover, as $a \in D_i(p')$ and $k_a < p'_a$ and $k_b > p'_b$ we must by price monotonicity have $(a, k_a) \succ_i (x, p_x)$ for $x \in \{b, 0\}$ as well. Hence, $D_i(k) = a$, which contradicts the gross substitutes condition since $b \notin w$ for any $w \in D_i(l)$.

Now we will show that $ab = D_i(p)$ implies that $(b, p'_b) \succ_i (0,0)$. Assume $(0,0) \succsim_i (b, p'_b)$, which by price monotonicity implies that $p'_b = p_b \geq v_b$. For some price vector k such that $k_b = p_b + \gamma$ and $k_a = p_a$ for some $\gamma > 0$ sufficiently small we must have $D_i(k) = ab$ as well. Let k' be a price vector where $k'_b = k_b$ and $k'_a > k_a$ kuch that $k'_b + k'_a > v_{ab}$. From the previous arguments we know that $a \notin D_i(k')$. Therefore, $0 = D_i(k')$. This however, violates the gross substitutes condition since $b \notin w$ for any $w \in D_i(k')$. Hence, $(b, k_b) \succ_i (0,0)$, which concludes the proof.

□

Lemma 9. $p^T \leq p^{min}$

Proof. To derive a contradiction assume that $p^t \leq p^{min}$ for some $t < T$ but $p_x^{t+1} > p_x^{min}$ for some $x \in ab$. Denote the unique minimal set in excess demand at time t by $O_*(p^t)$. We know that there must exist some t and $e \in [0, \varepsilon(t))$ such that $p'(e) = p^t + e\delta(p^t) \leq p^{min}$. As $e < \varepsilon(t)$, it follows that $O_*(p^t) = O_*(p'(e)) \neq 0$. Let $c(p) = \{x \in ab \mid p_x = p_x^{min}\}$ for any p. Moreover, let $c_1 = O_*(p'(e)) \cap c(p'(e))$ and $c_2 = O_*(p'(e)) \setminus c_1$. We start by noting that if $g(x, p'(e)) > 0$ for $x \in ab$, then $K_N(x,p) = \#\mathcal{D}_x(p'(e)) > q_x$. There are two cases to consider:

Case 1: $c_1 \neq \emptyset$. If $g(c_1, p'(e)) > 0$, then either $c_1 = ab$, in which case $p^{min} \notin \mathcal{P}$, or $c_1 \in ab$, which implies that $K_N(c_1, p'(e)) = \#\mathcal{D}_{c_1}(p'(e)) > q_{c_1}$. As $c_1 \subseteq w$ for all $w \in D_i(p'(e))$ for all $i \in \mathcal{D}_{c_1}(p'(e))$, it follows by Lemma 8 that $c_1 \subseteq w$ for all $w \in D_i(p^{min})$ for any such bidder i as well. Therefore, $K_N(c_1, p^{min}) \geq K_N(c_1, p'(e))$ and hence $g(c_1, p^{min}) > 0$, which contradicts that $p^{min} \in \mathcal{P}$.

