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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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Department of Economics
School of Economics and Management

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Tommy Andersson
Zaifu Yang
Dongmo Zhang

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How to Efficiently Allocate Houses under Price Controls?[1]

Tommy Andersson,[2] Zaifu Yang,[3] and Dongmo Zhang[4]

Abstract: Price controls are used in many regulated markets and well recognized as the cause of market inefficiency. This paper examines a practical housing market in the presence of price controls and provides a solution to the problem of how houses should be efficiently allocated among agents through a system of prices. We demonstrate that the dynamic auction by Talman and Yang (2008) always finds a core allocation in finitely many iterations, thus resulting in a Pareto efficient outcome.

Keywords: Ascending auction, assignment market, price control, Pareto efficiency, core.

JEL classification: C71, D44, D47.

1 Introduction

Price control refers to a situation where a local or central government authority restricts the prices of goods at specific markets. The intention behind this type of regulation is to make the goods affordable for the consumers, or to guarantee a minimum income for workers, or to provide certain necessary goods. A textbook example of such regulation is rent control which is a common practice in many countries including India, Luxembourg, Sweden, United Kingdom, and United States among others. In the state of New York, for example, the local government determines a maximal rent that a landlord can charge a tenant, and the landlords are periodically allowed to apply for increases within this maximal rent. The exact procedure for determining and updating the maximal rents differs from one municipality to another within the state but

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[2]Department of Economics, Lund University, Sweden, tommy.andersson@nek.lu.se

[3]Department of Economics and Related Studies, University of York, York, UK, zaifu.yang@york.ac.uk

[4]Intelligent Systems Laboratory, University of Western Sydney, Australia, d.zhang@uws.edu.au

in the city of New York, for example, the maximum rents should be calculated based upon the "Maximum Base Rent System" (MBRS) according to the "New York City Local Law 30 of 2012". MBRS is updated every two years to reflect changes in real estate taxes, operating and maintenance expenses, water and sewer charges, etc.[5]

In a classical paper, Friedman and Stigler (1946) observed that price controls may lead to an inefficient allocation of goods in the sense that the goods are not necessarily allocated to those that value them the most. More recently, similar observations and conclusions have been reached by Bulow and Klemperer (2012), Gleaser and Luttmer (2003), and Luttmer (2007). For instance, Bulow and Klemperer (2012) analyze several basic cases of how much consumer surplus can be lost due to price controls. However, none of these papers offer a solution to the problem of *how* the goods should be efficiently allocated among the agents in the presence of price controls. The main innovation of this paper is to examine a dynamic procedure that efficiently allocates a number of indivisible items like houses or apartments, to potential agents such as tenants, under price control. To our best knowledge, the current study is the first of its kind to offer a procedure that always guarantees to find a Pareto efficient outcome, although there are numerous studies on economic models under price control.[6]

The point of departure for the analysis is a practical housing market where several houses are going to be rented to finitely many potential tenants. Each tenant has a valuation on every house and is interested in renting at most one house. The landlord of each house has a reservation rent for her house and is not willing to let the house to any tenant if the rent falls short of this predetermined reservation rent. A local or central government authority imposes a ceiling rent on every house. These ceilings may be interpreted as a legislated rent control. The basic problem to be studied in this paper is *how* to efficiently allocate the houses among the tenants through a system of rents that satisfy both the reservation and ceiling rent constraints.

The housing market under price control we address in this paper reduces to the celebrated

[5]There are pros and cons of the price controls among economists and politicians. But we are not concerned with whether there should be price controls or not. Rather we accept them as an inescapable political and economic reality, because price controls including, e.g., laws on minimum wages, do exist in many developed and developing countries. As the title indicates, our aim is to make the best use of scarce resources in the presence of price controls.

