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2014

Document Version:
Other version

Link to publication

Citation for published version (APA):
Kratz, J. (2014). *Overlapping Multiple Assignments*. (Working Paper / Department of Economics, School of Economics and Management, Lund University; No. 44). Department of Economics, Lund University.

Total number of authors:
1

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Working Paper 2014:44

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December 2014

LUND UNIVERSITY

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Abstract

This paper studies an allocation problem with multiple assignments, indivisible objects, no endowments and no monetary transfers, where a single object may be assigned to several agents as long as the set of agents assigned the object satisfy a compatibility constraint. It is shown that, on the domain of complete, transitive and strict preferences, group-sorting sequential dictatorships are fully characterized by four different combinations of coalitional strategyproofness, strategyproofness, Pareto efficiency, non-bossiness, group-monotonicity and group-invariance. It is also demonstrated that the characterization in Pápai (2001) of sequential dictatorships for the case where assignments are not allowed to overlap is contained in the main result.

Keywords: Multiple assignments, overlapping assignments, sequential dictatorship, strategyproofness, compatibility.
JEL classification: D61, D63, D71.

1 Introduction

When there are people who seek to carry out various activities in different parts of a city, a conflict may arise if some activity is deemed incompatible with another activity, perhaps due to the presence of negative externalities. Some might consider it inadvisable to set up a garbage dump next to a major tourist attraction, or run a loud night club or a factory in a residential area. However, running a factory in the same area as a garbage dump might be deemed acceptable. One solution to this problem is zoning[1], the purpose of which is

*I want to thank Bettina Klaus, Szilvia Pápai, Jens Gudmundsson, Jim Ingebretsen-Carlsson and especially Tommy Andersson for their helpful comments and suggestions. Financial support from the "Jan Wallander and Tom Hedelius Foundation" (P2012-0107:1) is gratefully acknowledged.

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[1]Not to be confused with walk zones in the school choice literature (Dur et al., 2014).

to keep incompatible activities geographically separate. A city is divided into different zones and each zone has rules specifying which activities are allowed within the zone. The practical problem of how to divide a city into zones corresponds to the theoretical problem of how to assign bundles of objects, corresponding to different parts of a city, to agents. Each object may be assigned to several agents, provided that the agents assigned the same object are *compatible*. Since each agent may be assigned several objects, this means that the assignments of agents are allowed to overlap, without necessarily coinciding. This paper studies such an allocation problem with multiple assignments, indivisible objects, no endowments and no monetary transfers.

In the literature studying multiple assignments, each object may typically be assigned to at most one agent, both in problems with endowments (Konishi et al., 2001; Schummer and Vohra, 2013; Abizada and Schummer, 2013) and without endowments (Pápai, 2000, 2001; Klaus and Miyagawa, 2001; Ehlers and Klaus, 2003; Kojima, 2009, 2013). Allocation problems with multiple assignments, no endowments and no overlapping assignments are referred to as the *standard case* in this paper. These problems generalize the unit demand problem without endowments studied by e.g., Hylland and Zeckhauser (1979). An allocation problem can be solved by letting agents report their preferences over bundles of objects and adopting a *rule* that decides which bundles of objects should be assigned to which agents, given their reported preferences.

For the standard case, Pápai (2001) has shown that a rule is strategyproof, Pareto efficient and non-bossy if and only if it is a sequential dictatorship. *Sequential dictatorships* are a class of rules under which one agent, the first dictator, is assigned his most preferred bundle of objects. Given the first dictator's assignment, a second dictator is selected. The second dictator is assigned his most preferred bundle of objects out of all bundles that respect the first dictator's assignment. This process continues until all agents have been assigned some (possibly empty) set of objects. A sequential dictatorship can be said to adapt to an endogenous *priority structure* that assigns the highest priority to the first dictator, the second highest priority to the second dictator, and so on. Such a priority structure is uniform across all objects. Exogenous priority structures that are allowed to be non-uniform across objects have been studied by e.g., Ergin (2002).

Sequential dictatorships include the subclass known as *serial dictatorships* (Satterthwaite and Sonnenschein, 1981; Svensson, 1994, 1999). Serial dictatorships have many real world applications and have been studied extensively in, for instance, the house allocation literature (Abdulkadiroğlu and Sönmez, 1999; Andersson and Svensson, 2014). Ehlers and Klaus (2003) show that sequential dictatorships are characterized by the same properties as in Pápai (2001) on smaller preference domains as well. Furthermore, they demonstrate that serial dictatorships can be characterized by adding a fourth property called resource-monotonicity. Klaus and Miyagawa (2001) show that serial dictatorships are characterized by strategyproofness, Pareto efficiency and either consistency or population-monotonicity in the standard case. Pápai (2000) and Hatfield (2009) study similar allocation problems with multiple assignments, where the number of objects an agent may receive is given by a quota.

The main result in this paper is that when compatible agents may receive overlapping assignments, a rule satisfies coalitional strategyproofness, Pareto efficiency and a third property called group-monotonicity if and only if it belongs to a subclass of sequential dictatorships called group-sorting sequential dictatorships. It is also shown that this characterization still holds if coalitional strategyproofness is replaced by strategyproofness and non-bossiness. Furthermore, the characterization still holds if group-monotonicity is replaced by a property called group-invariance as well. It is not surprising that coalitional strategyproofness can be replaced by strategyproofness and nonbossiness as it is already known that every coalitionally strategyproof rule is strategyproof and nonbossy (Ehlers and Klaus, 2003). A similar relationship between the three properties is proven in Barberà and Jackson (1995).

A rule is *group-monotonic (group-invariant)* if it is never the case that some arbitrary agent i is strictly worse off (better or worse off) when some other agent compatible with i changes his reported preferences such that he receives a subset of the assignment he would have received otherwise. The restrictions imposed on the compatibility structure ensure that the set of agents may be partitioned into *groups* of compatible agents. A sequential dictatorship is *group-sorting* if the endogenous order in which agents are allowed to choose their preferred bundles of objects, called a *priority structure*, is sorted by groups. There is one exception to this requirement. Agents who can not feasibly be assigned any objects, given the assignments of all agents with higher priority, are allowed to violate this sorting. In other words, starting with the agent with highest priority and moving downwards in the priority structure, the priority structure is sorted by groups until every object has been assigned to at least one agent. In the context of zoning, a group-sorting sequential dictatorship allows all agents wishing to carry out a certain type of activity, type A, to select all areas in a city in which they would like to carry out said activities. The union of all selected areas constitute a zone, zone A, within which only activities of type A are allowed. Next, all agents wishing to carry out a different type of activity, type B, are allowed to select their most preferred areas that are not in zone A, and so on. If, at some point in this process, the map has been partitioned into zones in the sense that each part of the city belongs to a zone, then the relative priorities of the remaining agents have no impact on the assignment of any agent. The reason for this is that each area has already been appointed to some group, or type of activity, and the remaining agents may select any of the areas appointed to the group they belong to, regardless of their relative priorities.

Since all sequential dictatorships are group-sorting and all rules are group-monotonic in the standard case, the main result in this paper contains the characterization of sequential dictatorships provided by Pápai (2001) as a special case. It is also demonstrated that whenever assignments are allowed to overlap, neither group-sorting sequential dictatorships nor serial dictatorships satisfy consistency or population-monotonicity. Hence, the characterizations of serial dictatorships in Klaus and Miyagawa (2001) do not hold when assignments are allowed to overlap. Furthermore, it is shown that both in the standard case and in the more general case where assignments are allowed to overlap, there exists no Pareto efficient and resource-monotonic rule on the domain of complete, transitive and strict preferences.

In Section 2, the model is introduced and some different rules and properties are defined. In Section 3, the main result is presented and proven. In Section 4, some additional properties are defined. Furthermore, it is demonstrated that these properties are not satisfied by serial dictatorships or group-sorting sequential dictatorships in the problem studied in this paper. Section 5 contains some concluding remarks. Finally, there is an appendix containing the proof of one of the lemmas in Section 3.

2 Model and definitions

Let $N \subseteq \mathcal{N}$ be a finite set containing all $n \geq 2$ agents and let $A \subseteq \mathcal{A}$ be a finite set containing all indivisible objects, where $\mathcal{N}$ is the agent space and $\mathcal{A}$ is the object space. N and A are fixed throughout most of this paper. Subsets of A are occasionally referred to as bundles of objects. Each agent $i \in N$ has a *preference relation* $R_i \in \mathcal{R}$ over the power set $\mathcal{P}(\mathcal{A})$, where $\mathcal{R}$ is the domain of complete, transitive and strict preferences. In other words, each agent has preferences over all possible bundles of objects. If $S \subseteq N$,[2] let $R_S \equiv \{R_i \mid i \in S\}$ denote the preference relations of all agents in S, let $R_{-S} \equiv R_{N \setminus S}$ denote the preference relations of all agents not in S and let $R_{-i} \equiv R_{N \setminus \{i\}}$ denote the preference relations of all agents in N except i. Define R_i^B by letting $B \subseteq A$ be the most preferred bundle of objects, while letting the relative order of all other bundles of objects be the same as under R_i.

A *preference profile* is denoted by $R \in \mathcal{R}^n$, where $\mathcal{R}^n = \times_{i \in N} \mathcal{R}$. In other words, a

[2] $\subseteq$ is used to denote subsets and $\subset$ is used to denote proper subsets.

preference profile consists of one preference relation for each agent. A preference profile may or may not correspond to the true underlying preferences of the agents. Whenever a preference profile is written as a collection of more than one set of preference relations, as with e.g., (R'_S, R_{-S}), it is written inside parentheses. The exception to this rule is when the preference profile is already the only term within a pair of parentheses. Let aR_ib denote that agent i weakly prefers a to b under R_i and let aP_ib denote that agent i strictly prefers a to b under R_i. An *allocation* is defined as a function $\mu : N \to \mathcal{P}(A)$ and $\mu(i) \subseteq A$ is referred to as the *assignment* of i under μ. In other words, an allocation specifies which (possibly empty) bundle of objects each agent is assigned. Denote the set of agents assigned object a under μ by $\mu^{-1}(a) \equiv \{i \in N \mid a \in \mu(i)\}$. Two assignments $\mu(i)$ and $\mu(j)$ are said to *overlap* if $\mu(i) \cap \mu(j) \neq \emptyset$. That is, the assignments of two agents overlap if there exists at least one object assigned to both agents.

A *compatibility structure* is defined as a function $C : \mathcal{P}(\mathcal{N}) \to \{0, 1\}$. C is assumed to be fixed unless otherwise stated. If $C(S) = 1$ for some $S \subseteq N$, then the agents in S may receive overlapping assignments. The agents in S are then said to be *compatible*. If $C(S) = 0$, then the agents in S may not receive overlapping assignments. The agents in S are then said to be *incompatible*. That is, if $C(S) = 0$, no two agents $i, j \in S$ may receive assignments $\mu(i)$ and $\mu(j)$ such that $\mu(i) \cap \mu(j) \neq \emptyset$. Throughout this paper, the following restrictions are imposed on C:

(1) $C(\{i\}) = 1$ for all $i \in N$.

(2) If $C(S) = 1$, then $C(S') = 1$ for all $S' \subseteq S$.

(3) If $C(\{j\} \cup S) = 1$ and $C(\{k\} \cup S) = 1$ for some $j, k \in N$ and some non-empty $S \subseteq N$, then $C(\{j, k\} \cup S) = 1$.

Restriction (1) states that each agent is compatible with himself. Restriction (2) states that if the agents in some set S may receive overlapping assignments, then the agents in any subset of S may receive overlapping assignments as well. If agents $1, 2$ and 3 may be assigned object a simultaneously, then agents 1 and 2, agents 2 and 3 or agents 1 and 3 may also be assigned a simultaneously. Restriction (3) states that if there exists a non-empty set of agents S such that some agents i and j are both individually compatible with S, then they are also both jointly compatible with S. Restrictions (2) and (3) imply that if some agents i and j are both individually compatible with S, then they are also compatible with each other. It can also be noted that $C(\emptyset) = 1$, by restrictions (1) and (2).

