Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Andersson, Tommy; Ehlers, Lars; Svensson, Lars-Gunnar

2020

Document Version:
Other version

Link to publication

Citation for published version (APA):
Andersson, T., Ehlers, L., & Svensson, L.-G. (2020). *Non-Manipulable House Exchange under (Minimum) Equilibrium Prices*. (Working Papers; No. 2020:28).

Total number of authors:
3

Working Paper 2020:28

Department of Economics
School of Economics and Management

Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Tommy Andersson
Lars Ehlers
Lars-Gunnar Svensson

December 2020
Revised: Nov 2024

LUND UNIVERSITY

Non-Manipulable House Exchange under (Minimum) Equilibrium Prices

Lars-Gunnar Svensson

Department of Economics, Lund University

Tommy Andersson*

Department of Economics, Lund University

Lars Ehlers

Département de Sciences Économiques, Université de Montréal

November 4, 2024

Abstract

We consider a market with indivisible objects, called houses, and monetary transfers. Each house is initially occupied by one agent and each agent demands exactly one house. The problem is to identify the complete set of direct allocation mechanisms that can be used to reallocate the houses among the agents. On the one hand, for price equilibrium mechanisms, we show that the only strategy-proof mechanism is one with a minimum equilibrium price vector. The result is not true on the classical or the quasi-linear domains, but on reduced domains of preference profiles containing "almost all" profiles in the classical or the quasi-linear domain, respectively. On the other hand, while minimum price equilibrium mechanisms are not necessarily efficient (as prices are not zero), we show that no strategy-proof mechanism Pareto dominates a minimum price equilibrium mechanism, making them constrained efficient in the class of strategy-proof mechanisms.

JEL Classification: C71, C78, D47, D71, D78.

Key words: house allocation, initial endowments, minimum equilibrium prices, strategy-proofness, constrained efficiency.

*Corresponding author. Department of Economics, Lund University. Box 7082, 222 07 Lund, Sweden. Email: `tommy.andersson@nek.lu.se`. Phone: +46 (0)46 222 49 70.

1 Introduction

Market design has been a tremendous success. Researchers in the field have repeatedly demonstrated that analytically tractable models can be used to guide practitioners on how to solve important real-life problems involving objects like houses, kidneys, positions, school seats, electromagnetic spectrum, and many other things (see Sönmez, 2023, for a recent overview). The wide range of models can be classified in different dimensions, for example, whether agents have initial endowments or not, or whether prices are attached to the objects or not. This paper analyzes a model where agents have initial endowments and indivisible objects are allocated through price equilibrium mechanisms. Even though this class of models has received relatively little attention compared to, for example, classical exchange models without prices (Shapley and Scarf, 1974) and price equilibrium models without initial endowments (Vickrey, 1961), it has important practical applications. For instance, in the so-called U.S. Incentive Auction (officially known as FCC Auction 1001), previously allocated spectrum was repacked and reallocated via price mechanisms to free up electromagnetic spectrum for wireless communications (Milgrom and Segal, 2017). Another example is the U.K. Housing Act 1980, where eligible tenants are given the right to buy the homes they currently occupy (Andersson, Ehlers, and Svensson, 2016).[1]

One strand of the matching literature has provided a rationale for using equilibrium prices when agents have initial endowments and no monetary transfers are allowed. More precisely, as first demonstrated by Roth and Postlewaite (1977), Gale's top trading cycles mechanism (TTC mechanism, henceforth) selects the unique price equilibrium mechanism on the strict preference domain, and the "no-envy property" implied by price equilibria can justify why specific matchings are chosen in practical applications. In the other strand of the literature, agents have no initial endowments but buy objects at competitive prices. Here, Vickrey's (1961) minimum price equilibrium mechanism (MPE mechanism, henceforth) has been foundational when analyzing

[1]The Housing Act allows existing tenants to buy their homes at discounted, personalized prices below market value. As shown by Andersson, Ehlers, and Svensson (2016), adopting a price equilibrium mechanism would offer several advantages: all tenants would be at least as well off as they are under the current system, the public authority would generate higher revenues, and houses could be reallocated to better match tenants' needs as their circumstances evolve. Similar, though not identical, policies exist in other European countries, including Germany, Ireland, and Sweden. See Housing Europe (2021) for a recent report on the sale of social and public housing.

the price discovery problem on the classical preference domain. We consider price equilibrium mechanisms in a hybrid model of these two strands of the literature: agents have initial endowments (as in Roth and Postlewaite, 1977) and the focus is directed towards price equilibrium mechanisms (as in Vickrey, 1961). We provide a foundation for the MPE mechanism in this hybrid model.

Even though the TTC mechanism (with initial endowments and without monetary transfers) and the MPE mechanism (without initial endowments and with monetary transfers) are fundamentally different, their main properties turn out to be remarkably similar: they are both strategy-proof on their respective domains and they both generate outcomes that are individually rational,[2] and Pareto efficient (Ma, 1994; Roth, 1982; Vickrey, 1961). However, this result is sensitive to specific model assumptions and does not carry over to the model considered in the present paper. The reason is simply that when allowing for monetary transfers (i.e., prices), indifferences prevail. Once both indifferences and initial endowments are allowed, the findings in the above discussed strands of the literature diverge. Price equilibrium without monetary transfers is no longer unique and may be inefficient (Bogomolnaia, Deb, and Ehlers, 2005; Ehlers, 2014; Jaramillo and Manjunath, 2012), and when agents have initial endowments the MPE mechanism is no longer strategy-proof (Andersson, Ehlers, and Svensson, 2016). Together, these results imply that there is no strategy-proof price equilibrium mechanism that selects Pareto efficient outcomes on the classical domain when agents have initial endowments.

In an effort to restore the appeal of price equilibrium mechanisms in such models, Andersson, Ehlers and Svensson (2016) showed that MPE mechanisms are strategy-proof on a reduced domain of preference profiles that contains "almost all" profiles in the classical domain.[3] This result provides a strong argument in favor of using MPE mechanisms also in markets with initial endowments, even though they are manipulable on the classical preference domain. However, before such a recommendation can be made, several open and previously unexplored questions

[2]When there are no initial endowments, individual rationality means that each agent receives a non-negative utility.

[3]More precisely, let the classical and the reduced preference domains be given by $\mathcal{R}$ and $\tilde{\mathcal{R}}$, respectively, where $\tilde{\mathcal{R}} \subset \mathcal{R}$. Andersson and Svensson (2016, Appendix A) provided a measure on subsets of profiles in $\mathcal{R}$, and demonstrated that the set $\mathcal{R} - \tilde{\mathcal{R}}$ has measure zero. Consequently, $\mathcal{R} = \tilde{\mathcal{R}}$ a.e. (almost everywhere). In this sense, the reduced domain $\tilde{\mathcal{R}}$ contains "almost all" profiles in $\mathcal{R}$.

must be addressed. Specifically, reducing the domain of preference profiles might expand the set of strategy-proof price equilibrium mechanisms to include those that are not minimal. This raises a natural question: are there other strategy-proof price equilibrium mechanisms on the considered reduced domain? As demonstrated in this paper, the answer is no, and the reason is that MPE mechanisms uniquely characterize the class of strategy-proof price equilibrium mechanisms on the considered reduced domain. As we show, our foundation holds on a reduced domain of profiles of either the classical domain or the quasi-linear domain.

Here, it should be noted that characterizations of the MPE mechanism on the classical domain have previously been provided in the literature, for example, by Morimoto and Serizawa (2015), in models *without* initial endowments, but the analysis crucially hinges on the classical domain and it is not known whether a similar result can be obtained for the quasi-linear domain (see Sections 1.1 and 6 for additional remarks). To the best of our knowledge, MPE mechanisms have never been characterized in a model *with* initial endowments, and the literature on non-manipulable market design have not provided a foundation of any mechanism where the result does not hold on the full domain but on a reduced domain of preference profiles that contains "almost all" profiles in the full domain. This is the main distinguishing feature of our contribution.

Furthermore, no price equilibrium mechanism is both efficient and strategy-proof in the classical exchange model (recall the above discussions, and see also Hurwicz, 1972). Since our foundation does not explicitly address efficiency, another natural question arises: are there any strategy-proof mechanisms that Pareto dominate an MPE mechanism? As we show, the answer is again no. More precisely, by considering Pareto dominance within the class of strategy-proof mechanisms, we demonstrate that any mechanism, which Pareto dominates a MPE mechanism, must be manipulable. Moreover, on the considered reduced domain where MPE mechanisms remain strategy-proof, no strategy-proof mechanism Pareto dominates a MPE mechanism. Thus, MPE mechanisms are constrained efficient within the class of strategy-proof mechanisms. Similar results have been provided in other types of matching models. In particular, in a standard school choice setting, Abdulkadiroğlu, Pathak, and Roth (2009) and Kesten (2010) demonstrated that no other strategy-proof mechanism Pareto dominates the (student proposing) deferred acceptance algorithm.[4] Furthermore, in a pure exchange model with weak preferences and without transfers, Ehlers (2014) demonstrated that the TTC mechanism with fixed tie-breaking is not Pareto dominated by any strategy-proof mechanism. The latter results are parallel to our third main result where we show that MPE mechanisms cannot be Pareto dominated by a strategy-proof mechanism, so the result holds independently of if monetary transfers are allowed or not.

Our findings thus provide two new and compelling arguments in favor of using MPE mechanisms when agents have initial endowments and when restricting attention to the considered reduced domain where MPE mechanisms are strategy-proof: if attention is limited to strategy-proof price equilibrium mechanisms, MPE mechanisms are the only possible choice, and there is no other strategy-proof mechanism that Pareto dominates them.

1.1 Related Literature

When monetary transfers are allowed, agents become indifferent between consumption bundles at some prices. This causes significant problems for the model considered in this paper, forcing us to operate on a reduced domain to restore (constrained) efficiency and strategy-proofness. These types of problems are not unique to our model. In models without initial endowments where transfers are restricted by upper bounds, certain "indifference chains" destroy incentive properties on the classical domain (Andersson and Svensson, 2016). However, when considering a reduced domain that excludes such "indifference chains," positive results can be obtained in, for example, housing markets with rent control (Andersson and Svensson, 2014) and school choice environments with resource constraints and crowding (Phan, Tierney and Zhou, 2024). Similarly, in models with initial endowments where monetary transfers are not allowed, there are many permissive results on the strict preference domain when considering, for example, "hierarchical exchange rules" (Pápai, 2000) and "exchange rules with brokers" (Pycia and Ünver, 2017). As demonstrated by Ehlers (2002), once indifferences are allowed, also these rules break down and efficiency and strategy-proofness are no longer compatible.

Given the interest in (constrained) efficiency and strategy-proofness, this paper thus con-

[4] Part of the reason is, as shown by Erdil and Ergin (2008), that with equal priorities, no mechanism inherits non-manipulability, stability, and agent-optimality among stable assignments.

tributes to the general matching literature by demonstrating the necessity to define also MPE mechanisms on a reduced domain of preference profiles when agents have initial endowments. Furthermore, we obtain a foundation of MPE mechanisms for "almost all" preference profiles in the classical domain, in contrast to, for example, exchange models without monetary transfers where results related to, for example, efficiency and strategy-proofness break down once indifferences are permitted.

Our analysis of MPE mechanisms with initial endowments naturally relates to the literature on MPE mechanisms without initial endowments. This strand has expanded considerably since Vickrey's (1961) second-price auction with generalizations to more complex market structures and preference domains (see, e.g., Demange and Gale, 1985; Leonard, 1983; Sun and Yang, 2003).[5] Even in these generalized environments without initial endowments, MPE mechanisms continue to satisfy individual rationality, efficiency, and strategy-proofness on the classical domain. They have further been characterized by various axioms, such as those investigated by Miyake (1998), Morimoto and Serizawa (2015, 2018), and Svensson (2009). For instance, Morimoto and Serizawa (2015) demonstrate that an MPE mechanism is characterized by individual rationality, efficiency, strategy-proofness, and no subsidy for losers on the classical domain.[6] The main distinguishing feature in our model is that each agent initially owns one object, which destroys the compatibility of price equilibrium and strategy-proofness on the classical preference domain. Therefore, we must operate on a reduced domain. Our main results show that MPE mechanisms are constrained efficient and are characterized by strategy-proofness in the class of price equilibrium mechanisms, do not follow from the existing literature, simply because these models do not allow for initial endowments. Therefore, previous results cannot be used to infer properties of the MPE mechanism for the considered model and domain. Furthermore, different proof techniques are naturally required when allowing for initial endowments and when operating on a reduced domain (which we must operate on, in sharp contrast to the aforementioned

[5] For recent contributions on one-sided strategy-proofness in trading networks with monetary transfers, see Hatfield, Kojima, and Kominers (2017), Jagadeesan, Kominers, and Rheingans-Yoo (2018), and Schlegel (2018). See also Fleiner et al. (2019) for trading networks with frictions.

[6] The latter axiom ensures that agents who are not assigned an object (i.e., the "losers") are never subsidized. This property prevents uninterested agents from participating in auctions solely to exploit potential subsidies.

papers, to restore constrained efficiency and strategy-proofness).

The literature that is most directly aligned with our framework allows for both price mechanisms and initial endowments. In a seminal contribution, Miyagawa (2001) studied a housing market with monetary transfers, where each agent occupies one house but has the option to buy a more preferred one. He characterized mechanisms that are individually rational, strategy-proof, non-bossy,[7] and onto. The key difference between Miyagawa (2001) and our contribution lies in the imposition of non-bossiness, which implies that the considered mechanisms are characterized by a matrix of fixed prices, that is, prices unrelated to agents' preferences. In this framework, agent a pays p_{ah} when receiving house h, with budget-balance achieved when $p_{ah} = p_h - p_a$ for some price vector p. The assignment is then derived from the TTC mechanism, exactly as in Ma (1994), and TTC with fixed prices and tie-breaking satisfies all properties on the classical domain. Although non-bossiness is an appealing condition, it excludes a variety of useful mechanisms, including MPE mechanisms. As a result, non-bossiness does not preclude strategy-proofness and budget-balance, but it does rule out price equilibrium. In our contribution, price equilibrium and strategy-proofness are fundamental. Consequently, we must forgo non-bossiness and budget-balance. Thus, under fixed prices, allocation inefficiency prevails, while the absence of budget-balance leads to inefficiency in the form of "waste" of money.

1.2 Outline

The remaining part of the paper is organized as follows. Section 2 introduces the formal model together with assumptions and definitions that are used throughout the paper. Minimum equilibrium price mechanisms are introduced and discussed in Section 3. The main characterization theorem is provided in Section 3 together with the foundations for the use of minimum price equilibrium mechanisms on the considered restricted classical and quasi-linear preference domains. Section 5 is devoted to efficiency and considers Pareto dominance among strategy-proof mechanisms. Section 6 concludes the paper. The proof of several key lemmas, as well as the proofs of our main theorems, are relegated to Appendix A.

[7] A mechanism is non-bossy if an agent cannot change the outcome of other agents without altering her own outcome (Satterthwaite and Sonnenschein, 1981).

2 The Model

Let $A = \{1, 2, \ldots, n\}$ denote the finite set of *agents* and $H = \{1, 2, \ldots, n\}$ denote the finite set of *houses*. The endowment of agent $a \in A$ is house $h \in H$ if $h = a$. We consider a market where endowments are reallocated through a system of prices, i.e., a market where each agent can buy a more preferred house. In this market, there are two types of prices, a vector of *fixed* selling prices $\underline{p} \in \mathbb{R}^n_+$ and a vector $p \in \mathbb{R}^n_+$ of buying (equilibrium) prices. We define $\mathbb{R}^n_+$ to be the set of *feasible* price vectors.

The endowments can be interpreted in different ways, but here we use an interpretation that is consistent with a proposed extension of the U.K. Housing Act 1980 (see Andersson, Ehlers and Svensson, 2016, for all details). There is an owner of the houses, for example, a governmental authority, different from the agents in A, while each agent $a \in A$ rents house $h = a$. The owner would like to sell the houses to the group of renting agents, but not necessarily house $h = a$ to agent a. Agent a has a particular "right to stay" in the house she is renting: she can continue to rent her "own" house, but she has also the option to buy her "own" house for the fixed price $\underline{p}_a$. The fixed price vector $\underline{p}$ defines the owner's reservation prices, i.e., the lower bound on the feasible prices. Since the lower bound is fixed in the analysis, without loss of generality, we let $\underline{p}_h = 0$ for all $h \in H$.[8] The interpretation of the endowments is that continuing to rent or to buy the "own" house is a fixed alternative, in contrast to buying another house where the price that the agent has to pay depends on the market valuation. In a reallocation process, *individual rationality* means that an agent never is assigned an alternative that is worse than the most preferred of her two fixed alternatives.[9]

A reallocation of the endowments is denoted by *assignment* μ, which is a bijection $\mu : A \to H$. We sometimes write $\mu = (\mu_1, \ldots, \mu_n)$ where μ_a denotes the house obtained by agent a.

