Small Antenna Q and Gain – 80 Years of Progress

Mats Gustafsson
(based on collaboration with Lars, Marius, Arthur, Doruk, Casimir, Miloslav, Kurt, Lukas, Guy, Ben, Anja)

Electrical and Information Technology, Lund University, Sweden

Banff International Research Station 2019

Lund University 2015

Pioneering (theoretical) results 80 years ago

- Wheeler 1947 and Chu 1948

Physical Limitations of Omni-Directional Antennas*

L. J. Chu
Massachusetts Institute of Technology, Research Laboratory of Electronics, Boston, Massachusetts

(Received May 27, 1948)

The physical limitations of omni-directional antennas are considered. With the use of the spherical wave functions to describe the field, the directivity gain G and the Q of an unspecified antenna are calculated under idealized conditions. To obtain the optimum performance, three criteria are used, (1) maximum gain for a given complexity of the antenna structure, (2) minimum Q, (3) maximum ratio of G/Q. It is found that an antenna of which the maximum dimension is 2a has the potentiality of a broad band width provided that the gain is equal to or less than $4a/\lambda$. To obtain a gain higher than this value, the Q of the antenna increases at an astronomical rate. The antenna which has potentially the broadest band width of all omni-directional antennas is one which has a radiation pattern corresponding to that of an infinitesimally small dipole.

I. INTRODUCTION

An antenna system, functioning as a transmitter, provides a practical means of transmitting, to a distant point or points in space, a signal which appears in the form of r-f energy at the input terminals of the transmitter. The performance of such an antenna system is judged by the quality of transmission, which is measured by both the efficiency of transmission and the signal distortion. At a single frequency, trans-

* This work has been supported in part by the Signal Corps, the Air Materiel Command, and O.N.R.

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resolving power of a lens or a reflector is proportional to the ratio of the linear dimension to wave-length. Thus, over the entire frequency range, there seems to be a practical limit to the gain or the directivity of a radiating or focussing system.

From time to time, there arises the question of achieving a higher gain from an antenna of given size than has been obtained conventionally. Among published articles, Schelkunoff[1] has derived a mathematical expression for the current distribution along an array which yields higher directivity gain than that which has been usually obtained. It is mentioned at the end of this article that an array carrying this current distribution would have a narrow band width as well as high conduction loss. In 1943, LaPaz and Miller[2] obtained an optimum current distribution on a vertical antenna of given length

[1] S. A. Schelkunoff, Bell System Tech. J. **22**, 80–107 (1943).
[2] L. LaPaz and G. A. Miller, Proc. I.R.E. **31**, 214–232 (1943).

cality of supergain antennas. In his unpublished notes he derived the source distribution within a sphere of finite radius for any prescribed distribution of the radiation field in terms of a complete set of orthogonal, spherical, vector wave functions.[4] Mathematically, the series representing the source distribution diverges as the directivity gain of the system increases indefinitely. Physically, high current amplitude on the antenna, if it can be realized, implies high energy storage in the system, a large power dissipation, and a low transmission efficiency.

This paper presents an attempt to determine the optimum performance of an antenna in free space and the corresponding relation between its

[3] C. J. Bouwkamp and N. G. deBruijn, Philips Research Reports **1**, 135–158 (1946). This work was extended to the current distribution over an area by H. J. Riblet (Proc. I.R.E. **36**, 620–623 (1948)). Raymond M. Wilmotte, commenting on Riblet's work in a letter to the Editor (Proc. I.R.E. **36**, 878 (1948)), discussed the exceedingly low radiation resistance associated with discrete current distributions which have an abnormally high directivity.
[4] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941). Chap. 7, p. 392.

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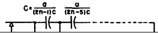
of pairs are reduced to $N+2$ including the input pair. It is interesting to observe that the instantaneous total energy density at any point outside the sphere is independent of time when Eq. (30) is satisfied. The difference between the mean electric energy density and the mean magnetic energy density is zero at any point outside the sphere enclosing the antenna. Furthermore, the instantaneous Poynting vector is independent of time. This implies that the power flow from the surface of the sphere enclosing a truly circularly polarized omni-directional antenna is a d.c. flow, and the instantaneous power is equal to the radiated power. These relationships are due to the dual nature of TE waves and TM waves as well as the 90° difference in time phase between the two sets of waves.

To obtain the Q of the antenna, it is convenient to combine the energies and dissipation in Z_n of the TM_n wave with that in Y_n of the TE_n wave and define a new Q_n as $2\omega W_n/P_n$ where W_n is the mean electric or magnetic energy stored in Z_n and Y_n, and P_n is the total power dissipated in both. Then

$$Q_n=\tfrac{1}{2}|\rho h_n|^2\rho\frac{dX_n}{d\rho}, \qquad (31)$$

where X_n is the imaginary part of Z_n. For $\rho=2\pi a/\lambda>n$, this Q_n is approximately equal to



III. FURTHER CONSIDERATIONS

A. Practical Limitations

The above analysis does not take into consideration many practical aspects of antenna design. In the following, a qualitative discussion will be given of some of the practical limitations.

It is assumed in the analysis that the antenna under consideration is located in free space. The results, with a minor modification are applicable to the problem of a vertically polarized antenna above a perfectly conducting ground plane. In practice, this condition can seldom be fulfilled. The performance of an antenna designed on the free-space basis will be modified by the presence of physical objects in the neighborhood. Currents will be induced on the objects. They will give rise not only to an additional scattered radiation field but also to a modification of the original current distribution on the antenna structure. Both the gain and Q of the antenna will be changed from their unperturbed values. The currents set up on the objects vary as the unperturbed field intensity at the locations of the objects. For the same power radiated, the r.m.s. amplitude of the unperturbed field intensity in the neighborhood of the antenna is approximately proportional to the square root of Q. In view of the rapid increase of Q as the gain of an antenna is increased above the normal value shown in Fig. 6, the disturbance of the field distribution in space by physical objects in the neighborhood of the antenna becomes increas-

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- maximal gain, minimum Q, maximal G/Q, circular polarization, surrounding structures, and bandwidth

the surface of the sphere. For a high-Q antenna, the ratio of the minimum conduction loss to the power radiated is therefore approximately proportional to the Q of the antenna computed in the absence of losses. Although this conduction loss is helpful in reducing the Q at the input terminals, it reduces the efficiency and the power gain of the antenna.

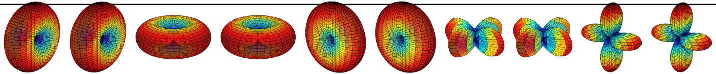
The condition of minimum energy storage within the sphere is not always realizable. On account of the unavoidable frequency sensitivities of the elements of the antenna structure or the matching networks, the Q of a practical antenna computed on the no conduction-loss basis will be usually higher than the one derived in this paper.

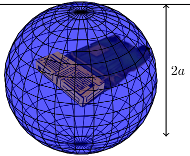
B. Band Width and Ideal Matching Network

We have computed the Q of an antenna from the energy stored in the equivalent circuit and the power radiated, and interpreted it freely as the reciprocal of the fractional band width. To be more accurate, one must define the band width in terms of allowable impedance variation or the tolerable reflection coefficient over the band. For a given antenna, the band width can be increased by choosing a proper matching network. The theoretical aspect of this problem has been dealt with by R. M. Fano.[6] Figure 11 given here through his courtesy illustrates the relations among the fractional band width, absolute amplitude of the reflection coefficient, and the parameter $2\pi a/\lambda$ of an antenna which has only the TM_1 wave outside the sphere. As shown in Section II, F this antenna has the lowest Q of all vertically polarized omni-directional antennas and its equivalent circuit is shown in Fig. 3. The curve of Fig. 11 is computed on the assumption that the input impedance of the antenna is equal to Z_1, and an ideal matching network is used to obtain a constant amplitude of the reflection coefficient over the band. The phase of the reflection coefficient, however, varies rapidly near the ends of the band.

[6] R. M. Fano, "Theoretical Limitations on the Broadband Matching of Arbitrary Impedances," R.L.E. Technical Report No. 41, January 2, 1948.