Define for each $i \in N$, $N_i \equiv \{j \in N \mid C(\{i, j\}) = 1\}$. N_i is then the set of all agents compatible with agent i. Note that $i \in N_i$ and $C(S) = 1$ for all $i \in N$ and all $S \subseteq N_i$. By the restrictions imposed on C, $N_i = N_j$ for all $i \in N$ and all $j \in N_i$. Hence, N_i and N_j are different labels for the same set whenever $j \in N_i$. Each such set is called a *group*. The restrictions imposed on C ensure that all groups are disjoint, allowing the set of agents to be partitioned by groups.[3] $C(\{i, j\}) = 1$, $j \in N_i$ and $i \in N_j$ are equivalent statements. Note that the intersection of two different groups is always empty and that $C(S) = 0$ whenever S contains agents belonging to different groups. If the additional restriction that $C(S) = 0$ whenever $|S| \geq 2$ is imposed, the problem studied is reduced to the allocation problem with multiple assignments that are not allowed to overlap, as studied by e.g., Klaus and Miyagawa (2001), Pápai (2001) and Ehlers and Klaus (2003). This is referred to as the *standard case*. This restriction is, in general, not imposed in this paper.

An allocation μ is *feasible* if $C(\mu^{-1}(a)) = 1$ for all $a \in A$. Since the allocation where every agent is assigned the empty set is feasible, there exists a feasible allocation for every possible compatibility structure. Furthermore, since $C(\{i\}) = 1$ for every $i \in N$, there exists a feasible allocation where every object is assigned to some agent for every possible compatibility structure. It should be noted, however, that it may be desirable to leave some objects unassigned, since preferences are not assumed to be monotonic. Let $\mathcal{M}$ denote the set of all feasible allocations, given some $N \subseteq \mathcal{N}$ and some $A \subseteq \mathcal{A}$. A *rule* selects a feasible allocation for each preference profile.

Formally, a rule is defined as a function $\varphi : \mathcal{R}^n \to \mathcal{M}$. If $\varphi(R) = \mu$, then $\varphi_i(R) \equiv \mu(i)$ and $\varphi_a^{-1}(R) \equiv \mu^{-1}(a)$. That is, if the preference profile R is reported, the rule φ assigns a set of objects $\varphi_i(R)$ to each agent $i \in N$. The rule assigns all agents in $\varphi_a^{-1}(R)$ bundles of objects containing the object $a \in A$. Denote the most preferred (not necessarily proper) subset of some bundle of objects S under R_i by $c(S, R_i)$. Formally, $c(S, R_i) = S'$ if $S' \subseteq S$ and $S' R_i S''$ for all $S'' \subseteq S$. Let $F : N \rightarrowtail\!\!\!\!\rightarrow \{1, 2, \ldots, n\}$ be a permutation of N. In other words, let F order the set of agents by assigning a unique integer between 1 and n to each agent in N, where $F(i)$ is the integer assigned to agent i. Let $\mathcal{F}$ denote the set of all such permutations, given N. A *priority structure* is a function $f : \mathcal{R}^n \to \mathcal{F}$ that selects a permutation of N for each preference profile. If $f(R) = F$, let $f_R(i) \equiv F(i)$ and $f_R^{-1}(i) \equiv F^{-1}(i)$. $F^{-1}(i)$ is well defined, since F is a bijection. Furthermore, $f_R(i)$ is referred to as the *priority* of agent i, where a lower value indicates a higher priority. For ease of notation, let $f(R) = (i_1, i_2, \ldots, i_n)$ denote that under $f(R)$, i_1 is the agent with

[3] Due to the restrictions imposed on the compatibility structure, it would be possible to leave out any references to it and simply assume that the set of agents is partitioned into groups of compatible agents. With some small adjustments to the paper, all results would still hold. The compatibility structure is kept in the paper in order not to obscure the relationship between this paper and the broader class of allocation problems with overlapping assignments, in which the same restrictions on the compatibility structure need not be imposed.

highest priority, i_2 is the agent with second highest priority and so on. For a given priority structure f and a given rule φ, define

$$S_R(i) \equiv \left\{a \in A \mid C\left((\varphi_a^{-1}(R) \cap \{i' \in N \mid f_R(i') < f_R(i)\}) \cup \{i\}\right) = 1\right\}.$$

$S_R(i)$ is the set of objects not assigned to agents that both have higher priority than i under $f(R)$ and are incompatible with i. For example, suppose that $f_R(i) = 1$, $f_R(j) = 2$ and $\varphi_i(R) = A' \subset A$. Then $S_R(j) = A$ if $C(\{i,j\}) = 1$ and $S_R(j) = A \backslash A'$ if $C(\{i,j\}) = 0$. This set is used to study certain rules in a context where assignments are allowed to overlap. In the standard case, $S_R(i)$ is reduced to the set of objects assigned to agents with a higher priority than i.

Definition 1. *A priority structure f is an* s-hierarchy network *associated with a rule φ if for each $R \in \mathcal{R}^n$,*

(1) $\varphi_i(R) = c(A, R_i)$ whenever $f_R(i) = 1$, and

(2) $\varphi_i(R)$ is defined recursively by $\varphi_i(R) = c\left(S_R(i), R_i\right)$ whenever $f_R(i) \geq 2$.

If there exists some s-hierarchy network associated with a rule φ, then φ is a *sequential rule*. A sequential rule φ with an associated s-hierarchy network f selects, for each $R \in \mathcal{R}^n$, some $\mu \in \mathcal{M}$ such that the agent with the highest priority under $f(R)$, i_1, is assigned his most preferred (not necessarily proper) subset of A, $\mu(i_1)$. The agent with the second highest priority under $f(R)$, i_2, is assigned his most preferred subset of A, $\mu(i_2)$, subject to the feasibility constraint that if $\mu(i_1) \cap \mu(i_2) \neq \emptyset$, then $C(\{i_1, i_2\}) = 1$. In general, the agent with priority k, i_k, is assigned his most preferred subset of A, $\mu(i_k)$, subject to the restriction that for all $a \in \mu(i_k)$, i_k is compatible with all agents that both: (1) are assigned a and (2) have higher priority than i_k under $f(R)$.

Definition 2. *An s-hierarchy network f associated with a rule φ is an* s-hierarchy tree *associated with φ if for all $R, R' \in \mathcal{R}^n$,*

(1) $f_R^{-1}(1) = f_{R'}^{-1}(1)$, and

(2) for all $j \in \{2, \ldots, n\}$, if $\varphi_{f_R^{-1}(i)}(R) = \varphi_{f_{R'}^{-1}(i)}(R')$ for all $i \in \{1, \ldots, j-1\}$, then $f_R^{-1}(j) = f_{R'}^{-1}(j)$.

If there exists an s-hierarchy tree associated with a rule φ, then φ is a *sequential dictatorship*. Note that every sequential dictatorship is a sequential rule. A sequential rule is a sequential dictatorship if (1) the identity of the agent with highest priority is the same for all preference profiles and (2) the identity of any other agent is determined exclusively by the assignments of agents with higher priority. For example, if the assignments of the agents with the highest and second highest priority are the same under two different preference profiles, then the identity of the agent with the third highest priority is also the same under both preference profiles. The process can be illustrated by a simple example.

Example 1. *Suppose* $N = \{i_1, i_2, i_3\}$, $A = \{a, b, c, d\}$, $N_{i_1} = \{i_1, i_2\}$ *and* $N_{i_3} = \{i_3\}$. *Let* f *be an s-hierarchy tree associated with* φ. *Then* φ *is a sequential dictatorship. Define* f *such that* $f(R) = (i_1, i_2, i_3)$ *and* $f(R'_{i_1}, R_{-i_1}) = (i_1, i_3, i_2)$. *Note that* $f_{\bar{R}}(i_1) = 1$ *for all* $\bar{R} \in \mathcal{R}^n$, *since* f *is an s-hierarchy tree. It can also be noted that the most preferred bundles of objects must be different under* R_{i_1} *and* R'_{i_1}. *That is, it must be the case that* $c(A, R_i) \neq c(A, R'_i)$. *If* $c(A, R_i) = c(A, R'_i)$, *then the agent with highest priority, agent* i_1, *is assigned the same bundle of objects under both* $\varphi(R)$ *and* $\varphi(R'_{i_1}, R_{-i_1})$. *Thus, since* f *is an s-hierarchy tree,* $f_R^{-1}(2) = f^{-1}_{(R'_{i_1}, R_{-i_1})}(2)$. *This contradicts the observation that* $f_R(i_2) = 2$ *and* $f_{(R'_{i_1}, R_{-i_1})}(i_2) = 3$. *Let the preferences be given by Table 1.*

Table 1:

R_{i_1}	R'_{i_1}	R_{i_2}	R_{i_3}
$\{a, b, c\}$	$\{a\}$	$\{b, c\}$	$\{c, d\}$
$\vdots$	$\vdots$	$\{a, b\}$	$\{d\}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Suppose the agents report R. *Since* $f_R^{-1}(1) = i_1$, *agent* i_1 *is assigned his most preferred bundle of objects under* R_i, $\varphi_{i_1}(R) = c(A, R_i) = \{a, b, c\}$. *Next, since* $f_R^{-1}(2) = i_2$, *agent* i_2 *is assigned his most preferred bundle of objects under* R_{i_2} *that respects the assignment of agent* i_1. *In other words, agent* i_2 *is assigned* $c(S_R(i_2), R_{i_2})$. *Since* $i_2 \in N_{i_1}$, i_1 *and* i_2 *are compatible and may receive overlapping assignments. Thus,* $S_R(i_2) = A$ *and* i_2 *is assigned* $\varphi_{i_2}(R) = c(S_R(i_2), R_{i_2}) = \{b, c\}$. *Finally, agent* $f_R^{-1}(3) = i_3$ *is assigned his most preferred bundle of objects under* R_{i_3} *that respects the assignments of agents* i_1 *and* i_2. *In other words, agent* i_3 *is assigned* $c(S_R(i_3), R_{i_3})$. *Since* $i_3 \notin N_{i_1}$, i_3 *may not be assigned a bundle of objects that contains any objects assigned to* i_1 *or* i_2. *Thus,* $S_R(i_3) = A \setminus (c(A, R_{i_1}) \cup c(S_R(i_2), R_{i_2})) = \{a, b, c, d\} \setminus \{a, b, c\} = \{d\}$. *Hence,* i_3 *may be assigned either* $\{d\}$ *or* $\emptyset$. *Since* $\{d\} R_{i_3} \emptyset$, i_3 *is assigned* $\varphi_{i_3}(R) = c(S_R(i_3), R_{i_3}) = \{d\}$.

Suppose the agents report (R'_{i_1}, R_{-i_1}). *Since* $f^{-1}_{(R'_{i_1}, R_{-i_1})}(1) = i_1$, i_1 *is assigned* $c(A, R'_{i_1}) = \{a\}$. *Next, since* $f^{-1}_{(R'_{i_1}, R_{-i_1})}(2) = i_3$ *and* $S_{(R'_{i_1}, R_{-i_1})}(i_3) = \{b, c, d\}$, i_3 *is assigned*

$c(S_{(R'_{i_1},R_{-i_1})}(i_3), R_{i_3}) = \{c, d\}$. *Finally, since* $f^{-1}_{(R'_{i_1},R_{-i_1})}(3) = i_2$ *and* $S_{(R'_{i_1},R_{-i_1})}(i_2) = \{a, b\}$, i_2 *is assigned* $c(S_{(R'_{i_1},R_{-i_1})}(i_2), R_{i_2}) = \{a, b\}$.

Definition 3. *An s-hierarchy tree associated with a rule* φ *is a* group-sorted *s-hierarchy tree associated with* φ *if* $f_R(i) < f_R(k) < f_R(j)$ *for some* $i, j \in N_i$, *some* $k \notin N_i$ *and some* $R \in \mathcal{R}^n$ *only if* $S_R(k) = \emptyset$.

If there exists a group-sorted s-hierarchy tree associated with some rule φ, then φ is a *group-sorting* sequential dictatorship. Every group-sorting sequential dictatorship is a sequential dictatorship. A sequential dictatorship is group-sorting if, starting with $f_R^{-1}(1)$ and moving downwards in the s-hierarchy, f is sorted by groups until each object has been assigned to some agent. Upon reaching that point, the s-hierarchy may be sorted in any arbitrary manner, since this sorting has no impact on the final allocation. Next, some properties of rules are defined.

