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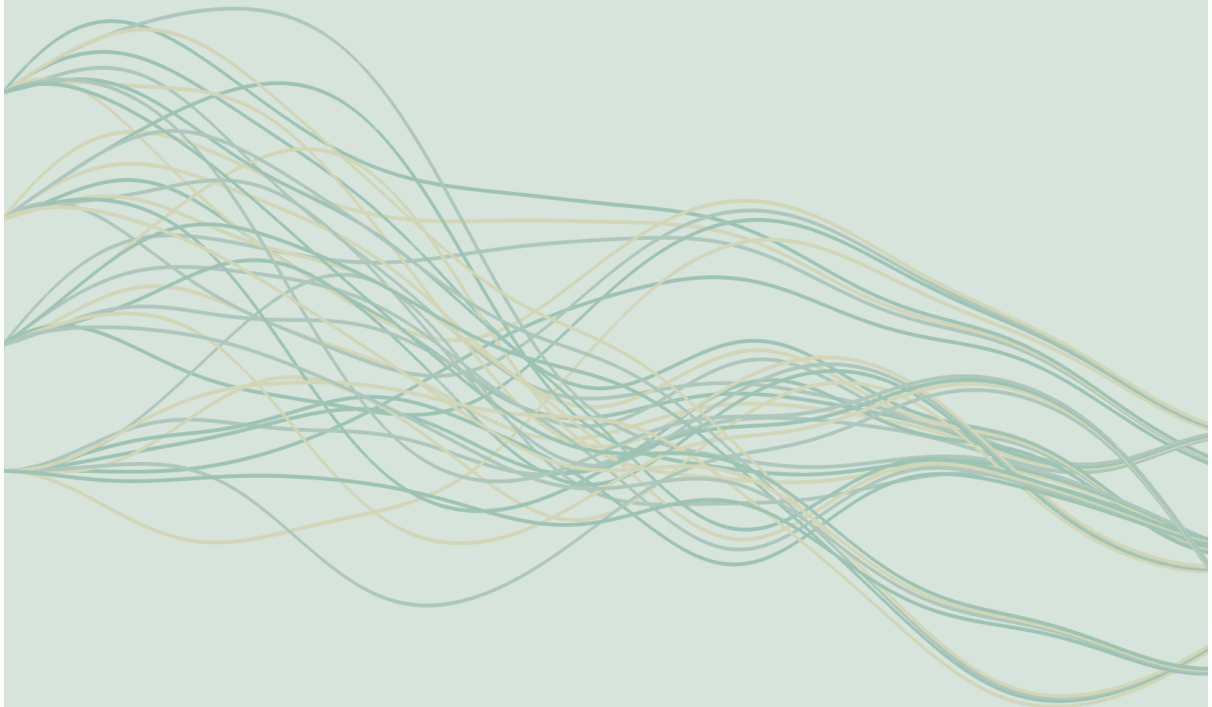

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Henrik Bengtsson

LUND
UNIVERSITY

DOCTORAL DISSERTATION

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Abstract
The excursion distribution of stochastic processes is important in many applications and is an important object in probability theory. The excursion distributions are defined as the time between a u-level up-crossing and the following down-crossing of the same level. While the notion of the excursion distribution might be intuitive, the general problem of describing it is not. Finding the excursion distribution continues to pose a problem and has been doing so during the last century. Due to the need to describe the excursion distribution in applications, several approximation methods have been developed.

One such method is the independent interval approximation. This method uses a stationary alternating renewal process to approximate the clipped process. The clipped process is computed by taking the sign of the process we are interested in. Which is usually a sufficiently smooth stationary Gaussian process. Then the interval distribution is found by matching the characteristics of the clipped processes to those of the stationary alternating renewal process.

In this thesis, a new approximation method is introduced, which is similar to the independent interval approximation. However, this method is based on the Slepian process, which models the behavior of a smooth stationary Gaussian process at the moment of an up-crossing of level u. The Slepian process is not stationary. Therefore, a non-stationary alternating renewal process is used to approximate the clipped Slepian process. This is the main idea behind the Slepian-based independent interval approximation. Paper II introduces this for zero-level excursion, and in Paper IV, it is extended to more general crossing levels. These two papers treat one of the two major topics within this thesis.

The second major topic of this thesis concerns the characteristics of alternating renewal processes. In particular, when the interval distributions can be recovered from characterizing functions such as the covariance or expected value functions. In Paper I, the symmetric case is treated, and it is shown that this is possible under monotonicity conditions on the expected value function. If this condition is satisfied, the interval distribution will be geometrically divisible. The asymmetric case is treated in Paper III, and similar conditions are derived.

The results concerning the recoverability of the interval distribution in Papers I and III have practical implications for the Slepian-based independent interval approximation. They ensure that the obtained approximation of the excursion distribution is a valid probability distribution, which is not obvious for the ordinary independent interval approximation.

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In dedication to my family, without you, this would not have been possible.

Contents

Acknowledgements

A mathematician is a device for turning coffee into theorems

—Alfréd Rényi

It is a great privilege to be able to spend one's time trying to understand the very nature of things. During the last several years, I have had this privilege ever since I set foot at the Department of Statistics as a lowly teaching assistant. The journey has not always been easy, especially when it felt like the world was trembling underneath me. Despite this, there have been many moments of absolute delight. Many of these moments have been with the people and friends I met during this journey.

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Henrik Bengtsson
27th of January, 2025

Abstract

The excursion distribution of stochastic processes is important in many applications and is an important object in probability theory. The excursion distributions are defined as the time between a u-level up-crossing and the following down-crossing of the same level. While the notion of the excursion distribution might be intuitive, the general problem of describing it is not. Finding the excursion distribution continues to pose a problem and has been doing so during the last century. Due to the need to describe the excursion distribution in applications, several approximation methods have been developed.

One such method is the independent interval approximation. This method uses a stationary alternating renewal process to approximate the clipped process. The clipped process is computed by taking the sign of the process we are interested in. Which is usually a sufficiently smooth stationary Gaussian process. Then the interval distribution is found by matching the characteristics of the clipped processes to those of the stationary alternating renewal process.

In this thesis, a new approximation method is introduced, which is similar to the independent interval approximation. However, this method is based on the Slepian process, which models the behavior of a smooth stationary Gaussian process at the moment of an up-crossing of level u. The Slepian process is not stationary. Therefore, a non-stationary alternating renewal process is used to approximate the clipped Slepian process. This is the main idea behind the Slepian-based independent interval approximation. Paper II introduces this for zero-level excursion, and in Paper IV, it is extended to more general crossing levels. These two papers treat one of the two major topics within this thesis.

The second major topic of this thesis concerns the characteristics of alternating renewal processes. In particular, when the interval distributions can be recovered from characterizing functions such as the covariance or expected value functions. In Paper I, the symmetric case is treated, and it is shown that this is possible under monotonicity conditions on the expected value function. If this condition is satisfied, the interval

distribution will be geometrically divisible. The asymmetric case is treated in Paper III, and similar conditions are derived.

The results concerning the recoverability of the interval distribution in Papers I and III have practical implications for the Slepian-based independent interval approximation. They ensure that the obtained approximation of the excursion distribution is a valid probability distribution, which is not obvious for the ordinary independent interval approximation.

Popular Science Summary

Suppose that you observe the amount of water that flows through a river over time. In many situations, it is of interest to know, for example, how long you will observe this amount of water increasing. This might give time to prepare for extreme events such as floods. Such a problem can be phrased as a level crossing problem for a random process, in this case, for the change or derivative of the amount of water flowing in the river.

Similar problems are ubiquitous in many fields, such as engineering, physics, and economics. Understanding how the random time between two level crossings is therefore important. However, describing these mathematically has proven to be a challenge for almost a century. As a consequence, approximation methods have been developed.

In this thesis, we introduce a new approximation method. This method is based on shifting the perspective from observing the random times from a distance to observing the level crossings from where they occur in time. By doing this, some problems that are in other approximation methods can be addressed in such a way that our approximation satisfies some mathematical conditions, for it has to be a reasonable approximation.

This is one of two major topics in this thesis. The second topic is the study of random processes that can only take two values, one and minus one. Two building blocks are used to construct these random processes. There are functions that describe these random processes.

Suppose that you only have the functions; in most situations, it is not clear that you can get back the building blocks. In this thesis, conditions are found such that when satisfied, allow us to find the building blocks from only these functions. It is important to find such conditions since they allow us to ensure that the previously mentioned approximation method gives reasonable approximations of the time between two-level crossings.

List of papers

This thesis is based on the following papers.

I **Characteristics of the switch process and geometric divisibility**
Henrik Bengtsson (2024). *Journal of Applied Probability*, 61(3), 802–809. doi:10.1017/jpr.2023.81

II **The Slepian model based independent interval approximation of persistency and zero-level excursion distributions**
Henrik Bengtsson and Krzysztof Podgórski (2024). Preprint, *arXiv*, doi.org/10.48550/arXiv.2401.01805. To be submitted for publication.

III **Characteristics of asymmetric switch processes with independent switching times**
Henrik Bengtsson and Krzysztof Podgórski (2024). Preprint, *arXiv*, doi.org/10.48550/arXiv.2409.05641. To be submitted for publication.

IV **Slepian model based independent interval approximation for the level excursion distributions**
Henrik Bengtsson and Krzysztof Podgórski (2024). Preprint, *arXiv*, doi.org/10.48550/arXiv.2410.06000. To be submitted for publication.

Introduction

Although our intellect always longs for clarity and certainty, our nature often finds uncertainty fascinating.

— Carl von Clausewitz

1 Background

The characterization of the excursion distribution of stochastic processes is a problem that has fascinated mathematicians and statisticians since the early 1940s. The easiest way to explain the idea of a level crossing is with an illustration. In Figure 1, a trajectory of a stationary Gaussian process can be seen, and its crossings of the level u The excursion distribution is defined as the distribution of the length between two level crossings of the same level. Hence, there are two such distributions associated with each process. One below and one above the level u.

These distributions are of interest due to their appearance in many fields, such as engineering and physics. For example, a heatwave can be defined as the excursion of temperature above some critical level. A heatwave is generally not a problem unless it lasts for a long time. Hence, there is an interest in understanding the tail behavior of the excursion distribution, since it can describe extreme events such as a long heatwave.

While the excursion distribution is an intuitive object, it has proven difficult to characterize based on the properties of the underlying process. This is why the problem has received considerable attention since the 1940s and the seminal work by Rice (1944, 1945). In the latter half of the 1980s, a mathematical formula was derived for the density of the level crossing distribution. This is the Durbin-Rychlik formula.[1] However, this formula is not explicit and is, therefore, difficult to use in practice.

[1] Named in honor after the works by Durbin (1985) and Rychlik (1987b)

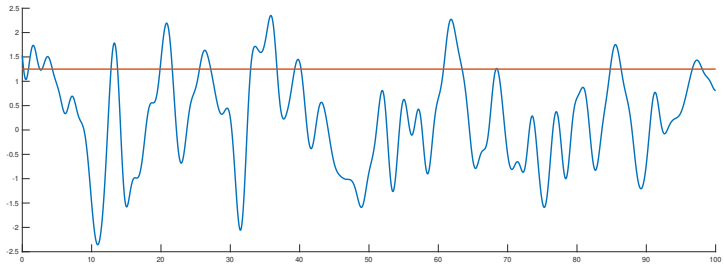

Figure 1: A trajectory of a stationary Gaussian process with the covariance function $C(t) = e^{-t^2/2}$, and a level $u = 5/4$.

Due to the difficulty and practical importance of the excursion distribution, several approximation methods have been developed. Such as methods based on Rice series,[2] and regression-based methods using the Slepian process.[3] However, in this thesis, we focus on the independent interval approximation, which traces its roots back to the work by McFadden (1956). This method has also seen a rise in popularity in physics during the last decades with the works by Majumdar et al. (1996) and Sire (2007, 2008).

This method is based on the clipped process. The clipped process is a binary stochastic process that takes the value one or minus one, depending on if the process we want to approximate is above or below u. The intervals of the clipped process will be dependent. However, if this dependence is disregarded and the intervals are treated as independent, the clipped process becomes an alternating renewal process. Hence, the alternating renewal process with independent intervals is used to approximate the clipped process. The name of the independent interval approximation is, therefore, very fitting.

To approximate the clipped process with the alternating renewal process, the two need to be connected. This is done by matching the covariance function, i.e., imposing the covariance of the clipped process onto the alternating renewal process. The interval distribution can then be written as a function of the covariance for the alternating renewal process. Thus, an approximation of the excursion distribution is found by using this relation after matching.

The issue with this approach is that there is no intuitive argument as to why this matching leads to an approximated excursion distribution that is a valid probability distribution. Not even for smooth stationary Gaussian processes. Addressing this issue

[2] See Lindgren (2019) for an overview.
[3] See Lindgren and Rychlik (1991)

is one of the main topics in this thesis.

The thesis can be divided into two main parts, each covering the main topics. The first is comprised of Papers II and IV, which introduce an alternative version of the independent interval distribution based on the Slepian process. The second part is comprised of Papers I and III. These deal with the recoverability of the interval distribution of the switch process[4] from the functions that characterize it, such as the expected value and covariance functions. This recovery of the interval distribution shows that the Slepian-based independent interval approximation is valid for a large class of Gaussian processes.

To provide a background to the subsequent papers, we cover three main topics in this introduction. We start with stationary Gaussian processes. Then move to renewal theory and the alternating renewal process. Lastly, we provide a short overview of geometric divisibility, which plays an important role in Paper I.

2 Stationary Gaussian processes and their excursions

Since we restrict ourselves to the excursion of sufficiently smooth stationary Gaussian processes, we give a brief introduction to them here. A stochastic process is a family of random variables indexed by an element belonging to some set, usually representing time. More formally let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process can be expressed as a function $X : \mathcal{T} \times \Omega \to \mathbb{R}$, such that $X(t, \cdot) : \Omega \to \mathbb{R}$ is a random variable for every $t \in \mathcal{T}$. The other argument can also be fixed. By fixing $\omega \in \Omega$, we obtain a function with t as an argument. Such a function is called a trajectory, and the theory for defining a distribution for the trajectories is well established. However, we limit ourselves to only mentioning the trajectories since they are important when defining the level crossings.

We are now ready to define a Gaussian process. If for any finite sequence of time points, $t_1, t_2, ..., t_k \in \mathcal{T}$, the vector $(X(t_1), X(t_2), ..., X(t_k))$ follows a multivariate normal distribution, we say that the stochastic process $X(t)$ is Gaussian.

With the definition of a Gaussian process in place, we define stationarity next. There are two versions of stationary, weak and strict. A process is strictly stationary if for all $\tau \in \mathbb{R}$ and time points $t_1, t_2, ..., t_k$,

$$X(t_1), X(t_2), ..., X(t_k) \stackrel{d}{=} X(t_1 + \tau), X(t_2 + \tau), ..., X(t_k + \tau).$$

For a stochastic process to be weakly stationary, two conditions need to be satisfied. The first is that the expected value is constant for all t, and the second is that the

[4] Which is an alternating renewal process with the two states one and minus one.

covariance between two points in time only depends on their distance in time. This can be expressed as

(i) $EX(t) = \mu$, for all $t \in \mathbb{R}$

(ii) $Cov(X(t), X(t+\tau)) = C(\tau)$, for all $t, \tau \in \mathbb{R}$,

where $\mu \in \mathbb{R}$ and $C(\tau)$ is a positive definite function, which we call the covariance function. We assume that the covariance function is normalized such that that $C(0) = 1$. For Gaussian processes, weak and strict stationarity coincide. This follows from the fact that the multivariate Gaussian distribution is uniquely characterized by its expected value and covariance matrix.

What remains to define is smoothness. We say that $X(t)$ is smooth if it is at least once quadratic mean differentiable. A stochastic process $X(t)$ is quadratic mean differentiable at a point t, if there exist a random variable $X'(t)$ such that

$$\lim_{h \to 0} E\left(\left(\frac{X(t+h) - X(t)}{h} - X'(t)\right)^2\right) = 0.$$

If this holds for all t we say that the process $X(t)$ is quadratic mean differentiable with the derivative $X'(t)$.

The quadratic mean derivative does not exist for all stationary Gaussian processes. However, Leadbetter et al. (1983) provides a useful condition: $X(t)$ is quadratic mean differentiable if and only if the second spectral moment is finite.[5] The second spectral moments is given by $\xi_2 = -C''(0)$. If ξ_2 is finite, the process $X'(t)$ is a zero mean stationary Gaussian process with the covariance function $C_{X'}(t) = -C''_X(t)$. It should also be noted that $(X'(t), X(t))$ are jointly normal and uncorrelated when evaluated at the same time point. This property will be used later when constructing the Slepian process.

For the remaining parts of this section, we will assume that the process $X(t)$ is a smooth stationary Gaussian process. We now move to defining some of the main objects of this thesis, namely level crossings and the excursion distribution.

2.1 Level crossings and excursions

In order to define the excursion distribution, we need to define a level crossing. In particularly a strict level crossing. Using the definition from Leadbetter et al. (1983), we say that a trajectory of a stochastic process $X(t, \omega_0)$[6], has an up-crossing of level

[5] The spectral moments are defined by $\xi_k = \int_{-\infty}^{\infty} |\xi|^k dF(\xi)$, where $F(\xi)$ is the spectral distribution function, i.e., $F(\xi)$ which satisfies: $C(t) = \int_{-\infty}^{\infty} e^{it\xi} dF(\xi)$.

[6] We write $X(t, \omega_0)$, $\omega_0 \in \Omega$ to indicate that we now are considering a trajectory.

u in the strict sense at time t_0 if there exist some $\varepsilon > 0$ such that $X(\tau, \omega_0) \leq u$ for all $\tau \in (t_0 - \varepsilon, t_0)$ and $X(\tau, \omega_0) > u$ for all $\tau \in (t_0, t_0 + \varepsilon)$. A down-crossing of the level u is defined analogously by changing the inequalities.

In Leadbetter et al. (1983), there exists a condition on the expected number of crossings of a bounded interval to ensure that the level crossings are strict. The level crossings are strict if the expected number of crossings in a bounded interval is finite. For a stationary Gaussian process the expected number of $u-$level crossings on $[0, 1]$ is given by

$$\begin{aligned} EN_u([0,1]) &= f_{X(0)}(u) E\left(|X'(0)| \,|X(0) = u\right) \\ &= \int_{-\infty}^{\infty} |\tau| f_{X(0),X'(0)}(u, \tau) d\tau. \end{aligned}$$

Where the joint density of $X(0)$ and $X'(t)$ is given by

$$f_{(X(0),X'(0))}(u, \tau) = \frac{1}{2\pi\sqrt{\xi_2}} e^{-\frac{u^2}{2} - \frac{\tau^2}{2\xi_2}}.$$

From this, we have

$$\begin{aligned} EN_u([0,1]) &= \int_{-\infty}^{\infty} |\tau| \frac{1}{2\pi\sqrt{\xi_2}} e^{-\frac{u^2}{2} - \frac{\tau^2}{2\xi_2}}(u, \tau) d\tau \\ &= \frac{1}{\pi}\sqrt{\xi_2} e^{-\frac{u^2}{2}}. \end{aligned}$$

This is the famous Rice formula for stationary Gaussian processes derived in Rice (1944, 1945). Several proofs of Rice's formula exist besides Rice's original proof; see, for example, Leadbetter et al. (1983) or Lindgren (2012).

The Rice formula holds even if ξ_2 is not finite. In this case, the process has an infinite crossing intensity. However, since we assume that $X(t)$ is quadratic mean differentiable, $\xi_2 < \infty$, and Theorem 7.3.2 in Leadbetter et al. (1983) states that the level crossings are strict.

We can now define the excursion distribution using the definition from Lindgren (1977) as the limit of the empirical distribution.[7] Let, $\tau_k, k \in \mathbb{N}$[8] be a increasing sequence of time points where $X(t), t \geq 0$ has $u-$level up-crossings, i.e., $0 < \tau_1 < \tau_2 < ..$ and let

$$\sigma_k = \inf\{t > 0 : X(\tau_k + t) < u\},$$

[7] For this, we need the additional assumption that process $X(t)$ is ergodic, which is equivalent to the spectral distribution having no atoms.

[8] To avoid confusion, let $\mathbb{N}$ be the set of positive integers $\{1, 2, 3, ...\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, i.e $\{0, 1, 2, 3, ...\}$.

i.e., the time from the up-crossing of level u at the τ_k to the subsequent down-crossing of the same level. From this, we define the excursion distribution by

$$P(\sigma < t) = \lim_{T \to \infty} \frac{\#\{\tau_k \in [0, T] : \sigma_k < t\}}{\#\{\tau_k \in [0, T]\}}.$$

The informal interpretation of this is that if we stand at a given up-crossing τ_k and measure the time until the process goes below the level u, we obtain the length of one excursion. By doing this for all the points τ_k, the excursion distribution evaluated at some t is obtained by simply counting the number of excursions with lengths less than t. Since we assume the process to be ergodic, we can find the limit of this distribution from a single trajectory.

The shift in perspective and viewing the process from the mark of an up-crossing is the essence of Palm theory, which traces its roots to the work by Palm (1943).[9] Palm theory is best associated with point processes, but it is of value when studying other stochastic processes, as we do here.

Despite the general difficulty of finding the excursion distribution, a mathematical expression exists for the density of σ. This is the Durbin-Rychlik[10] formula. Which is given for $t \geq 0$ by

$$f_\sigma(t) = f_{X(t)}(0) E\left(\mathcal{I}\left(X(\tau) < 0, \forall \tau \in (0, t)\right) \cdot \max(0, X'(t)) | X(t) = 0\right),$$

where $\mathcal{I}$ is the indication function. While giving the exact distribution, there is a problem with this formula. To evaluate the indicator function in the expected value, we need to evaluate an uncountably infinite number of random variables. This cannot be done by using the finite-dimensional distribution of the process, which limits its usefulness in practice.

In this section, we defined the level crossings. In the next section, we will look how the process behaves at such a level crossing.

2.2 The Slepian process

The Slepian process was introduced by Slepian (1963) and models the behavior of a process at the crossing instant of a typical up-crossing. Hence, it is the Palm version of the stationary Gaussian process when attached to an up-crossing.

We follow a similar construction as in Leadbetter et al. (1983). We start with the derivative at an up-crossing. Corollary 10.3.3 in Leadbetter et al. (1983) gives us the Palm

[9] The work by Khinchin (1969) should also be recognized.

[10] Derived for Gaussian process by Durbin (1985) and then generalized by Rychlik (1987b).

distribution of the derivative at a typical up-crossing, and this is given by a Rayleigh distribution, R, with the density

$$f_R(\tau) = \frac{\tau}{\xi_2} e^{-\frac{\tau^2}{2\xi_2}}$$

for $\tau \geq 0$. It should also be noted that this does not depend on the level u.