Bounds for spherical regions by Chu and Harrington





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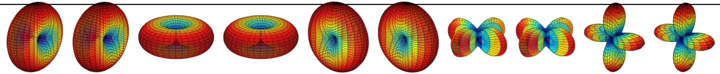

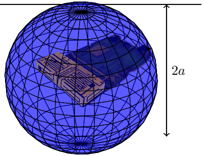

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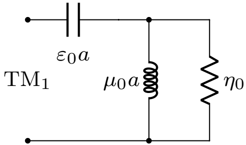

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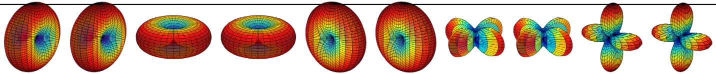

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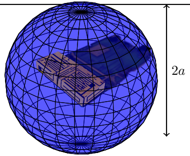

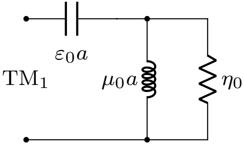

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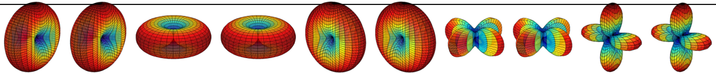

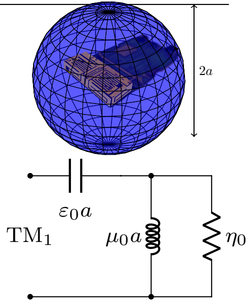

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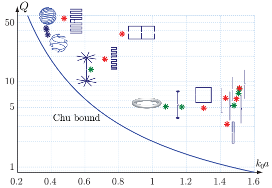

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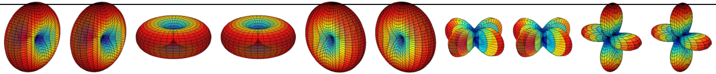

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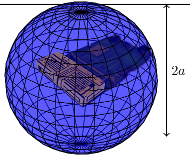
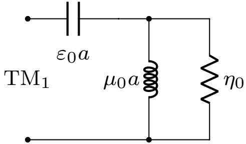
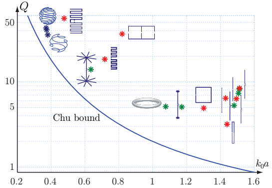
$$\text{Directivity:} \quad D \leq L^2 + 2L, \quad L \approx ka \text{ for } ka \geq 1$$







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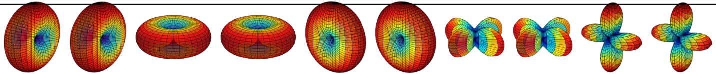

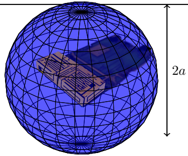

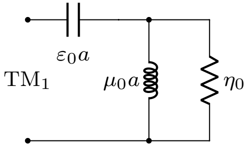

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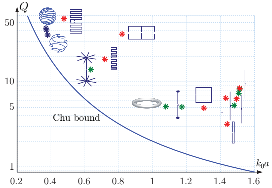

- Related investigations by *e.g.*, Hansen, Fante, Rhodes, Thal, McLean, Yaghjian, Kim, Sievenpiper, ...

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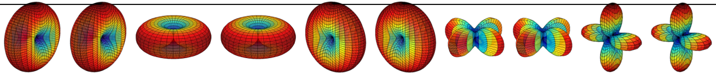

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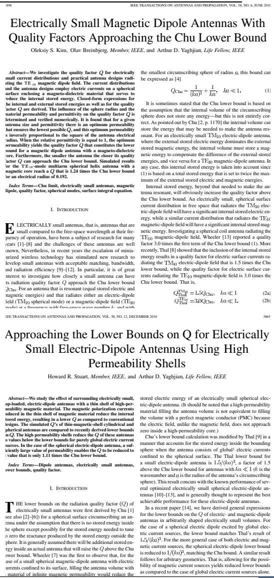

898 IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 58, NO. 6, JUNE 2010

Electrically Small Magnetic Dipole Antennas With Quality Factors Approaching the Chu Lower Bound

Oleksiy S. Kim, Olav Breinbjerg, *Member, IEEE*, and Arthur D. Yaghjian, *Life Fellow, IEEE*

Abstract—We investigate the quality factor Q for electrically small current distributions and practical antenna designs radiating the TE_{10} magnetic dipole field. The current distributions and the antenna designs employ electric currents on a spherical surface enclosing a magneto-dielectric material that serves to reduce the internal stored energy. Closed-form expressions for the internal and external stored energies as well as for the quality factor Q are derived. The influence of the sphere radius and the material permeability and permittivity on the quality factor Q is determined and verified numerically. It is found that for a given antenna size and permittivity there is an optimum permeability that ensures the lowest possible Q, and this optimum permeability is inversely proportional to the square of the antenna electrical radius. When the relative permittivity is equal to 1, the optimum permeability yields the quality factor Q that constitutes the lower bound for a magnetic dipole antenna with a magneto-dielectric core. Furthermore, the smaller the antenna the closer its quality factor Q can approach the Chu lower bound. Simulated results for the TE_{10}-mode multiarm spherical helix antenna with a magnetic core reach a Q that is 1.24 times the Chu lower bound for an electrical radius of 0.192.

Index Terms—Chu limit, electrically small antennas, magnetic dipole, quality factor, spherical modes, surface integral equation.

I. INTRODUCTION

...the smallest circumscribing sphere of radius a, this bound can be expressed as [4]

$$Q_{\text{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka}, \quad ka \ll 1. \quad (1)$$

It is sometimes stated that the Chu lower bound is based on the assumption that the internal volume of the circumscribing sphere does not store any energy—but this is not entirely correct. As pointed out by Chu [2, p. 1170] the internal volume can store the energy that may be needed to make the antenna resonant. For an electrically small TM_{10} electric-dipole antenna, where the external stored electric energy dominates the external stored magnetic energy, the internal volume must store a magnetic energy to compensate the difference of the external stored energies, and vice versa for a TE_{10} magnetic-dipole antenna. In any case, this internal stored energy is taken into account since (1) is based on a total stored energy that is set to twice the maximum of the external stored electric and magnetic energies.

Internal stored energy, beyond that needed to make the antenna resonant, will obviously increase the quality factor above the Chu lower bound. An electrically small, spherical surface current distribution in free space that radiates the TM_{10} electric-dipole field will have a significant internal stored electric energy, while a similar current distribution that radiates the TE_{10} ...

$$Q_{\text{Thal}}^{TM_{10}} = 1.5 Q_{\text{Chu}}, \quad ka \ll 1 \quad (2a)$$
$$Q_{\text{Thal}}^{TE_{10}} = 3.0 Q_{\text{Chu}}, \quad ka \ll 1. \quad (2b)$$

IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 58, NO. 12, DECEMBER 2010 3865

Approaching the Lower Bounds on Q for Electrically Small Electric-Dipole Antennas Using High Permeability Shells

Howard R. Stuart, *Member, IEEE*, and Arthur D. Yaghjian, *Life Fellow, IEEE*

Abstract—We study the effect of surrounding electrically small, top-loaded, electric-dipole antennas with a thin shell of high-permeability magnetic material. The magnetic polarization currents induced in the thin shell of magnetic material reduce the internal stored energy, resulting in a lower Q as compared to conventional designs. The simulated Q's of thin-magnetic-shell cylindrical and spherical antennas are compared to recently derived lower bounds on Q. The high permeability shells reduce the Q of these antennas to values below the lower bounds for purely global electric current sources. In the case of the spherical electric-dipole antenna, a sufficiently large value of permeability enables the Q to be reduced to a value that is only 1.11 times the Chu lower bound.

Index Terms—Dipole antennas, electrically small antennas, lower bounds, quality factor.

I. INTRODUCTION

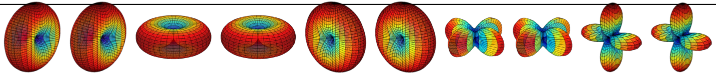
THE lower bounds on the radiation quality factor (Q) of electrically small antennas were first derived by Chu [1] ...

... stored electric energy of an electrically small spherical electric-dipole antenna. (It should be noted that a high-permeability material filling the antenna volume is not equivalent to filling the volume with a perfect magnetic conductor (PMC) because the electric field, unlike the magnetic field, does not approach zero inside a high-permeability core.)

Chu's lower bound calculation was modified by Thal [9] in a manner that accounts for the stored energy inside the bounding sphere when the antenna consists of global[1] electric currents confined to the spherical surface. The Thal lower bound for a small electric-dipole antenna is $1.5/(ka)^3$, a factor of 1.5 above the Chu lower bound for antennas with $ka \ll 1$ (k is the wavenumber and a is the radius of the antenna's circumscribing sphere). This result concurs with the known performance of several optimized electrically small spherical electric-dipole antennas [10]–[13], and is generally thought to represent the best achievable performance for these electric-dipole antennas.

In a recent paper [14], we have derived general expressions ...

Bounds for spherical regions by Chu and Harrington



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$$\text{Directivity:} \quad D \leq L^2 + 2L, \quad L \approx ka \text{ for } ka \geq 1$$

- Related investigations by *e.g.*, Hansen, Fante, Rhodes, Thal, McLean, Yaghjian, Kim, Sievenpiper, ...

Much knowledge but also many questions:

- How is bandwidth related to Q?
- How do materials affect the results?

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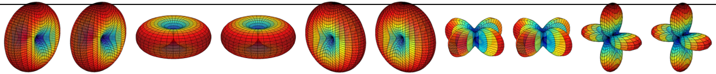

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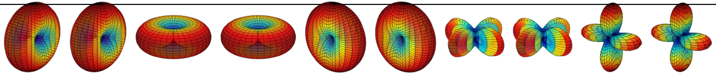

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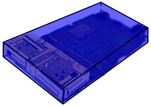
- How do non-spherical shapes perform? Why does the bandwidth depend on the thickness of a wire dipole?
- Can small antennas have $D > 3$?