[6]In the literature, price control, rent control, price rigidities, or regulated prices are often interchangeably used. Fixed prices are the extreme case of price rigidities.

assignment market that was first investigated by Koopmans and Beckmann (1957) and Shapley and Shubik (1971), when prices are totally flexible and no price controls are imposed. It is well-known that the assignment market has at least one Walrasian equilibrium (Koopmans and Beckmann, 1957) and that the set of Walrasian equilibrium price vectors forms a complete lattice (Shapley and Shubik, 1971). Due to price controls, Walrasian equilibria typically fail to exist. The traditional solution to such problems would be to consider weaker equilibrium concepts based on rationing schemes that curb consumers' demand or seller's supply on certain goods.[7] Even if rationing will help prices to facilitate the allocation of goods among the agents, it is unfortunately known that the use of a rationing scheme may generate Pareto inefficient solutions.[8]

While the literature on price control has focused almost entirely on economic models with divisible goods, Talman and Yang (2008) have recently studied the assignment market under price control where items for sale are inherently indivisible, such as houses or apartments. Modifying the processes of Crawford and Knoer (1981), and Demange, Gale and Sotomayor (1986), Talman and Yang (2008) propose an ascending auction with rationing (called the Talman-Yang auction, henceforth) that always generates a constrained equilibrium.[9] The constrained equilibrium consists of an assignment of items, a price vector and a rationing scheme. In this article, we adopt a more natural solution concept of a core allocation instead of constrained equilibrium. There are obvious advantages of this solution over the constrained equilibrium. Firstly, every core allocation is Pareto efficient and stable against any coalition deviation. Secondly, the core allocation is conceptually simpler, more intuitive and more straightforward than the constrained equilibrium as it consists of only an assignment of items and a supporting price vector for the assignment and does not use any rationing scheme. Thirdly, the core allocation has been widely used in general exchange economies and game theoretic models (see, e.g., Scarf, 1967, 1973). The main result of the current paper demonstrates that the Talman-Yang auction always finds a

[7] See Drèze (1975), Cox (1980), van der Laan (1980), Kurz (1982), Azariadis and Stiglitz (1983), Dehez and Drèze (1984), Weddepohl (1987), and Herings, Talman and Yang (1996) among many others.

[8] See, e.g., Böhm and Müller (1977), Herings and Konovalov (2009).

[9] Andersson and Svensson (2014) examined a similar problem and proposed a different constrained equilibrium (called a rationing price equilibrium) which depends on an exogenously given priority structure. Given this priority structure, they define an individual rational, equilibrium selecting and group non-manipulable allocation rule on a reduced preference domain that contains almost all preference profiles.

core allocation in finitely many iterations, thus resulting in a Pareto efficient outcome.

The rest of the paper proceeds as follows. Section 2 sets up the model. Section 3 presents the major results. Section 4 concludes.

2 The Model

A seller wishes to sell n heterogenous and indivisible *items*, denoted by $1, \ldots, n$, to m potential *bidders* indexed by $1, \ldots, m$.[10] We use the set $N = \{0, 1, \ldots, n\}$ to represent all items including a *dummy item* 0 and the set $M = \{1, \ldots, m\}$ to stand for all bidders. The dummy item 0 has no value but does no harm and can be assigned to any number of bidders. Every bidder $i \in M$ has a *valuation* $V^i(j) \in \mathbb{Z}_+$ in monetary units on each item $j \in N$ with $V^i(0) = 0$. The seller has a *reservation price* $\underline{p}_j \in \mathbb{Z}_+$ for each item $j \in N$ below which the item will not be sold. In addition, a central planner, say, the government imposes *price controls*. That is, each item $j \in N$ has a ceiling price $\overline{p}_j \in \mathbb{Z}_+$ above which the item is not permitted to sell. By convention $0 \leq \underline{p}_j < \overline{p}_j$ for each $j \in N$ and $\underline{p}_0 = \overline{p}_0 = 0$. A *price vector* $p \in \mathbb{R}^N_+$ indicates a price for every item, and the set of *admissible prices* is given by

$$P = \{p \in \mathbb{R}^N_+ \mid p_0 = 0,\ \underline{p}_j \leq p_j \leq \overline{p}_j,\ j = 1, \ldots, n\ \}.$$

When there are no price controls, i.e., when $\overline{p}_j = +\infty$ for every $j = 1, \ldots, n$, the model reduces to the classical assignment market model as studied, e.g., by Koopmans and Beckmann (1957) and Shapley and Shubik (1971).

