Domain decomposition methods for nonlinear elliptic and parabolic equations

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by Emil Engström

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Abstract
Nonoverlapping domain decomposition methods have been utilized for a long time to solve linear and nonlinear elliptic problems, and more recently, parabolic problems. Despite this, there is no convergence theory for nonlinear elliptic and parabolic equations on general Lipschitz domains in $\mathbb{R}^d$, $d \geq 2$. We therefore develop a Steklov–Poincaré theory for nonlinear elliptic and parabolic problems and study the properties of the Steklov–Poincaré operators. In the elliptic case, we show that the two standard methods, the Dirichlet–Neumann and Robin–Robin methods converge. We demonstrate with numerical results that the Neumann-Neumann method does not converge in some cases and instead develop two modified Neumann–Neumann methods and prove their convergence. In the parabolic case, we show convergence of the Robin–Robin method and introduce two modified Dirichlet–Neumann methods, for which we show convergence. We also discuss how the results can be applied to an equation after it has been discretized by a finite element method and give numerical results for each method.

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Nonoverlapping domain decomposition, Steklov–Poincaré operator, Convergence, Nonlinear elliptic equation, Nonlinear parabolic equation

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List of publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Convergence Analysis of the Nonoverlapping Robin–Robin Method for Nonlinear Elliptic Equations**
E. Engström, E. Hansen
SIAM J. Numer. Anal., 60(2):585–605, 2022

II **Modified Neumann–Neumann methods for semi- and quasilinear elliptic equations**
E. Engström, E. Hansen
Appl. Numer. Math., 206:322–33, 2024

III **Convergence of the Dirichlet-Neumann method for semilinear elliptic equations**
E. Engström
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IV **Linearly convergent nonoverlapping domain decomposition methods for quasilinear parabolic equations**
E. Engström, E. Hansen
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V **An abstract approach to the Robin–Robin method**
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Populärvetenskaplig sammanfattning på svenska

Partiella differentialekvationer används i stor utsträckning inom naturvetenskap för att beskriva system som är annorlunda i olika delar av rum och tid. En lösning till en partiell differentialekvation är alltså en funktion som beror på en eller flera rumsvariabler och möjligen en tidsvariabel. Några exempel är värmefördelning i ett hus vid jämviktsläge, vätskeflöde genom ett rör, spridning av luftföroreningar i atmosfären, och värmespridning under en industriell process, exempelvis varmvalsning.

De flesta partiella differentialekvationer går inte att lösa för hand. Vi måste istället uppskatta lösningar, men detta är ett väldigt beräkningstungt arbete och måste i många fall utföras på stora datorkluster. Sådana kluster använder sig av många beräkningsenheter för att utföra arbetet vilket kräver att beräkningsenheterna kan samarbeta effektivt.

I denna avhandling undersöks en metod som kallas områdesuppdelning. Vi delar upp vårt rum i flera mindre delområden där varje delområde ger upphov till ett mindre problem som kan lösas av en beräkningsenhet. Dessa problem är tyvärr inte oberoende av varandra så därför måste enheterna kommunicera sinsemellan. Denna kommunikation kan ske med olika typer av randvillkor, d.v.s. villkor som tillämpas på randen, eller gränsen, av varje delområde.

Det är viktigt att numeriska metoder är pålitliga. Helst vill man veta innan man använder metoden ifall den kommer konvergera, d.v.s., ifall metoden kommer ge en bättre uppskattning till lösningen på problemet desto mer arbete vi utför. Trots att områdesuppdelande metoder har studerats länge finns dessa teoretiska resultat inte för vissa ekvationer, bl.a. för ickelinjära och tidsberoende ekvationer.

Målet med denna avhandling är därför först att utveckla ett generellt ramverk för att analysera områdesuppdelande metoder för ickelinjära och tidsberoende partiella differentialekvationer. Detta ramverk används sedan för att bevisa konvergens av flera områdesuppdelande metoder tillämpade på ickelinjära och tidsberoende ekvationer. De teoretiska resultaten stödjs av numeriska resultat som visar exempel på hur denna konvergens ser ut när metoderna implementeras.

Chapter 1

Introduction

Partial differential equations are frequently used in natural sciences to describe various systems that differs in time and space. In this thesis we consider nonlinear elliptic and parabolic equations. These are large classes of partial differential equations that are applicable in many situations. Elliptic equation typically describe systems in equilibria and parabolic equations describe systems that evolve in time.

Numerically approximating solutions to partial differential equations requires an enormous amount of computations. Typically these computations are performed on distributed hardware. That is, hardware with multiple processors, each with their own memory. To take advantage of this we must construct numerical methods that can be run in parallel. Although there are many techniques for this, one of the most prominent ones is to employ a domain decomposition method. The domain is decomposed into several smaller subdomains, resulting in several smaller systems of equations. These smaller problems are dependent on each other through interface conditions and therefore an iteration is required in order to get a solution that also satisfies the interface conditions.

The first domain decomposition method was introduced as an analytical tool in [96] in order to show existence of solutions to the Poisson problem in more complicated domains than what was known at the time. The method proposed there is known as the alternating Schwarz method and it was suggested as a numerical method almost a hundred years later [88]. It was generalized in [77, 78, 79] with a complete convergence proof. Since then, many different domain decomposition methods have been suggested, see [91, 106] for an overview. A more recent development has been to apply the same idea to parabolic equations as well, see [43, Chapter 3] for an overview of this.

Domain decomposition methods are typically divided into two types. First, the nonoverlapping methods, sometimes referred to as iterative substructuring methods. These rely on decompositions that do not overlap and therefore the subdomains share an interface over which information is iteratively transferred. Second, the overlapping methods where each subdomain has an interface in the interior of other subdomains. In this thesis, we will only treat nonoverlapping domain decomposition methods.

In order to use the methods reliably, it is important to know that the method converges. It is also important to know what properties the convergence has, e.g., what method parameter choice is optimal or if it is affected by the discretization. There are two main approaches to analyzing this convergence. The first approach is to discretize the equation using a finite element method and then analyze the resulting matrices, see, e.g., [106]. One would then like to characterize how the convergence rate depends on the discretization parameter. It is, of course, preferable to have convergence rates that are independent of this. The second approach is to analyze the method on a continuous level, see, e.g., [91], which is the main approach taken in this thesis. That is, we analyze how the iteration behaves when applied to a partial differential equation. In many cases, once the continuous result has been established, the discrete results follow directly as a special case. This means that we can analyze both the continuous equation and the discrete method within the same framework.

The analysis of domain decomposition applied to linear elliptic equations have been studied for a long time and is well understood, see [91] for the nonoverlapping case and [106] for the overlapping case. For overlapping domain decompositions applied to nonlinear elliptic equations there are general convergence results, even in the continuous formulation [34, 78, 101, 102]. However, for the nonoverlapping methods such results are few with the exceptions [8, 9, 17], but these are restricted to one-dimensional domains and specific method parameter choices. Moreover, for parabolic equations the convergence of many methods have been studied on rectangular domains [44, 46, 47, 50, 72, 73], but for more general domains, less has been shown. A few exceptions that deal with general domains in $\mathbb{R}^d$, $d = 1, 2, 3$, can be found in [48, 55], but these use a type of analysis that is specific to the method studied and which is therefore difficult to generalize to other methods.

The goal of this thesis is thus to construct a general framework that can be used to analyze nonoverlapping domain decomposition methods applied to linear and nonlinear elliptic and parabolic equations. We use this framework to analyze several domain decomposition methods, with starting point in the three standard methods, namely, the Dirichlet–Neumann, Neumann–Neumann and Robin–Robin methods. The framework is then used to design several new efficient methods for both elliptic and parabolic equations. We strive to also show convergence for the discrete case with convergence rates that do not depend on the discretization. Moreover, we demon-

strate our convergence results with numerical results and compare the convergence of the methods.

The thesis is organized as follows. In Chapter 2 we discuss the underlying theory and notation that will be used in the thesis. We then recall the theory of nonoverlapping domain decomposition methods applied to linear elliptic equations in Chapter 3. Most of these results are taken from the literature, but we have made a few minor generalizations. This chapter is concluded by giving some numerical results that will be used for comparison in later chapters. In Chapter 4 we give our generalization of the linear elliptic theory to nonlinear elliptic equations, which is based on Papers I-III. We show that most of our methods have mesh independent convergence rates when applied to discrete equations and conclude with numerical results that verify our theoretical findings. In Chapter 5 we continue the generalization to parabolic equations, which is based on Papers IV and V. We begin by considering a linear equation and discuss the difficulties that arise when treating parabolic equations. We then consider a linear initial value problem and show how the analysis must be adapted for this case. Finally, we make the full generalization to nonlinear parabolic problems. We show how this framework can be used in the discrete context as well and provide numerical results for the studied methods applied to linear parabolic equations. Finally, we discuss future work in Chapter 6.

Chapter 2

Preliminaries

2.1 Banach spaces and operators

All analysis in this thesis is performed in Banach or Hilbert spaces using functional analysis. For an introduction to linear functional analysis we refer to [70] and for nonlinear functional analysis we refer to [93, 113]. We mainly consider four different frameworks based on four abstract results on bijectivity. These are the Lax–Milgram lemma, the Banach–Nečas–Babuška theorem, Zarantello's theorem, and the Browder–Minty theorem.

For a Banach space X, we denote the dual by X^* and the dual pairing by $\langle \cdot, \cdot \rangle_{X^* \times X}$, or simply by $\langle \cdot, \cdot \rangle$ if the spaces are obvious from the context. A linear operator $A : X \to X^*$ is said to be coercive if there exists a constant $c > 0$ such that

$$\langle Ax, x \rangle \geq c\|x\|_X^2 \quad \text{for all } x \in X.$$

For a proof of the following, see [70, Chapter 6, Theorem 6].

Theorem 2.1.1 (Lax–Milgram). *Let X be a Hilbert space and $A : X \to X^*$ be a linear operator. Suppose that A is bounded and coercive. Then A is bijective.*

Remark 2.1.1. *The Lax–Milgram lemma is typically stated for a bilinear form $a : X \times X \to \mathbb{R}$. This formulation is equivalent since we can define $A : X \to X^*$ by*

$$\langle Ax, y \rangle = a(x, y).$$

It can be verified that A is bounded and coercive if and only if a is bounded and coercive, i.e.,

$$\begin{aligned} |a(x, y)| &\leq C\|x\|_X\|y\|_X && \text{for all } x, y \in X \text{ and} \\ a(x, x) &\geq c\|x\|_X^2 && \text{for all } x \in X, \end{aligned}$$

respectively. When analyzing nonlinear equations the notation is simpler using A than a and therefore, in order to be consistent, we will use this notation in the linear case as well.

The linear operator $A : X \to X^*$ is said to be symmetric if

$$\langle Ax, y\rangle = \langle Ay, x\rangle \quad \text{for all } x, y \in X.$$

For two Hilbert spaces X, Y, and an operator $A : X \to Y$, we define the adjoint operator $A^* : Y^* \to X^*$ by

$$\langle A^* y, x\rangle_{X^*\times X} = \langle y, Ax\rangle_{Y^*\times Y}, \quad \text{for } x \in X, y \in Y^*.$$

In cases when A is not coercive, but satisfies a weaker condition, known as the inf-sup condition, one can still show bijectivity. This is the Banach–Nečas–Babuška theorem and it gives an equivalent characterization of bijective linear operators. For the proof we refer to [37, Corollary A.45].

Theorem 2.1.2 (Banach–Nečas–Babuška). *Let X and Y be Hilbert spaces and $A : X \to Y^*$ be a bounded linear operator. Then A is bijective if and only if A^* is injective and there exists a constant $c > 0$ such that*

$$\inf_{x\in X} \sup_{y\in Y} \frac{\langle Ax, y\rangle}{\|x\|_X \|y\|_Y} \geq c.$$

A (nonlinear) operator $G : X \to Y$ is Lipschitz continuous on a subset $D \subset X$ if there exists a constant $C > 0$ such that

$$\|Gx - Gy\|_Y \leq C\|x - y\|_X \qquad \text{for all } x, y \in D.$$

If $D = X$ we simply say that G is Lipschitz continuous. Moreover, we say that $G : X \to X^*$ is uniformly monotone if there exists a constant $c > 0$ such that

$$\langle Gx - Gy, x - y\rangle \geq c\|x - y\|_X^2 \qquad \text{for all } x, y \in X.$$

We now state Zarantello's theorem, our first nonlinear generalization of the Lax–Milgram lemma. This is a generalization since, for a linear operator, Lipschitz continuity is equivalent to being bounded and uniform monotonicity is equivalent to coercivity. Zarantello's theorem also naturally gives a linearly convergent iteration, which is useful for constructing numerical methods. By linear convergence of a sequence $\{z^n\}_{n=0}^{\infty}$ to $z \in X$ we mean that there exist constants $C > 0$ and $0 \leq L < 1$ such that

$$\|z^n - z\|_X \leq CL^n.$$

Since many of our results are based on generalizing Zarantello's theorem, we give a proof here for reference.

Theorem 2.1.3 (Zarantello). *Let X be a Hilbert space and $G : X \to X^*$ be a Lipschitz continuous and uniformly monotone operator. Then G is bijective.*

Furthermore, let $z^0 \in X$, $\chi \in X^*$, and $s_0 > 0$ be small enough. Then the iteration

$$z^{n+1} = z^n + s_0 P^{-1}(\chi - Gz^n) \tag{2.1}$$

converges linearly to the unique z such that $Gz = \chi$. Here, $P : X \to X^* : x \mapsto (x, \cdot)_X$ denotes the Riesz isomorphism.

Proof. We will show that the operator $K : X \to X^*$ defined by

$$Kx = x + s_0 P^{-1}(\chi - Gx)$$

is a contraction for any $\chi \in X^*$ and a small enough $s_0 > 0$. Consider $x, y \in X$. Then, by the Lipschitz continuity and uniform monotonicity of G, we have

$$\begin{aligned}
\|Kx - Ky\|_X^2 &= \|x - y - s_0 P^{-1}(Gx - Gy)\|_X^2 \\
&= \|x - y\|_X^2 - 2s_0 \big(P^{-1}(Gx - Gy), x - y\big)_X + s_0^2 \|P^{-1}(Gx - Gy)\|_X^2 \\
&= \|x - y\|_X^2 - 2s_0 \langle Gx - Gy, x - y\rangle + s_0^2 \|P^{-1}(Gx - Gy)\|_X^2 \\
&\leq \|x - y\|_X^2 - 2s_0 c \|x - y\|_X^2 + C s_0^2 \|x - y\|_X^2 \\
&= (1 - cs_0 + Cs_0^2)\|x - y\|_X^2.
\end{aligned}$$

It follows that, if $s_0 > 0$ is small enough, K is a contraction and therefore, by the Banach fixed point theorem [105, Theorem 2.1], the iteration (2.1) converges linearly to some $z \in X$ such that $Kz = z$. By the definition of K we then have $Gz = \chi$ and therefore G is bijective since $\chi \in X^*$ was arbitrary. □

For one of our generalizations of Zarantello's theorem we will need the Fréchet derivative of an operator. We say that an operator $G : X \to Y$ is Fréchet differentiable at a point $x \in X$ if there exists a linear operator $G'(x) : X \to Y$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|G(x + h) - G(x) - G'(x)h\|_Y}{\|h\|_X} = 0.$$

If G is Fréchet differentiable at all points $x \in D$ for some subset $D \subset X$ we say that G is Fréchet differentiable on D with Fréchet derivative $G' : D \to B(X, Y)$, where $B(X, Y)$ is the space of bounded linear operators from X to Y. For more details on the Fréchet derivative we refer to [104].

The previous results on bijectivity hold only for Hilbert spaces, but we will also require a framework for Banach spaces. Let X be a Banach space and consider the (nonlinear) operator $G : X \to X^*$. We say that G is monotone if

$$\langle Gx - Gy, x - y\rangle \geq 0 \qquad \text{for all } x, y \in X.$$

	Linear (Hilbert)	Nonlinear (Hilbert)	Nonlinear (Banach)
$X = Y$	Lax–Milgram	Zarantello	Browder–Minty
$X \neq Y$	Banach–Nečas–Babuška	-	-

Figure 2.1: The four analysis frameworks we employ in this thesis and whether they are applicable to Bubnov–Galerkin formulations ($X = Y$), or more generally to Petrov–Galerkin formulations ($X \neq Y$), in Hilbert or Banach spaces.

Moreover, G is coercive if

$$\lim_{\|x\|_X \to \infty} \frac{\langle Gx, x \rangle}{\|x\|_X} = \infty.$$

Finally, G is demicontinuous if

$$\langle Gx^k - Gx, y \rangle \to 0$$

as $x^k \to x$ for all $y \in X$. The proof of the following result can be found in [113, Theorem 26.A]. Note that this is a generalization of the bijectivity result in Zarantello's theorem since Lipschitz continuity implies demicontinuity and uniform monotonicity implies both monotonicity and coercivity.

Theorem 2.1.4 (Browder–Minty). *Let X be a reflexive Banach space and $G : X \to X^*$ be a monotone, coercive, and demicontinuous operator. Then G is bijective.*

We finish this section by summarizing the four results and their use cases in Figure 2.1. Notice, in particular, that we have not included a nonlinear version of the inf-sup condition. In fact, we are unaware of any such result at all.

2.2 Sobolev spaces

In this section we introduce the Sobolev spaces that will be used for the thesis. For an introduction to Sobolev spaces we refer to [38, Chapter 5] and [65, Chapter 5]. We let $1 < p < \infty$, denote by $U \subset \mathbb{R}^d$ any open set, and define the Sobolev space $W^{1,p}(U)$ in the usual manner, see [65, Chapter 5]. For $u \in W^{1,p}(U)$ we define the seminorm and norm as

$$\begin{aligned} |u|_{W^{1,p}(U)} &= \|\nabla u\|_{L^p(\mathbb{R})^d} \quad \text{and} \\ \|u\|^p_{W^{1,p}(U)} &= |u|^p_{W^{1,p}(U)} + \|u\|^p_{L^p(U)}, \end{aligned}$$

respectively. We will frequently work with fractional Sobolev space with differentiability $0 < s < 1$ on a Lipschitz manifold, typically a subset of the boundary of a domain, or in the case of time-dependent equations, $\mathbb{R}$ or $\mathbb{R}^+$. For a proper treatment of Lipschitz manifolds and, in particular how to define integrals and L^p-spaces on these, see [23, 89]. For a proper definition of Lipschitz domains, see [65, Section 6.2]. For a Lipschitz manifold $\mathcal{M}$ of dimension k and $0 < s < 1$ we define the fractional Sobolev space $W^{s,p}(\mathcal{M})$ as

$$\begin{aligned} W^{s,p}(\mathcal{M}) &= \{u \in L^p(\mathcal{M}) : |u|_{W^{s,p}(\mathcal{M})} < \infty\}, \\ \|u\|_{W^{s,p}(\mathcal{M})} &= \left(\|u\|^p_{L^p(\mathcal{M})} + |u|^p_{W^{s,p}(\mathcal{M})}\right)^{1/p}, \text{ and} \\ |u|^p_{W^{s,p}(\mathcal{M})} &= \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{|u(x) - u(y)|^p}{|x-y|^{k+sp}} \, \mathrm{d}s_x \, \mathrm{d}s_y. \end{aligned}$$

To keep the notation in this section simple we also write $W^{0,p}(\mathcal{M}) = L^p(\mathcal{M})$. The spaces $W^{s,p}(\mathcal{M})$ are reflexive and separable Banach spaces. This follows from [65, Theorem 6.8.4] in the case when $\mathcal{M}$ is a Lipschitz domain and from [65, p. 332] when $\mathcal{M}$ is the boundary of a Lipschitz domain. A similar argument shows that it holds for a general Lipschitz manifold $\mathcal{M}$. In the case $p = 2$ and $0 \leq s \leq 1$, $W^{s,p}(\mathcal{M})$ is a Hilbert space and we write $H^s(\mathcal{M}) = W^{s,2}(\mathcal{M})$. For $\mathcal{M} = \mathbb{R}$ and $p = 2$, we will use the equivalent definition

$$\begin{aligned} &H^s(\mathbb{R}) = \{v \in L^2(\mathbb{R}) : (1 + (\cdot)^2)^{s/2} \mathcal{F}v \in L^2_{\mathbb{C}}(\mathbb{R})\} \quad \text{with} \\ &\|v\|_{H^s(\mathbb{R})} = \|(1 + (\cdot)^2)^{s/2} \mathcal{F}v\|_{L^2_{\mathbb{C}}(\mathbb{R})}, \end{aligned}$$

see [103, Lemma 16.3] for a proof of equivalence. Here, $\mathcal{F} : L^2(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R})$ denotes the Fourier transform and $L^2_{\mathbb{C}}(\mathbb{R})$ is the space of complex valued functions on $\mathbb{R}$ that are square integrable. On this space we can define the fractional derivatives $\partial_{\pm}^{1/2} : H^{1/2}(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\partial_{\pm}^{1/2} = \mathcal{F}^{-1} M_{\pm} \mathcal{F}$$

with $M_+ u(\xi) = \sqrt{\mathrm{i}\xi} u(\xi)$ and $M_- u(\xi) = \overline{\sqrt{\mathrm{i}\xi}} u(\xi)$. We will also make frequent use of the Hilbert transform. Formally, the Hilbert transform of a function u is

$$\mathcal{H}u(t) = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \int_{|\tau| \geq \epsilon} \frac{1}{\tau} u(t - \tau) \, \mathrm{d}\tau.$$

However, we will use the definition

$$\mathcal{H}u = \mathcal{F}^{-1} M_{\mathrm{sgn}} \mathcal{F}u, \quad M_{\mathrm{sgn}} : u \mapsto -\mathrm{i}\mathrm{sgn}(\cdot) u,$$

from which it is easy to see that $\mathcal{H} : H^s(\mathbb{R}) \to H^s(\mathbb{R})$ is an isomorphism for $0 \leq s \leq 1$. For properties of the Hilbert transform we refer to [60]. In particular, the equivalence of the definitions is proven in [60, Chapter 5.2]. For our purposes we will require the following formulas, which are easy to verify using the Plancherel theorem

$$\partial_+^{1/2} = -\partial_-^{1/2}\mathcal{H}, \quad \mathcal{H}\mathcal{H} = -I,$$

and

$$\begin{aligned}(\partial_t u, v)_{L^2(\mathbb{R})} &= (\partial_+^{1/2} u, \partial_-^{1/2} v)_{L^2(\mathbb{R})} = -(\partial_-^{1/2} u, \partial_+^{1/2} v)_{L^2(\mathbb{R})} \\ &= -(u, \partial_t v)_{L^2(\mathbb{R})}.\end{aligned} \tag{2.2}$$

We now state the trace theorem, which classifies the boundary regularity of a function in a Sobolev space. For the proof we refer to [65, Theorems 6.8.13 and 6.9.2].

Lemma 2.2.1. *Let U be a bounded Lipschitz domain. Then there exists a bounded linear operator $T : W^{1,p}(U) \to W^{1-1/p,p}(\partial U)$ such that $Tu = u|_{\partial U}$ for all $u \in C^\infty(\bar{U})$. Moreover, there exists a bounded linear operator $R : W^{1-1/p,p}(\partial U) \to W^{1,p}(U)$ that is a right inverse to T.*

We denote by $W_0^{1,p}(U)$ the closure of $C_0^\infty(U)$ in $W^{1,p}(U)$. This is a closed subspace of $W^{1,p}(U)$ and can be identified as

$$W_0^{1,p}(U) = \{u \in W^{1,p}(U) : Tu = 0\}, \tag{2.3}$$

see [65, Theorem 6.6.4] for the proof.

We now consider a submanifold $\mathcal{N} \subset \mathcal{M}$ and the extension by zero operator $E : L^p(\mathcal{N}) \to L^p(\mathcal{M})$. We define the Lions–Magenes space as

$$\begin{aligned}W_{00}^{s,p}(\mathcal{N}) &= \{u \in L^p(\mathcal{N}) : Eu \in W^{s,p}(\mathcal{M})\}, \\ \|u\|_{W_{00}^{s,p}(\mathcal{N})} &= \|Eu\|_{W^{s,p}(\mathcal{M})}.\end{aligned}$$

This defines a reflexive and separable Banach space, see the proof of [Paper III, Lemma 4.1]. Note that we leave out the dependence on $\mathcal{M}$ in the notation. In fact, it can be shown that the space does not depend on $\mathcal{M}$, see [61, Theorem B.3] for $p = 2$. We conjecture that this result holds for $p > 2$ as well and for the sake of notation this is assumed to hold in the thesis. As shown in Paper III the analysis is not dependent on whether this conjecture is true or not.

2.3 Sobolev–Bochner spaces

For an introduction to Bochner and Sobolev–Bochner spaces we refer to [58, Chapter 2.5.d]. In this thesis we only use Sobolev–Bochner spaces that are Hilbert spaces. In

this case, they are equivalent to Hilbert tensor spaces, see [108, Chapter 3.4]. For a Hilbert space X and an interval $I = \mathbb{R}$ or $I = \mathbb{R}^+$ we denote the Bochner space by $L^2(I, X)$, see [58, Chapter 1.2.b] for a definition. We let $\otimes$ denote the algebraic tensor product and $\tilde{\otimes}$ the tensor product, see [108, Chapter 3.4]. We define the Sobolev–Bochner spaces through the Hilbert tensor spaces, e.g., $H^s(I, X) = H^s(I) \tilde{\otimes} X$. The Lions–Magenes space and other subsets of the Bochner space are defined similarly.

The following lemma is essential since it allows us to extend results on the spatial domain to the space-time domain. The proof can be found in [5, Section 12.4.1].

Lemma 2.3.1. *Let X_1, X_2, Y_1, and Y_2 be separable Hilbert spaces and consider the bounded linear operators $P_1 : X_1 \to Y_1$ and $P_2 : X_2 \to Y_2$. There exists a bounded linear operator $P_1 \tilde{\otimes} P_2 : X_1 \tilde{\otimes} X_2 \to Y_1 \tilde{\otimes} Y_2$ such that*

$$\big(P_1 \tilde{\otimes} P_2\big)(x_1 \otimes x_2) = P_1 x_1 \otimes P_2 x_2 \quad \text{for all } x_1 \in X_1, x_2 \in X_2.$$

In particular, using $X_1 = Y_1 = L^2(I)$, $X_2 = H^1(U)$, and $Y_2 = H^{1/2}(\partial U)$ together with the operators $P_1 = I$ and P_2 the trace operator as defined in Section 2.2, this gives the space time extension of the trace operator

$$T : L^2\big(I, H^1(U)\big) \to L^2\big(I, H^{1/2}(\partial U)\big).$$

When treating parabolic equations we will primarily work in an intersection space of the form

$$H^{1/2}\big(\mathbb{R}, L^2(U)\big) \cap L^2\big(\mathbb{R}, H^1(U)\big).$$

On this space, the trace operator defines a bounded operator

$$\begin{aligned} T : H^{1/2}\big(\mathbb{R}, L^2(U)\big) \cap L^2\big(\mathbb{R}, H^1(U)\big) \\ \to H^{1/4}\big(\mathbb{R}, L^2(\partial U)\big) \cap L^2\big(\mathbb{R}, H^{1/2}(\partial U)\big), \end{aligned} \tag{2.4}$$

and there exists a bounded linear right inverse

$$\begin{aligned} R : H^{1/4}\big(\mathbb{R}, L^2(\partial U)\big) \cap L^2\big(\mathbb{R}, H^{1/2}(\partial U)\big) \\ \to H^{1/2}\big(\mathbb{R}, L^2(U)\big) \cap L^2\big(\mathbb{R}, H^1(U)\big), \end{aligned} \tag{2.5}$$

see [24, Lemma 2.4 & Theorem 2.9]. From Lemma 2.3.1 it also follows that $\mathcal{H} : H^s(\mathbb{R}, X) \to H^s(\mathbb{R}, X)$ defines an isomorphism and arguing as in [Paper IV, Lemma 4.6] shows that this holds for

$$\mathcal{H} : H^{1/2}\big(\mathbb{R}, L^2(U)\big) \cap L^2\big(\mathbb{R}, H^1(U)\big) \to H^{1/2}\big(\mathbb{R}, L^2(U)\big) \cap L^2\big(\mathbb{R}, H^1(U)\big)$$

and

$$\begin{aligned} \mathcal{H} : H^{1/4}\big(\mathbb{R}, L^2(\mathcal{N})\big) \cap L^2\big(\mathbb{R}, H^{1/2}_{00}(\mathcal{N})\big) \\ \to H^{1/4}\big(\mathbb{R}, L^2(\mathcal{N})\big) \cap L^2\big(\mathbb{R}, H^{1/2}_{00}(\mathcal{N})\big). \end{aligned}$$

Other operators, such as ∇, ∂_t, and $\partial_{\pm}^{1/2}$ can be extended analogously.

2.4 Geometry of domain decomposition methods

In this section we introduce the notation, terminology, and assumptions used for our spatial domain decomposition. We will denote the spatial dimension by $d = 1, 2, \ldots$ and consider the spatial domain $\Omega \subset \mathbb{R}^d$. In this thesis, we will consider nonoverlapping domain decompositions with two subdomains Ω_1 and Ω_2 together with an interface Γ. That is, we have the following identities:

$$\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}, \quad \Omega_1 \cap \Omega_2 = \emptyset \quad \text{and} \quad \Gamma = (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega.$$

For examples of such decomposition, see Figures 2.2a and 2.2b. It is not strictly necessary that the subdomains Ω_i, $i = 1, 2$ are connected. In fact, we can also consider subdomains of the form

$$\Omega_i = \bigcup_{\ell=1}^{N_i} \Omega_{i\ell}, \quad N_i \in \mathbb{N}, \quad i = 1, 2,$$

where each $\Omega_{i\ell}$ is connected, e.g., Figure 2.2c. These decompositions are important for parallelization since the equation on Ω_i can then be divided into N_i independent equations on $\Omega_{i\ell}$, $\ell = 1, \ldots, N_i$. We do not, however, consider decompositions with crosspoints, i.e., points where two parts of Γ intersect, as in Figure 2.2d. These lead to more complex variational formulations with different test and trial spaces, see, e.g., [Paper I, Remark 6.5].

Throughout the thesis we will always assume that our domains are Lipschitz.

Assumption 2.4.1. *The domains Ω and Ω_i, $i = 1, 2$, are bounded and Lipschitz. Furthermore the sets Γ and $\partial\Omega_i$, $i = 1, 2$, are $(d-1)$-dimensional Lipschitz manifolds.*

Under this assumption we can define the outwards-pointing unit normals restricted to the interface ν_1 and ν_2 of Ω_1 and Ω_2, respectively. The normals satisfy $\nu_i \in L^\infty(\Gamma)$, see [65, Section 6.10.1].

For equations that are only monotone in seminorm, we also need to assume that each subdomain intersects the exterior boundary. This allows us to use the Poincaré inequality to achieve monotonicity in norm. Note that this assumption excludes decompositions of the form Figure 2.2b.

