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Abstract

We consider taxation of exchanges among a set of agents where each agent owns one object. Agents may have different valuations for the objects and they need to pay taxes for exchanges. We show that if a rule satisfies individual rationality, strategy-proofness, constrained efficiency, weak anonymity and weak consistency, then it is either the no-trade rule or a fixed-tax core rule. For the latter rules, whenever any agent exchanges his object, he pays the same fixed tax (a lump sum payment which is identical for all agents) independently of which object he consumes. Gale's top trading cycles algorithm finds the final assignment using the agents' valuations adjusted with the fixed tax if the induced preferences are strict.

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1 Introduction

A large literature on house exchange problems has been developed since the pioneering work of Shapley and Scarf (1974). These problems contain a finite set of agents, each of

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whom is endowed with a single house. Agents are willing to take part in cyclical exchanges if they are better off by such trades. The key assumption in the model is that monetary transactions are not allowed. In spite of its simplicity, house exchange models have been demonstrated to be powerful in many real-life applications. Maybe the best-known examples are the design of kidney exchange programs (Roth et al., 2004) and school choice mechanisms (Abdulkadiroğlu and Sönmez, 2003). However, even if it is natural to abstain from monetary transfers in some settings, it is very unnatural in others. In a real-life house exchange problem, for example, it is not unlikely that local authorities tax house exchanges.[1] This paper considers a house exchange problem where the monetary transfers are non-positive meaning that agents pay a non-negative tax whenever being involved in a house exchange. Before detailing the model and the main results of the paper, we provide a more general motivation for the type of problem considered here.

In recent years, many online services that facilitate house exchange have been developed. Even if most of these online services arrange temporary trades of vacation homes (e.g., `HomeExchange.com`), there are some alternative websites where persons are helped to perform permanent home swaps. For example, on the UK based site `EasyHouseExchange.com`, homeowners list their properties by, e.g., uploading photographs and detailed descriptions of their houses (including estimated market values), and state what housing they are looking for in return. The main idea is to create trading cycles among house owners. Most of these websites saw the light of the day in the global financial crises in 2008–2009. For example, Sergei Naumov who, in 2009, was the CEO of one of the largest US house exchange platforms `GoSwap.org` stated that:

> "*Since the housing market tanked, homeowners wishing to upgrade to bigger homes, downsize or relocate have become more open to the idea of making a home swap.*"[2]

The main reason for the increased popularity in permanent house exchanges during the financial crises was that some persons lived in houses that they no longer were able to afford. Because they also were unable to find a buyer, it was better to downsize to a smaller house than declaring bankruptcy even if this resulted in a financial loss. Persons with a more advantageous financial situation were not late to take advantage of this situation.

[1] Even in the absence of such tax, it is not unlikely that monetary transfers are needed to compensate for differences in house values.

[2] See `www.bankrate.com/finance/real-estate/home-swap-tough-market-1.aspx`. Retrieved June 1, 2018.

Consequently, permanent trading opportunities emerged. However, even if a homeowner is involved in a permanent cyclical trade, this does by no means imply that the homeowner can avoid paying taxes. For example, experts at `EasyHouseExchange.com` informed their customers that:

> "*Any transaction should be dealt with in the same way that a traditional sale or purchase would be.*"[3]

The framework analyzed in this paper can be thought of as a situation where a social planner attempts to design a tax schedule for house exchange. In relation to the `EasyHouseExchange.com` example and related online services, the findings in this paper can be applied to better understand exactly how to design a tax schedule for house owners involved in permanent home swaps. As will be apparent, these options are very limited, at least if the social planner is interested in a tax scheme satisfying a number of natural and desirable properties.

Each agent is endowed with a single indivisible object and has quasi-linear preferences over consumption bundles. Here, a consumption bundle is a pair consisting of one object and a tax attached to that object. The aim for the social planner is to define a mechanism or, equivalently, a tax schedule that, based on the self-reported preferences of the agents, determines the trades and the taxes. However, such a mechanism is not unique. Consequently, the social planner has the option to restrict the set of possible mechanisms by requiring that the outcome of the mechanism should satisfy a number of desirable properties. These properties are informally described below.

Individual rationality says that each agent weakly prefers his consumption bundle to his endowment and paying no tax. Strategy-proofness ensures that agents honestly report their true preferences over consumption bundles (to the social planner). Constrained efficiency says that the rule is efficient on its range of allocations. Consistency says that the rule is robust subject to the departure of a set of agents with their allotments when those coincide with their endowments. Anonymity says that whenever objects are reassigned, then the names of the agents do not matter. It turns out that both consistency and anonymity are too strong in our context as only the no-trade rule (where all agents always keep their endowments and pay zero tax) satisfies one of these axioms in conjunction with the other requirements and we will, therefore, define weaker versions of these axioms. These weakenings will be motivated in detail in Section 2.1, but the general idea behind them is

[3] See the article "Fair Trade?" in Financial Times (May 17, 2009).

that they should respect the core ideas in the stronger versions of the axioms, but only for the cases that are most relevant for the considered house allocation problem.

The main innovation of this paper is the introduction of a class of taxation rules called Gale's fixed-tax core rules. Here two ingredients play a key role. The first is the no-trade rule that prescribes that all agents keep their endowments and pay no taxes. The second is a given number $\alpha \geq 0$, henceforth referred to as Gale's fixed-tax (a lump sum payment which is identical for all agents). Given the number α, each agent's valuations induce a weak ordinal ranking over the objects. More precisely, an object is weakly preferred over another object if and only if the valuation of the first object minus Gale's fixed-tax is greater than or equal to the valuation of the second object minus Gale's fixed-tax. When the induced rankings are strict, Gale's fixed-tax core rule finds the assignment by applying Gale's top trading cycles mechanism (first defined in Shapley and Scarf, 1974), and agents who keep their endowments pay zero tax whereas agents exchanging their endowment pay the fixed-tax α. The main result of the paper (Theorem 1) shows that any rule satisfying the above properties (individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity) must be either the no-trade rule or a Gale's fixed-tax core rule. Existence of such rules is demonstrated (Theorem 2) by using a construction from Saban and Sethuraman (2013), called the Highest Priority Object (HPO) algorithm.[4]

Our model and the results derived from it relate to and extend previous work in the literature. As we study quasilinear preferences, the Clarke-Groves rules are focal (Clarke, 1971; Groves, 1973). These have the benefits of satisfying our properties *and* utilitarian optimality. Of course, as is well known, this latter property requires that the mechanism be able to sustain an arbitrarily high budget deficit. Possibly more problematic in practice is the fact that, in exchange settings, any individually rational Clarke-Groves rule will allow agents to receive a net transfer. While this is acceptable in some environments, it is infeasible in others. For example, it is illegal in all countries except Iran for persons to receive money for their organs, while patients typically do pay for organ transplants in some way (i.e., insurance copayments). It is therefore necessary for us to forego utilitarian optimality. Thus, we do not have access to the standard results in auction theory, and we must simultaneously deduce the assignment and payment rules. We discuss the technical consequences of this in Section 5.1, but it is worth noting here that, unlike the Clarke-Groves rules, Gale's fixed-tax core rules are defined on the full domain, including non-quasilinear preferences (see Section A, Remark 2). Sun and Yang (2003) also study quasilinear preferences in the

[4]This class of algorithms generalizes the procedure found by Jaramillo and Manjunath (2012).

presence of lower-bounds on payments, and in particular, object-wise lower bounds. They found that minimizing these payments subject to envy-freeness yields a strategy-proof and (Pareto) efficient rule.[5] However, this rule is not individually rational. Thus, it seems we must forego efficiency in general, and not just utilitarian optimality. Thankfully, we find that efficiency *on a restricted range* is possible.

Instead of insisting on pointwise efficiency, Sprumont (2013) showed that the max-med mechanisms are constrained optimal among all anonymous, strategy-proof, and envy-free mechanisms. Note that Gale's fixed-tax core rules are fundamentally different from the max-med mechanisms, and consistency cannot be applied to max-med mechanisms as they are defined only for two agents. Furthermore, neither the rule defined by Sun and Yang (2003) nor Clarke-Groves mechanisms satisfy consistency because they both generalize the "price externality" feature seen in second-price auctions (Vickrey, 1961). That is, the losing bidders determine the price paid by the winner, and by considering sub-populations containing the winner and some different sets of losing bidders, each population might generate a different price. However, Ehlers (2014) demonstrated that no efficient, individually rational, and strategy-proof rule can be consistent. Consequently, we do not aim for full consistency but rather for a conditional version. The above cited rules still fail this weaker condition while this paper uncovers a continuum of rules that satisfy it.

Miyagawa (2001) shows, in a setting where positive transfers to agents are allowed, that any mechanism satisfying individual rationality, strategy-proofness, ontoness, and non-bossiness must be a fixed price core rule. Under such rule, any agent has a personalized price for any object and if involved in an exchange, the transfer is equal to his personalized price of the object he consumes. By adding budget-balance, these personalized prices are represented by a price vector and an agent's transfer is equal to the difference between his personalized price and the price of the object he consumes. The ontoness axiom means that all exchanges are possible. Note that the mechanisms considered in this paper do not have any property pertaining to the thickness of its range. Furthermore, as it turns out, except for the no-trade rule, fixed-tax core rules, by satisfying constrained-efficiency, violate non-bossiness. This is also due to the fact that in showing the existence of fixed-tax core rules satisfying the above properties, we employ recent contributions on house exchange with indifferences and no monetary transfers. In this context Jaramillo and Manjunath (2012) have shown that there exist rules satisfying individual rationality, efficiency and

[5] These results were later proved on a more general preference domain by Andersson and Svensson (2008).

strategy-proofness.[6] The allocations in the range of a fixed-tax core rule correspond to all possible assignments in the house exchange model (because the same fixed-tax is always paid), and constrained efficiency in this model becomes efficiency in that model. Indeed, as demonstrated in this paper, the rules proposed by Jaramillo and Manjunath (2012) satisfy all of the above defined properties.

The remaining part of this paper is organized as follows. Section 2 presents the model and some desirable properties. Section 3 introduces Gale's fixed-tax core rules and states our main result. Section 4 shows the existence of rules satisfying our properties and a discussion about the implications for the presented results on a more general preference domain than the quasi-linear one. Section 5 contains some general remarks, e.g., a discussion of the Clarke-Groves mechanisms in our context for the two-agent economy.

2 Agents, Preferences and Allocations

Let $N = \{1, ..., n\}$ denote the finite universal set of agents. Agent i owns object i and N also denotes the set of indivisible objects. Let $e : N \to N$ denote the endowment vector such that $e_i = i$ for all $i \in N$. For all $N' \subseteq N$, let $e_{N'} = (e_i)_{i \in N'}$. Agent i's utility function $u_i \in \mathbb{R}^N$ assigns utility u_{ij} for receiving object j. We set $u_{ii} = 0$. Let $\mathcal{U}_i$ denote the set of all utility functions for i. For all $N' \subseteq N$, let $\mathcal{U}_{N'} = \times_{i \in N'} \mathcal{U}_i$. A consumption bundle is a tuple (j, t_i) where $j \in N$ and $t_i \in \mathbb{R}_+$, i.e. agent i pays the tax t_i for consuming j and his utility from consuming (j, t_i) is given by $u_{ij} - t_i$.[7]

Given $N' \subseteq N$, a list $u = (u_i)_{i \in N'}$ of individual utility functions (where $u_i \in \mathcal{U}_i$ for all $i \in N'$) is a (utility) profile (for N'). The set of utility profiles having the above properties is denoted by $\mathcal{U} = \cup_{N' \subseteq N} \mathcal{U}_{N'}$.

Given $N' \subseteq N$, a (feasible) assignment $\mathrm{a} : N' \to N'$ assigns every agent $i \in N'$ an object $j \in N'$ such that $\mathrm{a}_i \neq \mathrm{a}_j$ for all $i \neq j$ (where a_i denotes the object assigned to agent i). Note that any feasible assignment (for N') assigns every agent one object and all objects are assigned.

Given $N' \subseteq N$, an *allocation (for N')* consists of an assignment a and a tax vector $\mathrm{t} = (\mathrm{t}_i)_{i \in N'} \in \mathbb{R}_+^{N'}$, denoted by (a, t) for short. Here t_i denotes the tax agent i is paying in allocation (a, t) and $(\mathrm{a}_i, \mathrm{t}_i)$ denotes i's allotment in (a, t). Let $\mathcal{A}_{N'}$ denote the set of all allocations for N' and $\mathcal{A} = \cup_{N' \subseteq N} \mathcal{A}_{N'}$. An allocation rule φ is a pair (a^φ, t^φ) choosing for

[6]Independently Alcalde-Unzu and Molis (2011) have proposed another class of rules satisfying these properties.

[7]Note that t_i is not necessarily fixed.

each $N' \subseteq N$ and each utility profile $u \in \mathcal{U}_{N'}$ an allocation $(a^\varphi(u), t^\varphi(u)) \in \mathcal{A}_{N'}$. Whenever it is unambiguous and for ease of notation, we use the convention to write $\varphi = (a, t)$ instead of $\varphi = (a^\varphi, t^\varphi)$ and $\varphi_i(u) = (a_i(u), t_i(u))$ for any $i \in N' \subseteq N$ and $u \in \mathcal{U}_{N'}$. We say that two rules $\varphi = (a, t)$ and $\bar{\varphi} = (\bar{a}, \bar{t})$ are equivalent if for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$ we have $u_{ia_i(u)} - t_i(u) = u_{i\bar{a}_i(u)} - \bar{t}_i(u)$ for all $i \in N'$, i.e. for any utility profile the two chosen allocations are utility-equivalent for all agents. Sometimes we use the term mechanism instead of (allocation) rule.

Under the no-trade rule NT, each agent keeps his endowment and no taxes are paid, i.e. for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, $NT(u) = (e_{N'}, 0_{N'})$ (where $0_{N'} = (0, \ldots, 0)$).

2.1 Properties

In the following we introduce some basic properties for an allocation rule $\varphi = (a^\varphi, t^\varphi) \equiv (a, t)$. Individual rationality says that nobody should be worse off than keeping his endowment and paying no tax.

Individual Rationality: For all $N' \subseteq N$, all $u \in \mathcal{U}_{N'}$, and all $i \in N'$, $u_{ia_i(u)} - t_i(u) \geq 0$.

