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Andersson, Tommy; Svensson, Lars-Gunnar

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Department of Economics
School of Economics and Management

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Tommy Andersson
Lars-Gunnar Svensson

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Strategy-Proof Allocation of Objects: A Characterization Result*

Tommy Andersson† and Lars-Gunnar Svensson‡

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Abstract

This paper considers an allocation problem with a finite number of objects and unit-demand agents. The main result is a characterization of a class of strategy-proof price mechanisms on a general domain where preferences over pairs of objects and houses are rational, monotonic, and continuous. A mechanism belongs to this class if and only if the price space is restricted in a special way and, given this restriction, that the mechanism selects minimal equilibrium prices.

Keywords: Characterization, House allocation, Strategy-proofness, Multi-object auction.

JEL Classification: D44, D47, D63, D78, D82.

1 Introduction

We consider an allocation problem with a finite number of indivisible objects and a finite number of unit-demand agents. Since an agent's willingness to pay for an object is private information, we analyze mechanisms (i.e., allocation rules) that use agents' reported preferences to determine an allocation that consists of an assignment of objects to agents and a price vector that specifies how much an agent has to pay (or receive) for the object that she is assigned. A mechanism is strategy-proof if, for all preferences and all agents, it is a dominant strategy to report preferences truthfully. It is well-known that strategy-proofness is obtained if the mechanism always chooses an equilibrium allocation with the minimal equilibrium price vector. See, e.g., Vickrey (1961),

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†Department of Economics, Lund University, Sweden.

‡Department of Economics, Lund University, Sweden.

Leonard (1983), Demange and Gale (1986), Sun and Yang (2003), or Andersson and Svensson (2008).

This paper considers mechanisms defined on a general domain where preferences over pairs of objects and prices are rational, monotonic, and continuous. On this domain, it is proved that minimal prices are not only sufficient, but also necessary for mechanisms, the range of which are equilibrium allocations, to be strategy-proof. It is also demonstrated how the set of feasible prices has to be restricted in order to be consistent with strategy-proofness.

Characterizations of the set of strategy-proof mechanisms has previously been provided in the literature under various assumptions and conditions. A common result is the necessity of minimal prices for mechanisms to be consistent with strategy-proofness, while conditions on the price space are not generally analyzed in contrast to the present paper. Miyake (1998) considers a multi-object model where the price space is defined as all prices above exogenously given lower bounds (e.g., the reservation prices of the seller). A similar model is considered in Morimoto and Serizawa (2014). Also in their analysis, minimal prices follows from strategy-proofness. However, their result is proved without assuming the outcome to be price equilibria. It is rather a consequence of strategy-proofness and some other assumptions, e.g., that the outcomes of the mechanisms are assumed to be efficient allocations and that the number of agents is strictly greater than the number of objects. Svensson (2009) shows that not only prices have to be minimal, but also that price vectors p necessarily have to satisfy a restriction $p \geq \underline{p}$ (where $\underline{p}$ denotes some lower price bounds). In his model, the number of agents coincedes with the number of objects, whereas the present paper allows an arbitrary relation. Alternative characterizations, based on different axioms than the ones used in this paper, has been provided by, e.g., Mukherjee (2014) and Sakai (2008, 2013).[1]

The rest of this paper is organized as follows. Section 2 provides the formal model. The main results are derived in Section 3. A dynamic implementation of the results is suggested in Section 4, while Section 5 provides some concluding remarks and an informal discussion of the relationship between the concept of fairness and the main results of this paper.

2 The Formal Model

Let $A = \{1, 2, \ldots, n\}$ be a finite set of *agents* and $H = \{1, 2, \ldots, m\}$ a finite set of *indivisible objects*. The object could be interpreted as houses or jobs. In particular, negative prices may be interpreted as positive wages, so in that case the objects may be jobs. The objects will mostly be called *houses*, while consequences of the main result for the concept of fairness will be informally discussed in Section 5 for the job interpretation of the model. There is also an outside option, called *null house*, denoted by 0, the copies of which are unlimited. No agent in A owns a house in H. There may be one or several owners of the objects, but because the number of owners is

[1] See Remark 1 for additional notes on the related literature.

not important for the analysis, it will for simplicity be assumed that there is a single owner of all objects in H,

An *assignment* is a mapping $\mu : A \to H \cup \{0\}$ such that $\mu_a = \mu_{a'}$, for $a, a' \in A$ and $a \neq a'$, only if $\mu_a = 0$. The set of assignments is denoted by $\mathcal{A}$. *Prices* on houses are real numbers and a price vector is denoted by $p = (p_0, p_1, \ldots, p_m) \in \mathbb{R}^{m+1}$, where p_h is the price on house $h \in H \cup \{0\}$. A set $\mathbb{P} \subset \mathbb{R}^{m+1}$ is a set of *feasible* price vectors if conditions (i)–(iii) below are satisfied:

(i) For all $p \in \mathbb{P}$, $p_0 = 0$.