A *feasible assignment* π assigns every bidder $i \in M$ an item $\pi(i)$ such that no item in $N \setminus \{0\}$ is assigned to more than one bidder. Note that a feasible assignment may assign the dummy item to several bidders and a real item $j \neq 0$ may not be assigned to any bidder at all. An item $j \neq 0$ is *unassigned* at π if there is no bidder i such that $\pi(i) = j$. Let $U(\pi)$ denote the set of all unassigned items at π. A *feasible allocation* (π, p) consists of a feasible assignment π and an admissible price vector p such that $p_j = \underline{p}_j$ for every unassigned item $j \in U(\pi)$.

[10]Note that the results in this paper are independent of the assumption that there is one seller. All results would continue to hold even if there are multiple sellers, because each buyer demands only one item. We will also, without loss of generality, use seller, bidder, item and price, instead of landlord, tenant, house, and rent, respectively, in the rest of the paper.

The *demand set* of bidder $i \in M$ at a price vector $p \in \mathbb{R}^N_+$ is given by

$$D^i(p) = \{j \in M \mid V^i(j) - p_j \geq V^i(k) - p_k \text{ for every } k \in N\}.$$

A *Walrasian equilibrium* consists of a feasible assignment π and a price vector $p \in \mathbb{R}^N_+$ such that $\pi(i) \in D^i(p)$ for all $i \in M$ and $p_j = \underline{p}_j$ for every unassigned item $j \in U(\pi)$.

It is well-known from Koopmans and Beckmann (1957) and Shapley and Shubik (1971) that a Walrasian equilibrium exists in the economy when there are no price controls. In the case of price control, a Walrasian equilibrium may not exist since the equilibrium price vector may not be admissible. This can be easily seen from the following example. Suppose that a seller wishes to sell one item, called item 1, to two bidders, called 1 and 2. The seller's reservation price is zero and the ceiling price is $\overline{p}_1 = 3$. Bidder 1's valuation of item 1 is 5 and bidder 2's valuation of item 1 is 7. It is clear that at any admissible price, item 1 is in excess demand as both bidders demand it. Therefore it is impossible to find a Walrasian equilibrium.

To deal with price rigidities which fail the existence of a Walrasian equilibrium, we adopt the notion of *core* to the current model. This concept is more general than that of Walrasian equilibrium and is a widely used solution for general exchange economies and non-transferable utility games (see, e.g., Scarf, 1967, 1973).

Definition 2.1 *A feasible allocation (π, p) is* a core allocation *if there do not exist a coalition S of bidders besides the seller and another feasible allocation (ρ, q) such that $\rho(i) = 0$ for every $i \in M \setminus S$, and $V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}$ for every $i \in S$, and $\sum_{h \in N} q_h > \sum_{h \in N} p_h$.*

Clearly, every core allocation is Pareto efficient and robust against any possible coalition deviation. Such allocation is free of any rationing scheme. In the next section we will give a constructive proof of the existence of core allocation in the model via a dynamic auction.

3 The Talman-Yang Dynamic Auction

Talman and Yang's auction works roughly as follows: The auctioneer starts the auction at the reservation prices of the items for sale. Then the bidders respond with their demand sets. The auctioneer accordingly eliminates overdemanded items by increasing their prices or by a lottery to determine who should be assigned the item when its price has reached its ceiling price. The auction stops when there are no overdemanded items at which a core allocation will be shown to exist. To give a more precise description of the auction, some basic concepts must first be introduced.

For a set of real items $T \subseteq N \setminus \{0\}$, and a price vector $p \in \mathbb{R}^N_+$, define the *lower inverse demand* set of T at p by

$$D^-_T(p) = \{i \in M \mid D^i(p) \subseteq T\},$$

i.e., this is the set of bidders who demand only items in T at p. We also define the *upper inverse demand* of T at p by

$$D^+_T(p) = \{i \in M \mid D^i(p) \cap T \neq \emptyset\},$$

i.e., this is the set of bidders that demand at least one of the items in T at p. Clearly, the lower inverse demand set is a subset of the upper inverse demand set at given prices.