Obviously, if φ is individually rational and $a_i(u) = i$, then we have $u_{ii} - t_i(u) = -t_i(u) \geq 0$. As (by assumption) taxes are non-negative, we have $t_i(u) \geq 0$ and we obtain $t_i(u) = 0$. Thus, agent i pays *no tax* (or zero tax) if i keeps his endowment and φ is individually rational.

Strategy-proofness says that truth-telling is a weakly dominant strategy and because agents' preferences are private information, this property ensures that the mechanism's chosen allocations are based on the true preferences.

Strategy-Proofness: For all $N' \subseteq N$, all $u \in \mathcal{U}_{N'}$, all $i \in N'$ and all $u'_i \in \mathcal{U}_i$, $u_{ia_i(u)} - t_i(u) \geq u_{ia_i(u'_i, u_{-i})} - t_i(u'_i, u_{-i})$.

Constrained efficiency says that the rule is efficient on its range. Given $N' \subseteq N$, let $\mathcal{A}^\varphi_{N'}$ denote the range of rule φ for N', i.e. $\mathcal{A}^\varphi_{N'} = \{(a(u), t(u)) | u \in \mathcal{U}_{N'}\}$. Let $\mathcal{A}^\varphi = \cup_{N' \subseteq N} \mathcal{A}^\varphi_{N'}$.

Constrained Efficiency: For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\varphi(u) = (a(u), t(u))$, then there exists no $(\hat{\mathrm{a}}, \hat{\mathrm{t}}) \in \mathcal{A}^\varphi_{N'}$ such that for all $i \in N'$, $u_{i\hat{\mathrm{a}}_i} - \hat{\mathrm{t}}_i \geq u_{ia_i(u)} - t_i(u)$ with strict inequality holding for some $j \in N'$.

Note that any rule choosing for any set of agents a unique allocation is constrained efficient, and for example, the no-trade rule is constrained efficient.

Consistency requires that if the rule allocates the endowments of some set of agents $S \subseteq N$ among the agents in S and if the problem is restricted to only contain the agents in S and their endowments, then the agents in S must be assigned the same houses and pay the same taxes as in the problem containing all agents and all endowments.[8] In this sense, consistency is a notion of stability since the recommendation of the allocation rule is robust to population changes (Thomson, 1988).[9] It can be argued that consistency is a desirable property, because in the absence of specific tax revenue goals or balanced budget requirements it seems natural (a) that the tax of agent i should only depend on the exchange that the agent is part of and (b) that the tax of agent i should not depend on if some other agent j decides to enter or exit the housing market (unless agents i and j are involved in the same cyclical exchange).[10] The consistency axiom takes care of both (a) and (b).

To formally define consistency, consider a given set of agents $S \subseteq N' \subseteq N$ with corresponding utilities $u \in \mathcal{U}_{N'}$, and let $u|_S = (u_i)_{i \in S}$ and $\varphi_S(u) = (\varphi_i(u))_{i \in S}$.

Consistency: For all $S \subseteq N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\cup_{i \in S}\{a_i(u)\} = S$, then $\varphi(u|_S) = \varphi_S(u)$.

Unfortunately, as we show later in Corollary 1, consistency is very strong in conjunction with our other properties as basically only the no-trade rule will satisfy them.[11] Thus, we study instead a weaker version of the axiom.

The strong version of the consistency axiom, as defined in the above, was motivated using the stability arguments (a) and (b). But are these strong requirements really natural for the considered house exchange model with taxes? If, for example, agent i is involved in an exchange but is *indifferent* between his allotment and some other allotment outside of his own exchange cycle, then one may argue that it not unreasonable that the taxes of agent i potentially can be affected if this equally preferred allotment is removed from the problem. For example, in the classical work by, e.g., Guesnerie and Seade (1982), Stiglitz (1982) and Weymark (1986), the structure of the solution to the optimal taxation problem

[8] See Thomson (1992, 2009) for in-depth surveys of consistency.

[9] Other notions of stability is known to play a key-role for the long-term survival of matching markets, e.g., medical residency programmes (McKinney et al., 2005), school choice programmes (Abdulkadiroğlu et al., 2005), and labour markets for federal court clerkships (Roth and Xing, 1994).

[10] An alternative interpretation of (b) is that, because the consistency property holds for any subset of agents S, agents that not are assigned an endowment from an agent in S should not be able to destroy the trades among the agents in S simply by exiting from the housing market.

[11] This is not surprising. For instance, in the context of allocating indivisible objects (without monetary transfers), Ehlers and Klaus (2007) show that basically only mixed dictator-pairwise-exchange rules satisfy consistency in conjunction with strategy-proofness and efficiency.

involves indifferences, through a set of binding incentive compatibility constraints, and if a set of agents are removed from the problem, a different set of incentive compatibility constraints may bind, which ultimately affects the taxes that the agents are charged. Our weakening of the consistency axiom also captures this idea. More precisely, the type of stability property discussed in the above is only applied to sets of agents S that can be "isolated" from the rest of the agents in the economy in the sense that they *strictly* prefer their allotments to any allotment not assigned to an agent in S. Indifferences are therefore allowed to play a role in the weaker notion of consistency exactly as they are in the classical optimal taxation problem.

Weak Consistency: For all $S \subseteq N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\cup_{i \in S}\{a_i(u)\} = S$ and for all $i \in S$ and all $j \in N' \backslash S$, $u_{ia_i(u)} - t_i(u) > u_{ia_j(u)} - t_j(u)$, then $\varphi(u|_S) = \varphi_S(u)$.

Note that by definition, if $\cup_{i \in S}\{a_i(u)\} = S$, then consistency requires both $\varphi(u|_S) = \varphi_S(u)$ and $\varphi(u|_{N' \backslash S}) = \varphi_{N' \backslash S}(u)$ whereas weak consistency does not necessarily constrain the rule for $N' \setminus S$.

Anonymity says that the chosen allocations do not depend on the names of the agents, i.e., if the names of the agents and their endowments are permuted, their assignments should also be permuted in the same way. To formally define this property, let $\sigma : N' \to N''$ be a permutation. For any utility profile u for N', let $\sigma(u)$ denote the utility profile for N'' where both the names of the agents and their endowments are relabeled according to σ.[12] Similarly, σ is used for relabeling assignments and tax vectors.

Anonymity: For all $N', N'' \subseteq N$ with $|N'| = |N''|$, all $u \in \mathcal{U}_{N'}$ and all permutations $\sigma : N' \to N''$, if $\varphi(u) = (a(u), t(u))$, then $\varphi(\sigma(u)) = (\sigma(a(u)), \sigma(t(u)))$.

Note that in the context of exchange anonymity *does not imply* that agents with symmetric utility functions are treated equally: for instance, if two agents i and j have symmetric utility functions in the sense that $u_{ij} = u_{ji}$ and $u_{il} = u_{jl}$ for all $l \neq i, j$, anonymity *does not imply* that agents i and j are treated equally (unless i and j form a pairwise exchange), as for instance the no-trade rule satisfies anonymity.

Exactly as for the consistency axiom, the anonymity axiom is too strong in conjunction with the other properties of interest (see Corollary 1). Anonymity has been used earlier by, e.g., Miyagawa (2002) to characterize the core rule in house exchange problems. In his problem, however, no monetary transfers are allowed and agents have *strict preferences*. This also means that there are no exclusive (top-trading) exchange cycles, i.e., agents that

[12]Formally, for all $i, j, k \in N'$ we have $u_{ij} - t_i \geq u_{ik} - t'_i$ if and only if $u_{\sigma(i)\sigma(j)} - t_i \geq u_{\sigma(i)\sigma(k)} - t'_i$.

are involved in exchanges strictly prefer their allotment to any other attainable allotment. When *indifferences* are present, e.g., because an agent has identical valuations for several different objects or because the numeraire good compensates for differences in object valuations, exchange cycles can be exclusive. If so, the anonymity axiom can be violated even under very mild assumptions (see Example 1). To mitigate the effect that indifferences play in the considered house allocation problem with monetary transfers, we consider a weakening of the anonymity axiom that captures the central message of the strong version of the axiom.[13] Namely, the chosen allocation should not depend on the names of the agents, but only in cases where there are *no indifferences*, i.e., when indifferences cannot play a role (exactly as they don't play a role in Miyagawa, 2002). More precisely, weak anonymity says that when all agents strictly prefer their allotments to any other allotment (including the agent's own endowment), then the names of the agents should not matter.

Weak Anonymity: For all $N', N'' \subseteq N$ with $|N'| = |N''|$, all $u \in \mathcal{U}_{N'}$ and all permutations $\sigma : N' \to N''$, if $\varphi(u) = (a(u), t(u))$ and for all $i \in N'$ and all $j \in N' \backslash \{i\}$, $u_{ia_i(u)} - t_i(u) > \max\{0, u_{ia_j(u)} - t_j(u)\}$, then $\varphi(\sigma(u)) = (\sigma(a(u)), \sigma(t(u)))$.

Note that (i) consistency implies weak consistency and (ii) anonymity implies weak anonymity (but the reverse implications are not true). Similarly to consistency, as we show later in Corollary 1, anonymity will turn out to be too strong and basically only the no-trade rule will satisfy anonymity and our other properties. This is illustrated in the example below.

Example 1. Let $N = \{1, 2, 3\}$ and $u \in \mathcal{U}_N$ be such that $u_{12} = 1 = u_{32}$, $u_{13} = -1 = u_{31}$ and $u_{21} = u_{23} = 1$. Furthermore, let $\sigma : N \to N$ be the permutation such that $\sigma(1) = 3$, $\sigma(3) = 1$ and $\sigma(2) = 2$. Then $\sigma(u) = u$.

The no-trade rule NT satisfies anonymity for u as $NT(u) = NT(\sigma(u))$.

Let φ be an individually rational rule such that $\varphi(u) = (a(u), t(u)) \neq NT(u)$. Then either $a_1(u) \neq 1$ or $a_3(u) \neq 3$. If $a_1(u) \neq 1$, then by individual rationality of φ we have $a_1(u) = 2$, $t_1(u) \leq 1$, $a_3(u) = 3$, and $t_3(u) = 0$. Since $\sigma(u) = u$ and $\sigma(a(u)) \neq a(u)$, we obtain $\varphi(u) = \varphi(\sigma(u)) \neq (\sigma(a(u)), \sigma(t(u)))$, i.e. φ violates anonymity. Furthermore, note that $u_{3a_1(u)} - t_1(u) \geq 0 = u_{3a_3(u)} - t_3(u)$, i.e. the presumption for weak anonymity is not satisfied and φ does not violate weak anonymity at profile u. If $a_3(u) \neq 3$, then the same conclusions are drawn. In addition, weak consistency cannot be invoked for $S = \{1, 2\}$ as we have $u_{21} - t_2(u) \leq u_{23} - t_3(u)$.

[13]For a discussion of different weakenings of the anonymity axiom, see Hikaru (2019).

Now instead suppose that $u'_{21} = 1$, $u'_{23} = 0$, $u' = (u'_2, u_{-2})$ and $\varphi(u') = (a(u'), t(u')) = ((2, 1, 3), 0_N)$ (where this means $a_1(u') = 2$, $a_2(u') = 1$ and $a_3(u') = 3$). Then $u'_{12} - t_1(u') > u'_{13} - t_3(u')$ and $u'_{21} - t_2(u') > u'_{23} - t_3(u')$ and weak consistency invoked for $\varphi(u')$ and $S = \{1, 2\}$ implies $\varphi(u'|_S) = \varphi_S(u')$.

Similarly, suppose $u'' \in \mathcal{U}_N$ is such that $u''_{12} > u''_{13}$, $u''_{23} > u''_{21}$, $u''_{31} > u''_{32}$ and $\varphi(u'') = (a(u''), t(u'')) = ((2, 3, 1), 0_N)$. Then for all $i \in N$ we have $u''_{ia_i(u'')} - t_i(u'') > \max\{0, u''_{ia_j(u'')} - t_j(u'')\}$ for all $j \in N \backslash \{i\}$. Now if φ satisfies weak anonymity, then for all permutations $\sigma : N \to N$ we have $\varphi(\sigma(u'')) = (\sigma(a(u'')), \sigma(t(u'')))$. □

Furthermore, a natural weakening of constrained efficiency is constrained unanimity: this property only requires constrained efficiency when there is a unique constrained efficient allocation. In a private goods setting, this means that any agent prefers his allotment to any other agent's allotment. Now this is captured by our weakening of anonymity and of consistency.

3 Gale's Fixed-Tax Core Rules

In the following, we define (Gale's) fixed-tax core rules. Let $\alpha \geq 0$ be Gale's fixed tax (a lump sum payment which is identical for all agents). Given $i \in N' \subseteq N$ and $u \in \mathcal{U}_{N'}$, we define the relation $R_i(u_i, \alpha)$ over N' as follows: for all $j, k \in N' \backslash \{i\}$,

(i) $jR_i(u_i, \alpha)k \Leftrightarrow u_{ij} - \alpha \geq u_{ik} - \alpha$; and

(ii) $jR_i(u_i, \alpha)i \Leftrightarrow u_{ij} - \alpha \geq u_{ii}$.

Let $P_i(u_i, \alpha)$ denote the strict ranking associated with $R_i(u_i, \alpha)$. Given $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$, let $R_{N'}(u, \alpha) = (R_i(u_i, \alpha))_{i \in N'}$. Based on the fixed tax α, each utility profile induces "ordinal" rankings over the endowments. We say that $R_{N'}(u, \alpha)$ is strict (over acceptable objects) if for all distinct $i, j, k \in N'$, $jR_i(u_i, \alpha)kR_i(u_i, \alpha)i$ implies $jP_i(u_i, \alpha)kP_i(u_i, \alpha)i$ and $iR_i(u_i, \alpha)j$ implies $iP_i(u_i, \alpha)j$. Now if the induced preferences are strict, then we may apply Gale's top trading cycles algorithm[14] in order to find the unique core assignment. For strict $R_{N'}(u, \alpha)$, let $C(R_{N'}(u, \alpha))$ denote the unique core assignment.

Definition 1. A rule $\varphi = (a, t)$ is a (Gale's) fixed-tax core rule if there exists $\alpha \geq 0$ such that for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$,

[14]The Appendix defines the HPO-algorithm, which reduces to Gale's top trading cycles algorithm when the induced preferences are strict.

1. for all $i \in N'$, if $a_i(u) \neq i$, then $t_i(u) = \alpha$,

2. for all $i \in N'$, if $a_i(u) = i$, then $t_i(u) = 0$, and

3. if $R_{N'}(u, \alpha)$ is strict, then $a(u) = C(R_{N'}(u, \alpha))$.