(ii) $\mathbb{P}$ is a *closed* subset of $\mathbb{R}^{m+1}$ and *bounded from below,* i.e., there is a lower bound $\underline{p} \in \mathbb{R}^{m+1}$ such that $\mathbb{P} \subset \left\{p \in \mathbb{R}^{m+1} : p \geq \underline{p}\right\}$.

(iii) $\mathbb{P}$ is *monotonic*, i.e., if $p \in \mathbb{P}$ and $p' \in \mathbb{R}^{m+1}$, $p' \geq p$ and $p'_0 = 0$, then $p' \in \mathbb{P}$.

Each agent $a \in A$ has *rational* preferences R_a on houses and prices, i.e., on bundles of type $(h, p_h) \in (H \cup \{0\}) \times \mathbb{R}$. To simplify notation, let $(h, p) \equiv (h, p_h)$. That is, by (h, p) we mean house h at price p_h at the price vector p. Preferences are further assumed to satisfy the following assumptions:

- *Monotonicity.* For all houses $h \in H$ where $p^a_h \in \mathbb{R}$, $(h, p'_h)\, P_a\, (h, p_h)$ if $p'_h < p_h$.
- *Continuity.* For all houses $h \in H$, the sets $\{p_h \in \mathbb{R} : (h, p_h)\, R_a\, (h, p'_h)\}$ and $\{p_h \in \mathbb{R} : (h, p'_h)\, R_a\, (h, p)\}$ are closed for all $p'_h \in \mathbb{R}$.
- *Bounded desirability.* For each agent $a \in A$, there is a vector $q^a \in \overline{\mathbb{R}}^{m+1}$, where $\overline{\mathbb{R}}$ is the extended real line, with $q^a_0 = 0$ and $q^a_h \in [-\infty, \infty)$ for all $h \in H$, such that for all $h, h' \in H \cup \{0\}$ with $q^a_h, q^a_{h'} \in \mathbb{R}$, $(h, q^a)\, I_a\, (h', q^a)$. This means that q^a is an "indifference point" for the preferences R_a.[2]

A (preference) *profile* is a list $R = (R_a)_{a \in A}$ of agents' preferences. The set of profiles is $\mathcal{R} = \times_{a \in A} \mathcal{R}_a$, where agent a's preferences are in the set $\mathcal{R}_a$ of rational, monotonic, continuous, and boundedly desirable preferences on $(H \cup \{0\}) \times \mathbb{R}$.

A *state* is a pair $x = (\mu, p)$, where μ is an assignment and p a price vector. Here, $x_a = (\mu_a, p)$ for $a \in A$. The set of unassigned houses in state x is denoted by μ_0, i.e.:

$$\mu_0 = \{h \in H : \ \mu_a \neq h \text{ for all } a \in A\} \cup \{0\}.$$

Note that $0 \in \mu_0$ since there is an unlimited number of copies of the null house. The set of states is denoted by $\mathcal{S}$.

[2]Of course, $q^a_h = -\infty$ means that agent a cannot or will not consume house h at any price (or, in a job interpretation of the model, cannot or will not take job h at any wages).

Definition 1. For a profile $R \in \mathcal{R}$, a state $x = (\mu, p)$ is a (*price*) *equilibrium* if: (i) $p \in \mathbb{P}$, (ii) for all $a \in A$, $x_a R_a (h, p)$ for all $h \in H \cup \{0\}$, and (iii) for all $h \in \mu_0$ and $\varepsilon > 0$, $(p_h - \varepsilon, p_{-h}) \notin \mathbb{P}$. For a given profile $R \in \mathcal{R}$, the set of equilibria is denoted $\mathcal{E}_R$ and the set of corresponding price vectors Π_R.

The definition of equilibrium is the usual one. That is, each agent is assigned her most preferred bundle at the equilibrium price vector. Note that this is also an *individual rationality* condition since $x_a R_a (0, p)$. Further, prices on unassigned houses are on the lower bound of the price space.

Denote by $\mathbb{P}^r$ sets of the type $\mathbb{P}^r = \{h \in H \cup \{0\} : p_h \geq r_h\}$, where $r \in \mathbb{R}^{m+1}$ and $r_0 = 0$. Clearly $\mathbb{P}^r$ satisfies the conditions for a feasible set of price vectors and the vector r is a *lower bound* for $\mathbb{P}^r$. Note that for feasible sets of prices $\mathbb{P}^r$, the equilibrium condition (iii) in Definition 1 means that $p_h = r_h$ if $h \in \mu_0$. If r is a vector of the house owners' reservation prices and house h is not assigned to any agent at equilibrium, then its price is on its lower bound, i.e., $p_h = r_h$.

Definition 2. A *mechanism* is a mapping $f : \mathcal{R} \to \mathcal{S}$ of profiles to states. It is a *price mechanism* if for all $R \in \mathcal{R}$, the outcome $f(R) = (\mu, p)$ is an equilibrium state such that the number $|a \in A : \mu_a = 0|$ is minimal subject to (i) and (ii) in the definition of an equilibrium.[3]

The requirement of minimal number of assignments of null houses in a price mechanism is introduced to reduce the number of utility equivalent assignments. In particular, this is relevant when agents may be indifferent between a null house and a "real" house (i.e., a house in H). In that case, the assignment shall be the real house instead of the null house. Hence, the price mechanism prefers "trade" to "no trade." Obviously, it is in the interests of the owner that a house is sold to an agent even if the agent is indifferent between the null house and buying the real house.