Much progress over the years

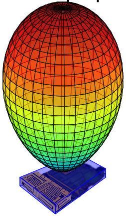
Theory and practice of small antennas have developed tremendous over the years with contributions from several researchers. This presentation is focused on theory. For more comprehensive overviews, design, and other perspectives, see *e.g.*, [Fuj+88; GTC15; Har68; HC11; VCF10]

- Bandwidth, Q-factor, and stored energy

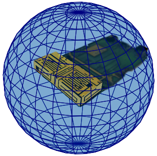
Arbitrarily shaped design regions



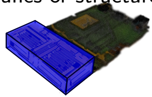
Radiation properties

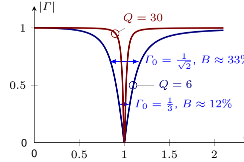


Spherical design region



Design regions close to ground planes or structures





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Much progress over the years

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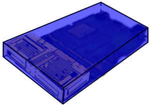
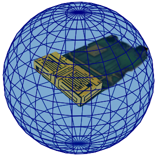
- Bandwidth, Q-factor, and stored energy
- Materials and temporal dispersion

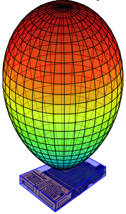
Arbitrarily shaped design regions



Spherical design region



Radiation properties



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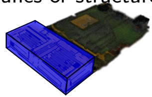

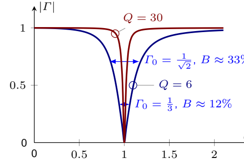

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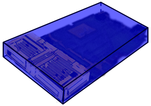
- Bandwidth, Q-factor, and stored energy
- Materials and temporal dispersion
- Arbitrary shapes

Arbitrarily shaped design regions



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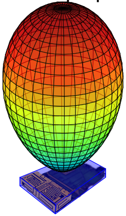
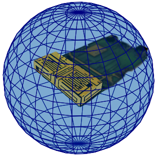
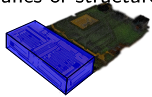
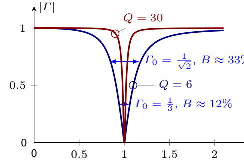


Spherical design region



Design regions close to ground planes or structures





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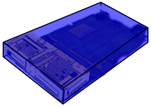
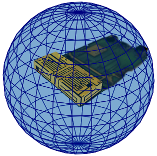
- Bandwidth, Q-factor, and stored energy
- Materials and temporal dispersion
- Arbitrary shapes
- Gain, directivity, and efficiency

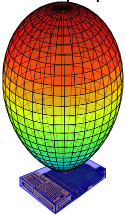
Arbitrarily shaped design regions



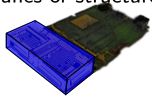
Spherical design region

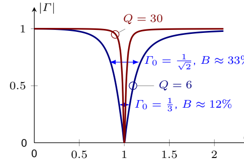


Radiation properties



Design regions close to ground planes or structures





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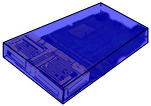
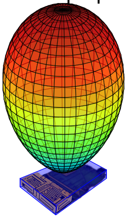
- Bandwidth, Q-factor, and stored energy
- Materials and temporal dispersion
- Arbitrary shapes
- Gain, directivity, and efficiency
- Optimization formulations

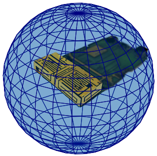
Arbitrarily shaped design regions



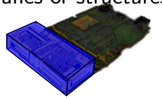
Radiation properties

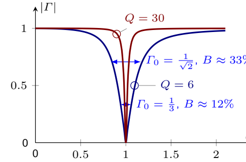


Spherical design region



Design regions close to ground planes or structures





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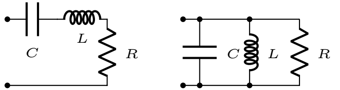
Stored energy and Q-factor

- Energy definition of the Q-factor from the ratio between the stored electric, W_{e}, and magnetic, W_{m}, energies and the dissipated power, *i.e.*,

$$Q = \frac{2\omega \max\{W_{\mathrm{e}}, W_{\mathrm{m}}\}}{P_{\mathrm{d}}}$$



$$W_{\mathrm{e}} = \frac{C|V|^2}{4} \quad \text{and } W_{\mathrm{m}} = \frac{L|I|^2}{4}$$

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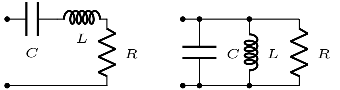
- Fractional bandwidth for single resonances [YB05]

$$B \approx \frac{2}{Q}\frac{\Gamma_0}{\sqrt{1-\Gamma_0^2}}$$



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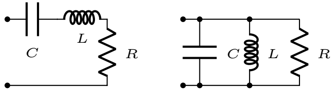

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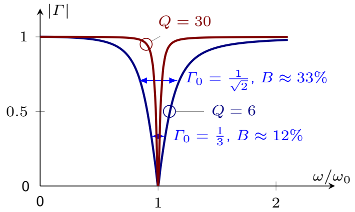

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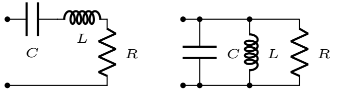

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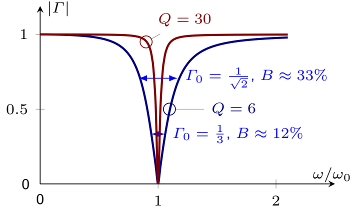

Yaghjian and Best 2005

- Influential, many results, 27 pages

Impedance, Bandwidth, and Q of Antennas

Arthur D. Yaghjian, *Fellow, IEEE,* and Steven R. Best, *Senior Member, IEEE*

***Abstract*—To address the need for fundamental universally valid definitions of exact bandwidth and quality factor (Q) of tuned antennas, as well as the need for efficient accurate approximate formulas for computing this bandwidth and Q, exact and approximate expressions are found for the bandwidth and Q of a general single-feed (one-port) lossy or lossless linear antenna tuned to resonance or antiresonance. The approximate expression derived for the exact bandwidth of a tuned antenna differs from previous approximate expressions in that it is inversely proportional to the magnitude $|Z_0'(\omega_0)|$ of the frequency derivative of the input impedance and, for not too large a bandwidth, it is nearly equal to the exact bandwidth of the tuned antenna at every frequency ω_0, that is, throughout antiresonant as well as resonant frequency bands. It is also shown that an appropriately defined exact Q of a tuned lossy or lossless antenna is approximately proportional to $|Z_0'(\omega_0)|$ and thus this Q is approximately inversely proportional to the bandwidth (for not too large a bandwidth) of a simply tuned antenna at all frequencies. The exact Q of a tuned antenna is defined in terms of average internal energies that emerge naturally from Maxwell's equations applied to the tuned antenna. These internal energies, which are similar but not identical to previously defined quality-factor energies, and the associated Q are proven to increase without bound as the size of an antenna is decreased. Numerical solutions to thin straight-wire and wire-loop lossy and lossless antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a lossy dispersive dielectric, confirm the accuracy of the approximate expressions and the inverse relationship between the defined bandwidth and the defined Q over frequency ranges that cover several resonant and antiresonant frequency bands.**

***Index Terms*—Antennas, antiresonance, bandwidth, impedance, quality factor, resonance.**

I. Introduction

THE primary purpose of this paper is twofold: first, to define a fundamental, universally applicable measure of bandwidth of a tuned antenna and to derive a useful approximate expression for this bandwidth in terms of the antenna's input impedance that holds at every frequency, that is, throughout the entire antiresonant as well as resonant frequency ranges of the antenna; and second, to define an exact antenna quality factor Q independently of bandwidth, to derive an approximate expression for this exact Q, and to show that this Q is approximately inversely proportional to the defined bandwidth.

The average "internal" electric, magnetic, and magnetoelectric energies that we use to define the exact Q of a linear antenna are similar though not identical to those of previous au-

and its relationship to Q are both more generally applicable and more accurate than previous formulas. As part of the derivation of the relationship between bandwidth and Q, exact expressions for the input impedance of the antenna and its derivative with respect to frequency are found in terms of the fields of the antenna. The exact Q of a general lossy or lossless antenna is also re-expressed in terms of two dispersion energies and the frequency derivative of the input reactance of the antenna. The value of the total internal energy, as well as one of these dispersion energies, for an antenna with an asymmetric far-field magnitude pattern, and thus the value of Q for such an antenna, is shown to depend on the chosen position of the origin of the coordinate system to which the fields of the antenna are referenced. A practical method is found to emerge naturally from the derivations that removes this ambiguity from the definition of Q for a general antenna.[1] The validity and accuracy of the expressions are confirmed by the numerical solutions to straight-wire and wire-loop, lossy and lossless tuned antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a frequency dependent dielectric material, over a wide enough range of frequencies to cover several resonant and antiresonant frequency bands. The remainder of the paper, many of the results of which were first presented in [9], is organized as follows.

Preliminary definitions required for the derivations of the expressions for impedance, bandwidth, and Q of an antenna are given in Section II.

