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# H-infinity Optimal Control for Infinite-Dimensional Systems with Strictly Negative Generator

Carolina Lidström, Anders Rantzer and Kirsten Morris

**Abstract**—A simple form for the optimal H-infinity state feedback of linear time-invariant infinite-dimensional systems is derived. It is applicable to systems with bounded input and output operators and a closed, densely defined, self-adjoint and strictly negative state operator. However, unlike other state-space algorithms, the optimal control is calculated in one step. Furthermore, a closed-form expression for the L2-gain of the closed-loop system is obtained. The result is an extension of the finite-dimensional case, derived by the first two authors. Examples demonstrate the simplicity of synthesis as well as the performance of the control law.

## I. INTRODUCTION

Infinite-dimensional models are often needed when the physical system of interest is both temporally and spatially distributed. For instance, heat conduction systems can be modelled by a parabolic partial differential equation known as the heat equation, see [1] for details on this equation. We consider  $H_\infty$  state feedback control of linear and time-invariant infinite-dimensional systems. The  $H_\infty$  control problem was first formulated for finite-dimensional systems, see [2] and the references therein. There are both state-space based and frequency domain based solutions to the  $H_\infty$  control problem for infinite-dimensional systems, as in the finite-dimensional case. In the frequency domain approach, see [3], one needs to determine the transfer function of the system, which in general can be hard. In the state-space based approach to this problem, the synthesis involves solving an infinite-dimensional operator-valued Riccati equation or inequality, see [4] and [5]. Closed-form solutions are generally hard or not possible to obtain. However, we show that for certain infinite-dimensional systems, it is not only possible to give an analytic solution to the infinite-dimensional operator-valued Riccati inequality, but also the resulting controller has a very simple form.

We consider infinite-dimensional systems with bounded input and output operators and where the state evolves on a separable Hilbert space. Moreover, the state operator is closed, densely defined, self-adjoint and strictly negative.

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Thus, it generates an exponentially stable strongly continuous semigroup. See [6] for further details. We give a simple form for an optimal  $H_\infty$  state feedback law applicable to these systems, given that the state and control input are penalized separately. More specifically, the control law is given by the product of the adjoint of the control input operator and the inverse of the state operator. Furthermore, we provide a closed-form expression for the  $L_2$ -gain of the closed-loop system's transfer function. The result is the analog to the result for finite-dimensional systems derived by the first two authors in [7]. The heat equation is an example of a system to which the derived control law is applicable. Examples are given in Section IV that show the simplicity of synthesis and the performance of the control law.

As mentioned earlier, closed-form solutions of the operator-valued Riccati equation are generally hard or impossible to obtain. Therefore, one common approach is to consider the state-space based synthesis problem for a finite-dimensional approximation of the original system. In this procedure one has to ensure that the controller synthesized for the approximated system stabilizes the original system and also provides performance that approaches optimal as the approximation order increases; see [8]. This can be problematic but there exist conditions under which this approach works, see [9] for  $H_\infty$  state feedback. However, the approximation order can be large and the multiple solutions of the Riccati equation required mean that computation can be intensive. Furthermore, it is difficult to determine the performance degradation resulting from the use of an approximated controller.

The result in this paper is important in several respects. First, for systems with self-adjoint generator to which the result directly applies, it provides an explicit characterization of the optimal controller. No iteration is required. This controller will be approximated in implementation, however the difference between the implemented and exact controller can be calculated. Furthermore, the result may be used in evaluation and benchmarking of algorithms for general systems.

The outline of this paper is as follows. Section II gives some mathematical preliminaries and the notation used. The main theorem is stated in Section III together with its proof. In Section IV, we illustrate the simplicity of synthesis and the performance of the derived control law by means of an example. Section IV also includes some further discussion. Concluding remarks are given in Section V.

## II. MATHEMATICAL PRELIMINARIES

The notations  $\mathbb{R}$  and  $\mathbb{C}$  stand for the set of real and complex numbers, respectively, while the set of nonnegative real numbers is denoted  $\mathbb{R}_+$ . The notation  $\operatorname{Re}(x)$  where  $x \in \mathbb{C}$  denotes the real part of  $x$ . We will only consider linear operators on separable Hilbert spaces, where we denote the inner product and norm by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ , respectively.