To derive the distribution of the Slepian process, we need to find its Palm distribution. This is given by Theorem 10.3.1 in Leadbetter et al. (1983). However, since we are in a Gaussian setting, the distribution can be found by considering the Gaussian vector $(X(t_1), X(t_2), X(0), X'(0))$ and conditioning on $(X(0) = u, X'(0) = \tau)$. From this, we obtain the following expected value of $X(t_1)$

$$E\left(X(t_1)|X(0) = u, X'(0) = \tau\right) = uC(t_1) - R\frac{C'(t_1)}{\xi_2}.$$

The expected value is of the same form for $X(t_2)$. The conditional covariance between $X(t_1), X(t_2)$ is given by

$$Cov\left(X(t_1), X(t_2)|X(0) = u, X'(0) = \tau\right) = C(t_1 - t_2) - C(t_1)C(t_2) - \frac{C'(t_1)C'(t_2)}{\xi_2}.$$

From these equations, we can now construct the Slepian process

$$\zeta_u(t) = uC(t) - R\frac{C'(t)}{\xi_2} + \kappa(t), \tag{1}$$

where $\kappa(t)$ is a non-stationary zero mean Gaussian process with the covariance function

$$C_\kappa(t, s) = C(t - s) - C(t)C(s) - \frac{C'(t)C'(s)}{\xi_2}.$$

The Slepian process decomposes the behavior at an up-crossing into two parts, one that depends on the level u and a residual process, $\kappa(t)$.

The Slepian process is not stationary due to conditioning on having an up-crossing at zero. Neither will it be Gaussian due to the Rayleigh random variable R and the non-stationary Gaussian process $\kappa(t)$. However, it should be noted that the effect of having an up-crossing at zero vanishes as time goes to infinity.

It should be remarked, as pointed out in Lindgren (1977), that the excursion distribution of $X(t)$ is the same as the first crossing distribution of $\zeta_u(t)$. The latter is defined by

$$\sigma = \inf\{t > 0 : \zeta_u(t) < u\}.$$

This follows from the definition of the excursion distribution and that the Slepian process is the Palm version of $X(t)$. Therefore, it is natural to use the Slepian process to approximate the excursion distribution of $X(t)$.

2.3 Independent interval approximation

Due to its importance for practical problems, several methods have been developed to approximate the excursion distribution. This section does not aim to provide an overarching overview of the existing methods. Instead, we will focus on the independent interval approximation. The roots of the method can be traced back to the work by McFadden (1956, 1958), but the methods have gained popularity in physics during the last decades. Starting with Majumdar et al. (1996), who used it to study Gaussian diffusion processes. An overview of the method can be found in Bray et al. (2013). The approximation has also been extended to non-zero-levels by Sire (2007, 2008). However, in this section, we focus on the independent interval approximation for zero-level excursions.

The method is based on the clipped process, which is defined for a stochastic process $X(t)$ by

$$Y(t) = \operatorname{sign}(X(t)).$$

The main idea of the independent interval approximation is to match the covariance of the clipped process to that of a stationary alternating renewal process. Then, the relation between the covariance function and the interval distribution for the alternating renewal process can be used to find an approximation of the excursion distribution. Figure 2 gives a schematic overview of the framework. If the process $X(t)$ is a smooth stationary Gaussian process with the covariance function $C(t)$ the covariance function of $Y(t)$ is given by

$$C_Y(t) = \frac{2}{\pi} \arcsin\left(C_X(t)\right).$$

This is the well-known arcsin relation derived by Van Vleck (1943).[11] The covariance can also be expressed through the probability that an interval $(0, t)$ contains n zero-level crossings. Let $p(n, t)$ that $(0, t)$ contains precisely n crossings. Then the covariance of the clipped process can also be written as[12]

$$C_Y(t) = EY(t)Y(0) = \sum_{n=0}^{\infty} (-1)^n p(n, t).$$

Finding the probability mass function of $p(n, t)$ is difficult due to the dependency of the excursion intervals.

[11] This technical report was declassified after the war, and for a later more accessible reference, see Van Vleck and Middleton (1966).

[12] The covariance only depends on the expected value of $Y(t)Y(0)$, which is either one or minus one depending on if the number of crossings in the interval $(0, t)$ is even or odd.

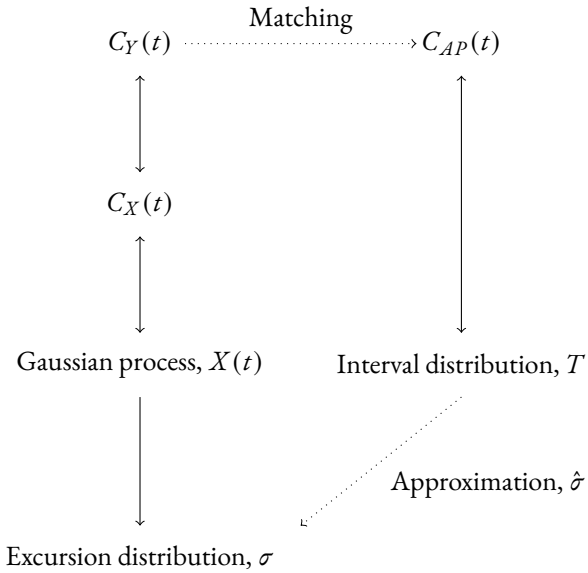

Figure 2: A schematic overview of the independent interval approximation for zero-level excursions. By moving from $X(t)$ to the interval distribution T through the matching, an approximation of the excursion distribution is obtained.

However, if we disregard the dependency between the intervals and treat them as independent, we have a stationary alternating renewal process. The covariance of this process can be written as a function of its interval distribution in the Laplace domain. Let T be the random variable associated with the intervals of the alternating renewal process, then the Laplace transform[13] of the covariance function $C_{AP}(t)$ is given by

$$\mathcal{L}(C_{AP})(s) = \frac{1}{s} - \frac{2}{s^2\mu}\frac{1-\psi_T(s)}{1+\psi_T(s)}.$$

where $\mathcal{L}(\cdot)$ denotes the Laplace transform and $\psi_T(s)$ is the Laplace transform associated

[13]The Laplace transform is defined by

$$\mathcal{L}(h)(s) = \int_0^\infty h(t)e^{-st}dt,$$

for $s \in \mathbb{C}$ for a exponentially bounded function $h(t), t \geq 0$. Let $f_X(t)$ be the density of a non-negative random variable X and $\psi_X(s) = \mathcal{L}(f_X)(s)$. For an overview of the Laplace transform, see Widder (1946) and for its use in probability theory Feller (1971).

with T. Solving this for $\psi_T(s)$ we obtain,

$$\psi_T(s) = \frac{1 - \frac{1}{2}\mu s(1 - s\mathcal{L}(C_{AP})(s))}{1 + \frac{1}{2}\mu s(1 - s\mathcal{L}(C_{AP})(s))}.$$

By substituting $C_{AP}(t)$ with $C_Y(t)$ in the equation, i.e. matching the covariance, we arrive at

$$\psi_{\hat{\sigma}}(s) = \frac{1 - \frac{1}{2}\mu s(1 - s\mathcal{L}(C_Y)(s))}{1 + \frac{1}{2}\mu s(1 - s\mathcal{L}(C_Y)(s))}. \tag{2}$$

which is thus the independent interval approximation of the excursion distribution σ of the stationary Gaussian process $X(t)$.

The dependency between the intervals of the clipped process has been investigated among others Longuet-Higgins (1962) and Rainal (1962). It was shown by Longuet-Higgins (1962) that the intervals cannot be independent for stationary Gaussian processes. The dependency between excursions has also been studied by Rychlik (1987a), Podgórski et al. (2000), and Lindgren (2019). However, it is generally difficult to implement this dependency within the independent interval approximation framework. Having a Markovian type of dependency between the intervals has been explored in McFadden (1958). However, it was shown by Rainal (1962) that imposing this dependence between the intervals is unsuitable for certain Gaussian processes.

An implicit assumption is made in the independent interval approximation by ignoring the dependency between the intervals of the clipped process. It is assumed that the covariance function $C_Y(t)$ is a valid covariance function for a stationary alternating renewal process, i.e., that there exists a stationary alternating renewal process with the covariance function $C_Y(t)$. There is no intuitive argument why it should be.

It is important to understand that if this assumption is violated, the obtained approximation $\hat{\sigma}$ might not even be a valid probability distribution. This follows from the fact that the covariance function uniquely characterizes the interval distribution of the alternating renewal process, when the intervals in each state follow the same distribution. The set of distributions that satisfies the conditions to be an interval distribution is vast. Hence, having an approximation that does not belong to this class is problematic when approximating the excursions.

There is another way to ensure that the obtained approximation is a valid probability distribution. This is to show that for a given covariance function of the clipped process $C_Y(t)$ the right-hand side of Equation (2) is completely monotone.[14] This is difficult from the definition and if it is not, then it cannot be the Laplace transform associated

[14] A function $h(s), s > 0$ is completely monotone if $(-1)^n h^{(n)}(s) \geq 0, \forall s > 0, n \in \mathbb{N}_0$.

with a probability distribution. This is a consequence of Bernstein's theorem; see Feller (1971). In Paper II, this implicit assumption is shown to be satisfied for a large class of stationary Gaussian processes. This follows from the equivalence between the Slepian-based and the ordinary independent interval approximation for the zero-level excursion distribution.

In physics, the independent interval approximation method has been heavily used to find the so-called persistency coefficient. This coefficient characterizes the tail of the excursion distribution. Under some mild conditions, the decay of the covariance function, see Slepian (1962), the tail of the excursion distribution is of the form $e^{-\theta t}$ where θ is the persistency coefficient.

These coefficients play an important role in the characterization of physical systems; see Bray et al. (2013) for an overview. They are also process-specific and generally difficult to find analytically. For example, the persistency coefficient for Gaussian diffusion in two dimensions was only derived in 2018 by Poplavskyi and Schehr (2018), and then only for the zero-level excursion distribution.

Within the independent interval framework, the persistency coefficient has been found by investigating the largest negative real pole of Equation (2) as mentioned in Sire (2007). However, for stationary Gaussian processes with oscillating covariance functions, this is unsuitable, as pointed out in Wilson and Hopcraft (2017) and Bengtsson and Podgórski (2024).

The main focus here is on the independent interval approximation. However, it should be mentioned that there are other ways to approximate the excursion distribution. For example, using a regression approach based on the Slepian process as in Lindgren and Rychlik (1991). Rice series has also been used to approximate the excursion, and a good overview of several different methods can be found in Lindgren (2019). Lastly, it should be mentioned that numerical methods have also been used to find the persistency coefficient; see Lindgren et al. (2022).

3 Renewal and regenerative processes

Processes that stochastically reset have long been a topic of interest in probability theory. Early work was done by Erlang (1909), who investigated the arrivals of calls. During the 1940s and 1950s, substantial contributions were made to the field of renewal theory. The contribution and impact of the early work done in renewal theory are summarized Smith (1958) and Cox (1967).

In the mid-1950s Smith (1955) introduced the regenerative process. Many well-known processes are regenerative, such as Markov chains and the alternating renewal process. Several more examples are given in the overview by Sigman and Wolff (1993) and Thorisson (2000). An important property of regenerative processes is that they

can be split into a sequence of cycles. The length of these are random and at the end of each cycle, the process stochastically resets.

Consider, for example, a Markov chain in continuous-time with a predefined reference state. A cycle, in this case, is the process between exiting the reference state and reentering it. Upon which it stochastically resets due to the Markov property.

We are now ready for the definition of a regenerative process.[15]

Definition 3.1. A stochastic process $X(t)$, $t \geq 0$ is called regenerative if there is a random variable $L_1 > 0$ such that the following two conditions are satisfied

i) $\{X(t + L_1) : t \geq 0\}$, is independent of $\{\{X(t) : t < L_1\}, L_1\}$

ii) $\{X(t + L_1) : t \geq 0\}$, is stochastically equivalent to $\{X(t) : t \geq 0\}$

As pointed out in Sigman and Wolff (1993), and proven in Sigman et al. (1994), if there exists a random variable L_1, then there exists an entire sequence $L_1, L_2, ..$ such that $X(t)$ regenerates at these times.

Before moving to the counting process, there are two important observations. The first is that the process can be delayed. In this case, the process will not have the same first cycle, but subsequent cycles will be stochastically equivalent. The second is that for a regenerative process $X(t)$, there is an embedded renewal counting process. This counting process simply counts the number of regenerations.

3.1 The counting process

To construct a renewal counting process, we need a sequence of random variables representing the times between the arrivals of some events. Let T_n, $n \in \mathbb{N}$ be a sequence of independent identically distributed (iid) non-negative random variables representing times between these events. We will assume, unless otherwise stated, that T_n are absolutely continuous with respect to the Lebesgue measure with support on $\mathbb{R}^+$, have a finite second moment, and $P(T_n > 0) = 1$, i.e., we have no atom at zero. These assumptions exclude the cases when the distribution is on a lattice. Under these assumptions we define the following sum

$$S_n = \sum_{k=1}^{n} T_k.$$

This sum is the random time until the nth arrival. It is also possible to shift perspective and count the number of arrivals in some interval, usually $(0, t]$. From this, we can

[15] We use the definition of classical regeneration here, as defined in Sigman and Wolff (1993).

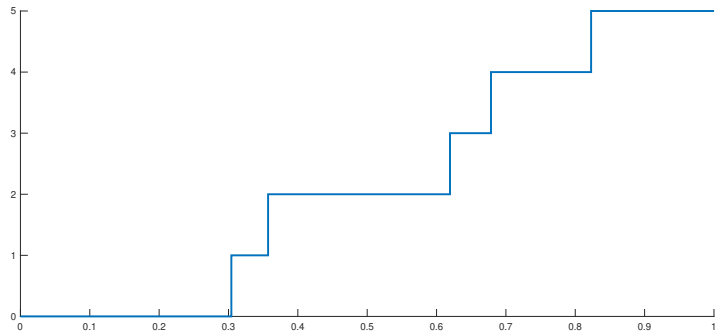

Figure 3: A trajectory of counting process with $T \sim Exp(4)$.

define the counting process. Let S_n be the sum defined above, then the counting process is defined for $t \geq 0$ by

$$N(t) = \sum_{n=1}^{\infty} \mathbb{I}(S_n < t).$$

where $\mathbb{I}(\cdot)$ is the indicator function. A realization of $N(t)$ can seen in Figure 3.

For a fixed t, $N(t)$ is a random variable, and its distribution can be found by exploiting the following equivalents between the two events

$$N(t) < n \Leftrightarrow S_n > t.$$

From this, the distribution function of $N(t)$ is given by

$$P(N(t) < n) = P(S_n > t) = 1 - F_{S_n}(t).$$

As a result, the probability mass function of $N(t)$ can be derived, which gives us

$$\begin{aligned} P(N(t) = n) &= P(N(t) < n+1) - P(N(t) < n) \\ &= F_{S_n}(t) - F_{S_{n+1}}(t). \end{aligned}$$

Hence, the distribution of $N(t)$ is completely characterized by $F_{S_n}(t)$.

There are different versions of the counting process that can be constructed by altering the properties of the sequence T_n. If T_n is an iid sequence that satisfies the previously mentioned assumption, we have an ordinary counting process.[16]

[16] We follow the naming convention in Cox (1967).

Besides $F_{S_n}(t)$, there is another function that describes the ordinary counting process. This is the renewal function, which is denoted by $H(t)$ and defined as

$$H(t) = EN(t),$$

and is related to the distribution of S_n by

$$H(t) = \sum_{n=1}^{\infty} P(S_n < t) = \sum_{n=1}^{\infty} F_{S_n}(t).$$

There is an intuitive interpretation of $H(t)$ as simply the expected number of arrivals on the interval $(0, t]$. This, however, is the distribution function of the renewal measure, and it should be noted that $H(t)$ is unbounded when t goes to infinity. It should also be noted that if the density of F_{T_n} exists, then the renewal density $H'(t)$ exists. This follows from Proposition 2.7 in Asmussen (2003).

Besides the distribution of the sequence T_n, $H(t)$ gives an alternative way to characterize the counting process. This is via the renewal function that was introduced by Feller (1941) during his study of Volterra-type equations on the form

$$U(t) = G(t) + \int_0^t U(t - \tau) dF(\tau).$$

where $G(t)$ and $F(t)$ are non-decreasing functions. Feller (1941) showed that if $F(t)$ and $G(t)$ are finite and right continuous, then there exists only one finite and non-decreasing function $U(t)$ for $t > 0$. It turns out that if $F(t) = G(t) = F_{T_n}(t)$, then the Laplace transform of $H(t)$ and $U(t)$ coincide. Hence, the renewal function is a solution to this integral equation, which can be written as

$$H(t) = F(t) + \int_0^t H(t - \tau) dF(\tau),$$

which is now more commonly known as the renewal equation.

In Feller (1941), several asymptotic properties were also derived for $H(t)$. We summarize some of the most important properties below for the ordinary counting process $N(t)$. For $\mu = ET_n$ and $\tau > 0$, we have following three limits

$$\begin{aligned}
&i) \quad \lim_{t\to\infty} \frac{H(t)}{t} = \frac{1}{\mu}, \\
&ii) \quad \lim_{t\to\infty} H(t + \tau) - H(t) = \frac{\tau}{\mu}. \\
&iii) \quad \frac{N(t)}{t} \xrightarrow{p} \frac{1}{\mu}, \text{ for } t \to \infty,
\end{aligned}$$

where $\xrightarrow{p}$ denotes convergence in probability. The first limit is the elementary renewal theorem; see Brémaud (2020) for more details. The second limit is known as Blackwell's theorem and was derived by Blackwell (1948). It should be remarked that, besides Blackwell's original proof, ε-coupling methods have been used by Thorisson (1987) to prove a more general version of Blackwell's theorem. The last limit is a law of large number type result for $N(t)$.

The last of the standard theorems in renewal theory is the Key renewal theorem. Under the previous assumption and notation, the key renewal theorem states that for an ordinary counting process $N(t)$ and a directly Riemann integrable function $g(t)$, we have

$$\lim_{t\to\infty} \int_0^t g(t-\tau)dH(\tau) = \frac{1}{\mu}\int_0^\infty g(\tau)d\tau.$$

The Key renewal theorem and Blackwell's theorem are equivalent. The interpretation of both is that the renewals even out when time passes to infinity, and thus, the renewal measure behaves as the Lebesgue measure scaled by the constant μ^{-1}.

We mentioned previously that other versions of the counting process exist besides the ordinary one. Such processes can be obtained by altering the properties of the sequence T_n. By letting T_1 have a different distribution, an equilibrium version of the ordinary counting process can be constructed. For this, we need the distribution of the forward and backward recurrence times. Fix t such that $S_n < t < S_{n+1}$ and define

$$\begin{aligned} U(t) &= t - S_n, \\ V(t) &= S_{n+1} - t. \end{aligned}$$

The backward recurrence time $U(t)$ is the time from the last arrival until t, and the forward recurrence time is the time from t until the next arrival. The distributions of $U(t)$ and $V(t)$ are well known and are given by

$$f_V(t) = f_U(t) = \frac{1-F(t)}{\mu},$$

for $t \geq 0$ and where $F(t)$ is the distribution function of T_n. The forward and backward recurrence distributions lead to an important observation. Fixing t in some arbitrary interval, far away from the origin, leads to a bias of the length of the interval which t is in. This is known as the inspection paradox.

The forward recurrence distribution can be used to construct an equilibrium version of the counting process. Let T_n be the same iid sequence as previously but change T_1 to have the density for $t \geq 0$

$$f_{T_1}(t) = \frac{1-F(t)}{\mu},$$

and denote this altered sequence by $T_n^{(e)}$, $n \geq 1$. Then, the equilibrium counting process $N_e(t)$ can be constructed in the same way as the ordinary counting process from the sum $S_n^{(e)}$ by letting

$$N_e(t) = \sum_{n=1}^{\infty} \mathbb{I}(S_n^{(e)} < t)$$

Altering the distribution of T_1 is important since this initializes the counting process in its equilibrium state. This means that the distribution of the number of arrivals in a fixed-length interval will not change with t, only the size of the interval. This is generally not the case with the ordinary counting process.

3.2 Alternating renewal processes

The alternating renewal process can be seen as a generalization of a continuous-time Markov chain with two states. In this section, we let these two states be one and minus one, making it consistent with the switch processes treated in papers I and III. It is also an example of a regenerative process if one of the two states is defined as the reference state, which we here set to one.

Under the same assumption in the previous subsection, let T_+ and T_- be two non-negative mutually independent random variables with the distribution functions $F_+(t)$ and $F_-(t)$, representing the time spent in each state. From these, we define

$$T_n \overset{d}{=} \begin{cases} T_+, & n \text{ is odd}, \\ T_-, & n \text{ is even}, \end{cases}$$

for $n \in \mathbb{N}$, we will denote the associated Laplace transforms [17] by $\psi_+(s)$ and $\psi_-(s)$. Let $T_n^{(a)}$ be an independent sequence of T_n. From this sequence, we first construct the sum

$$S_k^{(a)} = \sum_{n=1}^{k} T_n^{(a)},$$

which in turn defines the alternating counting process

$$N_a(t) = \sum_{k=1}^{\infty} \mathbb{I}(S_k^{(a)} < t).$$

[17] Recall that we denote the Laplace transform by $\mathcal{L}(\cdot)$ and the Laplace transform of a density by $\psi(s)$.

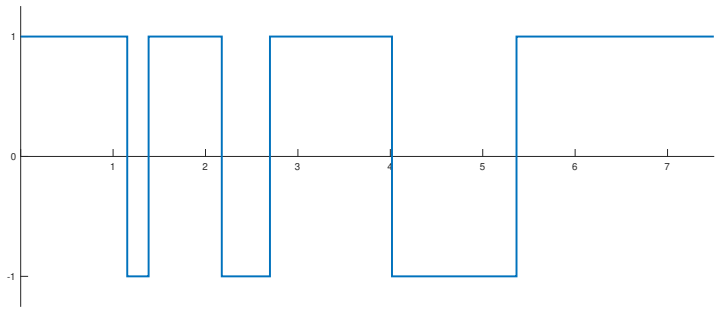

Figure 4: A trajectory of the alternating renewal process with the states one and minus one. With $T_+ \sim Exp(1)$ and $T_- \sim Exp(1/2)$.

The alternating renewal process can be defined by

$$X(t) = (-1)^{N_a(t)}.$$

A realization of $X(t)$ can be seen in Figure 4. The distribution of $X(t)$ is defined by the probability $P(X(t) = 1)$, and hence, several of its properties can directly be expressed as some function of this probability. Since $X(t) = 1$ is equivalent to the event, $N_a(t)$ is even, $P(X(t) = 1)$ can be found by summing $P(N_a(t) = k)$ over the even indices. To find $P(N_a(t) = k)$, we follow a similar way as for the ordinary counting process, and we have

$$P(N_a(t) = k) = F_{S_k^{(a)}}(t) - F_{S_{k+1}^{(a)}}(t).$$

However, due to the interlacing of T_+ and T_- the Laplace transform of $F_{S_{k+1}^{(a)}}(t)$ is not as elegant as for the ordinary counting process. However, it is given for $k \in \mathbb{N}_0$ by

$$\mathcal{L}\left(F_{S_k^{(a)}}\right)(s) = \begin{cases} \frac{1}{s}\psi_+(s)\left(\psi_+(s)\psi_-(s)\right)^{\frac{k-1}{2}}, & k \text{ is odd} \\ \frac{1}{s}\left(\psi_+(s)\psi_-(s)\right)^{\frac{k}{2}} & k \text{ is even,} \end{cases}$$

From this the Laplace transform of $P(N_a(t) = k)$ follows

$$\mathcal{L}\left(P(N_a(t) = k)\right)(s) = \begin{cases} \frac{1}{s}\left(\psi_+(s)\psi_-(s)\right)^{\frac{k-1}{2}}\left(\psi_+(s) - \psi_+(s)\psi_-(s)\right), & k \text{ is odd} \\ \frac{1}{s}\left(\psi_+(s)\psi_-(s)\right)^{\frac{k}{2}}\left(1 - \psi_+(s)\right), & k \text{ is even.} \end{cases}$$

for $k \in \mathbb{N}_0$.