In words, a rule is a fixed-tax core rule if there exists a fixed tax α such that for any utility profile, if an agent does not keep his endowment, then he pays the fixed tax α, the agents who keep their endowment pay zero, and for any utility profile that induces strict ordinal rankings, the core assignment of objects is chosen. We call α Gale's fixed tax as once the fixed tax is chosen, Gale's top trading cycles algorithm finds the unique core assignment if the induced preferences are strict.

Theorem 1. *If rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity, then φ is a fixed-tax core rule or φ is the no-trade rule.*

3.1 Proof of Theorem 1

Obviously, if $\varphi = NT$, then Theorem 1 is true. Let $\varphi \neq NT$. We need additional notation. Note that any assignment consists of cyclic exchanges or cycles. Formally, in assignment a, a cycle c is a sequence of distinct agents, $c = (i_1, i_2, \dots, i_k)$ such that $\mathrm{a}_{i_l} = i_{l+1}$ for all $l \in \{1, \dots, k-1\}$, and $\mathrm{a}_{i_k} = i_1$. Then k is the length of cycle c. We use the convention to write c for both the cycle c and the coalition of agents belonging to cycle c. Let $\mathcal{C}_k$ denote the set of all cycles of length k, and $\mathcal{C} = \cup_{k \in \{2,\dots,|N|\}} \mathcal{C}_k$ the set of all cycles of length at least two. Let $\mathcal{C}_k^{\varphi} = \{c \in \mathcal{C}_k :$ there exists $u \in \mathcal{U}_c$ such that $a(u) = c\}$.[15] Similarly we define $\mathcal{C}^{\varphi} = \cup_{k \in \{2,\dots,|N|\}} \mathcal{C}_k^{\varphi}$.

The roadmap of the proof of Theorem 1 consists of three parts.

The first part shows the following two basic facts. Lemma 1 shows that if a cycle of length k belongs to $\mathcal{C}_k^{\varphi}$, then all cycles of length k belong to $\mathcal{C}_k^{\varphi}$ and all agents must pay the same tax $\alpha(k)$ in any cycle of length k. Lemma 2 shows the non-emptiness of $\mathcal{C}^{\varphi}$ given that $\varphi \neq NT$.

The second part considers the case $\mathcal{C}_2^{\varphi} \neq \emptyset$. Lemma 3 shows that then we must have $\mathcal{C}_k^{\varphi} = \mathcal{C}_k$ for all $k \in \{3, \dots, |N|\}$ (meaning all cycles of length k are executed) and Lemma 4 shows that in any cycle of arbitrary length all agents pay the same fixed tax. Lemma 5 shows that any agent, who does not keep his endowment, pays this fixed tax. Finally, given

[15] Note that this implies $a_c(u) = a(u) = c$ as $u \in \mathcal{U}_c$.

the fixed tax, Lemma 6 shows that whenever the induced ordinal preferences are strict, then the core assignment has to be chosen.

The third part (Lemma 7) shows that if $\mathcal{C}^\varphi \neq \emptyset$, then $\mathcal{C}_2^\varphi \neq \emptyset$, i.e., that we always are in the second part of the proof when $\mathcal{C}^\varphi$ is non-empty.

Lemma 1. *Let $c = (1, 2, \ldots, k) \in \mathcal{C}_k^\varphi$. Then $\mathcal{C}_k^\varphi = \mathcal{C}_k$ and there exists $\alpha(k) \geq 0$ such that for all $c' \in \mathcal{C}_k$ and all $u \in \mathcal{U}_{c'}$, if $a(u) = c'$, then for all $i \in c'$, $t_i(u) = \alpha(k)$.*

Proof: Suppose that $(c, \mathrm{t}), (c, \mathrm{t}') \in \mathcal{A}_c^\varphi$ with $\mathrm{t} \neq \mathrm{t}'$. By constrained efficiency, for some $i, j \in c$, $\mathrm{t}_i < \mathrm{t}'_i$ and $\mathrm{t}_j > \mathrm{t}'_j$. Let $y = 1 + \max_{l \in \{1,\ldots,k\}} \{\mathrm{t}_l, \mathrm{t}'_l\}$. Let $u \in \mathcal{U}_c$ be such that for all $i \in c$, $u_{ii+1} = y$, $u_{ii} = 0$ and $u_{ij} = -1$ for $j \neq i, i+1$. By $(c, \mathrm{t}) \in \mathcal{A}_c^\varphi$ and constrained efficiency, $a(u) = c$. Let $i \in c$. We show that $u_{ii+1} - t_i(u) > 0$: suppose not; then by individual rationality and $u_{ii+1} = y$, $t_i(u) = y$; let $u'_i \in \mathcal{U}_i$ be such that $u'_{ii+1} = y - \frac{1}{2}$ and $u'_{ij} = -1$ for all $j \neq i, i+1$, and $u' = (u'_i, u_{-i})$. By strategy-proofness and individual rationality, $a_i(u') = i$ and $t_i(u') = 0$. Thus, by construction and individual rationality, $a(u') = e_c$ and $t_l(u') = 0$ for all $l \in c$. This is now a contradiction to constrained efficiency as $(c, \mathrm{t}) \in \mathcal{A}_c^\varphi$ and $u'_{ll+1} > \mathrm{t}_l$ for all $l \in c$.

Thus, for all $i \in c$, $u_{ii+1} - t_i(u) > 0$, and $a(u) = c$. By construction, $u_{ia_i(u)} - t_i(u) > \max\{0, u_{ia_j(u)} - t_j(u)\}$ for all $j \neq i$. Now by weak anonymity, for all $i, j \in c$, $t_i(u) = t_j(u) \equiv \alpha(k)$. Because $\mathrm{t} \neq \mathrm{t}'$, then either $\mathrm{t} \neq (\alpha(k))_{i \in c}$ or $\mathrm{t}' \neq (\alpha(k))_{i \in c}$, say $\mathrm{t}' \neq (\alpha(k))_{i \in c}$. If for all $i \in c$, $\mathrm{t}'_i \geq \alpha(k)$, then by constrained efficiency the allocation (c, t') can never be chosen for utility profiles of coalition c, which is a contradiction to $(c, \mathrm{t}') \in \mathcal{A}_c^\varphi$. Thus, for some $i \in c$, $\alpha(k) = t_i(u) > \mathrm{t}'_i$. But then using strategy-proofness and constrained efficiency yields a contradiction: i may report $u'_{ii+1} = \frac{1}{2}(t_i(u) + \mathrm{t}'_i)$ and $u'_{ij} = u_{ij}$ for $j \neq i+1$, and then by strategy-proofness and individual rationality, $a(u'_i, u_{-i}) = e_c$ and $t(u'_i, u_{-i}) = 0_c$, which is a contradiction to constrained efficiency by $(c, \mathrm{t}') \in \mathcal{A}_c^\varphi$.

Hence, for all $(c, \mathrm{t}), (c, \mathrm{t}') \in \mathcal{A}_c^\varphi$ we have $\mathrm{t} = \mathrm{t}'$, and $\alpha(k) = \mathrm{t}'_i = \mathrm{t}_i$ for all $i \in c$. Let $c' \in \mathcal{C}_k$ and $\sigma : c \to c'$ be a permutation. By weak anonymity and $a(u) = c$, we obtain $a(\sigma(u)) = c'$ and for all $i \in c'$, $t_i(\sigma(u)) = \alpha(k)$. Hence, $\mathcal{C}_k^\varphi = \mathcal{C}_k$. □

The next lemma finishes the first part of the proof.

Lemma 2. *If $\varphi \neq NT$, then $\mathcal{C}^\varphi \neq \emptyset$.*

Proof. Since $\varphi \neq NT$, there exist $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ such that $\varphi(u) \neq NT(u)$. By individual rationality, for all $i \in N'$, $a_i(u) = i$ implies $t_i(u) = 0$, and thus, $N'' = \{i \in N' : a_i(u) \neq i\} \neq \emptyset$. Let $u' \in \mathcal{U}_{N'}$ be such that (i) for all $i \in N''$, $u'_{ia_i(u)} - t_i(u) > 0 > u'_{ij}$ for all $j \in N' \backslash \{a_i(u), i\}$, and (ii) for all $i \in N' \backslash N''$, $u'_{ii} = 0 > u'_{ij}$ for all $j \in N' \backslash N''$.

Then by constrained efficiency, $a(u') \neq e_{N'}$ and $a(u')$ contains at least one cycle c (of length greater than or equal to two). Without loss of generality, let $c = (1, 2, \ldots, k)$ with $k \geq 2$. Hence, for all $i \in N'$, $u'_{ia_i(u')} - t_i(u') \geq 0 > u'_{ij} - t_j(u')$ for all $j \in N' \backslash c$ where the first inequality follows from individual rationality and the second from our construction and that transfers are non-negative. Thus, by weak consistency, $\varphi(u|_c) = \varphi_c(u)$ and we have $c \in \mathcal{C}_k^{\varphi}$, the desired conclusion. □

For the second part of the proof, let $\mathcal{C}_2^{\varphi} \neq \emptyset$. The next lemma shows then $\mathcal{C}^{\varphi} = \mathcal{C}$. By weak anonymity and strategy-proofness, if $\mathcal{C}_2^{\varphi} \neq \emptyset$, then $\mathcal{C}_2^{\varphi} = \mathcal{C}_2$ and in any pairwise trade agents pay the tax $\alpha(2)$ for two-agent utility profiles.

Let $c = (1, 2, \ldots, k)$. We say that $u \in \mathcal{U}_c$ is *c-cyclic* if for each $i \in c$, $u_{ii+1} > u_{ii-1} > \alpha(2)$ and $u_{ij} < 0$ for all $j \in c \backslash \{i-1, i, i+1\}$.

Lemma 3. *Let $C_2^{\varphi} \neq \emptyset$. Then for all $k \in \{3, \ldots, |N|\}$, $\mathcal{C}_k^{\varphi} = \mathcal{C}_k$.*

Proof: Suppose that for some $k \in \{3, \ldots, |N|\}$, $\mathcal{C}_k^{\varphi} = \emptyset$. Let $c = (1, 2, \ldots, k)$, $c' = (k, k-1, \ldots, 1)$, and fix a c-cyclic $u \in \mathcal{U}_c$. By $\mathcal{C}_k^{\varphi} = \emptyset$, we have $a(u) \neq c, c'$.

Because u is c-cyclic and from individual rationality, the only admissible trading arrangement is a mix of pairwise trading and keeping one's endowment. First, we show that no agent keeps his endowment by invoking weak consistency and the fact that in any pairwise trade agents pay the tax $\alpha(2)$ for two-agent utility profiles.

Suppose that there exists $i \in \{1, \ldots, k\}$ such that $a_i(u) = i$, say $i = 2$. By individual rationality, $a_2(u) = 2$ and $t_2(u) = 0$. Let $u'_2 \in \mathcal{U}_2$ be such that $u'_{21} = u_{21}$ and $u'_{2l} < 0$ for all $l \in c \backslash \{1, 2\}$. Let $u' = (u'_2, u_{-2})$. By individual rationality, $a_2(u') \in \{1, 2\}$. If $a_2(u') = 1$, then by strategy-proofness and both $a_2(u) = 2$ and $t_2(u) = 0$, $t_2(u') = u_{21}$. But then choose $u''_2 \in \mathcal{U}_2$ such that $u'_{21} > u''_{21} > \alpha(2)$ and $u''_{2l} = u'_{2l}$ for all $l \in c \backslash \{1\}$. Let $u'' = (u''_2, u_{-2})$. Then by strategy-proofness and individual rationality, $a_2(u'') = 2$ and $t_2(u'') = 0$. Thus, without loss of generality, we may suppose for u' that $a_2(u') = 2$ and $t_2(u') = 0$ (by individual rationality). By $\mathcal{C}_k^{\varphi} = \emptyset$, we have $a_1(u') \in \{1, k\}$.

If $a_1(u') = 1$, then let $u''_1 \in \mathcal{U}_1$ be such that $u''_{12} = u_{12}$ and $u''_{1l} < 0$ for all $l \in c \backslash \{1, 2\}$. Let $u'' = (u''_1, u'_{-1})$. By strategy-proofness, individual rationality and $\mathcal{C}_k^{\varphi} = \emptyset$, $a_1(u'') = 1$ and $t_1(u'') = 0$. Hence, by individual rationality, $a_2(u'') = 2$ and $t_2(u'') = 0$. But now by weak consistency, $a(u''_{\{1,2\}}) = (1, 2)$, which is a contradiction to constrained efficiency as $((2, 1), (\alpha(2), \alpha(2)) \in \mathcal{A}_{(2,1)}^{\varphi}$.

If $a_1(u') = k$, then using the same argument as above, strategy-proofness and weak consistency (because 1 may deviate as above, obtain 2 and pay $\alpha(2)$), we must have

$$u_{12} - \alpha(2) \leq u_{1k} - t_1(u') < u_{12} - t_1(u')$$

where the first inequality follows from strategy-proofness and the second one from $u_{1k} < u_{12}$. Thus, we have $t_1(u') < \alpha(2)$. Now let $u''_1 \in \mathcal{U}_1$ be such that $u''_{1k} - t_1(u') > 0 > u''_{1k} - \alpha(2)$ and $u''_{1l} < 0$ for all $l \in c\backslash\{1, k\}$. Let $u'' = (u''_1, u'_{-1})$. By strategy-proofness, $a_1(u'') = k$ and $t_1(u'') = t_1(u')$. By $\mathcal{C}^{\varphi}_k = \emptyset$, we have $a_k(u'') = 1$. But now by construction, $u''_{1k} - t_1(u'') > u''_{1a_j(u'')} - t_j(u'')$ for all $j \in c\backslash\{1, k\}$. If $u_{k1} - t_k(u'') > u_{ka_j(u'')} - t_j(u'')$ for all $j \in c\backslash\{1, k\}$, then by weak consistency, $\varphi(u''|_{\{1,k\}}) = \varphi_{\{1,k\}}(u'')$. Thus, $a_1(u''|_{\{1,k\}}) = k$ and $t_1(u''|_{\{1,k\}}) = t_1(u'') = t_1(u') < \alpha(2)$, which is a contradiction to the fact that in any pairwise trade for two-agent utility profiles agents pay the tax $\alpha(2)$. Otherwise ($u_{k1} - t_k(u'') \leq u_{ka_j(u'')} - t_j(u'')$ for some $j \in c\backslash\{1, k\}$), choose $u'''_k \in \mathcal{U}_k$ such that $u'''_{k1} = u_{k1} + 1$ and $u'''_{kl} < 0$ for all $l \in c\backslash\{1, k\}$, and let $u''' = (u'''_k, u''_{-k})$. Now by strategy-proofness, $a_k(u''') = 1$ and $t_k(u''') = t_k(u'')$. By $\mathcal{C}^{\varphi}_k = \emptyset$ and individual rationality, $a_1(u''') = k$ and $0 \leq t_1(u''') < \alpha(2)$. But now as above we use weak consistency to derive a contradiction to the fact that in any pairwise trade for two-agent utility profiles agents pay the tax $\alpha(2)$.