Definition 4. *A rule* φ *is* coalitionally strategyproof *if for all* $R \in \mathcal{R}^n$ *and all* $M \subseteq N$, *there exists no* $R' \in \mathcal{R}^n$ *such that* $\varphi_i(R'_M, R_{-M}) R_i \varphi_i(R)$ *for all* $i \in M$ *and* $\varphi_i(R'_M, R_{-M}) P_i \varphi_i(R)$ *for some* $i \in M$.

Under a coalitionally strategyproof rule, it is impossible for any coalition of agents to misrepresent their preferences in a way that weakly benefits all agents in the coalition and strictly benefits at least one agent in the coalition. Note that coalitional strategyproofness is a weaker property than that of *strict* coalitional strategyproofness. Under a strictly coalitionally strategyproof rule, no coalition of agents can misrepresent their preferences in a way that strictly benefits all agents in the coalition. Rules under which it is impossible for any agent to misrepresent his preferences in a way that strictly benefits himself are *strategyproof*. Since each agent is a coalition with cardinality one, every coalitionally strategyproof rule is strategyproof. A benefit of studying coalitionally strategyproof rules is that the reported preference profile can safely be assumed to reflect the agents' true preferences. This allows for efficiency evaluations of allocations and rules.

Definition 5. *A rule* φ *is* Pareto efficient *if, for all* $R \in \mathcal{R}^n$, *there exists no* $\mu \in \mathcal{M}$ *such that* $\mu(i) R_i \varphi_i(R)$ *for all* $i \in N$ *and* $\mu(i) P_i \varphi_i(R)$ *for some* $i \in N$.

This means that a rule is Pareto efficient if it always selects allocations such that no agent can be made strictly better off without making some other agent strictly worse off. If a rule is not Pareto efficient, then it may select allocations that are weakly worse than some other allocation for all agents and strictly worse for at least one agent.

Definition 6. *A rule φ is* non-bossy *if for all $R, R' \in \mathcal{R}^n$ and for all $i \in N$, $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ only if $\varphi(R) = \varphi(R'_i, R_{-i})$.*

If a rule does not satisfy non-bossiness, then it is possible for some agent to affect the assignments of other agents without affecting his own assignment. In Section 3, it is demonstrated that non-bossiness is implied by coalitional strategyproofness.

Definition 7. *A rule φ is* group-monotonic *if for all $i \in N$ and all $R, R' \in \mathcal{R}^n$, $\varphi_i(R'_i, R_{-i}) \subseteq \varphi_i(R)$ only if $\varphi_j(R'_i, R_{-i}) R_j \varphi_j(R)$ for all $j \in N_i \setminus \{i\}$.*

In other words, if some agent i changes his report such that he is assigned a subset of the assignment he would would have received otherwise, then no other agent in the group i belongs to is made worse off if the rule is group-monotonic. Similar to properties like population-monotonicity, which are defined in Section 4, this is a type of solidarity property. It ensures that no agent belonging to some group receives a strictly worse assignment if some other agents belonging to the same group changes his reported preferences such that he is assigned only some of the objects he would have been assigned otherwise, and no new objects.

Definition 8. *A rule φ is* group-invariant *if for all $i \in N$ and all $R, R' \in \mathcal{R}^n$, $\varphi_i(R'_i, R_{-i}) \subseteq \varphi_i(R)$ only if $\varphi_j(R'_i, R_{-i}) = \varphi_j(R)$ for all $i \in N_i \setminus \{i\}$.*

Group-invariance is a stronger property than group-monotonicity in the sense that group-monotonicity is implied by group-invariance. A rule is group-invariant if the assignment of no agent belonging to some group is affected whenever some other agent belonging to the same group changes his report such that he is assigned a subset of the assignment he would would have received otherwise.

3 Results

In this section, it is shown that group-sorting sequential dictatorships are characterized by different combinations of the properties introduced above. Specifically, Theorem 1 states that group-sorting sequential dictatorships are the only rules that satisfy coalitional strategyproofness, Pareto efficiency and group-monotonicity. Furthermore, this characterization still holds if group-monotonicity is replaced by group-invariance or if coalitional strategyproofness is replaced by strategyproofness and non-bossiness. The proof of Theorem 1 makes use of a number of lemmas, which are introduced and proven after the main results have been presented.

Theorem 1. *If φ is a rule, then the following statements are equivalent.*

(1) φ is a group-sorting sequential dictatorship.

(2) φ is coalitionally strategyproof, Pareto efficient and group-montonic.

(3) φ is coalitionally strategyproof, Pareto efficient and group-invariant.

(4) φ is strategyproof, Pareto efficient, non-bossy and group-monotonic.

(5) φ is strategyproof, Pareto efficient, non-bossy and group-invariant.

While Theorem 1 characterizes group-sorting sequential dictatorships for the allocation problem with overlapping multiple assignments, it can still provide some information relating to the standard case when assignments are not allowed to overlap. Recall that the allocation problem with overlapping multiple assignment is reduced to the standard case when the restriction that each agent only be compatible with himself is imposed. Imposing this restriction yields the following corollary.

Corollary 1 (Pápai, 2001)**.** *Whenever $C(S) = 0$ for all $S \subseteq N$ such that $|S| \geq 2$, a rule is coalitionally strategyproof and Pareto efficient, or strategyproof, Pareto efficient and non-bossy if and only if it is a sequential dictatorship.*

Proof. If $C(S) = 0$ for all $S \subseteq N$ such that $|S| \geq 2$, then $N_i \setminus \{i\} = \emptyset$ for all $i \in N$. This implies that every rule is group-monotonic and group-invariant. Furthermore, since $|N_i| = 1$ for all $i \in N$, every sequential dictatorship is group-sorting. □

Corollary 1 demonstrates that the result in Pápai (2001) showing that sequential dictatorships are characterized by strategyproofness, Pareto efficiency and non-bossiness is contained in Theorem 1. The intuition behind Corollary 1 is that whenever each agent is only compatible with himself, each agent constitutes a group. Properties like group-monotonicity and group-invariance then become meaningless in the sense that they impose no structure on the problem. Furthermore, any sorting is a sorting by groups, since no group contains more than one agent.

To prove the main result, a few lemmas are used. Lemma 1 states that coalitional strategyproofness implies both strategyproofness and nonbossiness. This relationship is pointed out by Ehlers and Klaus (2003). A proof of the lemma is included in this paper for completeness.

Lemma 1. *Every coalitionally strategyproof rule is strategyproof and non-bossy.*

Proof. Let φ be a coalitionally strategyproof rule. Then there exist no $R, R' \in \mathcal{R}^n$ and no $M \subseteq N$, such that $\varphi_i(R'_M, R_{-M})R_i\varphi_i(R)$ for all $i \in M$ and $\varphi_i(R'_M, R_{-M})P_i\varphi_i(R)$ for some $i \in M$. This implies that there exists no M, where $|M| = 1$, such that $\varphi_i(R'_M, R_{-M})P_i\varphi_i(R)$ for the unique agent $i \in M$. Hence, every coalitionally strategyproof rule is strategyproof.

Next, assume φ is bossy to reach a contradiction. Then there exist some $R, R' \in \mathcal{R}^n$ and some $i \in N$ such that $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ and $\varphi(R) \neq \varphi(R'_i, R_{-i})$. Strictness of preferences implies that there exists some agent $j \neq i$ for whom either $\varphi_j(R)P_j\varphi_j(R'_i, R_{-i})$ or $\varphi_j(R'_i, R_{-i})P_j\varphi_j(R)$. Let $R'_j = R_j$ and let $M = \{i, j\}$.

(1) If $\varphi_j(R'_i, R_{-i})P_j\varphi_j(R)$, then $\varphi_j(R'_M, R_{-M})P_j\varphi_j(R)$ and $\varphi_i(R'_M, R_{-M})R_i\varphi_i(R)$, which violates coalitional strategyproofness.

(2) If $\varphi_j(R)P_j\varphi_j(R'_i, R_{-i})$, then $\varphi_j(R)P_j\varphi_j(R'_M, R_{-M})$ and $\varphi_i(R)R_i\varphi_i(R'_M, R_{-M})$, which violates coalitional strategyproofness.

Hence, every coalitionally strategyproof rule is non-bossy. □

Barberà and Jackson (1995) have shown that strategyproofness and nonbossiness imply strict coalitional strategyproofness in a different context. It has already been demonstrated by Pápai (2001) that sequential dictatorships are strategyproof, Pareto efficient and non-bossy in the standard case. Ehlers and Klaus (2003) have also pointed out that sequential dictatorships are coalitionally strategyproof in the standard case, for the domain of complete, transitive, strict, responsive and separable preferences. Lemma 2 shows that sequential dictatorships are still coalitionally strategyproof, strategyproof, Pareto efficient and non-bossy on the domain of complete, transitive and strict preferences in the more general case where assignments are allowed to overlap.

Lemma 2. *Sequential dictatorships are coalitionally strategyproof, strategyproof, Pareto efficient and non-bossy.*

Proof. **Sequential dictatorships are coalitionally strategyproof.** This is a proof by contradiction. Consider a sequential dictatorship φ and assume that it is not coalitionally strategyproof. Then there exists some $R, R' \in \mathcal{R}^n$ and some $M \subseteq N$ such that $\varphi_i(R'_M, R_{-M})R_i\varphi_i(R)$ for all $i \in M$ and $\varphi_j(R'_M, R_{-M})P_j\varphi_j(R)$ for at least one $j \in M$. Consider one such $j \in M$. If $f_R(j) = 1$, then by the definition of a sequential dictatorship, $\varphi_j(R) = c(A, R_j)$ and $\varphi_j(R'_M, R_{-M}) = c(A, R'_j)$. Since $c(A, R'_j)P_jc(A, R_j)$ is a contradiction, $f_R(j) \geq 2$. This type of contradiction is encountered numerous times in this section. In general, $c(S, R'_i)P_ic(S, R_i)$ is a contradiction since both $c(S, R'_i)$ and $c(S, R_i)$ are subsets of S, and no subset of S can be strictly preferred to the most preferred subset of S. It can be demonstrated by induction on $f_R(k)$ that $\varphi_k(R'_M, R_{-M}) = \varphi_k(R)$ for all $k \in N$ such that $f_R(k) < f_R(j)$.

Induction basis: Let $f_R(k) = 1$ and note that $f_R^{-1}(1) = f_{R'}^{-1}(1)$ for all $R, R' \in \mathcal{R}^n$ by the definition of a sequential dictatorship. If $k \notin M$, then $\varphi_k(R) = \varphi_k(R'_M, R_{-M}) = c(A, R_k)$ by the definition of a sequential rule. If $k \in M$, then $\varphi_k(R'_M, R_{-M}) = c(A, R'_k)$ and $\varphi_k(R) = c(A, R_k)$. By assumption, $c(A, R'_k) R_k c(A, R_k)$. Since $c(A, R'_k) P_k c(A, R_k)$ is a contradiction and preferences are strict, $\varphi_k(R'_M, R_{-M}) = \varphi_k(R)$.

Suppose that $f_R(j) = 2$, then it follows from the induction basis that $S_{(R'_M, R_{-M})}(j) = S_R(j)$. To see this, note that if $C(\{j,k\}) = 1$, then $S_R(j) = A$ and if $C(\{j,k\}) = 0$, then $S_R(j)$ is the set of objects that have not been assigned to k. Since k is assigned the same set of objects under both $\varphi(R'_M, R_{-M})$ and $\varphi(R)$, $S_{(R'_M, R_{-M})}(j) = S_R(j)$ both when $C(\{j,k\}) = 1$ and $C(\{j,k\}) = 0$. By the definition of a sequential dictatorship, $\varphi_j(R'_M, R_{-M}) = c(S_R(j), R'_j)$ and $\varphi_j(R) = c(S_R(j), R_j)$. Since $c(S_R(j), R'_j) P_j c(S_R(j), R_j)$ is a contradiction, $f_R(j) \geq 3$.