We have now arrived at the Laplace transform of the probability that characterizes the process, namely

$$\mathcal{L}(P(X(t) = 1))(s) = \sum_{l=0}^{\infty} \mathcal{L}\left(P(N_a(t) = 2l)\right)(s) = \frac{1}{s} \frac{1 - \psi_+(s)}{1 - \psi_+(s)\psi_-(s)}$$

Due to the start in the state one, $P(X(t) = 1)$ will fluctuate over time. However, this effect will vanish with time, and we have the following limits that follow from the Key renewal theorem

$$\lim_{t\to\infty} P(X(t) = 1) = \frac{\mu_+}{\mu_+ + \mu_-}, \quad \lim_{t\to\infty} P(X(t) = -1) = \frac{\mu_-}{\mu_+ + \mu_-},$$

where $\mu_+ = ET_+$ and $\mu_- = ET_-$.

In Papers I and III, we characterize this process using the expected value function of $X(t)$. This function follows directly in the Laplace domain from the observation that $E(t) = EX(t) = 2P(X(t) = 1) - 1$; hence we have

$$\mathcal{L}(E)(s) = \frac{1 - 2\psi_+(s) + \psi_+(s)\psi_-(s)}{s(1 - \psi_+(s)\psi_-(s))}.$$

Due to starting in the state, one $X(t)$ will not be stationary. This is also evident from the fact that $E(t)$ varies with time. A stationary version can be constructed by changing the process's behavior around zero. This is similar to how the equilibrium version can be found for the ordinary counting process, with the additional complexity of having two distributions, T_+ and T_-.

For a fixed time point t, we can use the long-run probabilities to find the probability that the process is in either state. Conditioned on the state, the forward and backward recurrence times are the same as for the counting process. From this, we have that the forward and backward recurrence distribution is a mixture distribution of the form for $t \geq 0$

$$f_V(t) = f_U(t) = \delta \frac{1 - F_{T_+}(t)}{\mu_+} + (1 - \delta) \frac{1 - F_{T_-}(t)}{\mu_-},$$

where δ follows a Bernoulli distribution with $P(\delta = 1) = \mu_+/(\mu_+ + \mu_-)$, independent of T_+ and T_-.

The equilibrium alternating renewal process can now be construed. Let V_δ be the forward distribution conditioned on δ, and let $X_\alpha(t)$ be a modified version of $X(t)$. Such that $X_\alpha(t)$ starts from one when $\alpha = 1$ and minus one when $\alpha = 0$ while

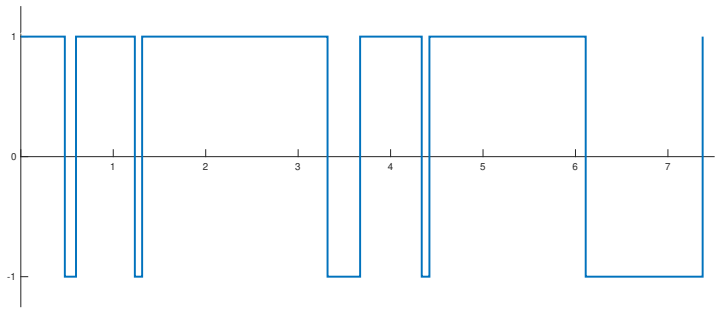

Figure 5: A trajectory of the stationary alternating renewal process with the states one and minus one, conditioned on $\delta = 1$. With $T_+ \sim Exp(1)$ and $T_- \sim Exp(1/2)$. Since it is delayed, the density of the first interval is $f_{V_\delta}(t) = (1 - F_{T_+}(t))/\mu_+$.

retaining the same distribution in each state as $X(t)$. From this, we can construct the equilibrium process for $t \geq 0$ by

$$Y(t) = \begin{cases} 2\delta - 1, & t < V_\delta, \\ X_{(1-\delta)}(t + V_\delta), & t \geq V_\delta. \end{cases}$$

A trajectory of the equilibrium alternating renewal process is shown in Figure 5.

An important property of the equilibrium process is that it is stationary. This follows from the relation between time and cycle stationarity derived in Thorisson (1995). By construction, the process $X(t)$ is cycle stationary, and since the expected cycle time $ET_+ + ET_- < \infty$, it is possible to construct a stationary version of $X(t)$. Our construction relies on inserting an interval right after zero instead of shifting the process as done in Thorisson (1992) and Thorisson (1995). However, this is equivalent to shifting due to T_+ and T_- being independent.

The properties of $Y(t)$ can be derived through the counting process in a manner similar to that for $X(t)$. We start with the sum

$$S_n^\delta = V_\delta + \sum_{k=1}^{n-1} T_{k+\delta}.$$

From S_n^δ we find the Laplace transform of its distribution function.[18]

$$\mathcal{L}(F_{S_n^\delta})(s) = \begin{cases} \frac{1}{s}, & n = 0, \\ \frac{1-\psi_\delta(s)}{s^2\mu_\delta}\left(\psi_+(s)\psi_-(s)\right)^{\frac{n-1}{2}}, & n \text{ is odd}, \\ \frac{1-\psi_\delta(s)}{s^2\mu_\delta}\psi_{(1-\delta)}(s)\left(\psi_+(s)\psi_-(s)\right)^{\frac{n-2}{2}}, & n \text{ is even}. \end{cases}$$

Which leads directly to the Laplace transform associated with $N_a^\delta(t)$

$$\mathcal{L}(P(N_a^\delta(t) = n|\delta))(s) = \\ \begin{cases} \frac{1}{s} - \frac{1-\psi_\delta(s)}{s^2\mu_\delta}, & n = 0, \\ \frac{1-\psi_\delta(s)}{s^2\mu_\delta}\left(\psi_{(1-\delta)}(s)\left(\psi_+(s)\psi_-(s)\right)^{\frac{n-2}{2}} - \left(\psi_+(s)\psi_-(s)\right)^{\frac{n}{2}}\right), & \text{n is even}, \\ \frac{1-\psi_\delta(s)}{s^2\mu_\delta}\left(\left(\psi_-(s)\psi_+(s)\right)^{\frac{n-1}{2}} - \psi_{(1-\delta)}(s)\left(\psi_-(s)\psi_+(s)\right)^{\frac{n-1}{2}}\right), & \text{n is odd}, \end{cases}$$

for $n \in \mathbb{N}_0$. By summing $P(N_a^\delta(t) = n|\delta)$ over the even and odd indices, respectively, we can find the Laplace transform of $P(Y(t) = 1|Y(0) = 2\delta - 1)$

$$\mathcal{L}(P(Y(t) = 1|Y(0) = 2\delta - 1))(s) = \begin{cases} \frac{1}{s} - \frac{(1-\psi_+(s))(1-\psi_-(s))}{s^2\mu_+(1-\psi_+(s)\psi_-(s))}, & \delta = 1, \\ \frac{(1-\psi_+(s))(1-\psi_-(s))}{s^2\mu_-(1-\psi_+(s)\psi_-(s))}, & \delta = 0. \end{cases}$$

We need these probabilities since the covariance function of $Y(t)$ can be written

$$C(t) = \frac{2}{\mu_+ + \mu-}\left(\mu_+ P(Y(t) = 1|Y(0) = 1) - \mu_- P(Y(t) = 1|Y(0) = -1) + \frac{\mu_+(\mu_- - \mu_+)}{\mu_+ + \mu-}\right)$$

Fortunately, this simplifies in the Laplace domain to

$$\mathcal{L}(C)(s) = \frac{4}{s(\mu_+ + \mu-)}\left(\frac{\mu_+\mu_-}{\mu_+ + \mu_-} - \frac{(1-\psi_+(s))(1-\psi_-(s))}{s(1-\psi_+(s)\psi_-(s))}\right)$$

We conclude this section with the following observation. When $T_+ \overset{d}{=} T_-$, knowing the covariance function is equivalent to knowing the distribution of $T_\pm$. When the distribution coincides, the equation above becomes

$$\mathcal{L}(C)(s) = \frac{1}{s} - \frac{2}{s^2\mu}\frac{1-\psi(s)}{1+\psi(s)}.$$

[18] To simplify notation let $\psi_\alpha(s)$ be $\psi_+(s)$ if $\alpha = 1$ and $\psi_-(s)$ if $\alpha = 0$.

Solving this for $\psi(s)$ we have

$$\psi(s) = \frac{1 - \frac{1}{2}\mu s(1 - s\mathcal{L}(C)(s))}{1 + \frac{1}{2}\mu s(1 - s\mathcal{L}(C)(s))}.$$

From these equations, it can be shown that if two alternating renewal processes have the same distribution, they will also have the same covariance and vice versa.

4 Geometric divisibility

Theorem 1 in Bengtsson (2024) provides conditions for the recovery of the interval distribution. If these conditions are met, the interval distribution belongs to a particular class, namely, the geometric divisible distributions. In this section, we will elaborate on this class and geometric summation. Geometric summation is the summation of ν_p random variables where ν_p follows a geometric distribution. Recall that a geometric random variable has the probability mass function

$$P(\nu_p = k) = (1 - p)^{k-1}p,\ k \in \mathbb{N},\ p \in (0, 1),$$

and the probability-generating function, for $|z| < (1 - p)^{-1}$

$$g(z) = \frac{pz}{1 - (1 - p)z}.$$

Geometric summation is present in several areas of probability theory and its application. For example, it is featured heavily in queuing theory. However, it traces its roots to the study of thinned renewal processes by Gnedenko (1972), Rényi (1956) and later Yannaros (1987). An early observation of its properties can be found in Feller (1971), which mentions that the geometric sum of two exponential distributions is again exponential. This was for a given p, and Arnold (1973) later showed that this holds for all $p \in (0, 1)$.

This notion was generalized in Klebanov et al. (1985) when a problem posed by Zolotarev was addressed. Zolotarev wondered if, for a random variable Y, there exists a random variable X_p independent of Y and ϵ_p, all $p \in (0, 1)$ such that

$$Y \overset{d}{=} X_p + \xi_p Y,$$

where ϵ_p is a Bernoulli random variable with parameter p. It was shown by Klebanov et al. (1985) that this is equivalent to characterizing the distribution Y such that for all

$p \in (0, 1)$

$$Y \overset{d}{=} \sum_{n=1}^{\nu_p} X_n.$$

where X_n is an iid sequence of random variables independent of ν_p. Random variables Y that allow for this stochastic representation are said to be geometrically infinitely divisible. This class is the geometric analog of the class of infinitely divisible distributions.[19] There are many well-known examples of these. For example, both the exponential and Laplace distributions belong to this class. Kozubowski and Podgórski (2010) have also shown that if the characteristic function can be expressed as the ratio of two second-order polynomials, the corresponding distribution is geometrically infinitely divisible.

Besides addressing Zolotarev's problem Klebanov et al. (1985) introduced a geometric analog of stable distributions.[20] We say that a random variable Y is geometrically stable if there for all $p \in (0, 1)$ exists a constant $\alpha_p > 0$ such that

$$Y \overset{d}{=} \alpha_p \sum_{n=1}^{\nu_p} X_n$$

where X_n is a sequence of iid random variables independent ν_p. It is well-known that stable distributions are also infinitely divisible, and this holds true for their geometric analogs. There are many interesting distributions that are geometrically stable. For example, the Mittag-Leffler and Linnik distributions as mentioned in Kozubowski (2010).[21]

We will turn to geometric divisibility. Geometrically divisible distributions extend the class of geometric infinitely divisible distributions by relaxing the condition on p.

Definition 4.1. Let ν_p be a geometric random variable, and $\{X_k\}_{k \geq 1}$ a sequence of iid random variables independent of ν_p. If a random variable Y has a stochastic representation

$$Y = \sum_{k=1}^{\nu_p} X_k,$$

[19] Infinitely divisible distributions are essential to characterize Lévy processes due to the well-known Lévy Khintchin representation theorem, see Satō (1999).

[20] Stable distribution received much attention in the 1920s and 1930s from, among others, Lévy, who introduced them, and Khintchin. For an overview of stable distributions, see Zolotarev (1986).

[21] This paper provides a more extensive survey of both geometric infinite divisibility and geometrically stable distributions.

then Y follows a p-geometric divisible distribution and is said to belong to the class $GD(p)$, $p \in (0,1)$, with the divisor X and we write $F_Y \in GD(p)$.

Hence, the stochastic representation only needs to hold for a given p. In contrast to geometrically infinitely divisible distribution, where it needs to hold for all $p \in (0,1)$.

From the definition of geometrically stable and infinitely divisible distributions, it is clear that they are also geometrically divisible. This class of distributions is large because they can easily be constructed by choosing a divisor.

Some elementary properties, such as expected value and variance, follow directly from Wald's equations. From Wald's first equation, we have

$$EY = E\nu_p EX = \frac{1}{p} EX.$$

From the second Wald's equation, we have

$$V(Y) = E\nu_p V(X) + (EX)^2 V(\nu_p) = \frac{V(X)}{p} + (EX)^2 \frac{1-p}{p^2}.$$

Since this class involves random sums, it is easiest to characterize it in the Laplace domain. The Laplace transform[22] associated with Y can be written as the probability generating function of ν_p with the argument substituted for the Laplace transform of the divisor[23]. This leads to

$$\psi_Y(s) = \frac{p\psi_X(s)}{1-(1-p)\psi_X(s)}.$$

This equation can be solved for $\psi_X(s)$ to derive a condition for a random variable Y to be geometrically divisible. If the right-hand side of

$$\psi_X(s) = \frac{\frac{1}{p}\psi_Y(s)}{1-(1-\frac{1}{p})\psi_Y(s)}$$

is a completely monotone function[24] then Y is geometrically divisible of order p.

This condition can be used to prove that if a random variable is geometrically divisible of order p, it is also geometrically divisible of order q for $0 < p \leq q < 1$.

[22] Recall that we denote the Laplace transform by $\mathcal{L}(\cdot)$ and the Laplace transform of a density by $\psi(s)$.

[23] Similar arguments can be made using the characteristic function instead in this section.

[24] Recall that a function h is completely monotone if $(-1)^n h^{(n)}(s) \geq 0, \forall s > 0, n \in \mathbb{N}_0$.

Suppose that Y belongs to $GD(p)$ with the divisor X. For Y to belong to $GD(q)$ with the same divisor X, the function

$$\frac{\frac{1}{q}\psi_Y(s)}{1-(1-\frac{1}{q})\psi_Y(s)}$$

must be completely monotone, from the condition mentioned above. Substituting the Laplace transform of Y into the equation above, we have

$$\frac{\frac{1}{q}\frac{p\psi_X(s)}{1-(1-p)\psi_X(s)}}{1-(1-\frac{1}{q})\frac{p\psi_X(s)}{1-(1-p)\psi_X(s)}} = \frac{\frac{p}{q}\psi_X(s)}{1-(1-\frac{p}{q})\psi_X(s)},$$

which is completely monotone for $\frac{p}{q} < 1$ or equivalently $p < q$ because it is the Laplace transform of a $GD(p/q)$ random variable with the divisor X. In essence, we can split a geometrically divisible distribution into fewer parts on average but not more.

This result allows us to relate the different classes of geometrically divisible distributions. Let GD_S denote the geometrically stable distribution and GD_∞ the geometrically infinitely divisible distribution. Then we have

$$GD_S \subseteq GD_\infty \subseteq GD(p) \subseteq GD(q) \subseteq GD_1,$$

for $0 < p < q \leq 1$ and where GD_1 is simply the set of probability distributions.

Exploring the properties of geometrically divisible distribution allows for the extension results for the distribution of one order to another. For example, the class of distributions in Theorem 1 in Bengtsson (2024) is of order $1/2$. However, if the interval distribution is of order $3/4$, then this switch process will also have a non-negative and decreasing expected value function.

5 Open problems

While several interesting results are presented within this thesis, there are several more open problems worthy of mentioning. The first is a question concerning the relation between the expected value and the covariance function. Theorem 2 in Bengtsson (2024) gives the following relation between the covariance and expected value for the stationary and non-stationary switch process

$$C'(t) = -\frac{2}{\mu}E(t).$$

There is a similar relation between the expected value of the clipped Slepian process $E_0(t)$ and the covariance of the clipped process, $C'_{cl(t)}(t)$. For the zero-level case, Proposition 2 in Bengtsson and Podgórski (2024) states

$$C'_{cl(t)}(t) = -\frac{2}{\pi} E_0(t)$$

Lastly, consider the expected value function of the Slepian process ζ_0 for the zero-level, which is

$$E\zeta_0(t) = -\sqrt{\frac{\pi}{2\xi_2}} C'(t)$$
$$C'(t) = -2\frac{\sqrt{2\pi}}{\mu} E\zeta_0(t)$$

where μ is one over the expected length between up-crossings i.e. $\mu = 2\pi/\sqrt{\xi_2}$. Here, we have a similar pattern for different processes, all relating their covariances to the expected value function. We, therefore, wonder if these results are examples of some more general relation relating to the characterizing function between time- and cycle-stationary processes.

The second unsolved problem is related to Paper IV. For higher u, there is a problem maintaining the monotonicity of both expected value functions. For higher levels, we are mostly interested in the excursion above and not below the level u. Therefore, we wonder if it is possible to recover one distribution while only one of the expected value functions is monotone.

The third relates to approximating the excursion distribution for stationary processes with oscillating covariance functions. As pointed out in Paper II, the largest negative real pole is a terrible way to approximate the persistency coefficient for the shift Gaussian covariance function. However, by finding the pair of complex poles with the largest real part and multiplying them, a reasonable approximation of the persistency coefficient can be found. This approximation also seems to be in agreement with the persistency coefficient obtained by simulating trajectories.

The last open problem is perhaps the simplest to express and the hardest to solve. That is, to fully characterize the relation between the expected value function and the interval distribution of the switch process. For example, if the interval distribution follows a gamma distribution, the expected value function will oscillate for certain parameter choices. This shows that the relationship between expected value functions and the interval distributions is not yet fully characterized.

6 Summary of the Papers

6.1 Paper I

In this paper, we study the relationship between the characterizing functions of the alternating renewal process and its interval distribution under symmetry. In essence, this poses the question of whether it is possible to find conditions on the characterizing functions such that the interval distribution can be recovered from these. This is a difficult question since this is equivalent to showing the complete monotonicity of a ratio of two non-standard functions.

Besides being an interesting problem in itself, it is also important since it provided the probabilistic foundation of the Slepian-based independent approximation, which is introduced in Paper II. We call the alternating renewal process with the two unique states, one and minus one, a switch process. Two versions exist: a stationary and a non-stationary one, the latter having a switch to one at the origin.

There are two important theorems within this paper. Theorem 1 states that the interval distribution[25] follows a geometric divisible distribution with $p = 1/2$ if and only if the expected value is non-negative and decreasing. This gives a partial characterization of when the interval can be recovered. The second result stated in Theorem 2 provides a relation between the covariance function of the stationary switch process and the non-stationary process on the form

$$C'(t) = -\frac{2}{\mu} E(t).$$

The latter theorem is interesting on its own. However, combined with Theorem 1, they lead to an alternative proof of the well-known sufficiency conditions for positive definiteness of a function provided by Pólya (1949). Further, any function satisfying these conditions is a covariance function of a stationary switch process constructed from a geometrically divisible distribution of order $1/2$.

6.2 Paper II

This paper introduces a new way of approximating excursion distributions using regenerative processes. This approximation is based on the Clipped Slepian process. As described in Section 2, the Slepian process models the behavior of a smooth Gaussian process at the crossing instance. The expected value of the clipped version of this process is used for the matching characteristic, similar to how the covariance is used

[25] Under symmetry, there is only one distribution of interest here since the interval distribution is the same in both states.

in the ordinary independent interval approximation by McFadden (1962), Majumdar et al. (1996) and Sire (2007, 2008).

Using Theorem 1 in Bengtsson (2024), conditions are derived, ensuring that the approximation is a valid probability distribution for a large class of stationary Gaussian processes. A somewhat surprising consequence of this is that it also addresses the question of the mathematical validity of the ordinary independent interval approximation, as discussed in Section 2. A consequence of Theorem 2 in Bengtsson (2024) is that matching the expected value function of the clipped Slepian process is equivalent to matching the covariance of the clipped process. Hence, the two approaches are equivalent, and for this large class of processes, the ordinary independent interval yields valid approximations, too.

While the Slepian-based approach approximates the entire excursion distribution, it also allows us to approximate the persistency coefficient. This coefficient characterizes the tail of the excursion distribution and is important for describing the properties of systems in statistical physics. The persistency coefficient is approximated for several processes and compared to both simulations of trajectories and the approximations given in Sire (2008).

An important limitation of the independent interval framework is also highlighted. An oscillating covariance function is considered in Wilson and Hopcraft (2017), and this is one of the covariances that does not satisfy the monotonicity conditions. Hence, the question of the mathematical validity of the independent interval approximation resurfaces.

For this case, both the independent interval approximation and the Slepian-based method are unsuitable for even approximating the persistency coefficient. The traditional approximation is the largest negative real pole of the Laplace transform of the approximated excursion distribution. For the oscillating covariance function considered by Wilson and Hopcraft (2017), the approximation error from the pole methods is of such a magnitude that the obtained coefficient is unsuitable even as an approximation. However, when the monotonicity condition is satisfied, the persistency coefficient from the Slepian-based method yields approximations that are not too dissimilar to the ones obtained from simulations.

6.3 Paper III

This paper builds on the work of Paper I. However, the need for symmetry in the interval distributions is removed. Hence allowing the intervals to have different distributions. To recover both distributions, we need to consider two non-stationary processes that switch to minus one or one at zero. If and only the corresponding expected value function are monotonic, then the two interval distributions T_+ and T_- have a stochastic

representation on the form

$$T_+ \stackrel{d}{=} X + \sum_{i=1}^{\nu_p - 1} Y, \quad T_+ \stackrel{d}{=} Y + \sum_{i=1}^{\nu_q - 1} X.$$

where ν_p follows a geometric distribution, and the distribution of X and Y are related to the expected value functions of the two non-stationary processes. This representation is a generalization of Theorem 1 in Paper I. However, under asymmetry, the derivative of the covariance function can be expressed as the linear combination of the two expected value functions. The implication of this is that covariance can not solely be used to recover both T_+ and T_-.