In Section III, the fractional conductance bandwidth and the fractional matched voltage-standing-wave-ratio (VSWR) bandwidth are defined and determined approximately for a general tuned antenna in terms of the input resistance and magnitude of the frequency derivative of the input impedance of the antenna. It is shown that the matched VSWR bandwidth is the more fundamental measure of bandwidth because, unlike the conductance bandwidth, it exists in general for all frequencies at which an antenna is tuned. (Throughout this paper, we are considering only the bandwidth relative to a change in the accepted power and not to any additional loss of bandwidth caused, for example, by a degradation of the far-field pattern of the antenna.)

In Section IV, the input impedance, its frequency derivative, the internal energies, and the Q of a tuned antenna are given in terms of the antenna fields, and the relationship between bandwidth and Q is determined. In particular, the frequency derivative of the input reactance is expressed in terms of integrals of

Yaghjian and Best 2005

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$$Q_{Z'_{\text{in}}} = \left|\omega Z'_{\text{in}}\right| / (2R_{\text{in}})$$

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$$Q_{Z'_{\mathrm{in}}} = \left|\omega Z'_{\mathrm{in}}\right| / (2R_{\mathrm{in}})$$

- Reactive (stored) energy [Fan69; Lev57]

$$W_{\mathrm{F,e}} = \frac{\epsilon_0}{4}\int_{\mathbb{R}^3_{\mathrm{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2}\,\mathrm{dV}$$

for complex materials and temporal dispersion, more in [Yag07]

Impedance, Bandwidth, and Q of Antennas

Arthur D. Yaghjian, *Fellow, IEEE,* and Steven R. Best, *Senior Member, IEEE*

***Abstract*—To address the need for fundamental universally valid definitions of exact bandwidth and quality factor (Q) of tuned antennas, as well as the need for efficient accurate approximate formulas for computing this bandwidth and Q, exact and approximate expressions are found for the bandwidth and Q of a general single-feed (one-port) lossy or lossless linear antenna tuned to resonance or antiresonance. The approximate expression derived for the exact bandwidth of a tuned antenna differs from previous approximate expressions in that it is inversely proportional to the magnitude $|Z_0'(\omega_0)|$ of the frequency derivative of the input impedance and, for not too large a bandwidth, it is nearly equal to the exact bandwidth of the tuned antenna at every frequency ω_0, that is, throughout antiresonant as well as resonant frequency bands. It is also shown that an appropriately defined exact Q of a tuned lossy or lossless antenna is approximately proportional to $|Z_0'(\omega_0)|$ and thus this Q is approximately inversely proportional to the bandwidth (for not too large a bandwidth) of a simply tuned antenna at all frequencies. The exact Q of a tuned antenna is defined in terms of average internal energies that emerge naturally from Maxwell's equations applied to the tuned antenna. These internal energies, which are similar but not identical to previously defined quality-factor energies, and the associated Q are proven to increase without bound as the size of an antenna is decreased. Numerical solutions to thin straight-wire and wire-loop lossy and lossless antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a lossy dispersive dielectric, confirm the accuracy of the approximate expressions and the inverse relationship between the defined bandwidth and the defined Q over frequency ranges that cover several resonant and antiresonant frequency bands.**

***Index Terms*—Antennas, antiresonance, bandwidth, impedance, quality factor, resonance.**

I. Introduction

THE primary purpose of this paper is twofold: first, to define a fundamental, universally applicable measure of bandwidth of a tuned antenna and to derive a useful approximate expression for this bandwidth in terms of the antenna's input impedance that holds at every frequency, that is, throughout the entire antiresonant as well as resonant frequency ranges of the antenna; and second, to define an exact antenna quality factor Q independently of bandwidth, to derive an approximate expression for this exact Q, and to show that this Q is approximately inversely proportional to the defined bandwidth.

The average "internal" electric, magnetic, and magnetoelectric energies that we use to define the exact Q of a linear antenna are similar though not identical to those of previous au-

and its relationship to Q are both more generally applicable and more accurate than previous formulas. As part of the derivation of the relationship between bandwidth and Q, exact expressions for the input impedance of the antenna and its derivative with respect to frequency are found in terms of the fields of the antenna. The exact Q of a general lossy or lossless antenna is also re-expressed in terms of two dispersion energies and the frequency derivative of the input reactance of the antenna. The value of the total internal energy, as well as one of these dispersion energies, for an antenna with an asymmetric far-field magnitude pattern, and thus the value of Q for such an antenna, is shown to depend on the chosen position of the origin of the coordinate system to which the fields of the antenna are referenced. A practical method is found to emerge naturally from the derivations that removes this ambiguity from the definition of Q for a general antenna.[1] The validity and accuracy of the expressions are confirmed by the numerical solutions to straight-wire and wire-loop, lossy and lossless tuned antennas, as well as to a Yagi antenna and a straight-wire antenna embedded in a frequency dependent dielectric material, over a wide enough range of frequencies to cover several resonant and antiresonant frequency bands. The remainder of the paper, many of the results of which were first presented in [9], is organized as follows.

Preliminary definitions required for the derivations of the expressions for impedance, bandwidth, and Q of an antenna are given in Section II.

In Section III, the fractional conductance bandwidth and the fractional matched voltage-standing-wave-ratio (VSWR) bandwidth are defined and determined approximately for a general tuned antenna in terms of the input resistance and magnitude of the frequency derivative of the input impedance of the antenna. It is shown that the matched VSWR bandwidth is the more fundamental measure of bandwidth because, unlike the conductance bandwidth, it exists in general for all frequencies at which an antenna is tuned. (Throughout this paper, we are considering only the bandwidth relative to a change in the accepted power and not to any additional loss of bandwidth caused, for example, by a degradation of the far-field pattern of the antenna.)

In Section IV, the input impedance, its frequency derivative, the internal energies, and the Q of a tuned antenna are given in terms of the antenna fields, and the relationship between bandwidth and Q is determined. In particular, the frequency derivative of the input reactance is expressed in terms of integrals of

Yaghjian and Best 2005

- Influential, many results, 27 pages
- Q-factor from the input impedance (tuned)

$$Q_{Z'_{\text{in}}} = \left|\omega Z'_{\text{in}}\right| / (2R_{\text{in}})$$

- Reactive (stored) energy [Fan69; Lev57]

$$W_{\text{F,e}} = \frac{\epsilon_0}{4} \int_{\mathbb{R}^3_{\text{r}}} |\boldsymbol{E}(\boldsymbol{r})|^2 - \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2}{r^2} \, \mathrm{dV}$$

  for complex materials and temporal dispersion, more in [Yag07]
- Coordinate dependence for non-symmetrical far-fields $\boldsymbol{F}$ [YB05]

$$\int_{4\pi} \hat{\boldsymbol{r}} |\boldsymbol{F}(\hat{\boldsymbol{r}})|^2 \, \mathrm{d}\Omega \neq \boldsymbol{0}$$

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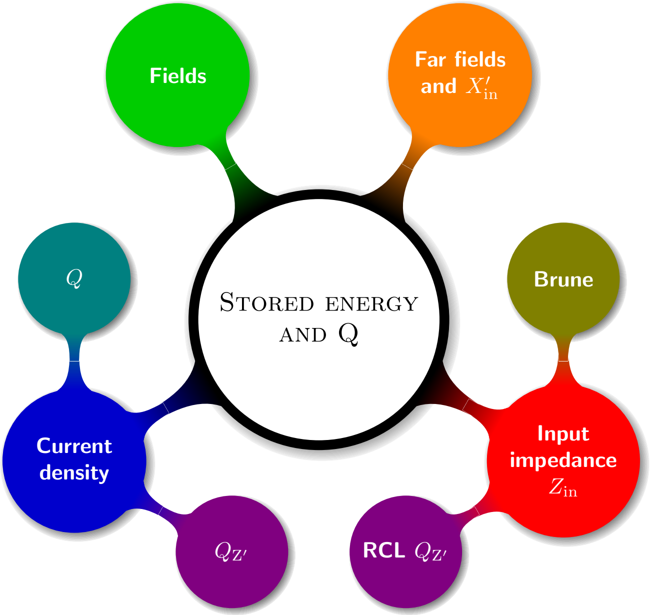
- Q-factor and stored (reactive) energy using
  - Fields: subtracting Poynting vector or far field
  - Currents and charges
  - Input impedance: differentiate or circuit fit
- similar (identical) results for electrically small structures
- some results for temporally dispersive media

K. Schab et al. "Energy stored by radiating systems". *IEEE Access* 6 (2018), pp. 10553 –10568



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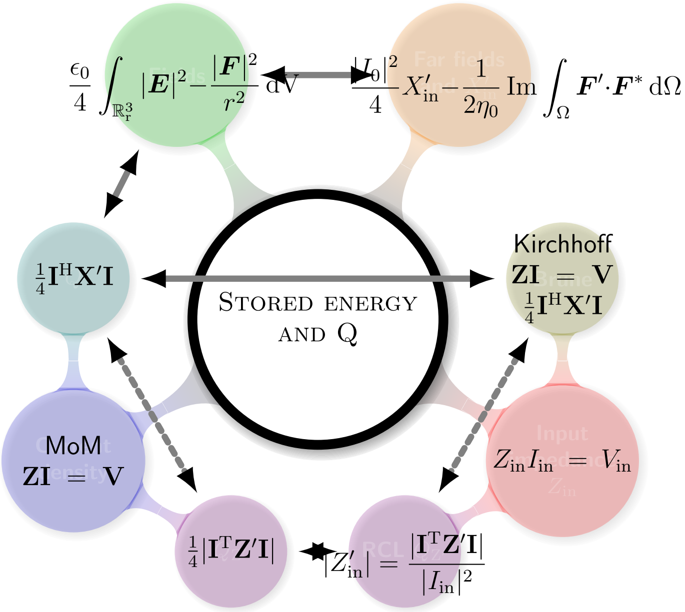