Definition 3. A mechanism f is *manipulable* at a profile $R \in \mathcal{R}$ by an agent $a \in A$ if there is a profile $(R'_a, R_{-a}) \in \mathcal{R}$ such that $x'_a P_a x_a$ for some $x' = f(R'_a, R_{-a})$ and some $x = f(R_a)$. A mechanism f is *strategy-proof* if no agent can manipulate at any profile.

3 The Class of Strategy-Proof Price Mechanisms

In this section, necessary conditions on a strategy-proof price mechanism are investigated. First, it is demonstrated that the set of feasible price vectors $\mathbb{P}$ has to be restricted to a set:

$$\mathbb{P}^r = \{p \in \mathbb{R}^{m+1} : p \geq r\} \text{ where } r \in \mathbb{R}^{m+1} \text{ and } r_0 = 0.$$

Second, it is demonstrated that for a price mechanism $f(R) = (\mu, p)$ to be strategy-proof, the price vector p chosen by the mechanism has to minimal in the set Π_R of equilibrium price vectors. Finally, it is concluded that minimal prices and the restricted domain for feasible price vectors also are sufficient for the price mechanism f to be strategy-proof.

[3]Throughout the paper, $|S|$ denotes the number of elements in a set S.

3.1 Restrictions on the Set of Feasible Price Vectors

It is first demonstrated how the set of feasible price vectors must be restricted. To show this, note that the set $\mathbb{P}^r$ is closed, bounded from below, and monotonic. It is, consequently, consistent with the definition of a set of feasible prices vectors.

Theorem 1. *Let $f : \mathcal{R} \to \mathcal{A} \times \mathbb{P}$ be a price mechanism. Then f is manipulable if $\mathbb{P} \neq \mathbb{P}^r$, $r \in \mathbb{R}^{m+1}$.*

Proof. To obtain a contradiction, suppose that f is strategy-proof and that $\mathbb{P} \neq \mathbb{P}^r$, $r \in \mathbb{R}^{m+1}$. Let $\underline{p}$ be a lower bound for $\mathbb{P}$. Then, since $\mathbb{P} \neq \mathbb{P}^r$, there are two vectors $r', r'' \in \mathbb{P}$, $r' \neq r''$, both minimal in $\mathbb{P}$. To see this, let r' be minimal in $\mathbb{P}$. There is such a vector since $\mathbb{P}$ is closed and bounded from below. Consider now the set $\mathbb{P}^{r'}$. Since $\mathbb{P}$ is monotonic, $\mathbb{P}^{r'} \subset \mathbb{P}$ and $\mathbb{P}^{r'} \neq \mathbb{P}$. Hence, there is a vector $r'' \in \mathbb{P} - \mathbb{P}^{r'}$. Now, r'' can be chosen minimal in $\mathbb{P}$. Then r' as well as r'' are minimal in $\mathbb{P}$ and $r' \neq r''$.

Consider now the two minimal vectors in $\mathbb{P}$, r' and r'', and note that they can be chosen so that there are houses h_1 and h_2 such that $r'_{h_1} < r''_{h_1}$ and $r'_{h_2} > r''_{h_2}$. Without loss of generality, assume that $h_1 = 1$ and $h_2 = 2$. Define now the preference profiles $R' = (R'_a)_{a \in A}$ and $R'' = (R''_a)_{a \in A}$ according to:

- Preferences R'_a are linear and represented by utility functions $u'_{ah}(p)$.
- $u'_{10}(p) = 0$ and $u'_{11}(p) = r'_1 - p_1$. For $h > 1$, $u'_{1h}(p) = -\infty$.
- $u'_{20}(p) = 0$ and $u'_{22}(p) = r'_2 - p_2$. For $h = 1$ and $h > 2$, $u'_{2h}(p) = -\infty$.
- $u'_{a0}(p) = 0$ and $u'_{ah}(p) = -\infty$ for each agent $a > 2$ and for all $h \in H$.

Preferences R''_a are defined analogously where u' is replaced with u'' and r' replaced with r''.

Clearly $x' = (\mu', p') \in \mathcal{E}_{R'}$ and $x'' = (\mu'', p'') \in \mathcal{E}_{R''}$ if $\mu' = \mu''$ and $\mu'_a = a$ for $a \leq 2$ and $\mu'_a = 0$ for $a > 2$. Moreover, $p'_h = r'_h$ for $h = 1, 2$ while for $h > 2$, p' must be chosen minimal in $\{p \in \mathbb{P} : p_h = r'_h \text{ for } h = 1, 2\}$. The price vector p'' is chosen analogously. Note that *all* equilibria in $\mathcal{E}_{R'}$ and $\mathcal{E}_{R''}$ must have this form. For instance, one cannot choose $p'_1 < r'_1$ because then $p' \notin \mathbb{P}$ by Definition 1(i), and if $p'_1 > r'_1$ then $1 \in \mu'_0$ which is not consistent with Definition 1(iii). Further, $p'_1 = r'_1$ and $\mu'_1 = 0$ is not consistent with the maximal trade requirement in Definition 2. Since all equilibria have this form, it also follows that $f(R') = x'$ and $f(R'') = x''$.