Now assume that $g(c_1, p'(e)) \leq 0$, which implies that $c_1 \in ab$ and $O_*(p'(e)) = ab$. To simplify let $c_1 = a$ and $c_2 = b$. By symmetry, the following arguments can be used when a and b are interchanged. It will now be shown that $g(a, p^{min}) > 0$. To see this we start by noting that as $a, b \in D_i(p'(e))$ for all $i \in \mathcal{D}_{a,b}(p'(e))$, it follows that $K_i(ab, p'(e)) = 1$ for any such bidder $i \in N$. Therefore, it follows that $g(ab, p'(e)) = \#\mathcal{D}_{a,b}(p'(e)) + g(a, p'(e)) + g(b, p'(e))$ and we know that $\#\mathcal{D}_{a,b}(p'(e)) \geq 1$ since $O_*(p'(e)) = ab$ and $g(a, p'(e)) \leq 0$. Moreover, as $O_*(p'(e)) = ab$ we know that $\#\mathcal{D}_{a,b}(p'(e)) + g(a, p'(e)) + g(b, p'(e)) > g(b, p'(e))$ or $\#\mathcal{D}_{a,b}(p'(e)) + g(a, p'(e)) > 0$. By gross substitutes and price monotonicity it must be that $K_i(a, p^{min}) \geq K_i(a, (p'(e)))$ for all $i \in N$. In particular, since $a, b \in D_i(p'(e))$ for all $i \in \mathcal{D}_{a,b}(p'(e))$, it follows that $K_i(a, p'(e)) = 0$ and by gross substitutes and price monotonicity that $K_i(a, p^{min}) = 1$ for any such bidder $i \in \mathcal{D}_{a,b}(p'(e))$. As $\#\mathcal{D}_{a,b}(p'(e)) + g(a, p'(e)) > 0$, it must be that $g(a, p^{min}) \geq \#\mathcal{D}_{a,b}(p'(e)) + g(a, p'(e)) > 0$, which is a contradiction.

Case 2: $c_1 = \emptyset$ and $c(p'(e)) \neq \emptyset$. As $c_1 = \emptyset$ and $c(p'(e)) \neq \emptyset$ it must be that $e, \delta(p^t)$ and $\varepsilon(t)$ are generated in step 3 of Process 1. Furthermore, $c_2 = O_*(p'(e)) \neq \emptyset$ and $O_*(p'(e)) \neq ab$ because if $O_*(p'(e)) = ab$, then $c_1 \neq \emptyset$. For simplicity we can let $c_2 = O_*(p'(e)) = a$ but symmetric arguments apply if $c_2 = b$. Let p'' be defined as $p''_b = p^{min}_b = p'_b(e)$ and $p''_a = p'_a(e) + \gamma$ for some $\gamma > 0$ sufficiently small such that $p''_a < p^{min}_a$. As e was generated in step 3 and $O_*(p^t) = a = O_*(p'(e))$, we know that $\delta_0 = 0$, $\delta_a(p^t) = 1$, and $\delta_b(p^t) = l_b(t)$, where $l_b(t) = \min\{\delta_b(p^t) \in \mathbb{R}_+ \mid \delta_0(p^t) = 0,\ \delta_a(p^t) = 1, \text{ and } \varepsilon(t) > 0\}$. More importantly, as $\varepsilon(t) = 0$ in step 2 of Process 1, $O_*(p'(e)) \neq O_*(p'')$.

Note that as $p''_b = p^{min}_b$ and $p''_a < p^{min}_a$, we know by Lemma 7 that $O_*(p'') \neq 0$. If $O_*(p'') = b$, then $p^{min} \notin \mathcal{P}$ as $g(b, p^{min}) > 0$ by the gross substitutes condition. Thus, $O_*(p'') = ab$, which implies that $g(ab, p'') > g(a, p'')$ or $\#\mathcal{D}_{a,b}(p'') + g(a, p'') + g(b, p'') > g(a, p'')$ and hence $\#\mathcal{D}_{a,b}(p'') + g(b, p'') > 0$. Since $a, b \in D_i(p'')$ for all $i \in \mathcal{D}_{a,b}(p'')$ we know by price monotonicity that $a \notin D_i(p^{min})$ and by Lemma 8 that $b \in D_i(p^{min})$ for all $i \in \mathcal{D}_{a,b}(p'')$ as well. Furthermore, $K_i(b, p^{min}) \geq K_i(b, p'')$ for any $i \in N \setminus \mathcal{D}_{a,b}(p'')$. Therefore, $g(b, p^{min}) > 0$, and/or $g(ab, p^{min}) > 0$, which contradicts that $p^{min} \in \mathcal{P}$. □

Theorem 1. *Process 1 always terminates at $p^T = p^{min}$.*

Proof. Lemma 7 and Lemma 9 together imply Theorem 1. □

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