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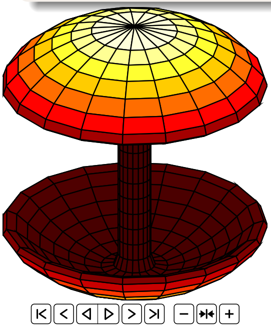
Stored energy expressed in **currents and charges**

Stored electric energy by Vandenbosch 2010 [Van10] (also [Gey03] in limit $ka \to 0$).

$$W_\mathrm{e} = \frac{\eta_0}{4\omega} \int_\Omega \int_\Omega \nabla_1 \cdot \boldsymbol{J}(\boldsymbol{r}_1) \nabla_2 \cdot \boldsymbol{J}^*(\boldsymbol{r}_2) \frac{\cos(k|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi k|\boldsymbol{r}_1 - \boldsymbol{r}_2|} - \frac{1}{2}\big(k^2 \boldsymbol{J}(\boldsymbol{r}_1) \cdot \boldsymbol{J}^*(\boldsymbol{r}_2) - \nabla_1 \cdot \boldsymbol{J}(\boldsymbol{r}_1) \nabla_2 \cdot \boldsymbol{J}^*(\boldsymbol{r}_2)\big) \frac{\sin(k|\boldsymbol{r}_1 - \boldsymbol{r}_2|)}{4\pi} \,\mathrm{dV}_1 \,\mathrm{dV}_2$$



- Can be derived from the subtracted far-field energy W_F ([GJ15])

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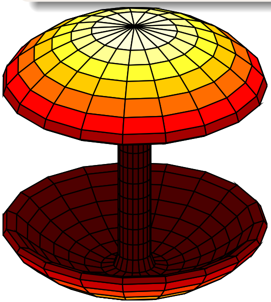

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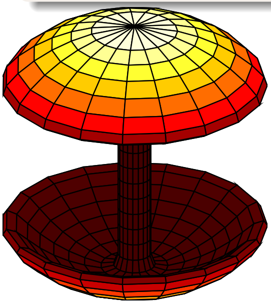

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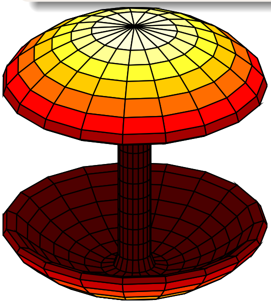

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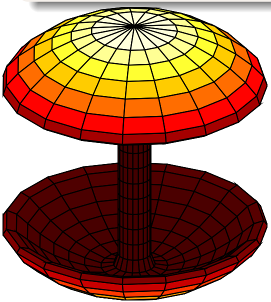

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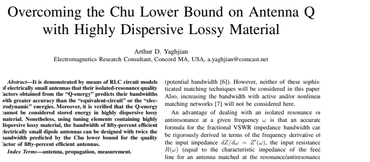

Overcoming the Chu Lower Bound on Antenna Q with Highly Dispersive Lossy Material

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***Index Terms*—antenna, propagation, measurement.**

(potential bandwidth [6]). However, neither of these sophisticated matching techniques will be considered in this paper. Also, increasing the bandwidth with active and/or nonlinear matching networks [7] will not be considered here.

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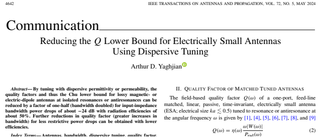

4642 IEEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 72, NO. 5, MAY 2024

Communication

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***Abstract*—By tuning with dispersive permittivity or permeability, the quality factors and thus the Chu lower bound for lossy magnetic- or electric-dipole antennas at isolated resonances or antiresonances can be reduced by a factor of one-half (bandwidth doubled) for input-impedance bandwidth power drops of about −24 dB with radiation efficiencies of about 50%. Further reductions in quality factor (greater increases in bandwidth) for less restrictive power drops can be obtained with lower efficiencies.**

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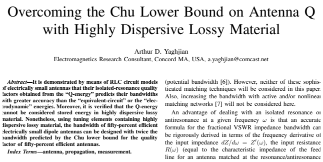

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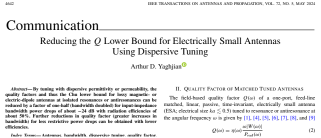

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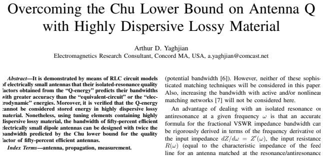

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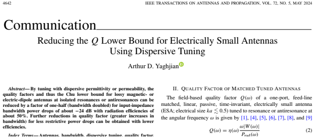

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II. Quality Factor of Matched Tuned Antennas

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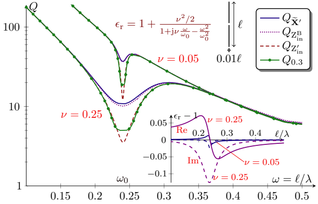

M. Gustafsson and C. Ehrenborg. "State-space models and stored electromagnetic energy for antennas in dispersive and heterogeneous media". *Radio Sci.* 52 (2017)

Material effects on Q and stored energy

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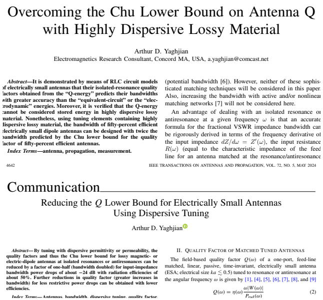
Overcoming the Chu Lower Bound on Antenna Q with Highly Dispersive Lossy Material

Arthur D. Yaghjian
Electromagnetics Research Consultant, Concord MA, USA, a.yaghjian@comcast.net

***Abstract*—It is demonstrated by means of RLC circuit models of electrically small antennas that their isolated-resonance quality factors obtained from the "Q-energy" predicts their bandwidths with greater accuracy than the "equivalent-circuit" or the "electrodynamic" energies. Moreover, it is verified that the Q-energy cannot be considered stored energy in highly dispersive lossy material. Nonetheless, using tuning elements containing highly dispersive lossy material, the bandwidth of fifty-percent efficient electrically small dipole antennas can be designed with twice the bandwidth predicted by the Chu lower bound for the quality factor of fifty-percent efficient antennas.**

***Index Terms*—antenna, propagation, measurement.**

(potential bandwidth [6]). However, neither of these sophisticated matching techniques will be considered in this paper. Also, increasing the bandwidth with active and/or nonlinear matching networks [7] will not be considered here.

An advantage of dealing with an isolated resonance or antiresonance at a given frequency ω is that an accurate formula for the fractional VSWR impedance bandwidth can be rigorously derived in terms of the frequency derivative of the input impedance $dZ/d\omega = Z'(\omega)$, the input resistance $R(\omega)$ (equal to the characteristic impedance of the feed line for an antenna matched at the resonance/antiresonance

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Communication

Reducing the Q Lower Bound for Electrically Small Antennas Using Dispersive Tuning

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***Abstract*—By tuning with dispersive permittivity or permeability, the quality factors and thus the Chu lower bound for lossy magnetic- or electric-dipole antennas at isolated resonances or antiresonances can be reduced by a factor of one-half (bandwidth doubled) for input-impedance bandwidth power drops of about −24 dB with radiation efficiencies of about 50%. Further reductions in quality factor (greater increases in bandwidth) for less restrictive power drops can be obtained with lower efficiencies.**

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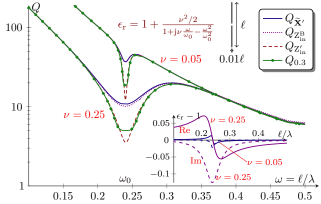

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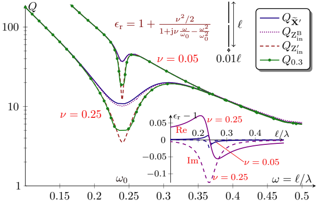

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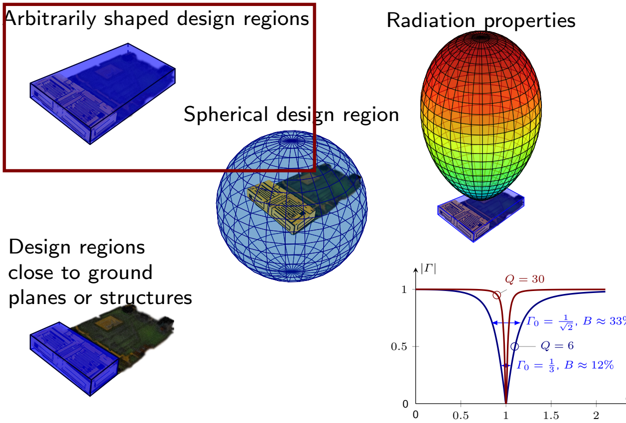
Much progress over the years