Assumption 2.4.2. *The domains Ω, Ω_i, $i = 1, 2$, are bounded and Lipschitz. Furthermore the sets Γ, $\partial\Omega_i$, and $\partial\Omega \setminus \partial\Omega_i$, $i = 1, 2$, are (non-empty) $(d-1)$-dimensional Lipschitz manifolds.*

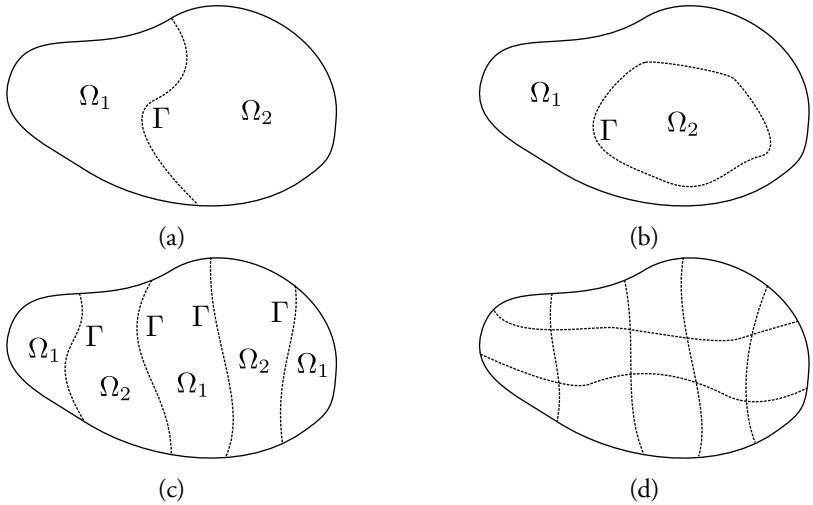

Figure 2.2: Examples of decompositions of Ω: (a) two subdomains with two intersection points; (b) two subdomains without intersection points; (c) multiple subdomains without crosspoints; (d) multiple subdomains with crosspoints.

We recall Poincaré's inequality. Under Assumption 2.4.2 there exists a constant C_p such that

$$\|u\|^p_{L^p(\Omega_i)} \leq C_p |u|^p_{W^{1,p}(\Omega_i)} \tag{2.6}$$

for all $u \in \{u \in W^{1,p}(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\}$.

In this thesis we will make use of the following three domain decompositions.

Decomposition 2.4.1. *Consider the domain $\Omega = (0,1) \subset \mathbb{R}$ together with the subdomains $\Omega_1 = (0, 1/2)$ and $\Omega_2 = (1/2, 1)$.*

Decomposition 2.4.2. *Consider the domain $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ together with the subdomains $\Omega_1 = (0, 1/2) \times (0, 1)$ and $\Omega_2 = (1/2, 1) \times (0, 1)$. The domain is illustrated in Figure 2.3.*

Decomposition 2.4.3. *Consider the domain $\Omega = (0,3) \times (0,2) \subset \mathbb{R}^2$ together with the subdomains as described in Figure 2.4.*

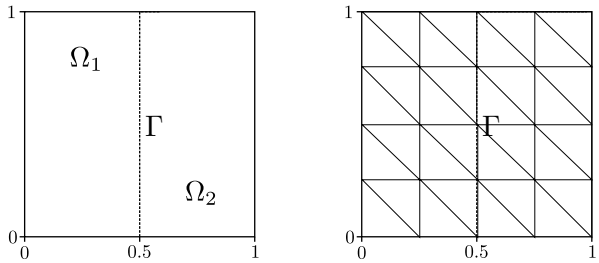

Figure 2.3: The domain decomposition used in Decomposition 2.4.2 (left) and the mesh used for discretization (right).

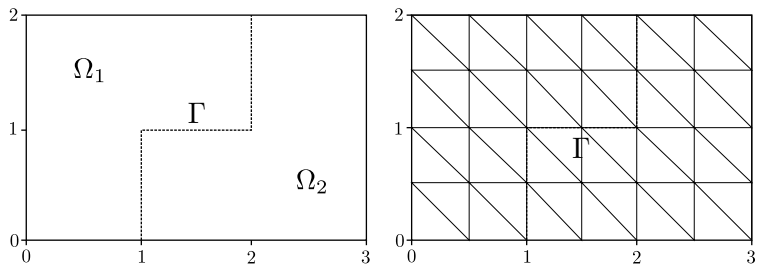

Figure 2.4: The domain decomposition used in Decomposition 2.4.3 (left) and the mesh used for discretization (right).

Chapter 3

Domain decomposition for linear elliptic equations

3.1 Introduction to the analysis of nonoverlapping domain decomposition methods

For a comprehensive introduction to linear elliptic partial differential equations we refer to [38, Chapter 6] and [51, 75]. In this chapter we will, for simplicity, only consider the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The theory of nonoverlapping domain decomposition methods for linear equations is well understood and a comprehensive introduction can be found in [33, 91, 106, 109]. The aim of this section is to explain the general idea of the analysis. This will be done formally without introducing the weak formulations that are necessary to make the theory rigorous.

The first step towards an analysis is to show that the equation is equivalent to a transmission problem. For a function u defined on Ω we define $u_i = u|_{\Omega_i}$. Conversely, for two functions (u_1, u_2) defined on (Ω_1, Ω_2) we can define the "glued" function $u = \{u_1 \text{ on } \Omega_1;\ u_2 \text{ on } \Omega_2\}$. Similarly, we introduce $f_i = f|_{\Omega_i}$. With this notation, the equation (3.1) can be reformulated as an equivalent transmission problem, which

is

$$\begin{cases} -\Delta u_i = f_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \text{ for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma, \\ \nabla u_1 \cdot \nu_1 = -\nabla u_2 \cdot \nu_2 & \text{on } \Gamma. \end{cases} \tag{3.2}$$

The first equation states that our subdomain solutions u_i satisfy the equation on the interior of the subdomains Ω_i and the second equation ensures that u_i satisfy the exterior boundary condition. The third equation, or the first transmission condition, says that the solution is continuous across the interface and guarantees that the glued solution is sufficiently smooth. Finally, the fourth equation, or the second transmission condition, makes sure that the normal derivatives agree on the interface. This condition ensures that the glued solution satisfies the original equation and the condition will therefore typically depend on the type of equation studied.

In order to reduce the transmission problem to a problem on the interface Γ we first consider the nonhomogeneous equations

$$\begin{cases} -\Delta u_i = g & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ u_i = \eta & \text{on } \Gamma. \end{cases} \tag{3.3}$$

We introduce the linear solution operators $F_i : \eta \mapsto u_i$ that solve (3.3) with $g = 0$ and $G_i : g \mapsto u_i$ that solve (3.3) with $\eta = 0$. With this notation, the transmission problem (3.2) can be written as only one equation

$$\nabla(F_1\eta + G_1 f_1) \cdot \nu_1 = -\nabla(F_2\eta + G_2 f_2) \cdot \nu_2 \qquad \text{on } \Gamma, \tag{3.4}$$

which we call the Steklov–Poincaré equation. The equivalence is given by the identity

$$u_i = F_i\eta + G_i f_i \quad i = 1, 2.$$

Notice that the first three conditions of (3.2) are all guaranteed to be satisfied by the definitions of the operators F_i and G_i. The fourth equation of (3.2) is exactly (3.4). If we introduce the notation

$$\begin{aligned} S_i &: \eta \mapsto \nabla F_i \eta \cdot \nu_i, \\ \chi_i &= -\nabla G_i f_i \cdot \nu_i \\ S &= S_1 + S_2, \quad \text{and} \\ \chi &= \chi_1 + \chi_2 \end{aligned}$$

then the Steklov–Poincaré equation can be written as

$$S\eta = \chi \qquad \text{on } \Gamma. \tag{3.5}$$

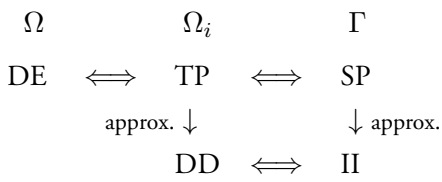

Figure 3.1: The basic idea for the analysis of a nonoverlapping domain decomposition method. The graph shows the connection between the differential equation (DE), the transmission problem (TP), the Steklov–Poincaré equation (SP), the domain decomposition method (DD), and the interface iteration (II).

The operators S_i, S are the Steklov–Poincaré operators. They are sometimes referred to as Dirichlet-to-Neumann operators, since they map Dirichlet data to Neumann data.

The reduction from the equation on Ω to the Steklov–Poincaré equation on Γ is summarized in the top row of Figure 3.1. The idea is now to approximate the solution to the transmission problem with a domain decomposition method and show that this is equivalent to approximating the solution to the Steklov–Poincaré equation with an interface iteration, see the bottom row of Figure 3.1.

Thus, a domain decomposition method approximates $(u_1, u_2) = (u|_{\Omega_1}, u|_{\Omega_2})$ by an iteration (u_1^n, u_2^n). The domain decomposition methods that we study here are the Dirichlet–Neumann, Neumann–Neumann, and Robin–Robin methods. The Dirichlet–Neumann method is one of the simplest convergent nonoverlapping domain decomposition methods that one can come up with and consists of alternating between a Dirichlet and a Neumann problem, corresponding to the first and second transmission conditions, respectively. The method has been extensively studied when applied to linear elliptic equations, see e.g. [11, 13, 41, 84, 85]. For a fixed method parameter $s_0 > 0$ and an initial guess η^0, the Dirichlet–Neumann method consists

of finding (u_1^n, u_2^n, η^n) such that

$$\begin{cases} -\Delta u_1^n = f_1 & \text{in } \Omega_1, \\ u_1^n = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ u_1^n = \eta^{n-1} & \text{on } \Gamma, \\ -\Delta u_2^n = f_2 & \text{in } \Omega_2, \\ u_2^n = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \nabla u_2^n \cdot \nu_2 = \nabla u_1^n \cdot \nu_2 & \text{on } \Gamma, \\ \eta^n = s_0\, u_2^n|_\Gamma + (1 - s_0)\eta^{n-1} & \text{on } \Gamma \end{cases}$$

for $n = 1, 2, \ldots$

Another common method is the Neumann–Neumann method, which was first introduced in [12]. For two fixed method parameters $s_1, s_2 > 0$ and an initial guess η^0, the method consists of finding $(u_1^n, u_2^n, w_1^n, w_2^n, \eta^n)$ such that

$$\begin{cases} -\Delta u_i^n = f_i & \text{in } \Omega_i, \\ u_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ u_i^n = \eta^{n-1} & \text{on } \Gamma, \text{ for } i = 1, 2, \\ -\Delta w_i^n = 0 & \text{in } \Omega_i, \\ w_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ \nabla w_i^n \cdot \nu_1 = \nabla u_1^n \cdot \nu_1 - \nabla u_2^n \cdot \nu_1 & \text{on } \Gamma, \text{ for } i = 1, 2, \\ \eta^n = \eta^{n-1} - (s_1\, w_1^n|_\Gamma - s_2\, w_2^n|_\Gamma) & \text{on } \Gamma \end{cases}$$

for $n = 1, 2, \ldots$

Finally, we consider the Robin–Robin method. For a fixed method parameter $s_3 > 0$ and an initial guess u_2^0, it consists of finding (u_1^n, u_2^n) such that

$$\begin{cases} -\Delta u_1^n = f_1 & \text{in } \Omega_1, \\ u_1^n = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \nabla u_1^n \cdot \nu_1 + s_3 u_1^n = \nabla u_2^{n-1} \cdot \nu_1 + s_3 u_2^{n-1} & \text{on } \Gamma, \\ -\Delta u_2^n = f_2 & \text{in } \Omega_2, \\ u_2^n = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \nabla u_2^n \cdot \nu_2 + s_3 u_2^n = \nabla u_1^n \cdot \nu_2 + s_3 u_1^n & \text{on } \Gamma. \end{cases} \tag{3.6}$$

for $n = 1, 2, \ldots$ Notice, in particular, that, since $\nu_2 = -\nu_1$, the two boundary conditions in (3.6) are two different linear combinations of the two transmission

conditions in (3.2). The Robin–Robin method was first introduced in [79] with a proof of convergence and has since been studied extensively [81, 110]. We note that the Robin–Robin method corresponds to the zeroth order optimized Schwarz method [42, 45, 82, 83, 111]. More modifications and generalizations of the Robin–Robin method can be found in [8, 20, 52]. The method has also been studied in the context of the Helmholtz equation [29, 49].

Each of these methods corresponds to an interface iteration. For the Dirichlet–Neumann, Neumann–Neumann, and Robin–Robin methods these are

$$\begin{aligned}
\eta^{n+1} &= \eta^n + s_0 S_2^{-1}(\chi - S\eta^n),\\
\eta^{n+1} &= \eta^n + (s_1 S_1^{-1} + s_2 S_2^{-1})(\chi - S\eta^n), \quad \text{and}\\
\eta^{n+1} &= (s_3 I + S_2)^{-1}\Big((s_3 I - S_1)(s_3 I + S_1)^{-1}\big((s_3 I - S_2)\eta^n + \chi\big) + \chi\Big),
\end{aligned}$$

respectively. Thus, in order to prove convergence of the methods, it is sufficient to prove convergence of the corresponding interface iteration.

3.2 Steklov–Poincaré theory for the Poisson equation

Since the interface iterations involve the Steklov–Poincaré operators, we begin our convergence analysis by studying the properties of these operators. In this section we will show how to define the Steklov–Poincaré operators rigorously and state their main properties, namely that they are bounded, coercive, and symmetric. This section follows closely the analysis provided in [91]. We recall the Sobolev spaces defined in Section 2.2 and introduce the spaces

$$\begin{gathered}
V = H_0^1(\Omega), \quad V_i^0 = H_0^1(\Omega_i), \quad V_i = \{v \in H^1(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\},\\
\text{and} \quad \Lambda = H_{00}^{1/2}(\Gamma) = \{\mu \in L^2(\Gamma) : E_i \mu \in H^{1/2}(\partial\Omega_i)\},
\end{gathered}$$

where $E_i : L^2(\Gamma) \to L^2(\partial\Omega_i)$ denotes the extension by zero. As we stated in Lemma 2.2.1, the trace operator is a bounded operator $T_{\partial\Omega_i} : H^1(\Omega_i) \to H^{1/2}(\partial\Omega_i)$ and has a bounded linear right inverse $R_{\partial\Omega_i} : H^{1/2}(\partial\Omega_i) \to H^1(\Omega_i)$. Since we are only interested in the boundary values on Γ we introduce the interface trace operator and corresponding right inverse by

$$T_i : V_i \to \Lambda : v \mapsto T_{\partial\Omega_i} v|_\Gamma\,, \qquad R_i : \Lambda \to V_i : \mu \mapsto R_{\partial\Omega_i} E_i \mu,$$

respectively.

We define the operators $A : V \to V^*$ and $A_i : V_i \to V_i^*$ by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{and} \quad \langle A_i u_i, v_i\rangle = \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \,\mathrm{d}x,$$

respectively. The weak formulation of (3.1) is to find $u \in V$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \qquad \text{for all } v \in V. \tag{3.7}$$

If Assumption 2.4.2 holds then the operators A and A_i are bounded and coercive. This follows immediately from the Poincaré inequality (2.6). In particular, according to Theorem 2.1.1, (3.7) has a unique solution for any $f \in V^*$.

To define the transmission problem for a general $f \in V^*$ we assume the following.

Assumption 3.2.1. *We have $f \in V^*$ and there exist $f_i \in V_i^*$ such that*

$$\langle f, v\rangle = \langle f_1, v|_{\Omega_1}\rangle + \langle f_2, v|_{\Omega_2}\rangle \qquad \textit{for all } v \in V.$$

Remark 3.2.1. *Most authors assume that $f \in L^2(\Omega)$, e.g., [91]. We are not aware of any other authors that have extended the theory to $f \in V^*$. The extension has to be done carefully, since we can not say that $V_i^* \subset (V_i^0)^*$. In order to solve the nonhomogeneous problem (3.9) we therefore consider $g = Kf_i$, where*

$$K : V_i^* \to (V_i^0)^* : f_i \mapsto f_i|_{V_i^0}\,.$$

Notice that the "inclusion" operator K is well defined, but not injective, and so we do not claim that $V_i^* \hookrightarrow (V_i^0)^*$. However, to keep the notation simple we write f_i for both $f_i \in V_i^*$ and $Kf_i \in (V_i^0)^*$.

The weak form of the transmission problem is to find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\begin{cases} \langle A_i u_i, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in V_i^0, i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i u_i, R_i \mu\rangle - \langle f_i, R_i \mu\rangle = 0 & \text{for all } \mu \in \Lambda. \end{cases} \tag{3.8}$$

The weak equation and the weak transmission problem are equivalent, see [91, Lemma 1.2.1] for a proof. Before defining the Steklov–Poincaré operators, we give a rigorous definition of the solution operators F_i and G_i. Consider the nonhomogeneous problem of finding $u_i \in V_i$ such that

$$\begin{cases} \langle A_i u_i, v\rangle = \langle g, v\rangle & \text{for all } v \in V_i^0, \\ T_i u_i = \eta. \end{cases} \tag{3.9}$$

We define $F_i : \Lambda \to V_i : \eta \mapsto u_i$ as the solution operator corresponding to the choice $g = 0$ and $G_i : (V_i^0)^* \to V_i^0 : g \mapsto u_i$ as the solution operator corresponding to $\eta = 0$. The fact that there exists a unique solution to (3.9) follows by shifting the problem to a homogeneous problem and then applying Theorem 2.1.1 after the

boundedness and coercivity of $A_i : V_i^0 \to (V_i^0)^*$ has been established. The Steklov–Poincaré operators $S_i : \Lambda \to \Lambda^*$ and corresponding source terms $\chi_i \in \Lambda^*$ are defined as

$$\langle S_i\eta, \mu\rangle = \langle A_i F_i \eta, R_i\mu\rangle \quad \text{and} \quad \langle \chi_i, \mu\rangle = \langle f_i - A_i G_i f_i, R_i\mu\rangle.$$

Remark 3.2.2. *The definition of the Steklov–Poincaré operators do not depend on the choice of extensions R_i. To see this, fix $i = 1, 2$ and suppose that there are two operators R_i and $\hat{R}_i$ that satisfy $T_i R_i = I$ and $T_i \hat{R}_i = I$, respectively. Then $T_i(R_i\mu - \hat{R}_i\mu) = 0$ for all $\mu \in \Lambda$ and thus $R_i\mu - \hat{R}_i\mu \in V_i^0$ according to (2.3). By the definition of F_i, this means that*

$$\langle A_i F_i\eta, R_i\mu\rangle = \langle A_i F_i\eta, \hat{R}_i\mu\rangle + \langle A_i F_i\eta, R_i\mu - \hat{R}_i\mu\rangle = \langle A_i F_i\eta, \hat{R}_i\mu\rangle$$

and therefore the Steklov–Poincaré operator S_i does not depend on the choice of extension. This is one the most fundamental properties of the operators since it allows us to write

$$\langle S_i\eta, \mu\rangle = \langle A_i F_i\eta, F_i\mu\rangle,$$

from which we can prove coercivity and symmetry. A similar argument shows that χ_i, $i = 1, 2$, are also independent of the choice of extension operators.

Introducing $S = S_1 + S_2$ and $\chi = \chi_1 + \chi_2$ the weak formulation of the Steklov–Poincaré equation is $S\eta = \chi$, or equivalently

$$\langle S\eta, \mu\rangle = \langle \chi, \mu\rangle \quad \text{for all } \mu \in \Lambda. \tag{3.10}$$

The weak formulations of the Steklov–Poincaré equation (3.10) and the transmission problem (3.8) are equivalent. The proof follows immediately from the definition of the Steklov–Poincaré operators. We now summarize the main properties of the Steklov–Poincaré operators corresponding to the Poisson equation. See [91, Chapter 1.2] for the proof.

Theorem 3.2.1. *Suppose that Assumption 2.4.2 hold. The Steklov–Poincaré operators $S_i, S : \Lambda \to \Lambda^*$ are bounded linear operators. Moreover, they are coercive and symmetric.*

Remark 3.2.3. *To prove coercivity of S_i we have used that $-\Delta$ is coercive, which follows by the Poincaré inequality under Assumption 2.4.2. If we instead of $-\Delta$ consider the coercive operator $-\Delta + I$, then Assumption 2.4.1 is sufficient to prove coercivity of the Steklov–Poincaré operators.*

3.3 Discrete Steklov–Poincaré operators

We now briefly discuss discrete Steklov–Poincaré operators. That is, Steklov–Poincaré operators corresponding to the equation (3.1) after it has been discretized by some

Galerkin method, e.g., a finite element method. For an introduction to the finite element method we refer to [22, 100]. Let $h > 0$ and consider the conforming finite dimensional subspaces $V^h \subset V$, $V_i^h \subset V_i$, $\Lambda^h \subset \Lambda$, and $V_i^{h,0} = V_i^h \cap V_i^0$. We make the following assumptions on the spaces.

Assumption 3.3.1. *The image of the trace operator on V_i^h is Λ^h. Moreover, we have a linear right inverse $R_i^h : \Lambda^h \to V_i^h$. The operators R_i^h, $h > 0$ are bounded independently of h. We also have $f^h \in (V^h)^*$ and there exist $f_i^h \in (V_i^h)^*$, $i = 1, 2$, such that*

$$\langle f^h, v\rangle = \langle f_1^h, v|_{\Omega_1}\rangle + \langle f_2^h, v|_{\Omega_2}\rangle \qquad \textit{for all } v \in V^h.$$

This assumption is satisfied if V^h, V_1^h, V_2^h and Λ^h are finite element spaces that arise from a regular family of triangulations that is shared between the domain and subdomains [91, Theorem 4.1.3].

Remark 3.3.1. *In implementation the discrete extension operator R_i^h is chosen as the extension by zero on all interior degrees of freedom, but this is not necessarily an h-independent extension. For the theory R_i^h is instead taken to be the discrete harmonic extension. Like in the continuous case, all equations are independent of this choice, see Remark 3.2.2, so it does not matter that we use a different extension in the implementation.*

We define the discrete variants A^h and A_i^h of A and A_i as

$$\begin{aligned}\langle A^h u, v\rangle &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \qquad \text{and}\\ \langle A_i^h u_i, v_i\rangle &= \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \,\mathrm{d}x,\end{aligned}$$

respectively. The discrete equation is then to find $u^h \in V^h$ such that

$$\langle A^h u^h, v\rangle = \langle f^h, v\rangle \quad \text{for all } v \in V^h.$$

As in the continuous case, the discrete equation is equivalent to a discrete transmission problem, which is given by finding $(u_1^h, u_2^h) \in V_1^h \times V_2^h$ such that

$$\begin{cases} \langle A_i^h u_i^h, v_i\rangle = \langle f_i^h, v_i\rangle & \text{for all } v_i \in V_i^{h,0},\ i = 1, 2,\\ \qquad T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i^h u_i^h, R_i^h \mu\rangle - \langle f_i^h, R_i^h \mu\rangle = 0 & \text{for all } \mu \in \Lambda^h. \end{cases} \tag{3.11}$$

Analogously to the operators F_i and G_i in the continuous case, we introduce the discrete solution operators $F_i^h : \Lambda^h \to V_i^h$ and $G_i : (V_i^{h,0})^* \to V_i^h$ as the solutions

u_i^h to the equations

$$\begin{cases} \langle A_i^h u_i^h, v\rangle = \langle g^h, v\rangle & \text{for all } v \in V_i^{h,0}, \\ T_i u_i = \eta \end{cases}$$

with $g^h = 0$ and $\eta = 0$, respectively. We can then define discrete Steklov–Poincaré operators $S_i^h : \Lambda^h \to (\Lambda^h)^*$ and corresponding source terms $\chi_i \in (\Lambda^h)^*$ as

$$\langle S_i^h \eta, \mu\rangle = \langle A_i^h F_i^h \eta, R_i^h \mu\rangle \quad \text{and} \quad \langle \chi_i^h, \mu\rangle = \langle f_i^h - A_i^h G_i^h f_i^h, R_i^h \mu\rangle,$$

respectively. We define $S^h = S_1^h + S_2^h$ and $\chi^h = \chi_1^h + \chi_2^h$. The discrete Steklov–Poincaré equation is to find $\eta^h \in \Lambda^h$ such that

$$\langle S^h \eta, \mu\rangle = \langle \chi^h, \mu\rangle \quad \text{for all } \mu \in \Lambda^h. \tag{3.12}$$

The main properties of the discrete Steklov–Poincaré operators are stated below.

Theorem 3.3.1. *Suppose that Assumptions 2.4.2 and 3.3.1 hold. The discrete Steklov–Poincaré operators $S_i^h, S^h : \Lambda^h \to (\Lambda^h)^*$ are bounded linear operators. Moreover, they are coercive and symmetric. The constants in the boundedness and coercivity conditions are independent of $h > 0$.*

3.4 Convergence of the Dirichlet–Neumann and Neumann–Neumann methods

We will now prove convergence of the Dirichlet-Neumann and Neumann–Neumann methods. We will first state the weak formulations of the methods and then the corresponding interface iterations. The convergence will then follow from a slight generalization of Zarantello's theorem, stated in Theorem 2.1.3, for the case of a linear operator G and the properties of the Steklov–Poincaré operators in Theorem 3.2.1. Finally, we will discuss the discrete variants of the methods and derive their convergence from the properties of the discrete Steklov–Poincaré operators in Theorem 3.3.1.

In its weak formulation, the Dirichlet–Neumann method consists of finding $(u_1^n, u_2^n, \eta^n) \in V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \langle A_1 u_1^n, v_1\rangle = \langle f_1, v_1\rangle & \text{for all } v_1 \in V_1^0, \\ T_1 u_1^n = \eta^{n-1}, & \\ \langle A_2 u_2^n, v_2\rangle = \langle f_2, v_2\rangle & \text{for all } v_2 \in V_2^0, \\ \langle A_2 u_2^n - f_2, R_2\mu\rangle + \langle A_1 u_1^n - f_1, R_1\mu\rangle = 0 & \text{for all } \mu \in \Lambda, \\ \eta^n = s_0 T_2 u_2^n + (1 - s_0)\eta^{n-1} & \end{cases} \tag{3.13}$$

for $n = 1, 2, \ldots$ Similarly the weak formulation of the Neumann–Neumann method consists of finding $(u_1^n, u_2^n, w_1^n, w_2^n, \eta^n) \in V_1 \times V_2 \times V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \langle A_i u_i^n, v_i \rangle = \langle f_i, v_i \rangle & \text{for all } v_i \in V_i^0, \\ T_i u_i^n = \eta^{n-1} & \text{for } i = 1, 2, \\ \langle A_1 w_1^n, v_1 \rangle = 0 & \text{for all } v_1 \in V_1^0, \\ \langle A_1 w_1^n - A_1 u_1^n + f_1, R_1 \mu \rangle = \langle A_2 u_2^n - f_2, R_2 \mu \rangle & \text{for all } \mu \in \Lambda \\ \langle A_2 w_2^n, v_2 \rangle = 0 & \text{for all } v_2 \in V_2^0, \\ \langle A_2 w_2^n + A_2 u_2^n - f_2, R_2 \mu \rangle = \langle -A_1 u_1^n + f_1, R_1 \mu \rangle & \text{for all } \mu \in \Lambda, \\ \eta^n = \eta^{n-1} - (s_1 T_1 w_1^n - s_2 T_2 w_2^n) \end{cases} \tag{3.14}$$

for $n = 1, 2, \ldots$

For the Dirichlet–Neumann method the interface iteration consists of finding $\eta^n \in \Lambda$ such that

$$\langle S_2 \eta^n, \mu \rangle = \langle S_2 \eta^{n-1}, \mu \rangle + s_0 \langle -S\eta^{n-1} + \chi, \mu \rangle \quad \text{for all } \mu \in \Lambda. \tag{3.15}$$

Similarly, the interface iteration of the Neumann–Neumann method is to find $(\lambda_1^n, \lambda_2^n, \eta^n) \in \Lambda \times \Lambda \times \Lambda$ such that

$$\begin{cases} \langle S_i \lambda_i^n, \mu \rangle = \langle -S\eta^{n-1} + \chi, \mu \rangle & \text{for all } \mu \in \Lambda,\ i = 1, 2, \\ \eta^n = \eta^{n-1} + s_1 \lambda_1^n + s_2 \lambda_2^n. \end{cases}$$

for $n = 1, 2, \ldots$ The convergence of the Dirichlet–Neumann method now follows by a slight generalization of Zarantello's theorem, Theorem 2.1.3, where P is replaced by S_2. The idea of the proof is to define a new norm $\|\mu\|_{S_2}^2 = \langle S_2 \mu, \mu \rangle$ and then show convergence in the same way as in Theorem 2.1.3. For this to be a norm we of course require the properties in Theorem 3.2.1. For more details we refer to [91, Theorem 4.2.2]

Theorem 3.4.1. *Suppose that Assumptions 2.4.2 and 3.2.1 hold. Let u_i^n be the iterates of the Dirichlet–Neumann method (3.13) with initial guess $\eta^0 \in \Lambda$ and some parameter $s_0 > 0$ that is small enough. Moreover, let (u_1, u_2) denote the solution to the transmission problem (3.8) and η the solution to the Steklov–Poincaré equation (3.10). Then there exist constants $L < 1$ and $C > 0$ such that*

$$\|u_i^n - u_i\|_{V_i} \leq C L^n \|\eta^0 - \eta\|_\Lambda, \quad i = 1, 2.$$

The result for the Neumann–Neumann method follows in a similar way, see [91, Theorem 4.2.5]

Theorem 3.4.2. *Suppose that Assumptions 2.4.2 and 3.2.1 hold. Let u_i^n be the iterates of the Neumann–Neumann method (3.14) with initial guess $\eta^0 \in \Lambda$ and some parameters $s_1, s_2 > 0$ that are small enough. Moreover, let (u_1, u_2) denote the solution to the transmission problem (3.8) and η the solution to the Steklov–Poincaré equation (3.10). Then there exist constants $L < 1$ and $C > 0$ such that*

$$\|u_i^n - u_i\|_{V_i} \leq CL^n \|\eta^0 - \eta\|_\Lambda, \quad i = 1, 2.$$

Remark 3.4.1. *For the Dirichlet–Neumann method, the assumption that S_2 is symmetric can be weakened to*

$$\langle S_2\mu, S_2^{-1}S\mu\rangle + \langle S\mu, \mu\rangle \geq c\|\mu\|_\Lambda^2 \quad \text{for all } \mu \in \Lambda,$$

which allows us to prove similar results for equations with non-symmetric terms that are small compared to the symmetric terms, such as advection-diffusion equations with dominant diffusion [91, Chapter 5.1]. For non-symmetric equations with large non-symmetric terms, such as advection-diffusion equations with dominant advection, one can prove convergence of modified methods [91, Chapter 6.4]. A similar observation also holds for the Neumann–Neumann method.