Now, consider the profile $R = (R''_1, R'_{-1})$ and let $f(R) = x = (\mu, p)$. Then, by the arguments above, $\mu = \mu'$ while $p_1 = r''_1$ and $p_2 = r'_2$. Then $p \in \mathbb{P}$ since $\mathbb{P}$ is monotonic. Now, it follows that $u_{22}(p) = u'_{22}(p) = r'_2 - r'_2 = 0$ while $u_{22}(p'') = u''_{22}(p) = r'_2 - r''_2 > 0$. Hence, agent 2 can manipulate at state x by changing preferences from R'_2 to R''_2, contradicting strategy-proofness. □

3.2 Minimal Equilibrium Prices

According to Theorem 1, the set of feasible price vectors has to be restricted to sets of the type $\mathbb{P}^r$ for some $r \in \mathbb{R}^{m+1}$ with $r_0 = 0$ in order for the mechanism to be strategy-proof. It is next demonstrated that strategy-proofness also requires the prices to be minimal equilibrium prices.

For a profile $R \in \mathcal{R}$, a price vector $p^m \in \Pi_R$ is *minimal* if for any $p \in \Pi_R$, $p \leq p^m$ only if $p = p^m$. The set Π_R of equilibrium prices is closed and bounded from below. Because the set Π_R also is a lattice if $\mathbb{P} = \mathbb{P}^r$, there is a unique minimal price vector in Π_R (see, e.g., Crawford and Knoer, 1981; Demange and Gale, 1985). A mechanism f is a *minimal price mechanism* if f is a price mechanism and $f(R) = (\mu^m, p^m)$, where p^m is the minimal price vector in Π_R.

The following result is the main characterization result of the paper (the proof can be found in Section 3.3).

Theorem 2. A price mechanism $f : \mathcal{R} \to \mathcal{A} \times \mathbb{P}$ is strategy-proof if and only if the set of feasible price vectors is $\mathbb{P} = \mathbb{P}^r$ for some r and f is a minimal price mechanism.

Remark 1. Special cases of Theorem 2 can be found in the existing literature. Svensson (2004, 2009) obtains the restriction on feasible prices (wages) as well as the necessity of minimal prices (wages) in a less general version of the model considered in this paper, e.g., because the preferences are quasi-linear, there is no outside option (i.e., null houses), and the number of objects and agents coincides. Miyake (1998) considers a model that is logically similar to the one considered in this paper, but for a somewhat smaller preference domain. Another main difference is that the set of feasible price vectors is assumed to be exogenously given and of type $\mathbb{P}^r$, $r_h \geq 0$. As in the present study, the characterization problem in Miyake (1998) assumes a price mechanism. This is not the case in Morimoto and Serizawa (2015) as they assume only that the range of the mechanism is the efficient states. Given this weaker assumption, the minimal prices follows from the strategy-proof condition. Compared to the present study, their considered preference domain is somewhat smaller than the one considered here and, importantly, $n > m$, i.e., the number of agents is strictly greater than the number of objects. The proof techniques in the above mentioned studies are also very different from the method used in this paper. □

3.3 The Proof of Theorem 2

This section is devoted to the proof of Theorem 2. A specific subset of preferences $\mathcal{R}'_a \subset \mathcal{R}_a$ will be crucial in the proof, so the first step is to define this subset. For this purpose, let $a \in A$ be an agent, $R \in \mathcal{R}$ and $x^m = (\mu^m, p^m) \in \mathcal{E}_R$ an equilibrium where p^m is the minimal price vector in Π_R. Let further the price difference for house h at price vectors p and p^m be given by $\delta_h \equiv p_h - p^m_h$. Preferences $R_a \in \mathcal{R}_a$ are represented by continuous and strictly decreasing utility functions $u_{ah}(\delta)$, $h \in H$, while $u_{a0}(\delta)$ is constant since $p_0 = 0$ for all δ. Denote by $\bar{u}_a$ the maximal utility at x^m, i.e., $\bar{u}_a = u_{a\mu^m_a}(\delta^m)$. By choice of unit, we let $u_{a0}(\delta) = 0$. Then, by individual rationality, it follows that $\bar{u}_a \geq 0$. Moreover, denote by S_a the set of houses that yields maximal utility at x^m to agent a, i.e.:

$$S_a = \{h \in H \cup \{0\} : (h, p_h) I_a x_a^m\}.$$

Then, by construction, $h = 0 \in S_a$ if and only if $\bar{u}_a = 0$.