Thus, for all $i \in \{1, \ldots, k\}$ we have $a_i(u) \neq i$. Hence, k is even and $a(u)$ must consist of $\frac{k}{2}$ pairwise exchanges (and $k \geq 4$). Without loss of generality, let $a_1(u) = k$. Note that $u_{12} > u_{1k}$ and for all $i \in c\backslash\{1\}$, $u_{ii+1} > u_{ii-1}$. Now starting with agent 1, 1 strictly prefers the pairwise trade with agent 2 to the pairwise trade with agent k when paying the tax $\alpha(2)$. We show below using individual rationality, strategy-proofness and weak consistency that agent 1 can induce a pairwise trade with agent 2. Now similarly, agent 2 strictly prefers the pairwise trade with agent 3 to the pairwise trade with agent 1 when paying the tax $\alpha(2)$, and using the same arguments, then agent 2 can induce a pairwise trade with agent 3. Then by induction we obtain a contradiction for the cycle $c = (1, \ldots, k)$.

Let $a_1(u) = k$. If $t_1(u) < \alpha(2)$, then let $u'_1 \in \mathcal{U}_1$ be such that $u'_{1k} - t_1(u) > 0 > u'_{1k} - \alpha(2)$ and $u'_{1l} < 0$ for all $l \in c\backslash\{1, k\}$. Let $u' = (u'_1, u_{-1})$. By strategy-proofness, $a_1(u') = k$ and $t_1(u') = t_1(u)$. Thus, by $\mathcal{C}^{\varphi}_k = \emptyset$, we have $a_k(u') = 1$. Now we derive a contradiction as above using weak consistency and the fact that in any pairwise trade for two-agent utility profiles agents pay $\alpha(2)$.

If $t_1(u) \geq \alpha(2)$, then let $u'_1 \in \mathcal{U}_1$ be such that $u'_{12} = u_{12}$ and $u'_{1l} < 0$ for all $l \in c\backslash\{1, 2\}$. Let $u' = (u'_1, u_{-1})$. By individual rationality, $a_1(u') \in \{1, 2\}$. If $a_1(u') = 2$, then by strategy-proofness, $u_{12} - t_1(u') \leq u_{1k} - t_1(u) < u_{12} - t_1(u)$ which implies $t_1(u') > t_1(u) \geq \alpha(2)$. But then choose $u''_1 \in \mathcal{U}_1$ such that $u''_{12} - t_1(u') < 0 < u''_{12} - \alpha(2)$ and $u''_{1l} < 0$ for all

$l \in c\backslash\{1,2\}$. Let $u'' = (u''_1, u_{-1})$. Now from strategy-proofness and individual rationality, it follows $a_1(u'') = 1$ and $t_1(u'') = 0$.

Thus, without loss of generality, let $t_1(u) \geq \alpha(2)$, $u' = (u'_1, u_{-1})$ be such that $u'_{12} > \alpha(2)$, $u'_{1l} < 0$ for all $l \in c\backslash\{1,2\}$, and both $a_1(u') = 1$ and $t_1(u') = 0$. Now by weak consistency and strategy-proofness, we cannot have $a_2(u') = 2$ (otherwise let $u''_2 \in \mathcal{U}_2$ be such that $u''_{21} = u_{21}$ and $u''_{2l} < 0$ for all $j \neq 1,2$, and then we use strategy-proofness and weak consistency for (u''_2, u'_{-2}) to derive a contradiction as above). Thus, by $\mathcal{C}^{\varphi}_k = \emptyset$, $a_2(u') = 3$ and $a_3(u') = 2$. Now we can use the same arguments as above for 3 in the role of 1 to deduce $t_3(u') \geq \alpha(2)$ and (without loss of generality) for $u''_3 \in \mathcal{U}_3$ such that $u''_{34} > \alpha(2)$ and $u''_{3l} < 0$ for all $l \in c\backslash\{3,4\}$ and $u'' = (u''_3, u'_{-3})$, we have $a_3(u'') = 3$ and $t_3(u'') = 0$ (and $a_2(u'') \neq 3$). Again by strategy-proofness and weak consistency, we cannot have $a_4(u'') = 4$. If $k = 4$, then $a_4(u'') = 1$ and by $\mathcal{C}^{\varphi}_k = \emptyset$, $a_1(u'') = k$, which is a contradiction to individual rationality (by $u'_{1k} < 0$). If $k > 4$, then by weak consistency and $\mathcal{C}^{\varphi}_k = \emptyset$, $a_4(u'') = 5$ and $a_5(u'') = 4$. Then using the same arguments as above for 5 in the role of 1, we obtain a contradiction since k is finite and even.

Thus, $\mathcal{C}^{\varphi}_k \neq \emptyset$. Now weak anonymity implies $\mathcal{C}^{\varphi}_k = \mathcal{C}_k$. □

Note that by Lemma 1, all agents pay in any cycle of length k the same fixed tax $\alpha(k)$. Let $\alpha(2) = \alpha$. We aim to show that whenever a cycle c of length k forms, all agents pay the fixed tax $\alpha(k) = \alpha$.

Lemma 4. *Let $C^{\varphi}_2 \neq \emptyset$. Then for all $k \in \{3, \ldots, |N|\}$, $\alpha(k) = \alpha$.*

Proof: Let $k \in \{3, \ldots, |N|\}$. Without loss of generality, let $c = (1, 2, \ldots, k) \in \mathcal{C}^{\varphi}_k$. By Lemma 1, all agents pay the fixed tax $\alpha(k)$ in cycle c for all $u \in \mathcal{U}_c$ such that $a(u) = c$, and by Lemma 3, $\mathcal{C}^{\varphi}_k = \mathcal{C}_k$.

We show $\alpha(k) = \alpha$. Suppose not, i.e. $\alpha(k) < \alpha$ or $\alpha(k) > \alpha$.

First, suppose $\alpha(k) < \alpha$. Consider the c-cyclic utility profile $u_c \in \mathcal{U}_c$ for the cycle $c = (1, \ldots, k)$ such that $\alpha(k) < u_{ii-1} < u_{ii+1} < \alpha$ for all $i \in c$. Let $c' = (k, \ldots, 1)$. Suppose that $a(u) \neq c, c', e_c$. Because in any cycle c all agents pay the fixed tax $\alpha(k)$, constrained efficiency implies $u_{ia_i(u)} - t_i(u) > u_{ii+1} - \alpha(k) > 0$ for some $i \in c$. But then $a_i(u) \neq i$ and

$$u_{ii+1} - t_i(u) \geq u_{ia_i(u)} - t_i(u) > u_{ii+1} - \alpha(k), \tag{1}$$

which implies $\alpha(k) > t_i(u)$. Let $u'_i \in \mathcal{U}_i$ be such that $\alpha(k) > u'_{ia_i(u)} > t_i(u)$ and $0 > u'_{il}$ for all $l \neq i, a_i(u)$. Let $u' = (u'_i, u_{-i})$. By strategy-proofness, $a_i(u') = a_i(u)$ and $t_i(u') = t_i(u)$. But then by individual rationality and $\alpha(k) > u'_{ia_i(u)}$, i is part of a pairwise exchange under u' with agent $j = a_i(u')$ (and $j \neq i$). Let $u''_j \in \mathcal{U}_j$ be such that $u''_{ji} = u_{ji}$ and $u''_{jl} = -1$ for all $l \neq i, j$. Let $u'' = (u''_j, u_{-j})$. By strategy-proofness, $a_j(u'') = a_j(u') = i$. By individual rationality, $a_i(u'') = j$ and $t_i(u'') < \alpha(k) < \alpha$. This is now a contradiction to weak consistency (since $u'_{ia_i(u')} < \alpha(k) < \alpha$ and in any pairwise trade for two-agent utility profiles agents pay α). Hence, $a(u) \in \{c, c', e_c\}$. By constrained efficiency and the fact that in any cycle of length k all agents pay the same fixed tax $\alpha(k)$, we have $a(u) = c$.

Let $u'_2 \in \mathcal{U}_2$ be such that $\alpha(k) < u'_{23} < u'_{21} < \alpha$ and $u'_{2l} = -1$ for $l \neq 1, 2, 3$. Let $u' = (u'_2, u_{-2})$. Suppose that $a(u') \neq c, c'$. If for some $i \in c \backslash \{2\}$, $u_{ia_i(u')} - t_i(u') > u_{ii+1} - \alpha(k)$, then we use the same arguments as above for (1) to derive a contradiction. Otherwise, by constrained efficiency, $u'_{2a_2(u')} - t_2(u') > u'_{23} - \alpha(k) > 0$ and 2 is involved in a pairwise trade under u'. Let $\hat{u} \in \mathcal{U}_c$ be such that for all $i \in c$, $\hat{u}_{ia_i(u')} = u'_{ia_i(u')}$ and $\hat{u}_{il} < 0$ for $l \neq i, a_i(u')$. If there is any trading under $\hat{u}$, then it must be pairwise as $a(u') \neq c, c'$. If there is any pairwise trade under $\hat{u}$, then using weak consistency gives together with Lemma 1 gives us a contradiction as for all $i, l \in c$ we have $\hat{u}_{il} < \alpha$. Thus, $a_i(\hat{u}) = i$ for all $i \in c$ which is a contradiction to constrained efficiency as $\hat{u}_{ia_i(u')} - t_i(u') \geq 0$ for all $i \in c$ and $\hat{u}_{2a_2(u')} - t_2(u') > 0$ (and $(a(u'), t(u')) \in \mathcal{A}^{\varphi}_c$). Hence, $a(u') \in \{c, c'\}$.

Suppose that $a(u') = c$. Because c is a cycle of length k, $t_2(u') = \alpha(k)$. Let $u''_2 \in \mathcal{U}_2$ be such that $u''_{21} = u'_{21}$ and $u''_{2l} = -1$ for $l \neq 1, 2$. Let $u'' = (u''_2, u_{-2})$. If $a(u'') = c'$, then $t_2(u'') = \alpha(k)$. But now we have $u'_{21} - \alpha(k) > u'_{23} - \alpha(k)$, a contradiction to strategy-proofness. Thus, $a(u'')$ consists of a mix of pairwise trading and keeping one's endowment (and $a(u'')$ contains at least one pairwise trade by constrained efficiency). But then using the same profile $\hat{u}$ as above we derive a contradiction using weak consistency and individual rationality as $u''_{ij} < \alpha$ for all $i, j \in c$.

Hence, $a(u') = c'$ and $a_1(u') = k$. Let $u''_1 \in \mathcal{U}_1$ be such that $u''_{12} = u_{12}$ and $u''_{1l} = -1$ for $l \neq 1, 2$. Let $u'' = (u''_1, u'_{-1})$. If $a(u'') = c$, then $t_1(u'') = \alpha(k)$. But now we have $u'_{12} - \alpha(k) > u'_{1k} - \alpha(k)$, a contradiction to strategy-proofness. Thus, $a(u'')$ consists of a mix of pairwise trading and keeping one's endowment (and $a(u'')$ contains at least one pairwise trade by constrained efficiency). But then using the same profile $\hat{u}$ as above we derive a contradiction using weak consistency and individual rationality as $u''_{ij} < \alpha$ for all $i, j \in c$. Hence, $\alpha(k) < \alpha$ is not possible.

Second, suppose $\alpha(k) > \alpha$. Consider the same type of utility profile u for the cycle $c = (1, \ldots, k)$ as in Lemma 3 such that $\alpha < u_{ii-1} < u_{ii+1} < \alpha(k)$ for all $i \in c$. Because in cycles of length k the fixed tax $\alpha(k)$ is paid, by individual rationality, $a(u)$ consists of a mix of pairwise trading and keeping one's endowment. Then using the same arguments as in the proof of Lemma 3 yields a contradiction. □

Next we show that if agent i does not keep his endowment, then i pays the fixed tax α.

Lemma 5. *Let $C_2^{\varphi} \neq \emptyset$. For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $a_i(u) \neq i$, then $t_i(u) = \alpha$.*

Proof. Let $i \in N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ be such that $a_i(u) \neq i$. We show $t_i(u) = \alpha$. Suppose $t_i(u) < \alpha$. Let $u'_i \in \mathcal{U}_i$ be such that $u'_{ia_i(u)} = \frac{1}{2}(\alpha + t_i(u))$ and $u'_{il} = -1$ for all $l \neq a_i(u), i$. Let $u' = (u'_i, u_{-i})$. By strategy-proofness, $a_i(u') = a_i(u)$ and $t_i(u') = t_i(u)$. Let $j \in N'$ be such that $a_j(u') = i$, and $u''_j \in \mathcal{U}_j$ be such that $u''_{ja_j(u')} = u_{ja_j(u')} + 1$ and $u''_{jl} = -1$ for all $l \neq a_j(u'), j$. Let $u'' = (u''_j, u'_{-j})$. By strategy-proofness, $a_j(u'') = i$ and $t_j(u'') = t_j(u')$. By individual rationality and our construction, $a_i(u'') = a_i(u')$ and $t_i(u'') < \alpha$. If $a_i(u'') = j$, then applying weak consistency yields a contradiction because in all pairwise exchanges the fixed tax α is paid. If $a_i(u'') \neq j$, then let $a_h(u'') = j$ and $u'''_h \in \mathcal{U}_h$ be such that $u'''_{hj} = u_{hj} + 1$ and $u'''_{hl} = -1$ for all $l \neq j, h$. Then we derive the same conclusions as above. At some point we arrive at a profile $\tilde{u}$ such that $a(\tilde{u})$ contains the cycle c, $i \in c$, $t_i(\tilde{u}) < \alpha$, and $\tilde{u}_{ha_h(\tilde{u})} - t_h(\tilde{u}) \geq 0 > \tilde{u}_{hj} - t_j(\tilde{u})$ for all $h \in c$ and all $j \in N' \backslash c$. Then applying weak consistency yields a contradiction because by Lemma 4 in the exchange c the fixed tax α is paid by all agents belonging to c.

