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Least Manipulable Envy-free Rules in Economies with Indivisibilities*

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Abstract

We consider envy-free and budget-balanced allocation rules for problems where a number of indivisible objects and a fixed amount of money is allocated among a group of agents. In "small" economies, we identify under classical preferences each agent's maximal gain from manipulation. Using this result we find the envy-free and budget-balanced allocation rules which are least manipulable for each preference profile in terms of any agent's maximal gain. If preferences are quasi-linear, then we can find an envy-free and budget-balanced allocation rule such that for any problem, the maximal utility gain from manipulation is equalized among all agents.

JEL Classification: C71, C78, D63, D71, D78.
Key Words: (Least) Manipulability, Envy-freeness, Budget-Balance, Indivisibilities.

1 Introduction

Several seminal papers have investigated the manipulability of competitive mechanisms in classical exchange economies. Hurwicz (1972) has shown that in "small" finite economies any competitive mechanism is manipulable, i.e. for some economies some agents' profit from mispresenting their true preferences may be substantial. Roberts and Postlewaite (1976) have shown as when a small finite economy is replicated, then under certain assumptions, any competitive mechanism becomes limiting incentive compatible.

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In this paper we consider economies with indivisible objects. Any agent's consumption bundle consists of an object and a monetary consumption. Such problems arise in rent division, job allocation, land distribution, and heritage division.[1] Specifically we are interested in investigating the manipulability of envy-free and budget-balanced allocation rules. In our setting Svensson (1983) has shown that there is a close relationship between competitive mechanisms and envy-free and budget-balanced rules. From Green and Laffont (1979) it is known that any such rule is manipulable. Not only this, as we show considering replica of economies with indivisible objects will neither change the set of envy-free and budget-balanced allocations nor alter the amount by which any agent is able to manipulate any envy-free and budget-balanced allocation rule. Therefore, we search for the rules which are "least" manipulable in the class of envy-free and budget-balanced allocation rules in small finite economies.

Specifically, we determine by how much any agent can profit from manipulation for any envy-free and budget-balanced rule. Namely, for any economy and any agent there exist envy-free and budget-balanced allocations which maximize his utility in this set. Then this agent's gain from (optimal) manipulation is equal to the utility difference between this maximizing allocation and the allocation chosen by the rule for this economy. This result then allows us to show the existence of envy-free and budget-balanced rules which are "least" manipulable in the following sense: for each preference profile, the amount by which any agent can manipulate is minimal among all profitable manipulations of all envy-free rules. Under quasi-linear utilities, we show that there exists an envy-free and budget-balanced allocation rule which for each utility profile equalizes the maximal utility gain from manipulation among all agents.

Envy-free allocation rules have been studied extensively in the literature, and consequently also issues related to (non-)manipulability. For example, Alkan, Demange, and Gale (1991) consider the case with two agents and two objects, and demonstrate that there is no envy-free and budget-balanced rule which is not susceptible to manipulation except in degenerate cases. Recently, Andersson, Ehlers, and Svensson (2010) and Fujinaka and Wakayama (2011) independently characterized the set of preference profiles for which an envy-free and budget-balanced allocation rule is non-manipulable by all agents. A different, but related, approach is taken by Tadenuma and Thomson (1995). They consider a model with one indivisible object (but where all agents receive monetary compensations) and study the direct revelation games associated with sub-solutions of the envy-free and budget-balanced solution. The main conclusion is that the set of Nash equilibrium allocations for any such sub-solution coincides with the set of envy-free and budget-balanced allocations of the true preferences. This finding has later been generalized by Beviá (2010), Fujinaka and Wakayama (2011), and Velez (2011). However, except for Fujinaka and Wakayama (2011), none of the above papers have attempted to search for "least" manipulable envy-free and budget-balanced allocation rules in the sense of minimizing the maximal gain from manipulation.

[1] See e.g. Andersson and Svensson (2008), Aragones (1995), Dufton and Larson (2011), Haake, Raith and Su (2000), Klijn (2000), Jaramillo, Kayı and Klijn (2012), Svensson (2009), and Sun and Yang (2003).

The remaining part of this paper is organized as follows. Section 2 introduces economies with indivisibilities and envy-free allocations. Section 3 characterizes any individual's maximizing envy-free allocations. Section 4 contains all our results regarding manipulation of envy-free allocation rules. Section 5 considers the replication of economies.

2 Agents, Preferences and Allocations

Let $N = \{1, ..., |N|\}$ denote the finite set of agents and $M = \{1, ..., |M|\}$ denote the finite set of objects. Throughout we assume $|M| = |N|$.[2] There is a finite amount $m \in \mathbb{R}$ of an infinitely divisible good called *money*. Each object $j \in M$ has a (monetary) compensation denoted by c_j. Let $c \in \mathbb{R}^M$ denote the compensation vector for all objects in M.