6.4 Paper IV

Paper IV extends the result of Paper II to non-zero level crossings using the Clipped Slepian process. The loss of symmetry leads to additional complexity since there are now two distributions of interest, the above and below excursion distributions. While only the excursion above a level u is of interest, both are approximated using this approach.

Two clipped processes are needed to approximate the excursion distributions. The first is a Slepian process, which has an up-crossing at $t = 0$, and the second one has a down-crossing at $t = 0$. Since we now have two expected value functions, we can approximate both excursion distributions.

To ensure that the approximations are proper probability distributions, we need to ensure that some monotonicity conditions are satisfied for the expected value functions. When these conditions are met, the approximated distributions have a stochastic representation of the same form as the equation above.

To illustrate this method, we again turn to the Gaussian diffusion process in two dimensions. In particular, the persistency coefficient is approximated since this is not known for this process except for zero-level excursions. Results are also compared to simulations of trajectories. By studying this particular case, an important caveat is found. When higher-level excursions are considered, the monotonicity of the clipped process coming from the down-crossing is lost. This leads to a problem ensuring that the approximation is a valid probability distribution for higher levels.

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Papers

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CHARACTERISTICS OF THE SWITCH PROCESS AND GEOMETRIC DIVISIBILITY

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Abstract

The switch process alternates independently between 1 and −1, with the first switch to 1 occurring at the origin. The expected value function of this process is defined uniquely by the distribution of switching times. The relation between the two is implicitly described through the Laplace transform, which is difficult to use for determining if a given function is the expected value function of some switch process. We derive an explicit relation under the assumption of monotonicity of the expected value function. It is shown that geometric divisible switching time distributions correspond to a non-negative decreasing expected value function. Moreover, an explicit relation between the expected value of a switch process and the autocovariance function of the switch process stationary counterpart is obtained, leading to a new interpretation of the classical Pólya criterion for positive-definiteness.

Keywords: Renewal theory; geometric divisibility; binary processes

2020 Mathematics Subject Classification: Primary 60K05
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1. Introduction

The study of binary stochastic processes has a long-standing tradition in probability theory. There exist many versions of such processes, for example the telegraph process in continuous time or the simple discrete-time Markov chain. These processes found applications in many fields, for example in renewal theory, signal processing [8], and statistical physics [2].

The focus of this paper is the switch process with independent switching times. More specifically, we consider a continuous-time stochastic process taking values in $\{-1, 1\}$, starting at 1 at the origin, and then switching according to an independent and identically distributed (i.i.d.) sequence of non-negative random variables. The switch process always starts from one and hence is not stationary; however, a convenient stationary counterpart can be defined. This counterpart will be referred to as the stationary switch process.

The expected value of the switch process is intrinsically connected with the switching time distribution. This is also the case for the covariance of the stationary switch process. Formalizing this connection is the main contribution of the paper, among other contributions such as formulating and deriving the underlying properties of the switch process. The connection also leads to a class of distributions that constitutes a proper sub-class of geometric infinite divisible distributions introduced in [6].

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The main results of this paper answer interesting questions related to renewal theory and signal processing. In the context of renewal theory, finding the original distribution given that we observe the thinned process was considered in [11]. From that perspective, Theorem 1 provides additional criteria for when such an inverse problem can be solved. In the signal processing context, Theorem 2 provides a partial solution to the classical difficult problem of obtaining the covariance function from the statistical properties of the point process used to construct binary random signals; see [8] for further discussion of such problems.

The structure of the paper is as follows. In Section 2, the basic concepts are defined. Section 3 contains the first main theorem relating expected value functions to the class of geometric divisible distributions. The second main theorem connecting the covariance of a stationary version of the switch process with the expected value of the switch process is presented in Section 4. In Section 5, a possible application for deriving results that can be used to approximate level-crossing distributions is elaborated on.

2. Preliminaries

2.1. The switch process and its expected value

Let T_k, $k = 1, 2, 3, \ldots$, be a sequence of i.i.d. non-negative random variables with the distribution function F, which is assumed to be absolutely continues with respect to the Lebesgue measure. Additionally, let the corresponding density f associated with F be bounded on any closed interval of the positive half-line. Define a renewal count process for $t \in [0, \infty)$ by

$$N(t) = \begin{cases} \sup\left\{n \in \mathbb{N}; \sum_{k=1}^{n} T_k \leqslant t\right\}, & t \geqslant T_1, \\ 0, & 0 \leqslant t < T_1. \end{cases}$$

In other words, $N(t)$ is the number of renewal events up to a time point t.

Definition 1. Let N(t), $t \geqslant 0$, be a count process. Then the switch process is defined by $X(t) = (-1)^{N(t)}$, $t \geqslant 0$.

The process $X(t)$ switches between the values 1 and -1 at each renewal event, hence the name.

One of the main objects of interest is the expected value function of the switch process $E(t) = \mathbb{E}X(t)$. The relation between E and the switching time distribution, F, is implicit in the time domain. There exist some elementary properties of $E(t)$, which are important but straightforward to derive; see, for example, [3]. First, we have the limiting results for $t \geqslant 0$, which follow from the key renewal theorem:

$$\lim_{t \to 0^+} E(t) = 1, \qquad \lim_{t \to \infty} E(t) = 0. \tag{1}$$

The existence of $E'(t)$ is of importance for the main results of the paper. Under the assumptions stated in this section and assuming $\sup_{t>0} f(t) < \infty$ on $F(t)$, $E'(t)$ exists and is well-defined. The last assumption can be relaxed. Specifically, there needs to exist $l \in \mathbb{N}$: $\sup_{u>0} f^{*l}(u) < \infty$, which allows for more general switching time distributions, e.g. those with unbounded densities at zero.

Let $\mathcal{L}(\cdot)$ denote the Laplace transform and, in particular, let $\Psi_F(s) = \mathcal{L}(f)(s)$, where f is the derivative of F when it exists. The Laplace transform of this probability-generating function is

well-known and has the following form for $s > 0$:

$$\mathcal{L}(E)(s) = \frac{1}{s}\frac{1-\Psi_F(s)}{1+\Psi_F(s)}. \tag{2}$$

This expression is easily solved for the switching time distribution,

$$\Psi_F(s) = \frac{1-s\mathcal{L}(E)(s)}{1+s\mathcal{L}(E)(s)}. \tag{3}$$

Although the above explicit relations tie the distribution of the switch process and its expected value, they do not provide an explicit condition when a function E is the expected value of a switch process. Naturally, by Bernstein's theorem, see [5, Theorem 1, p. 415], we could say that this is the case whenever the right-hand side of the above equation is a complete monotone function, but this condition is not easy to check in a concrete case. In Section 3, easy-to-check conditions for E are presented.

2.2. Geometric divisibility

The concept of geometric infinite divisibility was introduced in [6] and further treated in [1, 7]. It describes distributions that can be represented as a sum of i.i.d. random variables where the number of terms in the sum follows a geometric distribution with an arbitrary parameter $p \in (0, 1)$. The main focus here is on a weaker concept, defined next.

Definition 2. Let ν_p be a geometric random variable with $\mathbb{P}(\nu_p = k) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$, and $\{\tilde{W}_k\}_{k\geqslant 1}$ a sequence of i.i.d. non-negative random variables independent of ν_p. If the random variable W, with the distribution function F, has the stochastic representation $W = \sum_{k=1}^{\nu_p} \tilde{W}_k$, then W follows an r-geometric divisible distribution with $r = \mathbb{E}\nu_p$ and is said to belong to the class GD(r); we write $F \in GD(r)$.

The distribution of $\tilde{W}$ is then called the r-geometric divisor of the distribution of W. There are two important properties of a $GD(r)$ distribution, which are presented in the following propositions.

Proposition 1.

(i) *The Laplace transform of $F \in GD(r)$ is*

$$\Psi_F(s) = \frac{(1/r)\Psi_{\tilde{F}}(s)}{1-(1-(1/r))\Psi_{\tilde{F}}(s)}.$$

(ii) *The function*

$$\frac{r\Psi_F(s)}{1+(r-1)\Psi_F(s)}$$

is completely monotone if and only if $F \in GD(r)$.

The second important property is key to generalizing the main result in Section 3.

Proposition 2. *Let $u \in \mathbb{R}$: $1 < u \leqslant r < \infty$; then $GD(r) \subseteq GD(u)$.*

The proofs of both propositions follow using standard methods. However, it should be noted that the first result follows from Bernstein's theorem and the second result follows from the first by using r/u instead of $1/r$ in (ii).

3. Switch processes with monotonic expected value function

In this section we fully characterize switch processes with monotonic expected value functions. For the main result, we recall the assumptions on the switching time distribution: $F(t)$ has support on $(0, \infty)$ with a density $f(t)$ for which there exists $l \in \mathbb{N}$: $\sup_{t>0} f^{*l}(t) < \infty$ (a technical requirement for the existence of $E'(t)$).

Theorem 1. *Let $X(t)$ be a switch process with $E(t)$ its expected value function and the switching time distribution $F(t)$. Then the following conditions are equivalent:*

(i) $F(t) \in GD(2)$.

(ii) *$E(t)$ is non-negative and decreasing.*

Proof. (i) $\Rightarrow$ (ii): Since $F(t) \in GD(2)$ it has the following Laplace transform, as described in Section 2:

$$\Psi_F(s) = \frac{\frac{1}{2}\Psi_{\tilde{F}(s)}}{1 - \frac{1}{2}\Psi_{\tilde{F}(s)}}.$$

Substituting this into (2), we have

$$\mathcal{L}(E)(s) = \frac{1}{s} \frac{1 - \frac{\frac{1}{2}\Psi_{\tilde{F}}(s)}{1 - \frac{1}{2}\Psi_{\tilde{F}}(s)}}{1 + \frac{\frac{1}{2}\Psi_{\tilde{F}}(s)}{1 - \frac{1}{2}\Psi_{\tilde{F}}(s)}} = \frac{1}{s}(1 - \Psi_{\tilde{F}}(s)),$$

which is equivalent to $s\mathcal{L}(E)(s) - 1 = -\Psi_{\tilde{F}}(s)$. The existence of $E'(t)$ is needed in order to use the Laplace transform $\mathcal{L}(E')(s) = s\mathcal{L}(E)(s) - E(0)$. It follows from the stated assumptions by a rather standard although technical argument, see [4, Exercise 4.4.3]. Using the above-stated property of the Laplace transform and the limits of $E(t)$, $\mathcal{L}(-E')(s) = \Psi_{\tilde{F}}(s)$. By taking the inverse Laplace transform, this implies that $-E'(t)$ is a probability density function. Therefore, to satisfy the limiting results of (1), $E(t)$ must satisfy the conditions of (ii).

(ii) $\Rightarrow$ (i): Under the assumptions of (ii) and the limits of (1) we have

$$\int_0^\infty E'(t)\,\mathrm{d}t = \lim_{t\to\infty} E(t) - \lim_{t\to 0} E(t) = -1;$$

$-E'(t)$ is thus a probability density function. Combining this with the derivative property of the Laplace transform and the limits in (1), (3) becomes

$$\Psi_F(s) = \frac{1 - s\mathcal{L}(E)(s)}{1 + s\mathcal{L}(E)(s)} = \frac{\mathcal{L}(-E')(s)}{2 - \mathcal{L}(-E')(s)} = \frac{\frac{1}{2}\mathcal{L}(-E')(s)}{1 - \frac{1}{2}\mathcal{L}(-E')(s)}.$$

This is the Laplace transform of a $GD(2)$ distribution, as described in Section 2. Therefore, $F(t) \in GD(2)$ with the divisor $-E'(t)$, which yields (i). □

Remark 1. The switch process is a special case of the process $\alpha^{N(t)}$, where $\alpha = -1$. For any α not equal to minus one, the process will either diverge or converge to zero. For $\alpha \in [-1, 0)$, the expected value function is positive and decreasing if and only if the switching time distribution belongs to $GD(1 - \alpha)$. This is shown using an argument similar to the proof of Theorem 1.

Theorem 1 directly relates functional properties of the expected value of the switch process with the switching time distribution for the class of $GD(2)$ distributions. By combining Theorem 1 and properties of $E(t)$ derived in Section 2, a partial solution can be obtained for the case when the switching time distribution belongs to $GD(2)$. To highlight this partial characterization we have the following corollary, which follows from the second part of the proof of Theorem 1.

Corollary 1. *Let $E(t)$ be a function for $t \geqslant 0$ such that the following conditions are satisfied: $\lim_{t\to 0^+} E(t) = 1$, $\lim_{t\to\infty} E(t) = 0$, $E(t)$ is at least once differentiable on $(0, \infty)$, and $E'(t) \leqslant 0$ for all $t \geqslant 0$; then it is an expected value function of a switch process with a GD(2) switching time distribution.*

Corollary 2 gives an explicit representation of the distribution function and density for the 2-geometric divisor of the switching time distribution in terms of $E(t)$.

Corollary 2. *Let the switching time distribution, $F(t)$, belong to GD(2), with the divisor $\tilde{F}(t)$; then, for $t \geqslant 0$,*

$$E(t) = 1 - \tilde{F}(t), \qquad E'(t) = -\tilde{f}(t).$$

Proposition 2 can be used to extend the results of Theorem 1.

Corollary 3. *Let the switching time distribution be $GD(r)$, for some $r \geqslant 2$; then the corresponding expected value function of the switch process, $E(t)$, is non-negative and decreasing for $t \geqslant 0$.*

However, the opposite is not necessarily true, i.e. a non-negative and decreasing expected value function does not necessarily imply an r-geometric divisible switching time for $r > 2$.

Let us consider a switch process constructed from a count process $N(t)$ and satisfying the conditions of Theorem 1. Further, let $\tilde{N}(t)$ be a count process with the arrival times distributed according to the divisor of this switch process. The two count processes are related through thinning. More specifically, $N(t)$ is a thinning of $\tilde{N}(t)$, with the probability of thinning equal to $\frac{1}{2}$. Thus we have the following result.

Corollary 4. *A switch process $X(t)$ is $\frac{1}{2}$-thinned if and only if its expected value is non-negative and decreasing.*

From a given trajectory of $N(t)$, the trajectory of process $\tilde{N}(t)$ cannot be recovered, in general. However, it follows from Corollary 1 that the distribution of arrival times of $\tilde{N}(t)$ can be recovered. For further relations between geometric divisibility of the switching time distribution and the thinned renewal processes, see [11, 12].

4. The autocovariance of the stationary switch process

A stationary version of the switch process can be constructed by addressing the behavior around zero. Let $\mu < \infty$ be the expected value of the switching time distribution, and $((A, B), \delta)$ be non-negative random variables, mutually independent and independent of $X(t)$, such that δ takes values $\{-1, 1\}$ with equal probability and such that $f_{A,B}(a, b) = (1/\mu) f_T(a + b)$ so that the marginals of $f_{A,B}$ are $f_A(t) = f_B(t) = (1 - F_T(t))/\mu$.

Definition 3. Let $X_+(t)$ and $X_-(t)$ be two independent switch processes, and $((A, B), \delta)$ be as described above. Define

$$Y(t) = \begin{cases} -\delta, & -B < t < A, \\ \delta X_+(t-A), & t \geqslant A, \\ -\delta X_-(-(t+B)), & t \leqslant -B. \end{cases}$$

Then $\mathrm{Y}(t)$ is called a stationary switch process.

The stationarity of $Y(t)$ follows from standard results in renewal theory. In the same way as the switch process is characterized by its expected value function $E(t)$, the stationary switch process is characterized by its covariance function $C(t)$. There exists a relation between $E(t)$ and $C(t)$ presented in the next theorem.

Theorem 2. *Let $C(t)$ be the covariance of the stationary switch process, $E(t)$ be the expected value function of the switch process, and μ be the expected value of the switching time distribution. Then, for $t \geqslant 0$,*

$$C'(t) = -(2/\mu)E(t).$$

Proof. Starting with the covariance of $\mathrm{Y}(t)$, and utilizing symmetry, we have

$$(-Y(t) \mid \delta = 1) \overset{\mathrm{d}}{=} (Y(t) \mid \delta = -1)$$

and, for $t > 0$,

$$\begin{aligned} C(t) = \mathbb{E}(\mathbb{E}(Y(t)Y(0) \mid \delta)) &= -\mathbb{E}(Y(t) \mid \delta = 1) \\ &= -\int_0^\infty \mathbb{E}(Y(t) \mid \delta = 1, A = x) f_{A \mid \delta=1}(x)\,\mathrm{d}x \\ &= -\left(\int_0^t \mathbb{E}(\delta X(t-x) \mid \delta = 1, A = x) f_A(x)\,\mathrm{d}x + \int_t^\infty (-1) f_A(x)\,\mathrm{d}x \right). \end{aligned}$$

Since $E(t-x) = 0$, for $x > t$ we obtain $C(t) = 1 - F_A(t) - (E * f_A)(t)$. Using the above expression and (2),

$$\begin{aligned} \mathcal{L}(C)(s) = \frac{1}{s} - \mathcal{L}(f_A)(s)\left(\frac{1}{s} + \frac{1}{s}\frac{1-\Psi_F(s)}{1+\Psi_F(s)} \right) &= \frac{1}{s} - \frac{1-\Psi_F(s)}{\mu s}\left(\frac{2}{s}\frac{1}{1+\Psi_F(s)} \right) \\ &= \frac{1}{s}\left(1 - \frac{2}{\mu}\mathcal{L}(E)(s) \right). \end{aligned}$$

Using $\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$ and $C(0) = 1$,

$$s\mathcal{L}(C)(s) - 1 = -\frac{2}{\mu}\mathcal{L}(E)(s), \qquad C'(t) = -\frac{2}{\mu}E(t). \qquad \square$$

Theorem 2 allows us to use functional properties of the expected value of the switch process to investigate the covariance of the stationary switch process. In particular, combining

Theorem 2 with Theorem 1 and Proposition 1 yields a partial characterization of the covariance functions.

Corollary 5. *Let $C(t)$ be a symmetric function around zero, $t \in \mathbb{R}$, such that the following conditions are satisfied for all $t \in [0, \infty)$: $C(t) \geqslant 0$, $C'(t) \leqslant 0$, $C''(t) \geqslant 0$, and $C(0) = 1$. Then $C(t)$ is the covariance function of a stationary switch process with a GD(2) switching time distribution.*

Remark 2. Interestingly, the conditions of Corollary 5 are essentially equivalent to those in [9, Theorem 1]. Thus, the above corollary can be viewed as an alternative proof of the Pólya criterion of positive definiteness and consequently implying that characteristic functions satisfying the conditions in [9] can be characterized as covariance functions of stationary switch processes with $GD(2)$ switching time distribution.

By combining Theorem 1 and Theorem 2, the divisor's density and distribution can be derived from the covariance function.

Corollary 6. *Let $C(t)$ be the covariance of the stationary switch process and the switching time distribution belong to GD(2), with the divisor distribution $\tilde{F}$; then, for $t \geqslant 0$,*

$$1 + \frac{\mu}{2} C'(t) = \tilde{F}(t), \qquad \frac{\mu}{2} C''(t) = \tilde{f}(t),$$

where $\mu = -2C'(0^+)$.

The identification of μ does not require geometric divisibility, since it follows from the limits of $E(t)$ and Theorem 2.

Even if the switching time distribution does not belong to $GD(2)$ Theorem 2 is still applicable, as illustrated in the next example.

Example 1. Consider a switch process with $\Gamma(2, 2)$ switching time distribution. The expected value of this switch process, $E(t) = \sqrt{2} \sin((2t + \pi)/4) e^{-t/2}$ is oscillating so that the switching time distribution does not belong to GD(2). By Theorem 2, $C(t) = \cos(t/2) e^{-t/2}$ is the covariance of the stationary switch process.

5. Conclusions

To characterize which functions correspond to the expected value of the switch process is a difficult problem. By exploring the relationship between the functional properties of the expected value and the class of 2-geometric divisible distributions, a partial answer to the problem is given.

An explicit relation between the expected value function of the switch process and the covariance function of the stationary switch process is presented. It leads to corresponding relations between the 2-geometric divisible switching time distributions and the covariance of the stationary switch process. It enables the recovery of the switching time distribution from the covariance function under conditions that are easy to verify. This constitutes a partial solution to the well-known open problem of obtaining the switching time distribution from the covariance function of a continuous-time binary process. Complete answers to both the above-mentioned problems are still unknown.

Finally, it is apposite to mention the connection the presented results have to the persistence studies that are a long-standing and heavily investigated problem of statistical physics, see [2,

10]. There, the independent interval approximation (IIA) framework has been used to approximate the tail distribution and its tail index (persistency exponent). The results of this paper allow us to obtain an explicit IIA representation for many stochastic processes commonly used in statistical physics. This not only provides information about the tail behavior but also yields the explicit approximated distribution of excursions above or below zero. Explicit applications of the obtained results for the independent interval approximations of the level-crossing distributions is planned in future work.

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II

THE SLEPIAN MODEL BASED INDEPENDENT INTERVAL APPROXIMATION OF PERSISTENCY AND ZERO-LEVEL EXCURSION DISTRIBUTIONS

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Abstract. In physics and engineering literature, the distribution of the excursion time of a stationary Gaussian process has been approximated through a method based on a stationary switch process with independently distributed switching times. The approach matches the covariance of the clipped Gaussian process with that of the stationary switch process. By expressing the switching time distribution as a function of the covariance, the so-called independent interval approximation (IIA) is obtained for the excursions of Gaussian processes. This approach has successfully approximated the persistency coefficient for many vital processes in physics but left an unanswered question about when such an approach leads to a mathematically meaningful and proper excursion distribution. Here, we propose an alternative approximation: the Slepian-based IIA. This approach matches the expected values of the clipped Slepian process and the corresponding switch process initiated at the origin. It is shown that these two approaches are equivalent, and thus, the original question of the mathematical validity of the IIA method can be rephrased using the Slepian model setup. We show that this approach leads to valid approximations of the excursion distribution for a large subclass of the Gaussian processes with monotonic covariance. Within this class, the approximated excursion time distribution has a stochastic representation that connects directly to the covariance of the underlying Gaussian process. This representation is then used to approximate the persistency coefficient for several important processes to illustrate the Slepian-based IIA approach. Lastly, we argue that the ordinary IIA is ill-suited in certain situations, such as for Gaussian processes with a non-monotonic covariance.

1. Introduction

The distribution of excursion times for a stationary Gaussian process constitutes a long-standing and challenging problem in physics and applied probability theory. In its most generic form, the problem can be formulated as finding statistical properties of the excursion set $\mathcal{E}_u = \{t \in \mathbb{R} : X(t) > u\}$, where $X(t)$, $t \in \mathbb{R}$, is a stochastic process. This

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excursion is the exceedance above the level u, and thus, one often uses terms the *u-excursion sets* and the *excursion distributions* when probabilistic properties of such sets are sought.

While the idea of an excursion distribution is intuitive, the solution to the problem of fully characterizing it has so far been elusive despite considerable effort. This effort has, however, resulted in several significant results, such as the famous Rice formula. Due to the importance of excursion distributions in many fields and the general complexity of the problem, several approximation methods have been developed. Two methods have been a frequent topic in physics papers. The first is based on Rice series expansions, which use the moments of the distribution of the number of crossings. The second method is the independent interval approximation (IIA).