Thus, for all $i \in N'$, if $a_i(u) \neq i$, then $t_i(u) \geq \alpha$. Let $\hat{u} \in \mathcal{U}_{N'}$ be such that (i) $\hat{u}_{ia_i(u)} - \alpha > 0 > \hat{u}_{ij}$ for all $j \in N' \backslash \{i, a_i(u)\}$ and all $i \in N'$ with $a_i(u) \neq i$ and (ii) $0 > \hat{u}_{ij}$ for all $j \in N' \backslash \{i\}$ and all $i \in N'$ with $a_i(u) = i$. Then by constrained efficiency, weak consistency and Lemma 4, $a(\hat{u}) = a(u)$ and for all $i \in N$ with $a_i(u) \neq i$, $t_i(\hat{u}) = \alpha$. Thus, the assignment $a(u)$ together with everybody, who does not keep his endowment, paying α belongs to $\mathcal{A}^{\varphi}_{N'}$.

Suppose that $t_j(u) > \alpha$ for some $j \in N'$. Because $t_i(u) \geq \alpha$ for all $i \in N'$ such that $a_i(u) \neq i$, the assignment $a(u)$ together with everybody, who does not keep his endowment, paying α belongs to $\mathcal{A}^{\varphi}_{N'}$ and Pareto dominates $(a(u), t(u))$, which is a contradiction to constrained efficiency. □

The next lemma completes the second part of the proof of Theorem 1.

Lemma 6. *For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $R_{N'}(u, \alpha)$ is strict, then $a(u) = C(R_{N'}(u, \alpha))$.*

Proof. Let $c = (i_1, \ldots, i_k)$ be a top cycle in $R_{N'}(u, \alpha)$. Suppose that c is not part of $a(u)$, say $a_{i_k}(u) \neq i_1$. Because $R_{N'}(u, \alpha)$ is strict and in any exchange agents pay the fixed tax α, then $u_{i_k i_1} - \alpha > u_{i_k a_{i_k}(u)} - t_{i_k}(u)$. Let $u'_{i_k} \in \mathcal{U}_{i_k}$ be such that $u'_{i_k i_1} = u_{i_k i_1}$ and $u'_{i_k l} < 0$

for all $l \neq i_1, i_k$. By strategy-proofness and the fact that in cyclical exchanges the fixed tax α is paid, $a_{i_k}(u'_{i_k}, u_{-i_k}) = i_k$ and $t_{i_k}(u'_{i_k}, u_{-i_k}) = 0$. Note that $R_{N'}(u'_{i_k}, u_{-i_k}, \alpha)$ is strict and c remains a top cycle under $R(u'_{i_k}, u_{-i_k}, \alpha)$. Thus, $a_{i_{k-1}}(u'_{i_k}, u_{-i_k}) \neq i_k$. Similar as above $u_{i_{k-1}}$ can be replaced $u'_{i_{k-1}} \in \mathcal{U}_{i_{k-1}}$ such that $u'_{i_{k-1}i_k} = u_{i_{k-1}i_k}$ and $u'_{i_{k-1}l} < 0$ for all $l \neq i_{k-1}, i_k$. Then we arrive at a profile $u' = (u'_{\{i_1,\ldots,i_k\}}, u_{-\{i_1,\ldots,i_k\}})$ where c is still a top cycle under $R_{N'}(u', \alpha)$ but all agents in c receive their endowments, i.e. for $l \in \{i_1, \ldots, i_k\}$, $a_l(u') = l$ and $t_l(u') = 0$. By construction, for all $l \in \{i_1, \ldots, i_k\}$ and all $j \in N' \backslash \{i_1, \ldots, i_k\}$, $0 > u_{la_j(u')} - t_j(u')$. Thus, by weak consistency, for all $l \in \{i_1, \ldots, i_k\}$, $a_l(u'_{\{i_1,\ldots,i_k\}}) = l$ and $t_l(u'_{\{i_1,\ldots,i_k\}}) = 0$. This is a contradiction to constrained efficiency because c is top cycle under $R_{N'}(u', \alpha)$ and for all $l \in \{1, \ldots, k\}$, $u'_{i_l i_{l+1}} - \alpha = u_{i_l i_{l+1}} - \alpha > 0$.

Thus, c must be part of $a(u)$. Consider a top cycle in $N' \backslash c$, say $c' = (j_1, \ldots, j_m)$. If c' is not part of u, then we can do the same as above: let $u'_{j_m} \in \mathcal{U}_{j_m}$ be such that $u'_{j_m j_1} = u_{j_m j_1}$ and $u'_{j_m l} < 0$ for all $l \neq j_1, j_m$, and $u' = (u'_{j_m}, u_{-j_m})$. Then under u' the cycle c remains a top cycle in the strict $R_{N'}(u', \alpha)$, and thus by the above, $a(u')$ contains c. But then by strategy-proofness and the fact that in cyclical exchanges the fixed tax α is paid, $a_{j_m}(u') = j_m$ and $t_{j_m}(u') = 0$. Note that $R_{N'}(u', \alpha)$ is strict and c remains a top cycle under $R_{N'}(u', \alpha)$ and c' is a top cycle in $N' \backslash c$. Now the same arguments as above yield a contradiction to weak consistency and constrained efficiency. □

We have shown that if $\mathcal{C}_2^\varphi \neq \emptyset$, then by Lemma 4, Lemma 5 and Lemma 6, φ is a fixed-tax core rule.

Our final lemma shows the third part and completes the proof of Theorem 1.

Lemma 7. *If $\mathcal{C}^\varphi \neq \emptyset$, then $\mathcal{C}_2^\varphi \neq \emptyset$.*

Proof. Suppose that $\mathcal{C}_2^\varphi = \emptyset$. By $\mathcal{C}^\varphi \neq \emptyset$, let k be minimal such that $\mathcal{C}_k^\varphi \neq \emptyset$ and for all $l \in \{2, \ldots, k-1\}$, $\mathcal{C}_l^\varphi = \emptyset$. By weak anonymity, $\mathcal{C}_k^\varphi = \mathcal{C}_k$. Let $c = (1, \ldots, k)$ and $c' = (k, \ldots, 1)$. By Lemma 1, there exists a unique symmetric fixed tax $\alpha(k)$ for cycles of length k. Let $u \in \mathcal{U}_c$ be such that (i) $u_{21} > u_{23} > \alpha(k)$ and $u_{2l} = -1$ for $l \neq 1, 2, 3$ and (ii) for all $i \in c \backslash \{2\}$, $u_{ii+1} > u_{ii-1} > \alpha(k)$ and $u_{il} = -1$ for $l \neq i-1, i, i+1$. If $a(u) \neq c, c'$, then $a(u)$ is a mix of pairwise trading and keeping one's endowment. Then we use a similar argument as in Lemma 4, via a profile like $\hat{u}$, to deduce a contradiction to the hypothesis that $\mathcal{C}_2^\varphi = \emptyset$. By constrained efficiency and our choice of k, $a(u) \in \{c, c'\}$.

First, let $a(u) = c$. Then by Lemma 1, $t_2(u) = \alpha(k)$. Let $u'_2 \in \mathcal{U}_2$ be such that $u'_{21} = u_{21}$ and $u'_{2l} = -1$ for $l \neq 1, 2$. By constrained efficiency and our choice of k, $a(u'_2, u_{-2}) = c'$ and $t_2(u'_2, u_{-2}) = \alpha(k)$. But this is now a contradiction to strategy-proofness as $u_{21} - \alpha(k) > u_{23} - \alpha(k)$.

Second, let $a(u) = c'$. Let $u'_1 \in \mathcal{U}_1$ be such that $u'_{12} = u_{12}$ and $u'_{1l} = -1$ for $l \neq 1, 2$. By constrained efficiency and our choice of k, $a(u'_1, u_{-1}) = c$ and $t_1(u'_1, u_{-1}) = \alpha(k)$. But this is now a contradiction to strategy-proofness as $u_{12} - \alpha(k) > u_{1k} - \alpha(k)$. □

Note that Lemma 4, Lemma 5 and individual rationality imply 1. and 2. of Definition 1, and this together with Lemma 6 and Lemma 7 implies 3. of Definition 1.

Remark 1. For all $i \in N$, $u_i \in \mathcal{U}_i$ is a vector $u_i \in \mathbb{R}^N$ such that $u_{ii} = 0$. Then (strictly speaking) for $N' \subsetneq N$ and $u \in \mathcal{U}_{N'}$, the allocation $\varphi(u)$ may depend on the utilities of the agents in N' over $N \backslash N'$. Our definition of a rule did not exclude this. However, as one may check, the proof of Theorem 1 did not require this.[16]

Theorem 1 says very little when the induced rankings are not strict (and this is similar in Miyagawa (2001)). Then the rule φ satisfies the properties in Theorem 1 if φ makes the "right" choices (or tie-breaking decisions among several allocations). Below we show that there exist fixed-tax core rules satisfying all the properties in Theorem 1.

4 Existence

Theorem 1 showed if a rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity, then φ is a fixed-tax core rule or φ is the no-trade rule. Let φ be a fixed-tax core rule with fixed tax $\alpha \geq 0$. Note that the range of φ for $N' \subseteq N$ is given by $\mathcal{A}^{\varphi}_{N'} = \{(\hat{a}, \hat{t}) \in \mathcal{A}_{N'} | \hat{t}_i = 0 \text{ if } \hat{a}_i = i \text{ and } \hat{t}_i = \alpha \text{ otherwise}\}$. Let $\mathcal{W}_i$ denote the set of all weak ordinal rankings over N. Under φ, agent i's possible consumption bundles are $(i, 0)$ and (j, α) with $j \neq i$. Thus, agent i's utility functions induce *all* weak ordinal rankings over his consumption bundles or over N, i.e. $\{R_i(u_i, \alpha) | u_i \in \mathcal{U}_i\} = \mathcal{W}_i$.

Now in order to establish, for the fixed tax α, the existence of a rule satisfying our properties, we use a construction of Saban and Sethuraman (2013), called the *Highest Priority Object (HPO)* algorithm. This is a class of algorithms that generalizes the procedure found by Jaramillo and Manjunath (2012), which was together with Alcalde-Unzu and Molis (2011) the first to demonstrate the existence of individually rational, strategy-proof and efficient rules for the model of house exchange with indifferences and no monetary transfers.

[16] Indeed, if two agents are indifferent between a pairwise trade and keeping their endowments, our requirements do not pin down the chosen allocation, and in such situations either the agents keep their endowment and trade pairwise depending on the utilities over other houses.

Given $N' \subseteq N$, let $\mathcal{O}_{N'}$ denote the set of feasible assignments for N'. Let $f : \cup_{N' \subseteq N} \mathcal{W}_{N'} \to \cup_{N' \subseteq N} \mathcal{O}_{N'}$ be an assignment rule. Then

(i) f is assignment-individually-rational if and only if for all $N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, we have $f_i(R) R_i i$ for all $i \in N'$,

(ii) f is assignment-strategy-proof if and only if for all $N' \subseteq N$, all $R \in \mathcal{W}_{N'}$, all $i \in N'$ and all $R'_i \in \mathcal{W}_i$, $f_i(R) R_i f_i(R'_i, R_{-i})$, and

(iii) f is assignment-efficient if and only if for all $N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, there exists no feasible assignment $\mathrm{a} \in \mathcal{O}_{N'}$ such that $\mathrm{a}_i R_i f_i(R)$ for all $i \in N'$ with strict preference holding for some $j \in N'$.

Fix an assignment rule f belonging to the class of Highest Priority Object Algorithms (we define this class formally in the Appendix). Then by Saban and Sethuraman (2013), f is assignment-individually-rational, assignment-strategy-proof and assignment-efficient. Note that by Sönmez (1999), whenever the core is non-empty, any such assignment rule chooses a core allocation and all agents are indifferent among all core allocations.[17]

Given $\alpha \geq 0$ and an assignment rule f, Gale's fixed α-tax rule $\varphi^{\alpha,f} = (a^{\alpha,f}, t^{\alpha,f})$ based on f is defined as follows: for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, we have $a^{\alpha,f}(u) = f(R_{N'}(u, \alpha))$ and for all $i \in N'$, $t_i^{\alpha,f}(u) = 0$ if $a_i^{\alpha,f}(u) = i$ and $t_i^{\alpha,f}(u) = \alpha$ otherwise.

Theorem 2. *Let $\alpha \geq 0$ and f be an assignment rule belonging to the class of HPO-algorithms. Then the fixed α-tax rule $\varphi^{\alpha,f} = (a^{\alpha,f}, t^{\alpha,f})$ based on f satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and weak anonymity.*

Proof: It is straightforward to check that $\varphi^{\alpha,f}$ satisfies individual rationality, strategy-proofness and constrained efficiency because f is assignment-individually-rational, assignment-strategy-proof and assignment-efficient.

For weak consistency, let $S \subseteq N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ (setting $R = R_{N'}(u, \alpha)$) be such that $S = \cup_{i \in S}\{a_i^{\alpha,f}(u)\} = \cup_{i \in S}\{f_i(R)\}$ and $u_{if_i(R)} - t_i^{\alpha,f}(u) > u_{ij} - t_j^{\alpha,f}(u)$ for all $i \in S$ and all $j \in N' \backslash S$. This means that for all $i \in S$ and all $j \in N' \backslash S$, $f_i(R) P_i j$. But then in the HPO-algorithm, agents in S do not point to agents in $N' \backslash S$ (as agents always point to one of their most preferred objects) and any pointing from the agents in $N' \backslash S$ to

[17]Furthermore, by Saban and Sethuraman (2013, Theorem 3.2) for any $N' \subseteq N$ and $R \in \mathcal{W}_{N'}$, $f(R)$ belongs to the weak core of R. This means that there exists no feasible assignment $\mathrm{a} \in \mathcal{O}_{N'}$ such that for some $S \subseteq N'$, $\cup_{i \in S}\{\mathrm{a}_i\} = S$ and $\mathrm{a}_i P_i f_i(R)$ for all $i \in S$.

agents in S is irrelevant. Thus, $a^{\alpha,f}(u|_S) = f(R|_S) = f_S(R) = a^{\alpha,f}_S(u)$ and (by definition) $t^{\alpha,f}(u|_S) = t^{\alpha,f}_S(u)$. Hence, $\varphi^{\alpha,f}$ satisfies weak consistency.