A consumption bundle is a pair $(j, c_j) \in M \times \mathbb{R}$ (which stands for consuming object j and receiving compensation c_j). Agent i's preference over consumption bundles is represented by a continuous utility function $u_i : M \times \mathbb{R}^M \to \mathbb{R}$. Let $u_{ij}(c)$ denote the utility of agent i when consuming object j and receiving compensation c_j. The utility function u_i is supposed to satisfy the following three properties: for all $c, c' \in \mathbb{R}^M$, (i) *selfishness*, i.e., $u_{ij}(c) = u_{ij}(c')$ whenever $c_j = c'_j$, (ii) *monotonicity*, i.e., $u_{ij}(c) > u_{ij}(c')$ whenever $c_j > c'_j$, and (iii) *finite compensability*, i.e., for any two consumption bundles (j, c_j) and (k, c_k), there exists a number $\beta \in \mathbb{R}$ such that $u_{ij}(c) = u_{ik}(c')$ for $c'_k = c_k + \beta$ and $c'_l = c_l$ for all $l \neq k$. This means that no object is infinitely desirable or undesirable for any agent. The set of all utility functions having the above properties is denoted by $\mathcal{U}$. A list $u = (u_1, ..., u_n)$ of individual utility functions is a (utility) profile. We adopt the notational convention of writing $u = (u_i, u_{-i})$ for any $i \in N$. The set of all profiles is denoted by $\mathcal{U}^N$.

A feasible assignment $x : N \to M$ assigns every agent $i \in N$ exactly one object $j \in M$ and no object is assigned to more than one agent. Let x_i denote the object assigned to agent $i \in N$. An *allocation* consists of a compensation vector c and a feasible assignment x, and is denoted by (c, x). An allocation (c, x) is *budget-balanced* if $\sum_{j \in M} c_j = m$. For any given profile $u \in \mathcal{U}^N$, an allocation (c, x) is *efficient* if (i) (c, x) is budget-balanced and (ii) there is no other budget-balanced allocation (d, y) such that $u_{iy_i}(d) \geq u_{ix_i}(c)$ for all $i \in N$ with strict inequality holding for some $k \in N$.

Definition 1. At a given profile $u \in \mathcal{U}^N$, an allocation (c, x) is *envy-free* if (i) (c, x) is budget-balanced and (ii) for all $i \in N$ and all $j \in M$, $u_{ix_i}(c) \geq u_{ij}(c)$.

Let $\mathcal{F}(u)$ denote the set of envy-free allocations at a given profile $u \in \mathcal{U}^N$. From, e.g., Alkan, Demange and Gale (1991), Tadenuma and Thomson (1995), and Svensson (1983), it is known that $\mathcal{F}(u)$ is non-empty for each profile $u \in \mathcal{U}^N$ and that each allocation in $\mathcal{F}(u)$ is efficient.

A(n allocation) rule is a non-empty correspondence φ choosing for each profile $u \in \mathcal{U}^N$ a non-empty set of allocations $\varphi(u)$ such that $u_{ix_i}(c) = u_{iy_i}(d)$ for all $i \in N$ whenever $(c, x), (d, y) \in \varphi(u)$, i.e., all agents are indifferent between any two allocations selected by

[2] If $|N| > |M|$, then adding $|N| - |M|$ null objects does not alter our results.

the rule. Such a rule is called *essentially single-valued*. A rule φ is *envy-free* if $\varphi(u) \subseteq \mathcal{F}(u)$ for each profile $u \in \mathcal{U}^N$.

3 Individual Utility Maximizing Envy-free Allocations

For our purposes, it will turn out to be useful to characterize the utility maximizing envy-free allocations for any (individual) agent $k \in N$. Obviously, for any agent $k \in N$ and for each profile $u \in \mathcal{U}^N$, there exists an allocation in $\mathcal{F}(u)$ which maximizes the utility of agent k. This follows simply from the fact that the set $\mathcal{F}(u)$ is compact under our assumptions. For any profile $u \in \mathcal{U}^N$, let $\phi^k(u)$ denote the set of envy-free allocations which maximize the utility for agent $k \in N$. In the remaining part of the paper, let (c^k, x^k) stand for some element in $\phi^k(u)$ unless otherwise stated.

Given an allocation (c, x) and a profile $u \in \mathcal{U}^N$, for any $i, j \in N$ we write $i \rightarrow_{(c,x)} j$ if $u_{ix_i}(c) = u_{ix_j}(c)$, i.e., if agent i is indifferent between his consumption bundle and agent j's consumption bundle at allocation (c, x). Now, to characterize allocation (c^k, x^k) in more detail, the following concepts from Andersson, Ehlers and Svensson (2010) will be useful.

Definition 2. Let $u \in \mathcal{U}^N$ and (c, x) be an envy-free allocation.

(i) An indifference chain at allocation (c, x) consists of a tuple of distinct agents $g = (i_0, ..., i_t)$ such that $i_0 \rightarrow_{(c,x)} i_1 \rightarrow_{(c,x)} \cdots \rightarrow_{(c,x)} i_t$.

(ii) Agent $i \in N$ is linked to agent $k \in N$ at allocation (c, x) if there exists an indifference chain of type $(i_0, ..., i_t)$ at allocation (c, x) with $i = i_0$ and $i_t = k$.

(iii) Allocation (c, x) is agent-k-linked if each agent $i \in N$ is linked to agent k.

An indifference chain is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of linked agents. At agent-k-linked envy-free allocations each agent is linked to agent k through some indifference chain.

The following result establishes that for any profile, agent-k-linked envy-free allocations coincide with the set of envy-free allocations which maximize agent k's utility among all envy-free allocations.

Theorem 1. For each profile $u \in \mathcal{U}^N$, each agent $k \in N$ and each allocation $(c, x) \in \mathcal{F}(u)$, we have:

$(c, x) \in \phi^k(u)$ if and only if allocation (c, x) is agent-k-linked.