Induction hypothesis: Let $f_R(k) = t$. Assume that $\varphi_l(R) = \varphi_l(R'_M, R_{-M})$ for all $l \in N$ such that $f_R(l) \leq t$.

Induction step: Let $t < f_R(j) - 1$. It follows from the induction hypothesis and the definition of a sequential dictatorship that $f_R^{-1}(t+1) = f_{(R'_M, R_{-M})}^{-1}(t+1) \equiv i$ and, by extension, that $S_R(i) = S_{(R'_M, R_{-M})}(i)$. If $i \notin M$, then $\varphi_i(R) = c(S_R(i), R_i) = c(S_{(R'_M, R_{-M})}(i), R_i) = \varphi_i(R'_M, R_{-M})$. If $i \in M$, then $\varphi_i(R) = c(S_R(i), R_i)$ and $\varphi_i(R'_M, R_{-M}) = c(S_R(i), R'_i)$. By assumption, $c(S_R(i), R'_i) R_i c(S_R(i), R_i)$. Since $c(S_R(i), R'_i) P_i c(S_R(i), R_i)$ is a contradiction and preferences are strict, $\varphi_i(R'_M, R_{-M}) = \varphi_i(R)$.

Hence, the induction hypothesis holds for all $k \in N$ such that $f_R(k) < f_R(j)$. In other words, $\varphi_k(R'_M, R_{-M}) = \varphi_k(R)$ for all k such that $f_R(k) < f_R(j)$. By this result and the definition of a sequential dictatorship, $f_{(R'_M, R_{-M})}(j) = f_R(j)$. This implies that $S_{(R'_M, R_{-M})}(j) = S_R(j)$ and $c(S_{(R'_M, R_{-M})}(j), R'_j) = c(S_R(j), R'_j) = \varphi_j(R'_M, R_{-M})$. Since $j \in M$ and $\varphi_j(R'_M, R_{-M}) P_j \varphi_j(R)$ by assumption, it follows that $c(S_R(j), R'_j) P_j c(S_R(j), R_j)$, which is a contradiction. Sequential dictatorships are thus coalitionally strategyproof.

Sequential dictatorships are Pareto efficient. Let φ be a sequential dictatorship and define $f_R^{-1}(j) \equiv i_j$. Pareto efficiency can be proven by induction.

Induction basis: There exists no $\mu \in \mathcal{M}$ such that $\mu(i_1) P_{i_1} \varphi_{i_1}(R)$, since $\varphi_{i_1}(R) = c(A, R_{i_1})$.

Induction hypothesis: Let $t \geq 1$. Assume there exists no $\mu \in \mathcal{M}$ such that $\mu(i_s) R_{i_s} \varphi_{i_s}(R)$ for all $s \leq t$ and $\mu(i_s) P_{i_s} \varphi_{i_s}(R)$ for some $s \leq t$.

Induction step: By the definition of a sequential rule, $\varphi_{i_{t+1}}(R) = c(S_R(i_{t+1}), R_{i_{t+1}})$.

Suppose that there exists some allocation $\mu \in \mathcal{M}$ such that $\mu(i_{t+1})P_{i_{t+1}}\varphi_{i_{t+1}}(R)$ and $\mu(i_s)R_{i_s}\varphi_{i_s}(R)$ for all $s \leq t+1$. Note that $\mu(i_{t+1})P_{i_{t+1}}c(S_R(i_{t+1}), R_{i_{t+1}})$ implies that $\mu(i_{t+1}) \notin S_R(i_{t+1})$. This implies that there must exist at least one $i_j \in N$ such that $j \leq t$ and $\mu(i_j) \neq \varphi_{i_j}(R)$. By assumption, $\mu(i_j)R_{i_j}\varphi_{i_j}(R)$. Since preferences are strict, $\mu(i_j)P_{i_j}\varphi_{i_j}(R)$. This, together with the assumption that $\mu(i_s)R_{i_s}\varphi_{i_s}(R)$ for all $s \leq t+1$, contradicts the induction hypothesis. Hence, there exists no allocation $\mu \in \mathcal{M}$ such that $\mu(i_{t+1})P_{i_{t+1}}\varphi_{i_{t+1}}(R)$ and $\mu(i_s)R_{i_s}\varphi_{i_s}(R)$ for all $s \leq t+1$.

It has thus been shown by induction that there exist no $t \geq 1$ and $\mu \in \mathcal{M}$ such that $\mu(i_s)R_{i_s}\varphi_{i_s}(R)$ for all $s \leq t$ and $\mu(i_s)P_{i_s}\varphi_{i_s}(R)$ for some $s \leq t$. This implies that there exists no $\mu \in \mathcal{M}$ such that $\mu(i)R_i\varphi_i(R)$ for all $i \in N$ and $\mu(i)P_i\varphi_i(R)$ for some $i \in N$. Hence, φ is Pareto efficient.

Sequential dictatorships are strategyproof and non-bossy. This follows immediately from coalitional strategyproofness and Lemma 1. □

Lemma 3 demonstrates a relationship between the group-sorting property of sequential dictatorships, group-monotonicity and group-invariance. If a sequential dictatorship is group-sorting, then it must be both group-monotonic and group-invariant. Furthermore, if a sequential dictatorship is group-monotonic or group-invariant, then it must be group-sorting. Together, these results imply that if a sequential dictatorship is group-monotonic, then it must also be group-invariant and vice versa.

Lemma 3. *A sequential dictatorship is group-monotonic or group-invariant if and only if it is group-sorting.*

Proof. **Group-sorting sequential dictatorships are group-monotonic and group-invariant.** Let φ be a group-sorting sequential dictatorship. Then there exists an s-hierarchy tree f associated with φ, such that $f_R(i) < f_R(k) < f_R(j)$ for some $i, j \in N_i$, some $k \notin N_i$ and some $R \in \mathcal{R}^n$ only if $S_R(k) = \emptyset$. Consider one such f and some arbitrary $R \in \mathcal{R}^n$. It can be shown by induction that φ is both group-monotonic and group-invariant.

Induction basis: There exists some group $M_1 \subseteq N$ such that $f_R^{-1}(1) \in M_1$ and for all $i, j \in M_1$, $i \neq j$, and all $R' \in \mathcal{R}^n$, $S_R(i) = S_{(R'_j, R_{-j})}(i)$.

To see this, note that by the definition of a sequential dictatorship, $f_R^{-1}(1) = f_{R'}^{-1}(1) \equiv i$ and $S_R(i) = S_{R'}(i) = A$ for all $R, R' \in \mathcal{R}^n$. Let $N_i \equiv M_1$. Since φ is group-sorting, $S_R(j) = S_{R'}(j) = A$ for all $j \in M_1$ and all $R, R' \in \mathcal{R}^n$. The reason for this is that for any $k \in N$ with higher priority than some $j \in N_i$, it holds that either $k \in N_i$ or $S_R(k) = \emptyset$. If $k \in N_i$, then any objects assigned to k may still be assigned to j. If $S_R(k) = \emptyset$, then k will not be assigned any objects and will therefore prevent no objects from being assigned to j.

Induction hypothesis: Consider the group $M_t \subseteq N$, $t \geq 1$. Assume that for all $s \leq t$, all $i, j \in M_s$, $i \neq j$, and all $R' \in \mathcal{R}^n$, $S_R(i) = S_{(R'_j, R_{-j})}(i)$. Furthermore, let $M(t) \equiv \bigcup_{s=1}^{t} M_s$. Assume that there exists some $i \in M_t$ such that $f_R(i) < f_R(i')$ for all $i' \notin M(t)$.

The induction hypothesis is true for M_1 by the induction basis and the observation that $f_R^{-1}(1) \in M_1$.

Induction step: Suppose $|M(t)| = d - 1$ for some integer d.

(1) If there exists some $l \notin M(t)$ for which $f_R(l) \leq d-1$, then by the induction hypothesis and the fact that φ is group-sorting, $S_R(i) = S_{(R'_j, R_{-j})}(i) = \emptyset$ for all $i, j \in M_s$, $i \neq j$, all groups $M_s \nsubseteq M(t)$ and all $R' \in \mathcal{R}^n$. The reason for this is that the existence of the agent $l \notin M(t)$ implies that $f(R)$ is not sorted by groups for the set of agents with weakly lower priority than l and that l has higher priority than some agent in $M(t)$. Since f is a group-sorted s-hierarchy tree, this implies that all objects in A have been assigned to agents with higher priority than l. All agents assigned objects must also belong to $M(t)$, since any agent k with higher priority than l, who does not belong to $M(t)$ also has higher priority than some agent in $M(t)$. Hence, k is also subject to the requirement that all objects be assigned to agents with higher priority than k. Hence, $S_k(R) = \emptyset$ for any $k \notin M(t)$ with higher priority than some agent in $M(t)$ and for any $k \notin M(t)$ with lower priority than all agents in $M(t)$, since all objects have been assigned to agents in $M(t)$ and no agent not in $M(t)$ is compatible with any agent in $M(t)$. Therefore, $S_R(i) = S_{(R'_j, R_{-j})}(i) = \emptyset$ for all $i, j \in M_s$, $i \neq j$, all groups $M_s \nsubseteq M(t)$ and all $R' \in \mathcal{R}^n$. The induction hypothesis then holds for M_{t+1}, where M_{t+1} is the group containing the agent with highest priority in $N \setminus M(t)$.

(2) If there exists no $l \notin M(t)$ for which $f_R(l) \leq d - 1$, consider agent $i_d \equiv f_R^{-1}(d)$ and note that the induction hypothesis and the fact that φ is group-sorting imply that i_d is the agent in $N \setminus M(t)$ with highest priority. Thus, the definition of a sequential dictatorship implies that $f_R^{-1}(d) = f_{(R'_j, R_{-j})}^{-1}(d)$ and $S_R(i_d) = S_{(R'_j, R_{-j})}(i_d)$ for all $j \notin M(t) \cup \{i_d\}$ and all $R' \in \mathcal{R}^n$. Let $N_{i_d} \equiv M_{t+1}$. The fact that φ is a group-sorting sequential dictatorship implies that for all $i, j \in M_{t+1}$, $i \neq j$, and all $R' \in \mathcal{R}^n$, $S_R(i) = S_{(R'_j, R_{-j})}(i)$. Hence, the induction hypothesis holds for M_{t+1}.

The induction hypothesis thus holds for every t such that $M(t) \subseteq N$, including t such that $M(t) = N$. This means that $S_R(i) = S_{(R'_j, R_{-j})}(i)$ for all $i \in N$, all $j \in N_i$ and all $R' \in \mathcal{R}^n$. Recalling the definition of a sequential rule, this implies that for all $i \in N$, all $j \in N_i$ and all $R' \in \mathcal{R}^n$, $\varphi_i(R) = \varphi_i(R'_j, R_{-j}) = c(S_R(i), R_i)$. Hence, φ is group-monotonic and group-invariant.