For weak anonymity, suppose that for some $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ we have for all $i \in N'$ and all $j \in N' \backslash \{i\}$, $a^{\alpha,f}_i(u) - t^{\alpha,f}_i(u) > 0$ and $a^{\alpha,f}_i(u) - t^{\alpha,f}_i(u) > a^{\alpha,f}_j(u) - t^{\alpha,f}_j(u)$. Setting $R_{N'} = R_{N'}(u, \alpha)$, this means for all $i \in N'$, $f_i(R_{N'})P_i i$ and $f_i(R_{N'})P_i j$ for all $j \in N' \backslash \{i, f_i(R_{N'})\}$. But then $f(R_{N'})$ is the unique efficient assignment for $R_{N'}$. Now for any permutation $\sigma : N' \to N''$ (where $N'' \subseteq N$ and $|N''| = |N'|$), $\sigma(f(R_{N'}))$ remains the unique efficient assignment for $\sigma(R_{N'})$, and thus, by assignment-efficiency of f we have $a^{\alpha,f}(\sigma(u)) = f(\sigma(R_{N'})) = \sigma(f(R_{N'})) = \sigma(a^{\alpha,f}(u))$ and (by definition) $t^{\alpha,f}(\sigma(u)) = \sigma(t^{\alpha,f}(u))$. Hence, $\varphi^{\alpha,f}$ satisfies weak anonymity. □

The agents-optimal mechanism in Theorem 2 is the fixed-tax rule with $\alpha = 0$ (call it the zero-tax rule) and the agents-worst mechanism in Theorem 2 is the no-trade rule. Both these rules are worst for the mechanism designer (the government) in terms of monetary transfers from the agents to the mechanism because no taxes collected. Of course, this disregards consumer surplus and other welfare-enhancing considerations.

Below we strengthen constrained efficiency to efficiency and obtain a characterization of a class of fixed 0-tax rules.

Efficiency: For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, if $\varphi(u) = (a(u), t(u))$, then there exists no $(\hat{a}, \hat{t}) \in \mathcal{A}_{N'}$ such that for all $i \in N'$, $u_{i\hat{a}_i} - \hat{t}_i \geq u_{ia_i(u)} - t_i(u)$ with strict inequality holding for some $j \in N'$.

We need to introduce two other properties: ordinality says that the chosen allocation shall be invariant if the induced ordinal rankings over objects coincide with zero taxes; and assignment-weak-consistency defines this property for assignment rules.

Ordinality: For all $N' \subseteq N$ and all $u, u' \in \mathcal{U}_{N'}$, if $R_{N'}(u, 0) = R_{N'}(u', 0)$, then $\varphi(u) = \varphi(u')$.

Assignment-Weak-Consistency: For all $S \subseteq N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, if $\cup_{i \in S}\{f_i(R)\} = S$ and for all $i \in S$ and all $j \in N' \backslash S$, $f_i(R) P_i f_j(R)$, then $f(R|_S) = f_S(R)$.

Theorem 3. *A rule φ satisfies individual rationality, strategy-proofness, efficiency, weak consistency, weak anonymity and ordinality if and only if there exists an assignment rule f satisfying assignment-individual-rationality, assignment-strategy-proofness, assignment-efficiency, and assignment-weak-consistency such that φ is the fixed 0-tax rule based on f.*

Proof: First, let $\varphi = (a^{\varphi}, t^{\varphi})$ satisfy individual rationality, strategy-proofness, efficiency, weak consistency, weak anonymity and ordinality. By efficiency, $\varphi \neq NT$. Since efficiency implies constrained efficiency, Theorem 1 implies that φ is a fixed-tax core rule for some fixed $\alpha \geq 0$. Hence, by efficiency we must have $\alpha = 0$. Then it is straightforward that a^{φ} is a well-defined assignment rule (by ordinality) satisfying assignment-individual-rationality, assignment-efficiency, assignment-strategy-proofness and assignment-weak-consistency.

Conversely, let f be an assignment rule satisfying assignment-individual-rationality, assignment-strategy-proofness, assignment-efficiency, and assignment-weak-consistency. Consider the fixed 0-tax rule $\varphi^{0,f} = (a^{0,f}, t^{0,f})$ based on f. Note that by definition, $\varphi^{0,f}$ satisfies ordinality.

We show that $\varphi^{0,f}$ is efficient. If not, then there exists $N' \subseteq N$ and $u \in \mathcal{U}_{N'}$ such that for some $(\hat{\mathrm{a}}, \hat{\mathrm{t}}) \in \mathcal{A}_{N'}$, we have for all $i \in N'$, $u_{i\hat{\mathrm{a}}_i} - \hat{\mathrm{t}}_i \geq u_{ia_i^{0,f}(u)} - t_i^{0,f}(u)$ with strict inequality holding for some $j \in N'$. Without loss of generality, we may suppose that for all $i \in N'$, $\hat{\mathrm{t}}_i = 0$. Since $t_i^{0,f}(u) = 0$ for all $i \in N'$, setting $R = R_{N'}(u, 0)$ this implies (as $a^{0,f}(u) = f(R)$) $\hat{\mathrm{a}}_i R_i f_i(R)$ for all $i \in N'$ with strict preference holding for some $j \in N'$. But then f is not assignment-efficient, a contradiction. That $\varphi^{0,f}$ satisfies all the remaining properties follows as in the proof of Theorem 2. □

Note that the class of HPO-algorithms are assignment rules satisfying all properties in Theorem 3, and that without ordinality, our requirements do not determine the assignment when there are "many" indifferences.[18]

In the setting where positive transfers are allowed, Miyagawa (2001) shows that any rule satisfying his properties must be a fixed price core rule: under such rule any agent i has a personalized price p_{ij} for consuming object j and setting $p = (p_{ij})_{i,j \in N}$, any utility function u_i of agent i induces an ordinal ranking over objects as follows: for all $j, k \in N$, $jR_i(u_i, p)k \Leftrightarrow u_{ij} - p_{ij} \geq u_{ik} - p_{ik}$. Similar to us, he shows that for any utility profile, if the induced preferences are strict, then the fixed price core rule based on p must choose the unique core assignment of the induced preferences.

Most importantly, in showing existence of a fixed price core rule satisfying his properties, ties are broken exogenously if the induced preferences are weak to obtain a strict profile

[18] For instance, let f belong to the class of HPO-algorithms and $\varphi^{0,f}$ be the fixed 0-tax rule based on f. Let $\mu \in \mathcal{O}_N$ be an assignment such that $\mu_i \neq i$ for all $i \in N$. Now define the rule $\tilde{\varphi}$ as follows: (i) for any $u \in \mathcal{U}_N$ such that $u_{i\mu_i} = 0$ for all $i \in N$, $\tilde{\varphi}(u) = NT(u)$ if $\varphi^{0,f}(u) \neq NT(u)$, and $\tilde{\varphi}(u) = (\mu, 0_N)$ if $\varphi^{0,f}(u) = NT(u)$, and (ii) otherwise $\tilde{\varphi}(u) = \varphi^{0,f}(u)$. Then the modified rule $\tilde{\varphi}$ satisfies all properties in Theorem 3 except for ordinality. This is similar to Saban and Sethuraman (2013) who suggest that it is most likely impossible to characterize all assignment rules satisfying assignment-individual-rationality, assignment-strategy-proofness and assignment-efficiency.

and choose its unique core allocation. Those assignment rules are characterized by Ehlers (2014) and they do not satisfy efficiency but always choose a competitive allocation. Thus, even though Miyagawa (2001) and we show that when the induced preferences are strict, the unique core assignment is chosen, in establishing existence, Miyagawa's class and our class of assignment rules are disjunct: this is easily seen by the usual three agent example below.

Example 2. Let $N = \{1, 2, 3\}$ and agent i be endowed with house i. Let $R_1 : [23]1$ (which stands for $2I_1 3P_1 1$), $R_2 : 132$ and $R_3 : 123$.

Then for $R = (R_1, R_2, R_3)$ the assignments $(2, 3, 1)$ (which stands for agent 1 receiving house 2, agent 2 receiving house 3, and agent 3 receiving house 1) and $(3, 1, 2)$ are assignment-efficient.

The assignment $(2, 1, 3)$ is the core assignment after breaking the tie between houses 2 and 3 at R_1 in favor of house 2, i.e. $(2, 3, 1)$ is the unique core assignment for the strict profile $\hat{R}_1 : 231$, $R_2 : 132$ and $R_3 : 123$. The assignment $(3, 2, 1)$ is the core assignment after breaking the tie between houses 2 and 3 at R_1 in favor of house 3.

Fixed price core rules choose either $(2, 1, 3)$ or $(3, 2, 1)$ whereas any assignment-efficient rule chooses either $(2, 3, 1)$ or $(3, 1, 2)$. □

Remark 2 (General Preferences)**.** One may check that all our results remain true if agents have general preferences over consumption bundles: for all $i \in N$, let $\mathcal{B}_i = \{(i, 0)\} \cup \{(j, t_i) : j \in N \backslash \{i\} \text{ and } t_i \geq 0\}$ and $\mathcal{R}_i$ denote the set of all preference relations on $\mathcal{R}_i$ being (i) complete and transitive, (ii) monotonic: for all $j \in N\backslash\{i\}$ and all $0 \leq t_i < t'_i$ we have $(j, t_i)P_i(j, t'_i)$ and (iii) bounded: for all $j \in N\backslash\{i\}$, all $k \in N$ and all $t_i, t'_i \geq 0$, if $(j, t_i)P_i(k, t'_i)$, then there exists $t''_i > t_i$ such that $(k, t'_i)P_i(j, t''_i)$. As quasi-linear preferences are a subset of $\mathcal{R}_i$, it is easy to check that Theorem 1 and its proof remain true for general preferences. Similarly, Theorem 2 continues to hold on the general preference domain and existence is guaranteed. Note that here, instead of using for $i \in N$, $R_i(u_i, \alpha)$, we use

$$R_i|\{(i, 0)\} \cup \{(j, \alpha) : j \in N\backslash\{i\}\},$$

which is the restriction of R_i to the consumption bundles i may receive under Gale's fixed tax rule based on the fixed tax α.

5 Discussion

We end this paper with some remarks and discussions related to the main findings of the paper. Section 5.1 considers the case when the set N only contains two agents. This restriction enables us to characterize generalized versions of Clarke-Groves mechanisms using our considered axioms (except constrained efficiency). Section 5.2 demonstrates that the axioms used in the main characterization Theorem 1 are independent, i.e., by dropping any of them from Theorem 1, it is possible to construct a non-fixed-tax-rule that satisfies all of the other axioms. Finally, Section 5.3 shows that by strengthening weak consistency to consistency or weak anonymity to anonymity, no fixed-tax core rule satisfies our axioms and we are, consequently, only left with the no-trade rule.

5.1 Two Agents

Suppose in the remaining part of this subsection that the universal set N contains only two agents. This also means that they either exchange their houses with each other or keep their own houses. We will show that in this setting, the properties in Theorem 1 (without constrained efficiency) characterize the familiar Clarke-Groves payments (Clarke, 1971; Groves, 1973) satisfying our constraints.[19] In those payment schemes, the tax paid by an agent depends only on the valuation of the other agent and whether or not trade is executed. If we would impose efficiency, then trade occurs whenever $u_{12} + u_{21} > 0$ and, together with our other axioms, we easily deduce the pivotal rule in the family of Clarke-Groves rules (Green and Laffont, 1979; Moulin, 1986). We do not, however, impose this condition, and so we have access only to Roberts' (1979) theorem. However, this only tells us that agents are to trade when their valuations are sufficiently high. This leaves the work of deducing 1) what threshold should be set for trade and 2) which Clarke-Groves payment schemes satisfy our conditions. We show that there is a space of functions that work, but that this space is nonetheless constrained in subtle ways.

Referring briefly back to our general model, recall that our set of social alternatives there was exponential in the primitives: any subset of agents could be asked to trade amongst themselves. Thus, while Roberts' theorem does hold here, to use it would require us to pin down exponentially many constants. Anonymity might have allowed us to symmetrize the agent-wise constants, but *weak* anonymity does not immediately have this power. Similarly, some form of neutrality might have helped to simplify the outcome-wise constants, but weak

[19] Pápai (2003) studies fair prices in bidding settings.

consistency does not immediately yield such structure. Finally, our proof simultaneously deduces the assignment and payment rules, and restrictions on payments were important for deducing the assignment rule, and therefore, it is not clear whether a proof via Roberts' theorem is simpler.

In the following, we will introduce regular taxation schemes. Loosely speaking any such scheme is based on a non-increasing function. Let $\underline{x}, \overline{x} \in \mathbb{R}_+$ be such that $\underline{x} \leq \overline{x}$. Let $I = [\underline{x}, \overline{x}]$ denote the closed interval with endpoints $\underline{x}$ and $\overline{x}$. Let $g : I \to \mathbb{R}_+$ be a non-increasing function such that $g(\overline{x}) \geq \overline{x}$. Let $g(I) = \{g(x) : x \in I\}$ denote the image of g. Note that for any such function we have $I \leq g(I)$[20] and that g may contain points of discontinuities, i.e. for $x \in I$ we may have[21] $g(x-) \geq g(x) > g(x+)$ or $g(x-) > g(x) \geq g(x+)$ (where both inequalities may be strict and we set both $g(\underline{x}-) = +\infty$ and $g(\overline{x}+) = \overline{x}$). For our purposes, a function $g^{-1} : [\overline{x}, +\infty) \to I$ is an "inverse" of g if, for each $z \in [\overline{x}, +\infty)$, $g^{-1}(z) \in cl(\{x \in I : g(x-) \geq z \geq g(x+)\})$. Note that g^{-1} is not the inverse of g in the usual sense because g^{-1} is defined over $[\overline{x}, +\infty)$ and not only $g(I)$. Furthermore, for some $x', x'' \in I$ we may have $g(x') = g(x'') = z$ and $x' \neq x''$, i.e. $g^{-1}(z)$ may select x' or x'' (or possibly other elements in $cl(\{x \in I : g(x-) \geq z \geq g(x+)\})$). Let $\mathcal{G} = \{g : I \to \mathbb{R}_+ : g \text{ is non-increasing, } g(\overline{x}) \geq \overline{x}, \text{ and } g^{-1} \text{ is defined as above}\}$.