The two methods have discrete counterparts, which are given by replacing the spaces and operators in (3.13) and (3.14) by their discrete counterparts defined in Section 3.3. The interface iteration of the discrete Dirichlet–Neumann method is then to find $\eta^{n,h} \in \Lambda^h$ such that

$$\langle S_2^h \eta^{n,h}, \mu\rangle = \langle S_2^h \eta^{n-1,h}, \mu\rangle + s_0\langle -S^h\eta^{n-1,h} + \chi^h, \mu\rangle \quad \text{for all } \mu \in \Lambda^h.$$

for $n = 1, 2, \ldots$ Similarly, the interface iteration of the discrete Neumann–Neumann method is to find $(\lambda_1^n, \lambda_2^n, \eta^{n,h}) \in \Lambda^h \times \Lambda^h \times \Lambda^h$ such that

$$\begin{cases} \langle S_i^h \lambda_i^n, \mu\rangle = \langle -S^h\eta^{n-1,h} + \chi, \mu\rangle & \text{for all } \mu \in \Lambda^h,\ i = 1, 2, \\ \eta^{n,h} = \eta^{n-1,h} + s_1\lambda_1^n + s_2\lambda_2^n. \end{cases}$$

for $n = 1, 2, \ldots$ The fact that the discrete Steklov–Poincaré operators are bounded and coercive uniformly in $h > 0$ in Theorem 3.3.1 gives the following convergence results.

Theorem 3.4.3. *Suppose that Assumptions 2.4.2 and 3.3.1 hold. Let $u_i^{n,h}$ be the iterates of the discrete Dirichlet–Neumann method with initial guess $\eta^{0,h} \in \Lambda^h$ and some parameter $s_0 > 0$ that is small enough. Moreover, let (u_1^h, u_2^h) denote the solution to the discrete transmission problem (3.11) and η^h the solution to the discrete Steklov–Poincaré equation (3.12). Then there exists constants $L < 1$ and $C > 0$ independent of h such that*

$$\|u_i^{n,h} - u_i^h\|_{V_i} \leq CL^n \|\eta^{0,h} - \eta^h\|_\Lambda, \quad i = 1, 2.$$

A similar result holds for the discrete Neumann–Neumann method.

3.5 Convergence of the Robin–Robin method

The proof of the Robin–Robin method is typically given without employing the Steklov–Poincaré theory. Also, an important assumption on the solution of the equation is often left out. We will therefore give a rigorous proof here using Steklov–Poincaré operators. The idea of the proof is still the same as the standard proof, e.g., [91, Chapter 4.5], but the notation differs.

The weak formulation of the Robin–Robin method is to find $(u_1^n, u_2^n) \in V_1 \times V_2$ such that

$$\begin{cases} \langle A_1 u_1^n, v_1\rangle = \langle f_1, v_1\rangle & \text{for all } v_1 \in V_1^0, \\ \langle A_1 u_1^n - f_1, R_1\mu\rangle + \langle A_2 u_2^{n-1} - f_2, R_2\mu\rangle & \\ \quad = s_3(T_2 u_2^{n-1} - T_1 u_1^n, \mu)_{L^2(\Gamma)} & \text{for all } \mu \in \Lambda, \\ \langle A_2 u_2^n, v_2\rangle = \langle f_2, v_2\rangle & \text{for all } v_2 \in V_2^0, \\ \langle A_2 u_2^n - f_2, R_2\mu\rangle + \langle A_1 u_1^n - f_1, R_1\mu\rangle & \\ \quad = s_3(T_1 u_1^n - T_2 u_2^n, \mu)_{L^2(\Gamma)} & \text{for all } \mu \in \Lambda \end{cases} \tag{3.16}$$

for $n = 1, 2, \ldots$ Moreover, the weak formulation of the interface iteration corresponding to the Robin–Robin method is to find $(\eta_1^n, \eta_2^n) \in \Lambda \times \Lambda$ such that

$$\begin{cases} \langle (s_3 J + S_1)\eta_1^n - \chi_1, \mu\rangle = \langle (s_3 J - S_2)\eta_2^{n-1} + \chi_2, \mu\rangle & \text{for all } \mu \in \Lambda, \\ \langle (s_3 J + S_2)\eta_2^n - \chi_2, \mu\rangle = \langle (s_3 J - S_1)\eta_1^n + \chi_1, \mu\rangle & \text{for all } \mu \in \Lambda. \end{cases} \tag{3.17}$$

for $n = 1, 2, \ldots$ Here, J denotes the Riesz isomorphism on $L^2(\Gamma)$ defined by

$$J : L^2(\Gamma) \to L^2(\Gamma)^* : \eta \mapsto (\eta, \cdot)_{L^2(\Gamma)}. \tag{3.18}$$

Due to the presence of J we introduce the Steklov–Poincaré operators as unbounded affine operators on $L^2(\Gamma)$, i.e.,

$$\begin{aligned} D(\mathcal{S}_i) &= \{\mu \in \Lambda_i : S_i\mu - \chi_i \in L^2(\Gamma)^*\} \quad \text{and} \\ \mathcal{S}_i\mu &= J^{-1}(S_i\mu - \chi_i) \text{ for } \mu \in D(\mathcal{S}_i). \end{aligned}$$

The Robin–Robin method is then equivalent to finding $(\eta_1^n, \eta_2^n) \in D(\mathcal{S}_1) \times D(\mathcal{S}_2)$ such that

$$\begin{cases} (s_3 I + \mathcal{S}_1)\eta_1^n = (s_3 I - \mathcal{S}_2)\eta_2^{n-1}, \\ (s_3 I + \mathcal{S}_2)\eta_2^n = (s_3 I - \mathcal{S}_1)\eta_1^n \end{cases} \tag{3.19}$$

for $n = 1, 2, \ldots$, where $I : L^2(\Gamma) \to L^2(\Gamma)$ is the identity operator. This iteration is known as the Peaceman–Rachford iteration and was first introduced in [90]. Note

that the connection between the Robin–Robin method and the Peaceman–Rachford iteration has been observed before [1, 31] and has even been used to show that the discrete variant of the Robin–Robin method converges [81, 110]. The iteration is well defined in the sense that each step has a unique solution, see [Paper I, Corollary 8.5].

In order to prove convergence, we require that the solution of the Steklov–Poincaré equation (3.5) satisfies $\eta \in D(\mathcal{S}_i)$. We therefore make the following assumption, the interpretation being that

$$\nabla u_i \cdot \nu_i \in L^2(\Gamma).$$

Assumption 3.5.1. *Let $u \in V$ denote the solution to (4.2). Then the functionals*

$$\mu \mapsto \langle A_i(u|_{\Omega_i}) - f_i, R_i\mu\rangle, \quad i = 1, 2,$$

are elements in $L^2(\Gamma)^*$.

Remark 3.5.1. *The assumption is weaker than having a strong solution $u \in H^2(\Omega)$, which holds on convex Lipschitz domains if $f \in L^2(\Omega)$ [53, Theorem 9.1.22]. For instance, the regularity follows if we have $u \in H^{3/2+\epsilon}(\Omega)$ for some $\epsilon > 0$, since then each partial derivative $\partial_j u$ satisfies $\partial_j u \in H^{1/2+\epsilon}(\Omega)$ and therefore $\nabla u_i \cdot \nu_i \in L^2(\Gamma)$ since also $\nu_i \in L^\infty(\Gamma)$. The regularity $u \in H^{3/2+\epsilon}(\Omega)$ is true on smooth domains [59, Theorem 0.3]. For general Lipschitz domains, $u \in H^{3/2}(\Omega)$ has been shown if $f \in L^2(\Omega)$, see [59, Theorem B], but also shown to not necessarily hold if $f \in V^*$, see [59, Theorem 0.4]. Higher regularity than $H^{3/2}$ is not always guaranteed even if f is smooth; In [25] an example is given with a smooth f and a C^1 domain Ω such that the solution is not in $H^{3/2+\epsilon}(\Omega)$ for any $\epsilon > 0$. Even though the regularity is not generally true on Lipschitz domains, the assumption is often left out in the literature.*

Theorem 3.5.1. *Suppose that Assumptions 2.4.2, 3.2.1 and 3.5.1 hold. Let u_i^n denote the iterates of the Robin–Robin method (3.16) for some $s_3 > 0$ and $\eta^0 \in D(\mathcal{S}_2)$. Moreover, let (u_1, u_2) denote the solution to the transmission problem (3.8). Then*

$$\|u_i^n - u_i\|_{V_i} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

Proof. Let η be the solution to the Steklov–Poincaré equation (3.5) and η^n be the iterates of the Peaceman–Rachford iteration (3.19). The main observation is that we have the identity

$$\begin{aligned}\|\eta_i^n - \eta\|_\Lambda^2 &\leq (\mathcal{S}_i\eta_i^n - \mathcal{S}_i\eta, \eta_i^n - \eta)_{L^2(\Gamma)} \\ &= \frac{1}{4s_3}\big(\|(s_3 I + \mathcal{S}_i)\eta_i^n - (s_3 I + \mathcal{S}_i)\eta\|_{L^2(\Gamma)}^2 \\ &\quad - \|(s_3 I - \mathcal{S}_i)\eta_i^n - (s_3 I - \mathcal{S}_i)\eta\|_{L^2(\Gamma)}^2\big).\end{aligned}$$

We then apply the conditions $(s_3 I + \mathcal{S}_1)\eta_1^n = (s_3 I - \mathcal{S}_2)\eta_2^{n-1}$ and $(s_3 I + \mathcal{S}_2)\eta_2^n = (s_3 I - \mathcal{S}_1)\eta_1^n$ from (3.19). This yields

$$\begin{aligned}
\|\eta_1^n - \eta\|_\Lambda^2 + \|\eta_2^n - \eta\|_\Lambda^2 &\leq (\mathcal{S}_1\eta_1^n - \mathcal{S}_1\eta, \eta_1^n - \eta)_{L^2(\Gamma)} \\
&\quad + (\mathcal{S}_2\eta_2^n - \mathcal{S}_2\eta, \eta_2^n - \eta)_{L^2(\Gamma)} \\
&= \frac{1}{4s_3}\big(\|(s_3 I - \mathcal{S}_2)\eta_2^{n-1} - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2 \\
&\quad - \|(s_3 I - \mathcal{S}_1)\eta_1^n - (s_3 I - \mathcal{S}_1)\eta\|_{L^2(\Gamma)}^2 \\
&\quad + \|(s_3 I - \mathcal{S}_1)\eta_1^n - (s_3 I - \mathcal{S}_1)\eta\|_{L^2(\Gamma)}^2 \\
&\quad - \|(s_3 I - \mathcal{S}_2)\eta_2^n - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2\big) \\
&= \frac{1}{4s_3}\big(\|(s_3 I - \mathcal{S}_2)\eta_2^{n-1} - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2 \\
&\quad - \|(s_3 I - \mathcal{S}_2)\eta_2^n - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2\big).
\end{aligned}$$

Summing these inequalities yields

$$\begin{aligned}
\sum_{n=1}^{N} \|\eta_1^n - \eta\|_\Lambda^2 + \|\eta_2^n - \eta\|_\Lambda^2 &\leq \frac{1}{4s_3}\big(\|(s_3 I - \mathcal{S}_2)\eta_2^0 - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2 \\
&\quad - \|(s_3 I - \mathcal{S}_2)\eta_2^N - (s_3 I - \mathcal{S}_2)\eta\|_{L^2(\Gamma)}^2\big),
\end{aligned}$$

which shows that the sum is bounded. Therefore, the summands satisfy

$$\|\eta_1^n - \eta\|_\Lambda + \|\eta_2^n - \eta\|_\Lambda \to 0$$

as n tends to infinity. The convergence of the Robin–Robin method now follows from

$$u_i^n - u_i = F_i(\eta_i^n - \eta)$$

and the fact that F_i is bounded. □

The Robin–Robin method does not in general exhibit linear convergence. However, in certain cases this can be shown to be true. First note that the rate of convergence can be studied by examining the norms of the operators $(s_3 I - \mathcal{S}_i)(s_3 I + \mathcal{S}_i)^{-1}$, $i = 1, 2$, since, if we define $\mu^n = (s_3 I + \mathcal{S}_2)\eta_2^n$, we get the equivalent iteration

$$\mu^n = (s_3 I - \mathcal{S}_1)(s_3 I + \mathcal{S}_1)^{-1}(s_3 I - \mathcal{S}_2)(s_3 I + \mathcal{S}_2)^{-1}\mu^{n-1}.$$

Hence, if $(s_3 I - \mathcal{S}_i)(s_3 I + \mathcal{S}_i)^{-1}$ for $i = 1, 2$ are both contractions, we get linear convergence of (3.19). We therefore consider

$$\begin{aligned}
&\sup_{\substack{\eta,\mu\in L^2(\Gamma)\\ \eta\neq\mu}} \frac{\|(s_3 I - \mathcal{S}_i)(s_3 I + \mathcal{S}_i)^{-1}\eta - (s_3 I - \mathcal{S}_i)(s_3 I + \mathcal{S}_i)^{-1}\mu\|^2_{L^2(\Gamma)}}{\|\eta - \mu\|^2_{L^2(\Gamma)}} \\
&\quad = \sup_{\substack{\nu,\lambda\in D(\mathcal{S}_i)\\ \nu\neq\lambda}} \frac{\|(s_3 I - \mathcal{S}_i)\nu - (s_3 I - \mathcal{S}_i)\lambda\|^2_{L^2(\Gamma)}}{\|(s_3 I + \mathcal{S}_i)\nu - (s_3 I + \mathcal{S}_i)\lambda\|^2_{L^2(\Gamma)}} \\
&\quad = \sup_{\substack{\nu,\lambda\in D(\mathcal{S}_i)\\ \nu\neq\lambda}} \frac{\|(s_3 I - J^{-1}S_i)(\nu - \lambda)\|^2_{L^2(\Gamma)}}{\|(s_3 I + J^{-1}S_i)(\nu - \lambda)\|^2_{L^2(\Gamma)}} \\
&\quad = \sup_{\substack{\nu,\lambda\in D(\mathcal{S}_i)\\ \nu\neq\lambda}} \frac{s_3^2\|\nu - \lambda\|^2_{L^2(\Gamma)} - 2s_3\langle S_i(\nu - \lambda), \nu - \lambda\rangle + \|J^{-1}S_i(\nu - \lambda)\|^2_{L^2(\Gamma)}}{s_3^2\|\nu - \lambda\|^2_{L^2(\Gamma)} + 2s_3\langle S_i(\nu - \lambda), \nu - \lambda\rangle + \|J^{-1}S_i(\nu - \lambda)\|^2_{L^2(\Gamma)}}.
\end{aligned}$$

Here we have used the bijective identifications $\nu = (s_3 I + \mathcal{S}_i)^{-1}\eta$ and $\lambda = (s_3 I + \mathcal{S}_i)^{-1}\mu$ together with the fact that if $\nu, \lambda \in D(S_i)$ then $S_i(\nu - \lambda) \in L^2(\Gamma)^*$. It follows that, if we can find $\epsilon > 0$ such that

$$\begin{aligned}
&s_3^2\|\nu - \lambda\|^2_{L^2(\Gamma)} - 2s_3\langle S_i(\nu - \lambda), \nu - \lambda\rangle + \|J^{-1}S_i(\nu - \lambda)\|^2_{L^2(\Gamma)} \\
&\quad \leq (1 - \epsilon)(s_3^2\|\nu - \lambda\|^2_{L^2(\Gamma)} + 2s_3\langle S_i(\nu - \lambda), \nu - \lambda\rangle + \|J^{-1}S_i(\nu - \lambda)\|^2_{L^2(\Gamma)}),
\end{aligned}$$

we have a linearly convergent method. Rearranging shows that the inequality is equivalent to

$$\epsilon(s_3^2\|\nu - \lambda\|^2_{L^2(\Gamma)} + \|J^{-1}S_i(\nu - \lambda)\|^2_{L^2(\Gamma)}) \leq 2s_3(2 - \epsilon)\langle S_i(\nu - \lambda), \nu - \lambda\rangle. \quad (3.20)$$

This estimate can be used to show linear convergence of the Robin–Robin method in certain situations. For example, if $\Omega \subset \mathbb{R}$ then Γ is just a point and therefore $L^2(\Gamma)$ is a finite dimensional space. Thus $J^{-1}S_i : L^2(\Gamma) \to L^2(\Gamma)$ is bounded and (3.20) follows from the coercivity of S_i.

Similarly, if we consider the operator $(s_3 I - \mathcal{S}_i)(s_3 I + \mathcal{S}_i)^{-1}$ on the discrete subspace $\Lambda^h \subset \Lambda$ satisfying Assumption 3.3.1, then the same argument shows that the iteration converges linearly. However, the convergence rate will depend on the space Λ^h, unlike for the Dirichlet–Neumann and Neumann–Neumann methods. If Λ^h is a finite element space with mesh width $h > 0$ then the convergence rate will be $1 - O(\sqrt{h})$ for the optimal parameter $s_3 = O(h^{-1})$, see [52, 81, 110].

3.6 Numerical results

We now provide some new numerical results for our three domain decomposition methods applied to the Poisson equation. Although the results here are interesting in their own right, the aim of this section is not to give comprehensive numerical results for domain decomposition of linear elliptic equations, but rather to provide some context for numerical results of nonlinear equations. We only treat decompositions with two subdomains since, typically, for methods on a decomposition with multiple subdomains to be scalable, a global coarse problem is required, see, e.g. [106, Chapter 3] for details. Note that there are some exceptions to this, see, e.g., [16].

For all numerical results in this chapter we discretize using piecewise linear finite elements. The resulting linear systems are solved using UMFPACK, a direct solver for sparse matrices [27]. We do not use any iterative solvers since these would introduce an extra source of error. In practice, the iterative methods here are often used as a starting point to construct preconditioners for Krylov subspace methods, see [91, Chapter 3] for an explanation of this technique and [107, Part IV] for an introduction to Krylov subspace methods. For simplicity we will not study such iterations here.

We consider two natural ways to compute the error that differ slightly. The first is to compute the error by comparing directly to the finite element solution on the same mesh, which means that we are comparing to the solution of (3.11). We call this the discrete error and denote it

$$e_d = \frac{\|u_1^{n,h} - u_1^h\|_{V_1} + \|u_2^{n,h} - u_2^h\|_{V_2}}{\|u_1^h\|_{V_1} + \|u_2^h\|_{V_2}}.$$

In contrast, we can interpolate to a finer grid where we compare with the exact solution if available, otherwise a finite element solution on the finer grid. This means that we are comparing to the solution of (3.8). We call this the exact error and denote it

$$e_e = \frac{\|u_1^{n,h} - u_1\|_{V_1} + \|u_2^{n,h} - u_2\|_{V_2}}{\|u_1\|_{V_1} + \|u_2\|_{V_2}}.$$

In this section we consider the Poisson equation on Decomposition 2.4.1 together with $f(x) = 6x - 2$, Decomposition 2.4.2 together with $f(x, y) = (6x - 2)y(1 - y) + 2x^2(1 - x)$, and Decomposition 2.4.3 with $f(x, y) = 2y(2 - y) + 2x(3 - x)$. The initial guesses are always taken to be zero.

We first plot some iterations of the Robin–Robin method applied to Decomposition 2.4.1. We plot the function values $(u_1^{n,h}, u_2^{n,h})$, the pointwise discrete errors $(u_1^{n,h} - u_1^h, u_2^{n,h} - u_2^h)$ and the pointwise exact errors $(u_1^{n,h} - u_1, u_2^{n,h} - u_2)$. The

results are presented in Figure 3.2. We see that the first iteration is discontinuous, but after five iterations the discontinuity is very small with the discrete error around 10^{-5}. The exact error is close to the discrete error when $n = 1$ since the error of the domain decomposition method dominates, but when $n = 5$ the exact error is dominated by the error of the finite element method.

We now plot the errors against the number of iterations for the three methods on Decomposition 2.4.1. The mesh and method parameters are given in Table 3.1 and the results are presented in the top row of Figure 3.3. From the results we see that all three methods converge within machine accuracy in two iterations when considering the discrete error e_d. The exact error also converges in two iterations, but stabilizes around 10^{-4}, since then the error is dominated by the error of the finite element method to the exact solution.

We perform the same experiment on Decomposition 2.4.2. Again, the parameters are given in Table 3.1. The results are presented in the middle row of Figure 3.3. Note that we use two different parameters for the Robin–Robin method s_3 and s_3' corresponding to the notation RR and RR', respectively. From this we see that the Dirichlet–Neumann method and Neumann–Neumann method still converge in two iterations as in the one-dimensional case. For the Robin–Robin method, we see that the choice of parameter can lead to different convergence behavior when considering the different ways to compute the error. For the exact error, we still get convergence in two iterations using the parameter s_3, but much slower convergence using s_3'. However, in discrete error the other parameter choice s_3' leads to faster overall convergence, even though the choice s_3 yields faster convergence in the beginning of the iteration.

The same experiment is finally performed on Decomposition 2.4.3 with parameters as in Table 3.1. The results are presented in the bottom row of Figure 3.3. We see similar results as for Decomposition 2.4.2, but we no longer have convergence in two iterations of the Dirichlet–Neumann and Neumann–Neumann methods.

3.7 Parameter studies

In Section 3.4 we showed that the Dirichlet–Neumann and Neumann–Neumann methods are convergent and that the discrete variants have convergence factors that are independent of the mesh size h. We also argued in Section 3.5 that this was true for the Robin–Robin method for one dimensional problems and that it was untrue for two dimensional problems. Moreover, we saw that the Robin–Robin method converges for any $s_3 > 0$, while the Dirichlet–Neumann and Neumann–Neumann methods had conditions on the method parameters. The convergence behavior will of

course also depend on these parameters. The aim of this section is therefore to study the convergence dependence on the parameters h, s_0, s_1, s_2, s_3.

We define the experimental error reductions at iteration n as

$$q_d^n = \frac{e_d^{n+1}}{e_d^n}, \qquad q_e^n = \frac{e_e^{n+1}}{e_e^n}$$

and their geometric averages

$$q_d = \left(\prod_{n=1}^{N-1} q_d^n \right)^{\frac{1}{N-1}}, \qquad q_e = \left(\prod_{n=1}^{N-1} q_e^n \right)^{\frac{1}{N-1}},$$

where N is the number of iterations. In practice we only consider the errors that are above some tolerance, i.e., $e_d^{n+1} > \text{tol}_d$ and $e_e^{n+1} > \text{tol}_e$. The tolerance for the discrete error is always $\text{tol}_d = 10^{-8}$ and the tolerance for the exact error is chosen as approximately $\text{tol}_e = 10 e_{\text{fem}}$, where e_{fem} is the error of the finite element problem, i.e., the error at which the methods stagnate, see Figure 3.3.

We first consider the Dirichlet–Neumann method on Decompositions 2.4.1 to 2.4.3 and three mesh parameters h_1, h_2, h_3 for each decomposition, chosen as in Table 3.2. For each mesh we use 101 evenly spaced values of the parameters s_0 and compute both q_e and q_d. We plot q_e and q_d against the method parameters in Figure 3.4. As predicted by Theorem 3.4.3 we find that the convergence rate is independent of the mesh parameter h for all decompositions. However, the convergence rate depend heavily on the mesh itself. For Decompositions 2.4.1 and 2.4.2 the optimal parameter is $s_0 = 0.5$ and this choice yields convergence factors close to machine accuracy, which suggests that the method converges exactly in two iterations. On the other hand, for Decomposition 2.4.3 the optimal parameter choice is approximately $s_0 = 0.36$ and the convergence rates are worse. Note that for q_e it appears that the finer grid have better convergence properties, but this is simply due to the fact for finer grids, the minimum error is lower and therefore the minimal q_e is also lower.

We perform the same experiment for the Neumann–Neumann method with $s_1 = s_2$ and plot the results in Figure 3.5. We find similar results as for the Dirichlet–Neumann method, but with the optimal parameter choice approximately being $s_1 = s_2 = 0.25$ for Decompositions 2.4.1 and 2.4.2 and 0.21 for Decomposition 2.4.3. Note that it is possible that there are parameter choices with $s_1 \neq s_2$ that have better convergence rates.

Finally, the same experiment is performed on the Robin–Robin method and the results are presened in Figure 3.6. As discussed in Section 3.5 we find that the Robin–Robin method has mesh independent convergence rates for the one dimensional decomposition, but for the two-dimensional case the situation is more complex. The

discrete convergence rate q_d clearly increases as h decreases, but only for certain values of the parameter s_3. Moreover, the convergence rates for the exact error does not seem to have any significant dependence on h.

	Decomposition 2.4.1	Decomposition 2.4.2	Decomposition 2.4.3
h	1/8192	1/256	1/128
DN	0.5	0.5	0.36
NN	0.25	0.25	0.18
RR	2	4	2
RR'	-	62	42

Table 3.1: The method parameters used in Figure 3.3.

	Decomposition 2.4.1	Decomposition 2.4.2	Decomposition 2.4.3
h_1	1/8192	1/256	1/128
h_2	1/4096	1/128	1/64
h_3	1/2048	1/64	1/32

Table 3.2: The mesh parameters used in Figures 3.4 to 3.6.

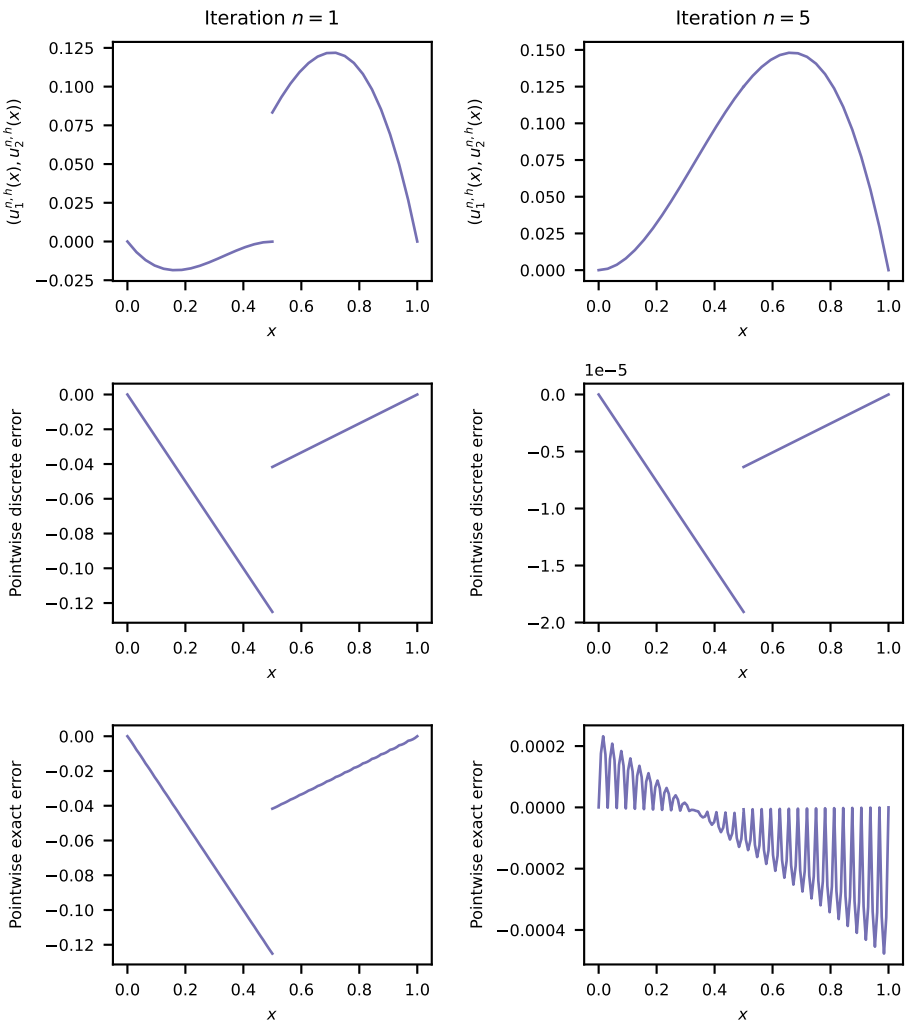

Figure 3.2: The values and errors after $n = 1, 5$ iterations of the Robin–Robin method applied to the Poisson equation for Decomposition 2.4.1. The parameters are $(h, s_3) = (1/32, 4)$.

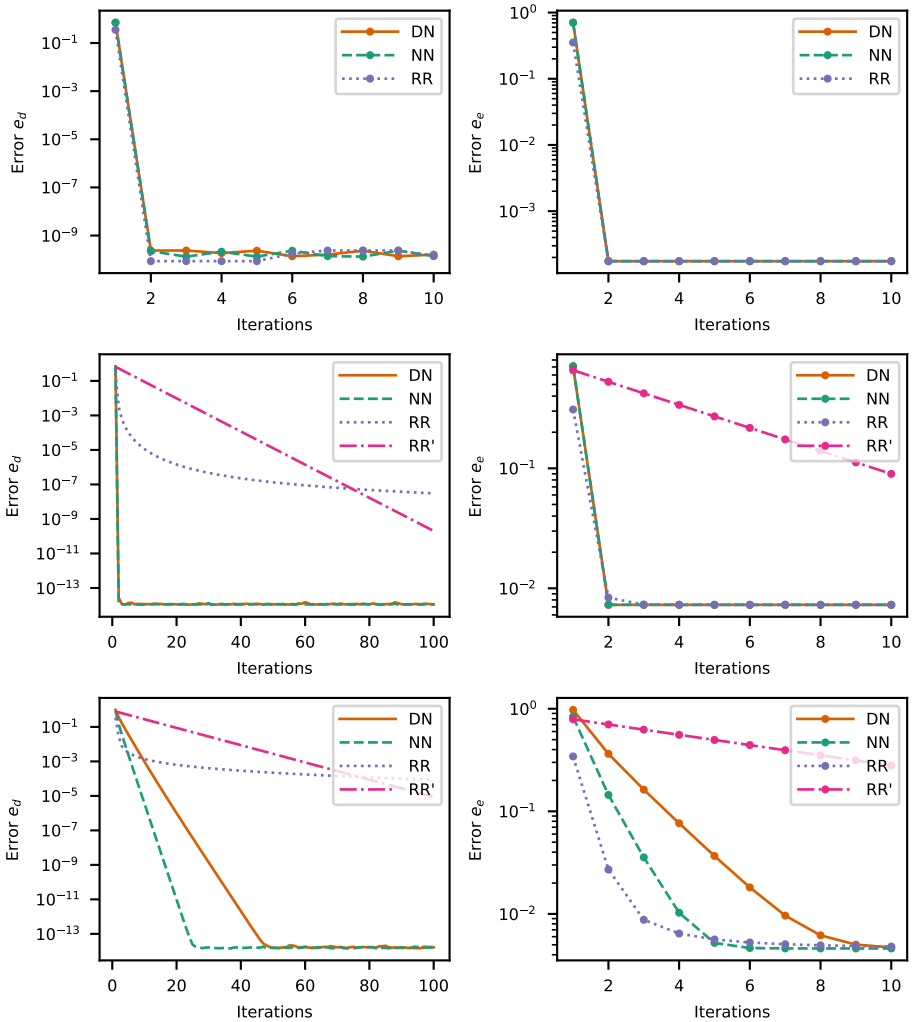

Figure 3.3: The errors of the Dirichlet–Neumann (DN), Neumann–Neumann (NN), and Robin–Robin (RR and RR') methods applied to the Poisson equation for Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The method and mesh parameters are as in Table 3.1.