Sequential dictatorships are not group-monotonic or group-invariant if they are not group-sorting. This is a proof by contradiction. Let φ be a sequential dictatorship that is not group-sorting. Then there exists no s-hierarchy tree f associated with φ, such that $f_R(i) < f_R(k) < f_R(j)$ for some $i, j \in N_i$, some $k \notin N_i$ and some $R \in \mathcal{R}^n$ only if $S_R(k) = \emptyset$. This implies that for each s-hierarchy tree f associated with φ, there exist some $R \in \mathcal{R}^n$, some $i, j \in N_i$ and some $k \notin N_i$ such that $f_R(i) < f_R(k) < f_R(j)$ and $S_R(k) \neq \emptyset$. Consider some s-hierachy tree f associated with φ and let k be the agent with highest priority for which the above holds. Note that $\varphi_i(R) \subset S_R(i) \equiv S$, for if $\varphi_i(R) = S_R(i)$, then $S_R(k) = \emptyset$. To see this, note that since k is the highest ordered agent for whom $f_R(i) < f_R(k) < f_R(j)$ for some $i, j \in N_i, N_i \neq N_k$ and some $R \in \mathcal{R}^n$, f is sorted by groups for all agents between agent $f_R^{-1}(1)$ and the agent with priority $f_R(k) - 1$. This means that there exists no agent in N_k with higher priority than agent k. Hence, if there exists some agent l such that $f_R(l) < f_R(k)$ and $\varphi_l(R) = S_R(l)$, then $S_R(k) = \emptyset$ since every object has already been assigned to agents not belonging to N_k. Let $S_R(i) \equiv S$ and note that by the definition of a sequential dictatorship, $\varphi_i(R_i^S, R_{-i}) = S$. Since $j \in N_i$, $\varphi_j(R^S_{\{i,j\}}, R''_k, R_{-\{i,j,k\}}) = S$ for all $R'' \in \mathcal{R}^n$. Define $R' \in \mathcal{R}^n$ such that any non-empty $S' \subseteq A$ is preferred to $\emptyset$ and note that $f_{(R'_k, R_{-k})}(k) = f_R(k)$. This implies that $f_{(R'_k, R_{-k})}(k) < f_{(R'_k, R_{-k})}(j)$ and consequently that $f_{(R^S_j, R'_k, R_{-\{k,j\}})}(k) < f_{(R^S_j, R'_k, R_{-\{k,j\}})}(j)$. Since $\varphi_l(R) = \varphi_l(R^S_j, R'_k, R_{-\{k,j\}})$ for all $l \in N$ such that $f_R(l) < f_R(k)$, $S_R(k) = S_{(R^S_j, R'_k, R_{-\{k,j\}})}(k)$. By $S_R(k) \neq \emptyset$ and the definition of R', it must be that under $\varphi(R^S_j, R'_k, R_{-\{k,j\}})$, k is assigned at least one object in $S \setminus \varphi_i(R^S_j, R'_k, R_{-\{k,j\}})$. This follows from the fact that $\varphi_i(R^S_j, R'_k, R_{-\{k,j\}})$ is a proper subset of S, which in turn is non-empty since if $S = \emptyset$, then $S_R(k) = \emptyset$, contradicting the assumption that $S_R(k) \neq \emptyset$. Therefore, $\varphi_j(R^S_j, R'_k, R_{-\{k,j\}}) \neq S$. This violates both group-invariance and group-monotonicity, as $\varphi_i(R^S_j, R'_k, R_{-\{k,j\}}) \subset \varphi_i(R^S_{\{i,j\}}, R'_k, R_{-\{i,k,j\}})$ and $\varphi_j(R^S_{\{i,j\}}, R'_k, R_{-\{i,k,j\}}) = S$, which is strictly preferred to $\varphi_j(R^S_j, R'_k, R_{-\{k,j\}})$. Hence, φ is not group-monotonic or group-invariant. □

The next lemma is just a technical lemma that is used to prove Lemma 5 further below. To understand Lemma 4, suppose that some strategyproof, Pareto efficient and non-bossy rule is implemented. Furthermore, suppose that M' is either empty or the union of one or more groups such that when the agents in M' report $R_{M'}$, they receive the same assignments regardless of the preference relations reported by the agents in $N \setminus M'$. Denote the set of all objects not assigned to the agents in M' when $R_{M'}$ is reported by B. Suppose the agents in M' report $R_{M'}$. Then Lemma 4 states that there must exist some group $M \subseteq N \setminus M'$ such that if the agents in M report that B is the only set of objects preferred to the empty set, then all agents in M are assigned B regardless of the preference relations reported by the agents not in M or M'.

Lemma 4. *Consider some strategyproof, Pareto efficient and non-bossy rule φ. Let $M' \subset N$ be either the empty set $\emptyset$, or the union of one or more groups such that $\varphi_i(R_{M'}, R'_{-M'}) = \varphi_i(R_{M'}, R''_{-M'})$ for all $i \in M'$ and all $R', R'' \in \mathcal{R}^n$. Let all preference relations in $\bar{R}$ rank $A \setminus \bigcup_{j \in M'} \varphi_j(R)$ first and $\emptyset$ second. Then there exists some group $M \subseteq N \setminus M'$ such that $\varphi_i(\bar{R}_M, R_{M'}, R'_{-M \cup M'}) = A \setminus \bigcup_{j \in M'} \varphi_j(R)$ for all $i \in M$ and all $R' \in \mathcal{R}^n$.*

The proof of Lemma 4 can be found in the appendix. It can be noted that in the special case where $M' = \emptyset$ and $C(S) = 0$ for all S such that $|S| \geq 2$, Lemma 4 corresponds to a similar observation used in the proof of the characterization of sequential dictatorships in Pápai (2001). Another similar finding is also used in the characterization of sequential dictatorships on a different preference domain in Ehlers and Klaus (2003).

Lemma 5 states that if a rule is strategyproof, Pareto efficient, nonbossy and group-monotonic, then it is a group-sorting sequential dictatorship. In other words, there exists no strategyproof, Pareto efficient, non-bossy and group-monotonic rule that is not a group-sorting sequential dictatorship. In the standard case, there exists no strategyproof, Pareto efficient and non-bossy rule that is not a sequential dictatorship (Pápai, 2001).

Lemma 5. *A rule is strategyproof, Pareto efficient, non-bossy and group-monotonic only if it is a group-sorting sequential dictatorship.*

Proof. Consider a rule φ satisfying strategyproofness, Pareto efficiency, nonbossiness and group-monotonicity. Let all preference relations in R' rank A first and $\emptyset$ second for all $i \in N$. By Lemma 4, there exists a group $M_1 \subseteq N$ such that $\varphi_i(R'_{M_1}, R_{-M_1}) = A$ for all $i \in M_1$ and all $R \in \mathcal{R}^n$. Fix a set of objects $B_i \subset A$, $B_i \neq \emptyset$ for each $i \in M_1$. Let all preference relations in $\bar{R}$ rank B_i first, A second and $\emptyset$ third. Suppose that there exist some $i \in M_1$ and $R \in \mathcal{R}^n$ such that $\varphi_i(\bar{R}_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) \neq B_i$. Then strategyproofness implies that $\varphi_i(\bar{R}_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) = A$, which violates Pareto efficiency. It violates Pareto efficiency, because the allocation under which all agents in $M_1 \setminus \{i\}$ are assigned A and i is assigned B_i is both feasible and Pareto dominates the allocation under which every agent in M_1 is assigned A. Since B_i is an arbitrary set of objects, there exists, for each $B \subseteq A$ and each $i \in M_1$, some $\bar{R}'_i \in \mathcal{R}$ such that $\varphi_i(\bar{R}'_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) = B$. Strategyproofness then implies that $\varphi_i(R_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) = c(A, R_i)$ for all $i \in M_1$ and all $R \in \mathcal{R}^n$. Furthermore, $\varphi_j(R_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) = A$ for all $j \in M_1 \setminus \{i\}$ by group-monotonicity, since $\varphi_i(R_i, R'_{M_1 \setminus \{i\}}, R_{-M_1}) \subseteq \varphi_i(R'_{M_1}, R_{-M_1})$. It can be shown by induction that $\varphi_i(R) = c(A, R_i)$ for all $i \in M_1$ and all $R \in \mathcal{R}^n$. The following induction basis has been demonstrated.

Induction basis: For all $i \in M_1$, all $j \in M_1 \backslash \{i\}$ and all $R \in \mathcal{R}^n$, $\varphi_i(R_i, R'_{M_1 \backslash \{i\}}, R_{-M_1}) = c(A, R_i)$ and $\varphi_j(R_i, R'_{M_1 \backslash \{i\}}, R_{-M_1}) = A$.

Induction hypothesis: Assume that for all $S \subseteq M_1$ such that $|S| \leq t$, all $i \in S$, all $j \in M_1 \backslash S$ and all $R \in \mathcal{R}^n$, $\varphi_i(R_S, R'_{M_1 \backslash S}, R_{-M_1}) = c(A, R_i)$ and $\varphi_j(R_S, R'_{M_1 \backslash S}, R_{-M_1}) = A$.

By the induction basis, the induction hypothesis holds for $t = 1$. To see this, let $S = \{i\}$ for some $i \in M_1$.

Induction step: To see that the induction hypothesis holds for any $S \subseteq M_1$ such that $|S| \leq t + 1$ as well, consider some $S' \subseteq M_1$ such that $|S'| = t$ and some $j \in M_1 \setminus S'$. By the induction hypothesis, $\varphi_j(R_{S'}, R'_{M_1 \backslash S'}, R_{-M_1}) = A$. By the same argument as above, strategyproofness and Pareto efficiency imply that $\varphi_j(R_{S' \cup \{j\}}, R'_{M_1 \backslash (S' \cup \{j\})}, R_{-M_1}) = c(A, R_j)$ for all $R_j \in \mathcal{R}$. Group-monotonicity and the induction hypothesis imply that $\varphi_k(R_{S' \cup \{j\}}, R'_{M_1 \backslash (S' \cup \{j\})}, R_{-M_1}) = A$ for all $k \in M_1 \setminus (S' \cup \{j\})$ and that $\varphi_k(R_{S' \cup \{j\}}, R'_{M_1 \backslash (S' \cup \{j\})}, R_{-M_1}) = c(A, R_k)$ for all $k \in S'$. Like before, the reason for this is that $\varphi_j(R_{S' \cup \{j\}}, R'_{M_1 \backslash (S' \cup \{j\})}, R_{-M_1}) \subseteq \varphi_j(R_{S'}, R'_{M_1 \backslash S'}, R_{-M_1})$. Let $S' \cup \{j\} \equiv S$. Then it has been demonstrated that for all $i \in S$ and all $j \in M_1 \backslash S$, $\varphi_i(R_S, R'_{M_1 \backslash S}, R_{-M_1}) = c(A, R_i)$ and $\varphi_j(R_S, R'_{M_1 \backslash S}, R_{-M_1}) = A$. Since S is an arbitrary subset of M_1 such that $|S| \leq t+1$, the induction hypothesis holds for any $S \subseteq M_1$ such that $|S| \leq t + 1$.

This implies that the induction hypothesis holds for any $S \subseteq M_1$, including $S = M_1$. Thus, $\varphi_i(R) = c(A, R_i)$ for all $i \in M_1$ and all $R \in \mathcal{R}^n$. This means that there always exists some group M_1 such that all agents belonging to M_1 are assigned their most preferred bundles of objects under all preference profiles. Set $M_0 = \emptyset, B_0 = \emptyset$ and recursively define $B_t \equiv \bigcup_{j \in M_t} c(A \setminus \bigcup_{l=0}^{t-1} B_l, R_j)$. To understand B_t, consider the group M_1. Since $B_0 = \emptyset$, B_1 is the set of objects that would be assigned to agents in M_1 if all agents in M_1 were assigned their most preferred bundles in A under R. Next, consider the group M_2. B_2 is the set of objects in $A \setminus B_1$ that would be assigned to agents in M_2 if all agents in M_2 were assigned their most preferred bundles in $A \setminus B_1$ under R. In general, B_t is the set of objects in $A \setminus (B_1 \cup B_2 \cup \cdots \cup B_{t-1})$ that would be assigned to agents in M_t if all agents in M_t were assigned their most preferred bundles in $A \setminus (B_1 \cup B_t \cup \cdots \cup B_{t-1})$ under R. Note that the identity of the group is unspecified for all groups except M_1. Let $M(t) \equiv \bigcup_{j=0}^{t} M_j$ and $B(t) \equiv \bigcup_{j=0}^{t} B_j$. It can be shown by induction that each agent $i \in N$ belongs to some group $M_k \subseteq N$ such that i is assigned $\varphi_i(R_{M(k-1)}, R'_{-M(k-1)}) = c\,(A \setminus B(k-1), R'_i)$ for all $R' \in \mathcal{R}^n$. The following induction basis has already been demonstrated.

Induction basis: There exists a group M_1 such that $\varphi_i(R') = c(A \setminus B(0), R'_i)$ for all $i \in M_1$ and all $R' \in \mathcal{R}^n$.

Induction hypothesis: Let $t \geq 1$. Assume that there exists some group $M_t \subseteq N \backslash M(t-1)$ such that $\varphi_i(R_{M(k-1)}, R'_{-M(k-1)}) = c\,(A \setminus B(k-1), R'_i)$ for all $i \in M_k$, all $k \in \{1, \ldots, t\}$

and all $R, R' \in \mathcal{R}^n$.