[8] For any positive reservation price $\underline{p}_h > 0$, we simply set $\underline{p}_h$ to be the "zero" when transferring house h to an agent distinct from agent h.

[9] For notational simplicity, this most preferred fixed alternative will be denoted by $(a, p) \equiv (a, 0)$ for each agent $a \in A$ since the price is zero for the agent independently of if the agent continues to rent or buy the house she currently occupies (since she chooses her most preferred option among the two). The most preferred option is then represented by $(a, 0)$ in the preferences. This is discussed in detail in Andersson, Ehlers and Svensson (2016, Lemma 1) where they show that, on the reduced domain defined in Section 4.1, it suffices to minimize the number of agents who keep their initial endowments.

The cardinality of μ indicates the number of agents who are not keeping their endowment, $|\mu| = |\{a \in A : \mu_a \neq a\}|$. A *state* is a pair $x = (\mu, p)$ of an assignment and a feasible price vector $p \in \mathbb{R}^n_+$. Here, $x_a = (\mu_a, p)$ means that agent a is assigned house μ_a at price p_{μ_a} in the price vector p. Let $\mathcal{X}$ denote the set of all states.

Each agent $a \in A$ possesses rational *preferences* R_a on houses and prices, i.e., R_a is complete and transitive on bundles of type $(h, p_h) \in H \times \mathbb{R}_+$. As usual, $(h, p_h)I_a(h', p_{h'})$ if and only if $[(h, p_h)R_a(h', p_{h'})$ and $(h', p_{h'})R_a(h, p_h)]$ and $(h, p_h)P_a(h', p_{h'})$ if and only if $[(h, p_h)R_a(h', p_{h'})$ and not $(h', p_{h'})R_a(h, p_h)]$. This means that preferences represent indirect preference over houses and prices, which is equivalent to consider preferences $(h, \beta_a - p_h)$ over houses and money where β_a denotes agent a's monetary endowment. To simplify notation, we also let $(h, p) \equiv (h, p_h)$.

Preferences are further assumed to be strictly *monotonic*, i.e., $(h, p'_h)P_a(h, p_h)$ if $p'_h < p_h$, for all houses $h \in H \backslash \{a\}$, while *constant* for the own house, i.e., $(a, p'_a)I_a(a, p_a)$ for all $p, p' \in \mathbb{R}^n_+$. The reason for assuming price independence of the own house is simply that an agent a always pays the reservation price $\underline{p}_a = 0$ for the own house independently of the values of the buying prices p. Finally, preferences are assumed to be *continuous* and *boundedly desirable.* Continuity means that for all $h \in H$, the sets $\{p_h \in \mathbb{R}_+ : (h, p_h)\ R_a\ (h, p'_h)\}$ and $\{p_h \in \mathbb{R}_+ : (h, p'_h)\ R_a\ (h, p)\}$ are closed for all $p'_h \in \mathbb{R}_+$. Bounded desirability means that if the price of a house is "sufficiently high," the agents will strictly prefer to keep the house they are currently living in rather than buying some other house, i.e., $(a, p_a)P_a(h, p_h)$ for each agent $a \in A$ and for each house $h \neq a$ for p_h "sufficiently high." Note that we do not exclude the case where an agent a does not demand a particular house $h \in H \backslash \{a\}$ for any price p_h, i.e., $(a, p_a)\ P_a\ (h, p_h)$ for all $p_h \in \mathbb{R}_+$.

For agent a, the set of rational, monotonic, continuous, and boundedly desirable preferences on $H \times \mathbb{R}_+$ is denoted $\mathcal{R}_a$, and called classical preferences. A (preference) *profile* is a list $R = (R_a)_{a \in A}$ of agents' preferences. The set of profiles is denoted by $\mathcal{R}$, where $\mathcal{R} = \times_{a \in A} \mathcal{R}_a$, and is referred to as the *classical preference domain*. The notation $\mathcal{R}_{-a}$ is used for the set $\mathcal{R}_{-a} = \times_{a' \in A \backslash \{a\}} \mathcal{R}_{a'}$.

As any agent $a \in A$ owns house $h = a$, individual rationality requires that any agent weakly prefers her bundle to her endowment. Given $R \in \mathcal{R}$, state $x = (\mu, p)$ is *individually rational* if for all $a \in A$, $x_a R_a (a, 0)$. As preferences are monotonic and $p \geq \underline{p} = 0$, this implies for all $a \in A$, $(\mu_a, 0) R_a (a, 0)$. We call an assignment individually rational under R if the latter condition holds for all agents $a \in A$. Let $\mathcal{A}_R$ denote the set of individually rational assignments for profile R, and let $\mathcal{X}_R$ denote the set of individually rational states for profile R.

We next define mechanisms together with concepts related to manipulability.

Definition 1. *A mechanism is a mapping $f : \mathcal{R} \to \mathcal{X}$ of profiles to states such that $f(R) \in \mathcal{X}$ for all $R \in \mathcal{R}$.*

A mechanism f is *manipulable* at profile $R \in \mathcal{R}$ by agent $a \in A$ if there is a profile $(R'_a, R_{-a}) \in \mathcal{R}$ such that for $f(R) = x$ and $f(R'_a, R_{-a}) = x'$, $x'_a P_a x_a$. Let $\breve{\mathcal{R}} \subset \mathcal{R}$ be a subset of profiles. The mechanism f is *strategy-proof* on domain $\breve{\mathcal{R}}$ if no agent can manipulate at any profile $R \in \breve{\mathcal{R}}$. Note that if f is strategy-proof on domain $\breve{\mathcal{R}}$ and $(R_a, R_{-a}) \in \breve{\mathcal{R}}$, then agent a cannot manipulate by using any preferences R'_a with $(R'_a, R_{-a}) \in \mathcal{R}$.

Finally, a mechanism f is individually rational if for any profile R, $f(R)$ is individually rational. Note that for $f(R) = x = (\mu, p)$, this implies $\mu \in \mathcal{A}_R$.

3 (Minimum) Price Equilibrium Mechanisms

In the previous section, states and mechanisms were allowed to choose prices in an arbitrary way for individually rational assignments. We next consider markets where houses are allocated through equilibrium prices, that is, where any agent can buy a most preferred house at the given (equilibrium) prices.

Definition 2. *For $R \in \mathcal{R}$, a state $x = (\mu, p)$ is a weak price equilibrium if (i) for all $a \in A$, $x_a R_a (h, p)$ for all $h \in H$ and (ii) $\mu_a = a$ and $p_a > 0$ only if $x_{a'} I_{a'} (a, p)$ for some $a' \neq a$. If, in addition, the cardinality of the assignment μ is maximal among all states with price vector p satisfying (i) and (ii), then x is a price equilibrium.*

Part (i) is the usual price equilibrium condition, that is, at prices p each agent has been assigned a most preferred alternative. This implies that the chosen state is individually rational as then $x_a R_a (a, p) I_a (a, 0)$. Part (ii) avoids trivial price vectors. Since an agent's welfare of her "own" house does not depend on the price of her house, any sufficiently high price would be an equilibrium price without (ii) when an agent prefers her "own" house to all other houses. That trade is socially preferred to no trade is reflected by the last part in the definition. This condition does not directly influence agents' welfare (as prices remain unchanged) but when there is an external owner of the houses and the agents are renting, then the profit of the owner weakly increases when a house is sold to a "non-renting" agent compared to when it is sold to a "renting" agent.

For a given profile $R \in \mathcal{R}$, the set of price equilibria is denoted by $\mathcal{E}_R$ and the set of corresponding equilibrium price vectors by Π_R. Hence, $p \in \Pi_R$ precisely when there is a state $(\mu, p) \in \mathcal{E}_R$. Moreover, the set of all price equilibriums is denoted $\mathcal{E}$, where $\mathcal{E} = \cup_{R \in \mathcal{R}} \mathcal{E}_R$. The sets $\mathcal{E}_R$ are nonempty.[10] Finally, a state $(\mu, p) \in \mathcal{E}_R$ is a *maximum trade equilibrium*, if $|\mu| \geq |\mu'|$ for all $(\mu', p') \in \mathcal{E}_R$. The requirement on a maximum trade equilibrium is that the number of agents who keep their endowment is minimal among *all* equilibrium price vectors.

We will demonstrate that a price equilibrium mechanism, which is strategy-proof, must be a minimum price equilibrium (MPE) mechanism. The result is not true on the classical preference domain $\mathcal{R}$. Instead, we consider a reduced domain of preference profiles which contains "almost all" profiles in $\mathcal{R}$, denoted by $\tilde{\mathcal{R}}$ where the result holds. Before this result can be formally stated, MPE mechanisms need to be defined.

Definition 3. *A price equilibrium mechanism is a mapping $f : \mathcal{R} \to \mathcal{E}$ of profiles to price equilibriums such that $f(R) \in \mathcal{E}_R$ for all $R \in \mathcal{R}$.*

The use of a minimum price equilibrium mechanism will be central in the main characterization result. Let $R \in \mathcal{R}$ and denote by $p^m \in \Pi_R$ a price vector that is minimal in Π_R, i.e., if $p \in \Pi_R$ and $p \leq p^m$ then $p = p^m$.

Definition 4. *A price equilibrium mechanism f is a minimum price equilibrium (MPE) mechanism on the domain $\breve{\mathcal{R}} \subset \mathcal{R}$ if for all $R \in \breve{\mathcal{R}}$, $f(R) = x \in \mathcal{E}_R$ and $x = (\mu, p^m)$ with p^m minimal in Π_R.*

Andersson, Ehlers and Svensson (2016, Example 2 and Proposition 2) demonstrated that if the number of agents is at least four, a minimal equilibrium price vector is not necessarily unique on

[10] See, for example, Proposition 1 in Andersson, Ehlers and Svensson (2016).

the domain $\mathcal{R}$. The multiplicity of a minimum equilibrium price vector is a direct consequence of the possibility for agents to "block" the trade of a house through their outside options to stay in the houses they are currently living in. As a consequence, any MPE mechanism is manipulable on the domain $\mathcal{R}$. For later purposes, we include the example below.

Example 1. Let $A = H = \{1, 2, 3, 4\}$ and $\underline{p} = (0, 0, 0, 0)$. For each agent $a \in A$, preferences over bundles (h, p) are represented by a quasi-linear utility function $u_{ah}(p) = v_{ah} - p_h$ for $h \neq a$ and $u_{aa}(p) = v_{aa}$ where the values v_{ah} are given by real numbers:

$$V = (v_{ah})_{a \in A, h \in H} = \begin{pmatrix} 0 & -2 & 0 & -2 \\ -2 & 0 & 0 & -2 \\ 2 & -2 & 0 & 1 \\ -2 & 2 & -2 & 1 \end{pmatrix}.$$

Let the profile $R \in \mathcal{R}$ denote the preferences that are represented by the above values.

In this case, both $p' = (1, 0, 0, 0)$ and $p'' = (0, 1, 0, 0)$ are minimum equilibrium price vectors. This follows since both $x' = (\mu', p')$ and $x'' = (\mu'', p'')$ are equilibrium states for $\mu' = (1, 3, 4, 2)$ and $\mu'' = (3, 2, 1, 4)$, respectively. At state x', agent 1 buys her house at its reservation price 0, and at state x'', agents 2 and 4 buy their houses at the reservation price 0. Hence, $|\mu'| > |\mu''|$, i.e., the minimum price equilibrium x'' is not a maximum trade equilibrium. It is easy to see that x' and x'' are the only minimum price equilibriums for profile R.[11]

Agent 3's utility at state x' is 1 whereas agent 3's utility at state x'' is 2. Similarly, agent 4's utility at state x' is 2 whereas agent 4's utility at state x'' is 1. In other words, agents 3 and 4 have opposed preferences over x' and x''. Due to this, and as we will demonstrate, if x' is chosen, agent 3 can profitably manipulate whereas when x'' is chosen, agent 4 can profitably manipulate.

[11] Consider the state $\hat{x} = (\hat{\mu}, \hat{p}) \in \mathcal{E}_R$ and suppose that $\hat{p}$ is a minimum equilibrium price vector. We first demonstrate that $\hat{p} = p' = (1, 0, 0, 0)$ or $\hat{p} = p'' = (0, 1, 0, 0)$. Since p' and p'' are minimum equilibrium price vectors, it is clear that either $\hat{p}_1 < 1$ or $\hat{p}_2 < 1$ because $\hat{p}$ is a minimum equilibrium price vector. Suppose that $\hat{p}_1 < 1$. Then $\hat{\mu}_3 = 1$ since $\hat{x}$ is a price equilibrium. Consequently, $\hat{\mu}_1 = 3$, $\hat{p}_3 = 0$ by individual rationality for agent 1, and it then follows that $\hat{\mu}_2 = 2$ by individual rationality for agent 2. But then it must be the case that $\hat{\mu}_4 = 4$ and $\hat{p}_2 \geq 1$, because otherwise agent 4 will envy agent 2 at state $\hat{x}$. Hence, $\hat{p} \geq p''$. But then $\hat{p} = p''$ and $\hat{x} = x''$, by definition, since $\hat{p}$ is a minimum equilibrium price vector by assumption. Analogous arguments can be used to show that $\hat{p} = p'$ if $\hat{p}_2 < 1$.

Let now f be minimum price mechanism on domain $\mathcal{R}$. Then $f(R)$ chooses either x' or x''. If $f(R) = x'$, then $\mu' = (1, 3, 4, 2)$ and agent 3's utility is equal to $v_{34} - p'_4 = 1$. Let R' denote the profile of quasi-linear preferences where the entry v_{32} in the matrix V is replaced by $v'_{32} = 2$. Obviously, $x' \notin \mathcal{E}_{R'}$ because $(2, p'_2)P'_3 x'_3$. On the other hand, it is easy to check that $x'' \in \mathcal{E}_{R'}$. Also, p'' is the unique minimum equilibrium price vector at profile R'. To see this, suppose that $\hat{x} = (\hat{\mu}, \hat{p}) \in \mathcal{E}_{R'}$ and that $\hat{p} \neq p''$ is a minimum equilibrium price vector at profile R'. Then $\hat{p}_2 < 1$, which implies that $\hat{\mu}_4 = 2$ and $\hat{\mu}_3 = 1$. But then individual rationality cannot be satisfied for both agents 1 and 2 at state x''. Thus, p'' must be chosen by $f(R')$. Then, by individual rationality for agents 1 and 2, it follows that agent 3 must receive house 1. Because $R' = (R'_3, R_{-3})$ and agent 3's utility from $(1, p'')$ under R_3 is equal to $v_{31} - p''_1 = 2 > 1$, agent 3 can profitably manipulate f at R.

If $f(R) = x''$, it can be demonstrated, by using identical arguments as in the above, that agent 4 can manipulate the mechanism by replacing the entry v_{41} in the matrix V by $v'_{41} = 2$. □

4 Foundation

4.1 Classical Preferences

The main reason that MPE mechanisms can be manipulated in Example 1 is that houses 1 and 2 are "connected by indifference" via agents 1 and 2 when the prices are zero, i.e., that $(1, 0)I_1(3, 0)I_2(2, 0)$. By considering a reduced domain $\tilde{\mathcal{R}}$ where no two houses are "connected by indifference,"[12] Andersson, Ehlers and Svensson (2016, Theorem 1 and Proposition 3) showed that a minimum equilibrium price vector is unique and achieves maximum trade. This uniqueness also destroys the possibility for agents to strategically "block trades" and, as a consequence, MPE mechanisms are strategy-proof on the domain $\tilde{\mathcal{R}}$ (Andersson, Ehlers and Svensson, 2016, Theorem 2).[13] Our first main result establishes that MPE mechanisms completely characterize

[12] This concept was first used by Andersson and Svensson (2014), but they used a slightly different version compared to the one used in this paper. See also Phan, Tierney and Zhou (2024) for additional and more recent discussions of the concept.

[13] Note that their result even allows for group manipulations which may result in preference profiles which do not belong to $\tilde{\mathcal{R}}$.

the class of strategy-proof price equilibrium mechanisms on the domain $\tilde{\mathcal{R}}$. Before stating this result formally, the reduced domain $\tilde{\mathcal{R}}$ needs to be defined and analyzed in somewhat greater detail.

Let $\mathcal{S}$ be the set of sequences $s = (h_j, a_j)_{j=1}^{r}$ of distinct houses $h_j \in H$ and distinct agents $a_j \in A$ such that $h_1 = a_1$, $h_j \neq a_j$ for all j such that $1 < j \leq r$ and $h_{j+1} \neq a_j$ for $j < r$ and $2 \leq r \leq n$.