A set of real items $O \subseteq N \setminus \{0\}$ is *overdemanded* at a price vector $p \in \mathbb{R}^N_+$ if $|D^-_O(p)| > |O|$. So for an overdemanded set O, the number of bidders who demand only items in O is strictly greater than the number of items in O. An overdemanded set O is said to be *minimal* if no strict subset of O is an overdemanded set. A set of real items $T \subseteq N \setminus \{0\}$ is *underdemanded* at price vector $p \in \mathbb{R}^N_+$ if $T \subseteq \{j \in N \setminus \{0\} \mid p_j > \underline{p}_j\}$ and $|D^+_T(p)| < |T|$. In other words, a set of real items $T \subseteq S$ is underdemanded at p if the price of every item j in T is strictly greater than its reservation price $\underline{p}_j$ and the number of bidders who demand at least one item in T is strictly less than the number of items in T.

Now we are ready to describe the Talman-Yang dynamic auction under price rigidities. Note that in the auction process, since the set of bidders and the set of items are weakly shrinking, the demand set of each bidder and the overdemanded sets need to be adapted accordingly.

The Talman-Yang Auction

Step 1: The auctioneer announces the set of items $N^0 = \{0, 1, \ldots, n\}$ for sale, the reservation price vector $\underline{p}$ and the ceiling price vector $\overline{p}$. The bidders, denoted by $M^0 = \{1, \ldots, m\}$, come to bid. Set the iteration counter to $t := 0$ and let $p^t := \underline{p}$. Go to Step 2.

Step 2: The auctioneer asks each bidders $i \in M^t$ to report their demand sets $D^i(p^t)$ for the items in the set N^t and checks whether there is any overdemanded set of items in N^t at p^t. If there is no overdemanded set of items, the auction stops. Otherwise, there is at least one overdemanded set in N^t. The auctioneer first chooses a minimal overdemanded set $O \subseteq N^t$ of items and next checks whether the price of any item in the set O has reached its ceiling price. Let $\overline{O} := \{j \in O \mid p^t_j = \overline{p}_j\}$. If $\overline{O}$ is empty, the auctioneer increases the price of each item in O by one unit and keeps the prices of all other items unchanged. Set $t := t + 1$ and return to Step 2. If $\overline{O}$ is nonempty, go to Step 3.

Step 3: The auctioneer picks an item at random from $\overline{O}$ and asks all bidders $i \in M^t$ with $D^i(p^t) \subseteq O$ who demand the item to draw lots for the right to buy the item. Then the (unique) winning bidder i^* gets the item j^* by paying its current price and exits from the auction. Set $M^{t+1} = M^t \setminus \{i^*\}$ and $N^{t+1} = N^t \setminus \{j^*\}$. If $M^{t+1} = \emptyset$ or $N^{t+1} = \emptyset$, the auction stops. Otherwise, set $t := t + 1$ and return to Step 2.

We remark that in the special case of this model when $\overline{p}_j = +\infty$ for all j as in the classical assignment market of Koopmans and Beckmann (1957) and Shapley and Shubik (1971), no bidder will demand any item with a price equal to the upper price limit as valuations are finite and because the dummy item 0 can be assigned to any number of bidders, i.e., $\overline{O} = \emptyset$ at any iteration. In this special case, the Talman-Yang auction is identical to the auction mechanism in Demange, Gale and Sotomayor (1986) and the outcome will always be a Walrasian equilibrium with the unique minimal equilibrium price vector.

Before stating the main result, we illustrate by means of an example how the auction actually operates, and after the example we provide some intuition behind the fact that the outcome of the auction is a core allocation.