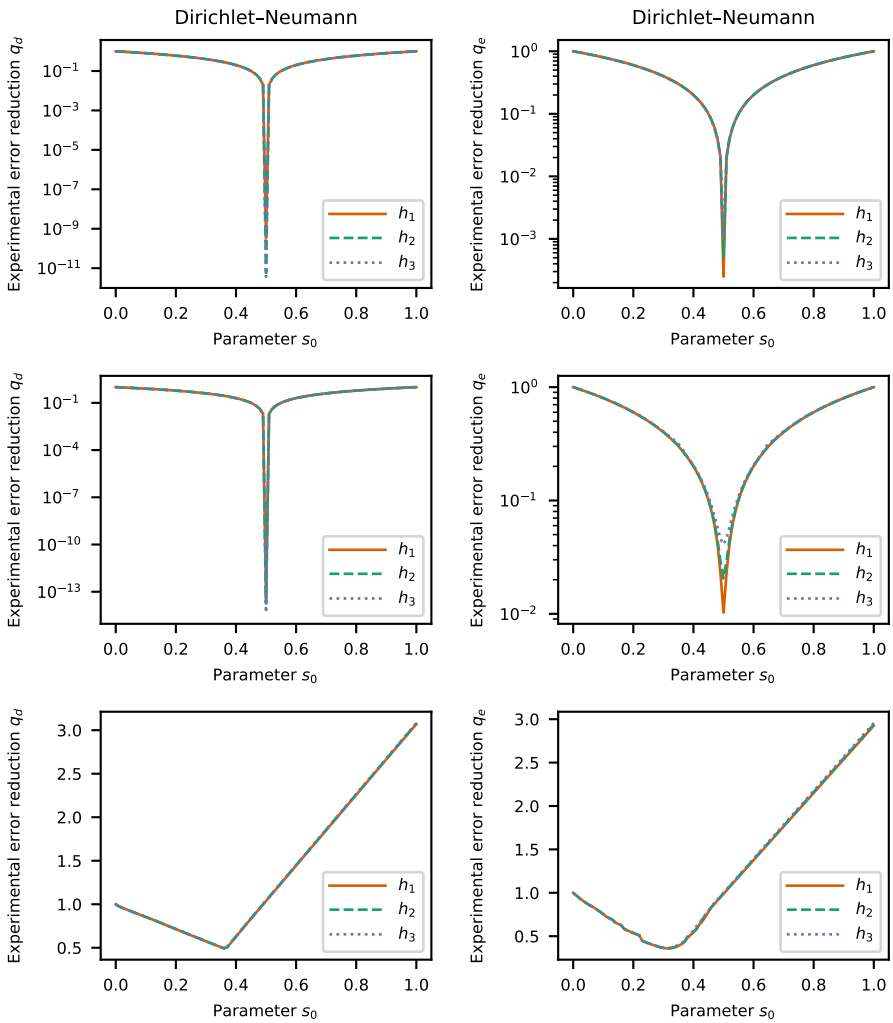

Figure 3.4: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Dirichlet–Neumann method applied to the Poisson equation on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 3.2.

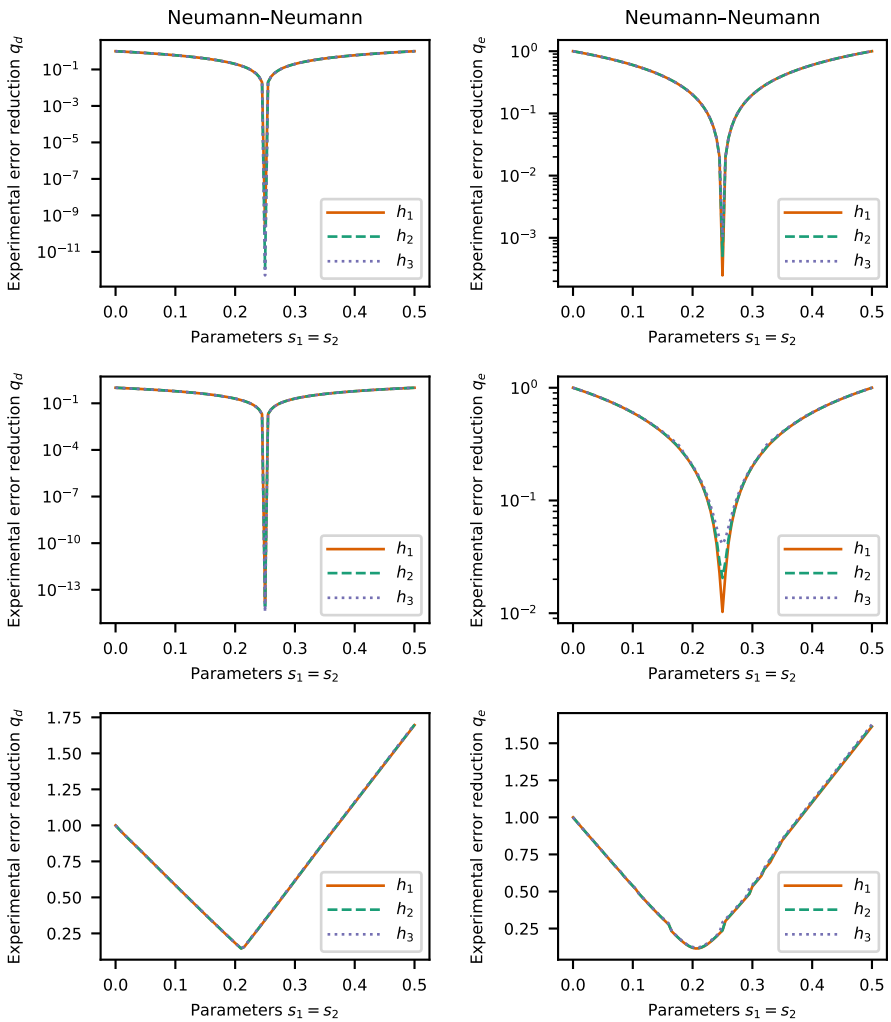

Figure 3.5: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Neumann–Neumann method applied to the Poisson equation on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 3.2.

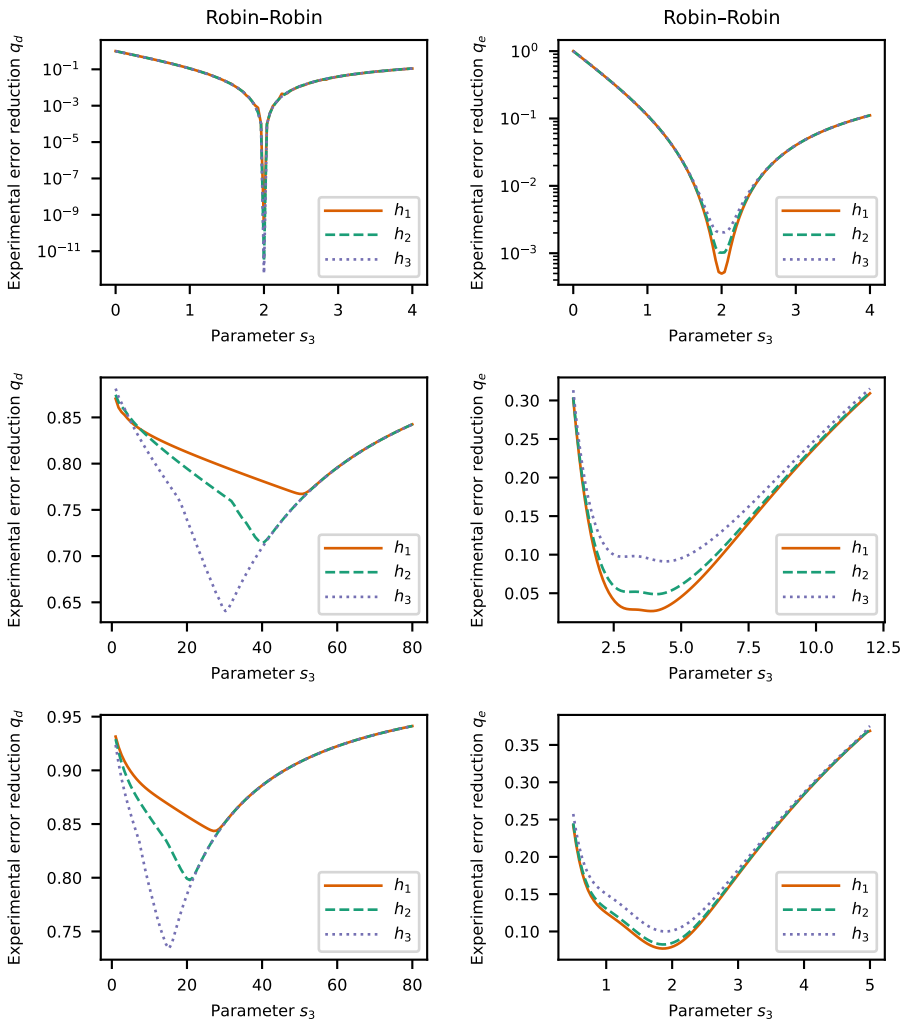

Figure 3.6: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Robin–Robin method applied to the Poisson equation on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 3.2.

Chapter 4

Domain decomposition for nonlinear elliptic equations

4.1 Nonlinear elliptic equations

In this section we give a brief overview of the theory of nonlinear elliptic equations. For a comprehensive treatment we refer to [93, 113]. We consider an abstract equation of the form

$$\begin{cases} \nabla \cdot \alpha(x, u, \nabla u) + \beta(x, u, \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

for some functions $\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. We let $p \geq 2$ and denote the Hölder conjugate of p by p', i.e.

$$p' = \frac{p}{p-1}.$$

Moreover, we let $1 < p^* < \infty$ denote any number such that

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

and let $p^{*\prime}$ denote its Hölder conjugate. Furthermore, we introduce the space

$$V = W_0^{1,p}(\Omega)$$

and define the operator $A : V \to V^*$ as

$$\langle A(u), v \rangle = \int_\Omega \alpha(x, u, \nabla u) \cdot \nabla v + \beta(x, u, \nabla u) v \, \mathrm{d}x.$$

The weak formulation of (4.1) is then to find $u \in V$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in V. \tag{4.2}$$

The following assumption is required to ensure the existence of a unique solution to the equation and define the Steklov–Poincaré operators. However, each method will require additional assumptions than these to prove that they are convergent. For details on this assumption, see [93, Theorem 2.36]. Note that, due to assuming strict monotonicity, we can allow for a weaker growth condition, see also [93, Remark 2.34]. We can also entirely disregard the coercivity assumption for the same reason.

Assumption 4.1.1. *Suppose that α and β are measurable in x, continuous in (y, z), and satisfy the following conditions:*

- *The function α satisfies the growth bound*

  $$|\alpha(x, y, z)| \leq h_1(x) + C|y|^{p^*/p'} + C|z|^{p-1}$$

  with $h_1 \in L^{p'}(\Omega)$.

- *The function β satisfies the growth bound*

  $$|\beta(x, y, z)| \leq h_2(x) + C|y|^{p^*-1} + C|z|^{p/p^{*\prime}}$$

  with $h_2 \in L^{p^{*\prime}}(\Omega)$.

- *We have the monotonicity bound*

  $$\begin{aligned}&(\alpha(x, y, z) - \alpha(x, y', z')) \cdot (z - z') + (\beta(x, y, z) - \beta(x, y', z')) \cdot (y - y')\\ &\quad \geq h_3(x)|z - z'|^p - h_4(x)|y - y'|^p,\end{aligned}$$

  where

  $$\inf_{x\in\Omega} h_3(x) > C_p \sup_{x\in\Omega} h_4(x)$$

  and C_p is the Poincaré constant defined in (2.6). For decompositions not fulfilling Assumption 2.4.2 we instead assume that α and β satisfy monotonicity bound

  $$\begin{aligned}&\big(\alpha(x, y, z) - \alpha(x, y', z')\big) \cdot (z - z') + \big(\beta(x, y, z) - \beta(x, y', z')\big) \cdot (y - y')\\ &\quad \geq c\big(|z - z'|^p + |y - y'|^{p^*}\big).\end{aligned}$$

Example 4.1.1. *Consider the semilinear equation given by*

$$\alpha(x, u, \nabla u) = \nabla u, \qquad \beta(x, u, \nabla u) = 10|u|u.$$

This equation satisfies Assumption 4.1.1 with $h_1 = h_2 = h_4 = 0$.

Example 4.1.2. *Let $p > 2$ and consider the quasilinear equation given by*

$$\alpha(x, u, \nabla u) = |\nabla u|^{p-2}\nabla u, \qquad \beta(x, u, \nabla u) = u.$$

This equation satisfies Assumption 4.1.1 with $h_1 = h_2 = 0$ and the stronger monotonicity requirement with $p^* = 2$.

The proof of the following result can be found in [93, Theorem 2.36]. The idea of the proof is to show that A satisfies the hypothesis of the Browder–Minty theorem, as stated in Theorem 2.1.4.

Lemma 4.1.1. *Suppose that Assumption 4.1.1 holds. Then $A : V \mapsto V^*$ is bijective.*

4.2 Domain decomposition methods for nonlinear elliptic equations

While the theory of domain decomposition methods for linear elliptic equations is well developed, for nonlinear domain decomposition the results are varied. As already stated in the introduction, for overlapping domain decomposition methods there are many convergence results, see e.g. [34, 78, 101, 102]. However, for nonoverlapping domain decomposition methods there are few results other than the ones presented here, with the exceptions [8, 9, 17].

In this section we will first introduce several nonlinear domain decomposition methods and develop the appropriate nonlinear Steklov–Poincaré theory. We will utilize two different nonlinear frameworks in order to prove convergence. The first is based on Zarantello's theorem, stated in Theorem 2.1.3, and allows us to study nonlinear equations in Hilbert spaces when the nonlinearity is Lipschitz. The second is based on the Browder–Minty Theorem and allows us to study a larger class of degenerate equations that are only well posed in Banach spaces. While this framework is more general it does not yield Lipschitz continuous operators.

Since we saw that the methods studied in Chapter 3 have good convergence properties, a natural starting point is to generalize these methods to the nonlinear case. While this is successful, both analytically and numerically, for the Dirichlet–Neumann and Robin–Robin methods, our generalization of the Neumann–Neumann method does not always converge and when it does, it often has poor convergence. To remedy this, we introduce two modifications to the Neumann–Neumann method that turns it into a convergent and efficient method.

Following the linear theory, the first step towards a Steklov–Poincaré theory is to define the transmission problem. In its strong form the transmission problem is to

find (u_1, u_2) such that

$$\begin{cases} \nabla \cdot \alpha(x, u_i, \nabla u_i) + \beta(x, u_i, \nabla u_i) = f_i & \text{in } \Omega_i, \\ u_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \text{ for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma, \\ \alpha(x, u_1, \nabla u_1) \cdot \nu_1 + \alpha(x, u_2, \nabla u_2) \cdot \nu_2 = 0 & \text{on } \Gamma. \end{cases}$$

Remark 4.2.1. *Notice that the Neumann condition in the linear transmission problem has been replaced with a nonlinear generalized Neumann condition. This is the natural condition corresponding to the equation (4.1), i.e., it arises naturally when using integration by parts to derive the weak formulation from the strong formulation. We will, however, refer to this as a Neumann condition. For example, the nonlinear Dirichlet–Neumann method alternates between a Dirichlet condition and a generalized Neumann condition.*

In order to study the weak formulation of the transmission problem we introduce the spaces

$$V_i^0 = W_0^{1,p}(\Omega_i), \quad V_i = \{v \in W^{1,p}(\Omega_i) : (T_{\partial\Omega_i} v)|_{\partial\Omega_i \setminus \Gamma} = 0\},$$
$$\Lambda = W_{00}^{1,1-1/p}(\Gamma)$$

and define the operators $A_i : V_i \to V_i^*$ as

$$\langle A_i u_i, v_i \rangle = \int_{\Omega_i} \alpha(x, u_i, \nabla u_i) \cdot \nabla v_i + \beta(x, u_i, \nabla u_i) v_i \, \mathrm{d}x.$$

As in the linear case, we assume that $f \in V^*$ and that there exist $f_i \in V_i^*$ such that

$$\langle f, v \rangle = \langle f_1, v|_{\Omega_1} \rangle + \langle f_2, v|_{\Omega_2} \rangle \qquad \text{for all } v \in V, \tag{4.3}$$

recall Remark 3.2.1. The weak equation (4.2) is equivalent to the weak transmission problem, which is to find $(u_1, u_2) \in V_1 \times V_2$ such that

$$\begin{cases} \langle A_i u_i, v_i \rangle = \langle f_i, v_i \rangle & \text{for all } v_i \in V_i^0, i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i u_i, R_i \mu \rangle - \langle f_i, R_i \mu \rangle = 0 & \text{for all } \mu \in \Lambda. \end{cases} \tag{4.4}$$

We define $F_i : \Lambda \to V_i : \eta \mapsto u_i$ as the solution operator to the nonhomogeneous problem

$$\begin{cases} \langle A_i u_i, v \rangle = \langle f_i, v \rangle & \text{for all } v \in V_i^0, \\ T_i u_i = \eta. \end{cases}$$

The operator F_i exists due to Assumption 4.1.1 and the Browder–Minty theorem. Note that this is slightly different from the linear case, since we include the source term f_i in the definition of F_i. We can now define the nonlinear Steklov–Poincaré operators $S_i, S : \Lambda \to \Lambda^*$ as

$$\langle S_i\eta, \mu\rangle = \langle A_iF_i\eta - f_i, R_i\mu\rangle \qquad \text{and}$$
$$\langle S\eta, \mu\rangle = \sum_{i=1}^{2}\langle A_iF_i\eta - f_i, R_i\mu\rangle.$$

Note that we also include f_i in the Steklov–Poincaré equation. This makes the notation simpler, but also means that S_i is nonlinear (affine) even if the problem is linear. The nonlinear Steklov–Poincaré equation is then $S\eta = 0$, or equivalently,

$$\langle S\eta, \mu\rangle = 0 \quad \text{for all } \mu \in \Lambda. \tag{4.5}$$

4.3 Dirichlet–Neumann method (Paper III)

The Dirichlet–Neumann method is a standard method for linear elliptic equations and, as will be demonstrated, exhibits fast convergence for both semilinear and quasilinear equations. While the method has been utilized and even shown to converge in specific situations [9, 17], there is no general theory of the nonlinear Dirichlet–Neumann method on Lipschitz decompositions in $\mathbb{R}^d$. In this section, which is based on Paper III, we therefore first generalize the Dirichlet–Neumann method to quasilinear equations and then prove convergence for semilinear equations on Lipschitz domains using the nonlinear Steklov–Poincaré theory developed in Section 4.1. For a fixed method parameter $s_0 > 0$ and an initial guess η^0, the Dirichlet–Neumann method consists of finding (u_1^n, u_2^n, η^n) such that

$$\begin{cases} -\nabla\cdot(\alpha(x,u_1^n,\nabla u_1^n)+\beta(x,u_1^n,\nabla u_i^n) = f_1 & \text{in } \Omega_1,\\ u_1^n = 0 & \text{on } \partial\Omega_1\setminus\Gamma,\\ u_1^n = \eta^{n-1} & \text{on } \Gamma,\\ -\nabla\cdot(\alpha(x,u_2^n,\nabla u_2^n)+\beta(x,u_2^n,\nabla u_2^n) = f_2 & \text{in } \Omega_2,\\ u_2^n = 0 & \text{on } \partial\Omega_2\setminus\Gamma,\\ \alpha(x,u_2^n,\nabla u_2^n)\cdot\nu_2 - \alpha(x,u_1^n,\nabla u_1^n)\cdot\nu_2 = 0 & \text{on } \Gamma,\\ \eta^n - s_0\, u_2^n|_\Gamma - (1-s_0)\eta^{n-1} = 0 & \text{on } \Gamma \end{cases} \tag{4.6}$$

for $n = 1, 2, \ldots$ We make the following assumption on the equation, which roughly translates to α being linear in z and independent of y, β being independent of z and twice differentiable in y, and satisfying Assumption 4.1.1.

Assumption 4.3.1. *The functions α and β are of the form $\alpha(x, y, z) = \alpha_z(x)z$ and $\beta(x, y, z) = \beta(x, y)$, respectively. The conditions on α, β and f are given in [Paper III, Assumption 3.2]. In particular, the function β differentiable in y and its derivative is denoted by β_y.*

The weak formulation of the Dirichlet–Neumann method is given in [Paper III, (5.1)]. We denote the iterates by $(u_1^n, u_2^n, \eta^n) \in V_1 \times V_2 \times \Lambda$ for $n = 1, 2, \ldots$. The interface iteration corresponding to the Dirichlet–Neumann method is to find $\eta^n \in \Lambda$ such that

$$\eta^n = (1 - s)\eta^{n-1} + sS_2^{-1}(0 - S_1\eta^{n-1}) \tag{4.7}$$

for $n = 1, 2, \ldots$

Remark 4.3.1. *The standard formulation of the interface iteration corresponding to the linear Dirichlet–Neumann method, given in (3.15), differs from the nonlinear formulation given here. It is easy to see however, that (3.15) is equivalent to (4.7) for linear problems. For nonlinear equations, the formulations are not equivalent and (4.7) corresponds to the Dirichlet–Neumann method (4.6).*

To our knowledge, there is no previous analysis of the iteration (4.7) when the operator S_2 is nonlinear. We have therefore proven our own convergence theorem under the condition that S_2 is locally Lipschitz continuous, see [Paper III, Theorem 2.1] for a full statement and proof. The idea is to generalize Zarantello's theorem (Theorem 2.1.3) to the case when P is nonlinear, but this comes with two main difficulties. The first is that the linear argument of defining a new norm $\|\eta\|_{S_2}^2 = \langle S_2\eta, \eta\rangle$ does not work when S_2 is nonlinear. This is solved by instead of considering $\|\eta - \mu\|_{S_2}^2$ simply considering

$$\langle S_2\eta - S_2\mu, \eta - \mu\rangle.$$

This is not a norm, but due to S_2 being uniformly monotone and Lipschitz continuous, it is equivalent to a norm, i.e.,

$$c\|\eta - \mu\|_\Lambda^2 \leq \langle S_2\eta - S_2\mu, \eta - \mu\rangle \leq C\|\eta - \mu\|_\Lambda^2.$$

The second issue is the symmetry of S_2. In fact, any operator that is symmetric in the sense that

$$\langle S_2\eta, \mu\rangle = \langle S_2\mu, \eta\rangle$$

is necessarily linear. However, if S_2 is Fréchet differentiable with symmetric Fréchet derivative, one can show that, for $\mu_1, \mu_2, \lambda_1, \lambda_2$, and the lines

$$\mu(t) = t\mu_1 + (1 - t)\mu_2 \quad \text{and} \quad \lambda(t) = t\lambda_1 + (1 - t)\lambda_2,$$

we have the symmetry relation

$$\langle S_2\mu_1 - S_2\mu_2, \lambda_1 - \lambda_2\rangle = \langle S_2\lambda_1 - S_2\lambda_2, \mu_1 - \mu_2\rangle + C(\mu_1, \mu_2, \lambda_1, \lambda_2), \quad \text{with}$$
$$\big|C(\mu_1, \mu_2, \lambda_1, \lambda_2)\big| \leq C \int_0^1 \|S_2'(\mu(t)) - S_2'(\lambda(t))\|_\Lambda \|\mu_1 - \mu_2\|_\Lambda \|\lambda_1 - \lambda_2\|_\Lambda \,\mathrm{d}t.$$

If S_2' is Lipschitz continuous the second term is bounded by

$$\big|C(\mu_1, \mu_2, \lambda_1, \lambda_2)\big| \leq C\big(\|\mu_1 - \mu_2\|_\Lambda + \|\lambda_1 - \lambda_2\|_\Lambda\big)^3$$

and can therefore be discarded if $\mu_1, \mu_2, \lambda_1, \lambda_2$ are close to each other. We will therefore show that the Steklov–Poincaré operators are Fréchet differentiable and study the properties of their Fréchet derivatives. The Fréchet derivatives of A_i are given by the operators $A_i' : V_i \to B(V_i, V_i^*)$ defined by

$$\langle A_i'(w_i)u_i, v_i\rangle = \int_{\Omega_i} \alpha_z(x)\nabla u_i \cdot \nabla v_i + \beta_y(x, w_i)u_i v_i \mathrm{d}x.$$

The Fréchet derivatives of the operators F_i are then given by $F_i' : \Lambda \to B(\Lambda, V_i)$ defined as the solution operators $u_i = F_i'(\nu)\eta$ of the problems

$$\begin{cases} \langle A_i'(F_i\nu)u_i, v\rangle = 0 & \text{for all } v \in V_i^0, \\ T_i u_i = \eta. \end{cases}$$

Finally, the Fréchet derivatives of the Steklov–Poincaré operators are given by the operators $S_i' : \Lambda \to B(\Lambda, \Lambda^*)$ defined by

$$\langle S_i'(\nu)\eta, \mu\rangle = \langle A_i'(F_i\nu)F_i'(\nu)\eta, R_i\mu\rangle.$$

The following theorem is essential for us to apply the abstract convergence theory since it gives the required Lipschitz condition of S_2'.

Theorem 4.3.1. *Let Assumptions 2.4.2 and 4.3.1 hold. Then the Steklov–Poincaré operators are differentiable with derivatives $S_i' : \Lambda \to B(\Lambda, \Lambda^*)$, where the operators S_i' are Lipschitz continuous on any bounded set $D \subset \Lambda$.*

We now state the convergence of the nonlinear Dirichlet–Neumann method. Unlike the linear proof there are some higher order error terms that must be made small, so we require that our initial guess is close to the solution. The proof is given in [Paper I, Corollary 5.2].

Theorem 4.3.2. *Let Assumptions 2.4.2 and 4.3.1 hold, $s_0 > 0$ be small enough, and suppose that η^0 is sufficiently close to η, the solution to (4.5). Then the iterates (u_1^n, u_2^n) of the Dirichlet–Neumann method (4.6) converge linearly to (u_1, u_2), the solution to the transmission problem (4.4), in $V_1 \times V_2$.*

Remark 4.3.2. *For the linear case, we remarked that the symmetry condition is not necessary to prove convergence, see Remark 3.4.1. Since the proof for the nonlinear case uses similar arguments to show convergence, we conjecture that it is sufficient to assume that the derivatives of α and β are close to symmetric.*

4.4 Neumann–Neumann and modified Neumann–Neumann methods (Paper II)

In the linear case, the Neumann–Neumann method's Steklov–Poincaré formulation is a preconditioned fixed point iteration. In the nonlinear case this becomes a nonlinear Steklov–Poincaré equation solved using a fixed point iteration with a nonlinear preconditioner. Since there is, to our knowledge, no convergence analysis for this type of iteration, and in fact our numerical results in Paper II indicate that it does not always converge, we instead consider linear preconditioners. We refer to these new methods as modified Neumann–Neumann methods.

The nonlinear Neumann-Neumann method is the following. Find $(u_1^n, u_2^n, w_1^n, w_2^n)$ such that

$$
\begin{cases}
-\nabla \cdot \alpha(x, u_i^n, \nabla u_i^n) + \beta(x, u_i^n, \nabla u_i^n) = f_i & \text{in } \Omega_i, \\
u_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\
u_i^n = \eta^{n-1} & \text{on } \Gamma, \text{ for } i = 1, 2, \\
-\nabla \cdot \alpha(x, w_i^n, \nabla w_i^n) + \beta(x, w_i^n, \nabla w_i^n) = 0 & \text{in } \Omega_i, \\
w_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\
\alpha(x, w_i^n, \nabla w_i^n) \cdot \nu_1 = & \\
\quad \alpha(x, u_1^n, \nabla u_1^n) \cdot \nu_1 - \alpha(x, u_2^n, \nabla u_2^n) \cdot \nu_1 & \text{on } \Gamma, \text{ for } i = 1, 2, \\
\eta^n - \eta^{n-1} + (s_1\, w_1^n|_\Gamma - s_2\, w_2^n|_\Gamma) = 0 & \text{on } \Gamma
\end{cases}
$$

for $n = 1, 2, \ldots$ Each step of the nonlinear Neumann–Neumann method consists of solving four equations before updating the interface values η^n. The first two are for u_1^n and u_2^n and will remain the same for the modified methods. We refer to the other two equations, the ones for w_1^n and w_2^n, as the auxiliary problems. The idea of our modified Neumann–Neumann methods is to replace the auxiliary problems with linear problems. The first auxiliary problem we suggest is simply solving Laplace equation with generalized Neumann conditions as follows

$$
\begin{cases}
-\Delta w_i^n = 0 & \text{in } \Omega_i, \\
w_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\
\nabla w_i^n \cdot \nu_1 = \alpha(x, u_1^n, \nabla u_1^n) \cdot \nu_1 - \alpha(x, u_2^n, \nabla u_2^n) \cdot \nu_1 & \text{on } \Gamma, \text{ for } i = 1, 2.
\end{cases}
$$

For notational simplicity we give the second method in the special case $\alpha(x, y, x) = \alpha(x, z)$ and $\beta(x, y, z) = \beta(x, y)$. We denote the derivatives of alpha with respect to z and β with respect to y by α_z and β_y, respectively. The second auxiliary problem we suggest is then the linearized equation

$$\begin{cases} -\nabla \cdot \left(\alpha_z(x, \nabla u_i^n)\nabla w_i^n\right) + \beta_y(x, u_i^n)w_i^n = 0 & \text{in } \Omega_i, \\ w_i^n = 0 & \text{on } \partial\Omega_i \setminus \Gamma, \\ \alpha_z(x, \nabla u_i^n)\nabla w_i^n \cdot \nu_1 + \alpha(x, \nabla u_2^n) \cdot \nu_1 = \alpha(x, \nabla u_1^n) \cdot \nu_1 & \text{on } \Gamma, \text{ for } i = 1, 2. \end{cases}$$

In order to prove that the first method converges we must make the following assumptions on the equation. This can be interpreted as α being Lipschitz continuous and β being Lipschitz continuous on any bounded set, as well as satisfying Assumption 4.1.1.