The domain of an operator  $T$  is denoted by  $D(T)$ , the adjoint of  $T$  is denoted by  $T^*$  and the inverse of  $T$ , if it exists, is denoted by  $T^{-1}$ . An operator  $T$  is called self-adjoint if  $T^* = T$  and  $D(T^*) = D(T)$ . The set of bounded linear operators from  $\mathcal{X}$  to  $\mathcal{Y}$  is denoted  $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ , and  $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$ . The norm of an operator  $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$  is defined as follows

$$\|T\| = \sup_{\substack{x \in D(T) \\ x \neq 0}} \frac{\|Tx\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}.$$

*Definition 1:* [6, p. 606, Def. A.3.71] A self-adjoint operator  $A$  on the Hilbert space  $\mathcal{Z}$  is nonnegative if  $\langle Az, z \rangle \geq 0$  for all  $z \in D(A)$ ,  $A$  is positive if  $\langle Az, z \rangle > 0$  for all nonzero  $z \in D(A)$  and  $A$  is strictly positive (coercive) if there exists an  $m > 0$  such that

$$\langle Az, z \rangle \geq m\|z\|^2 \text{ for all } z \in D(A).$$

We will use the notation  $A \succ 0$  for strict positivity of the self-adjoint operator  $A$ . We will use the terminology strictly negative denoted  $A \prec 0$  when  $-A \succ 0$ .

*Remark 1:* Let  $\mathcal{Z}$  be a Hilbert space and consider a self-adjoint strictly negative operator  $A$ . It is clear from the definition of strict negativity that  $A$  is injective, thus  $A^{-1}$  exists. Furthermore, it can be shown that it is bounded, positive and  $A^{-1} \in \mathcal{L}(\mathcal{Z})$ . See [6, Ex. A.4.2] for details on this.

*Definition 2:* [6, p. 15, Def. 2.1.2] A strongly continuous semigroup is an operator-valued function  $S(t)$  from  $\mathbb{R}_+$  to  $\mathcal{L}(\mathcal{Z})$  that satisfies the following properties

- 1)  $S(0) = I$ ,
- 2)  $S(t + \tau) = S(t)S(\tau)$  for  $t, \tau \geq 0$ ,
- 3)  $\lim_{t \rightarrow 0, t > 0} S(t)z = z$  for all  $z \in \mathcal{Z}$ .

*Definition 3:* [6, p. 215, Def. 5.1.1] A strongly continuous semigroup,  $S(t)$ , on a Hilbert space  $\mathcal{Z}$  is *exponentially stable* if there exist constants  $M, \alpha > 0$  such that  $\|S(t)\| \leq Me^{-\alpha t}$  for all  $t \geq 0$ .

*Definition 4:* [6, p. 20, Def. 2.1.8] The generator  $A : D(A) \rightarrow \mathcal{Z}$  of a strongly continuous semigroup  $S(t)$  on a Hilbert space  $\mathcal{Z}$  is defined by

$$D(A) = \left\{ z \in \mathcal{X} \mid \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{S(t)z - z}{t} \text{ exists} \right\}$$

$$Az = \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{S(t)z - z}{t} \quad \text{for all } z \in D(A).$$

*Remark 2:* If  $A$  is the generator of a strongly continuous semigroup as in Definition 4, then the domain of  $A$ , i.e.,  $D(A)$ , is dense in  $\mathcal{Z}$  and  $A$  is a closed operator, see [6, p. 21, Th. 2.1.10].

*Lemma 1:* [6, p. 33, Cor. 2.2.3] Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of a strongly continuous semigroup satisfying  $\|S(t)\| \leq e^{wt}$  are:

$$\begin{aligned} \operatorname{Re}(\langle Az, z \rangle) &\leq w\|z\|^2 & \text{for } z \in D(A), \\ \operatorname{Re}(\langle A^*z, z \rangle) &\leq w\|z\|^2 & \text{for } z \in D(A^*). \end{aligned}$$

*Remark 3:* If  $A$  is self-adjoint, then the sufficient condition becomes  $\langle Az, z \rangle \leq w\|z\|^2$  for  $z \in D(A)$ . Furthermore, if  $A$  is strictly negative by Definition 1, the condition clearly holds for some  $w < 0$ . Thus, by Definition 3,  $S(t)$  is exponentially stable. Hence,  $A$  is the generator of an exponentially stable strongly continuous semigroup.