Note that the induction hypothesis holds for $t = 1$ by the induction basis, since $(R_{M(k-1)}, R'_{-M(k-1)}) = R'$ when $k = 1$.

Induction step: Suppose $M(t) \neq N$. First note that the induction hypothesis implies that $B(t)$ is the set of objects assigned to agents in $M(t)$. Next, note that $M(t)$ is the union of one or more groups. Since groups are disjoint, $M(t) \cap N_i = \emptyset$ for all $i \in N \setminus M(t)$. This means that no $i \in N \setminus M(t)$ can feasibly be assigned any $a \in B(t)$, since all objects in $B(t)$ are assigned to agents not belonging to N_i. $B(t)$ can thus be interpreted as the the set of objects that are blocked by the agents in $M(t)$ when $R_{M(t)}$ is reported. Fix $R_{M(t)}$ and let all preference relations in $\bar{R}$ rank $A \setminus B(t)$ first and $\emptyset$ second. By the induction hypothesis, $\varphi_i(R_{M(t)}, R'_{-M(t)}) = \varphi_i(R_{M(t)}, R''_{-M(t)})$ for all $i \in M(t)$ and all $R', R'' \in \mathcal{R}^n$. Lemma 4 then implies that there exists some group $M_{t+1} \subseteq N \setminus M(t)$ such that $\varphi_i(\bar{R}_{M_{t+1}}, R'_{-M(t+1)}, R_{M(t)}) = A \setminus B(t)$ for all $i \in M_{t+1}$ and all $R' \in \mathcal{R}^n$. It can thus be shown by the same argument as before that $\varphi_i(R_{M(t)}, R'_{-M(t)}) = c(A \setminus B(t), R'_i)$ for all $i \in M_{t+1}$ and all $R' \in \mathcal{R}^n$. Then, by the induction hypothesis, $\varphi_i(R_{M(k-1)}, R'_{-M(k-1)}) = c\,(A \setminus B(k-1), R'_i)$ for all $i \in M_k$, all $k \in \{1, \ldots, t+1\}$ and all $R' \in \mathcal{R}^n$. In other words, the induction hypothesis holds for $t + 1$ as well.

Hence, each agent $i \in N$ belongs to some group $M_k \subseteq N$ such that i is assigned $\varphi_i(R_{M(k-1)}, R'_{-M(k-1)}) = c\,(A \setminus B(k-1), R'_i)$ for all $R' \in \mathcal{R}^n$. This implies that for all groups $M_k \subseteq N$, all agents $i \in M_k$ and all $R \in \mathcal{R}^n$, $\varphi_i(R) = c\,(A \setminus B(k-1), R_i)$. It also implies that φ is group-invariant. In other words, the reported preferences of one agent never affects the assignment of some other agent belonging to the same group. Note that the identity of each group M_k may differ depending on the preference profile. Let f_M be the set of priority structures f under which, for all $i, j \in N$ and all $R \in \mathcal{R}^n$, $j \in M_a$ and $k \in M_{b<a}$ only if $f_R(k) < f_R(j)$. Note that every $f \in f_M$ is sorted by groups and that f_M is guaranteed to be non-empty since the set of all groups M_k constitutes a partition of N. For all $f \in f_M$, all $i \in M_1$ and all $R \in \mathcal{R}^n$, $f_R^{-1}(1) \in M_1$ and $\varphi_i(R) = c\,(A, R_i)$ by the second induction basis. Thus, the first requirement in the definition of an s-hierarchy tree associated with φ is satisfied for all $f \in f_M$.

Recall that for all groups $M_k \subseteq N$, all agents $i \in M_k$ and all $R \in \mathcal{R}^n$, $\varphi_i(R) = c\,(A \setminus B(k-1), R_i)$. This implies that for all $f \in f_M$, all $M_k \subseteq N$ and all $i \in M_k$, $S_R(i) = A \setminus B(k-1)$. To see this, note that for any $i \in M_k$, every agent in $M(k-1)$ has higher priority than i, every agent in $N \setminus M(k)$ has lower priority than i and $S_R(i)$ is independent on the assignments of other agents in M_k. This means that for all $R \in \mathcal{R}^n$, $\varphi_i(R) = c(A, R_i)$ for $i = f_R^{-1}(1)$ and $\varphi_i(R) = c(S_R(i), R_i)$ for each $i \neq f_R^{-1}(1)$. Thus, every $f \in f_M$ is an s-hierarchy network associated with φ and φ is a sequential rule.

Furthermore, note that given some $R_{M(k-1)} \in \mathcal{R}^{|M(k-1)|}$, the identity of M_k is fixed for all $R_{-M(k-1)} \in \mathcal{R}^{|N \setminus M(k-1)|}$. In conjunction with the observation that every $f \in f_M$ is an s-hierarchy network associated with φ and that φ is group-invariant implies that the second requirement in the definition of an s-hierarchy tree associated with φ is satisfied for all $f \in f_M$. This means that φ is a sequential dictatorship. Finally, since every $f \in f_M$ is sorted by groups, every $f \in f_M$ is a group-sorted s-hierarchy tree and φ is a group-sorting sequential dictatorship. □

Theorem 1 follows almost immediately from the lemmas above.

Proof of Theorem 1. Let φ be a group-sorting sequential dictatorship. By Lemma 2 and Lemma 3, φ is coalitionally strategyproof, Pareto efficient, and group-invariant. Hence, statement (1) implies statement (3) in Theorem 1. Next, let φ be a coalitionally strategyproof, Pareto efficient and group-invariant rule. Since φ is group-invariant, it is also group-monotonic. This means that statement (3) implies statement (2). By Lemma 1, φ is strategyproof and non-bossy. Hence, statement (2) implies statement (4) and statement (3) implies statement (5). Furthermore, since group-invariance implies group-monotonicity, statement (5) implies statement (4). Finally, by Lemma 5, statement (4) implies statement (1). Hence, statements (1)-(5) are equivalent. □

4 Other properties

This section will focus on some properties featured in characterizations of sequential and serial dictatorships for the standard case and investigate the extent to which they are applicable when assignments are allowed to overlap. Up until this point, N and A have been fixed. To study the properties below, the model must be amended to let both $N \subseteq \mathcal{N}$ and $A \subseteq \mathcal{A}$ be variable. In this section, a rule is a collection of functions $\varphi = \{\varphi^{N,A} : \mathcal{R}^n \rightarrow \mathcal{M} \mid N \subseteq \mathcal{N}, A \subseteq \mathcal{A}\}$ rather than a single function. In other words, the rule φ selects an allocation $\varphi^{N,A}(R)$ for each pair $\{N, A\} \in \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{A})$ and each $R \in \mathcal{R}^n$. For convenience, $\varphi^{N,A}(R)$ is denoted by $\varphi(R, N, A)$.

Two properties that have been studied in the context of the standard case are *consistency* and *population-monotonicity*. Klaus and Miyagawa (2001) show that, for the standard case, serial dictatorships are characterized by strategyproofness, Pareto efficiency and consistency or strategyproofness, Pareto efficiency and population-monotonicity.

Definition 9. *A rule φ is* consistent *if for all $N \subseteq \mathcal{N}$, all $A \subseteq \mathcal{A}$, all $R \in \mathcal{R}$, all non-empty $S \subseteq N$ and all $i \in S$, $\varphi_i(R, N, A) = \varphi_i\left(R_S, S, \bigcup_{j \in S} \varphi_j(R, N, K)\right)$.*

To understand this definition, consider a rule that selects some allocation μ when the agents in N report R. If the same rule is applied to a subset of all agents $S \subseteq N$ and their assignments under μ, then consistency requires that all agents in S be assigned the same objects as under μ when they report R_S. In the standard case, consistency is often interpreted as a property requiring that the remaining agents be assigned the same objects if the rule is reapplied after some agents have been removed along with their assignments. This interpretation is not valid whenever assignments are allowed to overlap since it would allow for agents to be removed along with objects assigned to other agents, which is not in line with the formal definition of consistency.

Definition 10. *A rule is* population-monotonic *if for all $N \subseteq \mathcal{N}$, all $A \subseteq \mathcal{A}$, all non-empty $S \subseteq N$ and all $R \in \mathcal{R}^n$, either*

(1) $\varphi_i(R_S, S, A) R_i \varphi_i(R, N, A)$ for all $i \in S$, or

(2) $\varphi_i(R, N, A) R_i \varphi_i(R_S, S, A)$ for all $i \in S$.

In other words, if some agents are removed from the set of agents, then either all remaining agents are weakly better off or all remaining agents are weakly worse off. Next, let $\mathcal{F}^N$ be the set of all permutations of N, $F : N \rightarrowtail\!\!\!\!\rightarrow \{1, 2, \ldots, n\}$. Rather than a single function, a priority structure is redefined as a collection of functions $f = \{f^N : \mathcal{R}^n \to \mathcal{F}^N \mid N \in \mathcal{N}\}$ in this section. This means that a priority structure now selects an ordering of all agents $f^N(R)$ for each $N \in \mathcal{N}$ and each $R \in \mathcal{R}^n$. Denote the priority of agent $i \in N$ under $f^N(R)$ by $f_{R,N}(i)$.

Furthermore, the definitions of sequential rules, sequential dictatorships and group-sorting sequential dictatorships are amended such that the requirements in Definition 1, Definition 2 and Definition 3 must be satisfied for each pair $\{N, A\} \in \mathcal{P}(\mathcal{N}) \times \mathcal{P}(\mathcal{A})$. For example, if f is an s-hierarchy tree, then it is necessary that $f^{-1}_{R,N}(1) = f^{-1}_{R',N}(1)$ for all $N \subseteq \mathcal{N}$ and all $R, R' \in \mathcal{R}^n$. However, it is not necessarily the case that $f^{-1}_{R,N}(1) = f^{-1}_{R_S,S}(1)$ when $S \neq N$. Since consistency and population-monotonicity are featured in characterizations of serial dictatorships in the standard case, serial dictatorships will be formally defined to determine whether this might be the case when assignments are allowed to overlap as well.

Definition 11. *An s-hierarchy network f associated with some rule φ is an* exogenous s-hierarchy tree *associated with φ if for all $N, S \subseteq \mathcal{N}$,* *(1)* $f^N(R) = f^N(R')$ *for all* $R, R' \in \mathcal{R}^n$*, and*

(2) for all $i, j \in N \cap S$*,* $f_{R_S,S}(i) < f_{R_S,S}(j)$ *if and only if* $f_{R,N}(i) < f_{R,N}(j)$*.*

A rule φ is a *serial dictatorship* if there exists an exogenous s-hierarchy tree associated with φ. Under a serial dictatorship, it is impossible for any agent to affect the priority structure. For a given $N \subseteq \mathcal{N}$, the agent with priority k under some preference profile is the agent with priority k under all preference profiles. If, for some $N, S \subseteq \mathcal{N}$, there are two agents $i, j \in N \cap S$ such that i has higher priority than j when the set of all agents is given by N, then i has higher priority than j when the set of all agents is given by S as well. Note that a serial dictatorship is a sequential dictatorship. A serial dictatorship can be a group-sorting sequential dictatorship, but not all serial dictatorships are group-sorting sequential dictatorships. The first result in this section is that neither serial dictatorships nor group-sorting sequential dictatorships are consistent or population-monotonic.

Proposition 1. *Serial dictatorships and group-sorting sequential dictatorships are not consistent or population-monotonic.*

Proof. Suppose the set of all agents is given by $N = \{1, 2, 3\}$ and that the set of all objects is given by A. Let f be an exogenous s-hierarchy tree associated with some rule φ. Since $f^S(R) = f^S(R')$ for all $S \subseteq \mathcal{N}$ and all $R, R' \in \mathcal{R}^{|S|}$, $f^S(R)$ can be denoted by f^S for all $S \subseteq \mathcal{N}$ and all $R \in \mathcal{R}^{|S|}$. Suppose $N_1 = \{1, 3\}$, $S = \{2, 3\}$, $f^N = (1, 2, 3)$, and let all preference relations in R rank A first. By the definition of an exogenous s-hierarchy tree, $f^S = (2, 3)$. Since φ is a serial dictatorship, $\varphi_2(R, N, A) = \emptyset$, $\varphi_2(R_S, S, A) = A$, $\varphi_3(R, N, A) = A$ and $\varphi_3(R_S, S, A) = \emptyset$. Note that $\varphi_2(R_S, S, A)P_2\varphi_2(R, N, A)$, but $\varphi_3(R, N, A)P_3\varphi_3(R_S, S, A)$. This violates both consistency and population-monotonicity. Hence, serial dictatorships are not consistent or population-monotonic.