Given an agent $a \in A$ and a profile $R \in \mathcal{R}$, preferences R'_a are defined by utility functions $u'_{ah}(\delta)$, $h \in H \cup \{0\}$, satisfying the following conditions:

(a) $u'_{a0}(\delta) = \bar{u}_a = 0$ if $0 \in S_a$, otherwise $u'_{a0}(\delta) = -1$,

(b) $u'_{ah}(\delta) = \bar{u}_a - \alpha_a \delta_h$, $\alpha_a > 0$, for $h \in S_a$ and $h \neq 0$,

(c) $u'_{ah}(\delta) = \bar{u}_a - \alpha_a \delta_h - 100$, for $h \notin S_a \cup \{0\}$.

Let $\mathcal{R}'_a$ denote the subset of preferences in $\mathcal{R}_a$ that satisfies conditions (a)–(c). The following lemma reveals an invariance property of the minimal price vector.

Lemma 1. Let $\mathbb{P} = \mathbb{P}^r$, $r_h = 0$ for all $h \in H \cup \{0\}$. Let also $a' \in A$ be an agent, R a profile in $\mathcal{R}$, and $x^m = (\mu^m, p^m) \in \mathcal{E}_R$ an equilibrium where p^m is the minimal price vector in Π_R. Then p^m is also the minimal price vector in $\Pi_{R'}$, where $R' = (R'_a)_{a \in A}$, $R'_{a'} \in \mathcal{R}'_{a'}$ and $R'_a = R_a$ for $a \neq a'$.

Proof. Let, without loss of generality, $a' = 1$ and consider $u'_{1\mu_1^m}(\delta^m)$. Note first that by the definition of the set S_1, for all $h \in S_1$,

$$u'_{1\mu_1^m}(\delta^m) = \bar{u}_1 - \alpha_1 \delta^m_{\mu_1^m} = \bar{u}_1 = u'_{1h}(\delta^m),$$

while for $h \notin S_1$,

$$u'_{1\mu_1^m}(\delta^m) > u'_{1h}(\delta^m).$$

We start by proving that $x^m \in \mathcal{E}_{R'}$ and, consequently, that $p^m \in \Pi_{R'}$. To see this, note first that $x^m = (\mu^m, p^m) \in \mathcal{E}_R$. Now, for $R'_a = R_a$, we have $x_a^m R_a x_{a'}^m$ for all $a' \neq 1$ since $x^m \in \mathcal{E}_R$. For $a = 1$, we have $u'_{1\mu_1^m}(\delta^m) = \bar{u}_1 - \alpha_1 \delta^m_{\mu_1^m} = \bar{u}_1$ since $\mu_1^m \in S_1$. Further, $u'_{1h}(\delta^m) \geq \bar{u}_1 - \alpha_1 \delta_h^m$ for all $h \in H$. Hence, $x^m \in \mathcal{E}_{R'}$ and $p^m \in \Pi_{R'}$.

To prove that p^m is minimal in $\Pi_{R'}$, suppose that there is a vector $p' \in \Pi_{R'}$ such that $p' \leq p^m$ and $p' \neq p^m$. Let $x' = (\mu', p') \in \mathcal{E}_{R'}$ and define:

$$H' = \{h \in H : p'_h < p_h^m\} \text{ and } H'' = \{h \in H : p'_h = p_h^m\},$$
$$G' = \{a \in A : \mu'_a \in H'\} \text{ and } G'' = \{a \in A : \mu'_a \in H''\}.$$

Note that $H' \neq \emptyset$ by assumption. By definition of equilibrium, $H' \cap \mu_0^m = \emptyset$, i.e., if $h \in \mu_0^m$, then $p_h^m = 0$, and hence, $p'_h \geq p_h^m$, which contradicts our assumptions.

We next show that for all $a \in G''$, $x_a^m P_a(h, p^m)$ for all $h \in H'$. To obtain a contradiction, suppose that there is an agent $a \in G''$ and a house $h \in H'$ such that $(h, p^m) R_a x_a^m$. Then the following five statements hold:

- $(h, p^m) R_a x_a^m$ by assumption,
- $x_a^m R_a(h', p^m)$ by the equilibrium x^m, where $h' = \mu'_a$,
- $(h', p^m) = (h', p')$ since $a \in G''$ and then $p^m_{h'} = p'_{h'}$,
- $(h', p') R'_a(h, p')$ by the equilibrium x',
- $(h, p') P_a(h, p^m)$ by monotonicity, $p^m_h > p'_h$ since $h \in H'$.

If $a \neq 1$, we have $(h, p^m) R_a x_a^m R_a(h', p^m) R_a(h, p') P_a(h, p^m)$, which is a contradiction. If $a = 1$, then $(h', p') R'_a(h, p')$ means that $\bar{u}_1 - \alpha_1 \delta'_{h'} \geq \bar{u}_1 - \alpha_1 \delta'_h$ or $\bar{u}_1 - \alpha_1 \delta'_{h'} \geq \bar{u}_1 - \alpha_1 \delta'_h - 100$ since $h' \in S_1$. In both cases $\delta'_h \geq \delta'_{h'}$, which is a contradiction since $h \in H'$ and $h' \in H''$. Thus, the assumption $(h, p^m) R_a x_a^m$ always leads to a contradiction, so it must be the case that for all $a \in G''$, $x_a^m P_a(h, p^m)$ for all $h \in H'$.