If  $A$  is the generator of a strongly continuous semigroup  $S(t)$  on the Hilbert space  $\mathcal{Z}$ , then for all  $z_0 \in D(A)$ , the differential equation on  $\mathcal{Z}$

$$\frac{dz(t)}{dt} = Az(t), \quad z(0) = z_0,$$

has the unique solution  $z(t) = S(t)z_0$ . Consider an input  $u \in L_2(0, t; \mathcal{U})$ , where  $\mathcal{U}$  is a Hilbert space and  $L_p(\Omega; \mathcal{X})$  is the class of Lebesgue measurable  $\mathcal{X}$ -valued functions  $f$  with

$$\int_{\Omega} |f(t)|^p dt < \infty, \quad p \in [0, \infty].$$

Given  $u$  and an operator  $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ , the differential equation

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \quad z(0) = z_0,$$

has the following solution at any time  $t$

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds.$$

If we consider an output signal

$$y(t) = Cz(t) + Du(t)$$

where  $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$  and  $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ , the output at any time  $t$  given an input  $u$  is

$$y(t) = CS(t)z_0 + C \int_0^t S(t-\tau)Bu(\tau)d\tau + Du(t).$$

The Laplace transform of  $y(t)$  given  $z_0 = 0$  yields the transfer function of the system, denoted  $G$ , as follows

$$\hat{y}(s) = G(s)\hat{u}(s).$$

In what follows, the considered systems are assumed to be causal.

*Definition 5:* [8, p. 10, Def. 2.5] A system is *externally stable* or  *$L_2$ -stable* if for every input  $u \in L_2(0, \infty; \mathcal{U})$ , the output  $y \in L_2(0, \infty; \mathcal{Y})$ . If a system is externally stable, the maximum ratio between the norm of the input and the norm of the output is called the  *$L_2$ -gain*.

Define

$$H_{\infty} = \{G : \mathbb{C}_0^+ \rightarrow \mathbb{C} \mid G \text{ analytic and } \sup_{\operatorname{Re} s > 0} |G(s)| < \infty\},$$

where  $\mathbb{C}_0^+$  are all complex numbers with real part larger than zero, with norm

$$\|G\|_\infty = \sup_{\operatorname{Re} s > 0} \|G(s)\|.$$

The lemma below is stated for systems with finite-dimensional input and output spaces, e.g.,  $\mathcal{U}$  and  $\mathcal{V}$  are  $\mathbb{R}$ , but it generalises to infinite-dimensional ones. The notation  $M(H_\infty)$  stands for matrices with entries in  $H_\infty$ .

*Lemma 2:* [8, p. 10, Def. 2.6] A linear system is externally stable if and only if its transfer function matrix  $G \in M(H_\infty)$ . In this case,  $\|G\|_\infty$  is the  $L_2$ -gain of the system and we say that  $G$  is a *stable transfer function*.

*Definition 6:* [8, p. 10, Def. 2.9] The pair  $(A, B)$  is exponentially stabilizable if there exists a  $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$  such that  $A + BK$  generates an exponentially stable strongly continuous semigroup.

### III. MAIN THEOREM

Consider a linear time-invariant infinite-dimensional system

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Hd(t) \quad (1)$$

where the state  $z(t) \in \mathcal{Z}$  and  $\mathcal{Z}$  is a separable Hilbert space. The operator  $A$  is closed, densely defined, self-adjoint and strictly negative. Then by Lemma 1, a version of the Lumer-Philips Theorem,  $A$  is the generator of an exponentially stable strongly continuous semigroup on  $\mathcal{Z}$ . See Remark 3 for further comments on this statement. The state  $z(t)$  is assumed to be measurable with initial condition  $z(0) = 0$ . Furthermore, the control signal  $u(t) \in \mathcal{U}$  and the disturbance  $d(t) \in L_2(0, \infty; \mathcal{V})$ , where  $\mathcal{U}$  and  $\mathcal{V}$  are Hilbert spaces, and  $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$  and  $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ .

Consider  $H_\infty$  state feedback of (1) given unit cost on the state  $z(t)$  and control input  $u(t)$ , separately, i.e., the cost function is given by

$$\zeta(t) = \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}.$$

Given a stabilizing static state feedback controller  $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ , i.e.,  $u(t) = Kz(t)$ , the closed-loop system from the disturbance  $d(t)$  to the controlled output  $\zeta(t)$  is given by

$$\begin{aligned} \frac{dz(t)}{dt} &= (A + BK)z(t) + Hd(t) \\ \zeta(t) &= \begin{bmatrix} I \\ K \end{bmatrix} z(t) \end{aligned} \quad (2)$$

where  $A + BK$  generates an exponentially stable strongly continuous semigroup. We denote the Laplace transform of the closed-loop system given a controller  $K$  by  $G_K$ , i.e.,

$$\hat{\zeta}(s) = G_K(s)\hat{d}(s).$$

In the following theorem, we give a closed-form expression for a state feedback controller  $K$  that minimizes the  $L_2$ -gain of  $G_K$ . The optimal control law can be considered to be constant without restriction, see [8] for further details to

this statement. The notation  $B^*$  indicates the adjoint of the operator  $B$ .

*Theorem 1:* Consider the system (1) where  $A$  is closed, densely defined, self-adjoint and strictly negative,  $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$  and  $H \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ , where  $\mathcal{Z}$ ,  $\mathcal{U}$  and  $\mathcal{V}$  are Hilbert spaces. Then,  $\|G_K\|_\infty$  is minimized by the state feedback controller  $K_{\text{opt}} = B^*A^{-1}$  and the minimal value of the norm is given by  $\|H^*(A^2 + BB^*)^{-1}H\|^{1/2}$ .

*Proof:* The proof is divided into two parts. In the first part we show that

$$\|G_{K_{\text{opt}}}\| \leq \|H^*(A^2 + BB^*)^{-1}H\|^{1/2}.$$

In the second part of the proof, we show that no controller can achieve strict inequality. Hence, equality holds. In both parts of the proof, we use the following equivalence given by the strict bounded real lemma in infinite dimensions, see [10, Theorem 1.1], applied to (2): Given  $\gamma > 0$  and a controller  $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ , the following two statements are equivalent

- (i)  $A + BK$  generates an exponentially stable strongly continuous semigroup  $T(t)$  on the Hilbert space  $\mathcal{Z}$  and

$$\|G_K\|_\infty < \gamma.$$

- (ii) There exists a self-adjoint, *nonnegative* operator  $\tilde{P} \in \mathcal{L}(\mathcal{Z})$  such that

$$\begin{aligned} (A + BK)^*\tilde{P} + \tilde{P}(A + BK) \\ + I + K^*K + \gamma^{-2}\tilde{P}HH^*\tilde{P} \prec 0. \end{aligned} \quad (3)$$

First, as  $A$  is closed, densely defined, self-adjoint and strictly negative then by Lemma 1,  $A$  is the generator of an exponentially stable strongly continuous semigroup on  $\mathcal{Z}$ , denoted  $S(t)$ . Furthermore, we know that  $(A, B)$  is exponentially stabilizable as  $S(t)$  is exponentially stable. The domain of  $A + BK$ , i.e.,  $D(A + BK)$ , is equal to the domain of  $A$  as  $BK \in \mathcal{L}(\mathcal{Z})$ .

For the first part of the proof consider (ii) and set  $\tilde{P} = -A^{-1}$ ,  $K = K_{\text{opt}} = B^*A^{-1}$  and take any  $\gamma$  with

$$\|H^*(A^2 + BB^*)^{-1}H\|^{1/2} < \gamma.$$

It is possible to set  $\tilde{P} = -A^{-1}$  as  $A$  is self-adjoint and strictly negative, thus  $-A^{-1}$  is self-adjoint, nonnegative and  $A^{-1} \in \mathcal{L}(\mathcal{Z})$ , see Remark 1. Now, we will prove that  $\|G_{K_{\text{opt}}}\|_\infty < \gamma$  by the equivalence between (ii) and (i). First, notice that