Proof. Let $u \in \mathcal{U}^N$, $k \in N$ and $(c,x) \in \phi^k(u)$. First, we demonstrate that (c,x) is agent-k-linked. To obtain a contradiction, suppose that (c,x) is not agent-k-linked, i.e., that there is an agent $l \in N$ that is not linked to agent k. Let

$$\mathcal{G} = \{i \in N : \ i \text{ is linked to } k \text{ at } (c,x)\} \cup \{k\}.$$

Because $k \in \mathcal{G}$ and $l \in N \setminus \mathcal{G}$, both $\mathcal{G}$ and $N \setminus \mathcal{G}$ are non-empty. It follows by construction that $u_{ix_i}(c) > u_{ix_j}(c)$ if $i \in N \setminus \mathcal{G}$ and $j \in \mathcal{G}$. From the Perturbation Lemma in Alkan, Demange and Gale (1991) it then follows that there exists another $(d,y) \in \mathcal{F}(u)$ such that $d_{x_i} > c_{x_i}$ for all $i \in \mathcal{G}$. Then by Definition 1 and monotonicity of u_i it follows that

$$u_{iy_i}(d) \geq u_{ix_i}(d) > u_{ix_i}(c) \text{ for all } i \in \mathcal{G}.$$

Because $k \in \mathcal{G}$ it follows that $u_{ky_k}(d) > u_{kx_k}(c)$, which contradicts the fact that $(c,x) \in \phi^k(u)$ and (c,x) maximizes k's utility among all envy-free allocations. Hence, if $(c,x) \in \phi^k(u)$, then (c,x) is agent-k-linked.

In showing the other direction, let $u \in \mathcal{U}^N$, $k \in N$, and $(c,x),(d,y) \in \mathcal{F}(u)$ be two agent-k-linked envy-free allocations. By the first part of the proof, without loss of generality, we may suppose $(c,x) \in \phi^k(u)$. Obviously, if $c = d$, then for all $i \in N$, $u_{ix_i}(c) = u_{iy_i}(d)$ and $(d,y) \in \phi^k(u)$.

Suppose that $c \neq d$. Since $\sum_{i\in M} c_i = \sum_{i\in M} d_i = m$, the set $G = \{j \in M : c_j < d_j\}$ is non-empty.

We first show for all $i \in N$, if $x_i \in G$, then $y_i \in G$. To obtain a contradiction, suppose that $x_i \in G$ and $y_i \notin G$. But then by Definition 1 and monotonicity of u_i,

$$u_{ix_i}(d) > u_{ix_i}(c) \geq u_{iy_i}(c) \geq u_{iy_i}(d). \tag{1}$$

But this is a contradiction to $(d,y) \in \mathcal{F}(u)$. Hence, $y_i \in G$.

Let $H = \{i \in N : x_i \in G\}$. By $|N| = |M|$ we have $H \neq \emptyset$. Now for $i \in H$, it follows from condition (1) that $y_i \in G$.

First, let $k \in H$. Because (c,x) is agent-k-linked, there exist $i \in N \backslash H$ and $j \in H$ such that $i \rightarrow_{(c,x)} j$. But then we have

$$u_{iy_i}(d) \leq u_{iy_i}(c) \leq u_{ix_i}(c) = u_{ix_j}(c) < u_{ix_j}(d),$$

where the first inequality follows from $i \in N \backslash H$ and $d_{y_i} \leq c_{y_i}$, the second inequality from envy-freeness of (c,x), the equality from $i \rightarrow_{(c,x)} j$, and the last inequality from $x_j \in G$ and $c_{x_j} < d_{x_j}$. Now (d,y) is not envy-free, a contradiction.

If $k \in N \backslash H$, then we obtain similarly a contradiction to envy-freeness of (c,x) using the agent-k-linked allocation (d,y). □

Remark 1. Since agent k utility maximizing envy-free allocations exist for any profile $u \in \mathcal{U}^N$ (because $\mathcal{F}(u)$ is compact), it is clear that agent-k-linked envy-free allocations exist for any profile $u \in \mathcal{U}^N$. In addition, the second part of the proof of Theorem 1 demonstrates that if allocations (c,x) and (d,y) are agent-k-linked, then $c = d$ and all allocations belonging to $\phi^k(u)$ are utility equivalent. Hence, the compensation vector at the agent k utility maximizing allocations is unique. Note, however, that envy-free allocations maximizing the utility of agent k need not be unique because there may exist several assignments for such allocations.

4 Manipulability

Manipulability (and non-manipulability) in our context refers to the following.

Definition 3. A rule φ is manipulable at profile $u \in \mathcal{U}^N$ by agent $i \in N$ if there exists $\hat{u}_i \in \mathcal{U}$ and two allocations $(c, x) \in \varphi(u)$ and $(d, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_{iy_i}(d) > u_{ix_i}(c)$. If the rule φ is not manipulable by any agent at any profile $u \in \mathcal{U}^N$, then φ is said to be strategy-proof (or non-manipulable).[3]

It is well-known from Green and Laffont (1979) that any efficient rule is manipulable for some profile $u \in \mathcal{U}^N$. Since envy-free rules are efficient (Svensson, 1983), this result implies that any envy-free rule is manipulable for some profile $u \in \mathcal{U}^N$.

A natural weakening of strategy-proofness is ω-strategy-proofness where no agent can gain by more than ω from manipulation.

Definition 4. Let $\omega \geq 0$. A rule φ is ω-non-manipulable at profile $u \in \mathcal{U}^N$ if for all $i \in N$ and any $\hat{u}_i \in \mathcal{U}$, and any $(c, x) \in \varphi(u)$ and $(d, y) \in \varphi(\hat{u}_i, u_{-i})$, we have $u_{iy_i}(d) \leq u_{ix_i}(c) + \omega$. If φ is ω-non-manipulable at any profile $u \in \mathcal{U}^N$, then φ is said to be ω-strategy-proof.