Definition 5. *For a given profile $R \in \mathcal{R}$, two distinct houses, $h', h'' \in H$, are connected by indifference if there is a sequence $s \in \mathcal{S}$, and a corresponding price vector $p \in \mathbb{R}^n_+$, such that $h' = a_1$ and $h'' = a_r$, and $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < r$ and $(h_r, p) I_{a_r} (a_r, p)$. The subset of $\mathcal{R}$ where no two houses are connected by indifference, at any profile, is denoted by $\tilde{\mathcal{R}}$ and called the NCBI (Not-Connected-By-Indifference) domain.*

Note that for the type of price vectors considered in Definition 5, the prices p_{h_j} (with $1 < j \leq r$) are uniquely determined by continuity and monotonicity of the preferences.

We next informally argue that the subset of profiles that is removed from the classical domain $\mathcal{R}$ when constructing $\tilde{\mathcal{R}}$, that is, the profiles in the set $\mathcal{R} - \tilde{\mathcal{R}}$, is a "negligible set."[14] In this sense, the domain $\tilde{\mathcal{R}}$ contains "almost all" preference profiles in $\mathcal{R}$. For this purpose, let $R \in \mathcal{R}$ be a profile and $a \in A$ an agent, and denote by $\mathcal{R}^{con}_{aR}$ the set of preference relations $R'_a \in \mathcal{R}_a$ such that there are two houses $h', h'' \in H$, with $h'' = a$, that are "connected by indifference" at the profile (R'_a, R_{-a}). From Definition 5, it now follows that $\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R}'$, where:

$$\mathcal{R}' = \{R \in \mathcal{R} : R_a \in \mathcal{R}^{con}_{aR} \text{ for some } a \in A\}.$$

We can think of a profile in $\mathcal{R}'$ as the outcome in two steps of the natural lottery, where, for some agent a, the first outcome is R_{-a} and the second is $R_a \in \mathcal{R}^{con}_{aR}$. Now, if $\mathcal{R}^{con}_{aR}$ can be considered negligible, we can also consider $\mathcal{R}'$ negligible since there is only a finite number of agents and houses.

[14] The formal arguments for this claim are very similar to the ones provided by Andersson and Svensson (2016, Appendix A). There, a measure on subsets of profiles in $\mathcal{R}$ is provided, and it is demonstrated that the set of preference profiles where two houses are "connected by indifference" has measure zero. For the quasi-linear domain, see Section 4.2.

We next argue that $\mathcal{R}_{aR}^{con}$ can be considered negligible. Note first that if $R'_a \in \mathcal{R}_{aR}^{con}$, there is a sequence $s = (h_j, a_j)_{j=1}^q$ and a corresponding price vector $p \in \mathbb{R}_+^n$ such that $h' = a_1$, $h'' = a_q = a$, $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < q$ and $(h_q, p) I_a (a, p)$. The price p_{h_q} is uniquely determined by the profile R, but independent of preferences R_a. Let now preferences R'_a be represented by utility functions u'_{ah}, where $u'_{ah}(p)$ is the agent's willingness-to-pay for house $h \in H$. The indifference $(h_q, p) I'_a (a, p)$ prevails if and only if $u'_{aa}(p) = p_{h_q}$. When preferences are chosen by nature, it is reasonable to assume that $u'_{aa}(p) \neq p_{h_q}$ is the case for most preferences $R_a \in \mathcal{R}_a$, also as agent a's utility is constant for her "own" house at all prices. Since there is only a finite number of sequences $s \in \mathcal{S}$, the set $\mathcal{R}_{aR}^{con}$ is negligible. It then follows, by the above arguments, that $\mathcal{R}'$ is also considered negligible.

Our first main result shows that MPE mechanisms completely characterize the class of strategy-proof price equilibrium mechanisms on the domain $\tilde{\mathcal{R}}$.

Theorem 1. *Let f be a price equilibrium mechanism. Then f is strategy-proof on $\tilde{\mathcal{R}}$ if and only if f is an MPE mechanism on $\tilde{\mathcal{R}}$.*

Theorem 1 provides a foundation of the MPE mechanism on a subdomain of classical preference profiles, a domain that contains "almost all" preference profiles in the classical domain. In other words, even though MPE mechanisms are manipulable on the classical domain, this impossibility is not robust as it only holds on a negligible set of preference profiles. The further important difference to the literature of strategy-proof mechanisms is that once an impossibility is established, the analysis usually stops and no robustness checks are normally conducted. In contrast, our findings demonstrate that on a reduced preference domain that contains "almost all" profiles in the classical domain, any strategy-proof price equilibrium mechanism must select minimum equilibrium prices. That is, any such mechanism must be a MPE mechanism and by adopting it on the reduced domain, the impossibility result "vanishes."

In addition, the result in Theorem 1 may be used to support a (normative) definition of fairness. In a market model with private ownership, net trades are considered fair primarily because no agent "envies" any other agent's net trade. In general, however, no-envy is not sufficient as a fairness criterion. First, no-envy is not sufficient for obtaining a unique allocation and second, fairness should reasonably be based on agent's true preferences. According to Theorem 1, the outcome of the minimum price equilibrium mechanism is envy-free and non-manipulable. It is also the only price equilibrium mechanism that satisfies those two conditions. Hence, the outcome of the minimum price equilibrium mechanism is a strong candidate for a definition of (procedural) fairness.

4.2 Quasi-Linear Preferences

We next show that the result in Theorem 1 carries over to quasi-linear preferences. For each agent $a \in A$, preferences over bundles (h, p) are quasi-linear if there exist real numbers $(v_{ah})_{h \in H}$ such that R_a is represented by the utility function $u_{ah}(p) = v_{ah} - p_h$ for $h \neq a$ and $u_{aa}(p) = v_{aa}$. Let $\mathcal{Q}_a \subset \mathcal{R}_a$ denote the set of all quasi-linear preferences, and let $\mathcal{Q} = \times_{a \in A} \mathcal{Q}_a$ denote the set of profiles with quasi-linear preferences. Let now $\tilde{\mathcal{Q}} = \mathcal{Q} \cap \tilde{\mathcal{R}}$ and $\mathcal{Q}' = \mathcal{Q} \cap \mathcal{R}'$. Then $\tilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}'$. Below, we illustrate that $\mathcal{Q}'$ is a negligible subset of profiles with quasi-linear preferences. In this sense, $\tilde{\mathcal{Q}}$ contains "almost all" profiles in the quasi-linear domain $\mathcal{Q}$.

Note that any $R_a \in \mathcal{Q}_a$ has a representation of values $(v_{ah})_{h \in H}$ and adding the same constant to all these values induces the same quasi-linear preferences. We will use the canonical representation of R_a where $v_{aa} = 0$. Using this convention, $\mathcal{Q}_a$ corresponds to $\mathbb{R}^{n-1}$ and $\mathcal{Q}$ to $\mathbb{R}^{n(n-1)}$.

Let $R \in \mathcal{Q}$ be a profile of quasi-linear preferences. Suppose that houses h_1 and h_{q+1} are "connected by indifference," i.e., that there exist sequences of distinct agents $(a_1, \ldots, a_q)$ and distinct houses $(h_1, \ldots, h_{q+1})$, and a price vector p such that:

(i) $h_1 = a_1$ and $h_{q+1} = a_q$,

(ii) $v_{a_1 h_1} = v_{a_1 h_2} - p_{h_2}$ and $v_{a_q h_q} - p_{h_q} = v_{a_q h_{q+1}}$,

(iii) $v_{a_j h_j} - p_{h_j} = v_{a_j h_{j+1}} - p_{h_{j+1}}$ for $2 \leq j \leq q - 1$.

Summing all left-hand sides and all right-hand sides yields:

$$\sum_{j=1}^{q} (v_{a_j h_j} - v_{a_j h_{j+1}}) = 0. \tag{1}$$

Note that condition (1) is independent of the price vector p and that it represents a hyperplane in $\mathbb{R}^{n(n-1)}$ with "measure zero" in $\mathcal{Q}$. As the set of houses and the set of sequences is finite, it follows that $\mathcal{Q}'$ has measure zero in $\mathcal{Q}$. Hence, $\mathcal{Q}'$ is negligible (and $\tilde{\mathcal{Q}} = \mathcal{Q} - \mathcal{Q}'$). Furthermore, one may reduce each agent's preferences such that each profile belongs to $\tilde{\mathcal{Q}}$.[15]

Theorem 1 remains unchanged for the quasi-linear domain, that is, we obtain a foundation for MPE mechanisms on the subdomain of preference profiles $\tilde{\mathcal{Q}}$ that contains "almost all" profiles in the quasi-linear domain $\mathcal{Q}$.

Theorem 2. *Let f be a price equilibrium mechanism defined on $\mathcal{Q}$. Then f is strategy-proof on $\tilde{\mathcal{Q}}$ if and only if f is an MPE mechanism on $\tilde{\mathcal{Q}}$.*

We emphasize that Theorem 1 does not imply Theorem 2, as it is restricted to quasi-linear preferences, nor does Theorem 2 imply Theorem 1, as it is restricted to classical preferences. Nonetheless, in Appendix A, we present a unified proof of our two main results, explicitly constructing quasi-linear preferences.

5 Efficiency

In the context of assigning objects to agents with monetary transfers, a longstanding debate has focused on achieving efficiency alongside non-manipulability (see, e.g., Andersson and Svensson, 2014; Demange and Gale, 1985; Holmström, 1979; Sun and Yang, 2006). Our contribution to this literature demonstrates that, in a model with monetary transfers and initial endowments, MPE mechanisms are constrained efficient within the class of strategy-proof mechanisms. Although our main result (Theorem 3) holds on the general domain, we primarily consider quasi-linear preferences, specifically the domain $\mathcal{Q}$.

The strongest efficiency notion maximizes the sum of agents' utilities subject to individual rationality.

[15]Let $\mathbb{Q}$ denote the set of all rational numbers. For any agent $a \in A$, let $\mathcal{R}_a^q$ consist of all quasi-linear utility functions where $v_{ah} \in \mathbb{R}\backslash\mathbb{Q}$ for all $h \in H$ and $v_{ah} - v_{ah'} \in \mathbb{R}\backslash\mathbb{Q}$. Then it is easy to verify that $\mathcal{R}_1^q \times \cdots \times \mathcal{R}_n^q \subset \tilde{\mathcal{R}}$. This follows because the left-hand side belongs to $\mathbb{R} \setminus \mathbb{Q}$ in condition (1).

Definition 6 (Utilitarian efficiency). *Given $R \in \mathcal{Q}$ (where R_a is represented by u_a), a state $x \in \mathcal{X}_R$ is utilitarian efficient if there exists no state $x' \in \mathcal{X}_R$ such that $\sum_{a \in A} u_a(x') > \sum_{a \in A} u_a(x)$.*

Utilitarian efficiency implies that the sum of agents' values is maximized among individually rational assignments and if $\mu_a \neq a$, then $p_{\mu_a} = 0$. Hence, if x is utilitarian efficient, then x is welfare-equivalent to $(\mu, 0)$ where all prices are equal to zero.

As usual, for three or more agents strategy-proofness and utilitarian efficiency are incompatible on a non-negligible set of preference profiles. We omit the straightforward proof.[16]

Proposition 1. *For $|A| \geq 3$, any utilitarian efficient mechanism is manipulable on a non-negligible subset of the quasi-linear domain.*

Given the impossibility result in Proposition 1, we consider constrained efficiency among strategy-proof mechanisms, and more precisely Pareto dominance among mechanisms. Let f and g be two mechanisms. Then f (Pareto) dominates g, denoted by $f \gtrdot g$, if for all $R \in \mathcal{Q}$ we have $f_a(R) R_a g_a(R)$ for all $a \in A$, and for some $R \in \mathcal{Q}$ we have $f_a(R) P_a g_a(R)$ for some $a \in A$. Now, given a subdomain $\breve{\mathcal{Q}} \subseteq \mathcal{Q}$, we write $f \gtrdot |_{\breve{\mathcal{Q}}} g$, if the above holds for the subdomain $\breve{\mathcal{Q}}$ (instead of $\mathcal{Q}$).

Obviously, any MPE mechanism dominates any other price mechanism on the NCBI domain of quasi-linear preferences $\tilde{\mathcal{Q}}$ (as then there exists a unique minimum price). We are interested in Pareto dominance among strategy-proof mechanisms, and more precisely whether any MPE mechanism is undominated in the class of strategy-proof mechanisms. When there are at least four agents, the next results shows that no strategy-proof mechanism Pareto dominates an MPE mechanism as any such mechanism must be manipulable.

Proposition 2. *Let $|A| \geq 4$. If mechanism $\hat{f}$ dominates an MPE mechanism, then $\hat{f}$ is manipulable on the domain $\mathcal{Q}$.*

[16] Suppose that there are three agents, denoted by 1, 2, and 3. Assume also that agent 2 values house 1 at 100 and house 3 at -2, agent 3 values house 1 at 90 and house 2 at -1, whereas agent 1 values house 2 at 2 and house 3 at 3. Then utilitarian efficiency requires to choose $((2,1,3),0)$ (where $(2,1,3)$ denotes the chosen assignment and 0 denotes zero prices) but when agent 3 changes her report and values house 1 at 200 and house 2 at -1, utilitarian efficiency requires to choose $((3,2,1),0)$, which means that agent 3 can profitably manipulate. Note that these values can be varied continuously without changing these conclusions, i.e., the mechanism is manipulable on a non-negligible subset of the quasi-linear domain.

Proof. Let f be an MPE mechanism. Since f is individually rational and $\hat{f} \gtrdot f$, it follows that $\hat{f}$ is individually rational. Consider the profile R from Example 1. Then either $f(R) = x' = (\mu', p')$ where $\mu' = (1, 3, 4, 2)$ and $p' = (1, 0, 0, 0)$, or $f(R) = x'' = (\mu'', p'')$ where $\mu'' = (3, 2, 1, 4)$ and $p'' = (0, 1, 0, 0)$. Let $\hat{f}(R) = \hat{x} = (\hat{\mu}, \hat{p})$.

First, suppose that $f(R) = x'$. By $\hat{f} \gtrdot f$, $\hat{x}_a R_a x'_a$ for all $a \in A$. As agent 4's utility from x' is equal to 2 and $(2, 0)$ is the unique bundle with price zero that maximizes her preference, it follows that $\hat{x}_4 = (2, 0)$. Since agent 3's utility from x' is equal to 1, it follows from individual rationality of $\hat{x}$ that $\hat{x}_3 = (4, 0)$, $\hat{x}_2 = (3, 0)$ and $\hat{\mu}_1 = 1$, i.e., $\hat{\mu} = \mu'$ and $\hat{x}_a I_a x'_a$ for all $a \in A$. Now, consider a similar manipulation as in Example 1 where agent 3 changes the entry v_{31} in matrix V to $v'_{31} = 3$ and the entry v_{32} in matrix V to $v'_{32} = 2$, and for the obtained profile $R' = (R'_3, R_{-3})$ we have $f(R') = x''$ and $f_3(R') = (1, 0)P_3(4, 0)$. Because $\hat{f}_3(R')R'_3 f_3(R')$, it must be $\hat{f}_3(R') = (1, 0)$ since $(1, 0)$ is the unique bundle with price zero that maximizes R'_3, i.e., $\hat{f}_3(R')P_3\hat{f}_3(R)$ and $\hat{f}$ is manipulable.

Second, suppose that $f(R) = x''$. By $\hat{f} \gtrdot f$, $\hat{x}_a R_a x''_a$ for all $a \in A$. As agent 3's utility from x'' is equal to 2 and $(1, 0)$ is the unique bundle with price zero that maximizes her preference, it follows that $\hat{x}_3 = (1, 0)$. Since agent 4's utility from x'' is equal to 1, it follows from individual rationality of $\hat{x}$ that $\hat{x}_1 = (3, 0)$, $\hat{\mu}_2 = 2$ and $\hat{x}_4 = (4, 0)$, i.e., $\hat{\mu} = \mu''$ and $\hat{x}_a I_a x''_a$ for all $a \in A$. Now, consider a similar manipulation as in Example 1 where agent 4 changes the entry v_{41} in matrix V to $v'_{41} = 2$ and the entry v_{42} in matrix V to $v'_{42} = 3$, and for the obtained profile $R' = (R'_4, R_{-4})$ we have $f(R') = x'$ and $f_4(R') = (2, 0)P_4(4, 0)$. Because $\hat{f}_4(R')R'_4 f_4(R')$, it must be $\hat{f}_4(R') = (2, 0)$ since $(2, 0)$ is the unique bundle with price zero that maximizes R'_4, i.e., $\hat{f}_4(R')P_4\hat{f}_4(R)$ and $\hat{f}$ is manipulable. □

We next consider a domain which is slightly smaller than the NCBI domain by excluding also indifference chains where two houses are "connected by indifference" at the reservation prices zero.