Next, suppose the set of all agents is given by $N = \{1, 2, 3\}$ and that the set of all objects is given by A. Furthermore, suppose that $N_1 = \{1, 2\}$ and let f be defined such that $f^N(R) = (1, 2, 3)$ for all $R \in \mathcal{R}^3$ and $f^S(R_S) = (i, j)$ for all $R_S \in \mathcal{R}^2$ whenever $S \subset N$, $|S| = 2$ and $i > j$. Then f is a group-sorted s-hierarchy tree associated with some rule φ, since every sorting is a sorting by groups when there are only two agents. Consider such a group-sorting sequential dictatorship φ. Let $S = \{2, 3\}$ and let R rank A first and $\emptyset$ second. Then $\varphi_2(R, N, A) = A$, $\varphi_2(R, S, A) = \emptyset$, $\varphi_3(R, N, A) = \emptyset$ and $\varphi_3(R, S, A) = A$. This implies that $\varphi_2(R, N, A)P_2\varphi_2(R, S, A)$, while $\varphi_3(R, S, A)P_3\varphi_2(R, N, A)$, which violates both consistency and population-monotonicity. Hence group-sorting sequential dictatorships are not consistent or population-monotonic. □

An implication of this is that the results in Klaus and Miyagawa (2001) mentioned earlier do not hold in a more general context when assignments are allowed to overlap. Since both serial dictatorships and group-sorting sequential dictatorships are examples of sequential dictatorships, sequential dictatorships do, in general, not satisfy consistency or population-monotonicity.

Ehlers and Klaus (2003) show that serial dictatorships are characterized in the standard case, for the complete, transitive, responsive, separable and strict preference domain, by coalitional strategyproofness, Pareto efficiency and resource-monotonicity. Here, the definition of resource-monotonicity in Moulin and Thomson (1988) is used.

Definition 12. *A rule φ is* resource-monotonic *if for all $N \subseteq \mathcal{N}$, all $i \in N$, all $A, A' \subseteq \mathcal{A}$ and all $R \in \mathcal{R}^n$, $\varphi_i(R, N, A) R_i \varphi_i(R, N, A')$ whenever $A' \subseteq A$.*

In other words, a rule is resource-monotonic if all agents are weakly worse off when some objects are removed from the set of all objects. Ehlers and Klaus (2003) use a weaker definition of resource-monotonicity. However, since the result in Proposition 2 is negative, it is unproblematic to adopt a stronger definition. Ehlers and Klaus (2003) also show that sequential dictatorships are characterized by coalitional strategyproofness and Pareto efficiency. This implies that sequential dictatorships are, in general, not resource-monotonic on this preference domain. Since this preference domain is a subset of the domain of complete, transitive and strict preferences, sequential dictatorships are not resource-monotonic on the more general preference domain either. This is true for the standard case and, by extension, for the problem studied in this paper as well. To rule out the possibility that group-sorting sequential dictatorships or serial dictatorships are resource-monotonic, it is demonstrated that no Pareto efficient rule is resource-monotonic.

Proposition 2. *No Pareto efficient rule is resource-monotonic, even when the restriction is imposed that $C(S) = 0$ for all $S \subseteq \mathcal{N}$ such that $|S| \geq 2$.*

Proof. Suppose the set of all agents is given by $\mathcal{N} = \{1, 2\}$, that the set of all objects is given by $\mathcal{A} = \{a, b, c\}$ and that $C(\{1, 2\}) = 0$. Let φ be a Pareto efficient rule. Furthermore, let R_1 rank $\mathcal{A}$ first, $\{a, b\}$ second and $\emptyset$ third and let R_2 rank $\{b, c\}$ first and $\emptyset$ second. There are only two Pareto efficient allocations when R is reported: μ where $\mu(1) = \mathcal{A}$ and $\mu(2) = \emptyset$ and ν where $\nu(1) = \emptyset$ and $\nu(2) = \{b, c\}$. Suppose $\varphi(R, N, \mathcal{A}) = \mu$. Let $A' = \{b, c\}$ and note that $\varphi_2(R, N, A') = \{b, c\}$ by Pareto efficiency. Since $\{b, c\} P_2 \emptyset$ and $\{b, c\} \subset \mathcal{A}$, this violates resource-monotonicity. Suppose $\varphi(R, N, \mathcal{A}) = \nu$. Let $A' = \{a, b\}$ and note that $\varphi_1(R, N, A') = \{a, b\}$ by Pareto efficiency. Since $\{a, b\} P_1 \emptyset$ and $\{a, b\} \subset \mathcal{A}$, this violates resource-monotonicity. □

This means that whenever Pareto efficient rules are studied, resource-monotonicity need not be considered unless further restrictions are imposed on the preference domain. Hence, the the characterization in Ehlers and Klaus (2003) mentioned earlier does not hold on the domain of complete, transitive and strict preferences. This is true both for the problem studied in this paper and for the standard case when assignments are not allowed to overlap. Neither serial dictatorships nor group-sorting sequential dictatorships are resource-monotonic, since all sequential dictatorships are Pareto efficient.

5 Concluding remarks

In this paper, some characterizations of group-sorting sequential dictatorships were provided. It was also demonstrated that the characterization of sequential dictatorships by Pápai (2001) is a special case of Theorem 1. Finally, it was shown that some properties, on smaller preference domains or on the domain of complete, transitive and strict preferences, of serial dictatorships or sequential dictatorships in the standard case are not satisfied by serial dictatorships or group-sorting sequential dictatorships in the problem studied in this paper. It is still an open question whether sequential dictatorships may be characterized by coalitional strategyproofness and Pareto efficiency. For future research, it might be interesting to study the properties of group-sorting serial dictatorships, which satisfy some properties not satisfied by serial dictatorships or group-sorting sequential dictatorships. Restricting attention to problems with unit demand might make it easier to study allocation problems with overlapping assignments and endowments. Another possibility is to investigate an overlapping multiple assignment problem where a capacity constraint imposed, such that there is an upper limit for how many agents a single object may be assigned to.

A Appendix

A.1 Proof of Lemma 4

Proof. Let φ be a strategyproof, Pareto efficient and non-bossy rule. Let $M' \subset N$ be either the empty set, or the union of one or more groups such that $\varphi_i(R_{M'}, R'_{-M'}) = \varphi_i(R_{M'}, R''_{-M'})$ for all $i \in M'$ and all $R', R'' \in \mathcal{R}^n$. Let $A \setminus \bigcup_{j \in M'} \varphi_j(R)$ be denoted by B. B is the set of objects not assigned to agents in M' when $R_{M'}$ is reported. Let all preference relations in $\bar{R}$ rank B first and $\emptyset$ second. If M' is the union of one or more groups, no agent in $N \setminus M'$ may be assigned any objects in B when the agents in M' report $R_{M'}$. If $M' = \emptyset$, then $B = \emptyset$. Pareto efficiency implies that there exists a non-empty set of agents $M'' \subseteq N \setminus M'$ such that for all $i \in M''$, $\varphi_i(\bar{R}_{-M'}, R_{M'}) = B$. If $\varphi_i(\bar{R}_{-M'}, R_{M'}) = B$, then, by Pareto efficiency, $\varphi_{i'}(\bar{R}_{-M'}, R_{M'}) = B$ for all $i' \in N_i$ and $\varphi_{i'}(\bar{R}_{-M'}, R_{M'}) = \emptyset$ for all $i' \in N \setminus (M' \cup N_i)$. Let $M \equiv N_i$ for any $i \in M''$ and note that M is the largest set of agents in $N \setminus M'$ such that for all $i \in M$, $\varphi_i(\bar{R}_{-M'}, R_{M'}) = B$.

Consider an agent $j \in N\setminus(M' \cup M)$, some arbitrary $R_j \in \mathcal{R}$ and recall that R_j^B is defined by letting B be the highest ranked set of objects under R_j^B, while letting the relative order of all other sets of objects be the same as under R_j. By strategyproofness, there exists no $R_j \in \mathcal{R}$ such that $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = B$, since $\varphi_j(\bar{R}_{-M'}, R_{M'}) = \emptyset$ and $B \bar{R}_j \emptyset$. Again, strategyproofness implies that $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \varphi_j(R_j^B, \bar{R}_{-M' \cup \{j\}}, R_{M'})$. Non-bossiness implies that $\varphi(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \varphi(R_j^B, \bar{R}_{-M' \cup \{j\}}, R_{M'})$. It can be shown by contradiction that $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \emptyset$. Assume that $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) \neq \emptyset$. Pareto efficiency implies that any agent $k \in N\setminus(M' \cup \{j\})$ is assigned either B or $\emptyset$. If some M'' is the (possibly empty) set of agents assigned B under $\varphi(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'})$, then $C(\{j\} \cup M'') = 1$. Since it has already been established that $\varphi(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \varphi(R_j^B, \bar{R}_{-M' \cup \{j\}}, R_{M'})$, $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \varphi_j(R_j^B, \bar{R}_{-M' \cup \{j\}}, R_{M'}) \neq B$ violates Pareto efficiency, as j could feasibly be assigned B under $\varphi(R_j^B, \bar{R}_{-M' \cup \{j\}}, R_{M'})$ without affecting the assignment of anyone else. Hence, $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \emptyset$. Lemma 4 can then be proven by induction. Since $\varphi_j(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \emptyset$, non-bossiness implies the following induction basis.

Induction basis: $\varphi(R_j, \bar{R}_{-M' \cup \{j\}}, R_{M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$ for all $j \in N \setminus (M' \cup M)$ and all $R_j \in \mathcal{R}$.

Induction hypothesis: Assume that $\varphi(R_G, \bar{R}_{-M' \cup G}, R_{M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$ for all $G \subset N \setminus (M' \cup M)$ such that $|G| \leq l < n - |M' \cup M|$ and all $R_G \in \mathcal{R}^{|G|}$.

Let G be some $\{j\} \subset N \setminus (M' \cup M)$ and note that the induction hypothesis holds for $l = 1$ by the induction basis.

Induction step: It can be proven by contradiction that the induction hypothesis holds for $l + 1$ as well. Assume there exists some $H \subseteq N \setminus (M' \cup M)$ such that $|H| = l + 1$ and $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \mu \neq \varphi(\bar{R}_{-M'}, R_{M'})$. By strategyproofness and the induction hypothesis, $\mu(j) \neq B$ for all $j \in H$. To see this, note that $B \bar{R}_j B'$ for all $B' \subseteq B$ and all $j \in H$. If $\mu(j) = B$ for some $j \in H$, then $\mu(j) \bar{P}_j \varphi_j(R_{H\setminus\{j\}}, \bar{R}_{-M' \cup (H\setminus\{j\})}, R_{M'})$, since $\varphi_j(R_{H\setminus\{j\}}, \bar{R}_{-M' \cup (H\setminus\{j\})}, R_{M'}) = \emptyset$ by the induction hypothesis, as $|H \setminus \{j\}| \leq l$. This violates strategyproofness. Hence, $\mu(j) \neq B$ for all $j \in H$. If $\mu(j) = \emptyset$ for some $j \in H$, then $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R_{H\setminus\{j\}}, \bar{R}_{-M' \cup (H\setminus\{j\})}, R_{M'}) = \mu$ by non-bossiness. Since $|H \setminus \{j\}| \leq l$, the induction hypothesis implies that $\varphi(R_{H\setminus\{j\}}, \bar{R}_{-M' \cup (H\setminus\{j\})}, R_{M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$, which violates the assumption that $\varphi(\bar{R}_{-M'}, R_{M'}) \neq \mu$. Hence, $\mu(j) \neq$

$\emptyset$ and $\mu(j) \neq B$ for all $j \in H$. Strategyproofness and nonbossiness then imply that $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R_j^B, R_{H \setminus \{j\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu$ for all $j \in H$. If $C(H) = 1$, then the assignment ν under which there exists some $j \in H$ such that $\nu(j) = B$ and $\nu(i) = \mu(i)$ for all $i \in N \setminus \{i\}$ is feasible. Thus, $\varphi(R_j^B, R_{H \setminus \{j\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu$ violates Pareto efficiency, since ν Pareto dominates μ under $(R_j^B, R_{H \setminus \{j\}}, \bar{R}_{-M' \cup H}, R_{M'})$. Hence, $C(H) = 0$ and there exist some $j, k \in H$ such that $C(\{j, k\}) = 0$. Since it has already been established that $\mu(j) \neq \emptyset$ and $\mu(k) \neq \emptyset$, it must hold that $\mu(i) \neq B$ for all $i \in N \setminus M'$. Furthermore, $\mu(i) = \emptyset$ for all $i \in N \setminus (M' \cup H)$ by Pareto efficiency, since $\mu(i) \neq B$ and $\emptyset \bar{P}_i B'$ for all $B' \subset B$.