Assumption 4.4.1. *The functions α, β, and $f \in V^*$ satisfy the assumptions given in [Paper II, Assumption 2].*

The weak formulation of the first method (MNN1) is to find $(u_1^n, u_2^n, w_1^n, w_2^n, \eta^n) \in V_1 \times V_2 \times V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \langle A_i u_i^n, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in V_i^0, \\ T_i u_i^n = \eta^{n-1} & \text{for } i = 1, 2, \\ \langle \hat{A}_1 w_1^n, v_1\rangle = 0 & \text{for all } v_1 \in V_1^0, \\ \langle \hat{A}_1 w_1^n - A_1 u_1^n + f_1, R_1\mu\rangle = \langle A_2 u_2^n - f_2, R_2\mu\rangle & \text{for all } \mu \in \Lambda \\ \langle \hat{A}_2 w_2^n, v_2\rangle = 0 & \text{for all } v_2 \in V_2^0, \\ \langle \hat{A}_2 w_2^n + A_2 u_2^n - f_2, R_2\mu\rangle = \langle -A_1 u_1^n + f_1, R_1\mu\rangle & \text{for all } \mu \in \Lambda, \\ \eta^n - \eta^{n-1} = s_2 T_2 w_2^n - s_1 T_1 w_1^n \end{cases} \tag{4.8}$$

for $n = 1, 2, \ldots$ Here the operator $\hat{A}_i : V_i \to V_i^*$ is defined as

$$\langle \hat{A}_i u, v\rangle = \int_{\Omega_i} \nabla u \cdot \nabla v \, \mathrm{d}x.$$

We now formulate the interface iteration of this method. For this we define $\hat{F}_i$ as the solution operator corresponding to the problem

$$\begin{cases} \langle \hat{A}_i u, v\rangle = \displaystyle\int_{\Omega_i} \nabla u \cdot \nabla v \, \mathrm{d}x = 0 \quad \text{for all } v \in V_i^0, \\ T_i u = \eta. \end{cases}$$

The Steklov–Poincaré operators $P_i : \Lambda \to \Lambda^*$ corresponding to the auxiliary problem are then

$$\langle P_i \mu, \lambda \rangle = \langle \hat{A}_i \hat{F}_i \mu, \lambda \rangle.$$

The interface iteration of (4.8) then becomes

$$\eta^{n+1} = \eta^n + sP^{-1}(\chi - G\eta^n)$$

with $P = (s_1 P_1^{-1} + s_2 P_2^{-1})^{-1}$. The idea is now simply to mimic Zarantello's theorem, but with the modification as in the linear case, i.e., by defining the norm $\|\mu\|_P = \langle P\mu, \mu \rangle$. One remaining difficulty is that we do not have global Lipschitz continuity of G, but it is possible to generalize Zarentello's theorem to locally Lipschitz continuous operators, see [Paper II, Theorem 2] for details. We thus get the following result.

Theorem 4.4.1. *Let Assumptions 2.4.2 and 4.4.1 hold and $s_1, s_2 > 0$ be small enough. Then the iterates (u_1^n, u_2^n) of the modified Neumann–Neumann method (4.8) converge linearly to (u_1, u_2), the solution to the transmission problem (4.4) in $V_1 \times V_2$.*

Remark 4.4.1. *Unlike the nonlinear Dirichlet–Neumann method we do not assume any symmetry of the functions α and β appearing in the equation. The symmetry condition is instead satisfied by the linear preconditioner P.*

The weak formulation of the second modified Neumann–Neumann method (MNN2) is given by finding $(u_1^n, u_2^n, w_1^n, w_2^n, \eta^n) \in V_1 \times V_2 \times V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \langle A_i u_i^n, v_i \rangle = \langle f_i, v_i \rangle & \text{for all } v_i \in V_i^0, \\ T_i u_i^n = \eta^{n-1} & \text{for } i = 1, 2, \\ \langle A_1'(u_1^n) w_1^n, v_1 \rangle = 0 & \text{for all } v_1 \in V_1^0, \\ \langle A_1'(u_1^n) w_1^n - A_1 u_1^n + f_1, R_1 \mu \rangle = \langle A_2 u_2^n - f_2, R_2 \mu \rangle & \text{for all } \mu \in \Lambda \\ \langle A_2'(u_2^n) w_2^n, v_2 \rangle = 0 & \text{for all } v_2 \in V_2^0, \\ \langle A_2'(u_2^n) w_2^n + A_2 u_2^n - f_2, R_2 \mu \rangle = \langle f_1 - A_1 u_1^n, R_1 \mu \rangle & \text{for all } \mu \in \Lambda, \\ \eta^n - \eta^{n-1} = s_2 T_2 w_2^n - s_1 T_1 w_1^n \end{cases} \tag{4.9}$$

for $n = 1, 2, \ldots$ Here, A_i' is defined as in Section 4.3. This method requires an extra set of assumptions to show convergence, which can be interpreted as, in addition to satisfying Assumption 4.4.1, α being linear in z and independent of y and β being independent of z and differentiable in y with locally Lipschitz continuous derivative β_y.

Assumption 4.4.2. *The functions $\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, $\beta : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $f \in V^*$ satisfy the assumptions as in [Paper II, Assumptions 2 and 3].*

Theorem 4.4.2. *Let Assumptions 2.4.2 and 4.4.2 hold, $s_1, s_2 > 0$ be small enough, and suppose that η^0 is sufficiently close to η, the solution to the Steklov–Poincaré equation (4.5). Then the iterates (u_1^n, u_2^n) of the modified Neumann–Neumann method (4.9) converges linearly to (u_1, u_2), the solution to the transmission problem (4.4), in $V_1 \times V_2$.*

4.5 Robin–Robin method (Paper I)

The Robin–Robin method applied to nonlinear equations has been studied in [8, 9], but as for the other nonoverlapping nonlinear domain decomposition methods studied in this chapter, there is no general convergence theory on Lipschitz domains in $\mathbb{R}^d$. The aim of this section is therefore to prove convergence of the Robin–Robin method applied to a class of, possibly degenerate, quasilinear equations.

The analysis of the Robin–Robin method is different from the Dirichlet–Neumann and Neumann–Neumann iterations, even more so than in the linear case. While the analysis of those schemes relied on Zarantello's theorem, the analysis of the Robin–Robin method instead relies on the Browder–Minty theorem. This allows us to study a much larger class of quasilinear equations, such as the p-Laplace equation. On the other hand, as in the linear case, an extra regularity assumptions on the solution is required. Furthermore, as in the linear case, the convergence result is weaker since we do not necessarily have linear convergence.

For notational simplicity we assume that $\alpha(x, y, z) = \alpha(z)$ and $\beta(x, y, z) = \beta(y)$. The nonlinear Robin–Robin method is then to find (u_1^n, u_2^n) such that

$$\begin{cases} -\nabla \cdot \alpha(\nabla u_1^n) + \beta(u_1^n) = f_1 & \text{in } \Omega_1, \\ u_1^n = 0 & \text{on } \partial\Omega_1 \setminus \Gamma, \\ \alpha(\nabla u_1^n) \cdot \nu_1 + s_3 u_1^n = \alpha(\nabla u_2^{n-1}) \cdot \nu_1 + s_3 u_2^{n-1} & \text{on } \Gamma, \\ -\nabla \cdot \alpha(\nabla u_2^n) + \beta(u_2^n) = f_2 & \text{in } \Omega_2, \\ u_2^n = 0 & \text{on } \partial\Omega_2 \setminus \Gamma, \\ \alpha(\nabla u_2^n) \cdot \nu_2 + s_3 u_2^n = \alpha(\nabla u_1^n) \cdot \nu_2 + s_3 u_1^n & \text{on } \Gamma. \end{cases} \tag{4.10}$$

for $n = 1, 2, \ldots$ The weak form of the Robin–Robin method is given in [Paper I, (5.2)] and yields the iterates $(u_1^n, u_2^n) \in V_1 \times V_2$ for $n = 1, 2, \ldots$. In order to prove convergence we make the following assumption.

Assumption 4.5.1. *The functions $\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfy the assumptions as in [Paper I, Assumption 2.1]. The source term f can be decomposed as in (4.3).*

The properties of α and β yield the following properties of the nonlinear Steklov–Poincaré operators.

Theorem 4.5.1. *Suppose that Assumptions 2.4.1 and 4.5.1 hold. Then $S_i, S : \Lambda \to \Lambda^*$ are well defined and satisfy the monotonicity condition*

$$\langle S_i\eta - S_i\mu, \eta - \mu\rangle \geq c_i\big(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)}\big),$$

$$\langle S\eta - S\mu, \eta - \mu\rangle \geq c\sum_{i=1}^{2}\big(\|\nabla(F_i\eta - F_i\mu)\|^p_{L^p(\Omega_i)^d} + \|F_i\eta - F_i\mu\|^r_{L^r(\Omega_i)}\big)$$

for all $\eta, \mu \in \Lambda$, $i = 1, 2$, and some $r > 1$. Moreover, S_i and S are coercive and demicontinuous.

The interface iteration corresponding to the nonlinear Robin–Robin method is to find $\eta^n \in \Lambda$ such that

$$\begin{cases} \langle (s_3 J + S_1)\eta_1^{n+1}, \mu\rangle = \langle (s_3 J - S_2)\eta_2^n, \mu\rangle & \text{for all } \mu \in \Lambda, \\ \langle (s_3 J + S_2)\eta_2^{n+1}, \mu\rangle = \langle (s_3 J - S_1)\eta_1^{n+1}, \mu\rangle & \text{for all } \mu \in \Lambda. \end{cases}$$

for $n = 1, 2, \ldots$. Here, recall that J denotes the Riesz isomorphism (3.18). As in the linear case, we require more regularity in order to prove convergence, compare with Assumption 3.5.1 and Remark 3.5.1.

Assumption 4.5.2. *Let $u \in V$ denote the solution to (4.2). Then the functionals*

$$\mu \mapsto \langle A_i(u|_{\Omega_i}), R_i\mu\rangle, \quad i = 1, 2,$$

are elements in $L^2(\Gamma)^*$.

Remark 4.5.1. *In Paper I we assume that $\alpha(\nabla u) \in C(\overline{\Omega})^d$. As discussed in [Paper I, Remark 8.3] this is a stronger requirement than Assumption 4.5.2. If the boundary is $C^{1,\beta}$ then this stronger requirement is still fulfilled by Example 4.1.2 according to [74].*

The convergence now follows from the abstract result on the convergence of the Peaceman–Rachford iteration [80] in the same way as in Section 3.5.

Theorem 4.5.2. *Suppose that Assumptions 2.4.1, 4.5.1 and 4.5.2 hold. Let u_i^n be the iterates of the nonlinear Robin–Robin method (4.10) for some $s_3 > 0$ and let $(u_1, u_2) \in V_1 \times V_2$ denote the solution to the weak transmission problem (4.4). Then*

$$\|u_i^n - u_i\|_{V_i} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

4.6 Discrete nonlinear domain decomposition methods

In this section we introduce the discrete nonlinear Steklov–Poincaré operators analogous to the discrete linear Steklov–Poincaré operators defined in Section 3.3. We also discuss the consequences of these definitions for nonlinear domain decomposition methods on the discrete level. All the spatial discretizations in this chapter are finite element methods, for convergence results on these applied to nonlinear equations we refer to [6, 7, 21, 30, 35].

We consider a discretization satisfying Assumption 3.3.1 with the spaces replaced by the $W^{1,p}$ variants. We define the discrete nonlinear operators A^h and A_i^h of A and A_i in the same way as for the discrete linear operators. The discrete nonlinear equation is then to find $u^h \in V^h$ such that

$$\langle A^h u^h, v\rangle = \langle f^h, v\rangle \quad \text{for all } v \in V^h.$$

This equation is equivalent to the discrete nonlinear transmission problem, given by finding $(u_1^h, u_2^h) \in V_1^h \times V_2^h$ such that

$$\begin{cases} \langle A_i^h u_i^h, v_i\rangle = \langle f_i^h, v_i\rangle & \text{for all } v_i \in V_i^{h,0},\ i = 1, 2, \\ \quad T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i^h u_i^h, R_i^h \mu\rangle - \langle f_i^h, R_i^h \mu\rangle = 0 & \text{for all } \mu \in \Lambda^h. \end{cases} \tag{4.11}$$

Analogously to the operator F_i in the continuous case we introduce the discrete solution operator $F_i^h : \Lambda^h \to V_i^h$ as the solution u_i^h to the equation

$$\begin{cases} \langle A_i^h u_i^h, v\rangle = \langle f_i^h, v\rangle & \text{for all } v \in V_i^0, \\ \quad T_i u_i = \eta. & \end{cases}$$

We can then define discrete Steklov–Poincaré operators $S_i^h : \Lambda^h \to (\Lambda^h)^*$ as

$$\langle S_i^h \eta, \mu\rangle = \langle A_i^h F_i^h \eta - f_i^h, R_i \mu\rangle.$$

and define $S^h = S_1^h + S_2^h$. The discrete nonlinear Steklov–Poincaré equation is to find $\eta^h \in \Lambda^h$ such that

$$\langle S^h \eta, \mu\rangle = 0 \quad \text{for all } \mu \in \Lambda^h.$$

The discrete nonlinear operators have the same properties as the continuous variants, with constants that are independent of $h > 0$, if the corresponding assumption holds. This gives the same convergence results for the discrete variants of the nonlinear domain decomposition methods, the difference being that the discrete variants converge to the solution to (4.11). For the Dirichlet–Neumann and modified Neumann–Neumann methods we get convergence rates that are independent of the mesh parameter $h > 0$.

4.7 Numerical results for semilinear elliptic equations

In this section we consider the semilinear equation given in Example 4.1.1. We define the discrete and exact errors in the same way as in Section 3.6.

When solving nonlinear equations we always use Newton's method until the residual is no longer decreasing. The resulting linear equations are solved as in Section 3.6. As we did not discuss Krylov acceleration in the linear case, we do not discuss accelerations using Newton's method such as those in [32], but these observations are important for an efficient implementation.

We first consider the Dirichlet–Neumann, Neumann-Neumann, and Robin–Robin methods together with three decompositions Decompositions 2.4.1 to 2.4.3. On Decomposition 2.4.1 we use $f(x, y) = 6x-2$, on Decomposition 2.4.2 we use $f(x, y) = (6x - 2)y(1 - y) + 2x^2(1 - x)$, and on Decomposition 2.4.3 we use $f(x, y) = 2y(2-y)+2x(3-x)$. For each decomposition, the parameters $(h, s_0, s_1, s_2, s_3, s_3')$ are as in Table 4.1. The initial guesses are always taken to be zero. We plot the discrete and exact errors against the number of iterations in Figure 4.1. We find similar results as in Section 3.6. In particular, all methods converge in discrete and exact error. This is expected for the Robin–Robin method, but note that we have no proof of convergence for the Neumann–Neumann method and the equation does not satisfy Assumption 4.3.1 so there is no guarantee that the Dirichlet–Neumann method will converge. The main difference from the linear case is that we do not have exact convergence in two iterations on Decomposition 2.4.1 or for the Dirichlet–Neumann and Neumann–Neumann methods on Decomposition 2.4.2.

We now consider the modified Neumann–Neumann methods. We perform the same experiment and plot the results in Figure 4.2. The meshes and equations are the same and the method parameters are as in Table 4.1. Again, we find that both methods converge in discrete and exact norm, which is expected from Section 4.4. Moreover, we find that the modified methods perform similarly to the standard methods in most cases. However, for Decomposition 2.4.3 MNN2 converges significantly faster in discrete error than all other methods.

	Decomposition 2.4.1	Decomposition 2.4.2	Decomposition 2.4.3
h	1/8192	1/256	1/128
DN	0.54	0.5	0.37
NN	0.22	0.25	0.11
RR	2.3	3.8	4.5
RR'	-	58	58
MNN1	0.22	0.25	0.1
MNN2	0.25	0.25	0.21

Table 4.1: The method parameters used in Figures 4.1 and 4.2.

	Decomposition 2.4.1	Decomposition 2.4.2	Decomposition 2.4.3
h_1	1/8192	1/256	1/128
h_2	1/4096	1/128	1/64
h_3	1/2048	1/64	1/32

Table 4.2: The mesh parameters used in Figures 4.3 to 4.7 and 4.10 to 4.13.

For each method we perform the parameter study as in Section 3.7. That is, we compute the experimental error reduction for 101 evenly spaced values in an interval and plot the results in Figures 4.3 to 4.7. The mesh parameters are as in Table 4.2. We find that all the methods have convergence factors that are independent of the mesh size except for the Robin–Robin method on the two-dimensional decompositions. For the modified Neumann-Neumann method this is proven in Section 4.4 and for the Robin–Robin method this is the same behavior as in the linear case.

4.8 Numerical results for quasilinear elliptic equations

We perform the same experiments on the quasilinear equation in Example 4.1.2. We first plot the discrete and exact errors of our methods in Figures 4.8 and 4.9 for the parameters choices as in Table 4.3. We find that the Dirichlet–Neumann, Robin–Robin, and modified Neumann–Neumann methods converge, which is expected for the Robin–Robin method from Section 4.5, but not proven for the Dirichlet–Neumann and modified Neumann–Neumann methods applied to quasilinear equations. However, we also find that the Neumann–Neumann method does not converge.

We then compute the experimental error reduction for different parameter choices and Decompositions 2.4.1 to 2.4.3. The mesh parameters are described in Table 4.2. We do not consider the Neumann–Neumann method since we could not find a parameter choice for which it converges. The results are found in Figure 4.10 for the

Dirichlet–Neumann method, Figure 4.11 for the Robin–Robin method, Figure 4.12 for the MNN1 method, and Figure 4.13 for the MNN2 method. Again, we find that our methods have mesh independent convergence factors, except the Robin–Robin method, which behaves as in the linear case. Note that the MNN1 method does not converge for parameter choices greater than the ones depicted in Figure 4.12.

	Decomposition 2.4.1	Decomposition 2.4.2	Decomposition 2.4.3
h	1/8192	1/256	1/128
DN	0.49	0.44	0.37
NN	0.1	0.1	0.11
RR	1.5	1.5	3
RR'	-	23	46
MNN1	0.33	0.65	0.05
MNN2	0.25	0.25	0.21

Table 4.3: The method parameters used in Figures 4.8 and 4.9.

4.9 Comparison with linearized methods

A typical technique for applying domain decomposition methods to nonlinear equations is to first apply Newton's method to the full equation and then a domain decomposition method to the resulting linear equation. We will refer to these methods as linearized domain decomposition methods to separate them from our methods, in which we apply domain decomposition directly to the nonlinear equation and then use Newton's method on the subproblems. In this section we compare these two approaches for a specific example, which demonstrate the advantage of our methods from a computational perspective. Thus, the aim is not to provide a comprehensive comparison of these two approaches, but to show that there are examples where the nonlinear approach has advantages.

We consider the quasilinear equation in Example 4.1.2 on Decomposition 2.4.3. For simplicity we consider only the methods that exhibit linear convergence, i.e., the Dirichlet–Neumann method and modified Neumann–Neumann methods. We compare with a linearized Dirichlet–Neumann method. That is, the method obtained by performing Newton's method on (3.7) and then using the Dirichlet–Neumann method to solve the resulting linear equation. For the linearized Dirichlet–Neumann method we use the parameter $s_0 = 0.3$ and for the other methods the parameters are as in Table 4.3. We plot the discrete and exact errors against the number of linear solves required to achieve the error in Figure 4.14, from which we find that the nonlinear Dirichlet–Neumann method is significantly more efficient than the linearized

Dirichlet–Neumann method. Moreover, MNN2 is comparable to the Dirichlet–Neumann method, but converges slower than the Dirichlet–Neumann method in the beginning of the iteration.

Remark 4.9.1. *The amount of linear solves will depend on the number of iterations in the inner loop, i.e., the number of Newton iterations for our nonlinear methods or the number of decomposition method iterations for the linearized Dirichlet–Neumann method. These have been chosen to be optimal for each method. The number of inner iterations is* 1 *for the Dirichlet–Neumann method,* 4 *for the linearized Dirichlet–Neumann method,* 2 *for MNN*1*, and* 1 *for MNN*2*. For a more efficient method, the number of inner iterations should change from iteration to iteration depending on the residual, but this extra complication is not considered here.*

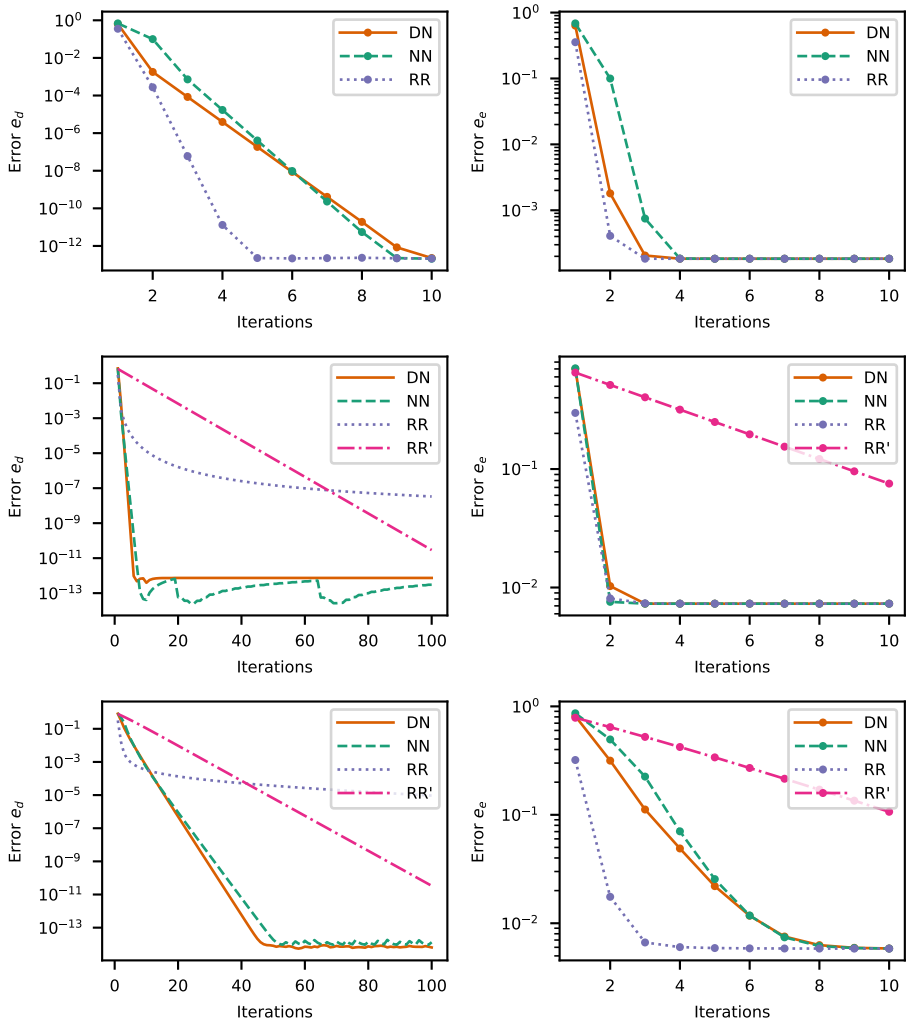

Figure 4.1: The errors of the Dirichlet–Neumann (DN), Neumann–Neumann (NN), and Robin–Robin (RR and RR') methods applied to Example 4.1.1 for Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The parameters are as in Table 4.1.

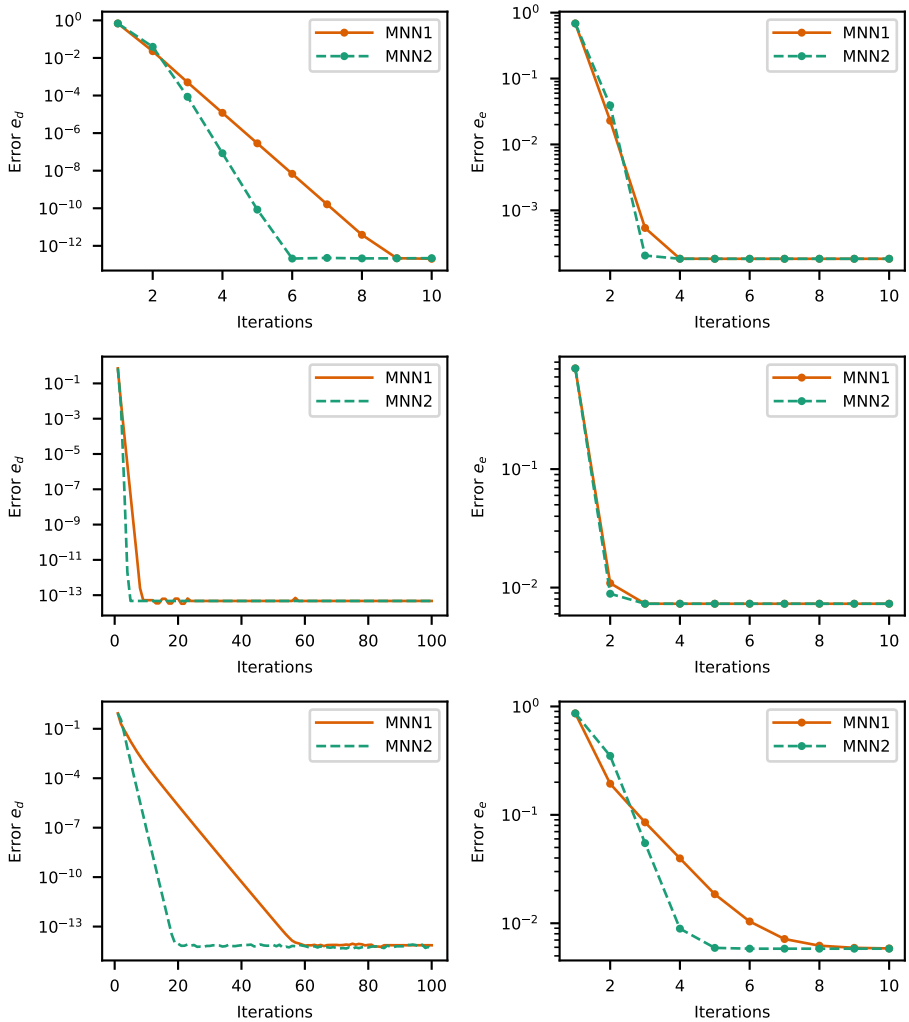

Figure 4.2: The errors of the MNN1 and MNN2 methods applied to Example 4.1.1 for Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The parameters are as in Table 4.1.

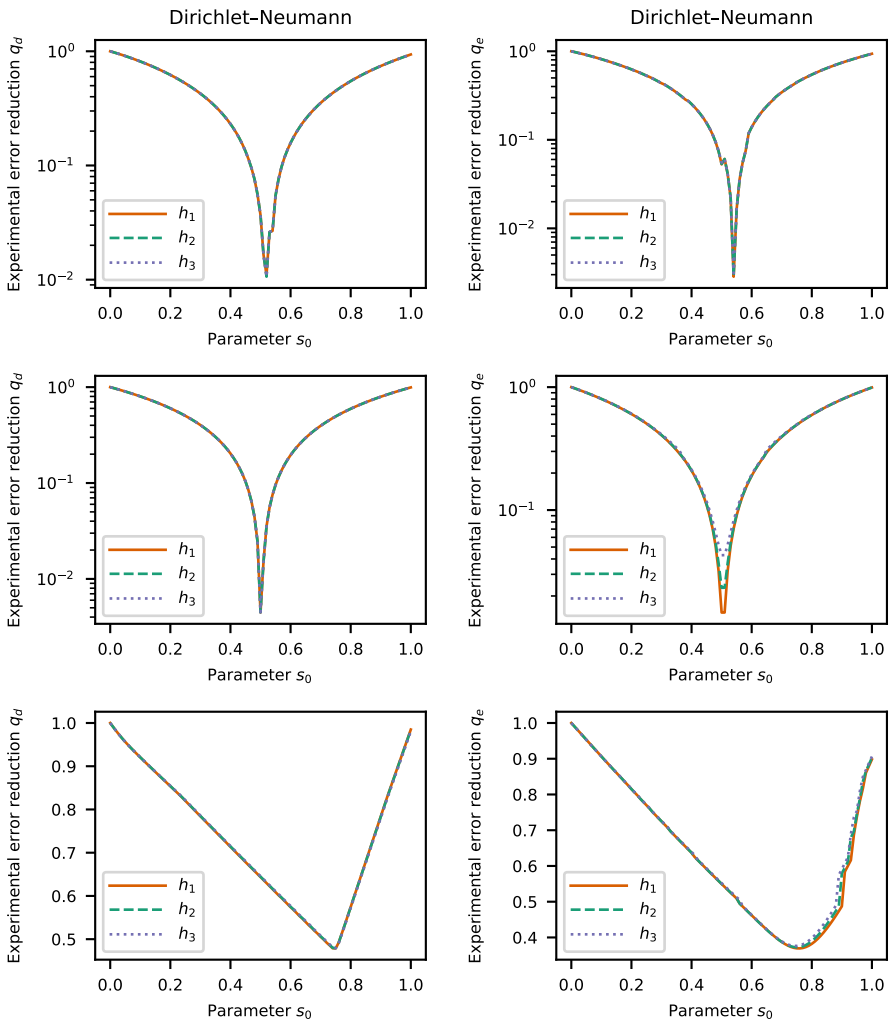

Figure 4.3: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Dirichlet–Neumann method applied to Example 4.1.1 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

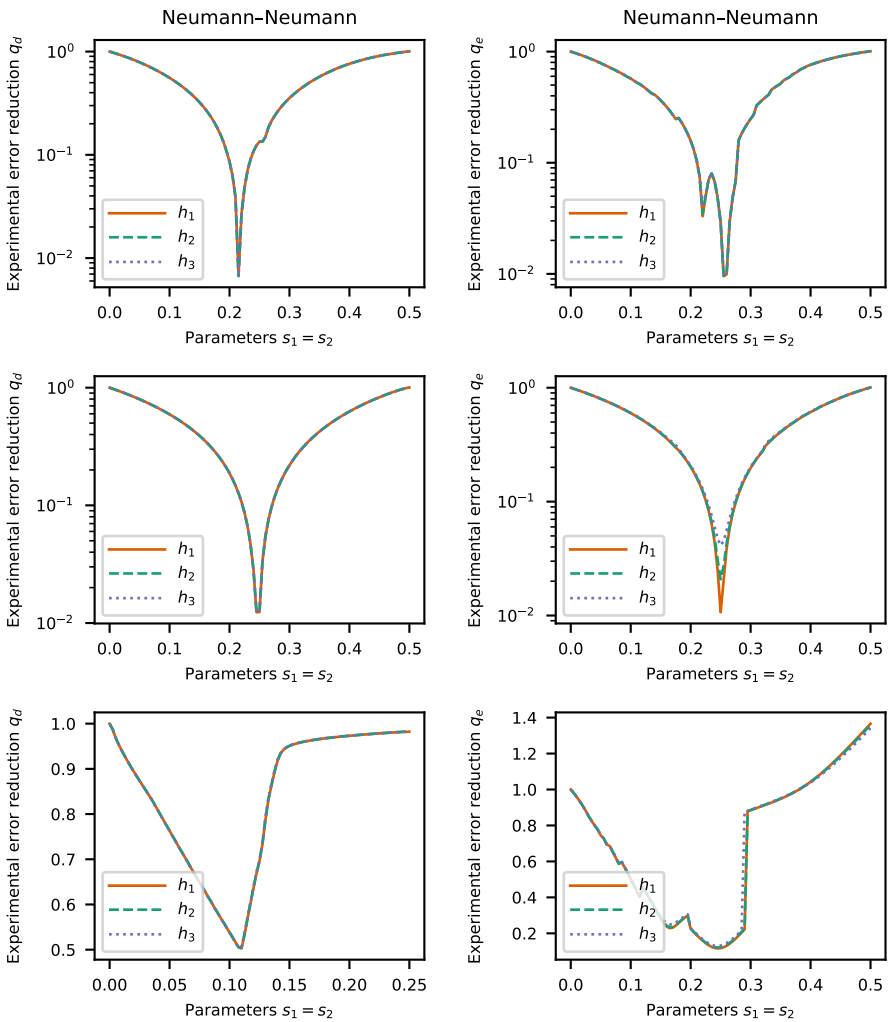

Figure 4.4: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Neumann–Neumann method applied to Example 4.1.1 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