Example 1: Suppose that there are five bidders and four real items, i.e., $M^0 = \{1, 2, 3, 4, 5\}$ and $N^0 = \{0, 1, 2, 3, 4\}$. The reservation and ceiling price vectors are given by $\underline{p} = (0, 5, 4, 1, 5)$ and $\overline{p} = (0, 6, 6, 4, 7)$. Bidders' values of the items are given in Table 1. The auction starts at the price vector $p^0 = (0, 5, 4, 1, 5)$. The reported demand sets are given by: $D^1(p^0) = \{3\}$, $D^2(p^0) = \{3\}$, $D^3(p^0) = \{3\}$, $D^4(p^0) = \{3\}$ and $D^5(p^0) = \{3\}$. The set $O = \{3\}$ is a minimal overdemanded set and the auctioneer adjusts p^0 to $p^1 = (0, 5, 4, 2, 5)$. The demand sets and price vectors and other relevant data generated by the auction are illustrated in Table 2. In iteration 3, the price of item 3 has reached its upper bound, i.e., $p_3^3 = 4$. The auctioneer assigns randomly item 3 to the bidders that demands it, i.e., bidders 1, 2, 3, 4 and 5. Say, that bidder 1 is assigned the item in the lottery and, therefore, pays 4 dollars and leaves the auction. Then, $p^4 = (p_0^4, p_1^4, p_2^4, p_4^4) = (0, 5, 4, 5)$, $M^4 = \{2, 3, 4, 5\}$ and $N^4 = \{0, 1, 2, 4\}$. Proceeding in this way, the auction terminates in iteration 6 when there is no overdemanded set of items. In the final outcome, bidder 1 gets item 3 and pays 4; bidder 2 gets item 2 and pays 5; bidder 3 gets item 0 and pays nothing; bidder 4 gets item 1 and pays 6; bidder 5 gets item 4 and pays 7. Hence, auction ends up with the allocation (π^*, p^*) where $p^* = (0, 6, 5, 4, 7)$ and $\pi^* = (3, 2, 0, 1, 4)$. □

Table 1: Bidders' values on each item.

Items	0	1	2	3	4
Bidder 1	0	4	3	10	7
Bidder 2	0	7	6	20	3
Bidder 3	0	5	5	40	7
Bidder 4	0	9	4	40	2
Bidder 5	0	6	2	40	10

We now provide a simple argument showing why the outcome from the above example, (π^*, p^*), is a core allocation. At (π^*, p^*), the seller's revenue is 22, bidder 1's profit (in terms of monetary units) is 6, bidder 2's profit is 1, bidder 3's profit is 0, bidder 4's profit is 3, and bidder 5's profit is 3. Suppose to the contrary that (π^*, p^*) is not a core allocation. Then there must exist a blocking coalition consisting of the seller and some bidders who can make themselves better off. In order for the seller to be better off, her revenue must be greater than 22. Notice that all prices in p^* except p_2^* have reached the ceiling prices. So the seller has to sell her items at a new price vector $q = (q_0, q_1, q_2, q_3, q_4)$ that must satisfy $q_0 = 0$, $q_1 = p_1^* = 6$, $p_2^* = 5 < q_2 \leq 6$, $q_3 = p_3^* = 4$, and $q_4 = p_4^* = 7$. It means that all items must be sold and the possible blocking coalition must have at least 4 bidders. Observe that item 2 can be sold only to bidder 2. But then bidder 2's profit will be less than what he gets from (π^*, p^*). This shows that (π^*, p^*) cannot be blocked by any coalition and thus must be a core allocation.