Let $g \in \mathcal{G}$. For each $u \in \mathcal{U}$, we define first a "hypothetical" (regular) tax $h(u)$ in order to check whether exchanging objects makes both agents better off. Below the tax $h_2(u)$ for agent 2 is defined in dependance of u_{12}, agent 1's valuation for 2's object.

(i) If $u_{12} \notin I \cup [\overline{x}, +\infty)$, then $h_2(u) = +\infty$.

(ii) If $u_{12} \in I$, then $h_2(u) = g(u_{12})$.

(iii) If $u_{12} \in [\overline{x}, +\infty)$, then $h_2(u) = g^{-1}(u_{12})$.

Note that for $u_{12} > g(I)$ we have $g^{-1}(u_{12}) = \underline{x}$. In a symmetric way we define $h_1(u)$. Note that this is actually a Clarke-Groves payment that satisfies our constraints.[22]

Now a regular tax rule checks first whether the agents' valuations for the other object exceed the hypothetical tax or not. If both valuations exceed the hypothetical taxes, then they exchange their objects and they pay these taxes. If not, then both agents keep their endowments and they pay no taxes. Formally, the regular tax rule $\phi^g = (a^g, t^g)$ is defined

[20]We use the usual convention that for two sets J and J' we write $J \leq J'$ if $z \leq z'$ for all $z \in J$ and all $z' \in J'$.

[21]Here we use the convention $g(x-) = \lim_{\epsilon \to 0} g(x - \epsilon)$ and $g(x+) = \lim_{\epsilon \to 0} g(x + \epsilon)$.

[22]See for instance Nisan (2007, Theorem 9.36) and Sprumont (2013, Lemma 1).

as follows: for all $u \in \mathcal{U}$, (i) if both $u_{12} \geq h_1(u)$ and $u_{21} \geq h_2(u)$, then $a^g(u) = (2,1)$ and $t^g(u) = h(u)$, and (ii) otherwise $a^g(u) = e$ and $t^g(u) = (0,0)$. The following is straightforward and left to the reader.

Proposition 1. *Let $N = \{1,2\}$. Any regular tax rule and the no-trade rule satisfy individual rationality, strategy-proofness, consistency and weak anonymity.*

Note that regular tax rules do not necessarily satisfy constrained efficiency.

Example 3. Let $I = [0,1]$ and for all $x \in I$, $g(x) = 2 - x$. If agents 1 and 2 report u^2 (with $u^2_{ii} = 0$ and $u^2_{ij} = 2$), then $a^g(u^2) = (2,1)$ and $t^g(u^2) = (0,0)$. If agents 1 and 2 report u^1 (with $u^1_{ii} = 0$ and $u^1_{ij} = 1$), then $a^g(u^1) = (2,1)$ and $t^g(u^1) = (1,1)$. Now for u^1, ϕ^g violates constrained efficiency because both agents strictly prefer $\phi^g(u^2)$ to $\phi^g(u^1)$. $\square$

5.2 Independence

Below we show that the properties in Theorem 1 are independent, i.e. by dropping any property in Theorem 1 we construct a rule satisfying all other properties and which is not a fixed-tax rule (Proposition 1 shows that constrained efficiency is independent from the other properties in Theorem 1).

Example 4 (Not individually rational)**.** Let $N = \{1,\ldots,n\}$. Use the same construction as for Theorem 2 just with the difference that any agent pays 2α when he keeps his endowment. Any such rule satisfies all properties of Theorem 1 except for individual rationality. Note that such a rule is not a fixed-tax rule because the assignment $C(R_{N'}(u,\alpha))$ is not necessarily chosen: agents pay for keeping their endowment and might instead prefer buying another house while for not paying any tax, then they keep their endowment. $\square$

Example 5 (Not strategy-proof)**.** Let $N = \{1,2,3\}$, and $0 < \alpha(3) < \alpha(2)$. In any cycle c of length 3, agents pay $\alpha(3)$, and in any cycle of length 2, agents pay $\alpha(2)$. For all $u \in \mathcal{U}_N$, if there exists a cycle $c = (i_1, i_2, i_3)$ of length 3 such that $u_{i_l i_{l+1}} - \alpha(3) \geq 0$ for all $l = 1,2,3$, then (choose some cycle of length 3, say c) $a(u) = c$ and for all $i \in N$, $t_i(u) = \alpha(3)$. Otherwise a two-cycle is chosen (and the other agent keeps his endowment). For $N' = \{i_1, i_2\}$, if $u_{i_1 i_2} - \alpha(2) \geq 0$ and $u_{i_2 i_1} - \alpha(2) \geq 0$, then $a(u) = (i_2, i_1)$ and $t(u) = (\alpha(2), \alpha(2))$ (and otherwise $a(u) = e_{\{i_1,i_2\}}$ and $t(u) = (0,0)$). Then φ satisfies all the properties in Theorem 1 except for strategy-proofness (because agents might disagree on which cycle of length 3 to choose, like in the proof of Lemma 4). $\square$

Example 6 (Not weakly consistent)**.** Let $N = \{1, \ldots, n\}$ and $\alpha > 0$. For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, (i) if $|N'|$ is odd, then $\varphi(u) = NT(u)$ and (ii) if $|N'|$ is even, let $\varphi(u)$ be the allocation chosen by an HPO-algorithm having as fixed tax α. Then this rule satisfies all properties of Theorem 1 except for weak consistency. □

Example 7 (Not weakly anonymous)**.** Let $N = \{1, \ldots, n\}$, $c = (1, 2, \ldots, n)$ and $\beta \in \mathbb{R}^N_+$ be a vector non-negative payments. For all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, (i) if $N' = N$ and for all $i \in N$, $u_{ii+1} - \beta_i \geq 0$, then $\varphi(u) = (c, \beta)$ and (ii) otherwise $a(u) = e_{N'}$ and $t_i(u) = 0$ for all $i \in N'$. Then this rule satisfies all properties of Theorem 1 except for weak anonymity. □

5.3 Anonymity and Consistency

As will be demonstrated in this subsection, if weak consistency is strengthened to consistency or if weak anonymity is strengthened to anonymity, then no fixed-tax core rules satisfies our properties and we are only left with the no-trade rule (assuming that there are at least 7 agents).

Corollary 1. *Let* $|N| \geq 7$*.*

(i) A rule φ *satisfies individual rationality, strategy-proofness, constrained efficiency, consistency and weak anonymity if and only if* φ *is the no-trade rule.*

(ii) A rule φ *satisfies individual rationality, strategy-proofness, constrained efficiency, weak consistency and anonymity if and only if* φ *is the no-trade rule.*

Proof: In showing (i), suppose that $\varphi = (a, t) \neq NT$. Then $\mathcal{C}^\varphi \neq \emptyset$. By the proof of Theorem 1, we then have $\mathcal{C}^\varphi = \mathcal{C}$. Then a must be an assignment rule satisfying assignment-individual-rationality, assignment-strategy-proofness, assignment-efficiency and assignment-consistency[23]. By Ehlers (2014, Proposition 2 (b)) no such rule exists. We repeat the example below:

Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $R \in \mathcal{W}_N$ be given as follows:

R_1	R_2	R_3	R_4	R_5	R_6	R_7
5	5	1, 6	2, 7	6, 7	3	4
2	1	3	4	5	6	7
1	2	4	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

.

[23]**Assignment-Consistency:** For all $S \subseteq N' \subseteq N$ and all $R \in \mathcal{W}_{N'}$, if $\cup_{i \in S}\{f_i(R)\} = S$, then $f(R|_S) = f_S(R)$.

Because either agent 1 or agent 2 does not receive object 5, it is easy to check that $\mu = (5, 1, 6, 2, 7, 3, 4)$ and $\mu' = (2, 5, 1, 7, 6, 3, 4)$ are the only individually rational and efficient assignments. Suppose that f is an assignment rule satisfying assignment-individual-rationality, assignment-strategy-proofnees, assignment-efficiency, and assignment-consistency. Then $f(R) = \mu$ or $f(R) = \mu'$.

Let $f(R) = \mu$. Then $\{f_3(R), f_6(R)\} = \{3, 6\}$ and by assignment-consistency, $f_2(R_{-3,6}) \neq 5$. Let $R'_2 \in \mathcal{W}_2$ be such that $R'_2 : 5, 2, \ldots$. By assignment-individual-rationality and assignment-strategy-proofness, $f_2(R'_2, R_{-2,3,6}) = 2$. Thus, by assignment-individual-rationality, $f(R'_2, R_{-2,3,6}) \in \{(1, 2, 7, 5, 4), (1, 2, 4, 5, 7)\}$. This contradicts assignment-efficiency because both assignments are Pareto dominated by $(1, 5, 2, 7, 4)$.

Let $f(R) = \mu'$. Then $\{f_4(R), f_7(R)\} = \{4, 7\}$ and by assignment-consistency, $f_1(R_{-4,7}) \neq 5$. Let $R'_1 \in \mathcal{W}_1$ be such that $R'_1 : 5, 1, \ldots$. By assignment-individual-rationality and assignment-strategy-proofness, $f_1(R'_1, R_{-1,4,7}) = 1$. Thus, by assignment-individual-rationality, $f(R'_1, R_{-1,4,7}) \in \{(1, 2, 6, 5, 3), (1, 2, 3, 5, 6)\}$. This contradicts assignment-efficiency because both assignments are Pareto dominated by $(5, 2, 1, 6, 3)$.

Hence, we must have $\varphi = NT$.

In showing (ii), suppose that $\varphi = (a, t) \neq NT$. Then $\mathcal{C}^{\varphi} \neq \emptyset$. By the proof of Theorem 1, we then have $\mathcal{C}^{\varphi} = \mathcal{C}$ and in any exchange the fixed tax $\alpha \geq 0$ is paid. Then a must be an assignment rule satisfying assignment-individual-rationality, assignment-strategy-proofness, assignment-efficiency and assignment-anonymity. Consider $N' = \{1, 2, 3\}$ and $u \in \mathcal{U}_{N'}$ such that $u_{12} = 2 + \alpha = u_{32}$, $u_{13} = 1 + \alpha = u_{31}$, and $u_{21} = 2 + \alpha = u_{23}$. Then u induces the following ordinal rankings: $R_1(u_1, \alpha) : 231$, $R_2(u_2, \alpha) : [13]2$ and $R_3(u_3, \alpha) : 213$. By (constrained) efficiency, $a(u) = (2, 3, 1)$ or $a(u) = (3, 1, 2)$. But then considering the permutation $\sigma : N' \to N'$ such that $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$ gives us a contradiction to anonymity because $\sigma(u) = u$ and $\sigma(R) = R$, but if $a(u) = (2, 3, 1)$, then $\sigma(2, 3, 1) = (3, 1, 2) \neq a(u)$, and if $a(u) = (3, 1, 2)$, then $\sigma(3, 1, 2) = (2, 3, 1) \neq a(u)$. Hence, $\varphi = NT$. □

Regarding Corollary 1, one might wonder whether keeping one of the two requirements (consistency or anonymity) and dropping the other one results in rules different than the no-trade rule.

The example below shows that serial dictatorship rules, where only pairwise exchanges with agent 1 are allowed, satisfy all properties in (i) of Corollary 1 except for weak anonymity.

Example 8. Let $|N| \geq 3$ and $\alpha > 0$. The rule below chooses from the set of allocations where only agent 1 can be part of a pairwise exchange with another agent (and they pay the tax α) and all other agents keep their endowment and pay nothing. For all $1 \in N' \subseteq N$, let $\hat{\mathcal{A}}_{N'} = \{(e_{N'}, 0_{N'})\} \cup \{(b, \mathrm{t}) | b_1 \neq 1, b_{b_1} = 1, \mathrm{t}_1 = \alpha = \mathrm{t}_{b_1}$ and for all $i \in N' \backslash \{1, b_1\}, b_i = i$ and $\mathrm{t}_i = 0\}$. Then for all $N' \subseteq N$ and all $u \in \mathcal{U}_{N'}$, (i) if $1 \notin N'$, then $\varphi(u) = (e_{N'}, 0_{N'})$; and (ii) if $1 \in N'$, then $\varphi(u)$ is chosen according to a serial dictatorship (with "right" tie-breaking): let $N' = \{i_1, \ldots, i_k\}$ with $i_1 = 1 < i_2 < \cdots < i_k$ and

$$X_{i_1}(u) = \{(c, \mathrm{t}') \in \hat{\mathcal{A}}_{N'} | u_{i_1 c_{i_1}} - \mathrm{t}'_{i_1} \geq u_{i_1 b_{i_1}} - \mathrm{t}_{i_1} \text{ for all } (b, \mathrm{t}) \in \hat{\mathcal{A}}_{N'}\},$$

and for $l \in \{2, \ldots, k\}$,

$$X_{i_l}(u) = \{(c, \mathrm{t}') \in X_{i_{l-1}}(u) | u_{i_l c_{i_l}} - \mathrm{t}'_{i_l} \geq u_{i_l b_{i_l}} - \mathrm{t}_{i_l} \text{ for all } (b, \mathrm{t}) \in X_{i_{l-1}}(u)\};$$

then $\varphi(u) = (a(u), t(u)) \in X_{i_k}(u)$ such that for $l \in \{1, \ldots, k\}$ with $a_{i_l}(u) = 1$, we have $b_{i_v} \neq 1$ for all $(b, \mathrm{t}) \in X_{i_k}(u)$ and all $v \in \{1, \ldots, l-1\}$. Then it is straightforward to verify that φ satisfies individual rationality, strategy-proofness, constrained efficiency and consistency (but not weak anonymity). □

Dropping weak consistency in (ii) of Corollary 1 we continue to get the no-trade rule for utility profiles with more than two agents.