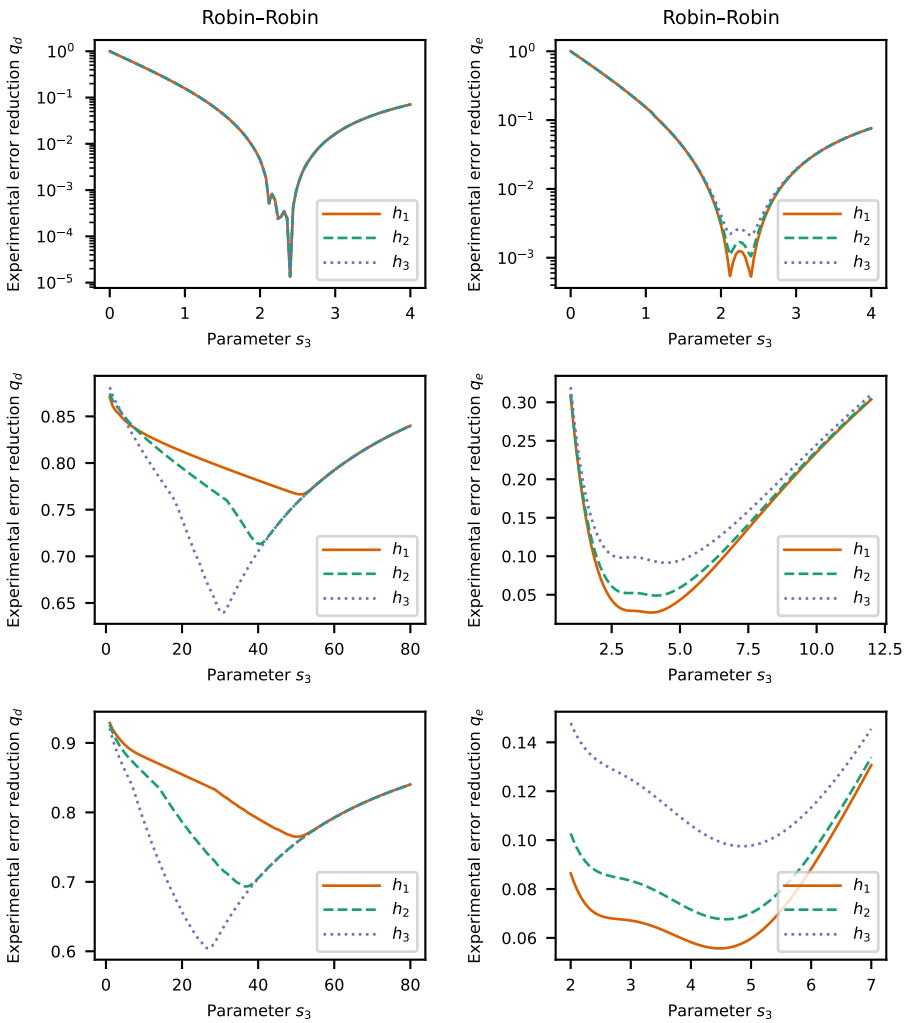

Figure 4.5: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Robin–Robin method applied to Example 4.1.1 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

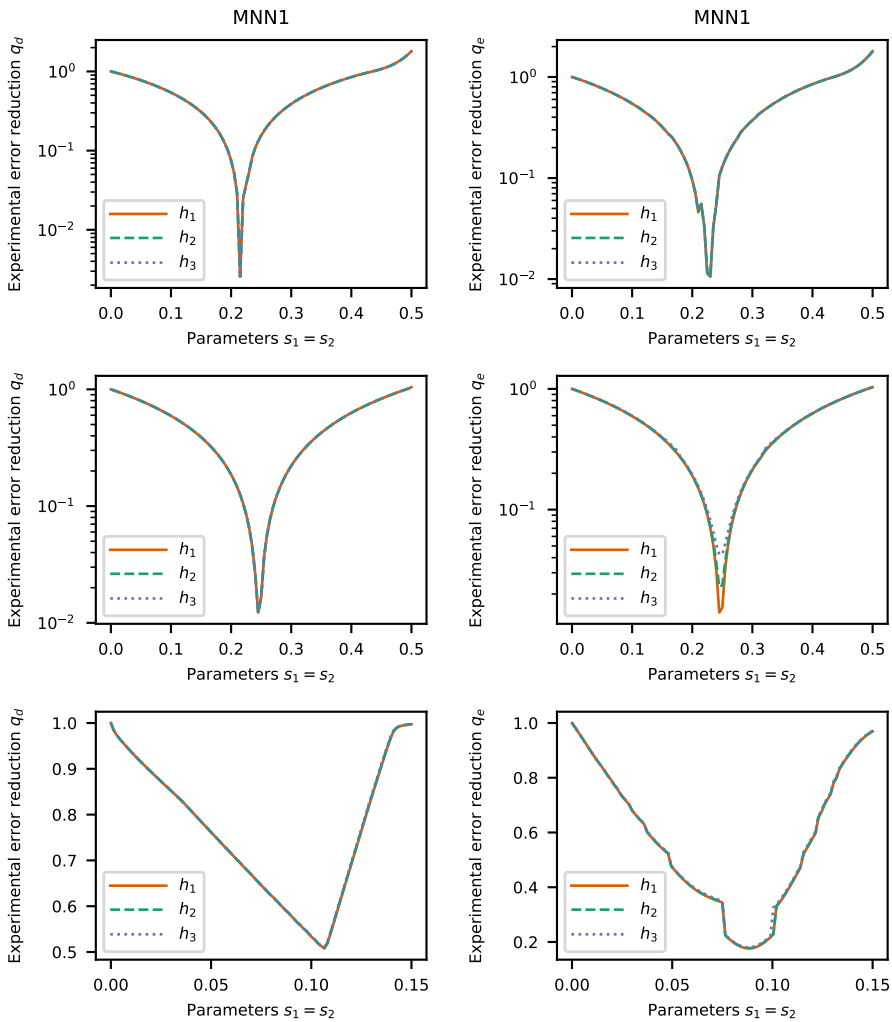

Figure 4.6: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the MNN1 method applied to Example 4.1.1 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

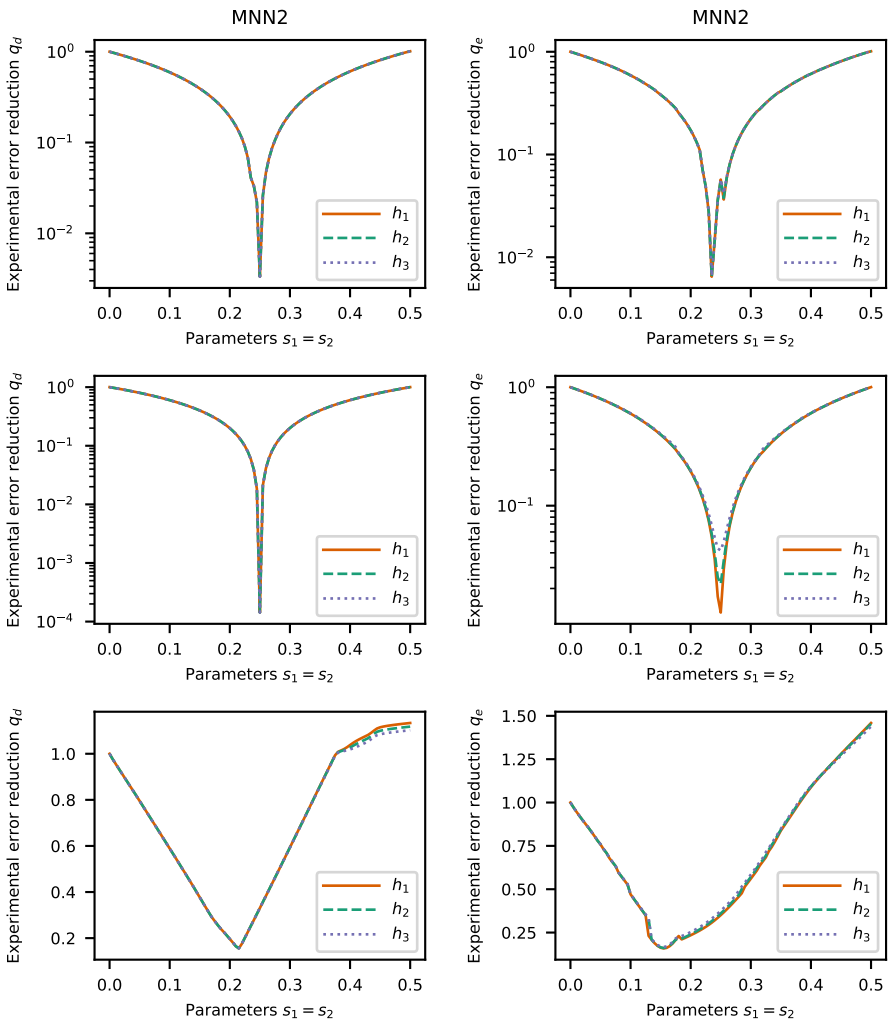

Figure 4.7: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the MNN2 method applied to Example 4.1.1 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

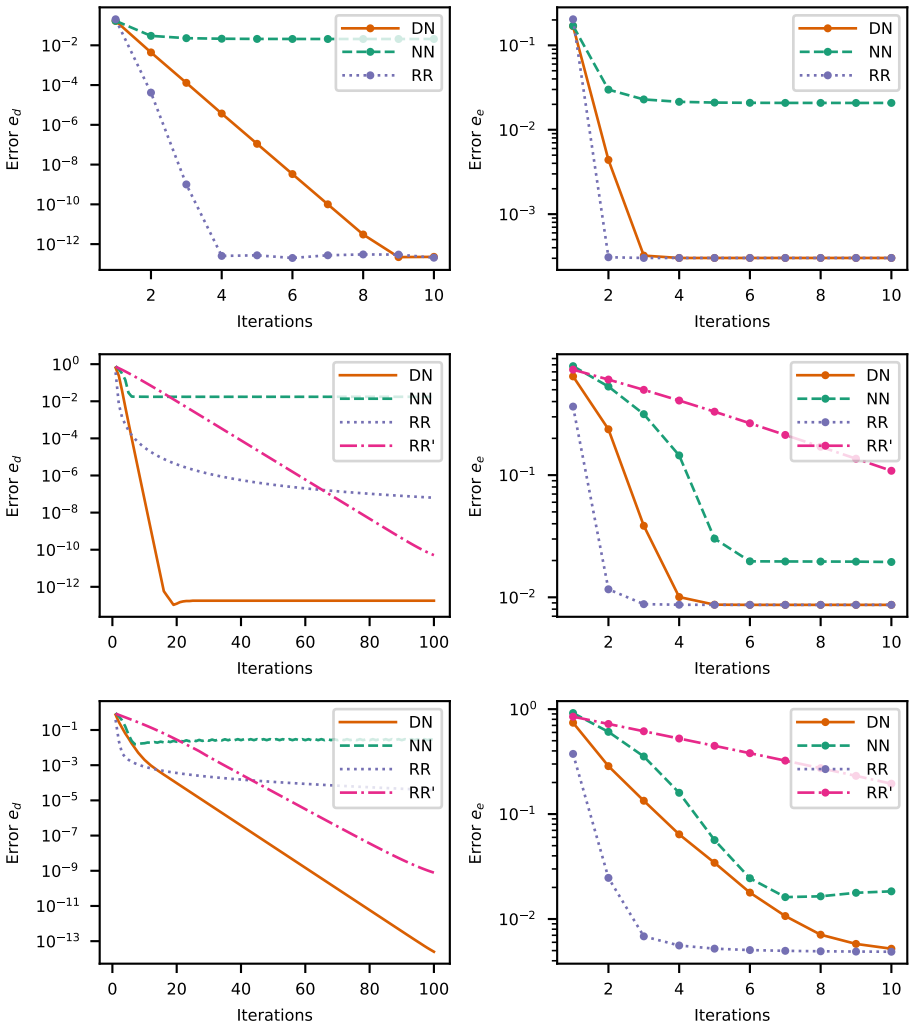

Figure 4.8: The errors of the Dirichlet–Neumann (DN), Neumann–Neumann (NN), and Robin–Robin (RR and RR') methods applied to Example 4.1.2 for Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The parameters are as in Table 4.3.

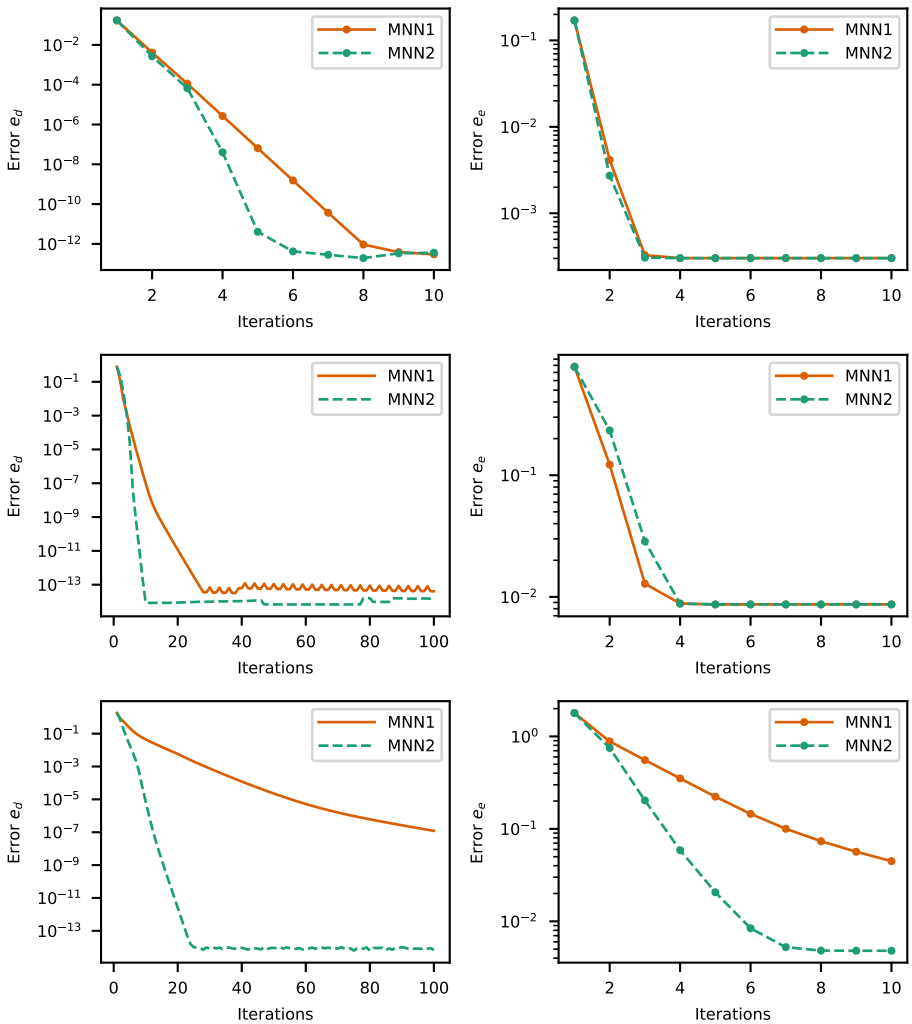

Figure 4.9: The errors of the MNN1 and MNN2 methods applied to Example 4.1.2 for Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The parameters are as in Table 4.3.

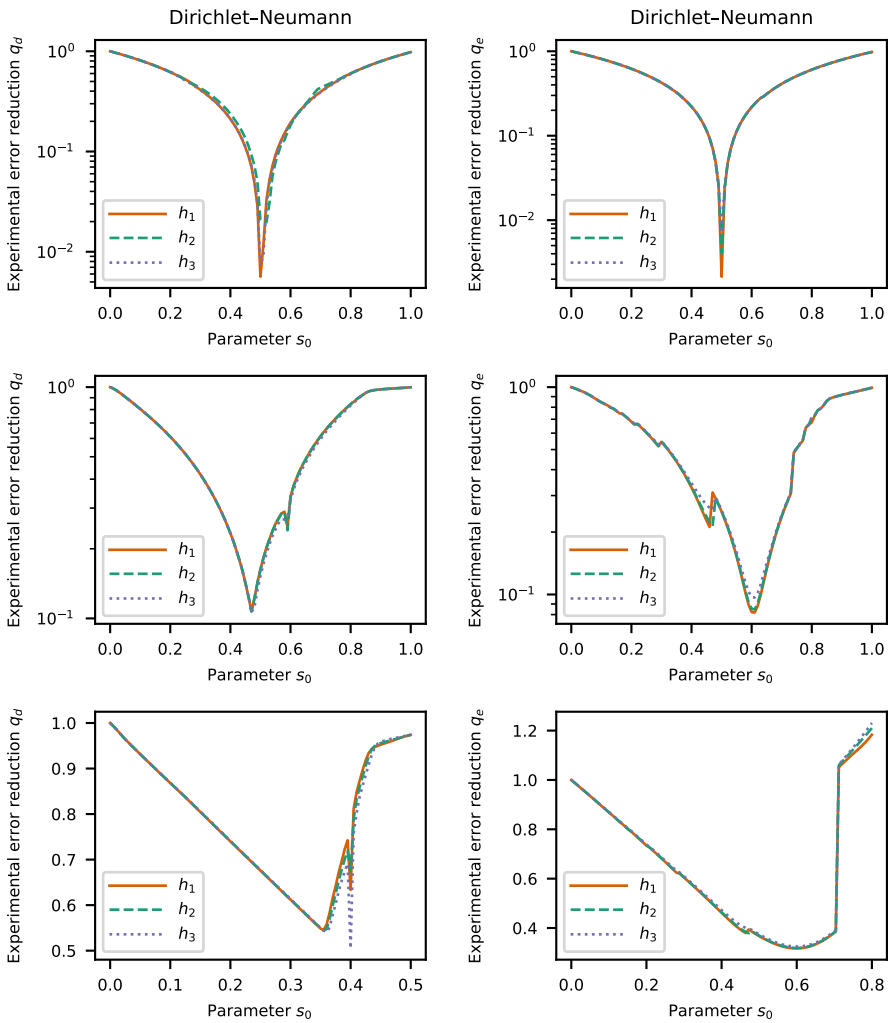

Figure 4.10: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Dirichlet–Neumann method applied to Example 4.1.2 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

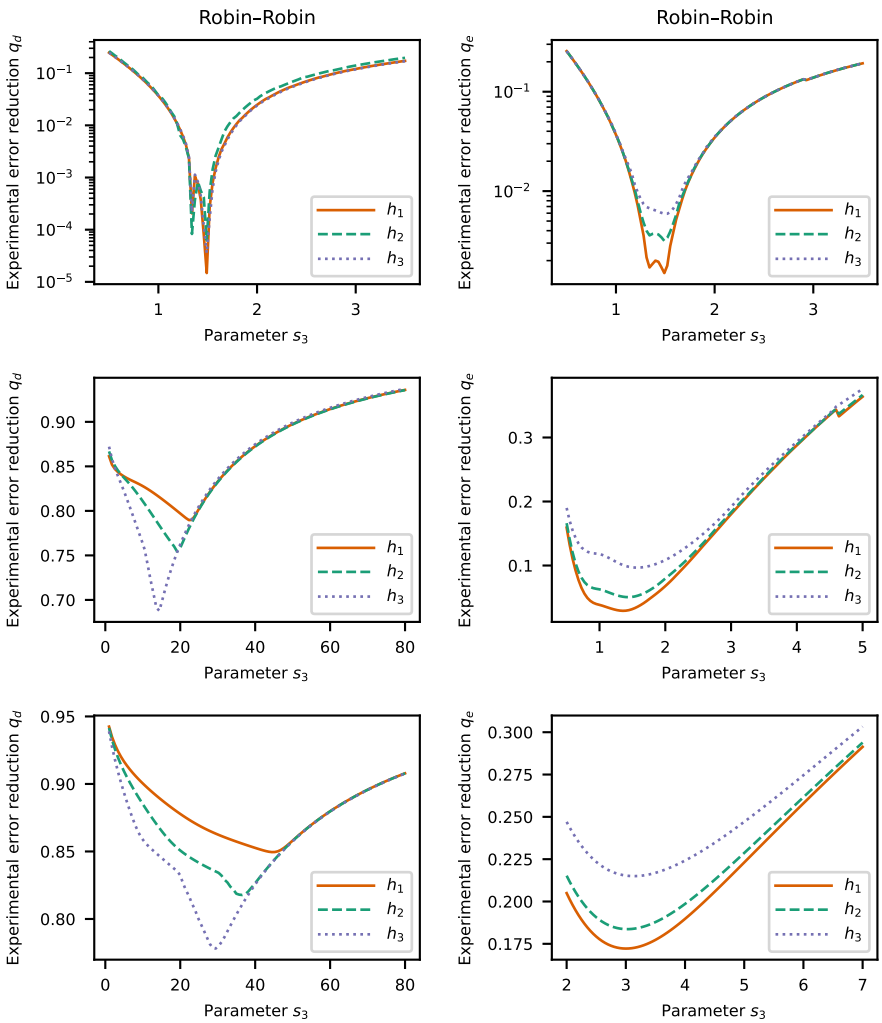

Figure 4.11: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Robin–Robin method applied to Example 4.1.2 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

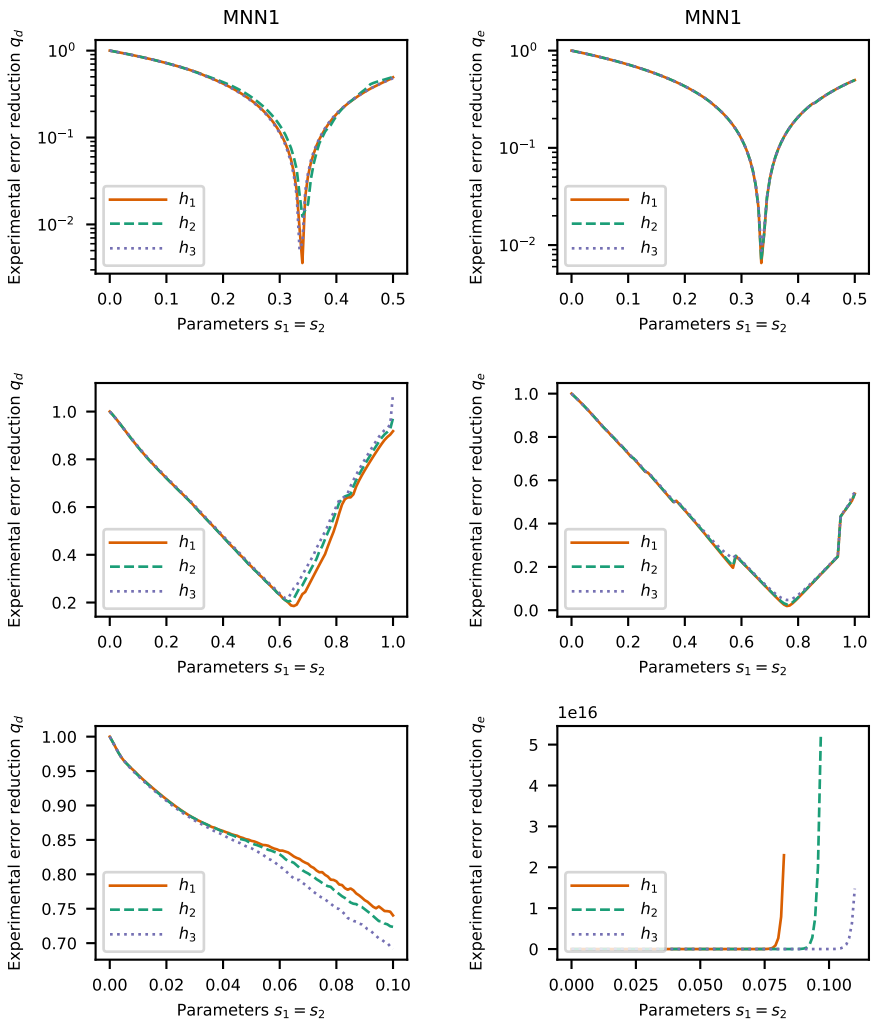

Figure 4.12: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the MNN1 method applied to Example 4.1.2 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

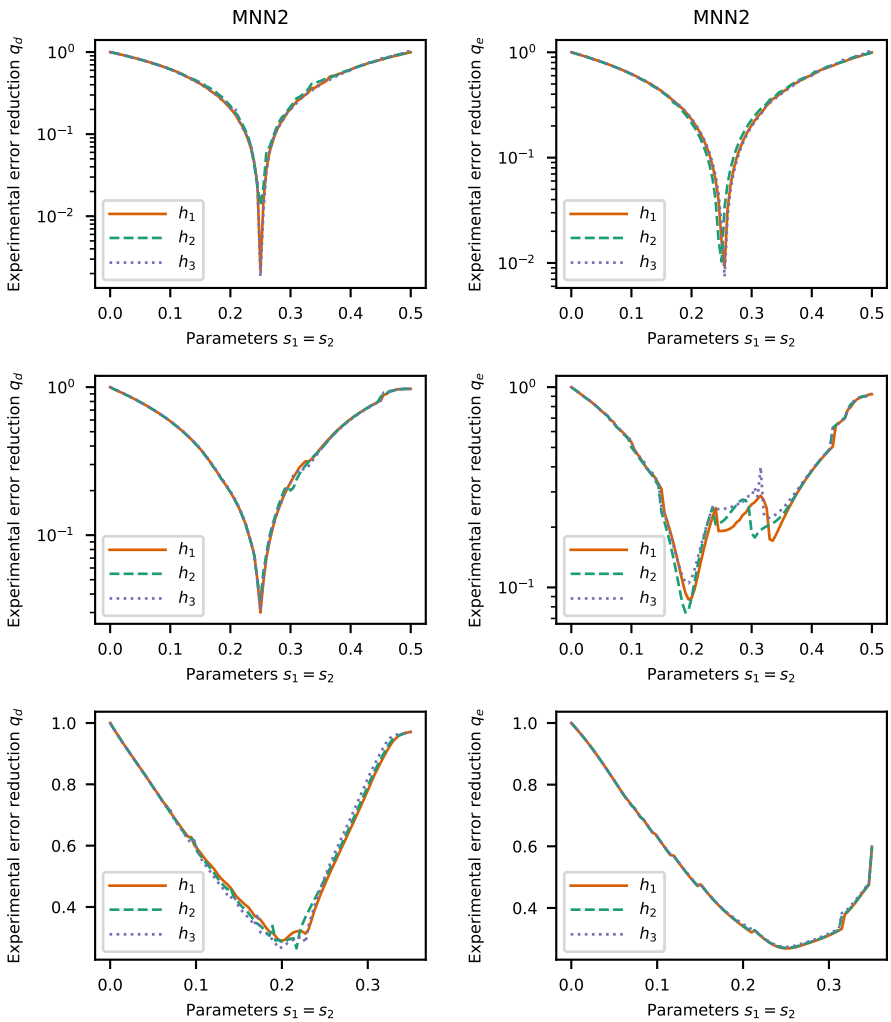

Figure 4.13: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the MNN2 method applied to Example 4.1.2 on Decomposition 2.4.1 (top), Decomposition 2.4.2 (middle), and Decomposition 2.4.3 (bottom). The mesh parameters are as in Table 4.2.

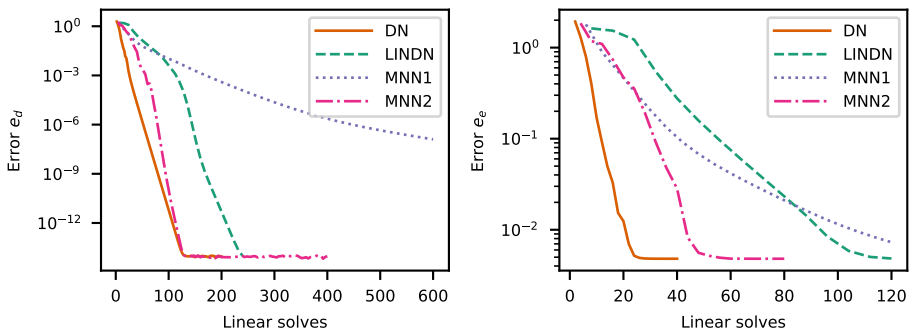

Figure 4.14: The errors of the Dirichlet–Neumann (DN), linearized Dirichlet–Neumann (LINDN), MNN1, and MNN2 methods applied to Example 4.1.2 for Decomposition 2.4.3 plotted against the number of linear solves.

Chapter 5

Domain decomposition for nonlinear parabolic equations

5.1 Domain decomposition for linear parabolic equations (Paper IV)

Domain decomposition methods have a long history of being applied to time dependent partial differential equations. There are numerous approaches to this. The simplest one is simply to use a time integrator and perform a domain decomposition method in each time step, see e.g. [86, Chapter 9]. Another approach is to employ domain decomposition based time integrators [36]. The approach studied here, however, is to decompose the space-time domain into subdomains and perform an iterative method on the space-time formulation of the partial differential equation. For an overview of this we refer to [43, Chapter 3]. Convergence has been proven for many methods and equations, but these proofs typically employ the Fourier transform in space and are therefore only valid on very simple domains [44, 46, 47, 54, 50, 72, 73]. Some exceptions to these are [48, 55], but these employ a type of analysis that is specific to the method studied and it is unclear how to generalize this to other methods. Space-time domain decomposition methods has also been studied in the context of subdomains with different material properties [2, 56].