The basis for the IIA method is the so-called clipped process. This process is obtained by computing the sign of a stochastic process, and McFadden (1956) noted that the interval lengths of the clipped process correspond to the zero-level excursions of the original process. However, due to the underlying dependency of the original process, the intervals of the clipped process will have a non-trivial dependency structure. Therefore, in the IIA framework, the core assumption is that the dependence between successive intervals is sufficiently small so that they can be treated as independent. Subsequently, the covariance of the clipped process can be matched with that of a stationary binary renewal process, for which the relation between covariance and interval distributions is known. Using this relation, an approximation of the excursion distribution is obtained as a function of the covariance of the clipped process. While this approach leads to an approximation, the validity of such a method needs to be investigated.

The validity of the IIA approach was further discussed in McFadden (1957, 1958), which also considered extending the approach to allow for Markovian types of dependency between intervals. This extension led to a study by Rainal (1962, 1963), which empirically examined both the IIA and the Markovian approximation. This study concluded that a Markovian type of dependency structure might not be suitable for approximating some Gaussian processes. Additionally, in Longuet-Higgins (1962), it was shown that the IIA is never exactly valid for Gaussian processes. These investigations imply that the independence assumptions have to be used with caution. While these problems were highlighted, it did not mean that the approximation could be adequate for many important models.

The popularity of the IIA has seen a resurgence during the last three decades. This is due to a greater interest in the tail of the excursion distribution in several areas such as optics, statistical physics, and more, (Brainina, 2013). To characterize the tail of this distribution, emphasis has been placed on the so-called *persistency coefficient*. This coefficient characterizes the tail of the excursion distribution and thus describes the probability of large excursions. A comprehensive overview of persistency coefficients and how they can be approximated from the IIA framework can be found in Bray et al. (2013).

Persistency coefficients are generally process-specific and often very difficult to derive analytically. Hence, they are only known explicitly for a handful of processes. The

persistency coefficient for the Ornstein-Uhlenbeck process was shown to be $\theta = 0.5$ by Bachelier (1900) using the explicit distribution of the Brownian motion excursion above $u < 0$, which is the Lévy distribution with the index of stability 0.5. Another process for which the persistency coefficient is known is the random acceleration process. It was shown by Sinai (1992) that the persistency coefficient for this process is $\theta = 0.25$. For fractional Brownian motions, which in its stationary version correspond to fractional Ornstein-Uhlenbeck processes, Molchan (1999) showed that the persistency coefficient for this process is $\theta = 1-H$, where H is the Hurst parameter. Another important class of processes in statistical physics is diffusion processes. It was only in 2018 that Poplavskyi and Schehr (2018) derived the persistency coefficient for the two-dimensional diffusion process ($\theta = 0.1875$). For this process, the IIA framework was used several times to approximate the persistency coefficient before it was derived. Estimates have been obtained by, among others Sire (2007, 2008); Schehr and Majumdar (2007); Majumdar et al. (1996), and they are relatively close to the persistency coefficient derived by Poplavskyi and Schehr (2018). The importance of the diffusion process and the recency of the result by Poplavskyi and Schehr (2018) shows that there is still a need for adequate approximation methods. Even for the zero-level excursions.

For the cases with analytically derived persistency coefficients, the IIA generally seems to produce reasonable approximations. However, several important questions still remain that have not been addressed so far. Since the approximated excursion distribution is given as a function of the covariance of the clipped process in the Laplace domain, this function needs to be completely monotone for it to correspond to a valid probability distribution. It is not apparent that the class of covariance functions of the clipped Gaussian process will always correspond to a probability distribution on a positive real line. This also raises questions about the validity of using the IIA to approximate the persistency coefficient.

The main idea of this paper is to base the IIA on the Slepian process, which describes the behavior at the crossing instance of the process we are interested in. The reasons for using the clipped Slepian process are twofold; firstly, by matching the clipped Slepian process to a non-stationary binary process, the so-called inspection paradox can be avoided since all intervals of the non-stationary binary process will have the same interval distribution. Secondly, in a recent paper by Bengtsson (2024), conditions were derived for when the interval distribution of a non-stationary binary can be uniquely obtained from the expected value function of this binary process. The latter result will be used to show that the approximated excursion distribution of the Slepian-based IIA is indeed a valid probability distribution for a large class of processes. This has direct implications for the ordinary IIA since it will later be shown that the ordinary IIA and the Slepian-based IIA are equivalent for zero-level excursions. Before introducing the paper's notation and outline of the paper, it should be noted that only processes with finite crossing intensities are treated unless otherwise stated. This is the same general assumption on crossing intensity that Bray et al. (2013) gives for the ordinary IIA.

	Clipped process	Binary process
	Stationary	
Process	$D_{cl}(t) \stackrel{\text{def}}{=} \text{sgn}(X(t))$	$D_s(t)$
Covariance	$R_{cl}(t) \stackrel{\text{def}}{=} \mathsf{Cov}\left(D_{cl}(t), D_{cl}(0)\right)$	$R(t) \stackrel{\text{def}}{=} \mathsf{Cov}\left(D_s(t), D_s(0)\right)$
	Slepian	
Process	$D_0(t) \stackrel{\text{def}}{=} \text{sgn}(X_0(t))$	$D(t)$
Expectation	$E_0(t) \stackrel{\text{def}}{=} \mathsf{E}\left(D_0(t)\right)$	$E(t) \stackrel{\text{def}}{=} \mathsf{E}\left(D(t)\right)$

Table 1. Notation for the characteristics that are used for the IIA matching clipped processes with switch processes in the stationary and attached at zero-crossing cases.

The structure of the paper is as follows: we start with an overview of the ordinary IIA and introduce the switch process in Section 2. Then, the Slepian-based IIA is the topic of Section 3. These sections focus on the approximate excursion distribution while Section 4 focuses on the persistency coefficient. In Section 5, these methods are used to obtain approximated persistency coefficients for several stochastic processes. Due to the idea of matching and clipping different processes, the notation used in the paper is summarized in Table 1.

2. The independent interval approximation

The instants of zero-level crossings of a smooth process $X(t)$ form an ordered sequence of points S_i, $i \in \mathbb{Z}$, leading to two sequences of interlaced intervals of the lengths: T_i^+, $i \in \mathbb{Z}$, for the excursions above the zero-level and T_i^-, $i \in \mathbb{Z}$, for the excursions below it. If a new process is constructed from this process such that it takes the value one when $X(t) \geqslant 0$ and minus one when $X(t) < 0$, we obtain the clipped version of $X(t)$. The clipped version can also be obtained directly from $X(t)$ by computing

$$D_{cl}(t) = \text{sign}(X(t)).$$

It is intuitive that the distributions of the intervals that D_{cl} takes the value one or minus one encapsulate the excursion behavior of $X(t)$. This has been known for a long time. See, for example, McFadden (1956).

While the notion of clipping a process $X(t)$ might be straightforward, we illustrate it with an example, in which we use a double peak spectral model for sea state data that accounts for dependence between the two sea systems: swell and wind. The model was developed based on measured spectra for Norwegian waters (Haltenbanken and Statfjord); see Torsethaugen and Haver (2004). The input to the model is significant wave height, H_s, for total sea and spectral peak period, T_p, for the primary (highest) peak. In our example, we have set $H_m = 6[m]$; $Tp = 8[s]$; and used WAFO-toolbox (WAFO-group, 2017) to

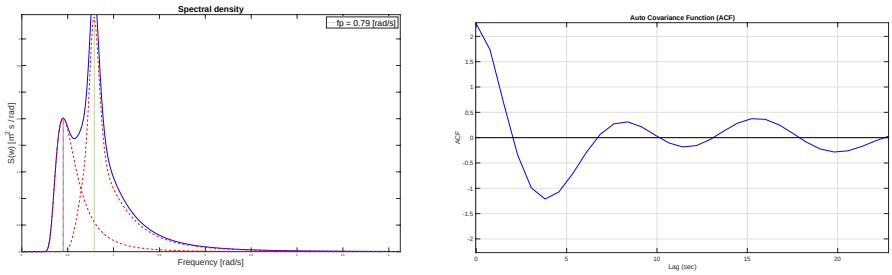

FIGURE 1. The Torsethaugen spectrum *(left)* and the corresponding correlation *(right)* that have been used for generating data that illustrate the introduced concepts.

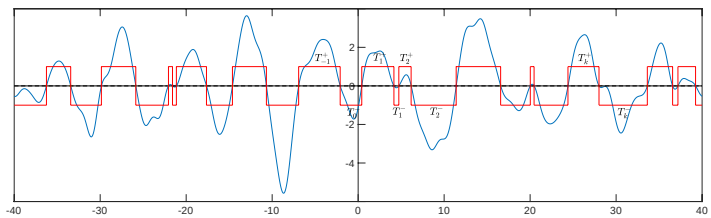

FIGURE 2. The excursion intervals of a Gaussian process $X(t)$ together with the corresponding clipped process.

make all computations. In Figure 1, we see the graphs of the two-peaked spectrum and the corresponding autocovariance. Unless otherwise stated, a covariance is assumed to be normalized to an autocovariance throughout the rest of the paper. A simulated trajectory from the model is shown in blue in Figure 2 together with the clipped process in red. In this example, the intervals will be dependent, which is generally the case for clipped processes, as mentioned previously. Therefore, the main idea behind the IIA is to overlook this complex dependency structure and approximate it with a binary process where the plus and minus interval distributions are independent and identically distributed (iid). Under this assumption, the relation between the plus and minus distribution directly and explicitly relates to the characteristics, such as the expected value or covariance. Therefore, the clipped process's characteristics can be used to derive an approximation of the excursion distribution through this simplified relation.

However, the key question is which characteristics of the clipped process should be used to construct this binary process. For example, the mean length of the intervals could be used and is given by the well-known formula from Rice (1944). However, since the expected value does not uniquely determine a distribution, a choice of distribution must still be made.

Another choice is to match the covariance of a stationary binary process to the one of the clipped. However, it follows from the inspection paradox that the interval containing zero of a stationary binary process will not have the same interval distribution as the other intervals. While the mathematical foundations and details have long been resolved, see Palm (1943), Khinchin (1955), and Ryll-Nardzewski (1961), these slight differences need to be accounted for when matching the covariance to a stationary binary process.

A way to address this issue is to use a non-stationary binary process and match expected value functions instead. This is the main idea of this paper and the topic of Section 3. Before presenting this approach, we introduce some important properties of the stationary and non-stationary binary processes used and some problems arising from using the covariance function in the ordinary IIA approach.

2.1. **The switch process.** The switch process is a binary process similar to the alternating renewal process but takes the specific values one and minus one. This process is used to approximate the clipped process. Therefore, we present the two versions and some properties of the switch process in this section. We start with the non-stationary version.

Consider the positive line and its origin as a starting reference point and define a random process taking values one and minus one over interlaced intervals of the lengths T_i^+, T_i^-, $i \in \mathbb{N}$. The distributions of these are referred to as the interval distributions to distinguish them from the excursion distribution. We put one over the interval $(0, T_1^+]$, after which we switch to minus one over the interval $(T_1^+, T_1^+ + T_1^-]$, then we switch again to one, and so on. More specifically,

$$D(t) = (-1)^{N(t)}, \tag{1}$$

where $N(t)$ is the renewal process constructed from T_i^+ and T_i^-, $i \in \mathbb{N}$. The resulting one/minus one process is denoted by $D(t)$, $t > 0$, and is referred to as the non-stationary switch process. In Figure 3 (*left*), a trajectory of such a process can be seen.

Since the excursion intervals, above and below zero, for a stationery Gaussian process are symmetric in distribution, we utilize this symmetry by letting $T_i^+ \stackrel{d}{=} T_i^-$ for all $i \in \mathbb{N}$. Further, denote the common cdf of T_i^+ and T_i^- by F for which we assume that the density function f exists and that the distribution has a finite expectation, i.e., $\mu = \mathsf{E}T_i^{\pm} < \infty$

Properties of $D(t)$, $t \geq 0$, can be determined from the distribution F of T_+ and T_-. However, it is important to note that the process $D(t)$ is not stationary due to the special role the origin plays in its definition. For example, the expected value function $E(t) = \mathsf{E}D(t)$ is not constant with time.

Next, we present some key properties of the switch process, which was derived in Bengtsson (2024). Since there is a wide use of the Laplace transform, denoted with $\mathcal{L}(\cdot)$, some properties of this transform are collected in Appendix A.

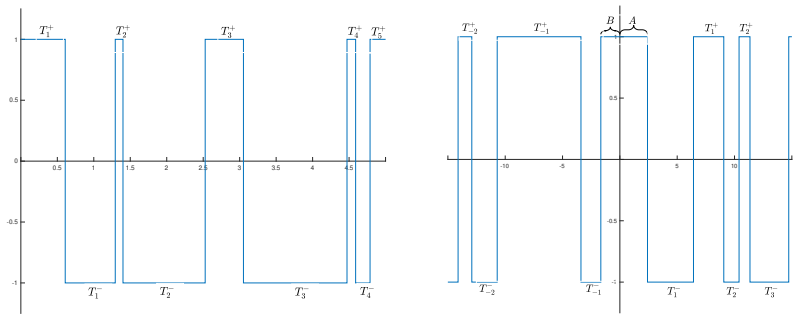

FIGURE 3. The non-stationary switch process *(left)* and the stationary version, *(right)* with $\delta = 1$. The inspection paradox states that the interval that contains zero has the distribution $A + B$, which is not the same as the other interval distributions.

Proposition 1. *Let $D(t)$ be a non-stationary switch process with the expected value function $E(t) = \mathsf{E}D(t)$. Then we have the relations for $s > 0$*

$$\mathcal{L}(E)(s) = \frac{1}{s}\frac{1-\Psi(s)}{1+\Psi(s)}, \tag{2}$$

$$\Psi(s) = \frac{1 - s\mathcal{L}(E)(s)}{1 + s\mathcal{L}(E)(s)}. \tag{3}$$

where $\Psi(s) = \mathcal{L}(f)(s)$.

This proposition provides an explicit link between the expected value function and the distribution of the intervals. This connection will later form the basis for the Slepian-based IIA.

It's clear from the expected value function that the process $D(t)$ is not stationary. However, we can extend the process to the entire real line, use a standard tool from renewal theory, and delay it forward and backward around zero. Then, these delaying distributions can be chosen so that the resulting process is stationary. Since the value of this process at zero will be random, we also need a Bernoulli random variable to determine the value of the process on the interval that contains zero.

Let A, B be the delays forward and backward and δ a Bernoulli random variable taking the value one and minus one. Then, we can define the delayed switch process by

$$D_s(t) = \begin{cases} -\delta, & -B < t < A, \\ \delta D_+(t - A), & t \geqslant A, \\ -\delta D_-(-(t + B)), & t \leqslant -B, \end{cases} \tag{4}$$

where $D_\pm$ are two independent non-stationary switch processes attached at the ends of the delaying interval $[-B, A]$ with the common interval distribution F. This construction

is illustrated in Figure 3 (*left*). We now need the distribution of A, B, δ such that the delayed switch process becomes stationary. For this, we have the following proposition, which follows from applying the key renewal theorem.

Proposition 2. *Let $D_+(t)$ and $D_-(t)$, $t > 0$ be two independent non-stationary switch processes with the interval distribution T, with the distribution F and density f, such that the expected value $\mathsf{E}T = \mu < \infty$. If A, B, δ are non-negative, mutually independent and independent of $D_\pm$ with a distribution is given by*

$$f_{A,B}(a,b) = \frac{f(a+b)}{\mu}$$
$$f_A(t) = f_B(t) = \frac{1-F(t)}{\mu}$$
$$\mathsf{P}(\delta = 1) = \mathsf{P}(\delta = -1) = \frac{1}{2}$$

Then, the delayed switch process in Equation (4) is stationary. We call a delayed switch process with this delay for the stationary switch process.

Remark 1. While it is not necessary that T has a density, for the proposition to hold, it is only necessary that $T_1^+ + T_1^-$ is not sitting on a lattice.

From Proposition 2, we observe the so-called inspection paradox. The interval containing zero will have a different distribution than the regular interval distribution T since it will have the length $A + B$. We illustrate this in Figure 3 (*left*).

The main characteristic of the stationary switch process is the covariance function, which, for the symmetric case, when $T_+ = T_-$ uniquely determines and characterizes the process. In the next proposition, we relate the covariance and the interval distribution.

Proposition 3. *For a fixed $t \geq 0$, the distribution of $D_s(t)$ is uniquely characterized by $P_\delta(t) = \mathsf{P}(D_s(t) = 1 | D_s(0) = \delta)$, $\delta = \pm 1$ that have the following Laplace transforms*

$$\mathcal{L}(P_\delta)(s) = \frac{1}{s}\begin{cases} 1 - \dfrac{1}{\mu s}\dfrac{1-\Psi(s)}{1+\Psi(s)} & ;\delta = 1, \\ \dfrac{1}{\mu s}\dfrac{1-\Psi(s)}{1+\Psi(s)} & ;\delta = -1, \end{cases}$$

where Ψ is the Laplace transform of the probability distribution of $T_i^\pm$'s. The covariance function $R(t) = \mathsf{Cov}(D_s(h), D_s(t+h))$ has the form

$$R(t) = P_1(t) - P_{-1}(t)$$

with the Laplace transform

$$\mathcal{L}(R)(s) = \frac{2}{s\mu}\left(\frac{\mu}{2} - \frac{1}{s}\frac{1-\Psi(s)}{1+\Psi(s)}\right).$$

This connection between the covariance and the interval distribution allows us to express the interval distribution as a function of the covariance in the following way

$$\Psi(s) = \frac{1 - \frac{\mu s}{2} + \frac{\mu s^2}{2}\mathcal{L}(R)(s)}{1 + \frac{\mu s}{2} - \frac{\mu s^2}{2}\mathcal{L}(R)(s)} \tag{5}$$

This equation agrees with formula (215) of Bray et al. (2013). In the next section, we will see how this equation is used to obtain an approximation of the excursion distribution in the ordinary IIA framework.

2.2. **The ordinary IIA.** In the previous section, we saw how the covariance characterizes the stationary process through its relation to the interval distribution. The ordinary IIA uses this relation to approximate this by substituting the covariances function in Equation (5) with the covariance of a clipped process; this is the essence of matching covariance.

An explicit relation exists between the covariance of a stationary Gaussian process and its clipped process for the zero level. The two are related by

$$R_{cl}(t) = \frac{2}{\pi}\arcsin(r(t)).$$

and then from this, we directly obtain the ordinary IIA approximated excursion distribution by combining the above equation with Equation (5) in the following way

$$\Psi_{\text{IIA}}(s) = \frac{1 - \frac{\mu s}{2} + \frac{\mu s^2}{2}\mathcal{L}(R_{cl})(s)}{1 + \frac{\mu s}{2} - \frac{\mu s^2}{2}\mathcal{L}(R_{cl})(s)}. \tag{6}$$

By doing this matching, we overlook the dependency structure and assume that it is sufficiently small to be ignored. However, two fundamental questions arise from this approach that have rarely been addressed in the past, see Lindgren et al. (2022).

The first question is: *Does $\mathcal{L}(R_{cl})(s)$ in equation (6) always lead to a valid probability distribution, i.e., if the IIA is mathematically sound?* This question is equivalent to finding under what conditions a Laplace transform corresponds to probability density functions. There is Bernstein's Theorem, which states if $\Psi_{\text{IIA}}(s)$ is completely monotone, then it's the Laplace transform of a probability distribution, see Widder (1946), Theorem 12a, p.160. To show that a function is completely monotone involves all the derivatives of a function. This is difficult in practice. In fact, even determining when a rational function is a Laplace transform of a probability density has been so far an unsolved problem despite many partial results, see Zemanian (1959, 1961); Sumita and Masuda (1987)

The second question, given an affirmative answer to the first one, is: *Is there any effective and explicit form of this distribution?* These questions will be approached, and partial answers will be obtained after an alternative approach is proposed. In this approach, one matches a non-stationary switch process with the clipped Slepian process with the origin attached to a zero-crossing, which is the topic of the next section.

3. The Slepian-based IIA

The Slepian-based IIA approximates the excursion distribution by independent intervals when the reference point is attached to a level crossing instead of the origin. Conceptually, it is more natural because the crossing instant is the excursion-relevant event while the origin is not. It also avoids the origin location bias, also known as the inspection paradox, as discussed above through the variables (A, B) in the section on the stationary switch process.

Compared to the ordinary IIA, the Slepian-based IIA uses different characteristics than the covariance. Instead of matching the covariances of the two stationary processes, it matches the expected value function of a switch process and the clipped Slepian process. The Slepian model was introduced by Slepian (1963) and models the statistical behavior at the instants of u level up-crossing of a stationary Gaussian process. This stationary Gaussian process needs to be sufficiently smooth to have well-defined level crossings and a covariance function that is at least twice continuously differentiable. For such a Gaussian process $X(t)$ with the covariance function $r(t)$ the Slepian process is given by

$$X_u(t) = u \cdot r(t) - R \cdot \frac{r'(t)}{\sqrt{-r''(0)}} + \Delta(t), \tag{7}$$

where R is a standard Rayleigh variable, with the density $f_R(s) = se^{-s^2/2}$ independent of the non-stationary Gaussian process Δ, with covariance

$$r_\Delta(t, s) = r(t - s) - r(t)r(s) + \frac{r'(t)r'(s)}{r''(0)}.$$

Since we focus on the zero-level crossing, this process reduces to

$$X_0(t) = -R \cdot \frac{r'(t)}{\sqrt{-r''(0)}} + \Delta(t). \tag{8}$$

From the above equation, we see the importance of having finite crossing density since R is scaled by $-r'(t)/\sqrt{-r''(0)}$. For more details on the Slepian process, see Leadbetter et al. (1983). With the Slepian process defined, we can now introduce the clipped Slepian process, which is defined by

$$D_0(t) = \operatorname{sgn}\left(X_0(t)\right),\ t \geq 0,$$

where X_0 is defined in (8). Since the Slepian process models an up-crossing, this equates to the clipped version starting from one. Hence, matching the clipped Slepian process with the switch process is a natural choice.

3.1. The approximated excursion distribution.

Due to the lack of stationarity of both the clipped Slepian process and the switch process, it is more appropriate to match the expected value instead of covariance. The expected value function has an explicit form and is given in the following proposition.

Proposition 4. *Let $E_0(t) = \mathsf{E}D_0(t)$ be the expectation of a Slepian process clipped at zero. Then for $t \geqslant 0$,*

$$E_0(t) = -\frac{1}{\sqrt{-r''(0)}} \frac{r'(t)}{\sqrt{1-r(t)^2}}, \tag{9}$$

where $r(t)$ is the covariance function of the original Gaussian process, for which it is assumed that $r''(t)$ is well defined for $t \geqslant 0$.