Note that 0-strategy-proofness is identical with strategy-proofness.

4.1 Maximal Gain from Manipulation

Because each rule φ that makes a selection from the set $\mathcal{F}(u)$ is manipulable, it is important to characterize exactly how much agents can gain from strategic misrepresentation. The following lemma states that if agent $k \in N$ profitably manipulates any envy-free rule, then the agent must be assigned an object whose compensation has increased.

Lemma 1. For any envy-free rule φ, for any profile $u \in \mathcal{U}^N$, for any agent $k \in N$ and for any $\hat{u}_k \in \mathcal{U}$, we have:

(i) If there exist $(c, x) \in \varphi(u)$ and $(d, y) \in \varphi(\hat{u}_k, u_{-k})$ such that $u_{ky_k}(d) > u_{kx_k}(c)$, then $d_{y_k} > c_{y_k}$.

(ii) If there exist $(c, x) \in \varphi(u)$ and $(d, y) \in \varphi(\hat{u}_k, u_{-k})$ such that $u_{ky_k}(d) \geq u_{kx_k}(c)$, then $d_{y_k} \geq c_{y_k}$.

[3] Note that for single-valued rules (which choose for each profile a unique allocation), Definition 3 may be rewritten as follows: φ is manipulable at $u \in \mathcal{U}^N$ by agent $i \in N$ if there exists $\hat{u}_i \in \mathcal{U}$ such that for $\{(c, x)\} = \varphi(u)$ and $\{(d, y)\} = \varphi(\hat{u}_i, u_{-i})$ we have $u_{iy_i}(d) > u_{ix_i}(c)$.

Proof. We only show (i) since (ii) can be shown similarly. Let $k \in N$, $u \in \mathcal{U}^N$, $\hat{u}_k \in \mathcal{U}$ and φ be any envy-free rule. Suppose that $(c, x) \in \varphi(u)$, $(d, y) \in \varphi(\hat{u}_k, u_{-k})$ and $u_{ky_k}(d) > u_{kx_k}(c)$. Then by envy-freeness,

$$u_{ky_k}(d) > u_{kx_k}(c) \geq u_{ky_k}(c).$$

This and monotonicity of u_i yields $d_{y_k} > c_{y_k}$ which concludes the proof. □

The following result shows that each agent $k \in N$ for each profile $u \in \mathcal{U}^N$ can manipulate any envy-free rule φ at most by receiving the utility from his utility maximizing envy-free allocation(s) (c^k, x^k) at profile $u \in \mathcal{U}^N$, i.e., the agent-k-linked envy-free allocation(s) at profile $u \in \mathcal{U}^N$ by Theorem 1.

For any profile $u \in \mathcal{U}^N$ and any $(c, x) \in \varphi(u)$, let

$$f_k(\varphi, u) = \sup_{\hat{u}_k \in \mathcal{U}} \max_{(d,y) \in \varphi(\hat{u}_k, u_{-k})} u_{ky_k}(d) - u_{kx_k}(c)$$

denote agent k's maximal gain from manipulation at u under φ.

Theorem 2. For any envy-free rule φ, for any profile $u \in \mathcal{U}^N$ and for any agent $k \in N$, we have for $(c^k, x^k) \in \phi^k(u)$ and $(c, x) \in \varphi(u)$, $f_k(\varphi, u) = u_{kx^k_k}(c^k) - u_{kx_k}(c)$.

Proof. Let $k \in N$, $u \in \mathcal{U}^N$, φ be an envy-free rule, and $(c, x) \in \varphi(u)$. Take some $\hat{u}_k \in \mathcal{U}$ and some $(d, y) \in \varphi(\hat{u}_k, u_{-k})$. If

$$u_{ky_k}(d) - u_{kx_k}(c) > u_{kx_k}(c^k) - u_{kx_k}(c),$$

then $u_{ky_k}(d) > u_{kx_k}(c^k)$ which would mean the agent-k-linked envy-free rule is manipulable by agent k, which is a contradiction to Andersson, Ehlers and Svensson (2010, Lemma 7)[4]. Thus, $f_k(\varphi, u) \leq u_{kx^k_k}(c^k) - u_{kx_k}(c)$.

It remains to show that $f_k(\varphi, u) \geq u_{kx^k_k}(c^k) - u_{kx_k}(c)$. Suppose that $(c^k, x^k) \in \phi^k(u)$. In the remaining part of the proof, let $\hat{u}^\varepsilon_k \in \mathcal{U}$ be such that for all $z \in \mathbb{R}^M$, $\hat{u}^\varepsilon_{kj}(z) = z_j - c^k_j$ for all $j \in M \setminus \{x^k_k\}$ and $\hat{u}^\varepsilon_{kx^k_k}(z) = z_{x^k_k} - c^k_{x^k_k} + \varepsilon$ for some "small" $\varepsilon > 0$.

Note first that $(c^k, x^k) \in \mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$. This follows since $(c^k, x^k) \in \mathcal{F}(u)$ and $\hat{u}^\varepsilon_{kx^k_k}(c^k) = \varepsilon > 0 = \hat{u}^\varepsilon_{kj}(c^k)$ for all $j \in M \setminus \{x^k_k\}$ by construction.