K. Fujimoto et al. *Small Antennas*. Antenna Series. Letchworth, England: Research Studies Press, 1988; M. Gustafsson, D. Tayli, and M. Cismasu. "Physical bounds of antennas". In: *Handbook of Antenna Technologies*. Ed. by Z. N. Chen. Springer-Verlag, 2015, pp. 197–233; R. F. Harrington. *Field Computation by Moment Methods*. New York, NY: Macmillan, 1968; R. C. Hansen and R. E. Collin. *Small Antenna Handbook*. Wiley, 2011; J. Volakis, C. C. Chen, and K. Fujimoto. *Small Antennas: Miniaturization Techniques & Applications*. New York, NY: McGraw-Hill, 2010

Q_{lb} for arbitrary shaped electrically small antennas

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Lower Bounds on the Q of Electrically Small Dipole Antennas

Arthur D. Yaghjian, *Life Fellow, IEEE*, and Howard R. Stuart, *Member, IEEE*

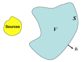

Fig. 1. Electric-dipole antenna divided into the antenna-proper V and sources of the incident fields.

MINIMUM Q FOR LOSSY AND LOSSLESS ELECTRICALLY SMALL DIPOLE ANTENNAS

Arthur D. Yaghjian[1,*], Mats Gustafsson[2], and B. Lars G. Jonsson[3]

[1]Electromagnetics Research Consultant, 115 Wright Road, Concord, MA 01742, USA

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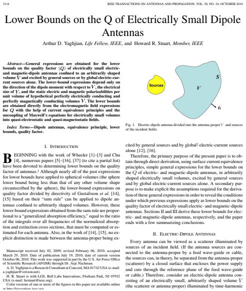

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Arthur D. Yaghjian, *Life Fellow, IEEE*, and Howard R. Stuart, *Member, IEEE*

***Abstract*—General expressions are obtained for the lower bounds on the quality factor (Q) of electrically small electric- and magnetic-dipole antennas confined to an arbitrarily shaped volume V and excited by general sources or by global electric-current sources alone. The lower-bound expressions depend only on the direction of the dipole moment with respect to V, the electrical size of V, and the static electric and magnetic polarizabilities per unit volume of hypothetical perfectly electrically conducting and perfectly magnetically conducting volumes V. The lower bounds are obtained directly from the electromagnetic field expressions for Q with the help of current equivalence principles and the uncoupling of Maxwell's equations for electrically small volumes into quasi-electrostatic and quasi-magnetostatic fields.**

***Index Terms*—Dipole antennas, equivalence principle, lower bounds, quality factor.**

Fig. 1. Electric-dipole antenna divided into the antenna-proper V and sources of the incident fields.

I. Introduction

BEGINNING with the work of Wheeler [1]–[3] and Chu [4], numerous papers [5]–[16], [37] (to cite a partial list) have been devoted to determining lower bounds on the quality factor of antennas.[1] Although nearly all of the past expressions for lower bounds have applied to spherical volumes (the sphere lower bound being less than that of any other volume shape circumscribed by the sphere), the lower-bound expressions on quality factor divided by directivity of Gustafsson *et al.* [14], [15] based on their "sum rule" can be applied to dipole antennas confined to arbitrarily shaped volumes. However, these lower-bound expressions obtained from the sum rule are proportional to a "generalized absorption efficiency," equal to the ratio of the integrals over all frequencies of the normalized absorption and extinction cross sections, that must be computed or estimated for each antenna. Also, in the work of [14], [15], no explicit distinction is made between the antenna-proper being excited by general sources and by global[2] electric-current sources alone [12], [16].

Therefore, the primary purpose of the present paper is to obtain through direct derivation, using surface current equivalence principles, simple general expressions for the lower bounds on the Q of electric- and magnetic-dipole antennas, in arbitrarily shaped electrically small volumes, excited by general sources and by global electric-current sources alone. A secondary purpose is to make explicit the assumptions required for the derivation of these general expressions in order to reveal the conditions under which previous expressions apply as lower bounds on the quality factor of electrically small electric- and magnetic-dipole antennas. Sections II and III derive these lower bounds for electric- and magnetic-dipole antennas, respectively, and the paper ends with a few summarizing conclusions.

II. Electric-Dipole Antennas

Every antenna can be viewed as a scatterer illuminated by sources of an incident field. (If the antenna sources are connected to the antenna-proper by a feed wave-guide or cable, the sources can, in theory, be separated from the antenna-proper (scatterer) by a closed surface that encloses the power supply and cuts through the reference plane of the feed wave-guide or cable.) Therefore, consider an electric-dipole antenna consisting of an electrically small, arbitrarily shaped volume V (the scatterer or antenna-proper) illuminated by time-harmonic

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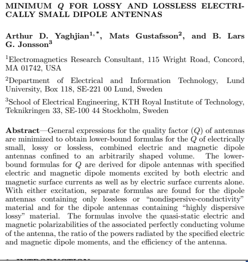

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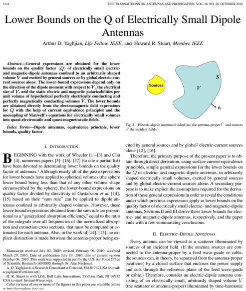
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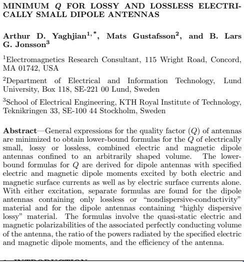
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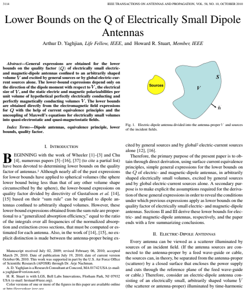

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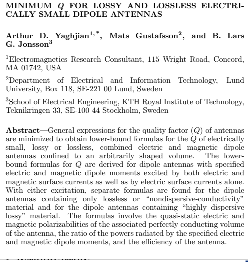

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[1]Electromagnetics Research Consultant, 115 Wright Road, Concord, MA 01742, USA

[2]Department of Electrical and Information Technology, Lund University, Box 118, SE-221 00 Lund, Sweden

[3]School of Electrical Engineering, KTH Royal Institute of Technology, Teknikringen 33, SE-100 44 Stockholm, Sweden

Abstract—General expressions for the quality factor (Q) of antennas are minimized to obtain lower-bound formulas for the Q of electrically small, lossy or lossless, combined electric and magnetic dipole antennas confined to an arbitrarily shaped volume. The lower-bound formulas for Q are derived for dipole antennas with specified electric and magnetic dipole moments excited by both electric and magnetic surface currents as well as by electric surface currents alone. With either excitation, separate formulas are found for the dipole antennas containing only lossless or "nondispersive-conductivity" material and for the dipole antennas containing "highly dispersive lossy" material. The formulas involve the quasi-static electric and magnetic polarizabilities of the associated perfectly conducting volume of the antenna, the ratio of the powers radiated by the specified electric and magnetic dipole moments, and the efficiency of the antenna.

M. Gustafsson, C. Sohl, and G. Kristensson. "Physical limitations on antennas of arbitrary shape". *Proc. R. Soc. A* 463 (2007), pp. 2589–2607; M. Gustafsson, C. Sohl, and G. Kristensson. "Illustrations of New Physical Bounds on Linearly Polarized Antennas". *IEEE Trans. Antennas Propag.* 57.5 (May 2009), pp. 1319–1327; C. Sohl and M. Gustafsson. "A priori estimates on the partial realized gain of Ultra-Wideband (UWB) antennas". *Quart. J. Mech. Appl. Math.* 61.3 (2008), pp. 415–430; A. D. Yaghjian, M. Gustafsson, and B. L. G Jonsson. "Minimum Q for Lossy and Lossless Electrically Small Dipole Antennas". *Progress In Electromagnetics Research* 143 (2013), pp. 641–673; A. D. Yaghjian and H. R. Stuart. "Lower Bounds on the Q of Electrically Small

Polarizability and EM antenna size

- Bounds for TM dipole radiation

$$Q_{\mathrm{lb,TM}} = \frac{6\pi}{k^3 \max \operatorname{eig} \boldsymbol{\gamma}_{\mathrm{e}}} \geq \frac{6\pi}{k^3 \max \operatorname{eig} \boldsymbol{\gamma}_{\infty}}$$

with high-contrast $\boldsymbol{\gamma}_{\infty}$ polarizability dyadic [GSK07; GSK09; YS10]