Definition 7. *For a given profile $R \in \mathcal{Q}$, two houses $h', h'' \in H$ with prices zero are connected by indifference if there is a sequence $s \in \mathcal{S}$, and a corresponding price vector $p \in \mathbb{R}^n_+$, such that $p_{h'} = 0$ and $p_{h''} = 0$, $(h_j, p)I_{a_j}(h_{j+1}, p)$ for $1 \leq j < r$ and $(h_r, p)I_{a_r}(h'', 0)$, and either*

$[h' \neq h'']$ or $[h' = h''$ and the sequence s contains at least two agents$]$. The subset of $\mathcal{Q}$ where no two houses with prices zero are connected by indifference, at any profile, is denoted by $\mathring{\mathcal{Q}}$ and called the $\underline{NCBI}$ domain.

Note that whereas NCBI rules out any indifference chains between two endowments, $\underline{NCBI}$ rules out indifference chains between two houses with reservation prices zero. As any agent is indifferent among all prices for her endowment, $\underline{NCBI}$ implies NCBI and $\mathring{\mathcal{Q}} \subseteq \tilde{\mathcal{Q}}$. Furthermore, no agent is indifferent between two houses at prices zero, i.e., if $R \in \mathring{\mathcal{Q}}$, then for all $a \in A$ and all distinct $h, h' \in H$, we have either $(h, 0)P_a(h', 0)$ or $(h', 0)P_a(h, 0)$. Using similar arguments as in Section 4.2, it follows that $\underline{NCBI}$ excludes a negligible set of preference profiles from the quasi-linear domain $\mathcal{Q}$.

Strict preferences are typically considered in models where objects are allocated without prices, with fixed (or discrete) prices, or in matching with contracts (see, e.g., Abdulkadiroğlu, Pathak and Roth, 2009; Abdulkadiroğlu and Sönmez, 1999; Hatfield and Milgrom, 2005; Ma, 1994; Shapley and Scarf, 1974). The question is if there exist strategy-proof mechanisms which dominate an MPE mechanism on the $\underline{NCBI}$ domain (where MPE mechanisms are strategy-proof). Related problems have been considered in other contexts such as school choice and random assignment.[17] To the best of our knowledge, no such result has been obtained in the context considered in this paper where agents have initial endowments and objects are assigned using non-discrete prices.

Theorem 3. *On the $\underline{NCBI}$ domain $\mathring{\mathcal{Q}}$, no strategy-proof mechanism dominates an MPE mechanism.*

We end this section by discussing three weaker efficiency notions: two where efficiency is required for the chosen assignment and one where efficiency is required for the chosen prices.

[17] In school choice, the deferred acceptance algorithm is inefficient, but Abdulkadiroğlu, Pathak and Roth (2009) show that no strategy-proof mechanism dominates it (see also Kesten, 2010). This finding is further generalized by Alva and Manjunath (2019) for arbitrary strategy-proof mechanism for the assignment of objects without monetary transfers and without initial endowments. For the assignment of objects without transfers and outside options, Erdil (2014) has shown that the random serial dictatorship mechanism is dominated in the class of strategy-proof mechanisms.

Definition 8 (Utilitarian assignment efficiency). *Given $R \in \mathcal{Q}$ (where R_a is represented by $(v_{ah})_{h \in H}$), a state $x = (\mu, p) \in \mathcal{X}_R$ is utilitarian assignment efficient if there exists no assignment $\mu' \in \mathcal{A}_R$ such that $\sum_{a \in A} v_{a\mu'_a} > \sum_{a \in A} v_{a\mu_a}$.*

Definition 9 (Price efficiency). *Given $R \in \mathcal{Q}$, a state $x = (\mu, p)$ is price efficient if there exists no price $p' \geq 0$ such that $(\mu_a, p')R_a(\mu_a, p)$ for all $a \in A$ and $(\mu_a, p')P_a(\mu_a, p)$ for some $a \in A$.*

Definition 10 (Assignment efficiency). *Given $R \in \mathcal{Q}$, a state $x = (\mu, p)$ is assignment efficient if there exists no assignment μ' such that $(\mu'_a, p)R_a(\mu_a, p)$ for all $a \in A$ and $(\mu'_a, p)P_a(\mu_a, p)$ for some $a \in A$.*

Obviously, utilitarian efficiency is equivalent to both utilitarian assignment efficiency and price efficiency, and utilitarian assignment efficiency implies assignment efficiency.

For the quasi-linear domain, any minimum price equilibrium $x = (\mu, p)$ satisfies utilitarian assignment efficiency when all houses with prices zero are (strictly) better than the initial endowment, i.e., when $(h, 0)P_a(a, 0)$ for all $h \neq a$,[18] but violates price efficiency whenever for some agent a we have $\mu_a \neq a$ and $p_{\mu_a} > 0$.

Requiring price efficiency means that the price of any house is zero. Suppose now that $p_h = 0$ for all $h \in H$, and let $\underline{H} = \{(h, 0) : h \in H\}$. For all $R \in \mathring{\mathcal{Q}}$, any R_a induces a strict (ordinal) preference on $\underline{H}$ (as indifferences among houses with prices zero are ruled out), which is the reduction $R_a|_{\underline{H}}$. Let $R|_{\underline{H}} = (R_a|_{\underline{H}})_{a \in A}$. Gale's top trading cycles mechanism (TTC mechanism, henceforth) may then be used to allocate the houses (with prices zero) among the agents. Call this the zero-price mechanism, i.e., for all $R \in \mathring{\mathcal{Q}}$, we have $f^0(R) = (\mu, 0)$ where μ is the outcome of the TTC mechanism when applied to $R|_{\underline{H}}$.

Informally, the TTC mechanism works as follows. For any profile, each agent "points" to her most preferred house with price zero. Because the set A is finite, there must be at least one (top)

[18] Let $R \in \mathring{\mathcal{Q}}$, $x = (\mu, p)$ be a minimum price equilibrium, and $\mu' \in \mathcal{A}_R$ (where, by assumption, now all assignments are individually rational). Then for all $i \neq h$, $v_{ih} > 0$. As (μ, p) is an MPE, there exists a house $\hat{h} \in H$ such that $p_{\hat{h}} = 0$. But now for all $i \neq \hat{h}$, it follows that $v_{i\mu_i} - p_{\mu_i} \geq v_{i\hat{h}} - p_{\hat{h}} = v_{i\hat{h}} > 0$, which implies $\mu_i \neq i$ (as $v_{ii} = 0$ by convention). For $j = \hat{h}$, we have $v_{j\mu_j} - p_{\mu_j} \geq v_{jh} - p_{jh}$ for all $h \in H$ independently of whether $\mu_j = \hat{h}$ or $\mu_j \neq \hat{h}$ since $p_{\hat{h}} = 0$. Then for all $i \in A$, we have $v_{i\mu_i} - p_{\mu_i} \geq v_{i\mu'_i} - p_{\mu'_i}$. Then, by taking sums over all agents, it follows that $\sum_{i \in A} v_{i\mu_i} \geq \sum_{i \in A} v_{i\mu'_i}$, which is the desired conclusion.

cycle and for each top cycle agents trade their initial endowments as specified by the cycle. The houses of these trading cycles are deleted from the preferences of the remaining agents and the same procedure is applied to the remaining agents and their reduced preferences, and so on.[19]

For any $R \in \mathcal{Q} \backslash \mathring{\mathcal{Q}}$, we may break indifferences in $R|_{\underline{H}}$ and apply again the TTC mechanism for the obtained strict preferences and set prices equal to zero. Now, zero-price mechanisms satisfy individual rationality, price efficiency, strategy-proofness (Roth, 1982), and assignment efficiency on the $\underline{NCBI}$ domain. Using Ma's (1994) result, it follows that any mechanism satisfying individual rationality, strategy-proofness, price efficiency and assignment efficiency on $\mathring{\mathcal{Q}}$ must coincide with the zero-price mechanism on the domain $\mathring{\mathcal{Q}}$.

Obviously, the zero price vector almost never coincides with equilibrium prices. If this is the case for some profile $R \in \mathring{\mathcal{Q}}$, then any agent must obtain her most preferred house, i.e., for $f^0(R) = (\mu, 0)$, for any agent $a \in A$ we have $(\mu_a, 0) P_a (h, 0)$ for all $h \neq \mu_a$, and the TTC mechanism would terminate after Step 1. Furthermore, if a public agency owns the objects (as in the U.K. Housing Act interpretation of the model), the agency would make zero profit (as prices are zero) and zero-price mechanisms have unbounded utilitarian assignment efficiency loss.[20] This is also the reason why, in our setting, no mechanism is strategy-proof if resale is allowed. From Andersson and Svensson (2016, footnote 7), it follows that any such mechanism must then have prices zero, but then again as shown above, the chosen assignment is typically utilitarian inefficient and "bribes" can be used to reallocate objects afterwards.[21,22] This is likely also the reason why the U.K. Housing Act prohibits resale for ten years after having acquired the public

[19]Formally, the TTC mechanism is defined as follows for $R|_{\underline{H}}$ (where we write h instead of $(h, 0)$ as all prices are zero). In Step 1, each agent $i \in A$ points to her most preferred house $top(R_i)$ in A. Then there exists at least one cycle $i_1 - i_2 - \cdots - i_t$ (where $top(R_{i_l}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$ and $top(R_{i_t}) = i_1$) and for any such cycle we set $f_{i_l}(R) = top(R_{i_l})$ for all $l \in \{1, \ldots, t\}$. Let C_1 denote the set of agents assigned in Step 1 and $N_1 = A \backslash C_1$. In Step $k+1$, each agent $i \in N_k$ points to her most preferred house $top(R_i|_{N_k})$ in N_k. Then there exists at least one cycle $i_1 - i_2 - \cdots - i_t$ (where $top(R_{i_l}|_{N_k}) = i_{l+1}$ for $l \in \{1, \ldots, t-1\}$ and $top(R_{i_t}|_{N_k}) = i_1$) and for any such cycle we set $f_{i_l}(R) = top(R_{i_l}|_{N_k})$ for all $l \in \{1, \ldots, t\}$. Let C_{k+1} denote the set of agents assigned in Step $k+1$ and $N_{k+1} = N_k \backslash C_{k+1}$. Stop when $N_k = \emptyset$.

[20]For instance, for three agents let $v_1 = (0, 2, 1)$, $v_2 = (2, 0, 1)$ and $v_3 = (100, 1, 0)$. Then $f^0(R) = ((2, 1, 3), 0)$, whereas any MPE mechanism chooses $((2, 3, 1), (1, 0, 0))$. Now by increasing v_{31} to infinity, one can see that the utilitarian assignment efficiency loss is unbounded.

[21]In the example in footnote 20, agent 3 may bribe agent 1 to report $(0, -1, -1)$ and afterwards to swap their initial endowments, making both agents better off. To avoid resale, the government would have to tax resale prices by 100 per cent which would then make agent 1 not to resell her house.

[22]Similar findings have been reported by Demange and Gale (1985, pp. 875–876) and Schummer (2000, p.307).

house. Similar policies have been enforced in other European countries, see Housing Europe (2021).

Remark 1. In a model with initial endowments and without monetary transfers, Roth and Postlewaite (1977) have shown that the outcome of the TTC mechanism chooses the unique price equilibrium for strict preferences: for any profile, higher prices are attached to houses in cycles executed earlier and prices within one cycle are identical. Then in a cycle, for any agent, the price of her house and the one assigned to her are identical. Nevertheless, prices can always be lowered infinitesimally without changing the price equilibrium conditions, and unless all agents have a different most preferred house, a minimum price equilibrium does not exist. For weak preferences where indifferences with the endowment are excluded, the NCBI condition holds but by applying the TTC mechanism with fixed tie-breaking, we always obtain a strategy-proof price equilibrium mechanism. Hence, uniqueness is lost as different tie-breaking results in different price equilibrium mechanisms (and again no MPE mechanism exists). This is in contrast to our foundation of the MPE mechanism on the NCBI domain where we obtain both uniqueness and existence. □

6 Conclusions

In a house allocation model with monetary transfers and initial endowments, we answered the fundamental question of characterizing the entire class of strategy-proof price equilibrium mechanisms. The main result showed that there is a domain that contains "almost all" preference profiles in the classical domain, such that the investigated class of price equilibrium mechanisms contains only one mechanism, namely the minimum price equilibrium mechanism. We further demonstrated that no strategy-proof mechanism Pareto dominates a minimum price equilibrium mechanism. All our main results hold on the classical domain as well as on the quasi-linear domain.

The paper has, in similarity with, e.g., Demange and Gale (1985) and Sun and Yang (2003), considered the classical preference domain. This domain is very natural if agents are risk averse, budget constrained, or experience wealth effects (see, e.g., Alaei, Kamal and Azarakhsh, 2016;

Baisa, 2017). However, because it may be difficult for agents to report non-quasi-linear preferences to a direct allocation mechanism, it is likely that an iterative version of the mechanism proposed in this paper is more transparent and easier to implement in real-life (another advantage of iterative mechanisms is that they do not necessarily require full preference revelation). A natural direction for future research is therefore to identify an iterative version of the direct price equilibrium mechanism considered in this paper. A first starting place for searching for such a mechanism are Andersson and Svensson (2018) and Morimoto and Serizawa (2015).

Zero-price mechanisms can be put into the framework of Morimoto and Serizawa (2015) without initial endowments: there are more agents than houses, i.e., $n = |A| > |H| = m$, $A = \{1, \dots, n\}$ and $H = \{h_1, \dots, h_m\}$. But then choose the first m agents from A, and let agent i "own" house h_i and apply the zero-price mechanism for those agents together with their initial endowments, and the other agents always receive $(0, 0)$, i.e., the "null object" (or outside option) at price zero. This mechanism satisfies in their main result (Theorem 2) strategy-proofness, individual rationality and no subsidy for losers (which means that payments shall be non-negative), and efficiency if transfers are required to be non-negative for allocations (which shows that their Theorem 2 does not hold if transfers are required to be non-negative for feasible allocations). It does not satisfy their efficiency notion in their context as transfers are allowed to be negative, one may make certain agents better off by giving them the null object together with a negative transfer (i.e., to "bribe" them to accept the null object).

Furthermore, for the quasi-linear domain without initial endowments, Vickrey-Clarke-Groves mechanisms (VCG mechanisms, henceforth) play an important role in the literature. Those mechanisms choose an allocation which maximizes the sum of the agents' utilities and determines individual payments such that truthful reporting is a weakly dominant strategy. On the one hand, VCG mechanisms are strategy-proof on the classical domain of quasi-linear preferences whereas MPE mechanisms are only strategy-proof on the domain of profiles where no two houses are "connected by indifference." On the other hand, VCG mechanisms ignore individual rationality (and an agent might be worse off compared to keeping her "own" house) whereas MPE mechanisms are individually rational, and VCG-prices might have to be negative.

A final remark is that our analysis is based on the assumption that there are equally many agents and houses. If there are more agents than houses, our analysis remains true by introducing a number of "dummy houses" (one for each agent not owning a house and with zero value to the agents) are introduced to the model to compensate for the difference. The case when there are strictly more houses than agents is more difficult to analyze, but we conjecture that the main results continue holds. The latter situation is unlikely to occur in real-life applications as there is usually a shortage of public housing and there are more agents than houses.

A Appendix: Proofs

This appendix contains the proofs of Theorems 1–3. Additional definitions, results, and lemmas will be introduced to facilitate the proofs.

A.1 Minimum Price Condition

Lemma 1 is an important consequence of a condition called the "minimum price condition" while Lemma 2 is a characterization of minimum price vectors.

Definition 11. *Let $R \in \tilde{\mathcal{R}}$ be a profile and $x = (\mu, p)$ a weak price equilibrium at the profile R. Then the state x satisfies the minimum price* (MP) *condition if for each nonempty set $S \subset \{h \in H : p_h > 0\}$, there is a house $h \in S$ and an agent $a \in A$, $a \neq h$, such that $\mu_a \notin S$ and $x_a I_a (h, p)$.*

Note that the MP condition is not satisfied at x if $p_h > 0$ for all $h \in H$.

Definition 12. *Let $R \in \mathcal{R}$ be a profile and $x = (\mu, p)$ and $x' = (\mu', p)$ two weak price equilibria where $p \in \Pi_R$. A sequence $(a_j)_{j=1}^{t+1}$ of agents $a_j \in A$, such that $a_j \neq a_{j+1}$ for $j \leq t$ but $a_{t+1} = a_1$, is called a* trading cycle *at x if $h_j = \mu_{a_j}$ and $\mu'_{a_j} = h_{j+1}$ for all j. If, in addition, also $h_{j+1} \neq a_j$ for all j and $h_j = a_j$ for some j, the trading cycle is* strong.