Remark 5.1.1. *There are also the related waveform relaxation methods, originating in circuit simulations, see [43, 71]. These are originally methods for solving ordinary differential equations, but they can be applied to a spatially discretized partial differential equation and may then be interpreted as space-time decomposition methods. Many authors use the term waveform relaxation synonymously to space-time decomposition. The convergence of*

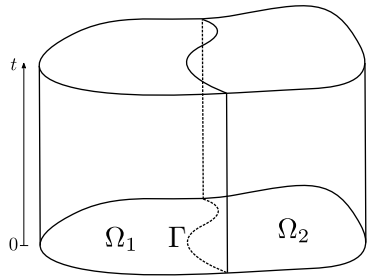

Figure 5.1: An example of a space-time decomposition.

such methods applied to systems of ordinary differential equations has been shown in [87].

For an introduction to linear parabolic equations we refer to [38, 69]. In this section we will consider a heat equation in heterogeneous media for all times $t \in \mathbb{R}$

$$\begin{cases} \partial_t u - \nabla \cdot \big(\alpha(x)\nabla u\big) = f & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{5.1}$$

We consider the usual spatial domain decomposition, which induces a space-time decomposition

$$\bar{\Omega} \times \mathbb{R} = (\bar{\Omega}_1 \times \mathbb{R}) \cup (\bar{\Omega}_2 \times \mathbb{R}).$$

An example of a space-time decomposition is illustrated in Figure 5.1.

The parabolic equation can then be rewritten as the transmission problem

$$\begin{cases} \partial_t u_i - \nabla \cdot \big(\alpha(x)\nabla u_i\big) = f_i & \text{in } \Omega_i \times \mathbb{R} & \text{for } i = 1, 2, \\ u_i = 0 & \text{on } (\partial\Omega_i \setminus \Gamma) \times \mathbb{R} & \text{for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma \times \mathbb{R}, & \\ \alpha(x)\nabla u_1 \cdot \nu_1 + \alpha(x)\nabla u_2 \cdot \nu_2 = 0 & \text{on } \Gamma \times \mathbb{R}. & \end{cases} \tag{5.2}$$

In this section we will mainly discuss the space-time Steklov–Poincaré operators and how they can be used to show convergence of the space-time Robin–Robin method. This method is given by finding (u_1^n, u_2^n) such that

$$\begin{cases} \partial_t u_1^n - \nabla \cdot \big(\alpha(x)\nabla u_1^n\big) = f_1 & \text{in } \Omega_1 \times \mathbb{R}, \\ u_1^n = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma) \times \mathbb{R}, \\ \alpha(x)\nabla u_1^n \cdot \nu_1 + s_3 u_1^n = \alpha(x)\nabla u_2^{n-1} \cdot \nu_1 + s_3 u_2^{n-1} & \text{on } \Gamma \times \mathbb{R}, \\ \partial_t u_2^n - \nabla \cdot \big(\alpha(x)\nabla u_2^n\big) = f_2 & \text{in } \Omega_2 \times \mathbb{R}, \\ u_2^n = 0 & \text{on } (\partial\Omega_2 \setminus \Gamma) \times \mathbb{R}, \\ \alpha(x)\nabla u_2^n \cdot \nu_2 + s_3 u_2^n = \alpha(x)\nabla u_1^n \cdot \nu_2 + s_3 u_1^n & \text{on } \Gamma \times \mathbb{R} \end{cases}$$

for $n = 1, 2, \ldots$

We will now construct the space-time Steklov–Poincaré theory of the linear parabolic equation. Since we wish to mimic the elliptic theory we seek a space-time variational formulation for parabolic equations. The standard one is given by the solution space

$$H^1(\mathbb{R}, H^{-1}(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega)),$$

but this space is unsuited for domain decomposition. The reason for this is that it is not always possible to glue two functions (u_1, u_2) defined on $\Omega_1 \times \mathbb{R}$ and $\Omega_2 \times \mathbb{R}$, respectively, into a function u on $\Omega \times \mathbb{R}$. A counterexample is given in [24, Example 2.14]. Instead, we consider the weaker solution space

$$H^{1/2}(\mathbb{R}, L^2(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega)).$$

This space has been used to treat parabolic equations [76] and for boundary element methods applied to parabolic equations [24]. Recently, the space has been become popular in the context of space-time finite element methods as well, see, e.g., [39, 67, 68, 99, 112]. While this space solves the gluing issue, we are still left with an equation with an operator that is coercive in the $L^2(\mathbb{R}, H^1(\Omega))$-norm, but not in the full $H^{1/2}(\mathbb{R}, L^2(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega))$-norm. Analogously, the Steklov–Poincaré operators are coercive only in the $L^2(\mathbb{R}, \Lambda)$-norm. This means that we can not apply the Lax–Milgram lemma directly. To solve this, we employ an idea from [40], which is to transform the equation into an equivalent equation where the operator is coercive. We apply this idea to the parabolic equation, but also directly on the Steklov–Poincaré equation. This requires a rigorous theory of the temporal Hilbert transform on the interface, which we have developed in Paper IV. Note that Steklov–Poincaré operators for time-dependent problems were first introduced in [55, 57], but without any analysis.

Finally, we only consider the convergence of the Robin–Robin method. This is due to the fact that the Steklov–Poincaré operators are not symmetric and, as we saw in Chapters 3 and 4, the convergence of the other standard methods require some form of symmetry. Modified methods are discussed in Section 5.3.

Remark 5.1.2. *As noted in Remark 3.4.1, convergence of the Dirichlet–Neumann and Neumann–Neumann methods only requires that our equation is close to symmetric. In the case of a parabolic equation, we have a symmetric term given by the spatial derivatives and an antisymmetric term given by the temporal derivative. Since these are bounded in different norms, the equation is not close to symmetric, even if α is very large.*

We recall the Bochner and Sobolev–Bochner spaces defined in Section 2.3 and define

the spaces

$$
\begin{aligned}
W &= H^{1/2}\big(\mathbb{R}, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}, V\big),\\
W_i^0 &= H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}, V_i^0\big),\\
W_i &= H^{1/2}\big(\mathbb{R}, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}, V_i\big),\\
Z &= H^{1/4}\big(\mathbb{R}, L^2(\Gamma)\big) \cap L^2\big(\mathbb{R}, \Lambda\big).
\end{aligned}
$$

We also recall the trace operator $T_{\partial\Omega_i}$ and its right inverse $R_{\partial\Omega_i}$ as in (2.4) and (2.5), respectively. As in the elliptic case, we are only interested in the trace on the interface $\Gamma \times \mathbb{R}$ and therefore define the space-time interface trace operators

$$
T_i : W_i \to Z : u_i \mapsto \left(T_{\partial\Omega_i} u_i\right)\big|_{\Gamma\times\mathbb{R}}.
$$

There is also a bounded linear map $R_i : Z \to W_i$ that is a right inverse to T_i, defined analogously to the elliptic case. For details on this, see [Paper IV, Lemma 4.4]. Moreover, we recall the operators $\partial_\pm^{1/2}$ and $\mathcal{H}$ defined in Section 2.3. In particular, we will make use of the two Hilbert transforms

$$
\mathcal{H}_i : W_i \to W_i \quad \text{and} \quad \mathcal{H}_\Gamma : Z \to Z.
$$

Before introducing the weak formulations we make the following assumption.

Assumption 5.1.1. *The function α satisfies $\alpha \in L^\infty(\Omega)$ and there exists a constant $c > 0$ such that*

$$
\alpha(x) \geq c, \quad \textit{for almost every } x \in \Omega.
$$

Moreover, $f \in W^*$ and there exist $f_i \in W_i^*$ such that

$$
\langle f, v\rangle = \langle f_1, v|_{\Omega_1\times\mathbb{R}^+}\rangle + \langle f_2, v|_{\Omega_2\times\mathbb{R}^+}\rangle \qquad \textit{for all } v \in W.
$$

The operators $A : W \to W^*$ and $A_i : W_i \to W_i^*$ are then defined as

$$
\begin{aligned}
\langle Au, v\rangle &= \int_{\mathbb{R}}\int_{\Omega} \partial_+^{1/2} u \partial_-^{1/2} v + \alpha(x)\nabla u\cdot\nabla v \,\mathrm{d}x\,\mathrm{d}t \qquad \text{and}\\
\langle A_i u_i, v_i\rangle &= \int_{\mathbb{R}}\int_{\Omega_i} \partial_+^{1/2} u_i \partial_-^{1/2} v_i + \alpha(x)\nabla u_i\cdot\nabla v_i \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

respectively. The operators A and A_i are bounded, but only coercive in the $L^2(\mathbb{R}, H^1)$-norm, i.e.,

$$
\langle Au, v\rangle \geq c\|u\|^2_{L^2(\mathbb{R},H^1(\Omega))}, \quad \langle A_i u_i, v_i\rangle \geq c\|u_i\|^2_{L^2(\mathbb{R},H^1(\Omega_i))}
$$

for all $u, v \in W$ and $u_i, v_i \in W_i$. However, if we define

$$\mathcal{H}_i^\varphi = \cos(\varphi) I - \sin(\varphi)\mathcal{H}_i, \tag{5.3}$$

then the operator $(\mathcal{H}_i^\varphi)^* A_i$ is coercive for small enough $\varphi > 0$, see [Paper IV, Lemma 5.1] for details.

The weak formulation of the parabolic equation is to find $u \in W$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in W. \tag{5.4}$$

Since A is not coercive we can not prove directly using the Lax–Milgram lemma, stated in Theorem 2.1.1, that (5.4) has a unique solution. This can be resolved by defining $\mathcal{H}^\varphi$ analogously to (5.3) and considering the bounded and coercive operator $(\mathcal{H}^\varphi)^* A$. By the Lax–Milgram lemma $(\mathcal{H}^\varphi)^* A$ is bijective and since $(\mathcal{H}^\varphi)^* : W^* \to W^*$ is a linear isomorphism, A is also bijective.

Remark 5.1.3. *Of course, if the solution has higher temporal regularity*

$$u \in H^1(\mathbb{R}, H^{-1}(\Omega))$$

then u also satisfies the variational formulation in

$$H^1(\mathbb{R}, H^{-1}(\Omega)) \cap L^2(\mathbb{R}, H^1(\Omega)).$$

This follows by extending (2.2) to the Bochner space $H^1\big(\mathbb{R}, H^{-1}(\Omega)\big)$.

The weak formulation of the space-time transmission problem is to find $(u_1, u_2) \in W_1 \times W_2$ such that

$$\begin{cases} \langle A_i u_i, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in W_i^0,\ i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i u_i - f_i, R_i \mu\rangle = 0 & \text{for all } \mu \in Z. \end{cases} \tag{5.5}$$

The equivalence of the weak formulations of the parabolic equation and the space-time transmission problem is shown in [Paper IV, Lemma 5.3].

Before defining the Steklov–Poincaré operators, we consider the nonhomogeneous problem for $g \in (W_i^0)^*$ and $\eta \in Z$. That is, to find $u_i \in W_i$ such that

$$\begin{cases} \langle A_i u_i, v\rangle = \langle g, v\rangle & \text{for all } v \in W_i^0, \\ T_i u_i = \eta. & \end{cases}$$

We introduce the bounded linear solution operators $F_i : Z \to W_i : \eta \mapsto u_i$ given by $g = 0$ and $G_i : (W_i^0)^* \to W_i : g \mapsto u_i$ given by $\eta = 0$. The space-time

Steklov–Poincaré operators $S_i : Z \to Z^*$ and corresponding source terms $\chi \in Z^*$ are defined as

$$\langle S_i\eta, \mu\rangle = \langle A_iF_i\eta, R_i\mu\rangle \quad \text{and} \quad \langle \chi_i, \mu\rangle = \langle f_i - A_iG_if_i, R_i\mu\rangle,$$

respectively. Moreover, we define the operator $S = S_1 + S_2$. The Steklov–Poincaré equation is then

$$S\eta = \chi.$$

Analogously to the operators $\mathcal{H}_i^\varphi$ we define

$$\mathcal{H}_\Gamma^\varphi = \cos(\varphi)I - \sin(\varphi)\mathcal{H}_\Gamma.$$

We have the following result on the properties of the space-time Steklov–Poincaré operators.

Theorem 5.1.1. *Suppose that Assumptions 2.4.2 and 5.1.1 hold. The Steklov–Poincaré operators $S_i, S : Z \to Z^*$ are well defined and bounded. Moreover, they are coercive in the norm $L^2(\mathbb{R}, \Lambda)$, i.e.,*

$$\langle S_i\eta, \eta\rangle \geq c\|\eta\|^2_{L^2(\mathbb{R},\Lambda)}, \quad \langle S\eta, \eta\rangle \geq c\|\eta\|^2_{L^2(\mathbb{R},\Lambda)} \quad \textit{for all } \eta \in Z, \quad i = 1, 2.$$

Finally, they are coercive-equivalent in Z, i.e., $(\mathcal{H}_\Gamma^\varphi)^* S_i$ and $(\mathcal{H}_\Gamma^\varphi)^* S$ are coercive for $\varphi > 0$ small enough.

The coercivity in $L^2(\mathbb{R}, \Lambda)$ is sufficient to show convergence of the Robin–Robin method and the coercive-equivalency is required to show that the Steklov–Poincaré operators are bijective, which, in turn, is necessary to show that the Robin–Robin method is well defined.

The weak formulation of the Robin–Robin method yields the iterates $(u_1^n, u_2^n) \in W_1 \times W_2$ for $n = 1, 2, \ldots$ The interface iteration corresponding to the Robin–Robin method is given by finding $(\eta_1^n, \eta_2^n) \in Z \times Z$ such that

$$\begin{cases} \langle (s_3J + S_1)\eta_1^n - \chi_1, \mu\rangle = \langle (s_3J - S_2)\eta_2^{n-1} + \chi_2, \mu\rangle & \text{for all } \mu \in Z, \\ \langle (s_3J + S_2)\eta_2^n - \chi_2, \mu\rangle = \langle (s_3J - S_1)\eta_1^n + \chi_1, \mu\rangle & \text{for all } \mu \in Z \end{cases}$$

for $n = 1, 2, \ldots$ Note that the interface iteration is of the same form as in the linear elliptic case (3.17), but the operators and spaces are different.

The convergence analysis of the Robin–Robin method now follows as in the linear elliptic case, given in Section 3.5, but by restricting the Steklov–Poincaré operators to $L^2(\Gamma \times \mathbb{R})$ instead of $L^2(\Gamma)$. This requires the following assumption.

Assumption 5.1.2. *Let $u \in W$ denote the solution to (5.4). Then the functionals*

$$\mu \mapsto \left\langle A_i\left(u|_{\Omega_i \times \mathbb{R}}\right) - f_i, R_i \mu \right\rangle, \quad i = 1, 2,$$

are elements in $L^2(\Gamma \times \mathbb{R})^*$.

As in the linear elliptic case, this assumption and the properties of the space-time Steklov–Poincaré operators yield the following convergence result of the Robin–Robin method.

Theorem 5.1.2. *Suppose that Assumptions 2.4.2, 5.1.1 and 5.1.2 hold. Let $s_3 > 0$, u_i^n denote the iterates of the Robin–Robin method, and (u_1, u_2) denote the solution to the transmission problem (5.5). Then*

$$\|u_i^n - u_i\|_{L^2(\mathbb{R}, V_i)} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

5.2 Domain decomposition for linear initial value problems (Paper V)

In Section 5.1 we gave an analysis for a parabolic problem posed for all times $t \in \mathbb{R}$. A more common problem is an initial value problem, where the solution is known at $t = 0$ and we are interested in finding the solution for all times $t \in \mathbb{R}^+$. We will first briefly discuss how an initial value problem can be recast as a problem on $\mathbb{R}^+$. We will then see that, while the analysis of Section 5.1 applies to this situation as well, it is unclear whether the domain decomposition methods preserve the initial value. We therefore propose a different analysis, based on a variational formulation with the trial space with temporal part $H_{00}^{1/2}(\mathbb{R}^+)$ and test space with temporal part $H^{1/2}(\mathbb{R}^+)$. Since the trial and test space are different we can not use the Lax–Milgram theorem as in Section 5.1, but we instead employ the Banach–Nečas–Babuška theorem, stated in Theorem 2.1.2.

The initial value problem is to find u such that

$$\begin{cases} \partial_t u - \nabla \cdot \left(\alpha(x) \nabla u\right) = f & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u = 0 & \text{in } \Omega \times \{0\}. \end{cases} \tag{5.6}$$

We can extend this to a problem on $\mathbb{R}$ and solve this instead. For this we first define $\hat{f}$ such that $\hat{f} = 0$ for $t < 0$ and $\hat{f} = f$ for $t \geq 0$. If $\hat{u}$ solves (5.1) with the source

term $\hat{f}$ then $u = u|_{\Omega\times\mathbb{R}^+}$ solves (5.6). This fact is obvious for the strong formulation, but not in the weak formulation. It is not even obvious how to interpret the initial value condition $u(0) = 0$ in $H^{1/2}(\mathbb{R})$ since pointwise evaluation is not a continuous operation on $H^{1/2}(\mathbb{R})$. One idea is to instead consider the Lions–Magenes space $H_{00}^{1/2}(\mathbb{R}^+)$. It has been shown in certain circumstances [40, Theorem 4.4], that if we have a weak solution u of (5.4) with $f = 0$ on $\mathbb{R} \setminus \mathbb{R}^+$ then $u \in H_{00}^{1/2}(\mathbb{R}^+, L^2(\Omega))$. However, we have not been able to apply this result to the iterates of our domain decomposition methods and we therefore consider a different approach, where we only work with spaces on $\mathbb{R}^+$.

The initial value problem is equivalent to the transmission problem

$$\begin{cases} \partial_t u_i - \nabla\cdot\big(\alpha(x)\nabla u_i\big) = f_i & \text{in } \Omega_i\times\mathbb{R}^+, \\ u_i = 0 & \text{on } (\partial\Omega_i\setminus\Gamma)\times\mathbb{R}^+, \\ u_i = 0 & \text{in } \Omega_i\times\{0\}, \quad \text{for } i = 1,2, \\ u_1 = u_2 & \text{on } \Gamma\times\mathbb{R}^+, \\ \alpha(x)\nabla u_1\cdot\nu_1 + \alpha(x)\nabla u_2\cdot\nu_2 = 0 & \text{on } \Gamma\times\mathbb{R}^+. \end{cases} \tag{5.7}$$

The Robin-Robin method for an initial value problem is given by finding (u_1^n, u_2^n) such that

$$\begin{cases} \partial_t u_1^n - \nabla\cdot\big(\alpha(x)\nabla u_1^n\big) = f_1 & \text{in } \Omega_1\times\mathbb{R}^+, \\ u_1^n = 0 & \text{on } (\partial\Omega_1\setminus\Gamma)\times\mathbb{R}^+, \\ u_1^n = 0 & \text{on } \Omega_1\times\{0\} \\ \alpha(x)\nabla u_1^n\cdot\nu_1 + s_3 u_1^n = \alpha(x)\nabla u_2^{n-1}\cdot\nu_1 + s_3 u_2^{n-1} & \text{on } \Gamma\times\mathbb{R}^+, \\ \partial_t u_2^n - \nabla\cdot\big(\alpha(x)\nabla u_2^n\big) = f_2 & \text{in } \Omega_2\times\mathbb{R}^+, \\ u_2^n = 0 & \text{on } (\partial\Omega_2\setminus\Gamma)\times\mathbb{R}^+, \\ u_2^n = 0 & \text{on } \Omega_2\times\{0\} \\ \alpha(x)\nabla u_2^n\cdot\nu_2 + s_3 u_2^n = \alpha(x)\nabla u_1^n\cdot\nu_2 + s_3 u_1^n & \text{on } \Gamma\times\mathbb{R}^+. \end{cases}$$

for $n = 1, 2, \ldots$ Note that this is the same as the Robin–Robin method on $\mathbb{R}$ as defined in Section 5.1, except for the inclusion of the initial value at $t = 0$. The same is true for the transmission problem (5.7).

For the analysis we introduce the Hilbert spaces

$$
\begin{aligned}
W &= H^{1/2}_{00}\big(\mathbb{R}^+, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}^+, V\big),\\
\tilde{W} &= H^{1/2}\big(\mathbb{R}^+, L^2(\Omega)\big) \cap L^2\big(\mathbb{R}^+, V\big),\\
W_i &= H^{1/2}_{00}\big(\mathbb{R}^+, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}^+, V_i\big),\\
\tilde{W}_i &= H^{1/2}\big(\mathbb{R}^+, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}^+, V_i\big),\\
W_i^0 &= H^{1/2}_{00}\big(\mathbb{R}^+, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}^+, V_i^0\big),\\
\tilde{W}_i^0 &= H^{1/2}\big(\mathbb{R}^+, L^2(\Omega_i)\big) \cap L^2\big(\mathbb{R}^+, V_i^0\big),\\
Z &= H^{1/4}\big(\mathbb{R}^+, L^2(\Gamma)\big) \cap L^2\big(\mathbb{R}^+, \Lambda\big).
\end{aligned}
$$

We point out two fundamental properties of these spaces that are essential for the analysis. First, the trial space W_i is a dense subset of the test space $\tilde{W}_i$. Second, the trace spaces of W_i and $\tilde{W}_i$ are both Z, see [Paper V, Lemma 2]. We define the interface trace operators

$$
T_i : \tilde{W}_i \to Z : u_i \mapsto T_{\partial\Omega_i} u_i|_{\Gamma\times\mathbb{R}^+}
$$

and denote their right inverses by $R_i : Z \to W_i$.

We define the operators $A : W \to \tilde{W}^*$ and $A_i : W_i \to \tilde{W}_i^*$ as

$$
\begin{aligned}
\langle Au, v\rangle &= \int_{\mathbb{R}^+}\int_{\Omega} \partial_t u\, v + \alpha(x)\nabla u\cdot\nabla v \,\mathrm{d}x\,\mathrm{d}t \quad \text{and}\\
\langle A_i u_i, v_i\rangle &= \int_{\mathbb{R}^+}\int_{\Omega_i} \partial_t u_i\, v_i + \alpha(x)\nabla u_i\cdot\nabla v_i \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

respectively. The above formulas actually only holds for smoother functions than $u \in W$ and $u_i \in W_i$. For a rigorous definition, we first consider smooth functions and then extend to the appropriate spaces as in [24]. Notice that, unlike the previous formulations in this thesis, the test and trial spaces are different.

Analogously to Assumption 5.1.1 we make the following assumption.

Assumption 5.2.1. *The function α satisfies $\alpha \in L^\infty(\Omega)$ and there exists a constant $c > 0$ such that*

$$
\alpha(x) \geq c, \quad \textit{for almost every } x \in \Omega.
$$

Moreover, $f \in \tilde{W}^*$ and there exist $f_i \in \tilde{W}_i^*$ such that

$$
\langle f, v\rangle = \langle f_1, v|_{\Omega_1\times\mathbb{R}^+}\rangle + \langle f_2, v|_{\Omega_2\times\mathbb{R}^+}\rangle \qquad \textit{for all } v \in \tilde{W}.
$$

The weak formulation of the initial value problem (5.6) is to find $u \in W$ such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in \tilde{W}. \tag{5.8}$$

This problem has a unique solution, see [95, Corollary 3.9]. Moreover, the initial value problem is equivalent to the weak formulation of the transmission problem, given by finding $(u_1, u_2) \in W_1 \times W_2$ such that

$$\begin{cases} \langle A_i u_i, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in \tilde{W}_i^0,\ i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i u_i - f_i, R_i \mu\rangle = 0 & \text{for all } \mu \in Z. \end{cases} \tag{5.9}$$

Note that the transmission problem is of the same form as (5.2). The only differences are the function spaces and the operators involved. However, note also that the test and trial spaces are different.

We define the Steklov–Poincaré operators $S_i : Z \to Z^*$ and the source terms $\chi \in Z^*$ as

$$\langle S_i \eta, \mu\rangle = \langle A_i F_i \eta, R_i \mu\rangle \quad \text{and} \quad \langle \chi_i, \mu\rangle = \langle f_i - A_i G_i f_i, R_i \mu\rangle,$$

respectively. Motivated by Section 5.1, we define the operator $\mathcal{H}_i^\varphi : Z \to Z$

$$\mathcal{H}_\Gamma^\varphi \eta = \left((\cos(\varphi) I - \sin(\varphi)\mathcal{H}_\Gamma) E_i \eta\right)|_{\Gamma\times\mathbb{R}^+},$$

where $E_i : Z \to H^{1/4}\big(\mathbb{R}, L^2(\Gamma)\big) \cap L^2\big(\mathbb{R}, \Lambda\big)$ is the extension by zero in time.

Theorem 5.2.1. *Suppose that Assumptions 2.4.2 and 5.2.1 hold. Then $S_i, S : Z \to Z^*$ are bounded and satisfy*

$$\langle S_i \eta, \eta\rangle \geq c\|\eta\|^2_{L^2(\mathbb{R}^+, V_i)} \quad \text{and } \langle S\eta, \eta\rangle \geq c\|\eta\|^2_{L^2(\mathbb{R}^+, V_i)}$$

respectively. Moreover, $(\mathcal{H}_\Gamma^\varphi)^* S_i$ and $(\mathcal{H}_\Gamma^\varphi)^* S$ are coercive for $\varphi > 0$ small enough. In particular, S_i and S are bijective.

Remark 5.2.1. *Notice that this is a weaker result than Theorem 5.1.1 since we do not know if $(\mathcal{H}_\Gamma^\varphi)^* : Z^* \to Z^*$ is an isomorphism. If we could show that $(\mathcal{H}_\Gamma^\varphi)^*$ is an isomorphism then the bijectivity of the Steklov–Poincaré operators would follow in the same way as in Section 5.1. Since this seems difficult to show we instead use the Banach–Nečas–Babuška theorem, stated in Theorem 2.1.2. This does not affect the convergence proof for the Robin–Robin method since this only relies on the coercivity in $L^2(\mathbb{R}^+, V_i)$.*

We also consider the weak formulation of the Robin–Robin method, which is to find $(u_1^n, u_2^n) \in W_1 \times W_2$ such that

$$\begin{cases} \langle A_1 u_1^n, v_1 \rangle = \langle f_1, v_1 \rangle & \text{for all } v_1 \in \tilde{W}_1^0, \\ \langle A_1 u_1^n - f_1, R_1 \mu \rangle + \langle A_2 u_2^{n-1} - f_2, R_2 \mu \rangle & \\ \quad = s_3 (T_2 u_2^{n-1} - T_1 u_1^n, \mu)_{L^2(\Gamma \times \mathbb{R})} & \text{for all } \mu \in Z, \\ \langle A_2 u_2^n, v_2 \rangle = \langle f_2, v_2 \rangle & \text{for all } v_2 \in \tilde{W}_2^0, \\ \langle A_2 u_2^n - f_2, R_2 \mu \rangle + \langle A_1 u_1^n - f_1, R_1 \mu \rangle & \\ \quad = s_3 (T_1 u_1^n - T_2 u_2^n, \mu)_{L^2(\Gamma \times \mathbb{R})} & \text{for all } \mu \in Z \end{cases}$$

for $n = 1, 2, \ldots$ The interface iteration corresponding to the Robin–Robin method is then to find $(\eta_1^n, \eta_2^n) \in Z \times Z$ such that

$$\begin{cases} \langle (s_3 J + S_1)\eta_1^n - \chi_1, \mu \rangle = \langle (s_3 J - S_2)\eta_2^{n-1} + \chi_2, \mu \rangle & \text{for all } \mu \in Z, \\ \langle (s_3 J + S_2)\eta_2^n - \chi_2, \mu \rangle = \langle (s_3 J - S_1)\eta_1^n + \chi_1, \mu \rangle & \text{for all } \mu \in Z \end{cases}$$

for $n = 1, 2, \ldots$ As in Sections 3.5 and 5.1 we make the following assumption in order to show the convergence of the Robin–Robin method for initial value problems.

Assumption 5.2.2. *Let $u \in W$ denote the solution to (5.8). Then the functionals*

$$\mu \mapsto \big\langle A_i\big(u|_{\Omega_i \times \mathbb{R}^+}\big) - f_i, R_i \mu \big\rangle, \quad i = 1, 2,$$

are elements in $L^2(\Gamma \times \mathbb{R}^+)^*$.

Theorem 5.2.2. *Suppose that Assumptions 2.4.2, 5.2.1 and 5.2.2 hold. Then the iterates of the Robin–Robin method (u_1^n, u_2^n) converge to the solution of the transmission problem (5.9) in $L^2(\mathbb{R}^+, V_1) \times L^2(\mathbb{R}^+, V_2)$.*

5.3 Domain decomposition for nonlinear parabolic equations (Paper IV)

The Robin–Robin method has been studied for semilinear parabolic equations on rectangular domains [15] and even for a class of quasilinear equations in [48]. Other than these two papers there are few theoretical studies on nonoverlapping domain decomposition treating nonlinear parabolic equations. For decompositions in time, semilinear hyperbolic equations have been studied in [63, 64].