Second, by $(c^k, x^k) \in \phi^k(u)$ and Theorem 1, (c^k, x^k) is agent-k-linked under profile u. But now (c^k, x^k) is agent-k-linked under $(\hat{u}^\varepsilon_k, u_{-k})$ and again by Theorem 1, $(c^k, x^k) \in \phi^k(\hat{u}^\varepsilon_k, u_{-k})$. Thus, ε is agent k's maximal utility in $\mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$.

Let $(d, y) \in \varphi(\hat{u}^\varepsilon_k, u_{-k})$. If $c^k_{x^k_k} < d_{x^k_k}$, then by envy-freeness, $\hat{u}^\varepsilon_{ky_k}(d) \geq \hat{u}^\varepsilon_{kx^k_k}(d) > \varepsilon$, which contradicts the fact that ε is agent k's maximal utility in $\mathcal{F}(\hat{u}^\varepsilon_k, u_{-k})$.

Thus, $c^k_{x^k_k} \geq d_{x^k_k}$. We show that $y_k = x^k_k$. Suppose that $y_k \neq x^k_k$. If $d = c^k$, then $\hat{u}^\varepsilon_{ky_k}(d) = 0 < \varepsilon = \hat{u}^\varepsilon_{kx^k_k}(d)$, a contradiction. Thus, $d \neq c^k$. By budget-balance, $c^k_{x^k_k} \geq d_{x^k_k}$

[4] Lemma 7 of Andersson, Ehlers and Svensson (2010) holds in our setting: first, Theorem 1 generalizes Theorem 6 of Andersson, Ehlers, and Svensson (2010) from quasi-linear utilities to our more general setting; and second, the proof of Lemma 7 in Andersson, Ehlers, and Svensson (2010) is not dependent on their quasi-linearity assumption and remains valid in our setting.

and $d \neq c^k$, for some $j \in M \backslash \{x_k^k\}$ we have $c_j^k < d_j$. Now by envy-freeness, $\hat{u}_{ky_k}^{\varepsilon}(d) \geq \hat{u}_{kj}^{\varepsilon}(d) > 0 = \hat{u}_{ky_k}^{\varepsilon}(c^k)$. Thus, $d_{y_k} > c_{y_k}^k$. Let $j \in N$ be such that $x_j^k = y_k$. Now we have

$$u_{jy_j}(d) \geq u_{jy_k}(d) > u_{jy_k}(c^k) \geq u_{jy_j}(c^k),$$

where the weak inequalities follow from envy-freeness and the strict inequality from $d_{y_k} > c_{y_k}^k$. Thus, $d_{y_j} > c_{y_j}^k$. Now again let $l \in N$ be such that $x_l^k = y_j$. Using the same arguments, it can be shown $d_{y_l} > c_{y_l}^k$. Continuing iteratively, now for some $h \in N$ we must (cycle and) have $y_h = x_k^k$. But now again $d_{y_h} > c_{y_h}^k$, or equivalently $d_{x_k^k} > c_{x_k^k}^k$, which is a contradiction to $c_{x_k^k}^k \geq d_{x_k^k}$.

Thus, $y_k = x_k^k$ and $c_{x_k^k}^k \geq d_{x_k^k}$. If $c_{x_k^k}^k = d_{x_k^k}$, then $f_k(\varphi, u) \geq u_{kx_k^k}(c^k) - u_{kx_k}(c)$, the desired conclusion. Let $c_{x_k^k}^k > d_{x_k^k}$. But now we have

$$\hat{u}_{kx_k^k}^{\varepsilon}(d) = d_{x_k^k} - c_{x_k^k}^k + \varepsilon \geq \max_{j \in M \backslash \{x_k^k\}} (d_j - c_j^k) > 0,$$

where the first inequality follows from envy-freeness and the construction of $\hat{u}_k^{\varepsilon}$ and the second inequality follows from $c_{x_k^k}^k > d_{x_k^k}$ and $\sum_{j \in M} d_j = m = \sum_{j \in M} c_j^k$. Thus, $d_{x_k^k} - c_{x_k^k}^k + \varepsilon > 0$. Now as $\varepsilon \to 0$, by $c_{x_k^k}^k > d_{x_k^k}$, we must have $d_{x_k^k} \to_{\varepsilon \to 0} c_{x_k^k}$. Thus, by $y_k = x_k^k$,

$$\lim_{\varepsilon \to 0} u_{ky_k}(d) = \lim_{\varepsilon \to 0} u_{kx_k^k}(d) = u_{kx_k^k}(c^k),$$

and $f_k(\varphi, u) \geq u_{kx_k^k}(c^k) - u_{kx_k}(c)$, the desired conclusion. □

Theorem 2 yields as corollary that if some agent i's maximal gain from manipulation is greater for one rule than for a second rule, then there is another agent j whose maximal gain from manipulation is smaller for the first rule than for the second one.

Corollary 1. For each profile $u \in \mathcal{U}^N$ and for any two envy-free rules φ and ψ, it holds that: if $f_i(\varphi, u) > f_i(\psi, u)$ for some $i \in N$, then $f_j(\varphi, u) < f_j(\psi, u)$ for some $j \in N$.