Clearly, if $(a_j)_{j=1}^{t+1}$ is a strong trading cycle at x, then x cannot be a price equilibrium since, in that case, trade cannot be maximal at all weak price equilibriums at prices p.

Lemma 1. *Let $R \in \tilde{\mathcal{R}}$ be a profile and x and x' two weak price equilibria, where the corresponding price vectors satisfy: $p, p' \in \Pi_R$ and $p' \leq p$, $p' \neq p$. Assume that x satisfies the* MP *condition. Then there exists a strong trading cycle $(a_j)_{j=1}^{t+1}$ at state x.*

Proof. Let $H' = \{h \in H : p'_h < p_h\}$ and $H'' = H - H'$. Then, $H' \neq \emptyset$ since $p' \neq p$. Furthermore, $H'' \neq \emptyset$ since $H'' = \emptyset$ means that $p_h > p'_h$ for all h, and hence, $p_h > 0$ for all h, which is not consistent with the MP condition (for $S = H$).

Let now $h_j = \mu_{a_j}$ and define a first part $(a_j)_{j=1}^{k}$ of the sequence $(a_j)_{j=1}^{t+1}$, where $h_j \in H'$ for all $1 \leq j < k \leq t$ while $h_k \in H''$, in the following way:

- Let h be an arbitrary house in H' and consider a sequence $(h'_i)_{i=1}^{k'}$, where $h'_1 = h$ and h'_i, $i < k'$, are different houses in H', while $h'_k \in H''$. Also let $(a'_i)_{i=1}^{k'}$ be the corresponding sequence of agents, where $\mu_{a'_i} = h'_i$. Further, the sequence has to satisfy: for each $q < k'$ and set $\{h'_i\}_{i=1}^{q}$, $x_{a'_{q+1}} I_{a'_{q+1}} x_{a'_j}$ for some $j \leq q$ and $h'_j \neq a'_{q+1}$. The sequence $(h'_i)_{i=1}^{k'}$ is obtained recursively in the following way:

- Let $h'_1 = h$. If we have obtained the sequence for q houses, i.e., the sequence $\{h'_i\}_{i=1}^{q}$, then, according to the MP condition, there is a house $h'_r \in \{h'_i\}_{i=1}^{q}$ and an agent, say a'_{q+1}, with $a'_{q+1} \neq h'_r$, such that $\mu_{a'_{q+1}} \notin \{h'_i\}_{i=1}^{q}$ and $x_{a'_{q+1}} I_{a'_{q+1}} (h'_r, p)$. Then let $h'_{q+1} = \mu_{a'_{q+1}}$. If $h'_{q+1} \in H''$ stop, and let $k' = q + 1$, otherwise continue. The sequence stops at some time k' since H is finite. Note that h'_{q+1} cannot stop in H' since the MP condition implies that $H'' \neq \emptyset$.

- The sequence $(h'_i)_{i=1}^{k'}$ clearly contains a subsequence $(h'_{i_j})_{j=1}^{k}$ such that $(a'_{i_j})_{j=1}^{k-1}$ are different agents and $x_{a'_{j+1}} I_{a'_{j+1}} x_{a'_j}$ for $1 \leq j < k$. Then, define the sequence $(h_j)_{j=1}^{k}$ as $h_j = h'_{i_j}$ and $a_j = a'_{i_j}$.

We next define the second part $(a_j)_{j=k}^{l}$ of the sequence $(a_j)_{j=1}^{t+1}$. To do this, denote the houses associated with $(a_j)_{j=k}^{l}$ by $(h_j)_{j=k}^{l}$, and note that $h_k \in H''$ by the above construction. Let now $a_{k+1} \in A$ be given by $\mu'_{a_{k+1}} = \mu_{a_k} = h_k \in H''$. Continue to define a_j, $j \geq k+1$ in a similar way, i.e., $\mu'_{a_{j+1}} = \mu_{a_j} = h_j \in H''$. The sequence ends at h_l if $h_l \in H'$.

Before continuing to define the sequence $(a_j)_{j=1}^{t+1}$, we note that $a_l \neq a_i$ for all i, $k \leq i < l$. To see this, assume that $a_l = a_i$ for some i, $k \leq i < l$. Then $i = k$ because of the rule $\mu'_{a_{j+1}} = \mu_{a_j}$. Furthermore, $x_{a_l} I_{a_l} x_{a_{l-1}}$ since $p'_{h_{l-1}} = p_{h_{l-1}}$. But then $x'_{a_l} I_{a_l} x_{a_l}$. Further, $x_{a_k} I_{a_k} x_{a_{k-1}}$, $p'_{h_{k-1}} < p_{h_{k-1}}$ and $a_k \neq h_{k-1}$ so, by monotonicity, $(h_{k-1}, p') P_{a_k} x_{a_k}$. But then $x'_{a_k} P_{a_k} x_{a_k}$, contradicting $x'_{a_l} I_{a_l} x_{a_l}$ when $a_l = a_k$. Hence, this case cannot prevail.

Note next that since the sequence $(h_j)_{j=k}^{l}$ ends at $h_l \in H'$, by construction, we can expand the sequence $(h_j)_{j=1}^{l}$ to a sequence $(h_j)_{j=1}^{k'}$, $k' > l$, in the same way as $(h_j)_{j=1}^{k}$ was constructed, where $h_j \in H'$ for $l \leq j < k'$ and $h_{k'} \in H''$. Moreover, all h_j are different for $1 \leq j \leq k$ and $l \leq j \leq k'$ by the construction.

Further expansion to $(h_j)_{j=1}^{l'}$, is obtained by the rule $\mu'_{a_{j+1}} = \mu_{a_j} = h_j \in H''$, for $k' \leq j \leq l'$. In this way, we further continue the expansion to a sequence $(h_j)_{j=1}^{r}$. The expansion of the sequence is stopped at the first agent a_r such that $a_r = a_i$ for some $i < r$. Then we have a cycle $(a_j)_{j=i}^{r}$ where all agents are different for $i \leq j < r$ and $a_r = a_i$. In addition, $x_{a_j} I_{a_j} x_{a_{j-1}}$, for $i < j \leq r$, and $x_{a_1} I_{a_1} x_{a_r}$. Moreover, the sequence satisfies:

1. If $h_j \in H'$ and $h_{j-1} \in H''$, then, by monotonicity, $h_j = \mu_{a_j} = a_j$.

2. $h_j \neq a_{j-1}$ for all j, since if $h_{j-1} = a_{j'}$ for some j' then $h_{j'}$ and $h_{j''}$ are connected by indifference where j'' satisfies $h_{j''} \in H'$ and $h_{j-1} \in H''$. This is not consistent with the "not connected by indifference condition."

Finally, given points 1 and 2 above, and after a renumbering, the cycle $(a_j)_{j=i}^{r}$ constitutes a strong trading cycle. $\square$

Lemma 2. *Let $R \in \tilde{\mathcal{R}}$ be a profile. A price vector p is minimal in Π_R, if and only if, for each equilibrium $(\mu, p) \in \mathcal{E}_R$, the* MP *condition holds.*

Proof. We first prove that the MP condition is a necessary condition. For this purpose, let $x = (\mu, p) \in \mathcal{E}_R$ be a price equilibrium and suppose that the MP condition is not satisfied at x. Then there is a nonempty set $S \subset \{h \in H : p_h > 0\}$ such that there is no $h \in S$ and $a \in A$, with $h \neq a$ and $\mu_a \notin S$, such that $x_a I_a (h, p)$. This means that all agents $a \in A$, with $\mu_a \notin S$ and $a \neq h$, strictly prefer x_a to (h, p) for all $h \in S$. On the other hand, for $a = h$ the utility of h

is independent of p_h. Then there is a price vector $p' \in \Pi_R$ such that $p' \leq p$, $p' \neq p$ (Alkan, Demange and Gale, 1991). Hence, p cannot be a minimum price vector in Π_R.

We next prove that the MP condition is a sufficient condition. Suppose that the MP condition is satisfied at a price equilibrium $x = (\mu, p) \in \mathcal{E}_R$ but that $p \in \Pi_R$ is not minimal in Π_R. Then there is a price equilibrium $x' = (\mu', p') \in \mathcal{E}_R$ such that $p' \leq p$, $p' \neq p$. Then, according to Lemma 1, there is a strong trading cycle $(a_j)_{j=1}^{t+1}$. But then trade cannot be maximal at x. To see this, let a state x'' be defined as: for all $a \notin \{a_j\}_{j=1}^{t}$ let $x''_a = x_a$, and for $a \in \{a_j\}_{j=1}^{t}$ let $x''_{a_{j+1}} = x'_{a_j}$ for $1 \leq j \leq t$. It then follows directly from Definition 12 that x'' is a weak price equilibrium and that trade is larger at x'' than at x. This is a contradiction to x being a price equilibrium. Hence, the MP condition is sufficient for p being a minimum price vector in Π_R. □

A.2 Proof of Theorems 1 and 2

First, by Andersson, Ehlers and Svensson (2016, Theorem 2), MPE mechanisms are strategy-proof on the domain $\tilde{\mathcal{R}}$.

Second, in showing the converse, by Andersson, Ehlers and Svensson (2016, Theorem 1) for any profile in $\tilde{\mathcal{R}}$ there exists a unique minimum equilibrium price vector.

We will use the following. Let $R \in \tilde{\mathcal{R}}$ be a profile and $x^m = (\mu^m, p^m)$ be a minimum price equilibrium under R. Let R_a be represented by u_a where $u_{ah}(p)$ for h is agent a's willingness to pay for house h under p. Let $\overline{u}_a = u_{a\mu_a^m}(p^m)$ denote agent a's utility from x^m and $\underline{u}_a = u_{aa}(p^m)$. Now for $v \in (\underline{u}_a, \overline{u}_a]$ let R_a^v be the preferences represented by utility function u_a^v such that (i) $u_{aa}^v(p) = v$ and (ii) $u_{ah}^v(p) = u_{ah}(p)$ for all $h \neq a$. Note that for all $a \in A$ such that $\underline{u}_a < \overline{u}_a$ we have:

$$\mu_a^m \neq a. \tag{2}$$

given R_a and $h \in H \backslash \{a\}$, let $r_a(h) \in \mathbb{R}$ be such that $(h, r_a(h)) I_a (a, 0)$ (if $(h, 0) R_a (a, 0)$) and $r_a(h) = -a$ (if $(a, 0) P_a (h, 0)$). Note that $r_a(h)$ is uniquely defined. Now if $R \in \tilde{\mathcal{R}}$, then for all

$h \in H$ and all distinct $a, a' \in A \backslash \{h\}$ we have:

$$r_a(h) \neq r_{a'}(h). \tag{3}$$

Now R_a^v changes the value of agent a's house to v while keeping all other values unchanged. It follows that $(R_a^v, R_{-a}) \in \tilde{\mathcal{R}}$ for almost all $v \in (\underline{u}_a, \overline{u}_a]$.

Lemma 3. *Let $R \in \tilde{\mathcal{R}}$ and let p^m be a minimal vector in Π_R. Then, for all $a \in A$ (where R_a is represented by u_a) such that $\underline{u}_a < \overline{u}_a$, $R' = (R_a^v, R_{-a}) \in \tilde{\mathcal{R}}$ for all $v \in (\underline{u}_a, \overline{u}_a]$, except for a finite set of values.*

Proof. Let $R \in \tilde{\mathcal{R}}$ and $a \in A$ be $\underline{u}_a < \overline{u}_a$. Let $v \in (\underline{u}_a, \overline{u}_a]$ and $R' = (R_a^v, R_{-a})$. Let $h', h'' \in H$ be two distinct houses and $s \in \mathcal{S}$ a sequence of distinct houses and agents. Let $p \in \mathbb{R}_+^n$ be a corresponding price vector such that $h' = a_1$ and $h'' = a_r$, and $(h_j, p) I_{a_j} (h_{j+1}, p)$ for $1 \leq j < r$. Thus, the houses h' and h'' are connected by indifference if and only if $(h_r, p) I_{a_r} (a_r, p)$. For $R \in \tilde{\mathcal{R}}$ this indifference cannot prevail (and for any sequence s which does not contain a).

Consider now the profile R'. Note that all prices in the sequence s are uniquely determined by R_{-a} (independently of whether a belongs to s or not). Now if a belongs to the sequence s, then by the above construction of the preference relation R_a^v we have:

(i) $a = a_j$ with $j \neq 1, r$, $u_{a_j h_j}(p) = u^v_{a_j h_j}(p) = u^v_{a_j h_{j+1}}(p) = u_{a_j h_{j+1}}(p)$, or

(ii) $a = a_1$, $v = u^v_{a_1 h_2}(p) = u_{a_1 h_2}(p)$, or

(iii) $a = a_r$, $v = u^v_{a_r h_r}(p) = u_{a_r h_r}(p)$.

Note that (i) cannot prevail as otherwise houses a_1 and a_r are not connected by indifference under R, a contradiction to $R \in \tilde{\mathcal{R}}$.

As all prices in the sequence s are uniquely determined by R_{-a} (independently of whether a belongs to s or not), (ii) and (iii) only hold for finite set of values of v. As the set of sequences is finite, $R' = (R_a^v, R_{-a}) \in \tilde{\mathcal{R}}$ except for a finite numbers of values of v. $\square$

The following shows that as we change R_a to R_a^v such that the resulting profile belongs to $\tilde{\mathcal{R}}$, the minimum price vector remains unchanged.

Lemma 4. *Let $R \in \tilde{\mathcal{R}}$ and let p^m be a minimal vector in Π_R. Then, for all $a \in A$ and all $v \in (\underline{u}_a, \overline{u}_a]$ (where R_a is represented by u_a) such that $(R_a^v, R_{-a}) \in \tilde{\mathcal{R}}$, p^m is a minimum price vector also in $\Pi_{(R_a^v, R_{-a})}$.*

Proof. Let $x^m = (\mu^m, p^m) \in \mathcal{E}_R$. Let $v \in (\underline{u}_a, \overline{u}_a]$ be such that $R' = (R_a^v, R_{-a}) \in \tilde{R}$. Since $x^m \in \mathcal{E}_R$, it follows directly from the definition of R_a^v that x^m is a weak price equilibrium given R' and, hence, $\hat{x} = (\hat{\mu}, p^m) \in \mathcal{E}_{R'}$ for some assignment $\hat{\mu}$. If $p_h^m = 0$ for all $h \in H$, then we are done. If $p_h^m > 0$ for some $h \in H$, then let $S \subseteq \{h \in H : p_h^m > 0\}$ be such that $S \neq \emptyset$. Such a set S exists since $p_h^m > 0$ for some $h \in H$. Then, by the necessary part of Lemma 2, the MP condition holds at x, i.e., there is a house $h \in S$ and an agent $a' \in A$, $a' \neq h$, such that $\mu_{a'}^m \notin S$ and $x_{a'} I_{a'}(h, p^m)$. If $a' = a$, then also $\hat{x}_a I_a'(h, p^m)$ by the construction of R_a^v and the fact that $v < \overline{u}_a$ implies $\mu_a^m \neq a \neq \hat{\mu}_a$. Hence, the MP condition is satisfied at $\hat{x} \in \mathcal{E}_{R'}$. Then, by the sufficiency part of Lemma 2, p^m is a minimal vector in $\Pi_{R'}$. □

Proof of Theorem 1. Suppose that the mechanism f is strategy-proof, but not a minimum price equilibrium mechanism. Then there is a profile $R \in \tilde{\mathcal{R}}$ such that $f(R) = x \equiv (\mu, p)$ and $p \geq p^m$, $p \neq p^m$, where p^m is minimal in Π_R. We consider two cases: either [$x_a^m P_a(a, 0)$ for all $a \in A$] or [$x_a^m I_a(a, 0)$ for some $a \in A$].

Case 1: $x_a^m P_a(a, 0)$ for all $a \in A$.

Then for all $a \in A$, $\underline{u}_a < \overline{u}_a$ and $\mu_a^m \neq a$. Hence, by $p \geq p^m$ and $p \neq p^m$, there is a house $h \in H$ such that $p_h > p_h^m \geq 0$. Without loss of generality, let $p_1 > p_1^m \geq 0$. Then there is an agent $a' \in A$, $a' \neq h = 1$, such that $x_{a'} I_{a'}(1, p)$ (where this follows from Definition 1(ii) if $\mu_1 = 1$ and otherwise we may choose a' such that $\mu_{a'} = 1$). Further, by monotonicity, it follows that $x_{a'}^m P_{a'} x_{a'}$ and $\mu_{a'}^m \neq a'$, since:

$$x_{a'}^m R_{a'}(1, p^m) P_{a'}(1, p) I_{a'} x_{a'} R_{a'}(\mu_{a'}^m, p).$$

Thus, $p_{\mu_{a'}^m} > p_{\mu_{a'}^m}^m \geq 0$.