But if the latter condition holds, it follows from the Perturbation Lemma in Alkan, Demange and Gale (1991) that prices p^m_h, $h \in H'$, can be decreased such that there is an equilibrium $x'' = (\mu'', p'') \in \mathcal{E}_R$ with $p \leq p^m$, $p \neq p^m$. But this is a contradiction to p^m being minimal in Π_R. Thus, p^m is minimal also in $\Pi_{R'}$. □

Recall from Theorem 1 that $\mathbb{P} = \mathbb{P}^r$ for some r is necessary for strategy-proofness. Assume now, without loss of generality, that $r_h = 0$ for all $h \in H \cup \{0\}$. Given Lemma 1, Theorem 1 can now be proved by contradiction. To obtain such contradiction, suppose that f is strategy-proof, but that there is a profile $R \in \mathcal{R}$ such that $f(R) = x \equiv (\mu, p)$ and $p \geq p^m$, $p \neq p^m$, where p^m is minimal in Π_R, i.e., $f(R) = (\mu, p)$ and $p \geq p^m$, $p \neq p^m$.

Given this assumption, it follows that $p_h > p^m_h$ for some $h \in H$, and $h \neq 0$ since $p_0 = 0$ for all p, and hence, $p_0 = p^m_0$. Without loss of generality, assume that $p_1 > 0$ and $\mu_1 = 1$. Then, $x_1^m P_1 x_1$ since $x_1^m R_1(\mu_1, p^m)$ by equilibrium, and $(\mu_1, p^m) P_1(\mu_1, p)$ by monotonicity. Moreover, $0 \notin S_1$, so $\mu_1^m \neq 0$.

Let $R' = (R'_1, R_{-1}) \in \mathcal{R}$ be a profile defined by $R'_1 \in \mathcal{R}'_1$ according to conditions (a)–(c) from the above. Then by Lemma 1, p^m is minimal in $\Pi_{R'}$. Let $x' = (\mu', p') = f(R')$ and $\mu'_1 = h'$. Now, $x' \in \mathcal{E}_{R'}$, so $x'_1 R'_1(0, p')$, i.e., $u'_{1h'}(\delta') \geq u'_{10}(\delta')$. Hence, if $0 \in S_1$ then $u'_{1h'}(\delta') \geq 0$, and if $0 \notin S_1$ then $u'_{1h'}(\delta') \geq -1$. It is next demonstrated that $h' = 0$ for α_1 "sufficiently large" by analyzing the following two cases:

Case (i), $h' \neq 0$. If $h' \in S_1$, then $u'_{ah'}(\delta) = \bar{u}_1 - \alpha_1 \delta'_{h'} \geq u'_{10} = -1$. Hence, $\delta'_{h'} \leq (\bar{u}_1 + 1)/\alpha_1$. This also means that $\delta'_{h'} \to 0$ as $\alpha_1 \to \infty$.

Case (ii), $h' \notin S_1 \cup \{0\}$. If $h' \notin S_1 \cup \{0\}$, then $u'_{ah'}(\delta) = \bar{u}_1 - \alpha_1 \delta'_{h'} - 100 \geq u'_{10} = -1$. Hence, $\delta'_{h'} \leq (\bar{u}_1 - 99)/\alpha_1$. This also means that $\delta'_{h'} \to 0$ as $\alpha_1 \to \infty$.

To see that $h' = 0$ for α_1 sufficiently large in both Cases (i) and (ii), let $\Delta_{\alpha_1} = u_{1h'}(\delta') - u_{11}(\delta)$. Here, $u_{1h'}(\delta') \to \bar{u}_1$ as $\delta'_{h'} \to 0$ since $u_{1h'}$ is continuous and $\delta^m_h = 0$ for all $h \in S_1$. But then $\Delta_{\alpha_1} > 0$ for α_1 sufficiently large. This is true in both Cases (i) and (ii). However, $\Delta_{\alpha_1} > 0$ is not consistent with f being strategy-proof. Hence, for α sufficiently large, $h' = 0$ must be the case.

To summarize, for each profile $R \in \mathcal{R}$ such that $f(R) = x \equiv (\mu, p)$ and $p \geq p^m$, $p \neq p^m$, where p^m is minimal in Π_R, there is an agent $a \in A$ such that $x^m_a P_a x_a$. If $R_a \notin \mathcal{R}'_a$ then agent a can make a similar substitution as in the above, and the profile R is changed to $R' = (R'_a, R_{-a})$. If $f(R') = (\mu', x')$, then $\mu'_a = 0$. Repeat now this type of substitution process as many times as possible. This process stops at a profile R when, for $f(R) = (\mu, p)$, there is no agent a such that $x^m_a P_a x_a$ and $R_a \notin \mathcal{R}'_a$.