Proof. Let $(c, x) \in \varphi(u)$ and $(d, y) \in \psi(u)$. Suppose that the statement is not true, i.e., that for all $l \in N$, $f_l(\varphi, u) \geq f_l(\psi, u)$. By Theorem 2, we have then for all $l \in N$, $u_{lx_l}(c) \leq u_{ly_l}(d)$, and $u_{ix_i}(c) < u_{iy_i}(d)$. But now allocation (c, x) is not efficient, which contradicts the fact that all envy-free allocations are efficient (Svensson, 1983). □

Theorem 2 characterizes the exact amount by which an agent may manipulate any envy-free rule. In applications, we may want to minimize the gains from manipulation for all agents in the spirit of Definition 4, i.e., identifying a smallest global bound and an envy-free rule such that for any given profile no agent can manipulate by more than this bound (and we cannot find another envy-free rule with a smaller bound). Of course, by Theorem 2 this approach is fruitless because utilities are arbitrary and for any fixed $\omega \geq 0$ there does not exist an envy-free rule which is ω-strategy-proof (or ω-non-manipulable for all profiles). Instead we follow below a local bound approach, i.e., where the bound is dependent on the given profile and we minimize this bound for any profile.

4.2 Existence

Because each agent can manipulate any envy-free allocation rule it is natural to ask if there is an allocation rule that is better than other from the viewpoint of manipulability. In some recent papers this issue has been investigated by minimizing the number of profiles in $\mathcal{U}^N$ for which the rule is manipulable (see Aleskerov and Kurbanov, 1999; Maus, Peters and Storcken, 2007a,b), by minimizing the domain (with respect to inclusion) on which the rule is manipulable (Pathak and Sönmez, 2013), and by finding rules that prevent the most agents and coalitions of agents by gaining from misrepresentation (Andersson, Ehlers and Svensson, 2010). Here, we have a somewhat different approach and instead search for rules that minimize the maximal gain that any agent can obtain by strategic misrepresentation (see also Fujinaka and Wakayama, 2011). This maximal gain is given by the functions of type f previously given in Theorem 2. Hence, the aim is to identify a rule satisfying the following:

Definition 5. Let $\varpi : \mathcal{U}^N \to \mathbb{R}_+$.

(i) A rule φ is ϖ-strategy-proof if for any profile $u \in \mathcal{U}^N$ we have $\max_{i \in N} f_i(\varphi, u) \leq \varpi(u)$.

(ii) An envy-free rule ψ is least manipulable if for any $\varpi : \mathcal{U}^N \to \mathbb{R}_+$ and any envy-free rule φ which is ϖ-strategy-proof, ψ is ϖ-strategy-proof.

Alternatively, Definition 5 means finding an envy-free rule ψ such that for all $u \in \mathcal{U}^N$:

$$\psi = \arg \min_{\varphi \text{ is envy-free}} \max_{i \in N} f_i(\varphi, u). \tag{2}$$

The following theorem establishes the existence of such rule.

Theorem 3. There exists a least manipulable envy-free rule for each profile $u \in \mathcal{U}^N$.

Theorem 3 follows simply from the fact that for any profile $u \in \mathcal{U}^N$, $\mathcal{F}(u)$ is compact and therefore, by Theorem 2, there exist envy-free allocations $(c, x) \in \mathcal{F}(u)$ which solve

$$\arg \max_{(c,x) \in \mathcal{F}(u)} \min_{k \in N} \left(u_{kx_k^k}(c^k) - u_{kx_k}(c) \right).$$

Now, simply let the rule choose any such allocation for the profile u. Here the important consequence of Theorem 2 is that the maximal gain from manipulation is independent of which envy-free rule is considered.

4.3 Quasi-Linear Utilities

To obtain more specific results, we shall consider the subclass of quasi-linear utility functions $\mathcal{U}_q^N \subset \mathcal{U}^N$, i.e., $u \in \mathcal{U}_q^N$ if and only if for each $i \in N$ there exists $v_i \in \mathbb{R}^M$ such that for all $c \in \mathbb{R}^M$ and all $j \in M$:

$$u_{ij}(c) = v_{ij} + c_j.$$

Under quasi-linear utility functions, the following result from Svensson (2009, Proposition 2) will be useful.[5]

Lemma 2. For each profile $u \in \mathcal{U}_q^N$ and all $(c, x), (d, y) \in \mathcal{F}(u)$, we have $(c, y), (d, x) \in \mathcal{F}(u)$.

Using Theorem 2 and Lemma 2, we can simplify the maximal manipulation possibility of agent k under quasi-linear utilities. More specifically, for any arbitrary envy-free rule φ, any agent $k \in N$, and any $u \in \mathcal{U}_q^N$, let $(c, x) \in \varphi(u)$ and $(c^k, x^k) \in \phi^k(u)$. Now by Lemma 2 we have $(c, x^k) \in \mathcal{F}(u)$ and $u_{kx_k}(c) = u_{kx_k^k}(c)$. Quasi-linearity then yields

$$f_k(\varphi, u) = c^k_{x_k^k} - c_{x_k^k}.$$

The following result establishes that the sum of maximal gains from manipulation (given by the function f), at a given profile, is independent of which envy-free rule is considered. Therefore, if the maximal gain from manipulation decreases for one agent, then maximal gain from manipulation must increase for some other agent.