For ease of notation, let $R^0 = R$, $f(R^0) = x^0 = (\mu^0, p^0)$, $a(1) = a'$ and $x^m = x^{m_0} = (\mu^{m_0}, p^{m_0})$. Now choose $R_{a(1)}^1$ such that we have (i) $R^1 = (R_{a(1)}^1, R_{-a(1)}^0) \in \tilde{\mathcal{R}}$, (ii) $(a(1), 0) P_{a(1)}^1 (h, 0)$

for all $h \in H \backslash \{a(1), \mu^{m_0}_{a(1)}\}$ and (iii) $p^0_{\mu^{m_0}_{a(1)}} > r^1_{a(1)}(\mu^{m_0}_{a(1)}) > p^{m_0}_{\mu^{m_0}_{a(1)}}$ such that $(\mu^{m_0}_{a(1)}, r^1_{a(1)}(\mu^{m_0}_{a(1)}))P^0_{a(1)}x^0_{a(1)}$.

Suppose now that agent $a(1)$ manipulates by using preferences $R^1_{a(1)}$ as defined above. Then for $f(R^1) = x^1 = (\mu^1, p^1)$, we have by construction, $\mu^1_{a(1)} \in \{\mu^{m_0}_{a(1)}, a(1)\}$. If $\mu^1_{a(1)} = \mu^{m_0}_{a(1)}$, then $p^0_{\mu^{m_0}_{a(1)}} > r^1_{a(1)}(\mu^{m_0}_{a(1)}) \geq p^1_{\mu^{m_0}_{a(1)}}$ implying $x^1_{a(1)}P^0_{a(1)}x^0_{a(1)}$, a contradiction to strategy-proofness. Thus, $\mu^1_{a(1)} = a(1)$ and $p^1_{\mu^{m_0}_{a(1)}} \geq r^1_{a(1)}(\mu^{m_0}_{a(1)}) > p^{m_0}_{\mu^{m_0}_{a(1)}}$.

Note that x^{m_0} is a weak price equilibrium under R^1. Thus, for any MPE $x^{m_1} = (\mu^{m_1}, p^{m_1})$ for R^1, we have (i) $p^{m_0} \geq p^{m_1}$ (by $R^1 \in \tilde{\mathcal{R}}$), (ii) for all $a \in A$, $\mu^{m_1}_a \neq a$ (from (i) and $\mu^{m_0}_a \neq a$), and (iii) $\mu^{m_1}_{a(1)} = \mu^{m_0}_{a(1)}$ (from (ii) and the construction of $R^1_{a(1)}$). Hence, for all $a \in A$, $x^{m_1}_a R^1_a x^{m_0}_a P^1_a (a, 0)$.

Thus, for $h = \mu^{m_0}_{a(1)}$, we have $p^1_h > p^{m_0}_h \geq p^{m_1}_h$. Let $a' \in A$ be such that $\mu^1_{a'} = h$. Then $a' \neq a(1)$, and either [$a' = h$, $x^{m_1}_{a'} P_{a'} x^1_{a'}$ and $p^1_{\mu^{m_1}_{a'}} > p^{m_1}_{\mu^{m_1}_{a'}}$] or [$a' \neq h$, $x^{m_1}_{a'} P_{a'} x^1_{a'}$ and $p^1_{\mu^{m_1}_{a'}} > p^{m_1}_{\mu^{m_1}_{a'}}$].

In both cases we set $a' = a(2)$ and choose $R^2_{a(2)}$ such that we have (i) $R^2 = (R^2_{a(2)}, R^1_{-a(2)}) \in \tilde{\mathcal{R}}$, (ii) $(a(2), 0)P^2_{a(2)}(h', 0)$ for all $h' \in H \backslash \{a(2), \mu^{m_1}_{a(2)}\}$ and (iii) $p^1_{\mu^{m_1}_{a(2)}} > r^2_{a(2)}(\mu^{m_1}_{a(2)}) > p^{m_1}_{\mu^{m_1}_{a(2)}}$ such that $(\mu^{m_1}_{a(2)}, r^2_{a(2)}(\mu^{m_1}_{a(2)}))P^1_{a(2)}x^1_{a(2)}$.

Now by induction, suppose that for $l > 1$ we have (i) $R^l = (R^l_{a(l)}, R^{l-1}_{-a(l)}) \in \tilde{\mathcal{R}}$, (ii) $(a(l), 0)P^l_{a(l)}(h', 0)$ for all $h' \in H \backslash \{a(l), \mu^{m_{l-1}}_{a(l)}\}$ and (iii) $p_{\mu^{m_{l-1}}_{a(l)}} > r^l_{a(l)}(\mu^{m_{l-1}}_{a(l)}) > p^{m_{l-1}}_{\mu^{m_{l-1}}_{a(l)}}$ such that $(\mu^{m_{l-1}}_{a(l)}, r^l_{a(l)}(\mu^{m_{l-1}}_{a(l)}))P^{l-1}_{a(l)}x^{l-1}_{a(l-1)}$.

Note that $x^{m_{l-1}}$ is a weak price equilibrium under R^l. Thus, for any MPE $x^{m_l} = (\mu^{m_l}, p^{m_l})$ for R^l, we have (i) $p^{m_{l-1}} \geq p^{m_l}$ (by $R^l \in \tilde{\mathcal{R}}$), (ii) for all $a \in A$, $\mu^{m_l}_a \neq a$ (from (i) and $\mu^{m_{l-1}}_a \neq a$), and (iii) for all $k \in \{1, \ldots, l\}$, $\mu^{m_l}_{a(k)} = \mu^{m_{k-1}}_{a(k)}$ (from (ii) and the construction of $R^k_{a(k)}$). Hence, for all $a \in A$, $x^{m_l}_a R^l_a x^{m_{l-1}}_a P^l_a (a, 0)$.

As above, if agent $a(l)$ manipulates by using preferences $R^l_{a(l)}$ as defined above (from R^{l-1}), then for $f(R^l) = x^l$, we have by construction, $\mu^l_{a(l)} = a(l)$ and $p^l_{\mu^{m_{l-1}}_{a(l)}} \geq r^l_{a(l)}(\mu^{m_{l-1}}_{a(l)}) > p^{m_{l-1}}_{\mu^{m_{l-1}}_{a(l)}} \geq p^{m_l}_{\mu^{m_{l-1}}_{a(l)}} \geq 0$.

Let $T = \{a(1), \ldots, a(l)\}$. Let $i_1 \in A$ be such that $\mu^l_{i_1} = \mu^{m_{l-1}}_{a(l)}$. Now if $i_1 \notin T$, then we set $a(l+1) = i_1$ and we continue as above; and otherwise $i_1 \in T \backslash \{a(l)\}$ and by construction, $\mu^{m_l}_{i_1} \neq \mu^{m_{l-1}}_{a(l)}$ and $i_1 = \mu^l_{i_1}$. But then by $x^{m_l}_{i_1} P^l_{i_1}(i_1, 0)$, we have $p^l_{\mu^{m_l}_{i_1}} > p^{m_l}_{\mu^{m_l}_{i_1}} \geq 0$. Let $i_2 \in A$ be such that $\mu^l_{i_2} = \mu^{m_l}_{i_1}$. Now if $i_2 \notin T$, then we set $a(l+1) = i_1$ and we continue as above; and otherwise $i_2 \in T \backslash \{i_1\}$ and by construction, $\mu^{m_l}_{i_2} \neq \mu^{m_l}_{i_1}$ and $i_2 = \mu^l_{i_2}$. But then we find i_3, and either [we find $a(l+1) \notin T$] or [we find $\hat{T} \subseteq T$ such that (i) for all $a \in \hat{T}$, $\mu_a^l = a$ and $p_a^l > p_a^{m_l} \geq 0$ and (ii) $\hat{T} = \cup_{a \in \hat{T}} \{\mu_a^{m_l}\}$].

But then for all $a' \notin T$, we have $x_{a'}^l P_{a'}^l x_a^l$ for all $a \in \hat{T}$ (as otherwise we have found $a(l+1)$) and by construction and $\hat{T} = \cup_{a \in \hat{T}} \{\mu_a^{m_l}\}$, we have for all $a' \in T \backslash \hat{T}$, $x_{a'}^l R_{a'}^l (a', 0) P_{a'}^l x_a^l$ for all $a \in \hat{T}$. But then by Definition 1 (ii), for any $a \in \hat{T}$ there exists $a' \in \hat{T} \backslash \{a\}$ such that $x_{a'}^l I_{a'}^l x_a^l$; and there exists in $\hat{T}$ an indifference cycle $i(1), \ldots, i(k)$ such that $x_{i(1)}^l I_{i(1)}^l x_{i(2)}^l \cdots x_{i(k)}^l I_{i(k)}^l x_{i(1)}^l$ and $\{i(1), \ldots, i(k)\} \subseteq \hat{T}$ which implies that trade is not maximal at x^l, a contradiction. This finishes the proof of Case 1.

Case 2: $x_a^m I_a(a, 0)$ for some $a \in A$.

Now let T be maximal with respect to set inclusion such that for (ease of notation) $R' = (R'_T, R_{-T})$ we have $f(R') = x = (\mu, p)$ with $p \geq p^m$ and $p \neq p^m$ (where for all $a \in T$, $x_a^m P_a(a, 0)$ and we changed R_a to R'_a as above in Case 1). Then for all $a \in T$, $x_a^m P'_a(a, 0)$, $\mu_a^m \neq a$ and $\mu_a \in \{a, \mu_a^m\}$. Note also for the last agent a in T who changed her preferences from R_a to R'_a we have $\mu_a = a$ and $p_{\mu_a^m} > p_{\mu_a^m}^m$, i.e., $p \geq p^m$ and $p \neq p^m$ is satisfied.

Note that as $p \neq p^m$, (μ, p) violates the MP price condition under R', i.e., there exists $S \subseteq \{h \in H : p_h > 0\}$ such that for all $a \in A$ with $\mu_a \notin S$, we have $x_a P'_a(h, p)$ for all $h \in S$. Without loss of generality, let S be minimal with respect to set inclusion. If $|S| > 1$, then for all $a \in A$ such that $\mu_a \in S$, there exists $h \in S \backslash \{\mu_a\}$ such that $x_a I'_a(h, p)$. But then by our choice of S, S must consist of indifference cycles $i(1), \ldots, i(k)$ such that $x_{i(1)} I'_{i(1)} x_{i(2)} \cdots x_{i(k)} I'_{i(k)} x_{i(1)}$ and $\{\mu_{i(1)}, \ldots, \mu_{i(k)}\} \subseteq S$, and any $a \in A$ such that $\mu_a \in S$ must belong to at least one such cycle (and no cycle can be disjoint from all other cycles). Now we must have for all $a \in A$ such that $\mu_a \in S$, $\mu_a \neq a$ as otherwise $\mu_a = a$ and by choosing a cycle in S to which a belongs to either trade is not maximal at x or two houses are connected by indifference (which contradicts $R' \in \tilde{\mathcal{R}}$). Note that $|S| = 1$ and $a = \mu_a \in S$ are impossible by Definition 1 (ii) as $p_{\mu_a} > 0$ and our choice of S. Thus, if $p_{h'} = p_{h'}^m$ for some $h' \in S$, then $p_h = p_h^m$ for all $h \in S$ (again by our choice of S and all houses in S can be connected via indifference cycles). As x^m satisfies the MP condition, there exists $a' \in A$ such that $\mu_{a'}^m \notin S$ and for some $h \in S$, $x_{a'}^m I'_{a'}(h, p^m)$. Let $\hat{A} = \{a'\} \cup \{a \in A : \mu_a^m \in S\}$. Obviously, $|\hat{A}| > |S|$ and there exists $a \in \hat{A}$ such that

$\mu_a \notin S$. But then $\mu_a^m \in S$ implies $(\mu_a^m, p^m) R'_a (\mu_a, p^m) R'_a x_a$ and as $(\mu_a^m, p) I'_a (\mu_a^m, p^m)$, this is a contradiction to our choice of S; and $\mu_a = \mu_a^m$ implies $p_a = p_a^m$ which is again a contradiction to our choice of S. Thus, for all $h \in S$, $p_h > p_h^m \geq 0$ and $\mu_h \neq h$. But then for all $a \in A$ such that $\mu_a \in S$, we have $a \in T$ (as otherwise we have found $a \notin T$ and the deviation in Case 1 is possible).

Now let $a' \in A$ be such that $\mu_{a'} \in S$. Then $a' \in T$, $\mu_{a'} = \mu_{a'}^m \neq a'$ and $p_{\mu_{a'}} > p_{\mu_{a'}}^m$. Furthermore, for all $a \in A$ with $\underline{u}_a = \overline{u}_a$ we have $a \notin T$ (and thus, $\mu_a \notin S$) and $(a, 0) P'_a x_{a'}$ implying $\mu_{a'} \neq a$. Now let a' manipulate with $R''_{a'}$ such that (i) $R'' = (R''_{a'}, R'_{-a'}) \in \tilde{\mathcal{R}}$, (ii) $(a', 0) P''_{a'} (h', 0)$ for all $h' \in H \backslash \{a', \mu_{a'}^m\}$ and (iii) $p_{\mu_{a'}} > r''_{a'}(\mu_{a'}) > p_{\mu_{a'}}^m$.

Note that x^m then satisfies the MP price condition under R'', and x^m is an MPE under R''. Let $f(R'') = x' = (\mu', p')$. Furthermore, as in Case 1 it follows $\mu'_{a'} = a'$ and $p'_{\mu_{a'}} \geq r''_{a'}(\mu_{a'}) > p_{\mu_{a'}}^m$. If $p'_{\mu_{a'}} > r''_{a'}(\mu_{a'})$, then $\mu_{a'} = \mu'_a \neq \mu_a$ implies $a \notin T$ (as otherwise $a \in T$ and $a \neq \mu_a^m \neq \mu_{a'}^m = \mu_{a'}$, a contradiction) and by the maximality of T, $\mu'_a = a$. But then from Definition 1 (ii) there exists a'' with $x'_{a''} I'_{a''} (a, p')$ which is impossible for $a'' \in T$ and $a'' \notin T$ would again contradict the maximality of T. Thus, $p'_{\mu_{a'}} = r''_{a'}(\mu_{a'}) > p_{\mu_{a'}}^m$.

Note that a' belongs to a trading cycle $i_1 - i_2 - \cdots - i_k$ from μ to μ' such that $\mu'_{i_l} = \mu_{i_{l-1}}$ for $l = 2, \ldots, k$ and $\mu'_{i_1} = \mu_{i_k}$. Let $a' = i_2$. Then $\mu_{i_2} = \mu_{i_2}^m$ and $\mu_{i_1} = \mu'_{i_2} = i_2$.

We show (I) $\mu_{i_l} \neq i_l$ for all $l \in \{1, \ldots, k\}$ and (II) $x''_{i_2} I''_{i_2} (\mu_{i_2}, p')$ and $x'_{i_l} I'_{i_l} (\mu_{i_l}, p')$ for all $l \in \{1, \ldots, k\} \backslash \{2\}$. Note that (II) yields the desired contradiction as $\mu'_{i_2} = i_2$ and trade is not maximal under x'.

Note that (I) $\mu_{i_2} \neq i_2$ and by $p'_{\mu_{i_2}} = r''_{i_2}(\mu_{i_2}) > p_{\mu_{i_2}}^m$, (II) $x'_{i_2} I''_{i_2} (\mu_{i_2}, p')$. Then $\mu_{i_2} = \mu'_{i_3} \neq i_3$ is not possible as $i_3 \notin T$ would imply $x_{i_3}^m P'_{i_3} x'_{i_3}$ (and we found $i_3 \notin T$, a contradiction to the maximality of T) or $i_3 \in T$ would imply $(i_3, 0) P'_{i_3} (\mu'_{i_3}, p')$ (as $\mu_{i_3}^m \neq \mu_{i_2}^m = \mu_{i_2}$). Thus, $\mu_{i_2} = \mu'_{i_3} = i_3$ and (I) $\mu_{i_3} \neq i_3$. If $i_3 \in T$, then as above $\mu_{i_3} = \mu_{i_3}^m$ and $p'_{\mu_{i_3}} = r'_{i_3}(\mu_{i_3}) > p_{\mu_{i_3}}^m$ and thus, (II) $x'_{i_3} I'_{i_3} (\mu_{i_3}, p')$. If $i_3 \notin T$ and $x_{i_3}^m P'_{i_3} (i_3, 0)$, then we have found $i_3 \notin T$, a contradiction to the maximality of T.