Proof. We have

$$\mathsf{E}(D_0(t)) = \mathsf{E}\big(I(X_0(t) > 0) - I(X_0(t) \leqslant 0)\big) = 2\mathsf{P}(X_0(t) > 0) - 1,$$

the probability of $X_0(t)$ being above zero, depends on the sign of $r'(t)$, and we have

$$\mathsf{P}(X_0(t) > 0) = \begin{cases} \mathsf{P}\left(R < \Delta(t)\frac{\sqrt{-r''(0)}}{r'(t)}\right) & : r'(t) > 0, \\ \mathsf{P}(\Delta(t) > 0) & : r'(t) = 0, \\ 1 - \mathsf{P}\left(R \leq \Delta(t)\frac{\sqrt{-r''(0)}}{r'(t)}\right) & : r'(t) < 0. \end{cases}$$

The probability of a Rayleigh random variable being less than a normal random variable is easy to obtain, and regardless of the sign of $r'(t)$, we have

$$\mathsf{P}(X_0(t) > 0) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{-r''(0)}} \frac{r'(t)}{\sqrt{1-r(t)^2}}\right),$$

which leads to (9). □

From the proposition, it's clear that the behavior of the expected value function E_0 of the clipped Slepian process is largely determined by the derivative of $r(t)$. This thus serves as a direct link to the covariance of the process we are interested in and the expected value function.

However, matching in the time domain is cumbersome, which is why it's done in the Laplace domain. This is done by substituting the expected value of the clipped Slepian process from Proposition 4 into the second equation of Proposition 1. Hence, we directly obtain an expression for the approximated excursion distribution in the Laplace domain. We summarize this in the next proposition.

Proposition 5. *Let E_0 be the expected value of the Clipped Slepian process, then the Slepian-based IIA approximation of the excursion distribution denoted by $\widehat{T}$ is given in the Laplace domain by*

$$\Psi_{\widehat{T}}(s) = \frac{1 - s\mathcal{L}(E_0)(s)}{1 + s\mathcal{L}(E_0)(s)}.$$

The approximated excursion distribution can be obtained in the time domain by inverting the above Laplace transform. This can be done either analytically or numerically. However, there are still several important questions that need to be answered. For example, what conditions on E_0 guarantee that the approximated excursion distribution is

indeed a valid probability distribution? This is difficult to verify for arbitrary expected value functions due to Bernstein's theorem, as previously mentioned. Further, what is the relation between the Slepian-based IIA and the ordinary IIA for the approximating zero-level excursions? The next section deals with these questions and properties of the Slepian-based IIA.

3.2. **Properties of the Slepian-based IIA.** One of the main advantages of the Slepian-based IIA is that approximated excursion distribution can be retrieved explicitly for a large class of stationary Gaussian processes in terms of a stochastic representation. The following theorem on this representation thus serves as the probabilistic foundation for the Slepian-based IIA and follows directly from Theorem 1 and Corollary 2 in Bengtsson (2024).

Theorem 1. *If the expected value function $E_0(t)$ of the clipped Slepian process is differentiable with a non-positive derivative for $t \geqslant 0$. Then, the Slepian-based IIA approximation of the excursion distribution has the following stochastic representation*

$$\widehat{T} = \sum_{i=1}^{\nu} \widetilde{T}_i. \tag{10}$$

Where $\widetilde{T}_i$ is an iid sequence of positive random variables having the survival function equal to $E_0(t)$ and ν follows a geometric distribution with the probability mass function $\mathsf{P}(\nu = k) = (1/2)^k$, $k \in \mathbb{N}$.

A random variable $\widehat{T}$, and its distribution with the stochastic representation of Theorem 1 is called 2-geometric divisible, and the distribution of $\tilde{T}_i$'s is referred to as a geometric divisor of the distribution of $\widehat{T}$. For a brief introduction to this class of distributions, see Bengtsson (2024). The representation of Theorem 1 lacks usefulness unless there is a way to express $\widetilde{T}$ in terms of some characteristic. Luckily, there is such a way, which we highlight in the next corollary.

Corollary 1. *The cdf of the divisor $\widetilde{T}$ is given by*

$$F_{\widetilde{T}}(t) = 1 - E_0(t).$$

The implication of this is that there are several methods to obtain $\widehat{T}$ from $\widetilde{T}$. The choice of method can, therefore, be tailored depending on the particular properties of $\widehat{T}$ we are interested in. For example, the expected value of $\widehat{T}$ follows directly from Wald's equation and is $\mathsf{E}\widehat{T} = 2\mathsf{E}\widetilde{T}$.

If the distribution of $\widehat{T}$ is sought, the standard method is through the Laplace transform since we deal with sums of random variables. In the Laplace domain, then we have the following relation between $\widehat{T}$ and $\widetilde{T}$

$$\Psi_{\widehat{T}} = \frac{\frac{1}{2}\Psi_{\widetilde{T}}}{1 - \frac{1}{2}\Psi_{\widetilde{T}}}. \tag{11}$$

If the analytical inverse Laplace transform of Equation (11) is unwieldy, the representation still allows for numerical methods. It should also be noted that the stochastic representation in Theorem 1 allows for straightforward sampling of values from $\widehat{T}$, opening up the use of Monte Carlo methods to approach this problem.

We have seen how Theorem 1 provides the theoretical foundation for the Slepian-based IIA and has several practical consequences for obtaining the approximated excursion distribution. However, one question remains: what is the relation to the ordinary IIA framework?

The core idea of both the ordinary IIA and Slepian-based IIA are very similar, namely the idea of matching characteristics of a clipped process to a version of the switch process. It turns out that the two approaches coincide for the zero level, which we show in the following Theorem.

Theorem 2. *Let $R(t)$ be the covariance of a stationary switch process and $E(t)$ be the expectation of the corresponding non-stationary switch process. Then*

$$R'(t) = -\frac{2}{\mu}E(t),$$

where μ is the first moment of the interval distribution.

Similarly, if $R_{cl}(t)$ is the covariance of a clipped smooth stationary Gaussian process and $E_0(t)$ is the expected value of the clipped Slepian process for the level $u = 0$, then

$$R'_{cl}(t) = -\frac{2}{\mu}E_0(t),$$

where $\mu = \frac{\pi}{\sqrt{-r''(0)}}$ represents also the average length of the excursion interval.

Proof. Theorem 2 in Bengtsson (2024) establishes the first part of the result. For the second part we have

$$\begin{aligned} R'_{cl}(t) &= \frac{d}{dt}\left(\frac{2}{\pi}\arcsin r(t)\right) \\ &= \frac{2}{\pi}\frac{r'(t)}{\sqrt{1-r(t)^2}}. \end{aligned}$$

Let $\mu = \frac{\pi}{\sqrt{-r''(0)}}$, then, by Proposition 4,

$$R'_{cl}(t) = -\frac{2}{\mu}E_0(t).$$

Where μ is the average interval length of the excursion intervals is a well-known fact following from the ergodic theorem. □

Remark 2. In Theorem 2, two seemingly unrelated binary processes have the same relation between expected value and covariance when the process is attached to a jump. It is an interesting question if this relation generalizes to all binary processes with finite jump intensity on any compact set such that the intervals between jumps are well defined.

We obtain the following rather unexpected equivalence between the two IIA approaches.

Corollary 2. *Matching the covariance of the clipped process to the covariance of a stationary switch process is equivalent to matching the expected value of the clipped Slepian process with the expected value of a non-stationary switch process.*

Additionally, Theorem 2 also has implications for Theorem 1. Since any covariance function $R(t)$ that leads to the clipped covariance function $R_{cl}(t)$ having a non-positive derivative for $t > 0$ will correspond to an expected value function that satisfies the conditions of Theorem 1, the theorem also provides a foundation for the ordinary IIA framework for a large class of Gaussian processes.

We end this section with a caveat on when both the Slepian-based and ordinary IIA approaches encounter technical difficulties. If $\mathsf{E}\tilde{T}_i = \infty$ this implies that $\mathsf{E}\widehat{T} = \infty$ in (10), which leads to a limitation for the choice of $E_0(t)$ in order to have a sensible IIA.

Corollary 3. *If $S(t)$ is a survival function of a distribution such that $\int_0^\infty S(t)\, dt = \infty$, then it cannot be the expected value function of any stationary switch process. In particular, if a covariance function $r(t)$ of a Gaussian process yields a non-increasing and non-integrable expected value function*

$$E_0(t) = -\frac{1}{\sqrt{-r''(0)}} \frac{r'(t)}{\sqrt{1 - r^2(t)}},$$

then there is no switch process that can be used for the purpose of matching.

With this caveat in place, we will next treat the general problem of estimating the persistency coefficients within the IIA framework.

4. Persistency coefficient through the IIA

The persistency coefficient is defined by Bray et al. (2013) as the asymptotic decay of the first persistence probability. This is the probability that a zero mean process $X(t)$, $t \geq 0$ starting at zero does not change the sign between $(0, \tau)$. It has been argued in Sire (2008) that if $|r(t)| < C/t$, then the tail of this probability decays exponentially.

In physics and engineering, the exponent, or coefficient, is an important characteristic of stochastic systems since it provides insight into how long a system can stay in a certain state. The problem is, as pointed out by Bray et al. (2013), that this coefficient is process-specific and generally hard to find analytically. See Bray et al. (2013) for an overview of both the persistency coefficient and approximation methods for it.

In the IIA framework, the persistency coefficient can be approximated by investigating the tail of $\widehat{T}$. If the conditions of Theorem 1 are satisfied, this can be done through inverting Equation (11) or the equation in Proposition 5. Since this is the form of a ratio of two functions, this can present some difficulty.

In physics, this difficulty has been circumvented by instead using the pole of Equation (5) with the largest negative real part as the approximation for the persistency coefficient. There are several challenges to such an approach. Firstly, the Laplace transform

considered needs to correspond to a probability distribution, which is not obvious, as it has already been discussed in previous sections.

Another challenge is whether finding a pole with the largest negative real value guarantees the corresponding exponential tail behavior. This can be justified if the distribution can be expressed in terms of a finite mixture of exponential distributions, Sumita and Masuda (1987), so its Laplace transform is a rational function. Alternatively, one can assume the rational form of the Laplace transform and determine the behavior by the partial fraction decomposition. However, for most stationary Gaussian processes used in physics, the covariance of the clipped process does not lead to a rational function. Moreover, the poles with the largest negative real part do not necessarily lay on the real axis. In general, the mathematical justification of asymptotic behavior through the poles is not automatic. Thus, the use of the poles to find the persistency coefficient should be viewed as a heuristic one. However, when applicable, there is a simple criterion to find the pole within the Slepian-based IIA, which follows directly from Proposition 5.

Proposition 6. *Let $\Psi_{\widehat{T}}$ be the Slepian-based IIA approximation from Proposition 5. Then, the largest negative real pole of $\Psi_{\widehat{T}}$ is found as the largest negative real solution to the equation*

$$s\mathcal{L}(E_0)(s) + 1 = 0.$$

Additionally, if the conditions of Theorem 1 are satisfied, then the largest negative real pole of the $\Psi_{\widehat{T}}$ can be found by investigating its divisor. The largest negative real pole of $\Psi_{\widehat{T}}$ corresponds to the largest negative real solution to the equation $\Psi_{\tilde{T}}(s) = 2$. It is clear that the tail of a geometrically divisible random variable is heavier than that of its divisor. However, there is no strict relation between the two tails, as the following example shows. Nevertheless, in the next proposition, we connect the exponential bounds of the tails.

Example 1. Let $\tilde{T}_i = \alpha$, for $\alpha > 0$, and ν geometric random variable with $p = 1/2$. Then, by direct evaluation, $T = \sum_{i=1}^{\nu} \tilde{T}_i$ has the rate (persistency) $\log 2/\alpha$.

Proposition 7. *Let T have the stochastic representation of Theorem 1 with the divisor $\widetilde{T}$ and assume that there is some $\beta > 0$, then we have the two statements.*

i) If $\mathsf{P}(\tilde{T} > \tau) \leq e^{-\beta\tau}$, then $\mathsf{P}(T > \tau) \leq e^{-\beta\tau/2}$.
ii) If $\mathsf{P}(\tilde{T} > \tau) \geq e^{-\beta\tau}$, then $\mathsf{P}(T > \tau) \geq e^{-\beta\tau/2}$

Proof. The proof follows by induction, and we will prove i) in full and ii) can be obtained by changing the direction of the inequalities used in the proof. For $k \in \mathbb{N}$ independent copies $\tilde{T}_i$, we show that

$$\mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > \tau) \leq e^{-b\tau} \sum_{j=0}^{k-1} \frac{b^j \tau^j}{j!}.$$

Assume that the above is valid for a certain $k \in \mathbb{N}$, and F_k is the cdf of $\tilde{T}_1 + \cdots + \tilde{T}_k$, then

$$\begin{aligned}
\mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_{k+1} > \tau) &= \int_0^\tau \mathsf{P}(\tilde{T}_{k+1} > \tau - s)\ dF_k(s) + \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > \tau) \\
&\leq \int_0^\tau e^{-b(\tau-s)}\ dF_k(s) + \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > \tau) \\
&= \left(\Big(1 - \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > s)\Big)\, e^{-b(\tau-s)} \right) \Bigg|_{s=0}^{\tau} \\
&\quad - \int_0^\tau \Big(1 - \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > s)\Big)\, be^{-b(\tau-s)}\ ds \\
&\quad + \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > \tau) \\
&= 1 - \int_0^\tau be^{-b(\tau-s)} ds + \int_0^\tau \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > s) be^{-b(\tau-s)}\ ds \\
&= e^{-b\tau} \left(1 + b \int_0^\tau \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > s) e^{bs}\ ds \right) \\
&\leq e^{-b\tau} \left(1 + \sum_{j=0}^{k-1} \frac{b^{j+1}}{j!} \int_0^\tau s^j\ ds \right) = e^{-b\tau} \left(1 + \sum_{j=0}^{k-1} \frac{b^{j+1}}{(j+1)!} \tau^{j+1} \right),
\end{aligned}$$

which concludes the induction proof. The result follows from

$$\mathsf{P}(T \geq \tau) = \sum_{k=1}^{\infty} \frac{1}{2^k} \mathsf{P}(\tilde{T}_1 + \cdots + \tilde{T}_k > \tau) \leq e^{-b\tau} \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{j=0}^{k-1} \frac{b^j \tau^j}{j!} = e^{-b\tau} \sum_{j=0}^{\infty} \frac{b^j \tau^j}{j!} \frac{1}{2^j}.$$

□

As stated previously, the relation between the tail of the divisor and the full distribution does not follow a general structure. This is why simulation-based methods might be preferable for approximating the persistency coefficient. Regardless if trajectories are simulated or samples are obtained using Theorem 1, there is still a need to approximate the persistency coefficient from these samples. This can be easily done under assumptions on the tail behavior.

Suppose that the tail has an exponential form, i.e., for large t, there is a $\theta > 0$ such that

$$\mathsf{P}(\widehat{T} > t) \sim e^{-\theta t}, \tag{12}$$

the persistency coefficient can be approximated using the ordinary least squares estimator for θ on the empirical survival function since

$$\ln(\mathsf{P}(\widehat{T} > t)) \approx a + \theta t. \tag{13}$$

Of course, more elaborate tail estimators could be used in this context, but since reasonably large samples are obtainable, striving for efficiency is not essential here. The core idea of estimating tail coefficients using linearization and least squares on the empirical distribution function or a QQ plot is well established. Early results of this idea were

presented by Schultze and Steinebach (1996) (ECDF) and Kratz and Resnick (1996) for the QQ plot. However, in actual computation $\mathsf{P}(\widehat{T} > t)$ is replaced by $1 - \hat{F}(t)$, where $\hat{F}$ is the empirical CDF of the sample from $\widehat{T}$. This method alleviates the problems that follow from estimating persistency coefficients using the poles of the Laplace transform.

We conclude this section with a discussion of cases where the pole method is ill-suited to approximate the persistency coefficient and where the IIA simply fails to produce meaningful approximations. Not much is known about persistency coefficients when the covariance function oscillates. This is also one of the cases where using the pole method leads to problems. Wilson and Hopcraft (2017) studied processes with the covariance function

$$r(t) = \cos(\alpha t)\, e^{-\frac{t^2}{2}},$$

for which a large discrepancy was found between the largest negative real pole obtained using the IIA framework and the persistency coefficient obtained from simulations. It seems that care needs to be taken when applying the IIA framework to processes with oscillating covariance functions. This has direct consequences for practical application if one wants to model sea states using the Torsethaugen model, for example.

It is not only processes with oscillating covariance functions that pose a problem for the IIA. There is an entire class of processes where the IIA approach fails, which we illustrate with a counterexample. Recall the classical result of Theorem 2 from Newell and Rosenblatt (1962).

Theorem 3. *If $X(t)$ is a Gaussian stationary process with $EX(t) = 0$ and $r(t) < Ct^{-\alpha}$ for some $\alpha > 0$, $C > 0$ and all $t > 0$, then for some $K > 0$:*

$$\mathsf{P}\left(X(t) \geq 0, t \in [0, \tau]\right) < \begin{cases} e^{-K\tau}; & 1 < \alpha, \\ e^{-K\tau/\log\tau}; & \alpha = 1, \\ e^{-K\tau^{\alpha}}; & 0 < \alpha < 1. \end{cases}$$

From this theorem, we obtain the following lemma needed for the counterexample.

Lemma 1. *Let $X(t)$, $t \in \mathbb{R}$ be a stationary Gaussian process with twice differentiable covariance function, $r(t)$ such that for a certain $\beta > 1$*

$$\lim_{t\to\infty} r'(t) = O(t^{-\beta}),$$

and $X_0(t)$, $t \geq 0$ be the Slepian process associated with $X(t)$. If the process satisfies also the assumptions of Theorem 1, then

$$\mathsf{P}(\widehat{T} \geq \tau) \geq O(\tau^{-\beta}),$$

and there exists no $\beta' > 1$ such that

$$\mathsf{P}(X_0(t) > 0, t \in (0, \tau)) \geq O(\tau^{-\beta'}).$$

Proof. We have

$$\mathsf{P}(\widehat{T} \geq \tau) \geq \mathsf{P}(\tilde{T} \geq \tau) = -\frac{1}{\sqrt{r''(0)}} \frac{r'(\tau)}{\sqrt{1-r^2(\tau)}} = O\left(\tau^{-\beta}\right).$$

For the second part, if there is $\beta' > 1$ such that

$$\mathsf{P}(X_0(t) > 0, t \in (0,\tau)) \geq O(\tau^{-\beta'}),$$

then

$$\begin{aligned}\mathsf{P}\left(X(t) \geq 0, t \in [0,\tau]\right) &= \frac{1}{2\mu} \int_{\tau}^{\infty} \mathsf{P}\left(X_0(t) > 0, t \in (0,u)\right)\ du \\ &\geq O(\tau^{-(\beta'-1)},\end{aligned}$$

which contradicts Theorem 3. □

From the above result, we see that for the processes satisfying Lemma 1, the tail behavior of $\widehat{T}$ based on the IIA is of the power order, and this cannot be the case for the actual distribution. Thus, under these circumstances, using the IIA to approximate the persistency coefficient or the tail behavior is not valid. The next example illustrates this.

Example 2. Consider a stationary Gaussian process with the covariance function

$$r(t) = \left(1 + \frac{t^2}{2}\right)^{-\alpha},\ \alpha > 0.$$

We note that this is a valid covariance function since it is the characteristic function of a generalized symmetric Laplace distribution. The corresponding covariance function of the clipped process is

$$R_{cl}(t) = \frac{2}{\pi} \arcsin\left(\left(1 + \frac{t^2}{2}\right)^{-\alpha}\right).$$

and thus

$$E_0(t) = \frac{\sqrt{\alpha} t \left(1 + t^2/2\right)^{-(1+\alpha)}}{\sqrt{1 - \left(1 + t^2/2\right)^{-2\alpha}}}.$$

This function satisfies the condition of Theorem 1 and yields a divisor with finite expectation. Hence, the approximated excursion distribution $\widehat{T}$ will also have finite expectation. However, we observe that Theorem 3 can be applied, and the tail behavior of $\widehat{T}$ has nothing to do with the tails of the excursion times.

5. Applications and examples

There are many covariance functions leading to positive and decreasing expected value functions of the clipped Slepian process. In these cases, Theorem 1 allows for both the approximation of the persistency coefficient, denoted by θ throughout this section, and the full approximate excursion distribution. A cautionary example is also presented to

highlight the danger of naively using the IIA framework to approximate the persistency coefficient. We summarize the cases considered in this section in Table 2.

TABLE 2. Processes and their characteristics.

PROCESS $X(t)$	COVARIANCE $r(t)$	EXPECTED VALUE $E_0(t)$
Diffusion in $\mathbb{R}^d$, $d \in \mathbb{N}$	$\dfrac{1}{\cosh^{d/2} t/2}$	$\dfrac{1}{\cosh t/2}\sqrt{\dfrac{d}{2}\cdot\dfrac{\cosh^2 t/2-1}{\cosh^d t/2-1}}$
Random acceleration	$\left(3-e^{-\lvert t\rvert}\right)e^{-\lvert t\rvert/2}/2$	$\sqrt{\dfrac{3}{4e^t-1}}$
Shifted Gaussian, $\alpha \in \mathbb{R}$	$\cos\alpha t\cdot e^{-t^2/2}$	$\dfrac{1}{\sqrt{1+\alpha^2}}\dfrac{\alpha\sin\alpha t+t\cos\alpha t}{\sqrt{1-\cos^2\alpha t\cdot e^{-t^2}}}\cdot e^{-t^2/2}$
Matérn, $\nu \geq 2$	$C_\nu\cdot t^\nu K_\nu(t)$	$\sqrt{2(\nu-1)}\dfrac{t^\nu K_{\nu-1}(t)}{\sqrt{C_\nu^{-2}-t^{2\nu}K_\nu^2(t)}}$, $C_\nu=\frac{2^{1-\nu}}{\Gamma(\nu)}$

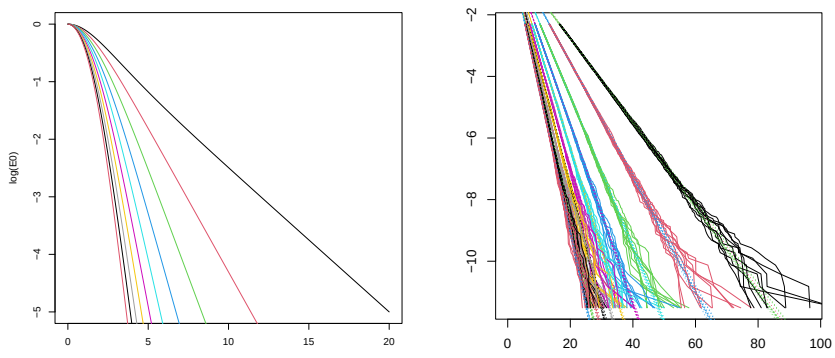

FIGURE 4. *(right)* The logarithm of E_0 for $d = 1, ..., 10$. The steeper curves are associated with higher dimensions. *(left)* Logarithm of the empirical cdf from the approximated excursion distribution for the diffusion process, which is used for estimating the persistency coefficient.