Table 2: The data generated by the auction for Example 1.

t	p^t	N^t	M^t	O	$D^1(p^t)$	$D^2(p^t)$	$D^3(p^t)$	$D^4(p^t)$	$D^5(p^t)$
0	$(0,5,4,1,5)$	N	M	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
1	$(0,5,4,2,5)$	N	M	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
2	$(0,5,4,3,5)$	N	M	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
3	$(0,5,4,4,5)$	N	M	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$	$\{3\}$
4	$(0,5,4,5)$	$N \setminus \{3\}$	$M \setminus \{1\}$	$\{4\}$	–	$\{1,2\}$	$\{4\}$	$\{1\}$	$\{4\}$
5	$(0,5,4,6)$	$N \setminus \{3\}$	$M \setminus \{1\}$	$\{1,2,4\}$	–	$\{1,2\}$	$\{2,4\}$	$\{1\}$	$\{4\}$
6	$(0,6,5,7)$	$N \setminus \{3\}$	$M \setminus \{1\}$	$\emptyset$	–	$\{1,2\}$	$\{0,2,4\}$	$\{1\}$	$\{4\}$

To formally prove that the outcome of the Talman-Yang auction is a core allocation, we need to invoke two previous results from the literature. The first lemma is due to van der Laan and Yang (2008, Lemma 3.2) and states that a set of items O is minimal overdemanded at prices p if it is overdemanded and if the number of items in each strict subset T of O is strictly smaller than the number of bidders that demand some item in T and in addition only demand items in O.

Lemma 3.1 *Let O be a minimal overdemanded set of items at a price vector p. Then, for every nonempty subset T of O, we have*

$$|D_T^+(p) \cap D_O^-(p)| > |T|.$$

The second lemma is due to Mishra and Talman (2010, Theorem 1) and gives a necessary and sufficient condition for the existence of a Walrasian equilibrium in the assignment market.

Lemma 3.2 *There is a Walrasian equilibrium at $p \in \mathbb{R}_+^N$ if and only if there is neither overdemanded set of items nor underdemanded set of items at p.*

The next lemma shows that the Talman-Yang auction does not generate any underdemanded set of items at any iteration.

Lemma 3.3 *There is no underdemanded set of items in each iteration of the Talman-Yang auction.*

Proof: Observe that there is no underdemanded set of items at $p^0 = \underline{p} \in \mathbb{R}^N_+$ as the set $\{j \in N \setminus \{0\} \mid p_j > \underline{p}_j\}$ is empty. Suppose next that there is no underdemanded set of items at p^t for $t > 0$. Without loss of generality, we may assume that $p^t_j > \underline{p}_j$ for all $j \in N \setminus \{0\}$. We will prove that there is no underdemanded set of items at p^{t+1}. Let O be the minimal overdemanded set of items at p^t chosen by the auctioneer. We need to consider the following two cases:

Case 1. Suppose that no item j in the set O has reached its ceiling price $\overline{p}_j$ at iteration t. Then, $p^{t+1}_k = p^t_k + 1$ for every $k \in O$ and $p^{t+1}_k = p^t_k$ for every $k \in N^t \setminus O$ by the rules of the auction. Because there is no underdemanded set of items at p^t, we have $|D^+_T(p^t)| \geq |T|$ for any set T of real items in N^t. For any $T \subseteq O$, by Lemma 3.1 the number of bidders in the lower inverse demand set $D^-_O(p^t)$ that demand at least one item of T at p^t is at least one more than the number of items in T. Since the valuations are integers and the increment for items in O is one, the number of bidders that demand at least one item of T at p^{t+1} will be the same as the number of bidders that demanded at least one of T at p^t. In all other cases, for any $T \subseteq N^t \setminus \{0\}$, the number of bidders that demand at least one item of T at p^{t+1} will be at least as big as the number of bidders that demanded at least one item of T at p^t. So there is no underdemanded set of items at p^{t+1}.