If $i_3 \notin T$ and $x_{i_3}^m I'_{i_3} (i_3, 0)$, then $p_{\mu_{i_3}} = p_{\mu_{i_3}}^m$. We then also show $p'_{\mu_{i_3}} = p_{\mu_{i_3}}^m$ and (II) $x'_{i_3} I'_{i_3} (\mu_{i_3}, p')$. If $p'_{\mu_{i_3}} > p_{\mu_{i_3}}^m$, then [$\mu'_{i_4} = \mu_{i_3} \neq i_4$ would imply $i_4 \in T$, $\mu_{i_3} = \mu_{i_4}^m$, $\mu_{i_4} = i_4$, and $x_{i_3} P'_{i_4} x_{i_4}$, a contradiction] and [$\mu'_{i_4} = \mu_{i_3} = i_4$ would imply there exists $a \in A$ with $x'_a I'_a (i_4, p')$, then again $a \in T$, $i_4 = \mu^m_a$, $\mu_a = a$, and $x_{i_3} P'_a x_a$, a contradiction].

Now we continue by induction: let $l \in \{2, \ldots, k\}$ and suppose we have (I) $\mu_{i_t} \neq i_t$ for all $t \in \{2, \ldots, l\}$ and (II) $x''_{i_2} I''_{i_2}(\mu_{i_2}, p')$ and $x'_{i_t} I'_{i_t}(\mu_{i_t}, p')$ for all $t \in \{3, \ldots, l\}$.

In showing (I) $\mu_{i_{l+1}} \neq i_{l+1}$ and (II) $x'_{i_{l+1}} I'_{i_{l+1}}(\mu_{i_{l+1}}, p')$, we consider three subcases for i_l: [1] $i_l \in T$, [2] $i_l \notin T$ and $x^m_{i_l} I'_{i_l}(i_l, 0)$, and [3] $i_l \notin T$ and $x^m_{i_l} P'_{i_l}(i_l, 0)$.

For [1], if $i_l \in T$, then $\mu_{i_l} = \mu^m_{i_l}$, $\mu'_{i_l} = i_l$ and $p'_{\mu_{i_l}} = r'_{i_l}(\mu_{i_l}) > p^m_{\mu_{i_l}}$. Then $\mu_{i_l} = \mu'_{i_{l+1}} \neq i_{l+1}$ is impossible as otherwise $i_{l+1} \in T$ and $\mu_{i_l} \neq \mu^m_{i_{l+1}}$. Thus, $\mu_{i_l} = \mu'_{i_{l+1}} = i_{l+1} \neq \mu_{i_{l+1}}$ (which yields (I) for i_{l+1}) and either [$i_{l+1} \in T$ yielding $\mu_{i_{l+1}} = \mu^m_{i_{l+1}}$ and $p'_{\mu_{i_{l+1}}} = r'_{i_{l+1}}(\mu_{i_{l+1}})$] or [$i_{l+1} \notin T$ yielding $(i_{l+1}, 0) I'_{i_{l+1}}(\mu_{i_{l+1}}, p)$ and as above, $p_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}} = p'_{\mu_{i_{l+1}}}$]. In both cases, (II) $x'_{i_{l+1}} I'_{i_{l+1}}(\mu_{i_{l+1}}, p')$.

For [2], if $i_l \notin T$ and $x^m_{i_l} I'_{i_l}(i_l, 0)$, then $p_{\mu_{i_l}} = p^m_{\mu_{i_l}}$ by $\mu_{i_l} \neq i_l$. Then it follows as above $p'_{\mu_{i_l}} = p^m_{\mu_{i_l}}$. But then $i_{l+1} \in T$ and $\mu_{i_l} \neq i_{l+1}$ is not possible as otherwise $\mu_{i_l} = \mu^m_{i_l}$ and $x_{i_l} P'_{i_{l+1}} x_{i_{l+1}}$ (where $\mu_{i_{l+1}} = i_{l+1}$ by construction of $R'_{i_{l+1}}$); thus, $i_{l+1} \in T$ implies $\mu_{i_l} = i_{l+1}$ and $\mu_{i_{l+1}} = \mu^m_{i_{l+1}} \neq i_{l+1}$ (which yields (I) for i_{l+1}) and then again $p'_{\mu_{i_{l+1}}} = r'_{i_{l+1}}(\mu_{i_{l+1}})$ (which yields (II) $x'_{i_{l+1}} I'_{i_{l+1}}(\mu_{i_{l+1}}, p')$). If $i_{l+1} \notin T$ and $\mu_{i_{l+1}} = i_{l+1}$, then we have contradiction to that no two houses are connected by indifference at p' from (II) (as we can take the maximal $t \geq 2$ such that $\mu'_{i_t} = i_t$ (which exists by $\mu'_{i_2} = i_2$). This yields (I) $\mu_{i_{l+1}} \neq i_{l+1}$ and $p_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}}$. But then $\mu_{i_l} = i_{l+1}$ implies as above also $p_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}} = p'_{\mu_{i_{l+1}}}$ and (II) $x'_{i_{l+1}} I_{i_{l+1}}(\mu_{i_{l+1}}, p')$; and $\mu_{i_l} \neq i_{l+1}$ implies $p'_{\mu_{i_l}} = p^m_{\mu_{i_l}}$ and $x^m_{i_{l+1}} P'_{i_{l+1}}(i_{l+1}, 0)$ (as otherwise we have again a contradiction that no two houses are connected by indifference). We show again $p'_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}}$. If $p'_{\mu_{i_{l+1}}} > p^m_{\mu_{i_{l+1}}}$, then $\mu_{i_{l+1}} \neq i_{l+2}$ is not possible as above for $i_{l+2} \notin T$ or $i_{l+2} \in T$. Thus, $\mu_{i_{l+1}} = i_{l+2}$ and by Definition 1 (ii) for some $a \neq i_{l+2}$ we have $x'_a I'_a(i_{l+2}, p'_{i_{l+2}})$ which is not possible for either $a \in T$ or $a \notin T$. Thus, $p'_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}}$ and (II) $x'_{i_{l+1}} I'_{i_{l+1}}(\mu_{i_{l+1}}, p')$.

For [3], if $i_l \notin T$ and $x^m_{i_l} P'_{i_l}(i_l, 0)$, then $\mu_{i_l} \neq i_l$ and $p_{\mu_{i_l}} = p^m_{\mu_{i_l}}$, and $\mu'_{i_l} \neq i_l$ and $p'_{\mu_{i_l}} = p^m_{\mu_{i_l}}$. Then it follows as above $p'_{\mu_{i_l}} = p^m_{\mu_{i_l}}$ and we draw the same conclusions as above at the end in [2] yielding $p'_{\mu_{i_{l+1}}} = p^m_{\mu_{i_{l+1}}}$ and (II) $x'_{i_{l+1}} I'_{i_{l+1}}(\mu_{i_{l+1}}, p')$. □

Proof of Theorem 2. Start as above with a profile $R \in \tilde{\mathcal{Q}}$. Let $a \in A$ be such that $\underline{u}_a < \overline{u}_a$. Observe that for all $v \in (\underline{u}_a, \overline{u}_a]$ by construction R^v_a is quasi-linear, and $(R^v_a, R_{-a}) \in \tilde{\mathcal{Q}}$ for almost all $v \in (\underline{u}_a, \overline{u}_a]$. Furthermore, given quasi-linear R_a and $h \in H\backslash\{a\}$, $(h, r_a(h))I_a(a,0)$ and $(h,0)R_a(a,0)$ imply $v_{ah} - r_a(h) = v_{aa}$ and $r_a(h) = v_{ah} - v_{aa}$.

In order to show that the proof of Theorem 1 also proves Theorem 2, it remains to show that any deviation can be chosen to be quasi-linear. It suffices to consider the first deviation in Case 1 (as all later deviations are analogous for the other agents).

Let $R^0 = R$, $f(R^0) = x^0 = (\mu^0, p^0)$, $a(1) = a'$ and $x^m = x^{m_0} = (\mu^{m_0}, p^{m_0})$. For ease of notation we use below a instead of $a(1)$ and $x^m = (\mu^m, p^m)$ instead of x^{m_0}.

Note that R_a^0 is quasi-linear and $p^0_{\mu_a^m} > p^m_{\mu_a^m}$. Let R_a^0 be represented by u_a^0 with values $(v^0_{ah})_{h\in H}$. Then $x_a^m P_a^0 x_a^0 R_a^0(a,0)$ implies $v^0_{a\mu_a^m} - p_a^m > u_{a\mu_a^0}(p^0) \geq v^0_{aa}$. Now we choose R_a^1 to be quasi-linear with representation u_a^1 and values $(v^1_{ah})_{h\in H}$ such that:

(I) $v^1_{a\mu_a^m} = v^0_{a\mu_a^m}$,

(II) $v^1_{aa} \in (u_{a\mu_a^0}(p^0), v^0_{a\mu_a^m} - p^m_{\mu_a^m})$ such that $p^0_{\mu_a^m} > v^1_{a\mu_a^m} - v^1_{aa} > p^m_{\mu_a^m}$,

(III) for all $h \in H\backslash\{a, \mu_a^m\}$ and all p, $u^1_{ah}(p) = u^0_{ah}(p) - (1 + |v^0_{aa}| + \max_{h'\in H} |v^0_{ah'}|)\alpha$ where $\alpha \in (1,2)$.

Note that the choice v^1_{aa} in (II) is possible by (I) together with $p^0_{\mu_a^m} > p^m_{\mu_a^m}$ and $v^0_{a\mu_a^m} - p_a^m > u_{a\mu_a^0}(p^0)$. Also $v^1_{aa} < v^0_{a\mu_a^m} - p^m_{\mu_a^m}$ always implies (by (I)) $r_a^1(\mu_a^m) = v^1_{a\mu_a^m} - v^1_{aa} > p^m_{\mu_a^m}$, and for v^1_{aa} close enough to $v^0_{a\mu_a^m} - p^m_{\mu_a^m}$, $r_a^1(\mu_a^m)$ is arbitrarily close to $p^m_{\mu_a^m}$.

By (II) and (III), it follows that (ii) $(a,0)P_a^1(h,0)$ for all $h \in H\backslash\{a, \mu_a^m\}$. Finally, (iii) $p^0_{\mu_a^m} > r_a^1(\mu_a^m) > p^m_{\mu_a^m}$ follows from (II) as $r_a^1(\mu_a^m) = v^1_{a\mu_a^m} - v^1_{aa}$. By quasi-linearity of R_a^0, we have:

$$u^0_{a\mu_a^m}(r_a^1(\mu_a^m)) = v^0_{a\mu_a^m} - (v^1_{a\mu_a^m} - v^1_{aa}) = v^1_{aa}$$

where the first equality follows from $r_a^1(\mu_a^m) = v^1_{a\mu_a^m} - v^1_{aa}$ and the second from (I). Now as $v^1_{aa} > u_{a\mu_a^0}(p^0)$, we obtain $(\mu_a^m, r_a^1(\mu_a^m))P_a^0 x_a^0$, which is the second part of (iii).

Now from the same arguments as in the proof of Lemma 3 there exist v^1_{aa} and $\alpha \in (1,2)$ such that (i) $R^1 = (R_a^1, R^0_{-a}) \in \tilde{\mathcal{Q}}$.[23]

[23] Note that by $(a,0)P_a^1(h,0)$ for all $h \in H\backslash\{a, \mu_a^m\}$ in the proof of Lemma 3 in (ii) we must have $a = a_1$ and

A.3 Proof of Theorem 3

Let g be an MPE mechanism. We show that on the $\underline{NCBI}$ domain $\mathring{\mathcal{Q}}$ there exists no strategy-proof mechanism f dominating g. As for any agent a with preferences R_a, $(a,p)I_a(a,0)$ for all prices $p \geq 0$, below we use the convention to sometimes write a instead of (a,p) or $(a,0)$.

Suppose not, i.e., f is strategy-proof and $f \gtrdot|_{\mathring{\mathcal{Q}}} g$. But then for all $R \in \mathring{\mathcal{Q}}$ it follows that $f_a(R) R_a g_a(R) R_a (a,0)$ for all $a \in A$ as g is individually rational, and for $f(R) = (\mu, p)$, $g(R) = (\nu, \overline{p})$ and all $h \in H$ for agent a with $\mu_a = h \neq a$, we have:

$$(h, p_h) R_a g_a(R) R_a (h, \overline{p}_h),$$

which implies $p_h \leq \overline{p}_h$. Without loss of generality, we use the convention $p_h = \overline{p}_h$ if $h = \mu_a = a$, i.e., if agent a keeps her endowment under $f(R)$, which together with the above implies $0 \leq p \leq \overline{p}$.

Suppose for some $R \in \mathring{\mathcal{Q}}$ and $a \in A$ (where $f(R) = (\mu, p)$ and $g(R) = (\nu, \overline{p})$) we have $f_a(R) = (\mu_a, p) P_a (\nu_a, \overline{p})$. From $p \leq \overline{p}$ and $(\nu_a, \overline{p}) R_a (\mu_a, \overline{p})$ we obtain $0 \leq p_{\mu_a} < \overline{p}_{\mu_a}$ and $\mu_a \neq a$.

Now let R'_a be such that $(\mu_a, p) P_a a P_a (\overline{\mu}_a, \overline{p})$ and $a P_a (h, 0)$ for all $h \in H \backslash \{\mu_a, a\}$, and $R' = (R'_a, R_{-a}) \in \mathring{\mathcal{Q}}$. By strategy-proofness and individual rationality of f and g on the $\underline{NCBI}$ domain we have $f_a(R') = (\mu_a, p_{\mu_a})$ and $g_a(R') = a$. Because g is an MPE mechanism, for $g(R') = (\nu', \overline{p}')$ we have $p_{\mu_a} < \overline{p}'_{\mu_a}$. Note that we continue to have $f_a(R') P'_a g_a(R')$.

If there exists another agent $a' \neq a$ for whom $f_{a'}(R') P_{a'} g_{a'}(R')$, then we do the same as above. Let S denote the agents for whom we changed preferences as above. By construction for any $a \in S$ there exists $h_a \in H \backslash \{a\}$ such that $a P_a (h, 0)$ for all $h \in H \backslash \{h_a, a\}$.

Thus, we arrive at a profile R such that (where $f(R) = (\mu, p)$ and $g(R) = (\nu, \overline{p})$) for all $a \in A$, $f_a(R) P_a g_a(R)$ implies $a \in S$, $\mu_a = h_a$, $0 \leq p_{h_a} < \overline{p}_{h_a}$, and either $g_a(R) = a$ or $g_a(R) = (h_a, \overline{p})$. Let $T = \{a \in A : f_a(R) P_a g_a(R)\}$. But now for the last agent a who changed

$h_2 = \mu_a^m$ and in (iii) we must have $h_r = \mu_a^m$ and $a_r = a$ (and in both these cases $v^1_{aa} = u^1_{a\mu_a^m}(p)$ and v^1_{aa} can be adjusted); and in (i) we must have $a = a_j$, $h_j = \mu_a^m \neq h_{j+1}$ or $h_j \neq h_{j+1} = \mu_a^m$ (and either $u^1_{a\mu_a^m}(p) = u^1_{ah_j}(p)$ or $u^1_{a\mu_a^m}(p) = u^1_{ah_{j+1}}(p)$ and in both cases α can be adjusted).

her preference we have $g_a(R) = a$, $0 \le p_{h_a} < \overline{p}_{h_a}$ and $a \in T \subseteq S$.

As $g(R) = (\nu, \overline{p})$ is an MPE and $R \in \mathring{\mathcal{Q}} \subseteq \tilde{\mathcal{Q}}$, the MP price condition ensures that there exists $h \in H$ such that $\overline{p}_h = 0$. Hence, from $0 \le p \le \overline{p}$, we obtain $p_h = 0$. Let $j \in A$ be such that $f_j(R) = (h, 0)$. But then:

$$(h, 0) R_j g_j(R) R_j (h, 0),$$

where the first relation follows from $f_j(R) R_j g_j(R)$ and the second one from the fact that $g(R)$ is an MPE. Hence, $f_j(R) = (h, 0) I_j g_j(R) = (\nu_j, \overline{p}_{\nu_j})$. Let $H_0 = \{h \in H : \overline{p}_h = 0\}$ and $N_0 = \{i \in A : \mu_i \in H_0\}$. By the above argument, we have $p_h = 0$ for all $h \in H_0$ and $f_i(R) I_i g_i(R)$ for all $i \in N_0$. Furthermore, for the last agent a for whom we changed preferences, we have $\overline{p}_{\mu_a} > p_{\mu_a} \ge 0$ and $\mu_a = h_a \neq a = \nu_a$, i.e., $H_0 \neq H$.