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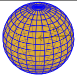
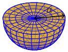
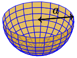
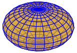
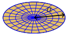
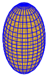
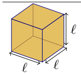
- Analytical expressions for many shapes [GTC15]

geometry	high contrast polarizability dyadic $\boldsymbol{\gamma}_\infty$
	Sphere with radius a [82] $\boldsymbol{\gamma}_\infty = \gamma_{\mathrm{sph}}\mathbf{1} = 4\pi a^3\mathbf{1} \approx 12.57a^3\mathbf{1}$
	Solid hemisphere with radius a [82] $\boldsymbol{\gamma}_\infty = 4\pi\big(2-\frac{59}{27\sqrt{3}}\big)a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{4}{27\sqrt{3}}\big(\frac{64}{3}-\frac{25}{16}(\sqrt{3}+1)\big)a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 9.28a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 4.59a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.74\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.36\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Hemispherical shell with radius a [82] $\boldsymbol{\gamma}_\infty = \big(2\pi+\frac{8}{3}\big)a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \big(2\pi-\frac{16}{3}+\frac{4\pi}{2+\pi}\big)a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 8.95a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 3.39a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.71\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.27\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Oblate spheroid with width $2a$ and height $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e = \sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{4\pi\xi e^3}{3(e-\xi\arccos\xi)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{8\pi e^3}{3(\arccos\xi-\xi e)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Circular disc with radius a [44] $\boldsymbol{\gamma}_\infty = \frac{16}{3}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 5.33a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 0.42\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}})$
	Prolate spheroid with height $2a$ and width $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e = \sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{8\pi e^3}{3(\ln\frac{1+e}{1-e}-2e)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{16\pi\xi^2e^3}{3(2e-\xi^2\ln\frac{1+e}{1-e})}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx \frac{4\pi}{3(\ln 2-\ln\xi-1)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \mathrm{O}(\xi^2)$ as $\xi \to 0$
	Cube with side lengths $\ell = 2a/\sqrt{3}$ [66, 98] $\boldsymbol{\gamma}_\infty \approx 3.644305190268\ell^3\mathbf{1} \approx 5.61a^3\mathbf{1} \approx 0.45\gamma_{\mathrm{sph}}\mathbf{1}$

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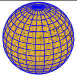
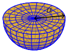
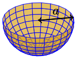
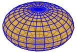
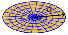
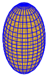
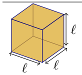
	Sphere with radius a [82] $\boldsymbol{\gamma}_\infty = \gamma_{\mathrm{sph}}\mathbf{1} = 4\pi a^3\mathbf{1} \approx 12.57a^3\mathbf{1}$
	Solid hemisphere with radius a [82] $\boldsymbol{\gamma}_\infty = 4\pi\big(2-\frac{59}{27\sqrt{3}}\big)a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{4}{27\sqrt{3}}\big(\frac{64}{3}-\frac{25}{16}(\sqrt{3}+1)\big)a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 9.28a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 4.59a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.74\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.36\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Hemispherical shell with radius a [82] $\boldsymbol{\gamma}_\infty = \big(2\pi+\frac{8}{3}\big)a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \big(2\pi-\frac{16}{3}+\frac{4\pi}{2+\pi}\big)a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 8.95a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 3.39a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.71\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.27\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Oblate spheroid with width $2a$ and height $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e=\sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{4\pi\xi e^3}{3(e-\xi\arccos\xi)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{8\pi e^3}{3(\arccos\xi-\xi e)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Circular disc with radius a [44] $\boldsymbol{\gamma}_\infty = \frac{16}{3}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 5.33a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 0.42\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}})$
	Prolate spheroid with height $2a$ and width $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e=\sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{8\pi e^3}{3(\ln\frac{1+e}{1-e}-2e)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{16\pi\xi^2 e^3}{3(2e-\xi^2\ln\frac{1+e}{1-e})}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx \frac{4\pi}{3(\ln 2-\ln\xi-1)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \mathrm{O}(\xi^2)$ as $\xi\to 0$
	Cube with side lengths $\ell = 2a/\sqrt{3}$ [66, 98] $\boldsymbol{\gamma}_\infty \approx 3.644305190268\ell^3\mathbf{1} \approx 5.61a^3\mathbf{1} \approx 0.45\gamma_{\mathrm{sph}}\mathbf{1}$

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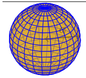
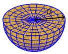
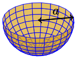
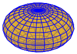
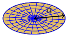
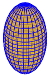
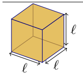
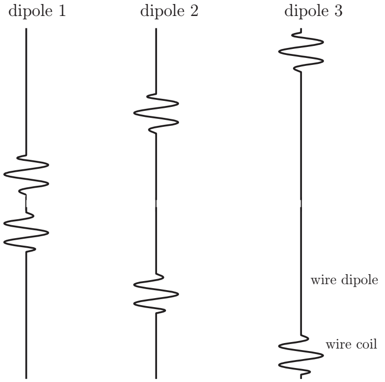
	Sphere with radius a [82] $\boldsymbol{\gamma}_\infty = \gamma_{\mathrm{sph}}\mathbf{1} = 4\pi a^3\mathbf{1} \approx 12.57a^3\mathbf{1}$
	Solid hemisphere with radius a [82] $\boldsymbol{\gamma}_\infty = 4\pi(2-\frac{59}{27\sqrt{3}})a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{4}{27\sqrt{3}}(\frac{64}{3}-\frac{25}{16}(\sqrt{3}+1))a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 9.28a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 4.59a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.74\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.36\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Hemispherical shell with radius a [82] $\boldsymbol{\gamma}_\infty = (2\pi+\frac{8}{3})a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + (2\pi-\frac{16}{3}+\frac{4\pi}{2+\pi})a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx 8.95a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 3.39a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} \approx 0.71\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + 0.27\gamma_{\mathrm{sph}}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Oblate spheroid with width $2a$ and height $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e = \sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{4\pi\xi e^3}{3(e-\xi\arccos\xi)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{8\pi e^3}{3(\arccos\xi-\xi e)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$
	Circular disc with radius a [44] $\boldsymbol{\gamma}_\infty = \frac{16}{3}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 5.33a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) \approx 0.42\gamma_{\mathrm{sph}}(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}})$
	Prolate spheroid with height $2a$ and width $2b$, where $b \leq a$ [82, 103]. Set $\xi = b/a$ and $e = \sqrt{1-\xi^2}$ $\boldsymbol{\gamma}_\infty = \frac{8\pi e^3}{3(\ln\frac{1+e}{1-e}-2e)}a^3(\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}+\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}) + \frac{16\pi\xi^2 e^3}{3(2e-\xi^2\ln\frac{1+e}{1-e})}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$ $\approx \frac{4\pi}{3(\ln 2-\ln\xi-1)}a^3\hat{\boldsymbol{z}}\hat{\boldsymbol{z}} + \mathrm{O}(\xi^2) \quad \text{as } \xi \to 0$
	Cube with side lengths $\ell = 2a/\sqrt{3}$ [66, 98] $\boldsymbol{\gamma}_\infty \approx 3.644305190268\ell^3\mathbf{1} \approx 5.61a^3\mathbf{1} \approx 0.45\gamma_{\mathrm{sph}}\mathbf{1}$

Polarizability $\gamma = \hat{e} \cdot \boldsymbol{\gamma}_{\mathrm{e}} \cdot \hat{e}$: Interpretation

Wire dipoles (length $\ell \approx 2a$) with coils

Geometries of the three wire dipoles



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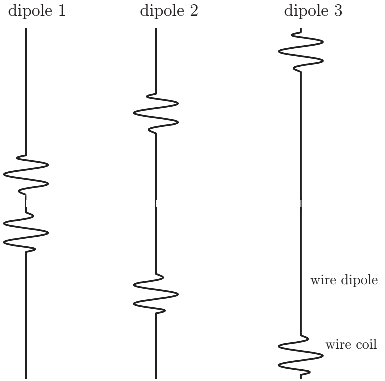

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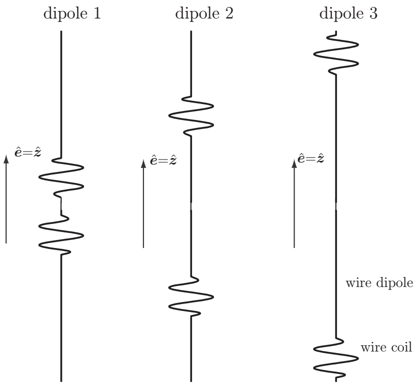
External electrostatic field along the dipoles



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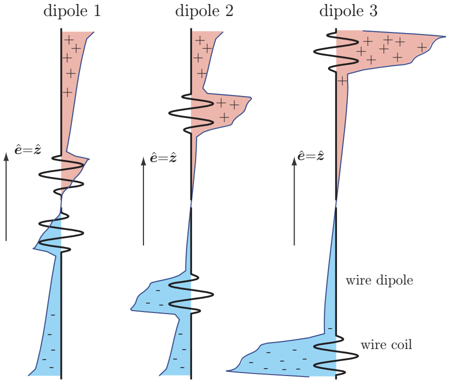
Induced charge density on the wire



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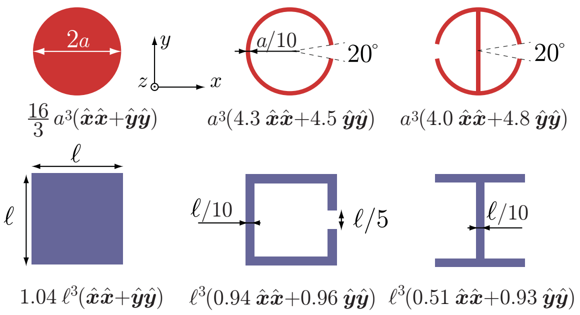
Separation of charge for large polarizability.