At the profile R where the process stops, it is possible to construct a cycle $(a_j)_{j=1}^{k+1}$ of agents $a_j \in A$, all different except $a_1 = a_{k+1}$ (i.e., “the first” and “the last” agent in the cycle). Further, for $h_j = \mu_{a_j}$, let a_{j+1} be recursively defined by $h_{j+1} = \mu_{a_{j+1}} = \mu^m_{a_j}$. Finally, let a_1 be the last agent where the preferences were replaced by preferences in $\mathcal{R}'_a$. This means that $h_1 = 0$ and $h_1 \notin S_{a_1}$.

Consider now this cycle. Note first that, by equilibrium, $u_{a_1 h_1}(\delta) \geq u_{a_1 h_2}(\delta)$. Furthermore, $R_{a_1} \in \mathcal{R}'_{a_1}$, and then, since $h_1 \notin S_{a_1}$ and $h_2 \in S_{a_1}$, it follows that:

$$u_{a_1 h_1}(\delta) = u_{a_1 0}(\delta) = -1 \geq \bar{u}_{a_1} - \alpha_{a_1} \delta_{h_2} \text{ where } h_2 = \mu^m_{a_1}.$$

Then, $\delta_{h_2} \geq (\bar{u}_{a_1} + 1)/\alpha_{a_1} > 0$. Thus, $\delta_{h_2} > \delta^m_{h_2} = 0$.

Now, $\mu_{a_2} = h_2$ and $\delta_{h_2} > 0$ entail that $x^m_{a_2} P_{a_2} x_{a_2}$ and, hence, $R_{a_2} \in \mathcal{R}'_{a_2}$. By repeating the above arguments, we obtain:

$$u_{a_2 h_2}(\delta) = \bar{u}_{a_2} - \alpha_{a_2} \delta_{h_2} \geq \bar{u}_{a_2} - \alpha_{a_2} \delta_{h_3} \text{ where } h_3 = \mu^m_{a_2}.$$

Then, $\delta_{h_3} \geq \delta_{h_2}$. By recursion, we have $\delta_{h_{j+1}} \geq \delta_{h_j}$ for $1 < j \leq k$ and $\delta_{h_{k+1}} \geq \delta_{h_k}$. However, $\delta_{h_{k+1}} = \delta_{h_1}$, which is a contradiction since the sequence $(\delta_j)_{j=1}^{k+1}$ of prices is increasing with $\delta_{h_1} = 0$, and $\delta_{h_{k+1}} > 0$ since $\delta_{h_2} > 0$. Hence, the original assumption, $p \geq p^m$, $p \neq p^m$, cannot be true, and $p = p^m$ must be the case.

Finally, we note that a minimal price mechanism is strategy-proof, see, e.g., Leonard (1983), Demange and Gale (1985), or Andersson and Svensson (2008).

4 Dynamic Implementation

The minimal price mechanism analysed in the preceding section is a direct mechanism where the agents are asked to report their complete preference relations. In Andersson and Svensson

(2018), a dynamic mechanism is constructed and analyzed where only partial preferences are required to reach the minimal price equilibrium. A simplified version of that mechanism can also be applied for the problem considered in this paper.[4]

It the following, it is, without loss of generality, assumed that the feasible prices are the vectors in $\mathbb{P} = \{p \in \mathbb{R}^{m+1}_{+} : p_0 = 0\}$. The price space is partitioned by a grid where each box is of type $S_\alpha \subset \mathbb{P}$ satisfying:[5]

$$\delta\mathbb{N}^m = \{\alpha \in \mathbb{R}^m_+ : \alpha_j = \delta k_j \text{ for some } k_j \in \mathbb{N}\},$$
$$\text{for each } \alpha \in \delta\mathbb{N}^m,\ S_\alpha = (0,0,\ldots,0) \times (\times_{h\in H}\,[\alpha_h, \alpha_h + \delta)).$$

The vector α is the minimal corner in a box and $\delta > 0$ defines the size of the boxes. The outcome of the dynamic mechanism is a finite increasing sequence $(p^t)_{t=1}^T$ of price vectors, called an *English Price Sequence*, and a corresponding finite increasing sequence of boxes, where no price is increased more than δ. In each box, only prices on over-demanded houses (see below for a definition) are raised. Agents' demands are required only in the various boxes in the sequence, and the sequence of boxes ends when the unique minimal price equilibrium is obtained. The sequences are formally defined in the following way.

Let $a \in A$ and $p \in \mathbb{P}$, and denote by $d_a^p \subset H \cup \{0\}$ the (reported) *demand set* at the price vector p, i.e.:

$$d_a^p = \{h \in H \cup \{0\} : (h,p)\, R_a\, (h',p) \text{ for all } h' \in H \cup \{0\}\}.$$

A set $H' \subset H$ of houses is *over-demanded* if $|a \in A : d_a^p \subset H'| > |H'|$ and a *minimal over-demanded* set if there is no over-demanded set $H'' \subset H'$, $H'' \neq H'$.