Note that from μ to ν houses are exchanged in cycles. Let $c = (i_1, \ldots, i_t)$ be such that $\mu_{i_l} = \nu_{i_{l-1}}$ for all $l = 2, \ldots, t$ and $\mu_{i_1} = \nu_{i_t}$. Abusing notation, let c also denote the coalition of agents in this cycle, and where we choose c such that $a \in c$ (where a is the last agent who changed her preference). Thus, $t \ge 2$. Let $H_c = \{\mu_i : i \in c\}$ be the houses exchanged in cycle c. By definition, $H_c = \{\nu_i : i \in c\}$.

We distinguish the following two cases.

Case 1: $H_c \cap H_0 \neq \emptyset$.

Let $h \in H_c \cap H_0$. Then $p_h = \overline{p}_h = 0$. Let $i_l \in c$ be such that $\mu_{i_l} = h$. As $f_{i_l}(R) R_{i_l} g_{i_l}(R)$, $(\nu, \overline{p})$ is an MPE and $\overline{p}_h = 0$, we obtain $(h, 0) I_{i_l} (\nu_{i_l}, \overline{p})$. By $\underline{NCBI}$, $\nu_{i_l} \neq i_l$ and $\overline{p}_{i_l} > 0$.

By the MP condition, there exists $k \in A \backslash \{i_l\}$ with $(\nu_k, \overline{p}) I_k (\nu_{i_l}, \overline{p})$. Again by $\underline{NCBI}$, $\nu_k \neq k$ and $\overline{p}_{\nu_k} > 0$.

Now consider houses $\{\nu_{i_l}, \nu_k\}$. Since $\overline{p}_{\nu_{i_l}} > 0$ and $\overline{p}_{\nu_k} > 0$ the MP condition ensures the existence of $m \in A \backslash \{i_l, k\}$ such that $(\nu_m, \overline{p}) I_m (\nu_k, \overline{p})$ or $(\nu_m, \overline{p}) I_m (\nu_{i_l}, \overline{p})$. Independently of whether $m \in \{\nu_k, \nu_{i_l}\}$ or $m \notin \{\nu_k, \nu_{i_l}\}$, $\underline{NCBI}$ yields $\nu_m \neq m$ and $\overline{p}_{\nu_m} > 0$.

Note that we can always continue the chain with an agent distinct from the previously chosen ones and a house distinct from the previously chosen ones, which is a contradiction to finiteness of A and H.

Case 2: $H_c \cap H_0 = \emptyset$.

Thus, by $a \in H_c$, $\overline{p}_a > 0$. Now applying the MP condition to H_c yields $i \in A \backslash c$ such that $(\nu_i, \overline{p}) I_i (\hat{h}, \overline{p})$ for some $\hat{h} \in H_c$. If $\overline{p}_{\nu_i} > 0$, then we consider $H_c \cup \{\nu_i\}$ and apply again the MP condition. Now iteratively applying the MP condition and as $H_0 \neq \emptyset$, we obtain a shortest path from H_0 to H_c:

$$(h, 0) = (\nu_{j_0}, \overline{p}) \xrightarrow{j_0} (\nu_{j_1}, \overline{p}) \xrightarrow{j_1} (\nu_{j_2}, \overline{p}) \xrightarrow{j_2} \cdots \xrightarrow{j_u} (\hat{h}, \overline{p}), \tag{4}$$

where $(\nu_{j_l}, \overline{p}) I_{j_l} (\nu_{j_{l+1}}, \overline{p})$ for $l \in \{0, \ldots, u-1\}$, and $(\nu_{j_u}, \overline{p}) I_{j_u} (\hat{h}, \overline{p})$ and $\hat{h} \in H_c$. Note that as this is a shortest path from H_0 to H_c, we have $\{j_0, j_1, \ldots, j_u\} \cap c = \emptyset$, $\{\nu_{j_0}, \nu_{j_1}, \ldots, \nu_{j_u}\} \cap H_c = \emptyset$ and $\overline{p}_{\nu_{j_l}} > 0$ for all $l = 1, \ldots, u$. Furthermore, by $\underline{NCBI}$ we have $\nu_{j_l} \neq j_{l-1}$ for all $l = 1, \ldots, u$, and $\hat{h} \neq j_u$ (and $\overline{p}_{\hat{h}} > 0$).

Let $i_l \in c$ be such that $\nu_{i_l} = \hat{h}$. Then either $f_{i_l}(R) P_{i_l} g_{i_l}(R)$ or $f_{i_l}(R) I_{i_l} g_{i_l}(R)$.

Subcase 2.1: $f_{i_l}(R) P_{i_l} g_{i_l}(R)$.

As $i_l \in c$ and $|c| \geq 2$, we have $\mu_{i_l} \neq \nu_{i_l}$ which together with $i_l \in S$ implies $g_{i_l}(R) = i_l = \hat{h}$, $\mu_{i_1} = h_{i_l}$ and $\overline{p}_{h_{i_l}} > p_{h_{i_l}} \geq 0$. Without loss of generality, $i_l P_{i_l} (h_{i_l}, \overline{p})$.[24] Let R'_{i_l} be such that $(h_{i_l}, p) P'_{i_l} (h, 0) P'_{i_l} i_l P'_{i_l} (h', 0)$ for all $h' \in H \backslash \{h_{i_l}, h, i_l\}$, and $R' = (R'_{i_l}, R_{-i_l}) \in \mathring{\mathcal{Q}}$. But then executing the cycle $(j_0, j_1, \ldots, j_u, i_l)$ in $(\nu, \overline{p})$ yields a weak price equilibrium for R'. Hence, $p \in \Pi_{R'}$ and for $g(R') = (\nu', \overline{p}')$ we obtain $0 \leq \overline{p}' \leq \overline{p}$, $\overline{p}'_h = 0$ and (by $(h, 0) P'_{i_l} i_l$) $\nu'_{i_l} \neq i_l$.

By strategy-proofness of f and g, and the construction of R'_{i_l}, we still obtain $f_{i_l}(R') = (h_{i_l}, p_{h_{i_l}}) P'_{i_l} g_{i_l}(R')$, and for $f(R') = (\mu', p')$, we have $\overline{p}'_{h_{i_l}} > p'_{h_{i_l}} = p_{h_{i_l}}$. Furthermore, we may suppose $\nu'_{i_l} = h$ as otherwise $\nu_{i_1} = h_{i_l}$ and we change R'_{i_l} to R''_{i_l} such that:

$$(h_{i_l}, p) P''_{i_l} (h, 0) P''_{i_l} i_l P''_{i_l} (h_{i_l}, \overline{p}'_{i_l}) P''_{i_l} (h', 0),$$

for all $h' \in H \backslash \{h_{i_l}, h, i_l\}$, and obtain the same conclusions for (R''_{i_l}, R_{-i_l}) and R as we did above for R' and R.

But then consider the cycle c' according to which agents exchange houses from μ' to ν' with

[24]If $i_l I_{i_l} (h_{i_l}, \overline{p})$, then we change R_{i_l} to R'_{i_l} such that $(h_{i_l}, p) P'_{i_l} i_l P'_{i_l} (h_{i_l}, \overline{p})$ and obtain the same conclusions.

agent i_l belonging to it. But then $\nu'_{i_l} = h$ and $\overline{p}'_h = 0$, and we obtain a contradiction as in Case 1.

Subcase 2.2: $f_{i_l}(R) I_{i_l} g_{i_1}(R)$.

By $\underline{NCBI}$ (starting the chain from $(h, 0)$ as in (4)), we have $\mu_{i_l} \neq i_l$ and $\overline{p}_{\mu_{i_l}} > 0$ (as otherwise $\overline{p}_{\mu_{i_l}} = 0 = p_{\mu_{i_l}}$). Now consider agent i_{l-1}. If $f_{i_{l-1}}(R) P_{i_{l-1}} g_{i_{l-1}}(R)$, then as $i_{l-1} \in c$ and $|c| \geq 2$, $\mu_{i_{l-1}} = h_{i_{l-1}} \neq i_{l-1} = g_{i_{l-1}}(R)$ and $\overline{p}_{h_{i_{l-1}}} > p_{h_{i_{l-1}}} \geq 0$. Then change $R_{i_{l-1}}$ to $R'_{i_{l-1}}$ in the same way as we did in Subcase 2.1 for R_{i_l} by:

$$(h_{i_{l-1}}, p) P'_{i_{l-1}} (h, 0) P'_{i_{l-1}} i_{l-1} P'_{i_{l-1}} (h', 0),$$

for all $h' \in H \backslash \{h_{i_{l-1}}, h, i_{l-1}\}$ and obtain as in Subcase 2.1 the contradiction. If $f_{i_{l-1}}(R) I_{i_{l-1}} g_{i_{l-1}}(R)$, then by $\underline{NCBI}$, we have $\mu_{i_{l-1}} \neq i_{l-1}$ and $\overline{p}_{\mu_{i_{l-1}}} > 0$ (as otherwise $\overline{p}_{\mu_{i_{l-1}}} = 0 = p_{\mu_{i_{l-1}}}$). Now we continue as above. Since $a \in c$ and $f_a(R) P_a g_a(R)$, at some point we find $i_{l-v} \in c$ such that $f_{i_{l-v}}(R) P_{i_{l-v}} g_{i_{l-v}}(R)$ and obtain a contradiction as in Subcase 2.1.

Acknowledgements

Andersson acknowledges financial support from the Jan Wallander and Tom Hedelius Foundation under project P22–0087. Ehlers acknowledges SSHRC (Canada) for financial support under Insight Grant 435-2023-0129.

References

Abdulkadiroğlu, A., Sönmez, T., 1999. House allocation with existing tenants, *Journal of Economic Theory* 88, 233–260.

Abdulkadiroğlu, A., Sönmez, T., 2003. School choice: A mechanism design approach, *American Economic Review* 93, 729–747.

Abdulkadiroğlu, A., Pathak, P., Roth, A.E., 2009. Strategy-proofness versus efficiency in matching with indifferences: redesigning the NYC High School Match. *American Economic Review* 99, 1954–1978.

Alaei, S., Kamal, J., Azarakhsh, M., 2016. Competitive equilibrium in two sided matching markets with general utility functions. *Operations Research* 64, 638–645.

Alkan, A., Demange, G., and Gale, D., 1991. Fair allocation of indivisible objects and criteria of justice, *Econometrica* 59, 1023–1039.

Alva, S., and Manjunath, V., 2019. Strategy-proof Pareto-improvement, *Journal of Economic Theory* 181, 121–142.

Andersson, T., and Svensson, L.-G., 2014. Non-manipulable house allocation with rent control, *Econometrica* 82, 507–539.

Andersson, T., and Svensson, L.-G., 2016. Strategy-proof house allocation with price restrictions, *Journal of Economic Theory* 163, 167–177.

Andersson, T., and Svensson, L.-G., 2018. Sequential rules for house allocation with price restrictions, *Games and Economic Behavior* 107, 41-59.

Andersson, T., Ehlers, L., and Svensson, L.-G., 2016. Transferring ownership of public housing to existing tenants: A market design approach, *Journal of Economic Theory* 165, 643–671.

Baisa, B., 2017. Auction design without quasilinear preferences, *Theoretical Economics* 12, 53–78.

Bogomolnaia, A., Deb R., and Ehlers, L. 2005. Strategy-proof assignment on the full preference domain, *Journal of Economic Theory* 123, 161–186.

Demange, G., 1982. Strategyproofness in the assignment market game. Unpublished manuscript.

Demange, G., and Gale, D., 1985. The strategy structure of two-sided matching markets, *Econometrica* 53, 873–888.

Ehlers, L., 2002. Coalitional strategy-proof house allocation, *Journal of Economic Theory* 105, 298–317.

Ehlers, L., 2014. Top trading with fixed tie-breaking in markets with indivisible goods, *Journal of Economic Theory* 151, 64–87.

Erdil, A., 2014. Strategy-proof stochastic assignment, *Journal of Economic Theory* 151, 146–162.

Erdil, A., and Ergin, H., 2008. What's the matter with tie-breaking? Improving efficiency in School Choice, *American Economic Review* 98, 669–689.

Fleiner, T., Jagadeesan, R., Jankó, Z., and Teytelboym, A., 2019. Trading networks with frictions, *Econometrica* 87, 1633–1661.

Hatfield, J.W., Kojima, F., and Kominers, S.D., 2017. Strategy-proofness, investment efficiency, and marginal returns: An equivalence, Working Paper.

Hatfield, J.W., and Milgrom P., 2005. Matching with Contracts, *American Economic Review* 95, 913–935.

Holmström, B.,1979. Groves' scheme on restricted domains, *Econometrica* 47, 1137–1144.

Housing Europe (2021). The selling of social and public housing in Europe. Available online at `https://www.housingeurope.eu/resource-1515/the-sale-of-social-and-public-housing-in-europe`.

Hurwicz, L., 1972. On informationally decentralized systems, in *Decision and Organisation: A Volume in Honor of Jacob Marschak* ed. by C.B. McGuire and Roy Radner, Amsterdam: North-Holland.

Jagadeesan, R., Kominers, S.D., and Rheingans-Yoo, R., 2018. Strategy-proofness of worker-optimal matching with continuously transferable utility, *Games and Economic Behavior* 108, 287–294.

Jaramillo, P., and Manjunath, V., 2012. The difference indifference makes in strategy-proof allocation of objects, *Journal of Economic Theory* 147, 1913–1946.

Kesten, O., 2010. School Choice with Consent, *The Quarterly Journal of Economics* 125, 1297–1348.

Leonard, H.B., 1983. Elicitation of honest preferences for the assignment of individuals to positions, *Journal of Political Economy* 91, 461–479.

Ma, J., 1994. Strategy-proofness and the strict core in a market with indivisibilities,*International Journal of Game Theory* 23, 75–83.

Miyagawa, E. 2001, House allocations with transfers, *Journal of Economic Theory* 100, 329–355

Milgrom, P., and Segal, I. 2017. Designing the US Incentive Auction. In Handbook of Spectrum Auction Design, edited by M. Bichler and J. Goeree, chap. 37. Cambridge: Cambridge University Press.

Miyake, M., 1998. On the incentive properties of multi-item auctions, *International Journal of Game Theory* 27, 1–19.

Morimoto, S., and Serizawa, S., 2018. Strategy-proofness and efficiency for non-quasi-linear and common-tiered-object preferences: Characterization of minimum price rule, *Games and Economic Behavior* 109, 327–363.

Morimoto, S., and Serizawa, S., 2015. Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule, *Theoretical Economics* 10, 445–487.

Pápai, S., 2000. Strategyproof assignment by hierarchical exchange, *Econometrica* 68, 1403–1433.

Phan, W., Tierney, R., Zhou, Y., 2024. Crowding in school choice, *American Economic Review* 114, 2526–52.

Pycia, M., and Ünver, M. U., 2017. Incentive compatible allocations and exchange of discrete resources, *Theoretical Economics* 12, 287–329.

Roth, A.E., 1982. Incentive compatibility in a market with indivisible goods. *Economics Letters* 9, 127–132.

Roth, A. E., Postlewaite, A. 1977. Weak versus strong domination in a market with indivisible goods, *Journal of Mathematical Economics* 4, 131–137.

Roth, A. E., Sönmez, T., Ünver, M. U., 2004. Kidney exchange, *Quarterly Journal of Economics* 119, 457–488.

Satterthwaite, M.A., and Sonnenschein, H., 1981. Strategy proof allocation mechanisms at differentiable points, *Review of Economic Studies* 48, 587–597.

Schlegel, J.C., 2022. The structure of equilibria in trading networks with frictions, *Theoretical Economics* 17, 801–839.

Schummer, J., 2000. Eliciting preferences to assign positions and compensation, *Games Economic Behavior* 30, 293–318.

Shapley, L., and Scarf, H., 1974. On cores and indivisibility, *Journal of Mathematical Economics* 1, 23–37.

Sönmez, T., 2023. Minimalist market design: A framework for economists with policy aspirations. Boston College Working Paper.

Sun, N., and Yang, Z., 2003. A general strategy proof fair allocation mechanism, *Economics Letters* 81, 73–79.

Sun, N., and Yang, Z., 2006. Equilibria and indivisibilities: Gross substitutes and complements, *Econometrica* 74, 1385–1402.

Svensson, L.-G., 2009. Coalitional strategy-proofness and fairness, *Economic Theory* 40, 227–245.

Vickrey, W., 1961. Counterspeculation, auctions, and competitive sealed tenders, *Journal of Finance* 16, 8–37.