Case 2. Suppose that some item j^* in the set O has reached its ceiling price $\overline{p}_j$. Then by the rules of the auction, the winning bidder $i^* \in M^t$ with $j^* \in D^{i^*}(p^t) \subseteq O$ leaves the market with the item j^* by paying $p^t_{j^*} = \overline{p}_{j^*}$. As $p^{t+1}_j = p^t_j$ for all items in N^{t+1} by the rules of the auction, we can draw the following three conclusions. First, for any bidder $i \in M^{t+1}$ with $j^* \notin D^i(p^t)$, it holds that $D^i(p^{t+1}) = D^i(p^t)$. Second, for any bidder $i \in M^{t+1}$ with $j^* \in D^i(p^t)$ and $|D^i(p^t)| > 1$, it holds that $D^i(p^{t+1}) = D^i(p^t) \setminus \{j^*\}$. Third, for any bidder $i \in M^{t+1}$ with $D^i(p^t) = \{j^*\}$, it holds that $D^i(p^{t+1}) \subseteq N^t \setminus \{j^*\}$ (note that there can be at most one such bidder by Lemma 3.1 and the definition of a minimal overdemanded set). As the demand-set of bidder $D^{i^*}(p^t)$ is a subset of O by the rules of the auction, it follows from the above three conclusions that if a subset T of N^{t+1} is underdemanded at prices p^{t+1}, then T must be a subset of $O \setminus \{j^*\}$ where $i^* \in D^+_T(p^t)$.

Let $O' = O \setminus \{j^*\}$ and consider an arbitrary subset T of O' at p^{t+1} where $i^* \in D^+_T(p^t)$. Note that bidder i^* does not belong to M^{t+1} by the rules of the action. From the above three conclusions it is then clear that all bidders except i^* that demand some item in T at p^t also demand some item from the set T at p^{t+1}. Hence

$$D_T^+(p^{t+1}) \cap D_{O'}^-(p^{t+1}) = (D_T^+(p^t) \cap D_O^-(p^t)) \setminus \{i^*\}. \tag{3.1}$$

Since T is a subset of O at p^t, it follows from Lemma 3.1 that

$$|D_T^+(p^t) \cap D_O^-(p^t)| > |T|. \tag{3.2}$$

The above observations and the fact that $|D_T^+(p^{t+1})| \geq |D_T^+(p^{t+1}) \cap D_{O'}^-(p^{t+1})|$ give

$$|D_T^+(p^{t+1})| \geq |D_T^+(p^{t+1}) \cap D_{O'}^-(p^{t+1})| \tag{3.3}$$

$$= |D_T^+(p^t) \cap D_O^-(p^t)| - 1 \tag{3.4}$$

$$> |T| - 1. \tag{3.5}$$

Hence, $|D_T^+(p^{t+1})| \geq |T|$. But then the set T cannot be underdemanded at p^{t+1} by definition. □

The main theorem establishes that the allocation found by the Talman-Yang auction is always in the core.

Theorem 3.4 *The Talman-Yang auction always finds a core allocation in a finite number of iterations.*

Proof: We first show that the auction terminates in finitely many iterations. This follows immediately from the fact that the auction is ascending and the valuation of every bidder on each item is finite. Let t^* be the iteration when the auction terminates. Because there is no underdemanded set of items and no overdemanded set of items when the auction terminates by the construction of the auction and Lemma 3.3, there is a Walrasian equilibrium (π^*, p^{t^*}) for the bidders in M^{t^*} and items in N^{t^*} by Lemma 3.2. Together with their winning items and those bidders who paid ceiling prices before iteration t^*, we obtain a feasible allocation (π, p).

It remains to prove that (π, p) is a core allocation. Suppose to the contrary that (π, p) were not a core allocation. Then there would exist a coalition S of bidders besides the seller and an allocation (ρ, q) blocking (π, p). So for the seller, we have

$$\begin{aligned} \textstyle\sum_{j \in N} q(j) &= \textstyle\sum_{i \in S} q_{\rho(i)} + \sum_{j \in U(\rho)} \underline{p}_j \\ &> \textstyle\sum_{i \in M} p_{\pi(i)} + \sum_{j \in U(\pi)} \underline{p}_j \\ &= \textstyle\sum_{j \in N} p_j. \end{aligned}$$