In each box S_α, $x = (\mu, p)$ is a *temporary state* if $p \in S_\alpha$ and $\mu_a \in d_a^p$. Given S_α and a temporary state $x = (\mu, p)$, $p \in S_\alpha$, a *price regime* $\Pi^x \subset S_\alpha$ is defined according to: $p' \in \Pi^x$ if and only if there is a temporary state $x' = (\mu', p')$, with $p' \in S_\alpha$, such that $p'_h = p_h$ if h is not in a minimal over-demanded set at p and for such an h, $\mu'_a = \mu_a = h$. Let $x = (\mu, p)$ be a temporary state and let $\xi(x) = \sup \Pi^x$. Note that $\xi(x)$ need not be a singleton. Now, we can recursively define a price sequence in the following way.

Definition 4. Given the partition of the set of feasible price vectors $\{S_\alpha : \alpha \in \delta\mathbb{N}^m\}$, a sequence $(p^t)_{t=1}^T$ of price vectors constitutes an *English Price Sequence* (EPS, henceforth) if there is a sequence $(x^t)_{t=1}^T$ of supporting temporary states, with $x^t = (\mu^t, p^t)$, such that $p^{t+1} \in \xi(x^t)$. The starting point is p^1 with $p_h^1 = 0$ for all $h \in H \cup \{0\}$. The EPS terminates at step T if $p^T \neq p^{T-1}$ and $p^{T+1} = p^T$.

Note that there can be several price changes in one and the same box, so the corresponding sequence of boxes can have fewer steps than T. It can be proved that $T < \infty$ and that the end-

[4] For proofs of the results in this section, see Andersson and Svensson (2018).

[5] $\mathbb{N} = \{0, 1, 2, \ldots\}$.

point p^T is the unique minimal price vector, while the sequence $(p^t)_{t=1}^T$ is not necessarily unique. Let $(S_{\alpha^j})_{j=1}^{T'}$, $T' \leq T$, be the sequence of boxes containing a price vector from the EPS $(p^t)_{t=1}^T$. Then the measure of the set $\cup_j S_{\alpha^j} \to 0$ as $\delta \to 0$, so the part of the price space where agents recursively report their demand can be arbitrarily small by choosing δ small. The EPS can be seen as the outcome of an auction rule defined as follows:

The Iterative English Auction Rule. Initialize the price vector to p^1. For each Step $t := 1, \ldots, T$:

1. Each agent $a \in A$ reports his demand set $d_a^{p^t}$ at prices p^t.
2. Calculate a supporting temporary state $x^t = (\mu^t, p^t)$.
3. Define a small price regime Π^{x^t} and calculate $p^{t+1} \in \xi(x^t)$.
4. If $p^{t+1} = p^t$, stop. Otherwise, set $t := t + 1$ and continue.

If the reported demand sets d_a^p in the Iterative English Auction Rule are consistent with rational preferences $R \in \mathcal{R}$, i.e., a reported demand set d_a^p can be derived from some preference ordering $R_a \in \mathcal{R}_a$, then bidding truthfully is an ex post Nash equilibrium.

5 Concluding Remarks and a Fair Wage Interpretation

We have considered an allocation problem with a finite number of objects and unit-demand agents, and provided a characterization of a class of strategy-proof price mechanisms on a general preference domain. But there is also an appealing fairness interpretation of the results that relates to the version of the model with jobs and wages. More precisely, any definition of fairness is faced with two fundamental problems, namely the uniqueness problem and the implementation problem. For instance, if fairness requires an allocation to be envy-free (Foley, 1967), there are in general many allocations satisfying this condition. In the jobs and wages interpretation of the model, Theorem 2 may be useful in analyzing those two problems.

To see this, let H be a set of various jobs, and negative prices be wages, e.g., $w_j = -p_j$, $j \in H$. Feasible wages are $\mathbb{W} \subset \mathbb{R}^{m+1}$, where $w \in \mathbb{W}$ if and only if $w = -p$ for some $p \in \mathbb{P}^r$. Suppose now that necessary for fairness is a state $x = (\mu, w)$ with no envy, i.e., $x_a R_a x_{a'}$ for all agents $a, a' \in A$. Since the calculation of such a state requires agents' private information, and that information can be obtained by using a strategy-proof mechanism, Theorem 2 shows that the degree of freedom in choosing a fairness criterion is reduced to the choice of a feasible set $\mathbb{W}^r$ of wages of the type, $\mathbb{W}^r = -\mathbb{P}^r \subset -\mathbb{P}$, where $-\mathbb{P}$ is the exogenously given set of production possibilities. Hence, the choice for the mechanism designer is one choice of a feasible wage structure r which determines $\mathbb{W}^r$, and then there is only one mechanism that solves the uniqueness problem and the implementation problem, and that is the "maximal wage mechanism," i.e., the version of the minimal price mechanism with positive prices. In this model, the vector r may be interpreted as a vector of reservation wages of the employers, and the mechanism results in a non-manipulable rule for the employees, while the employers may manipulate by their choice of r.

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