Corollary 2. *If a rule φ satisfies individual rationality, strategy-proofness, constrained efficiency, and anonymity, then for all $N' \subseteq N$ with $|N'| > 2$ and all $u \in \mathcal{U}_{N'}$, $\varphi(u) = NT(u)$.*

Proof: Let $\varphi = (a, t)$. Suppose that for some $N' \subseteq N$ with $|N'| > 2$ and some $u \in \mathcal{U}_{N'}$, $\varphi(u) = (a(u), t(u)) \neq NT(u)$. Without loss of generality, let $N' = N$. Then $a(u)$ contains a cycle c of length greater than or equal to two, say $c = (1, \ldots, k)$. Without loss of generality, let c be a cycle of smallest length greater than or equal to two contained in any allocation belonging to $\mathcal{A}^{\varphi}_{N}$. Note that Lemma 1 does not use weak consistency and the same parallel arguments show that whenever $(a(u), \mathrm{t})$ and $(a(u), \mathrm{t}')$ belong to $\mathcal{A}^{\varphi}_{N}$, then $\mathrm{t} = \mathrm{t}'$ and all agents belonging to the same cycle pay the same tax. Let $\alpha(k) = t_i(u)$ for all $i \in c$. Now by anonymity, it follows that for any permutation $\sigma : N \to N$ such that $\sigma(i) = i$ for all $i \in N \backslash c$, we have $\varphi(\sigma(u)) = (\sigma(a(u)), \sigma(t(u)))$. In particular, for all $i \in c$, $\varphi_i(\sigma(u)) = (c_{\sigma(i)}, \alpha(k))$.

We distinguish two cases depending on whether $k \geq 3$ or $k = 2$.

First, let $k \geq 3$. Then we construct $\hat{u} \in \mathcal{U}_N$ inducing the following ordinal profile over objects: $R_1 : 231[N\backslash\{1,2,3\}]$, $R_2 : [13]2[N\backslash\{1,2,3\}]$, $R_3 : 243[N\backslash\{1,2,3\}]$, $R_i : [c\backslash\{i\}]i[N\backslash c]$ for all $i \in c$, and $R_j : a_j(u)j[N\backslash\{j\}]$. For agent 1, we choose $\hat{u}_1 \in \mathcal{U}_1$ such that $\hat{u}_{12}-\alpha(k) = 2 > 1 = \hat{u}_{13}-\alpha(k) > 0 > -1 = \hat{u}_{1j}-t_j(u)$ for all $j \in N\backslash\{1,2,3\}$. In addition, for agent 2: $\hat{u}_2 \in \mathcal{U}_2$ is such that $\hat{u}_{21} - \alpha(k) = 1 = \hat{u}_{23} - \alpha(k) > 0 > -1 = \hat{u}_{1j} - t_j(u)$ for all $j \in N\backslash\{1,2,3\}$, for agent 3: $\hat{u}_3 \in \mathcal{U}_3$ is such that $\hat{u}_{32} - \alpha(k) = 2 > 1 = \hat{u}_{31} - \alpha(k) > 0 > -1 = \hat{u}_{1j} - t_j(u)$ for all $j \in N\backslash\{1,2,3\}$, for each agent $i \in c\backslash\{1,2,3\}$: $\hat{u}_i \in \mathcal{U}_i$ is such that $\hat{u}_{il} - \alpha(k) = 1 > 0 > -1 = \hat{u}_{1j} - t_j(u)$ for all $l \in c\backslash\{i\}$ and all $j \in N\backslash c$, and for each agent $j \in N\backslash c$: $\hat{u}_j \in \mathcal{U}_j$ is such that $\hat{u}_{ja_j(u)} - a_j(u) = 1 > 0 > -1 = \hat{u}_{jl} - t_l(u)$ for all $l \in N\backslash\{j, a_j(u)\}$.

Let $\sigma : N \to N$ be such that $\sigma(1) = 3$, $\sigma(3) = 1$ and $\sigma(i) = i$ for all $i \in N\backslash\{1,3\}$. By our choice of c and constrained efficiency, $a(u) = c$ or $a(u) = \sigma(c)$, say $a(u) = c$. But then $\sigma(u) = u$, and thus, $\varphi(\sigma(u)) = \varphi(u)$ which is a contradiction to anonymity as $c \neq (\sigma(a(u)))_{i\in c}$ (as agent 2 cannot be assigned the same object since $c_2 = 3$ and $\sigma_2(a(u))) = 3$).

Second, let $k = 2$. If cycles of length 3 are not possible, then the same arguments as above yield a contradiction. If cycles of length 3 are possible, then we do the same as in the proof of (ii) of Corollary 1. □

APPENDIX

A The HPO Algorithms

In order to establish, for some flat tax α, the existence of a rule satisfying our axioms, we use a construction of Saban and Sethuraman (2013), called the *Highest Priority Object (HPO)* algorithm. In this section, we elaborate on these algorithms for the sake of completeness.

We shall first describe the algorithm in words. Note that efficient exchange in the presence of indifferences is much more complicated than in their absence. Any allocation will decompose into trading cycles, but these will not be the simple "top cycles" used in the classical algorithm. Rather than find these cycles directly, the literature has employed a familiar, simpler strategy: having agents trade until all gains are exhausted. That is, unlike in the Top Trading Cycles algorithm, agents are required to stay in the market even after they have traded. This is because trading within their thick indifference set may benefit others while not harming them.

There are two phases in each step of the generic HPO algorithm, *removal and update* and *improvement*. During *removal and update*, the algorithm removes the agents who are holding one of their favorite objects, among those remaining, and whose participation in further trading cycles cannot benefit others. These agents are then permanently assigned the object they hold and sent away. The remaining agents update their preferences, given that some objects are no longer available. Any agent holding an object they value at least as much as all remaining objects is called *satisfied*.

During *improvement*, trading cycles are executed. A single agent may trade several times, and hold several different objects, before finally being removed in the *removal and update* phase.

We first make formal the *removal and update* phase. Because agents may hold several different objects before leaving the algorithm, we can no longer conflate agents with objects. Let Ω be the set of (remaining) objects and $\mu : N \to \Omega$ a one-to-one assignment of agents to objects. Given μ and preference profile R, the **ttc graph**, denoted $\mathcal{G}(N, \mu, R)$, has vertices N and directed edges $\{(i, j) : \forall \omega \in \Omega, \mu(j)\ R_i\ \omega\}$. Note that the ttc graph may have loops, as agents may hold their favorite object and remain in the algorithm. As in the body text, we write $(i, i^2, i^3, \ldots, j)$ to refer to the directed path $\{(i, i^2), (i^2, i^3), \ldots, (i^{n-1}, j)\}$. A **sink**, S, of a generic directed graph $\mathcal{G}$ is a (strongly) connected component: for each $i, j \in S$, there is a directed path $(i, i^2, i^3, \ldots, j) \subseteq \mathcal{G}$ with $\{i, i^2, i^3, \ldots, i^{n-1}, j\} \subseteq S$, and for each $i \in S$, $j \notin S$, there is no such path from i to j. A **terminal sink** S^T is a sink with the property that for each $i \in S^T$, $(i, i) \in \mathcal{G}$. Agents in a terminal sink of $\mathcal{G}(N, \mu, R)$ are satisfied. Moreover, they do not belong to any (directed) circuit that includes someone who *is not* satisfied. Thus, they cannot contribute to any Pareto improving trades and so are permanently assigned the objects they hold and are removed. The remaining agents have their preferences updated so that the objects just removed are no longer in their preference ranking.

The *improvement* phase consists mainly of selecting a simple graph, in which each node has out-degree 1, from the starting graph $\mathcal{G}(N, \mu, R)$. Let L be the possibly-empty set of labeled agents. The phase may begin with some agents labeled, depending on the previous step in the algorithm. In the first step, no agents begin labelled. If there are any labeled agents, they select the same agent they pointed to in the last step. Thus, for these agents, ties are broken by history. Next, agents who are not satisfied break ties based on the name of the objects, with everyone using a common order $\prec$. Finally, satisfied agents break ties in a more complicated manor. Label all agents whose ties are already broken, so at this point, all the previously-labelled and unsatisfied agents are labelled. Recursively, perform the following operations: 1) Select an unlabelled agent who is pointing to a labelled agent, breaking ties in this selection by the name of the object each holds; 2) Break the selected agent's ties first by eliminating all objects held by unlabelled agents and then using the name of the objects (and $\prec$); 3) Label the selected agent.

The set of labelled agents will expand to the entire set of agents, at which point all ties have been broken, and we are left with a simple subgraph $\mathcal{G} \subseteq \mathcal{G}(N, \mu, R)$. Now remove all labels, and execute all trading cycles. We must apply labels again for use in the next step. For each unsatisfied agent, j, who *did not* just trade, identify the longest path of satisfied agents $(i_1, i_2, \ldots, j) \subseteq \mathcal{G}(N, \mu, R)$. Label each of these satisfied agents. This completes one step of the algorithm. Proceed now to the next step, beginning with *removal and update*.

We also give a complete, formal description of the algorithm in two figures. Algorithm 1 contains subroutines necessary to run HPO, while the HPO algorithm is Algorithm 2. The algorithms are written in pseudocode, so the meaning of "=" is what it means in programming: "set the name on the left hand side to refer to the value stored on the right." Thus, the potentially confusing sentence $N = N \setminus S$ means, "henceforth, symbol N refers to what was *previously* meant by $N \setminus S$." The order $\prec$ is on the names of the objects.

The core of the algorithm consists of the repeated application of three subroutines, Prune(), Subgraph(), and Trade(). Prune() is run first, and removes all terminal sinks from consideration, making the sub-allocation for those agents final. Subgraph() performs the tie-breaking described above. Finally, Trade() executes trading cycles, and should only be passed graphs that are the output of Subgraph(), as it cannot process overlapping cycles.

Given $R \in \mathcal{W}_N$, $\Omega = N$, and for the strict priority order $\prec$ on Ω, let $f^{\prec}(R)$ denote the output of the HPO Algorithm. For all $N' \subseteq N$ and $R' \in \mathcal{W}_{N'}$, let $\Omega' = N'$ and $\prec|_{\Omega'}$ denote the restriction of $\prec$ to Ω', and let $f^{\prec}(R')$ denote the output of the HPO Algorithm when applied to R' and $\prec|_{\Omega'}$. Now by Saban and Sethuraman (2013), $f^{\prec}$ is assignment-individually-rational, assignment-strategy-proof and assignment-efficient. The argument in the proof of Theorem 2 shows that $f^{\prec}$ satisfies weak anonymity.

In order to see weak consistency of $f^{\prec}$, let $S \subseteq N' \subseteq N$ and $R \in \mathcal{W}_{N'}$ be such that $\cup_{i \in S}\{f_i^{\prec}(R)\} = S$ and for all $i \in S$ and all $j \in N' \backslash S$, $f_i^{\prec}(R) P_i f_j^{\prec}(R)$. By $\cup_{i \in S}\{f_i^{\prec}(R)\} = S$, the last condition implies for all $i \in S$ and all $j \in N' \backslash S$, $f_i^{\prec}(R) P_i j$. Thus, if $i \in S$ belongs to the terminal sink S^T in the HPO Algorithm applied to R and $\prec|_{N'}$, then $S^T \subseteq S$. But then applying the HPO algorithm to R_S and $\prec|_S$ yields the same terminal

Algorithm 1 Subroutines

procedure PRUNE(N, Ω, μ, R)
    $G = \mathcal{G}(N, \mu, R)$
    while there is a terminal sink S of G **do**
        $(N, \Omega, \mu, R) = \big(N \setminus S,\ \Omega \setminus \mu(S),\ \mu|_{N \setminus S},\ (R_i|_{\mu(N \setminus S)})_{i \in N \setminus S}\big)$ ▷ Remove and update
        $G = \mathcal{G}(N, \mu, R)$ ▷ Repeat with new graph
    end while
    return (N, Ω, μ, R)
end procedure

L is the set of labelled agents, possibly empty.
G^L is a graph storing the edges that were previously selected for the labelled agents. It might also be empty.
The subgraph selection automatically chooses G^L, and then builds upon it.

procedure SUBGRAPH$(N, \Omega, \mu, R, L, G^L)$
First the unsatisfied agents point, with ties broken by $\prec$
    for $i \in N \setminus L, (i, i) \notin \mathcal{G}(N, \mu, R)$ **do**
        $\omega = \min_{\prec} \max_{R_i} \Omega$ ▷ Break i's ties with $\prec$
        $G^L = G^L \cup \{(i, \mu^{-1}(\omega))\}$ ▷ i points to whomever holds ω
        $L = L \cup \{i\}$.
    end for
Now the satisfied agents point
    while there is an unlabelled agent, $i \in N \setminus L$ **do**
        $G = \mathcal{G}(N, \mu, R)$
        $A = \{i \in N \setminus L : \exists j \in L, (i, j) \in G\}$ ▷ Agents pointing to labelled agents. This set is not empty, for otherwise there would be a terminal sink.
        $i = \mu^{-1}\Big[\min_{\prec} \mu(A)\Big]$ ▷ i holds the highest priority object among A
        $\Omega = \mu(L)$ ▷ i will point to the object of a labelled agent
        $\omega = \min_{\prec} \max_{R_i} \Omega$ ▷ Break ties with $\prec$.
        $G^L = G^L \cup \{(i, \mu^{-1}(\omega))\}$ ▷ Increase the simple subgraph G^L
        $L = L \cup \{i\}$ ▷ i is labelled
    end while
    return G^L ▷ The output is the simple subgraph G^L
end procedure

procedure TRADE(G, μ)
    for circuits $(i^1, i^2, \ldots, i^n, i^1) \subseteq G$ **do**
        for $k \in \{1, \ldots, n\}$ **do**
            $\mu(i^k) = \mu(i^{k+1}) \mod n$
        end for
    end for
    return μ
end procedure

Algorithm 2 The HPO Algorithm

$L = \emptyset$
$G^L = \emptyset$
$\mu = e$
while $N \neq \emptyset$ **do**
  $(N, \Omega, \mu, R) = \textsc{Prune}(\mathcal{G}(N, \mu, R))$
  $L = L \setminus N$
  $G^L = G^L|_N$
  $G = \textsc{Subgraph}(N, \Omega, \mu, R, L, G^L)$
  $\alpha = \textsc{Trade}(G, \mu)$
  $L = \{i \in N : (i,j) \in G, (i,i) \in \mathcal{G}(N, \mu, R), \alpha(j) = \mu(j)\}$ ▷ Agents labelled for next step
  $G^L = \{(i,j) : i \in L, (i,j) \in G\}$ ▷ Labeled agents' tie-breakers stored
  $\mu = \alpha$ ▷ Trade updated
end while

sinks for the agents belonging to S (because they do not point to any objects in $N' \setminus S$)[24] and hence, $f^{\prec}(R_S) = f^{\prec}_S(R)$, which is the desired conclusion.

[24] The careful reader may check that Saban and Sethuraman (2013, Claim 1) is here useful.

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