5.1. **The d-dimensional diffusion.** The d-dimensional diffusion is often referred to the scalar field $\phi(\mathbf{x}, t)$, with the location $\mathbf{x} \in \mathbb{R}^d$ and time $t > 0$ that is the solution to

$$\frac{\partial\phi}{\partial t} = \Delta\phi(\mathbf{x}, t) = \nabla^2\phi(\mathbf{x}, t),$$

where the initial condition at $\tau = 0$ is set to the d-dimensional (in argument) Gaussian noise. Then, for large t, the system is approximately described by a stationary Gaussian field

$$X(t) = \frac{\phi(\mathbf{0}, e^t)}{\sqrt{\mathsf{E}(\phi^2(\mathbf{0}, e^t)}},$$

TABLE 3. Approximations of persistency coefficients for diffusions. For $d = 1, 2$, the true persistency is 0.1203 and 0.1875, respectively.

DIMENSION	Persistency coefficient	RIND2022	NL2001
1	0.1360 ± 0.0012	0.1206	0.1205
2	0.1858 ± 0.0017	0.1874	0.1875
3	0.2441 ± 0.0014	0.2382	0.2382
4	0.2901 ± 0.0011	0.2805	0.2806
5	0.3286 ± 0.0016	0.3171	0.3173
6	0.3618 ± 0.0025	*	*
7	0.3915 ± 0.0033	*	*
8	0.4195 ± 0.0034	*	*
9	0.4446 ± 0.0030	*	*
10	0.4668 ± 0.0034	0.4589	0.4587

with the covariance function

$$r(t) = \frac{1}{\cosh^{d/2}(t/2)}.$$

The corresponding expected value function can be found in Table 2 and it is clear that $E_0(t)$ is a strictly decreasing function in t for each $d \in \mathbb{N}$ and thus satisfies the condition of Theorem 1. It is also easy to observe that the tail of E_0 is almost exponential, which is illustrated in Figure 4 (*left)*.

To obtain an approximation for the persistency coefficient, we simulate from the divisor using the outlined method in Appendix B and the stochastic representation of Theorem 1. The resulting persistency coefficient estimates can be found in Table 3. Each 95% confidence interval is based on 10 estimates of the persistency coefficient, which are in turn based on sampling 10^5 observation from the divisor. The 10^4 largest of these was then used for the least squares estimate of the persistency coefficients. In the table, we also include the results from a large and well-controlled simulation experiment of over 10^8 realizations of first crossing events reported Newman and Loinaz (2001). The exact computation method based on the generalized Rice formula and its implementation is given in the routing RIND and presented in Lindgren et al. (2022).

The conclusion from the simulation study is that the persistency coefficient obtained using the Slepian-based IIA is reasonable in comparison to the analytically derived values and the ones obtained by Newman and Loinaz (2001) and from using RIND. However, the method's limited accuracy should also be noted.

5.2. **Random acceleration process.** The random acceleration process models a particle on the line whose acceleration is subject to a zero mean Gaussian noise force. In exponential time, i.e., $T = e^t$, the process is defined through the following equation of motion

$$\frac{d^2Y}{dT^2} = \eta(T),$$

where $\eta(T)$ is a white noise Gaussian process, i.e., the derivative of a Brownian motion (Wiener process) or a so-called δ-correlated Gaussian white noise process, where δ stands for the Dirac's delta. More explicitly, $Y(T) = \int_0^T B(u)\ du$, where B is the standard Brownian motion. It is easy to check that the Gaussian process $X(t) = \sqrt{3/2}e^{-3t/2}Y(e^t)$ is a stationary Gaussian process with the covariance function

$$r(t) = \frac{3}{2}e^{-\frac{|t|}{2}} - \frac{1}{2}e^{-\frac{3|t|}{2}}.$$

By elementary computation the first and second derivative of $r(t)$ are well defined with $r'(0) = 0$ and $r''(0) = -3/4$. However, the third derivative has a discontinuity at zero, which Sire (2007, 2008) argued leads to a poor approximation of the persistency coefficient. However, the process is sufficiently smooth since it has finite crossing intensity to construct a Slepian process. Hence, the expected value function of the clipped Slepian process for this example becomes

$$E_0(t) = \frac{e^{-t/2}(1-e^{-t})\sqrt{3}}{\sqrt{4-(3-e^{-t})^2e^{-t}}} = \sqrt{\frac{3}{4e^t-1}}.$$

This function satisfies the conditions in Theorem 1, and we use this observation to sample from the approximated excursion distribution through the divisor; see Appendix B.

Taking a similar approach as for the diffusion example, we estimate the persistency coefficient based on a sample size of 10^7 out of the 10^5 largest where used for the least square estimate. This approach was repeated 10 times which yielded the following 95% confidence bound

$$\hat{\theta} = 0.2647 \pm 0.00083.$$

This approximation is in line with the estimates from Sire (2007) ($\theta_{\text{app}} = 0.2647$) using the pole method and which should be compared to the true value of $\theta = 0.25$ derived by Sinai (1992).

5.3. **Shifted and non-shifted Gaussian covariance.** Consider a stochastic process with the covariance function

$$r(t) = \cos(\alpha t)\ e^{-t^2/2},$$

for some $\alpha \geqslant 0$. This case has been treated extensively by Wilson and Hopcraft (2017), which serves as a good overview of this case. We consider this example for three main reasons. The first is that it is the limiting case of the Matérn covariance when the smoothness parameter goes to infinity. Secondly, the true persistency coefficient is not known for this process. Thirdly, for any $\alpha > 0$, the covariance function is oscillating. Hence, we will first treat the case when $\alpha = 0$ and then, after this, consider the case $\alpha = 2$.

The expected value function E_0 can be found in Table 2 and for $\alpha = 0$, it satisfies the conditions of Theorem 1 and we use this observation when sampling from the approximated excursion distribution. The details can be found in Appendix B, and we sample

10^7, out of which we use the $2 \cdot 10^6$ largest to estimate the persistency coefficient. This is repeated 10 times, leading to the following 95% confidence interval

$$\hat{\theta} = 0.4116 \pm 0.00017.$$

To compare, we simulate trajectories using the WAFO package (WAFO-group, 2017). The length of each trajectory is 10^7, and 10^3 is simulated. From these, we obtain an estimated persistency coefficient of $\theta_{\text{sim}} = 0.4199 \pm 0.00058$. Comparing these two estimates to the IIA using the pole methods by Sire (2008) ($\theta_{\text{IIA}} = 0.4115$), we observe that they are all reasonably close.

Consider the case where $\alpha = 2$. For this case, E_0 oscillates, and hence Theorem 1, can not be relied on to ensure the validity of the IIA approach. However, if one naively applies the IIA framework and uses the pole method to approximate the persistency coefficient, an estimate of $\theta_{\text{IIA}} = 2.3522$ is obtained. Using the previously mentioned setup for simulating trajectories, an approximation of $\theta_{\text{sim}} = 1.3795 \pm 0.0012$ is obtained. Hence, it is clear that the use of the IIA and finding the pole is not suitable for approximating the persistency coefficient.

We only consider the largest negative real pole here. However, if a numerical search for the complex poles, a pair of conjugate poles is located at $1.05 \pm 0.53\ i$. While the real part of this pair is closer to the approximated persistency coefficient using simulation, it is still not an exact approximation. However, by multiplying this pair of complex conjugate poles, the value 1.3834 is obtained, which is remarkably close to the value from simulated trajectories. This seems to hold for several $\alpha > 0$. We, however, can't necessarily see the reason why these values are close to the persistency coefficients obtained by simulating trajectories. In conclusion, the case when $\alpha > 0$ shows that care is needed since the IIA approach might break down for certain classes of covariance functions.

5.4. **Matérn covariance.** The Matérn covariance

$$r(t) = \frac{2^{1-\nu}}{\Gamma(\nu)} \cdot t^{\nu} K_{\nu}(t),$$

$\nu > 0$ is widely used in spatial statistics. The parameter ν determines the smoothness of the process. Several well-known covariance functions are special cases of this class such as the exponential ($\nu = 1$) and the previously treated Gaussian covariance ($\nu \to \infty$). In Figure 5 *(Left)*, the covariances of the underlying Gaussian processes are presented for selected values of ν.

To ensure that the process has a finite crossing density, we limit ourselves to $\nu \geq 2$, thus allowing for the construction of the Slepian process. A key observation is that in this case, the expected value of the clipped Slepian process E_0 will satisfy the conditions of Theorem 1 and can be found in Table 2. We present the survival functions of the divisor in Figure 5 *(Middle)* in which we see that the stronger dependence, as shown in the covariance r, translates to heavier tails in the divisor.

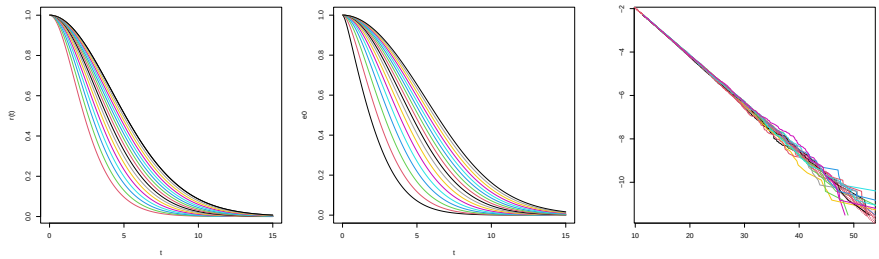

FIGURE 5. The Matérn covariance case. *Left:* The covariances r for $\nu = l/2$, $l = 4, 5, \ldots, 20$, the larger ν the stronger dependence. *Middle:* The survival functions E_0 of the geometric divisor, the larger ν the heavier tail.

Remark 3. A re-parameterized version of the Matérn covariance is often used in spatial statistics. By letting $t = \sqrt{2\nu}\ d/\rho$, where ρ is an additional parameter and $d \geqslant 0$ is the new argument. The link between the persistency coefficients in the two parameterizations can be derived by observing that if $P(T > t) \approx C\ e^{-\theta t}$, then $P(T > \sqrt{2\nu}\ d/\rho) \approx C\ e^{-\theta\sqrt{2\nu}\ d/\rho}$. The re-parameterized persistency coefficient can be obtained by simply scaling the estimate by $\sqrt{2\nu}\ d/\rho$.

Generating random variables from these distributions can be numerically implemented, although the presence of the modified Bessel function may constitute a numerical challenge. For the special cases, $\nu = k + 1/2$, $k \in \mathbb{N}$, we have the explicit formulas for the Bessel functions, and this can be utilized in random number generations, as discussed in Appendix B.

In Figure 5 *(Right)*, the results of simulations in the case of $\nu = 5/2$ are presented. This value was chosen because the random simulation from the geometric divisor could be devised by a simple inversion of E_0. Simulations for this case lead to the IIA persistency coefficient $\hat{\theta} = 0.2188 \pm 0.0011$. The methodology of the obtained approximation has been described in previous sections. This is remarkably close to the persistency coefficient derived using WAFO. Using the same method for WAFO simulations as before, we obtain a 95% interval of $\theta_{\text{sim}} = 0.2184 \pm 0.00026$.

6. Conclusion

The IIA method of obtaining the excursion distributions has been explored, and its benefits and limitations have been determined. Several important questions have been posed, such as when the IIA framework is mathematically sound and whether explicit expressions of the approximated excursion distribution exist. By introducing the clipped Slepian process and matching its expected value function with that of the switch process, the validity of the IIA framework is shown for a large class of covariance functions. For this class, an explicit representation of the approximated excursion distribution exists. This representation allows additional ways to estimate persistency coefficients using Mote Carlo methods.

A natural extension of this approach to non-zero level crossings will be investigated in the future.

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Appendix A. Laplace transform

Here, we establish the notation of the Laplace, which is the main tool used in many of our arguments. Let us recall that for a positive and bounded function $P(t)$, $t > 0$, its Laplace transform $\mathcal{L}(P)(s)$, $s > 0$, is defined through

$$\mathcal{L}(P)(s) = \int_0^\infty P(t)e^{-ts}\, dt.$$

The Laplace transform can also be extended for $s < 0$, and since it is a decreasing function of s, then there exists the smallest number $s_0 \geq -\infty$ for which $\mathcal{L}(P)(s)$ is finite for all $s > s_0$. This value is referred to as the largest negative pole of the respective Laplace transform. For a distribution on the positive half line given by a cdf F, its Laplace transform Ψ is given by

$$\Psi(s) = \int_0^\infty e^{-ts}\, dF(t).$$

so that if the distribution has a density f, then

$$\mathcal{L}(f)(s) = \Psi(s).$$

Similarly, we have the Laplace transform of the CDF

$$\mathcal{L}(F)(s) = \Psi(s)/s.$$

Appendix B. Sampling from the Geometric divisor

Here, we provide details of the simulators of various geometric divisors utilized in the IIA approximations based on Theorem 1.

B.1. **Diffusion.** Let us start from the diffusion in $d = 2$. Since $\operatorname{arccosh}(x) = \log(x + \sqrt{x^2 - 1})$, $x > 1$, we have the following generator of the geometric divisor

$$\tilde{T}_2 = 2\ln\left(1 + \sqrt{1 - U^2}\right) - 2\ln U. \tag{14}$$

For the diffusion in dimension $d = 1$, the survival function can be written as

$$E_{0,1}(t) = \frac{\sqrt{2}}{2}\sqrt{\cosh^{-1}(t/2) + \cosh^{-2}(t/2)},$$

which has the following survival inverse simulator

$$\tilde{T}_1 = 2\ln\left(1 + \sqrt{1 + 8U^2} + \sqrt{2}\sqrt{1 + 4U^2 - \sqrt{1 + 8U^2}}\right) - 4(\ln U + \ln 2).$$

For $d > 2$, we have

$$E_{0,d}(t) = E_{0,d-1}(t)\sqrt{\frac{d(\cosh^{d-1}(t/2) - 1)}{(d-1)(\cosh^{d}(t/2) - 1)}}.$$

It is easy to verify that

$$G_d(t) = \frac{d(\cosh^{d-1}(t/2) - 1)}{(d-1)(\cosh^d(t/2) - 1)}$$

is also a survival function; thus if one can obtain its inverse G_d^{-1}, then the recursive formula for a sampler from $E_{0,d}$ is

$$\tilde{T}_d = \min\left(\tilde{T}_{d-1}, G_d^{-1}(U^2)\right),$$

where the uniform variable U is independent of $\tilde{T}_{d-1}$. To find the inverse G_d^{-1}, let us consider the function $b_d(a)$, $a > 1$, $d \geq 3$ that is the inverse to the convex polynomial function on the positive half-line

$$a_d(b) = 1 + b + \cdots + b^{d-1}.$$

Then

$$G_d^{-1}(g) = 2\ln\left(1/b_d\left(\frac{d}{d - g(d-1)}\right) + \sqrt{1/b_d^2\left(\frac{d}{d - g(d-1)}\right) - 1}\right).$$

Since $a_d(b)$, $b > 0$ is a convex polynomial function, one can efficiently evaluate its inverse using, for example, the Newton–Raphson method.

B.2. **Random acceleration process.** Obtaining a random number from the divisor for the random acceleration process is straightforward by inverting the survival function given by E_0, leading to

$$\tilde{T} = \ln(2/U^2 + 1) - 2\ln 2.$$

This allows for fast simulation of the divisor and, hence, the full approximated excursion distribution through the stochastic representation of Theorem 1.

B.3. **Shifted Gaussian.** We consider only the Gaussian case ($\alpha = 0$) in this section since Theorem 1 does not apply to the shifted Gaussian covariance function. Compared to the two previous examples, the Gaussian case does not allow for easy inversion sampling for the divisor. However, the target density is explicit and can be evaluated at a low computational cost. Hence, random numbers can be simulated using rejection sampling. The target density denoted by f is

$$-\frac{d}{dt}E_0(t) = \frac{e^{t^2}(t^2-1)+1}{(e^{t^2}-1)^{\frac{3}{2}}}.$$

Proposals are sampled iid from a Rayleigh distribution with the parameter $\sigma = 1.3$, which has the density

$$g(t) = \frac{t}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}},$$

for $t \geqslant 0$. The probability of accepting a proposed sample from g, denoted x_p, is

$$\frac{f(x_p)}{1.18\ g(x_p)}.$$

If the proposal is rejected, a new proposal is generated until it is accepted. Then, the stochastic representation of Theorem 1 is used to obtain a sample from the full distribution.

B.4. **Matérn covariance.** The formula for the survival function E_0 is given in Table 2. One way to simulate is to obtain the inverse of this function E_0^{-1} and simulate $E_0^{-1}(U)$, where U is a uniform random variable on $(0,1)$. Since the derivative of E_0 is readily available

$$E_0'(t) = -\sqrt{2(\nu-1)}C_\nu t^{\nu-1}\left(\frac{tK_{\nu-2}(t) - K_{\nu-1}(t)}{\left(1 - C_\nu^2 t^{2\nu}K_\nu^2(t)\right)^{1/2}} + \frac{C_\nu^2 t^{2\nu+1}K_{\nu-1}(t)K_\nu^2(t)}{\left(1 - C_\nu^2 t^{2\nu}K_\nu^2(t)\right)^{3/2}}\right),$$

one can seek an importance sampling bound for the density or invert numerically E_0 using the Newton–Raphson method. One can also observe that for $\nu = r+1/2$, $r = 0, 1, 2, 3, \ldots$ the Bessel functions have explicit form

$$K_\nu(u) = \sqrt{\frac{\pi}{2u}}e^{-u}\sum_{k=0}^{r}\frac{(r+k)!}{(r-k)!k!}(2u)^{-k}$$

which significantly reduces the computational cost. For illustration, consider $\nu = 5/2$. Then

$$\begin{aligned} K_{\nu-2}(u) &= \sqrt{\frac{\pi}{2u}}e^{-u}, \\ K_{\nu-1}(u) &= \sqrt{\frac{\pi}{2u}}e^{-u}\left(1 + \frac{1}{u}\right), \\ K_\nu(u) &= \sqrt{\frac{\pi}{2u}}e^{-u}\left(1 + \frac{3}{u} + \frac{3}{u^2}\right). \end{aligned}$$

This leads to

$$E_0(t) = \sqrt{3}\frac{t^2+t}{\sqrt{9e^{2t} - (t^2+3t+3)^2}}.$$

and

$$E_0'(t) = E_0(t)\left(\frac{2t+1}{t^2+t} + \frac{(t^2+3t+3)(2t+3) - 9e^{2t}}{9e^{2t} - (t^2+3t+3)^2}\right).$$

Based on this evaluation, the Newton–Raphson method of finding the solution to $E_0(t) = u$ has been implemented. A slight modification was needed to ensure the stability of the inversion. It has been used to generate the persistency coefficient approximation presented in Subsection 5.4.

III

CHARACTERISTICS OF ASYMMETRIC SWITCH PROCESSES WITH INDEPENDENT SWITCHING TIMES

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Abstract. The asymmetric switch process is a binary stochastic process that alternates between the values one and minus one, where the distributions of the time in these states may differ. Two versions of the process are considered: a non-stationary version that starts with a switch at time zero and a stationary one constructed from the non-stationary one. Characteristics of these two processes, such as the expected values and covariance, are investigated. The main results show an equivalence between the monotonicity of the expected value functions and the distribution of the intervals having a stochastic representation in the form of a sum of random variables, where the number of terms follows a geometric distribution. This representation has a natural interpretation as a model in which switching attempts may fail at random. From these results, conditions are derived when these characteristics lead to valid interval distributions, which is vital in applications.

1. Introduction

The study of binary stochastic processes has a long and rich history. Throughout this history, various particular forms have received considerable attention, from elementary Markov chains to renewal processes. The attention garnered by these processes is understandable due to their many applications in fields such as queuing theory Erlang (1909), signal processing (Picinbono, 2016), statistical physics (Bray et al., 2013), and approximating the excursion distribution for stochastic processes (McFadden, 1956) and Bengtsson and Podgórski (2024).

To study the excursion of stochastic processes McFadden (1956) introduced the notion of the clipped process. The clipped process is obtained from a sufficiently smooth process $W(t)$ by computing $Z(t) = \text{sign}(W(t) - u)$. The process $Z(t)$ is a binary process taking the values one and minus one, and the time spent in the two states represents the excursion times of $W(t)$, i.e., the times $W(t)$ is above or below the value u. Connecting characteristics, such as the covariance function of $Z(t)$ to the excursion distributions of $W(t)$ is still a notoriously tricky problem, despite having received considerable attention during the 19th century by, among others, Rice (1944) and McFadden (1956, 1958).

The difficulty stems from the complex dependency structures of the intervals of the clipped process $Z(t)$. One way to address this difficulty is through approximation methods. One such method is the independent interval approximation (IIA), and for an overview of this method, see Sire (2007, 2008) and Bray et al. (2013) for extensive

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overviews of this method in the context of applications in physics. This method uses a switch process to approximate $Z(t)$ and, through this, approximate the distributions of the $u-$level excursions of $W(t)$.

The switch process is a binary process similar to the clipped process. However, the times the process takes the value one and minus one are independent. This independence significantly reduces the complexity of linking the process's characteristics to the distribution of the intervals. Thus, characteristics of $Z(t)$ can be imposed on a switch process, and then the distributions of the interval lengths are deduced from these. This deduced distribution then serves as an approximation for the excursion distribution of $W(t)$.

This application motivates the study of the relationship between the switch process's characteristics and the distribution of the intervals. A partial characterization was obtained in Bengtsson (2024) for when the interval lengths have the same distribution for both states. In this paper, we allow these interval lengths to have different distributions. In this sense, the switch process is called asymmetric, in contrast to the symmetric one studied in Bengtsson (2024). This extension has implications for the previously mentioned IIA framework when approximating non-zero-level excursions.

The outline of the paper is as follows. The two versions of the switch process are introduced in Section 2. The main results are found in Section 3. The paper concludes with some illustrating examples in Section 4. Some supporting results and proofs have been collected in the Appendix to keep the paper somewhat self-contained.

2. Switch process

Two versions of the switch process are constructed from two independent switching time distributions, which are the distributions of the length of the intervals that the process spends in each state. Let T_+ and T_- denote the random variables associated with these distributions. Where T_+ is the length of the intervals for which the process takes the value one, and T_- is the length of the intervals for which the process takes the value minus one. While the details on how they are used for constructing the switch processes will be discussed later, the two versions are illustrated in Figure 1 and 2.

For the construction of the processes, we need some basic assumptions on the distributions of T_+ and T_-. Unless otherwise stated, it will be assumed that they are non-negative, have no atom at zero, are absolutely continuous with support on the entire non-negative real line, and have finite expectation. Additionally, we assume that there exists a $m \in \mathbb{N} : \sup_{t>0}(f_+ * f_-)^{*m}(t) < \infty$, where f_+ and f_- are the densities associated with T_+ and T_- and $*$ denotes convolution of functions. This condition is necessary for the existence of the derivatives of some characteristics, such as the expected value functions.