Theorem 4. Let φ and ψ be two envy-free rules. Then for each profile $u \in \mathcal{U}_q^N$ it holds that:

$$\sum_{i \in N} f_i(\varphi, u) = \sum_{i \in N} f_i(\psi, u). \tag{3}$$

Proof. Let $i, j \in N$, $(c^i, x^i) \in \phi^i(u)$ and $(c^j, x^j) \in \phi^j(u)$. By Lemma 2, we have $(c^j, x^i) \in \mathcal{F}(u)$ and obviously $(c^j, x^i) \in \phi^j(u)$. Thus, without loss of generality, we may assume $x^i = x^j$ for all $i, j \in N$. From the definition of f we obtain

$$\sum_{i \in N} f_i(\varphi, u) = \sum_{i \in N} (c^i_{x_i^i} - c_{x_i^i}) = \sum_{i \in N} c^i_{x_i^i} - \sum_{i \in N} c_{x_i^i},$$
$$\sum_{i \in N} f_i(\psi, u) = \sum_{i \in N} (c^i_{x_i^i} - d_{x_i^i}) = \sum_{i \in N} c^i_{x_i^i} - \sum_{i \in N} d_{x_i^i}.$$

Thus, (3) holds by essentially single-valuedness and Remark 1 if

$$\textstyle\sum_{i \in N} c_{x_i^i} = \sum_{i \in N} d_{x_i^i}. \tag{4}$$

We have $x_i^i \neq x_j^j$ for any $j \in N$ where $j \neq i$. Hence, by Definition 1, feasibility and budget-balance,

$$\textstyle\sum_{i \in N} c_{x_i^i} = \sum_{i \in N} d_{x_i^i} = m. \tag{5}$$

This together with (4) yields the desired conclusion. □

[5] Furthermore, by Svensson (2009, Proposition 3), Definition 3 may be rewritten as follows on the domain of quasi-linear utilities: φ is manipulable at $u \in \mathcal{U}_q^N$ by agent $i \in N$ if there exists $\hat{u}_i \in \mathcal{U}_q$ such that for all $(c, x) \in \varphi(u)$ and all $(d, y) \in \varphi(\hat{u}_i, u_{-i})$ we have $u_{iy_i}(d) > u_{ix_i}(c)$.

Theorem 4 establishes that the sum of maximal individual gains from manipulation (given by the function f), at a given profile, is independent of which envy-free rule is considered. This insight has the same flavor as Andersson, Ehlers, and Svensson (2010, Theorem 1 and Proposition 1): they show that it is impossible to distinguish any two envy-free rules by their "degree of manipulability" if the measure is based on the number (or the set) of profiles at which a given envy-free rule is manipulable (by some agent). Theorem 4 demonstrates that the very same result holds if the (cardinal) measure is based on the sum of maximal individual gains from manipulation. Hence, like in Andersson, Ehlers, and Svensson (2010), a "finer" (cardinal) measure is needed to distinguish envy-free rules. In this sense, Theorem 4 can be seen as a motivation for adopting Definition 5 for a least manipulable envy-free rule (in the quasi-linear domain). Theorem 3 demonstrates that there exists a least manipulable envy-free rule for each profile. Below we show that under quasi-linear preferences this rule selects envy-free allocations such that the maximal gain from manipulation is equal for all agents.

Theorem 5. There exists an envy-free rule φ such that for each profile $u \in \mathcal{U}_q^N$, we have:

$$f_i(\varphi, u) = f_j(\varphi, u) \text{ for all } i, j \in N. \tag{6}$$

Proof. Let $u \in \mathcal{U}_q^N$. By Lemma 2, if $(c, x) \in \mathcal{F}(u)$ and $(d, y) \in \mathcal{F}(u)$, then $(c, y) \in \mathcal{F}(u)$ and $(d, x) \in \mathcal{F}(u)$. For this reason we shall assume in the remaining part of the proof, without loss of generality, that the feasible assignment is identical and given by x for all envy-free allocations in $\mathcal{F}(u)$.

We need to show that there exists a envy-free allocation $(c, x) \in \mathcal{F}(u)$ such that

$$c^i_{x_i} - c_{x_i} = c^j_{x_j} - c_{x_j} \text{ for all } i, j \in N.$$

Consider now the utility maximizing compensation vectors $c^1_{x_1}, ..., c^n_{x_n}$ for agents $1, ..., n$, respectively, at profile $u \in \mathcal{U}_q^N$ and note that they are unique by Remark 1. Since preferences are quasi-linear, we now have for all $i \in N$ and all $(c, x) \in \mathcal{F}(u)$:

$$c^i_{x_i} \geq c_{x_i}. \tag{7}$$

Thus, for any $(c, x) \in \mathcal{F}(u)$, we have $\sum_{i \in N} c^i_{x_i} \geq \sum_{i \in N} c_{x_i} = m$. Now, obviously there exists $\varepsilon \geq 0$ such that

$$\sum\nolimits_{i \in N} (c^i_{x_i} - \varepsilon) = m. \tag{8}$$

Let $c^\varepsilon \in \mathbb{R}^M$ be the compensation vector where $c^\varepsilon_{x_i} = c^i_{x_i} - \varepsilon$ for each $i \in N$ such that (8) holds. To complete the proof we need to demonstrate that $(c^\varepsilon, x) \in \mathcal{F}(u)$. To obtain a contradiction, suppose that $(c^\varepsilon, x) \notin \mathcal{F}(u)$. Then

$$v_{ix_i} + c^\varepsilon_{x_i} < v_{ix_j} + c^\varepsilon_{x_j} \text{ for some } i, j \in N. \tag{9}$$

From the definition of c^ε we obtain that

$$v_{ix_i} + c^i_{x_i} - \varepsilon < v_{ix_j} + c^j_{x_j} - \varepsilon,$$

i.e. (using $c^j_{x_i} \leq c^i_{x_i}$ from (7)):

$$v_{ix_i} + c^j_{x_i} \leq v_{ix_i} + c^i_{x_i} < v_{ix_j} + c^j_{x_j},$$

which contradicts $(c^j, x) \in \mathcal{F}(u)$. Hence, $(c^\varepsilon, x) \in \mathcal{F}(u)$, the desired conclusion. □