In this section we will therefore discuss the extension of the Steklov–Poincaré theory in Section 5.1 to nonlinear problems. Due to the mesh dependence of the convergence

factor of the Robin–Robin method we seek new methods that are provably convergent with mesh independent convergence factors. We suggest two modifications of the Dirichlet–Neumann method that fit this description. Since the convergence of the Robin–Robin method follows by the same observations as in Sections 4.5 and 5.1 we will mainly consider these modified Dirichlet–Neumann methods here and only briefly discuss the Robin–Robin method. We consider the nonlinear parabolic equation

$$\begin{cases} \partial_t u - \nabla \cdot \alpha(x, u, \nabla u) + \beta(x, u, \nabla u) = f & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \end{cases}$$

for some functions $\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. The corresponding transmission problem is to find (u_1, u_2) such that

$$\begin{cases} \partial_t u_i - \nabla \cdot \alpha(x, u_i, \nabla u_i) \\ \quad + \beta(x, u_i, \nabla u_i) = f_i & \text{in } \Omega_i \times \mathbb{R} & \text{for } i = 1, 2, \\ u_i = 0 & \text{on } (\partial\Omega_i \setminus \Gamma) \times \mathbb{R} & \text{for } i = 1, 2, \\ u_1 = u_2 & \text{on } \Gamma \times \mathbb{R}, \\ \alpha(x, u_1, \nabla u_1) \cdot \nu_1 = -\alpha(x, u_2, \nabla u_2) \cdot \nu_2 & \text{on } \Gamma \times \mathbb{R}. \end{cases}$$

For a fixed method parameter $s_0 > 0$ and an initial guess η^0, the nonlinear space-time Dirichlet–Neumann method is to find (u_1^n, u_2^n, η^n) such that

$$\begin{cases} \partial_t u_1^n - \nabla \cdot \alpha(x, u_1^n, \nabla u_1^n) \\ \quad + \beta(x, u_1^n, \nabla u_1^n) = f_1 & \text{in } \Omega_1 \times \mathbb{R}, \\ u_1^n = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma) \times \mathbb{R}, \\ u_1^n = \eta^{n-1} & \text{on } \Gamma \times \mathbb{R}, \\ \partial_t u_2^n - \nabla \cdot \alpha(x, u_2^n, \nabla u_2^n) \\ \quad + \beta(x, u_2^n, \nabla u_2^n) = f_2 & \text{in } \Omega_2 \times \mathbb{R}, \\ u_2^n = 0 & \text{on } (\partial\Omega_2 \setminus \Gamma) \times \mathbb{R}, \\ \alpha(x, u_2^n, \nabla u_2^n) \cdot \nu_2 = -\alpha(x, u_1^n, \nabla u_1^n) \cdot \nu_1 & \text{on } \Gamma \times \mathbb{R}, \\ \eta^n = s_0\, u_2^n|_{\Gamma \times \mathbb{R}} + (1 - s_0)\eta^{n-1} & \text{on } \Gamma \times \mathbb{R} \end{cases}$$

for $n = 1, 2, \ldots$ Similarly, the nonlinear space-time Robin–Robin method is to find

(u_1^n, u_2^n) such that

$$\begin{cases} \partial_t u_1^n - \nabla \cdot \alpha(x, u_1^n, \nabla u_1^n) + \beta(x, u_1^n, \nabla u_1^n) = f_1 & \text{in } \Omega_1 \times \mathbb{R}, \\ \qquad\qquad u_1^n = 0 & \text{on } (\partial\Omega_1 \setminus \Gamma) \times \mathbb{R}, \\ \alpha(x, u_1^n, \nabla u_1^n) \cdot \nu_1 + s_3 u_1^n & \\ -\alpha(x, u_2^{n-1}, \nabla u_2^{n-1}) \cdot \nu_1 - s_3 u_2^{n-1} = 0 & \text{on } \Gamma \times \mathbb{R}, \\ \partial_t u_2^n - \nabla \cdot \alpha(x, u_2^n, \nabla u_2^n) + \beta(x, u_2^n, \nabla u_2^n) = f_2 & \text{in } \Omega_2 \times \mathbb{R}, \\ \qquad\qquad u_2^n = 0 & \text{on } (\partial\Omega_2 \setminus \Gamma) \times \mathbb{R}, \\ \alpha(x, u_2^n, \nabla u_2^n) \cdot \nu_2 + s_3 u_2^n & \\ -\alpha(x, u_1^n, \nabla u_1^n) \cdot \nu_2 - s_3 u_1^n = 0 & \text{on } \Gamma \times \mathbb{R}. \end{cases}$$

for $n = 1, 2, \ldots$

In order to construct the nonlinear space-time Steklov–Poincaré theory we make the following assumption, which means that our functions α and β are Lipschitz continuous.

Assumption 5.3.1. *The functions $\alpha : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfy the assumptions as in [Paper IV, Assumption 2]. Moreover, $f \in \tilde{W}^*$ and there exist $f_i \in \tilde{W}_i^*$, $i = 1, 2$, such that*

$$\langle f, v \rangle = \langle f_1,\, v|_{\Omega_1 \times \mathbb{R}^+} \rangle + \langle f_2,\, v|_{\Omega_2 \times \mathbb{R}^+} \rangle \qquad \textit{for all } v \in \tilde{W}.$$

Recall the spaces W, W_i^0, W_i, Z and the operators $T_i, R_i, \mathcal{H}_\Gamma^\varphi$ defined in Section 5.1. The operators $A : W \to W^*$ and $A_i : W_i \to W_i^*$ are defined as

$$\langle Au, v \rangle = \int_{\mathbb{R}} \int_\Omega \partial_+^{1/2} u \partial_-^{1/2} v - \alpha(x, u, \nabla u) \cdot \nabla v + \beta(x, u, \nabla u) v \,\mathrm{d}x \,\mathrm{d}t \quad \text{and}$$
$$\langle A_i u_i, v_i \rangle = \int_{\mathbb{R}} \int_{\Omega_i} \partial_+^{1/2} u_i \partial_-^{1/2} v_i - \alpha(x, u_i, \nabla u_i) \cdot \nabla v_i + \beta(x, u_i, \nabla u_i) v_i \,\mathrm{d}x \,\mathrm{d}t,$$

respectively. The weak formulation of the parabolic equation is to find $u \in W$ such that

$$\langle Au, v \rangle = \langle f, v \rangle \quad \text{for all } v \in W. \tag{5.10}$$

This is equivalent to the weak formulation of the nonlinear space-time transmission problem, which is to find $(u_1, u_2) \in W_1 \times W_2$ such that

$$\begin{cases} \langle A_i u_i, v_i \rangle = \langle f_i, v_i \rangle & \text{for all } v_i \in W_i^0,\ i = 1, 2, \\ T_1 u_1 = T_2 u_2, & \\ \sum_{i=1}^2 \langle A_i u_i - f_i, R_i \mu \rangle = 0 & \text{for all } \mu \in Z. \end{cases} \tag{5.11}$$

Before defining the nonlinear space-time Steklov–Poincaré operators, we consider the nonhomogeneous equation for $g \in (W_i^0)^*$ and $\eta \in Z$. That is, to find $u_i \in W_i$ such that

$$\begin{cases} \langle A_i u_i, v\rangle = \langle g, v\rangle & \text{for all } v \in W_i^0, \\ T_i u_i = \eta. \end{cases}$$

We denote the corresponding solution operator by $F_i : Z \to W_i$ and define the Steklov–Poincaré operators $S_i : Z \to Z^*$ as

$$\langle S_i\eta, \mu\rangle = \langle A_i F_i \eta - f_i, R_i\mu\rangle.$$

If we define $S = S_1 + S_2$, the nonlinear space-time Steklov–Poincaré operator is to find $\eta \in Z$ such that $S\eta = 0$. We have the following properties of the nonlinear space-time Steklov–Poincaré operators.

Theorem 5.3.1. *Suppose that Assumptions 2.4.2 and 5.3.1 hold. The nonlinear space-time Steklov–Poincaré operators $S_i, S : Z \to Z^*$ are Lipschitz continuous and uniformly monotone in $L^2(\mathbb{R}, \Lambda)$. Moreover, the operators $(\mathcal{H}_\Gamma^\varphi)^* S_i$ and $(\mathcal{H}_\Gamma^\varphi)^* S$ are uniformly monotone for $\varphi > 0$ small enough.*

The weak formulation of the nonlinear space-time Dirichlet–Neumann method is to find $(u_1^n, u_2^n, \eta^n) \in V_1 \times V_2 \times \Lambda$ such that

$$\begin{cases} \langle A_1 u_1^n, v_1\rangle = \langle f_1, v_1\rangle & \text{for all } v_1 \in W_1^0, \\ T_1 u_1^n = \eta^{n-1}, & \\ \langle A_2 u_2^n, v_2\rangle = \langle f_2, v_2\rangle & \text{for all } v_2 \in W_2^0, \\ \langle A_2 u_2^n - f_2, R_2\mu\rangle + \langle A_1 u_1^n - f_1, R_1\mu\rangle = 0 & \text{for all } \mu \in Z, \\ \eta^n = s_0 T_2 u_2^n + (1 - s_0)\eta^{n-1} & \text{on } \Gamma \end{cases}$$

for $n = 1, 2, \ldots$ Similarly, the weak formulation of the nonlinear space-time Robin–Robin method is to find $(u_1^n, u_2^n) \in V_1 \times V_2$ such that

$$\begin{cases} \langle A_1 u_1^n, v_1\rangle = \langle f_1, v_1\rangle & \text{for all } v_1 \in W_1^0, \\ \langle A_1 u_1^n - f_1, R_1\mu\rangle + \langle A_2 u_2^{n-1} - f_2, R_2\mu\rangle & \\ \quad = s_3(T_2 u_2^{n-1} - T_1 u_1^n, \mu)_{L^2(\Gamma\times\mathbb{R})} & \text{for all } \mu \in Z, \\ \langle A_2 u_2^n, v_2\rangle = \langle f_2, v_2\rangle & \text{for all } v_2 \in W_2^0, \\ \langle A_2 u_2^n - f_2, R_2\mu\rangle + \langle A_1 u_1^n - f_1, R_1\mu\rangle & \\ \quad = s_3(T_1 u_1^n - T_2 u_2^n, \mu)_{L^2(\Gamma\times\mathbb{R})} & \text{for all } \mu \in Z \end{cases} \tag{5.12}$$

for $n = 1, 2, \ldots$ Since the convergence proof of the Dirichlet–Neumann method require symmetry of the Steklov–Poincaré operators it is not possible to generalize

this to the nonlinear case, compare with Remark 5.1.2. We instead use the following generalization of Zarantello's theorem to motivate the construction of two modified Dirichlet–Neumann methods. These are analogous to the modified Neumann–Neumann methods in Section 4.4, but with the extra complication that we are not considering a coercive equation. The theorem also yields the bijectivity of the Steklov–Poincaré operators.

Theorem 5.3.2. *Let X be a Hilbert space and $G : X \to X^*$ be a (nonlinear) operator. Moreover, suppose that $Q : X \to X$ is a linear isomorphism such that $Q^*G : X \to X^*$ is Lipschitz continuous and uniformly monotone. Then G is bijective.*

Furthermore, let $\eta^0 \in X$, $\chi \in X^*$, $P : X \to X^*$ be any symmetric and coercive linear operator, and $s_0 > 0$ be small enough. Then the iteration

$$\eta^{n+1} = \eta^n + s_0 P^{-1} Q^* (\chi - G\eta^n)$$

converges linearly to the unique η such that $G\eta = \chi$.

This generalization suggests a method of the form

$$\eta^n = \eta^{n-1} + s_0 P^{-1} (\mathcal{H}_\Gamma^\varphi)^* (0 - S\eta^{n-1}),$$

where P is a coercive and symmetric operator. The obvious choice is $P : Z \to Z^* : \mu \mapsto (\mu, \cdot)_Z$, but, while this method converges according to Theorem 5.1.2, it is unclear how to implement it. Instead, we consider the method corresponding to the choice $P = P_1$, where $P_1 : Z \to Z^*$ is the operator defined as

$$\langle P_1 \eta, \mu \rangle = \langle \hat{A} \hat{F} \eta, R_2 \mu \rangle.$$

Here, $\hat{A} : W_2 \to W_2^*$ is

$$\langle \hat{A} u, v \rangle = \int_{\mathbb{R}} \int_{\Omega_2} \partial_+^{1/2} u \partial_+^{1/2} v + \nabla u \cdot \nabla v \, \mathrm{d}x \mathrm{d}t$$

and $\hat{F}$ is the solution operator of the nonhomogeneous problem to find $u_2 \in W_2^0$ such that

$$\begin{cases} \langle \hat{A} u_2, v \rangle = 0 & \text{for all } v \in W_2^0, \\ T_2 u_2 = \eta. \end{cases}$$

In other words, P_1 is the Steklov–Poincaré operator corresponding to $\hat{A}$.

Remark 5.3.1. *The operator $\hat{A}$ can be interpreted as the weak formulation of the operator $\sqrt{-\partial_{tt}} - \Delta$. This can be seen by using Parseval's formula as follows*

$$\int_{\mathbb{R}} \partial_+^{1/2} u \partial_+^{1/2} v \, \mathrm{d}t = \int_{\mathbb{R}} \sqrt{i\xi} \hat{u} \overline{\sqrt{i\xi} \hat{v}} \, \mathrm{d}\xi = \int_{\mathbb{R}} \sqrt{\xi^2} \hat{u} \bar{\hat{v}} \, \mathrm{d}\xi =: \int_{\mathbb{R}} \sqrt{-\partial_{tt}} u v \, \mathrm{d}t.$$

Another option is to choose a Steklov–Poincaré operator corresponding to the Laplace equation and add a half derivative. That is, the choice $P = P_2 : Z \to Z^*$ with

$$\langle P_2\eta, \mu\rangle = \int_{\mathbb{R}}\int_{\Gamma} \partial_+^{1/4}\eta\partial_+^{1/4}\mu \,\mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}}\int_{\Omega_2} \nabla H_2\eta \cdot \nabla H_2\mu \,\mathrm{d}x\mathrm{d}t.$$

Here, H_2 is the solution operator corresponding to $-\Delta$, see Chapter 3, and $\partial_+^{1/4}$ is defined analogously to $\partial_+^{1/2}$.

For the sake of completeness, we now give the weak formulations of these methods. The first method (MDN1) is to find $(u_1^n, u_2^n, \eta^n) \in W_1 \times W_2 \times Z$ such that

$$\begin{cases} \langle A_i u_i^n, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in W_i^0, \\ T_i u_i^n = \eta^{n-1} & \text{for } i = 1, 2, \\ \langle \hat{A} w_2^n, v_2\rangle = 0 & \text{for all } v_2 \in W_2^0, \\ \langle \hat{A} w_2^n, R_2\mu\rangle = \langle -A_1 u_1^n + f_1, R_1\mu\rangle & \\ \qquad\qquad + \langle -A_2 u_2^n + f_2, R_2\mu\rangle & \text{for all } \mu \in Z, \\ \eta^n = \eta^{n-1} + s_0 T_2 w_2^n & \text{on } \Gamma \end{cases} \tag{5.13}$$

for $n = 1, 2, \ldots$ The second method (MDN2) is to find $(u_1^n, u_2^n, \eta^n) \in W_1 \times W_2 \times Z$ such that

$$\begin{cases} \langle A_i u_i^n, v_i\rangle = \langle f_i, v_i\rangle & \text{for all } v_i \in W_i^0, \\ T_i u_i^n = \eta^{n-1} & \text{for } i = 1, 2, \\ \langle \hat{A} w_2^n, v_2\rangle = 0 & \text{for all } v_2 \in W_2^0, \\ \langle \hat{A} w_2^n, R_2\mu\rangle = \langle -A_1 u_1^n + f_1, R_1\mu\rangle & \\ \qquad\qquad + \langle -A_2 u_2^n + f_2, R_2\mu\rangle & \text{for all } \mu \in Z, \\ \eta^n = \eta^{n-1} + s_0 T_2 w_2^n & \text{on } \Gamma \end{cases} \tag{5.14}$$

for $n = 1, 2, \ldots$ Note that the only difference between (5.13) and (5.14) is the meaning of the operator $\hat{A}$. The following convergence result holds for our modified Dirichlet–Neumann methods.

Theorem 5.3.3. *Suppose that Assumptions 2.4.2 and 5.3.1 hold. Let $s_0 > 0$ and $\varphi > 0$ be small enough, u_i^n denote the iterates of either of the modified Dirichlet–Neumann methods (5.13) and (5.14), and u_i denote the solution to (5.11). Then (u_1^n, u_2^n) converges linearly to (u_1, u_2) in $W_1 \times W_2$.*

The convergence of the Robin–Robin method follows as in the nonlinear elliptic case under the following assumption.

Assumption 5.3.2. *Let $u \in W$ denote the solution to (5.10). Then the functionals*

$$\mu \mapsto \left\langle A_i\left(u|_{\Omega_i \times \mathbb{R}}\right) - f_i, R_i \mu \right\rangle, \quad i = 1, 2,$$

are elements in $L^2(\Gamma \times \mathbb{R})^*$.

Theorem 5.3.4. *Suppose that Assumptions 2.4.2, 5.3.1 and 5.3.2 hold. Let $s_3 > 0$, u_i^n denote the iterates of the nonlinear space-time Robin–Robin method (5.12), and (u_1, u_2) denote the solution to (5.11). Then*

$$\|u_i^n - u_i\|_{L^2(\mathbb{R}, V_i)} \to 0, \quad i = 1, 2,$$

as n tends to infinity.

5.4 Discrete linear space-time domain decomposition

A common method to discretize parabolic equations is to treat the spatial and temporal space differently, e.g., using a finite element method for the spatial discretization and a time-stepping method for the temporal discretization, see [69]. However, when using a variational space-time framework it is perhaps more natural to employ space-time finite element methods, see [66] for an introduction. This also means that our convergence results in the continuous case can be directly applied to the discrete case, under some conditions on the discrete space. Space-time finite element methods result in large sparse systems that can be solved using iterative methods [97, 98] or direct methods [67]. The latter uses a variant of the Hilbert transform to achieve a stable method.

Since we are not aware of any efficient and provably convergent space-time finite element method for nonlinear equations, we restrict our discussion to the case of linear parabolic equations on $\mathbb{R}$. We assume that our spatial discretization satisfies Assumption 3.3.1. We will construct a space-time discretization as the tensor space $W^h = U_N^\tau \tilde{\otimes} V^h$. Moreover, our decomposed spaces will be defined as

$$W_i^h = U_N^\tau \tilde{\otimes} V_i^h, \quad W_i^{h,0} = U_N^\tau \tilde{\otimes} V_i^{h,0}, \quad \text{and } Z^h = U_N^\tau \tilde{\otimes} \Lambda^h.$$

Here, V_i^h is the spatial discretization space as in Section 3.3 and U_N^τ is our temporal discretization space. The difficulty with constructing a temporal discretization space is that in order to show convergence of our discrete modified Dirichlet–Neumann methods, we must have that the Hilbert transform is an isomorphism on the discrete space. Moreover, the Hilbert transform must be easily implemented on this space, or approximated by something that is easily implemented. Since the goal is mainly to study the convergence of our domain decomposition methods we do not wish to

introduce another approximation and we therefore seek a discrete space for which the Hilbert transform can be computed analytically. Such a finite element space has been introduced in [26] and we give a brief explanation of the method here.

Let $N > 0$ and $\tau > 0$ and consider the interval $(-N\tau, N\tau)$ discretized with the points $\omega_j = j\tau$, $j = -N, \dots, N$. We consider the basis elements $\{\psi_j\}_{j=-N}^{N}$ defined through their Fourier transforms, which are defined as the functions $\{\mathcal{F}\psi_j\}_{j=-N}^{N}$ that satisfy

$$\mathcal{F}\psi_j(\omega_\ell) = \begin{cases} 1 & \text{if } j = \ell \text{ or } j = -\ell, \\ 0 & \text{otherwise.} \end{cases}$$

and are linear between the points ω_ℓ. To this set of basis elements we then append their Hilbert transforms, i.e.,

$$\tilde{\psi}_j = \mathcal{H}\psi_j.$$

In total, we have $2N+2$ basis elements that span the subspace U_N^τ, which is invariant under the Hilbert transform since $\mathcal{H}^2 = -I$. It can be shown that the basis elements are

$$\begin{aligned}
\psi_0(t) &= \frac{1}{\pi t^2 \tau}\big(1 - \cos(t\tau)\big), \\
\tilde{\psi}_0(t) &= \frac{t\tau - \sin(t\tau)}{\pi t^2 \tau}, \\
\psi_j(t) &= \frac{2}{\pi t^2 \tau}\big(1 - \cos(t\tau)\big)\cos(tj\tau), \quad j = 1, \dots, N, \\
\tilde{\psi}_j(t) &= \frac{2}{\pi t^2 \tau}\big(1 - \cos(t\tau)\big)\sin(tj\tau), \quad j = 1, \dots, N.
\end{aligned}$$

These functions are not localized in time, but since they are localized in Fourier space, we can assemble our matrices using Parseval's formula. Moreover, the involved functions are simple enough that the integrals can be computed exactly.

Remark 5.4.1. *It is possible to define the discrete space-time Steklov–Poincaré operators for other discretizations than the one shown here. The issue is that, if we use the more standard space-time finite element methods, we have different test and trial spaces and a more general analysis than the one presented in Section 5.1 must be used.*

With our discrete spaces defined, we can introduce discrete variants of the weak equation, transmission problem, and Steklov–Poincaré operators, which yields discrete convergence results corresponding to our results in Sections 5.1 and 5.3. This is discussed in more detail in [Paper IV, Section 9]. In particular, the discrete variants of the modified Dirichlet–Neumann methods have convergence factors that are independent of $h, \tau, N > 0$.

5.5 Numerical results for the heat equation

We consider the discrete and exact errors e_d and e_e, respectively, analogous to the errors defined in Section 3.6, but using the $L^2(\mathbb{R}, H^1)$-norm. We consider the heat equation on Decomposition 2.4.1 with f chosen such that the solution is given by

$$u(t,x) = \begin{cases} (e^{-t/2} - e^{-t})(x^2 - x^3) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

We plot the errors of the modified Dirichlet–Neumann methods and the Robin–Robin method against the number of iterations in Figure 5.2. The method parameters are $s_0 = 0.4, \varphi = 0.02\pi$ for MDN1, $s_0 = 0.5, \varphi = 0.02\pi$ for MDN2, and $s_3 = 4$ for the Robin–Robin method. The mesh parameters are $h = 1/512$, $N = 256$, and $\tau = 0.4$. The initial guesses are always taken to be zero. We find that all three methods converge and the Robin–Robin method converges significantly faster than the modified Dirichlet–Neumann methods.

For each of our method we perform the parameter study as described in Section 3.7. We use the mesh parameters $(h_1, h_2, h_3) = (1/512, 1/256, 1/128)$ together with $(N_1, N_2, N_3) = (256, 128, 64)$. The experimental convergence orders can be found in Figure 5.3. We find that the modified Dirihlet–Neumann methods have mesh independent convergence factors for small $s_0 > 0$ and the optimal parameters are approximately $s_0 = 0.5$ and $s_0 = 0.4$ for MDN1 and MDN2, respectively. We also see that the Robin–Robin method has convergence factors that depend on the discretization parameter. Note that this is true even though it is a (spatially) one-dimensional decomposition, likely due to the fact that the temporal discretization is also different for the different meshes.

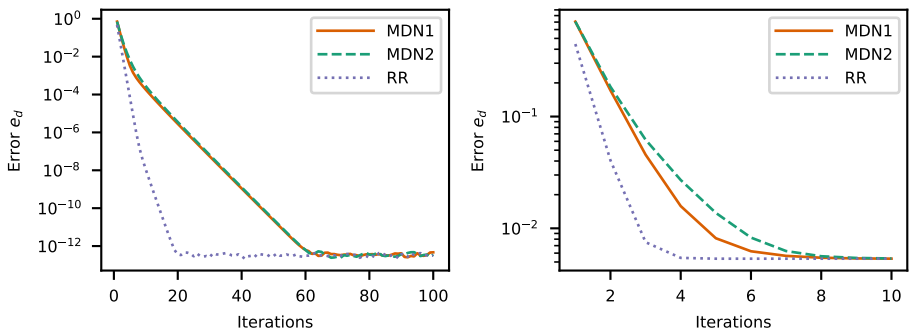

Figure 5.2: The errors e_d (left) and e_e (right) of the modified Dirichlet–Neumann methods (MDN1 and MDN2) and the Robin–Robin (RR) method applied to the heat equation on Decomposition 2.4.1.

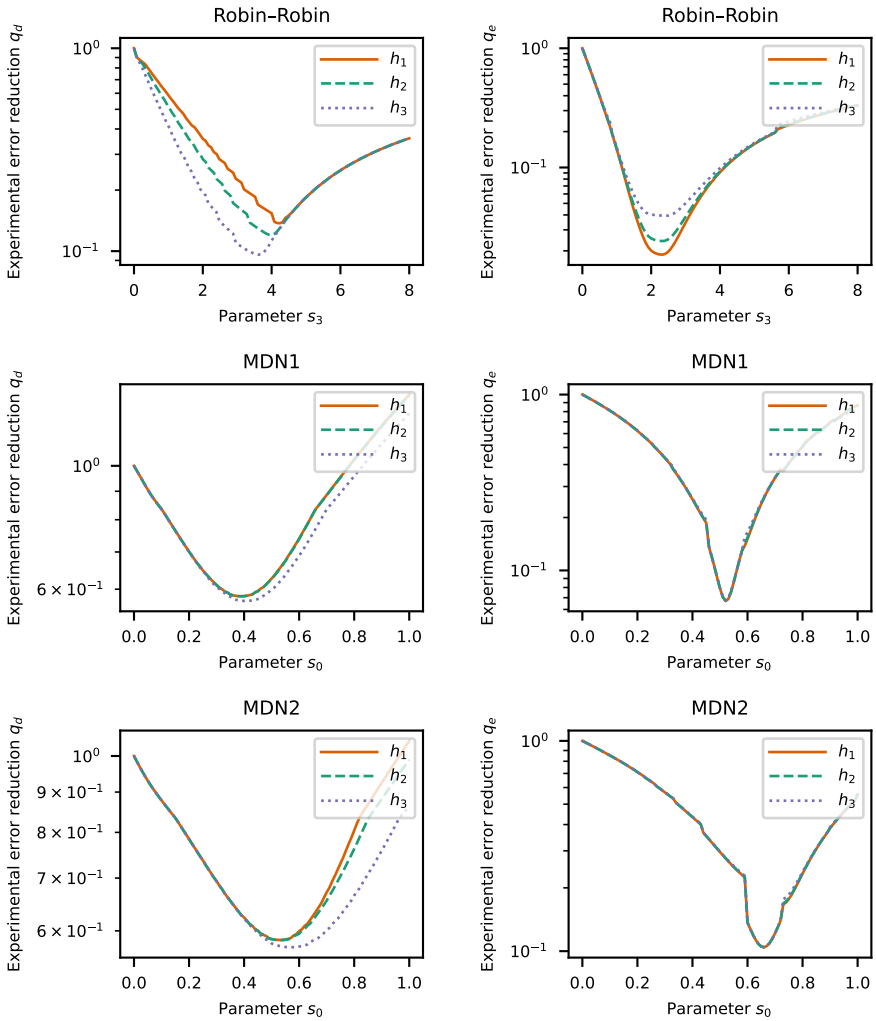

Figure 5.3: The experimental error reductions q_d (left) and q_e (right) plotted against the method parameter for the Robin–Robin method (top), the MDN1 method (middle), and the MDN2 method (bottom) applied to the heat equation. The decomposition is as in Decomposition 2.4.1 and the three different mesh parameters are $(h_1 = 1/512, h_2 = 1/256, h_3 = 1/128)$.

Chapter 6

Outlook

6.1 Summary and conclusions

In this thesis we have studied domain decomposition methods for nonlinear elliptic and parabolic problems. In both cases, we have constructed Steklov–Poincaré operators and studied the properties of these operators. We have then used this theory to prove convergence of some standard domain decomposition methods. Moreover, this theory has been used to construct convergent methods when the standard methods do not, or can not be proven to, converge. We have also discussed the effect of the discretization on the speed of convergence and found that, with the exception of the Robin–Robin method, the convergence is independent of this.

Furthermore, we compared the convergence speed of the methods numerically and demonstrated that, other than the Neumann–Neumann method, these are viable options for parallelization of numerical solvers for nonlinear partial differential equations.

There are still many questions left to study, both analytically and numerically. We therefore briefly discuss some ideas for future work.

6.2 Parallel implementations

We are still lacking proper parallel implementations and numerical results for our domain decomposition methods applied to decompositions with more than two subdomains. It is also interesting to study how the methods scale, that is, how the con-

vergence factors depend on the number of subdomains. Note that one-level methods, such as the ones presented here, typically have convergence factors that depend heavily on the number of subdomains [16]. It is therefore important to use global coarse solvers, see, e.g. [106, Chapter 3], but it unclear how to analyze these in a continuous setting.

6.3 Continuous analysis with cross points

In this thesis we did not consider decompositions with cross points such as in Figure 2.2d. These decompositions are important since they allow for further parallelization of the numerical solver.

There are numerous results concerning domain decompositions with cross points for discrete equations when the partial differential equation has been discretized using finite elements, see e.g. [106, Chapter 5]. In the continuous case, however, we are only aware of a few results. First, transmission problems can be formulated and are equivalent to the weak formulation of the equation. This holds for a large class of nonlinear elliptic equations [94]. However, it has been shown that in certain cases the Dirichlet–Neumann method is not well posed in the presence of cross points [19]. The same has also been shown for the Neumann–Neumann method [18], but can be resolved by using a specific coarse space correction that can be formulated at the continuous level. Cross points have also been studied in the context of nonlinear equations in [10].

6.4 Space-time finite elements for nonlinear parabolic equations

In Section 5.3 we showed that much of the theory of domain decomposition methods can be generalized to nonlinear parabolic problems. However, since there is a lack of theoretical results for nonlinear space-time finite element methods it is still unclear how to discretize these problems using variational methods, which means that it is difficult to construct numerical examples that demonstrate the theory.

6.5 Robin–Robin method for degenerate parabolic equations

In Section 5.3 we assumed that the equation was nondegenerate, but we know from Section 4.5 that the Robin–Robin method can be proven to converge for degener-

ate problems as well. However, even though degenerate parabolic equations with $H^{1/2}$-temporal regularity has been studied in [40], the theoretical tools for this is still lacking. One of the main difficulties remaining is to show that there is a trace operator

$$\begin{aligned} T : H^{1/2}\big(\mathbb{R}, L^p(U)\big) \cap L^2\big(\mathbb{R}, W^{1,p}(U)\big) \\ \to H^{1/4}\big(\mathbb{R}, L^p(\partial U)\big) \cap L^2\big(\mathbb{R}, W^{1-1/p,p}(\partial U)\big) \end{aligned}$$

for Lipschitz domains U. Moreover, since the spaces involved are not Hilbert spaces, one can not use the equivalence to tensor Hilbert spaces that is employed heavily in Paper IV.

6.6 Domain decomposition for initial value problems on bounded time intervals

Although infinite time intervals can be seen as more general than bounded time intervals, there are some questions that are interesting in the case of bounded time intervals. For instance, it is well known that in certain cases the convergence factor is dependent on the size of the time interval, see [46, Theorems 2.2 and 2.3].

Variational formulations with $H^{1/2}$-temporal regularity on finite intervals have been studied in the context of space-time finite element methods in [99]. Their analytical tools are similar to ours, utilizing a discrete Hilbert transform.

6.7 Domain decomposition for moving domains

A natural extension of this theory is to equations on domains that are moving or decompositions that are moving. A good starting point is [14], which generalizes the results in [24] to moving domains. A more general framework for analyzing partial differential equations on moving domains can be found in [4]. It is also possible to consider domains and surfaces where the evolution depends on an equation [62].

This also serves as a first step towards a rigorous theory of fluid-structure interaction, see [92] for an overview. Steklov–Poincaré operators have been introduced for these problems in [3, 28] after the problem has been discretized by a timestepping method, but no space-time analysis exists to our knowledge.

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Scientific publications

Author contributions

Paper I: Convergence Analysis of the Nonoverlapping Robin–Robin Method for Nonlinear Elliptic Equations

Esil Hansen and I did the analysis together and I wrote most of section 3.

Paper II: Modified Neumann–Neumann methods for semi- and quasilinear elliptic equations

I did most of the analysis and Eskil Hansen and I wrote the paper together. In particular, I extended the analysis from Lipschitz equations to locally Lipschitz equations. I also performed the numerical experiments.

Paper III: Convergence of the Dirichlet-Neumann method for semilinear elliptic equations

I am the sole contributor to the paper.

Paper IV: Linearly convergent nonoverlapping domain decomposition methods for quasilinear parabolic equations

Eskil Hansen and I did the analysis and wrote the paper together. I performed the numerical experiments.

Paper V: An abstract approach to the Robin–Robin method

Eskil Hansen and I did the analysis together and I wrote the paper.