$$\tilde{P}(A + BK) = -A^{-1}(A + BB^*A^{-1}) = -I - K^*K.$$

Thus, (3) can be equivalently written as

$$-I - K^*K + \gamma^{-2}A^{-1}HH^*A^{-1} \prec 0. \quad (4)$$

Inequality (4) holds if and only if

$$\begin{bmatrix} I + K^*K & -A^{-1}H \\ -H^*A^{-1} & \gamma^2I \end{bmatrix} \succ 0 \quad (5)$$

by the Schur Complement Lemma for bounded linear operators, see [11, Def. 3.1 and Lem. A.1]. Again, by the same Lemma, inequality (5) is equivalent to

$$\gamma^2 I - H^*(A^2 + BB^*)H \succ 0. \quad (6)$$

where we have used that

$$\begin{aligned} \gamma^2 I - H^* A^{-1} (I + K^* K)^{-1} A^{-1} H \\ = \gamma^2 I - H^* (A^2 + BB^*) H. \end{aligned}$$

Inequality (6) is true by the definition of  $\gamma$ . Hence,  $\|G_{K_{\text{opt}}}\| < \gamma$  by the equivalence between (ii) and (i).

For the second part of the proof, consider again (3). Given a self-adjoint, nonnegative operator  $\tilde{P}$  that solves (3). We can construct a self-adjoint, strictly positive operator  $P_\epsilon \succ 0$  by  $P_\epsilon = \tilde{P} + \epsilon I$ , where  $\epsilon > 0$  is some small real number. Then, we can define

$$\begin{aligned} M_\epsilon = (A + BK)^* P_\epsilon + P_\epsilon (A + BK) \\ + I + K^* K + \gamma^{-2} P_\epsilon H H^* P_\epsilon \end{aligned}$$

and we know that  $M_0 \prec 0$ . Furthermore,

$$\begin{aligned} M_\epsilon = M_0 + \epsilon 2A + \epsilon (K^* B^* + BK) \\ + I + K^* K + \gamma^{-2} (P_\epsilon H H^* P_\epsilon - P_0 H H^* P_0). \end{aligned}$$

The right-hand side of this equality is negative for small  $\epsilon$  as  $2A \prec 0$  and  $K, B, P$  and  $H$  are bounded. Thus,  $M_\epsilon \prec 0$ , i.e., the following holds

$$(A + BK)^* P + P (A + BK) + I + K^* K + \gamma^{-2} P H H^* P \prec 0$$

for some  $P \succ 0$ . This  $P$  is invertible and we can rewrite the inequality further as

$$\begin{aligned} P^{-1} (A + BK)^* + (A + BK) P^{-1} \\ + P^{-2} + P^{-1} K^* K P^{-1} + \gamma^{-2} H H^* \prec 0. \end{aligned}$$

We perform the change of variables

$$(P^{-1}, K P^{-1}) \rightarrow (X, Y),$$

thus  $X \in \mathcal{L}(\mathcal{Z})$  and  $Y \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ , and sum of squares to write the inequality as follows

$$(X + A)^2 + (Y^* + B)(Y^* + B)^* - A^2 - BB^* + \gamma^{-2} H H^* \prec 0.$$

The first two terms of the operator expression are always non-negative and thus no controller can satisfy a bound  $\gamma$  smaller than  $\|H^*(A^2 + BB^*)^{-1}H\|^{\frac{1}{2}}$ . Hence the controller constructed in the first part is optimal and the proof is complete. ■

#### IV. CONTROL OF THE HEAT EQUATION

In this section, we illustrate the simplicity in synthesis of the control law given by Theorem 1. The example concerns control of the heat equation, see (7) below, which describes the distribution of heat, or variation in temperature, in a region over time. The equation also describes other types of diffusion, such as chemical diffusion.



Fig. 1. Rod of length  $l$  with one-dimensional spatial coordinate  $x$ .

Consider the following partial differential equation that models heat propagation in a rod of length  $l$

$$\frac{\partial z}{\partial t}(x, t) = \frac{\partial^2 z}{\partial x^2}(x, t) \quad 0 < x < l, t \geq 0. \quad (7)$$

The temperature at time  $t$  at position  $x$  is  $z(x, t) \in \mathcal{Z} = L_2(0, l)$ . See Figure 1 for a depiction of the rod.