Recall that $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \mu$ and consecutively replace each $R_i \in R_H$ with R_i^B. Strategyproofness and nonbossiness then imply that $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R_H^B, \bar{R}_{-M' \cup H}, R_{M'})$. Let all preference relations in R'_H rank B first, $\mu(i)$ second and $\emptyset$ third. Next, consecutively replace each $R_i^B \in R_H^B$ with R'_i. Again, strategyproofness and nonbossiness imply that $\varphi(R_H^B, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R'_H, \bar{R}_{-M' \cup H}, R_{M'}) = \mu$. Consider the same $j, k \in H$ as before and recall that $C(\{j, k\}) = 0$. Let all preference relations in R'' rank B first, $\mu(j)$ second, $\mu(k)$ third and $\emptyset$ fourth. If $\varphi_k(R''_k, R'_{H \setminus \{k\}}, \bar{R}_{-M' \cup H}, R_{M'}) \in \{B, \mu(j)\}$, then $\varphi_j(R''_k, R'_{H \setminus \{k\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \emptyset$ by feasibility and Pareto efficiency. This violates non-bossiness, since $\varphi(\bar{R}_j, R''_k, R'_{H \setminus \{j,k\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$ by the induction hypothesis and both j and k are assigned $\emptyset$ under $\varphi(\bar{R}_{-M'}, R_{M'})$.

In general, non-bossiness is violated if the agents in M' report $R_{M'}$, the agents in $N \setminus (M' \cup H)$ report $\bar{R}_{-M' \cup H}$ and there exist $i, i' \in H$ such that i receives a non-empty assignment and i' is assigned $\emptyset$. This observation is used several times in this proof. Strategyproofness implies that $\varphi_k(R''_k, R'_{H \setminus \{k\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu(k)$ and non-bossiness implies that $\varphi(R''_k, R'_{H \setminus \{k\}}, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R'_H, \bar{R}_{-M' \cup H}, R_{M'}) = \mu$. Let $D_i \equiv N_i \cap H$. That is, let D_i be the set of agents in H that are compatible with some agent i. Let all preference relations in $\hat{R}_{D_k}$ rank B first, $\mu(j)$ second, $\mu(k)$ third, $\mu(i)$ fourth if $\mu(i) \neq \mu(k)$ and $\emptyset$ after $\mu(i)$. Note that $j \notin D_k$. Non-bossiness implies that $\varphi_j(R''_k, \tilde{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu(j)$ for all $\tilde{R} \in \mathcal{R}^n$, since j would otherwise be assigned $\emptyset$ by Pareto efficiency. It can not be the case that $\varphi_j(R''_k, \tilde{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = B$, as k would then be assigned $\emptyset$ by feasibility, violating non-bossiness. Similarly, non-bossiness and Pareto efficiency imply that $\varphi_k(R''_k, \tilde{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu(k)$ for all $\tilde{R} \in \mathcal{R}^n$. Hence, $\varphi_i(R''_k, \hat{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu(k)$ for all $i \in D_k$ by Pareto efficiency.

Furthermore, since no $i \notin M' \cup D_k$ prefers any $S \notin \{B, \mu(i)\}$ above $\emptyset$, Pareto efficiency implies that $\varphi_i(R''_k, \hat{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = \mu(i)$ for all $i \notin D_k$. To see this, first note that every $i \in M'$ is assigned $\mu(i)$ whenever $R_{M'}$ is reported by the agents in the possibly empty set M'. Next, note that since no agent is assigned B and B is the only set of objects preferred to $\emptyset$ by the agents in $N \setminus (M' \cup H)$ under $\bar{R}$, Pareto efficiency implies that every agent in $N \setminus (M' \cup H)$ is assigned $\emptyset$. Finally, note that every agent in $i \in H \setminus D_k$ may feasibly be assigned $\mu(i)$ since $\mu(i) \cap \mu(i') = \emptyset$ for all $i \notin H \setminus D_k$ and all $i' \in H \setminus D_k$. Hence, every $i \in D_k$ is assigned $\mu(k)$ and every $i \notin D_k$ is assigned $\mu(i)$ under $\varphi(R''_k, \hat{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'})$. Consecutively replace $\hat{R}_i$ with R''_i for all $i \in D_k \setminus \{k\}$. Non-bossiness and strategyproofness imply that $\varphi(R''_{D_k}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R''_k, \hat{R}_{D_k \setminus \{k\}}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'})$. Furthermore, non-bossiness and strategyproofness imply that $\varphi(R''_{D_k \cup \{j\}}, R'_{H \setminus (D_k \cup \{j\})}, \bar{R}_{-M' \cup H}, R_{M'}) = \varphi(R''_{D_k}, R'_{H \setminus D_k}, \bar{R}_{-M' \cup H}, R_{M'})$. Let all preference relations in $\hat{R}'$ rank B first, $\mu(j)$ second, $\mu(i)$ third if $\mu(i) \neq \mu(j)$ and $\emptyset$ after $\mu(i)$. Consecutively replace R'_i with $\hat{R}'_i$ for all $i \in D_j \setminus \{j\}$ and let $\varphi(R''_{D_k \cup \{j\}}, \hat{R}'_{D_j \setminus \{j\}}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) = \lambda$. The first agent $i' \in D_j \setminus \{j\}$ for whom $R'_{i'}$ is replaced with $\hat{R}'_{i'}$ must be assigned either $\mu(i')$ or $\mu(j)$ by strategyproofness and non-bossiness. Non-bossiness ensures that i' can not be assigned B. If i' is assigned $\mu(i')$ and $\mu(i') \neq \mu(j)$, then the assignments are unchanged for all agents by non-bossiness. This would violate Pareto efficiency, since $C(\{i', j\}) = 1$, j is assigned $\mu(j)$ and $\mu(j) \hat{P}'_{i'} \mu(i')$. Furthermore, j can not be assigned $\mu(k)$, since every agent in D_k would then be assigned $\emptyset$ by Pareto efficiency, which violates non-bossiness. Therefore, i' and j are assigned $\mu(j)$, every agent in D_k is assigned $\mu(k)$ and every agent $i \in H \setminus (D_k \cup \{i', j\})$ is assigned $\mu(i)$ by Pareto efficiency. This means that the same argument can be repeated for every $i \in D_j \setminus \{j, i'\}$. Hence, $\lambda(i) = \mu(j)$ for all $i \in D_j$ and $\lambda(i) = \mu(k)$ for all $i \in D_k$. By Pareto efficiency, $\lambda(i) = \mu(i)$ for all $i \notin D_j \cup D_k$. Consecutively replace $\hat{R}'_i$ with R''_i for all $i \in D_j \setminus \{j\}$. Recall that $\mu(j)$ is preferred to $\mu(k)$ under R''. Thus, strategyproofness and non-bossiness imply that

$$\varphi(R''_{D_k \cup D_j}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) = \lambda \tag{1}$$

Let all preference relations in $\bar{R}'$ rank B first, $\mu(k)$ second, $\mu(j)$ third and $\emptyset$ fourth. Consecutively replace R''_i with $\bar{R}'_i$ for all $i \in D_j$. By strategyproofness, the first $i' \in D_j$ for whom $R''_{i'}$ is replaced with $\bar{R}'_{i'}$ is assigned either $\mu(j)$ or $\mu(k)$, where B can be ruled out by non-bossiness. As demonstrated above, since i' is not assigned $\emptyset$, non-bossiness implies that each $i \in H$ is assigned some $B' \neq \emptyset$, where B' is preferred to $\emptyset$ by Pareto efficiency. This argument can be repeated for all $i \in D_j \setminus \{i'\}$. Hence, $\varphi_i(R''_{D_k}, \bar{R}'_{D_j}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) \neq \emptyset$ for all $i \in H$. Let $\varphi(R''_{D_k}, \bar{R}'_{D_j}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) \equiv \gamma$. Note that for all $B' \subset B$, $\mu(j) R''_i B'$ for all $i \in D_k$, $\mu(k) \bar{R}'_i B'$ for all $i \in D_j$, $\mu(i) R'_i B'$ for all $i \in H \setminus (D_k \cup D_j)$ and $\emptyset \bar{R}_i B'$ for all $i \in N \setminus (M' \cup H)$. Hence, Pareto efficiency implies that $\gamma(i) = \mu(k)$ for all $i \in D_j$, $\gamma(i) = \mu(j)$ for all $i \in D_k$ and $\gamma(i) = \mu(i)$ for all $i \in N \setminus (D_j \cup D_k)$. Let all preference relations in $\bar{R}''$ raznk B first, $\mu(j)$ second and $\emptyset$ third. Strategyproofness and non-bossiness imply that $\varphi(\bar{R}''_k, R''_{D_k \setminus \{k\}}, \bar{R}'_{D_j}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) = \gamma$. Recall that $\mu(j)$ is preferred to $\mu(k)$ under R'', while $\mu(k)$ is preferred to $\mu(j)$ under $\bar{R}'$. The ordering of all other bundles of objects are the same under R'' and $\bar{R}'$. Consecutively replace $\bar{R}'_i$ with R''_i for all $i \in D_j$. If any $i \in D_j$ is assigned $\mu(j)$, then k is assigned $\emptyset$, which violates non-bossiness. Hence, strategyproofness and non-bossiness imply that $\varphi(\bar{R}''_k, R''_{D_j \cup (D_k \setminus \{k\})}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) = \gamma$. Now, note that if $\bar{R}''_k$ is replaced by R''_k, then k is assigned $\lambda(k) = \mu(k)$ by equation (1). Since $\mu(j) P''_k \mu(k)$, or equivalently,

$$\varphi(\bar{R}''_k, R''_{D_j \cup (D_k \setminus \{k\})}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}) P''_k$$
$$\varphi(R''_k, R''_{D_j \cup (D_k \setminus \{k\})}, R'_{H \setminus (D_k \cup D_j)}, \bar{R}_{-M' \cup H}, R_{M'}),$$

this violates strategyproofness. Hence, there exists no $H \subseteq N \setminus (M' \cup M)$ such that $|H| = l + 1$ and $\varphi(R_H, \bar{R}_{-M' \cup H}, R_{M'}) = \mu \neq \varphi(\bar{R}_{-M'}, R_{M'})$. This means that the induction hypothesis holds for $l + 1$ as well.

It has thus been shown by induction that $\varphi(R_G, \bar{R}_{-M' \cup G}, R_{M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$ for all $G \subseteq N \setminus (M' \cup M)$ and all $R_G \in \mathcal{R}^{|G|}$. When $G = N \setminus (M' \cup M)$, this implies that $\varphi(\bar{R}_M, R_{M'}, R'_{-M \cup M'}) = \varphi(\bar{R}_{-M'}, R_{M'})$ for all $R' \in \mathcal{R}^n$. It has already been demonstrated that $\varphi_i(\bar{R}_{-M'}, R_{M'}) = B = A \setminus \bigcup_{j \in M'} \varphi_j(R)$ for all $i \in M$. Hence, there exists some group $M \subseteq N \setminus M'$ such that $\varphi_i(\bar{R}_M, R_{M'}, R'_{-M \cup M'}) = A \setminus \bigcup_{j \in M'} \varphi_j(R)$ for all $i \in M$ and all $R' \in \mathcal{R}^n$. □

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