Properties of the polarizability dyadics

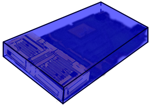
Removal of metal from circular and square plates



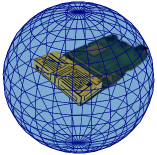
- The polarizability decrease (or is unchanged) if you remove material
- The region in the center of the structure does not contribute much to the polarizability
- Volume (and large area) is not needed for a large polarizability
- Important to be able to support a large separation of charge
- Quantifies 'Fat is good' for the bandwidth of small antennas
- AntennaQ.m on matlabcentral fileexchange

Much progress over the years

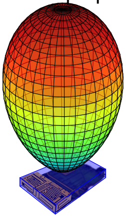
Arbitrarily shaped design regions



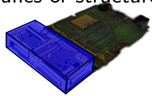
Spherical design region

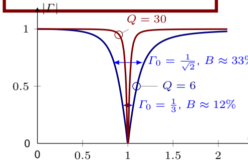


Radiation properties



Design regions close to ground planes or structures





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Gain and directivity effects on Q_{lb}

- Harrington [Har58] $D \leq L^2 + 2L = 3$ for $L = 1$

EEE TRANSACTIONS ON ANTENNAS AND PROPAGATION, VOL. 53, NO. 8, AUGUST 2005 2653

A Monopole Superdirective Array

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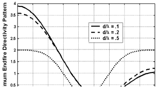
***Abstract*—In principle, the end-fire directivity of a linear pe-iodic array of N isotropic radiators can approach N^2 as the pacing between elements decreases, provided the magnitude nd phase of the input excitations are properly chosen. Thus, the irectivity of a two-element array of isotropic radiators would pproach a value of four, that is, 6 dB higher than that of a single sotropic radiator. We have conducted a theoretical, computa-ional, and experimental study for a two-element superdirective rray of resonant monopoles. In agreement with the theoretical nd computational curves, the measured gain of the monopole rray does indeed continually increase with decreasing spacing f the monopoles, provided the relative magnitudes and phases re maintained. However, for very small separation, maximum**



Electrically small supergain end-fire arrays

A. D. Yaghjian,[1] T. H. O'Donnell,[1] E. E. Altshuler,[1] and S. R. Best[2]

Received 14 September 2007; revised 29 January 2008; accepted 6 March 2008; published 14 May 2008.

[1] The theory, computer simulations, and experimental measurements are presented for electrically small, two-element supergain arrays with near-optimal end-fire gains of 7 dB. We show how the difficulties of narrow tolerances, large mismatches, low radiation efficiencies, and reduced scattering of electrically small parasitic elements are overcome by using electrically small resonant antennas as the elements in both separately driven and singly driven (parasitic), two-element, electrically small supergain end-fire arrays. Although rapidly increasing narrow tolerances prevent the practical realization of the maximum theoretically possible end-fire gain of electrically small arrays with many

M. Gustafsson and S. Nordebo. "Optimal Antenna Currents for Q, Superdirectivity, and Radiation Patterns Using Convex Optimization". *IEEE Trans. Antennas Propag.* 61.3 (2013), pp. 1109–1118

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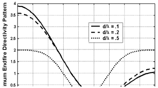

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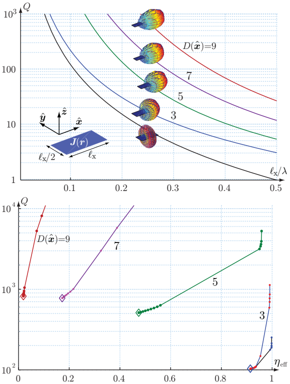
$$\begin{aligned} &\text{minimize} && \text{stored energy} \\ &\text{subject to} && \text{far field}(\hat{\boldsymbol{k}}, \hat{\boldsymbol{e}}) = 1 \\ &&& \text{radiated power} \leq k^3/D_0 \end{aligned}$$



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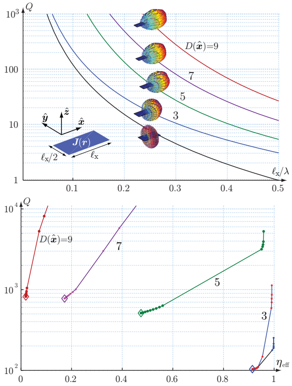
- These problems are often convex or non-convex QCQPs (quadratically constrained quadratic program) and *e.g.*, solved using duality [BV04]



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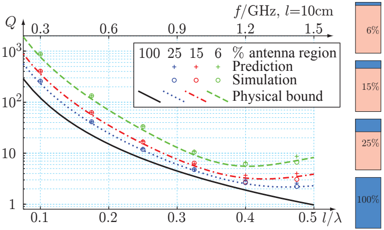
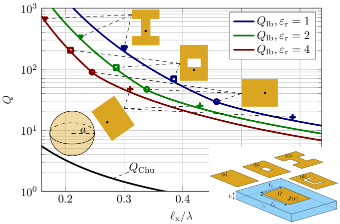
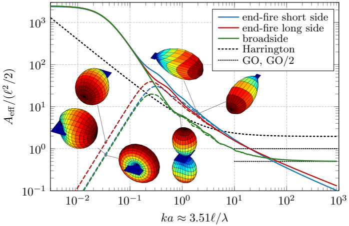
Antenna limits based on convex optimization, QCQP, and duality







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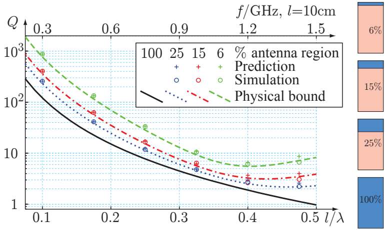

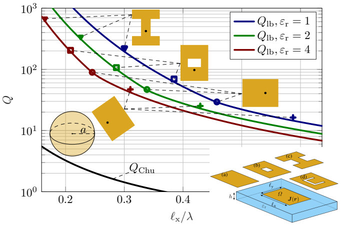

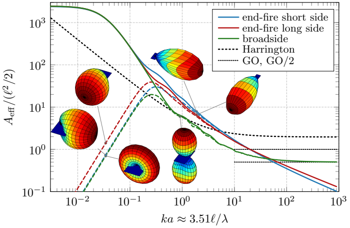

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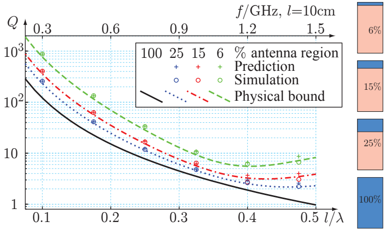

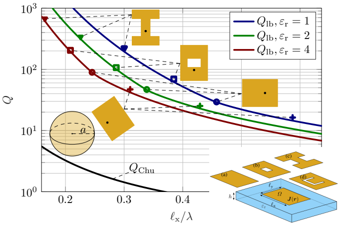

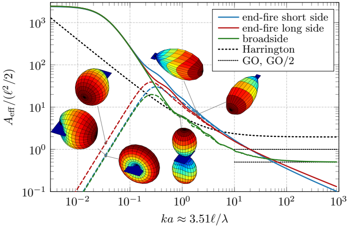

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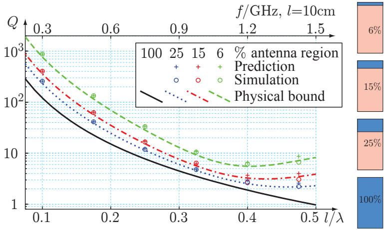

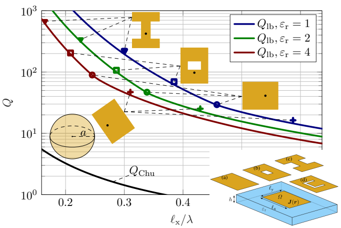

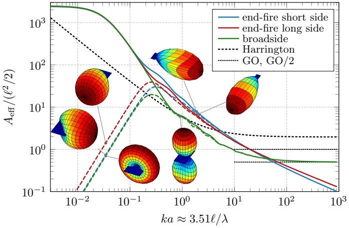

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- Many results and investigations the last decade. Also extended to scattering

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Conclusions

- Our understanding of limitations and possibilities concerning small antennas has evolved significantly since the classical publications by Wheeler (1947) [Whe47] and Chu (1948) [Chu48]
- Have significant knowledge about the impact of size, shape, and materials on antenna performance in relation to bandwidth, efficiency, gain, directivity, and capacity
- Many fundamental results by A. Yaghjian
- I am very happy for collaboration, discussions, and insights over the years
- Still many open theoretical and practical questions concerning small antennas

Less knowledge and many open questions on, *e.g.*,

- multiband antennas
- multiport antennas

Banff International Research Station 2019

Lund University 2015

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