To fully determine the temperature of the rod, the initial temperature profile as well as the boundary conditions have to be specified. As we consider  $H_\infty$  control, the initial temperature is set to zero. We will consider Dirichlet boundary conditions, i.e.,

$$z(0, t) = 0, \quad z(l, t) = 0.$$

Define the operator  $A$  as

$$A = \frac{d^2 z}{dx^2}$$

with domain

$$\begin{aligned} D(A) = \{z \in L_2(0, l) \mid z, \frac{dz}{dt} \text{ locally absolutely continuous,} \\ \frac{d^2 z}{dx^2} \in L_2(0, l) \text{ with } z(0) = 0, z(l) = 0\}. \end{aligned}$$

This operator fulfils the requirements for Theorem 1, i.e., it is closed, densely defined, self-adjoint and strictly negative. For a proof of this see [12, pp. 92-94]. Thus, by Lemma 1,  $A$  generates an exponentially stable strongly continuous semigroup  $S(t)$  on  $L_2(0, l)$ , the state  $z$  evolves on the space  $L_2(0, l)$  and we can write (7) as

$$\dot{z}(t) = Az(t), \quad z(x, 0) = 0.$$

Now, suppose the temperature is controlled by an input  $u(t)$  and affected by a disturbance  $d(t)$  as follows

$$\dot{z}(t) = Az(t) + Bu(t) + Hd(t), \quad z(x, 0) = 0,$$

where  $B, H \in \mathcal{L}(\mathbb{R}, L_2(0, l))$ ,  $u \in L_2(0, \infty; \mathbb{R})$  and the disturbance  $d \in L_2(0, \infty; \mathbb{R})$ . Given the properties stated for the system, Theorem 1 is applicable. We will now, given some explicit examples of operators  $B$  and  $H$ , write down the closed-form expression for the control law given by Theorem 1.

The structure of the optimal control law, i.e.,  $K_{\text{opt}} = B^* A^{-1}$  is not dependent upon the operator  $H$ , as can be seen in Theorem 1. We will only consider

$$(Hd)(x) = d(t) \text{ for all } 0 < x < l.$$

In other words, the disturbance is uniformly distributed along the entire rod. We will treat operators  $B$  defined by

$$Bu = \chi_{[0,\alpha]}(x)u(t) \quad (8)$$

where  $0 < \alpha \leq l$  and

$$\chi_{[0,\alpha]}(x) = \begin{cases} 1 & \text{if } 0 < x < \alpha \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for  $\alpha = l$  the control input is uniformly distributed along the entire rod while for instance for  $\alpha = l/2$  it is only distributed in  $0 < x < l/2$  while it is zero for the remaining part of the rod. The adjoint of operator  $B$  defined in (8) is

$$B^*y(x, t) = \int_0^\alpha y(x, t)dx \text{ for } y \in L_2(0, l).$$

Consider the following equality, as a step towards explicitly stating the optimal control law  $u(t) = K_{\text{opt}}z(x, t) = B^*A^{-1}z(x, t)$ ,

$$z(x, t) = Ay(x, t), \quad y \in D(A).$$

The function  $y(x, t)$  can be written as

$$y(x, t) = \int_0^l G(x, s)z(s, t)ds$$

where

$$G(x, s) = \begin{cases} \frac{(s-l)x}{l} & \text{if } 0 < x < s \\ \frac{s}{l}(x-l) & \text{if } s < x < l \end{cases}$$

is the Green's function of  $A$ . Note that  $G(x, s)$  is piece-wise linear in  $x$  with  $G(0, s) = G(l, s) = 0$ . Now, if  $\alpha = 1$  in (8), then

$$\begin{aligned} u(t) &= B^*A^{-1}z(x, t) = \int_0^l \int_0^l G(x, s)z(s, t) ds dx \\ &= \int_0^l \underbrace{\left[ \int_0^l G(x, s) dx \right]}_{:=f(s)} z(s, t) ds \quad (9) \end{aligned}$$

where

$$f(s) = \frac{s(s-l)}{2}.$$

The control input is thus a weighted integral of the deviation in temperature along the spatial coordinate. The quadratic weight  $f(s)$  determines the scalar signal for controlling the temperature profile, as a compromise between the deviation in temperature from zero and the cost for changing the temperature. The general form of the control signal, i.e., without any specific value on  $\alpha$ , is similarly given by

$$\begin{aligned} u(t) &= \int_0^l \left( \int_0^\alpha G(x, s) dx \right) z(s, t) ds \\ &= \int_0^\alpha f_1(s)z(s, t) ds + \int_\alpha^l f_2(s)z(s, t) ds \end{aligned}$$

where

$$f_1(s) = \frac{s(s-l)}{2} + \frac{s(l-\alpha)^2}{2l} \text{ and } f_2(s) = \frac{\alpha^2}{2l}(s-l).$$