The envy-free rule described in the previous proof can be related to the constrained equal losses rule (Aumann and Maschler, 1985; Hokari and Thomson, 2003; Thomson, 2003). To see this connection, suppose that all agents in N are asked by the mechanism designer to select an envy-free allocation at profile $u \in \mathcal{U}^N_q$. Obviously, each agent $k \in N$ would suggest (or claim) an allocation $(c^k, x^k) \in \phi^k(u)$, i.e., an envy-free allocation that maximizes agent k's utility. Again, as above we can fix an assignment x. Obviously $(c^k, x) \in \phi^k(u)$ and, without loss of generality, we may set $x^k = x$ for all $k \in N$. Now we simply let for all $k \in N$:

$$c_{x_k} = c^k_{x_k} - \lambda(u) \text{ and } \lambda(u) \text{ is chosen so that } \sum_{k \in N} (c^k_{x_k} - \lambda(u)) = m.$$

In this sense each agent incurs an equal loss of $\lambda(u)$ between the chosen envy-free allocation and the allocations that maximize his utility among all envy-free allocations. Setting $\varpi(u) = \lambda(u)$ for any $u \in \mathcal{U}^N_q$, Theorem 4 implies that the above rule is least manipulable in the class of envy-free rules on the domain of quasi-linear preferences.

5 Replication of Economies

Roberts and Postlewaite (1976) have shown as when a small finite economy is replicated, then under certain assumptions, any competitive mechanism becomes limiting incentive compatible. More precisely, for any given $\varepsilon > 0$, there is a large enough economy such that the gains from manipulation do not exceed ε. Several subsequent papers have examined different qualifications of the result by Roberts and Postlewaite (1976).[6]

In our model, Svensson (1983) introduced a connection between envy-free allocations and competitive allocations in the following sense: For an envy-free allocation (c, x), the vector $-c = (-c_j)_{j \in M}$ is interpreted as prices which means that agent i pays the price $-c_{x_i}$ for receiving object x_i. Then envy-freeness readily translates to the fact that given the prices $-c$, agent i weakly prefers object x_i to any other object.

One of our motivations for our paper was that replicating the economy does not alter the gains from manipulation of envy-free allocation rules. In other words, even as the economy becomes large leaves the manipulation possibilities unchanged and in determining the least manipulable envy-free rules we need to do this for small economies. To formalize this point, let $E = (N, M, u)$ denote the original economy. Let $E^{<t>}$ denote the t-replica of E with tN agents (each agent $i \in N$ is replicated t times), with tM objects (each object $j \in M$ is replicated t times) and each replica of agent i has i's utility function u_i (where agents have

[6] Among others, Jackson (1992), Manelli and Jackson (1997), Cordoba and Hammond (1998) and Kovalenkov (2002).

identical utilities for an object j and its replicas). Similarly, for an allocation (c, x) of E, let $(c, x)^{<t>}$ stand for the allocation of $E^{<t>}$ where any replica of agent $i \in N$ receives the replica of i's consumption bundle (x_i, c_{x_i}) in (c, x) (and i receives (x_i, c_{x_i})).

The following observations are straightforward: if (c, x) is an envy-free allocation in E, then $(c, x)^{<t>}$ is an envy-free allocation in $E^{<t>}$. Thus, for any agent k, the utility of his maximizing envy-free allocations in E is smaller than or equal to the utility of his maximizing envy-free allocations in $E^{<k>}$. In fact, these utilities must be equal as the following argument shows[7]:

Consider E and $E^{<2>}$ and suppose that some allocation (c, x) in $E^{<2>}$ maximizes agent k's utility among all envy-free allocations in $E^{<2>}$. Note that (c, x) does not need to be a 2-replica of some allocation in E. Since (c, x) is envy-free, now any two agents who receive the replica of the same object must receive identical compensations. Setting $2M = M \cup \{j' : j \in M\}$, we have $c_j = c_{j'}$ for all $j \in M$. But then by budget-balance of (c, x) we must have $\sum_{j \in M} c_j = m$. Now we construct from (c, x) an allocation for E as follows (again setting $2N = N \cup \{i' : i \in N\}$): for any agent $i \in N$, if his replica receives the same consumption bundle as i, then just drop i' and his consumption bundle; otherwise choose the agent (l or l') who receives the same consumption bundle as i' and assign to l the consumption bundle of i' and drop l' and one consumption bundle of i'; now l or l' received an object different than i' and we repeat the procedure for this object; at some point there will be a cycle (going back to i) and we simply keep i's consumption bundle unchanged. Now this gives us an allocation for E which is envy-free. Since we chose an allocation with maximal utility of agent k in the set of envy-free allocations in $E^{<2>}$, now this utility must be identical as in $\phi^k(u)$.

Of course, the above argument is true for E, $E^{<2>}$, $E^{<4>}$, $E^{<8>}, \ldots, E^{<2^t>}, \ldots$, i.e. using the first fact, in E and in all replicas $E^{<t>}$ the maximal utility of agent k among all envy-free allocations is identical. Hence, Theorem 2 applies and the gains from manipulation remain unchanged for envy-free rules in E and $E^{<t>}$.

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[7] We omit the formal details which are available from the authors upon request.

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