It is clear that there exists some $j^* \in N$ such that

$$q_{j^*} > p_{j^*}. \tag{3.6}$$

This means that some bidder $i^* \in S$ must be assigned item j^* at ρ, i.e., $\rho(i^*) = j^*$. On the other hand, for every bidder $i \in S$, we have

$$V^i(\rho(i)) - q_{\rho(i)} > V^i(\pi(i)) - p_{\pi(i)}. \tag{3.7}$$

For bidder i^*, it follows from conditions (3.6) and (3.7) that

$$\begin{aligned} V^{i^*}(j^*) - p_{j^*} &> V^{i^*}(j^*) - q_{j^*} \\ &> V^{i^*}(\pi(i^*)) - p_{\pi(i^*)}. \end{aligned}$$

But then this inequality would imply that at prices p, bidder i^* should have rejected item $\pi(i^*)$ in favor of item j^*, yielding a contradiction. Observe that this argument is valid, because item j^* cannot be any item that has reached its upper price level $\overline{p}_{j^*}$ and has been sold before the auction stops, for otherwise, then we would have $q_{j^*} = p_{j^*} = \overline{p}_{j^*}$ which then contradicts inequality (3.6). This demonstrates that (π, p) is indeed a core allocation. □

4 Conclusion

The present paper has examined an assignment market under price control and has thereby extended the classical assignment market as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), and Demange, Gale and Sotomayor (1986) among many others. Like the classical assignment market, the current model allows every buyer to have different valuations for the items and the buyers are interested in acquiring at most one item, but unlike the classical assignment market, the presence of price rigidities often fails the existence of a Walrasian equilibrium. As well-recognized in the literature (see Friedman and Stigler (1946), Dreze (1975), van der Laan (1980), Kurz (1982), and more recently Bulow and Klemper (2012)), a central issue with price rigidities is that they can always cause the loss of market efficiency. In this paper, we have demonstrated somehow surprisingly that the dynamic auction proposed by Talman and Yang (2008) can resolve this issue.

As shown in Theorem 3.4, the Talman-Yang auction always finds a core allocation in finitely many iterations, thus yielding a Pareto efficient outcome. This result can be seen as a novel generalization of the celebrated equilibrium theorem of Koopmans and Beckmann (1957) and Shapley and Shubik (1971) in the new context with price rigidities. This dynamic auction works in an intuitive way and is based on a familiar ascending auction format with several new features. Starting with the reserve price of each item, in each round the auctioneer asks each bidder to report his demand at the current prices and increases the price of each item in a minimal overdemanded set or assigns by a lottery an item to a bidder if the price of the item reaches its ceiling price and the bidder demands the item. The auction terminates with a core allocation when there is no overdemanded set. The core allocation consists of an assignment of items and its supporting price vector. It is interesting to point out that the speed of the auction's convergence to a core allocation can actually be improved by using the "maximal set in excess demand" instead of the more familiar notion of minimal overdemanded set. The maximal set in excess demand set is explored in Andersson, Andersson and Talman (2013) and is shown to be unique and can be found in polynomial time.

The current study leaves us with some open questions. As mentioned earlier, we have focused on the efficiency problem but ignored the incentive problem. Both problems are important and difficult. As often is the case, an analysis of the case with price-taking agents is the first important and necessary step towards the study of the case with strategic agents. As pointed out previously, in the absence of price rigidities the Talman-Yang auction coincides with the well-known auction of Demange, Gale and Sotomayor (1986) and thus terminates at the unique minimal Walrasian equilibrium price vector which will induce every bidder to act truthfully. In the presence of price rigidities, the Talman-Yang auction will typically end up with a price vector even below the minimal Walrasian equilibrium price vector of the assignment market without price rigidities and should therefore intuitively create less incentives for bidders to misrepresent their bids. However, the issue is far more difficult to analyze as it might appear otherwise, because the price rigidities and the lottery in the auction will create too many 'final' price vectors and thus lose the existence of a unique "minimal supporting price vector" which is crucial for the incentive compatibility property for the assignment market model; see Demange and Gale (1985), and Demange, Gale and Sotomayor (1986). A second open question concerns how to generalize the current core existence result to the more general models where each bidder is allowed to demand multiple items but the price of each item may not be completely flexible. Such models have previously been studied by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000, 2004), and Ausubel (2004, 2006) in the absence of price controls.

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