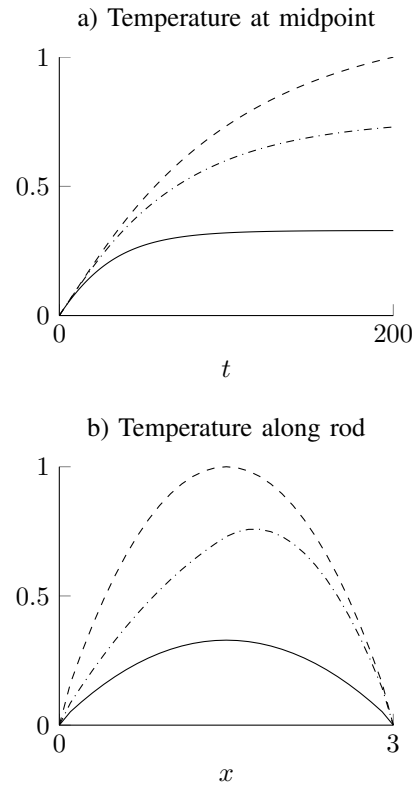


Fig. 2. Response from unit disturbance for system with  $B = 0$  is given by the dashed lines, with  $B$  defined in (8) with  $\alpha = l$  is given by the solid lines and with  $\alpha = l/2$  is given by the dashed dotted lines. a) Temperature at  $x = l/2$  over time, b) temperature along the rod at  $t = 200$ .

The weighting function is altered dependent on if the spatial coordinate is less than or larger than  $\alpha$ , to account for the asymmetry in  $B$ . Notice that  $f_1(s) = f(s)$  and the second integral is zero when  $\alpha = l$  as expected from the firstly derived expression for  $u(t)$  in (9).

Given a constant disturbance  $d(t) = 1$  for  $t \geq 0$ , the state  $z(x, t)$  is determined numerically in MATLAB, see [13], by the finite element method for 200 time steps with interval length 0.01 and spatial segments of length 0.1, with  $l = 3$ . The integrals in the expression of the control law are approximated numerically by the trapezoidal rule.

In Figure 2a, the time trajectory of the temperature at the midpoint, i.e.,  $x = l/2$ , is shown for the control input operators  $B$  defined by (8) given  $\alpha = l$  and  $\alpha = l/2$  as well as  $B := 0$ . Clearly, when  $\alpha = l$  we get the best disturbance attenuation as shown by the solid line. When  $\alpha = l/2$ , the controller is not able to attenuate the disturbance as effectively and of course with  $B = 0$  the system evolves only according to the heat equation with a disturbance. In Figure 2b we show the temperature distribution of the rod at the final time  $t = 200$ . Here one can see that the temperature distribution given with  $\alpha = l/2$  is not symmetric along  $x$ . This is due to that the control input operator  $B$  in this case is asymmetric in  $x$ . The temperature distributions are normalized such that  $z(200, l/2)$  given  $\alpha = 0$  is equal to 1.

## V. CONCLUSIONS

We give a closed-form expression for an optimal  $H_\infty$  state feedback controller applicable to systems with bounded input and output operators and closed, densely defined, self-adjoint and strictly negative state operator. We demonstrate, by means of an example, the simplicity of synthesis of the control law as well as its performance. The control law may be used in evaluation and benchmarking of general purpose algorithms for  $H_\infty$  controller synthesis. Future work includes comparison of a finite-dimensional approximation of the optimal controller to a controller derived by a general purpose algorithm. Further, to investigate possible benefits of having a closed-form expression for an optimal controller in the synthesis of controllers for large scale systems.

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