To study these functions, we use the Laplace transform, which is the standard tool in renewal theory. We denote the Laplace transform by $\mathcal{L}(\cdot)$ and in particular the Laplace we denote the Laplace transform of a probability density by $\Psi(s)$. Some elementary results and the definition can be found in the appendix. With the notation of the Laplace transform in place, we are now ready to define the non-stationary switch process.

The non-stationary switch process. The construction of the non-stationary switch process starts with the binary random variable δ, independent of the switching time distributions, with $\mathsf{P}(\delta = 1) = p$ and $\mathsf{P}(\delta = -1) = 1 - p$. The two values of δ are denoted by $+$ and $-$ when δ is used as a subscript. This random variable δ determines the starting value of the process at the origin. We then interlace random intervals of lengths distributed as T_+ and T_-, respectively, starting from the origin. Since we now consider sequences of switching time distributions, we denote these by T_i^+, $i \in \mathbb{N}$ and T_i^-, $i \in \mathbb{N}$.

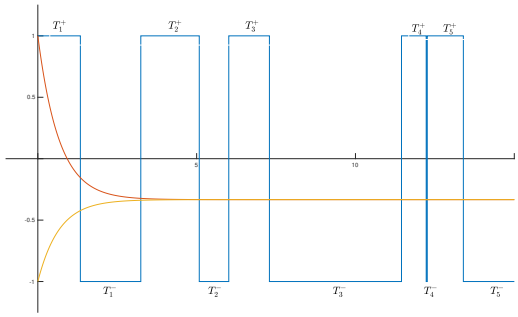

FIGURE 1. A realization of a non-stationary switch process (blue), $D(t)$, with exponential switching times and $\delta = 1$. Together with $E_+(t)$ (red) and $E_-(t)$ (orange).

If $\delta = 1$, the non-stationary switch process takes the value one over the interval $[0, T_1^+)$, and then switches to minus one on the interval $[T_1^+, T_1^+ + T_1^-)$, then it switches back to one and so forth. If $\delta = -1$, the process takes the value minus one on the interval $[0, T_1^-)$ and then it switches to one on $[T_1^-, T_1^- + T_1^+)$ and so on when time passes. The role δ, T_+, and T_- play in the construction of the non-stationary switch process is perhaps best illustrated in Figure 1, which shows a realization of the process.

We have so far not placed any assumptions on the dependency structure of T_i^+, $i \in \mathbb{N}$, T_i^-, $i \in \mathbb{N}$. However, we only consider the simple cases where T_i^+ and T_i^- constitute two mutually independent sequences of independent identically distributed (iid) random variables. A binary stochastic process taking the values $\{-1, 1\}$ and is constructed under the mentioned assumption by interlacing intervals, as described previously, is called a non-stationary switch process and is denoted by $D(t)$, $t \geq 0$.

By having a switch at zero, the process will not be stationary in the traditional sense; it will, however, be cycle stationary. This notion of stationarity is defined by the cycles (T_i^+, T_i^-), $i \in \mathbb{N}$, having the same distribution, i.e $(T_i^+, T_i^-) \stackrel{d}{=} (T_j^+, T_j^-)$ for all $i, j \in \mathbb{N}$, where $\stackrel{d}{=}$ denotes equality in distribution. It should be noted that cycle stationery will be preserved by letting T_i^+ and T_i^- be dependent. However, this case will not be investigated further in this paper.

The lack of stationary in the traditional sense makes the covariance function depend on two arguments, which is harder to utilize. This leads us to investigate the expected value function conditioned on the starting value δ. The relation between the expected value functions and the distributions of T_+ and T_- in the Laplace domain is given next.

Proposition 1. *The Laplace transform of* $P_\delta(t) = \mathsf{P}\big(D(t) = 1 \big| \delta\big)$, $t > 0$, *is given by*

$$\mathcal{L}(P_\delta)(s) = \frac{1 - \Psi_+(s)}{s(1 - \Psi_+(s)\Psi_-(s))} \cdot \begin{cases} 1 & ;\delta = 1, \\ \Psi_-(s) & ;\delta = -1, \end{cases}$$

where Ψ_+ *and* Ψ_- *are Laplace transformation of probability distributions corresponding to* T_+ *and* T_-, *respectively. Moreover, for the expected value function* $E_\delta(t) = \mathsf{E}(D(t)\big|\delta)$,

$t > 0$ we have

$$\mathcal{L}(E_\delta(s)) = \frac{\Psi_-(s) - \Psi_+(s) + \delta(1-\Psi_-(s))(1-\Psi_+(s))}{s(1-\Psi_+(s)\Psi_-(s))}.$$

In addition to this proposition, we have the following corollary on the derivatives of E_+ and E_-, which will be important when investigating switch processes with monotone expected value functions.

Corollary 1. *Let E_+ and E_- be the expected value functions of a non-stationary switch process with switching time distributions such that there exists a $m \in \mathbb{N}$:* $\sup_{t>0}(f_+ * f_-)^{*m}(t) < \infty$. *Then we have the following relations*

$$\mathcal{L}(E'_+)(s) = -2\Psi_+(s)\frac{1-\Psi_-(s)}{1-\Psi_+(s)\Psi_-(s)},\ \mathcal{L}(E'_-)(s) = 2\Psi_-(s)\frac{1-\Psi_+(s)}{1-\Psi_+(s)\Psi_-(s)}.$$

This proposition and corollary focuses only on the expected value functions, its derivatives, and $P_\delta(t)$. More general distributional properties are difficult to derive. For example, obtaining an explicit form for the covariance function is difficult. A more straightforward function to derive is the variance of $D(t)$. Since $\mathsf{E}(D(t)^2) = 1$, we have

$$\mathsf{V}(D)(t) = 1 - (\mathsf{E}D(t))^2.$$

From the above equation, it is clear that the variance diminishes closer to zero since the probability that a switch has occurred is fairly small.

The limiting behavior of the expected value functions is important for describing the process's behavior and characterizing the expected value functions. While these limits follow standard results on alternating renewal processes, we present a rigorous treatment in Lemma 4, which can be found in the appendix. From this lemma and the observation that $E_\pm(t) = 2P_\pm(t) - 1$ we have the following limits

$$\begin{aligned} &\lim_{t\to 0^+} E_+(t) = 1, \quad \lim_{t\to 0^+} E_-(t) = -1, \\ &\lim_{t\to\infty} E_+(t) = \lim_{t\to\infty} E_-(t) = \frac{\mu_+ - \mu_-}{\mu_+ + \mu_-}. \end{aligned} \tag{1}$$

From these limits, it is clear that the initial effect of the switch placed at zero will diminish when t goes to infinity.

Until now, we have only considered the non-stationary switch process on the non-negative part of the real line. However, the process can be extended to the entire real line, similar to how it was constructed for the non-negative part. Suppose $\delta = 1$ then the extended process would take the value minus one on the interval $[-T^-_{-1}, 0)$ and then switch to one on $[-(T^+_{-1} + T^-_{-1}), -T^-_{-1})$ and so forth. If $\delta = -1$ the process becomes one on $[-T^+_{-1}, 0)$ and minus one on $[-(T^+_{-1} + T^-_{-1}), -T^+_{-1})$. Hence, for a given sequence of pairs $\mathcal{T} = (T^-_i, T^+_i)_{i\in\pm\mathbb{N}}$, and initial choice of the sign δ, can be used to extend the non-stationary switch process on the entire real line.

The attachment of a switch at zero results in the process not being stationary in the traditional sense. In the subsequent subsection, a standard technique from renewal theory is used to construct a stationary version by delaying the process backward and forward around zero.

The stationary switch process. The stationary switch process is constructed from the non-stationary switch process. Since this process is cycle stationary, it follows from Thorisson (1995) that such a construction is possible. This construction is done by delaying the process forward and backward so there is no longer a switch at the origin. Hence,

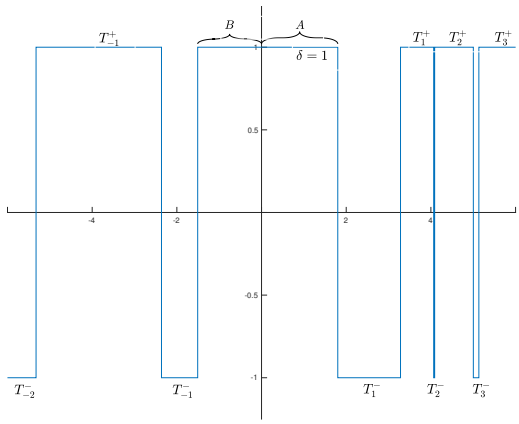

FIGURE 2. A realization of a stationary switch process, $\tilde{D}$ with exponential switching times and $\delta = 1$. A and B are the forward and backward delays.

the interval containing the origin will follow a different distribution than T_+ or T_-. In essence, we reintroduce the well-known inspection paradox for which the mathematical details have been known since Palm (1943).

For the construction, we follow a similar argument as in Lindgren et al. (2022). Let $[-B, A]$ be the interval containing the origin, and δ be the value of the process on this interval. From the ends, $-B$ and A non-stationary switch processes are attached with the initial values $-\delta$. The process with these delays is called the delayed switch process and is denoted by $\tilde{D}(t)$, $t \in \mathbb{R}$. A realization of $\tilde{D}(t)$ can be seen in Figure 2, and the problem now becomes finding the distribution of (A, B, δ) such that the delayed process becomes stationary. The solution is presented in the next proposition and follows from a standard application of the key renewal theorem, which can be found in the appendix.

Proposition 2. *Let $\mu_+ = ET_+$, $\mu_- = ET_-$ and f_+, f_- be the densities of T_+ and T_-, respectively. If the distribution of (A, B, δ) is given through*

$$\mathsf{P}(\delta = 1) = \frac{\mu_+}{\mu_- + \mu_+}, \; \mathsf{P}(\delta = -1) = \frac{\mu_-}{\mu_- + \mu_+}$$
$$f_{A,B|\delta}(a, b|1) = \frac{f_+(a+b)}{\mu_+}, \; f_{A,B|\delta}(a, b| -1) = \frac{f_-(a+b)}{\mu_-},$$

where $f_{A,B|\delta}$ stands for the conditional density, then the delayed switch process $\tilde{D}(t)$ is stationary in time, with the expected value $(\mu_+ - \mu_-)/(\mu_+ + \mu_-)$. Additionally its distribution is uniquely characterized by $\tilde{P}_\delta(t) = \mathsf{P}(\tilde{D}(t) = 1|\tilde{D}(0) = \delta)$ that has the Laplace transform of the following form

$$\mathcal{L}\tilde{P}_\delta(s) = \frac{1}{s}\begin{cases} 1 - \dfrac{1}{\mu_+ s}\dfrac{(1-\Psi_+(s))\,(1-\Psi_-(s))}{1-\Psi_+(s)\Psi_-(s)} & ;\delta = 1, \\ \dfrac{1}{\mu_- s}\dfrac{(1-\Psi_+(s))\,(1-\Psi_-(s))}{1-\Psi_+(s)\Psi_-(s)} & ;\delta = -1, \end{cases}$$

where Ψ_+ and Ψ_- are Laplace transformation of probability distributions corresponding to T_+ and T_-, respectively. The autocovariance function for this process, $R(t) =$

$\mathsf{Cov}(\tilde{D}(u), \tilde{D}(u+t))$, *is on the form*

$$R(t) = \frac{2}{\mu_+ + \mu_-}\left(\tilde{P}_+(t)\mu_+ - \tilde{P}_-(t)\mu_- + \mu_+\frac{\mu_- - \mu_+}{\mu_+ + \mu_-}\right).$$

Lastly, the Laplace transform of $R(t)$, $t > 0$, *is given by*

$$\mathcal{L}(R)(s) = \frac{4}{s\,(\mu_+ + \mu_-)}\left(\frac{\mu_+\mu_-}{\mu_+ + \mu_-} - \frac{1}{s}\frac{(1-\Psi_+(s))(1-\Psi_-(s))}{1-\Psi_-(s)\Psi_+(s)}\right)$$

A delayed switch process with $[-B, A]$ and δ as described in Proposition 2 will, from this point onwards, be called a stationary switch process and denoted by $\tilde{D}(t)$. While we assume that the densities of T_+ and T_- exist, this is not a technical necessity, as seen in the following remark.

Remark 1. We should note from the proof that it is not required for the distribution of the switching times to have a density. It is sufficient that the distribution of $T_1^+ + T_2^-$ is not sitting on a lattice for the key renewal theorem to hold. In fact, the stationary distribution of (A, B, δ) is given by

$$\mathsf{P}(\delta = 1, A \le a, B > b) = \frac{\int_0^a \bar{F}_+(u+b)\,du}{\mu_+ + \mu_-}$$

We note two simple consequences of this fact. First, the distributions of the delays A and B are identical and always continuous with respect to the Lebesgue measure with their densities conditionally on δ given by

$$f_{A|\delta}(x|\delta) = f_{B|\delta}(x|\delta) = \frac{\bar{F}_\delta(x)}{\mu_\delta},$$

which in the Laplace domain becomes

$$\Psi_{A|\delta}(s) = \Psi_{B|\delta}(s) = \frac{1 - s\mathcal{L}F_\delta(s)}{s\mu_\delta} = \frac{1-\Psi_\delta(s)}{s\mu_\delta}.$$

using Equation (4). Second, we note that the distribution of the interval containing the origin is given by the density

$$f_{A+B|\delta}(x|\delta) = \frac{x f_\delta(x)}{\mu_\delta}.$$

We have now constructed a stationary switch process $\tilde{D}(t)$ from the non-stationary switch process $D(t)$. In the next section, we connect the characterizing function to the properties of the switching time distributions.

3. Characteristics of the switch process

We have seen how the distributions of T_+ and T_- give rise to the following functional characteristics: the expected value functions E_+, E_-, and P_+, P_- of the non-stationary version, and the autocovariance R, $\tilde{P}_+$, and $\tilde{P}_-$ for the stationary counterpart. Now, we reverse the question: What functional characteristics lead to valid distributions, and what are their properties? To address these questions, we start with a proposition showing that the distributions of T_+ and T_- are identifiable from the expected value functions of the non-stationary switch process.

Proposition 3. *The distributions T_+ and T_- are uniquely identifiable from E_+ and E_- in the Laplace domain through the equations*

$$\Psi_+(s) = \frac{s\mathcal{L}(E_+)(s) - 1}{s\mathcal{L}(E_-)(s) - 1},$$
$$\Psi_-(s) = \frac{s\mathcal{L}(E_-)(s) + 1}{s\mathcal{L}(E_+)(s) + 1}.$$

Proposition 3 shows that the distribution of T_+ and T_- can be uniquely determined from the expected value functions. The subsequent proposition provides the basis to examine if this is also possible from the covariance function.

Proposition 4. *Using the definitions of Proposition 1 and 2, we have the following relations between characteristics of the stationary and non-stationary switch process:*

$$\mathcal{L}(\tilde{P}'_+) = \frac{\mathcal{L}(P_-) - \mathcal{L}(P_+)}{\mu_+},\ \mathcal{L}(\tilde{P}'_-) = \frac{\mathcal{L}(P_+) - \mathcal{L}(P_-)}{\mu_-}.$$
$$\mathcal{L}(R') = 4\,\frac{\mathcal{L}(P_-) - \mathcal{L}(P_+)}{\mu_+ + \mu_-} = 2\,\frac{\mathcal{L}(E_-) - \mathcal{L}(E_+)}{\mu_+ + \mu_-}.$$

The following remark presents the implications of this proposition for the recoverability of T_+ and T_- from the covariance function.

Remark 2. The probabilities $\tilde{P}_+$ and $\tilde{P}_-$ can not be uniquely obtained from P_+ and P_-. Subsequently any relationship between P_+, P_- and T_+, T_- can not be used to derive $\tilde{P}_+$ and $\tilde{P}_-$. This lack of identifiability is shown by considering the following system of equations, which follows directly from the top equations of Proposition 4

$$\begin{bmatrix}\mathcal{L}(\tilde{P}'_+)(s)\\ \mathcal{L}(\tilde{P}'_-)(s)\end{bmatrix} = \begin{bmatrix}-\frac{1}{\mu_+} & \frac{1}{\mu_+}\\ \frac{1}{\mu_-} & -\frac{1}{\mu_-}\end{bmatrix}\begin{bmatrix}\mathcal{L}(P'_+)(s)\\ \mathcal{L}(P'_-)(s)\end{bmatrix}.$$

Since the above 2×2 matrix is singular, there is no unique solution such that $\mathcal{L}(P'_+)$ and $\mathcal{L}(P'_-)$ can expressed by $\mathcal{L}(\tilde{P}'_+)$ and $\mathcal{L}(\tilde{P}'_-)$.

A consequence of the previous remark is that if one wants to recover T_+ and T_- from characteristics of the stationary switch process, it cannot be done purely from R, $\tilde{P}_+$ and $\tilde{P}_+$. Thus, additional characteristics need to be considered. One might consider the expected value of the number of switches from one to minus one and vice versa. However, these will not provide additional information to identify the distributions of T_+ and T_- since they are linear in time due to the stationarity of $\tilde{D}$. Therefore, it is a nontrivial task to retrieve the switching time distributions from the characteristics of the stationary switch process.

In contrast to the previously mentioned difficulties, Proposition 3 seems to provide a straightforward way to recover the switching time distributions. However, this is not the case. For a proposed pair of expected value functions, it has to be verified that the right-hand side of the two equations of Proposition 3 are completely monotone functions. Recall that a function ϕ on $[0, \infty)$ is completely monotone if it has finite limits, is infinitely differentiable, and satisfies the condition for all $n \in \mathbb{N}$

$$(-1)^n \phi^{(n)}(s) \geq 0,\ s > 0.$$

Hence, to show that the two equations of Proposition 3 are completely monotone is arduous at best and almost impossible at worst. This difficulty is a consequence of Bernstein's theorem (see Theorem 1, page 439 in Feller (1970)), which states that a

function is the Laplace transform of a probability distribution if and only if it is completely monotone and one at zero. Therefore, recovering the switching time distributions from Proposition 3 remains a problem. While the following proposition does not resolve the issue, it characterizes the properties of the expected value functions.

Proposition 5. *The following conditions are necessary and sufficient for differentiable functions E_+ and E_- on $[0,\infty)$ with values in $[-1,1]$ to define a valid switch process $D(t)$ for which they correspond to $\mathsf{E}\,(D(t)|D(0)=1)$ and $\mathsf{E}\,(D(t)|D(0)=-1)$, respectively,*

i) $\lim_{t\to 0^+} E_+(t) = 1$, $\lim_{t\to 0^+} E_-(t) = -1$, $\lim_{t\to\infty} E_+(t) = \lim_{t\to\infty} E_-(t) = \gamma$, *for some* $\gamma \in (-1,1)$,

ii) the functions $\dfrac{\mathcal{L}(E_+')(s)}{\mathcal{L}(E_-')(s)-2}$ *and* $\dfrac{\mathcal{L}(E_-')(s)}{\mathcal{L}(E_+')(s)+2}$ *are completely monotone.*

For the class of non-stationary switch processes with monotone expected value functions, this problem is solvable, and this will be the focus of the remainder of this section. The solution has two components: the first is an observation regarding the behavior of E_+' and E_-' under scaling. The second one is random variables, which have a stochastic representation in terms of a sum of random variables, where the number of terms follows a geometric distribution. We start with the first component. Suppose that E_+' and E_-' exists then we have

$$\int_0^\infty E_+'(t)\,dt = -\frac{2\mu_-}{\mu_+ + \mu_-}, \quad \int_0^\infty E_-'(t)\,dt = \frac{2\mu_+}{\mu_+ + \mu_-}.$$

If we additionally assume that E_+ and E_- are monotone functions, then the functions

$$-\frac{\mu_+ + \mu_-}{2\mu_+}E_+'(t), \quad \frac{\mu_+ + \mu_-}{2\mu_-}E_-'(t)$$

are probability density functions.

The second component consists of the class of geometric divisible distributions. This generalizes the notion of geometric infinitely divisible, which was introduced by Klebanov et al. (1985). The link between the geometric divisible distribution and non-stationary switch processes was noted in Bengtsson (2024); this was, however, only for the case where $T_+ \stackrel{d}{=} T_-$. We provide the following definition for completeness.

Definition 1. *Let ν_p be a geometric random variable with the probability mass function $p_{\nu_p}(k) = (1-p)^{k-1}p$ for $k = 1, 2...$ and $\{V_k\}_{k\geq 1}$ a sequence of iid non-negative random variables independent of ν_p. If a random variable has a stochastic representation*

$$W = \sum_{k=1}^{\nu_p} V_k,$$

then W follows a r-geometric divisible distribution with $r = 1/p$ and said to belong to the class $GD(r)$, with the divisor V and we write $F \in GD(r)$.

The notion that V is a divisor of W is natural since V divides W into smaller parts. From this definition, the density of W can be expressed as $f_W = f_V^{*\nu_p}$. This form is unwieldy, and we therefore present the more useful expression in the Laplace domain

$$\Psi_W(s) = \frac{\frac{1}{r}\Psi_V(s)}{1-(1-\frac{1}{r})\Psi_V(s)}. \tag{2}$$

The main difference between the concepts of geometric divisibility and geometric infinite divisibility is for which $p \in (0,1)$ the stochastic representation of Definition 1 needs to

hold. A random variable is said to be geometric infinitely divisible if this property is satisfied for *all* $p \in (0,1)$, while for a random variable to be geometrically divisible, this property only needs to hold for *one* specific $p \in (0,1)$. A simple consequence of this is that any geometric infinitely divisible distribution is also geometrically divisible.

The next theorem connects non-stationary switch processes having monotone expected value functions with the class of geometrically divisible distributions.

Theorem 1. *Let $D(t)$, $t \in \mathbb{R}$, be a non-stationary process with expected value functions such that $-E'_+(t)$ and $E'_-(t)$ are non-negative for $t > 0$. Then, the time from a switch until it returns to that state has the the stochastic representation.*

$$T_- + T_+ \stackrel{d}{=} \sum_{k=1}^{\nu_\beta} X_k + \sum_{k=1}^{\nu_\alpha} Y_k$$

where ν_α and ν_β are independent and geometrically distributed with the parameters

$$\alpha = \frac{\mu_-}{\mu_+ + \mu_-}, \ \beta = \frac{\mu_+}{\mu_+ + \mu_-}$$

and the densities of X_i and Y_i are given by

$$f_X(t) = -\frac{1}{2\alpha} E'_+(t), \ f_Y(t) = \frac{1}{2\beta} E'_-(t),$$

respectively. The sequences of X_i and Y_i are mutually independent sequences of iid random variables, which are also independent of ν_α and ν_β.

Proof. From Proposition 3, the relation $\alpha = 1 - \beta$, and the following property of the Laplace transform $\mathcal{L}(h')(s) = s\mathcal{L}(h) - h(0)$ we have

$$\begin{aligned}
\Psi_+(s)\Psi_-(s) &= \frac{s\mathcal{L}(E_+)(s) - 1}{s\mathcal{L}(E_-)(s) - 1} \frac{s\mathcal{L}(E_-)(s) + 1}{s\mathcal{L}(E_+)(s) + 1}, \\
&= \frac{-\mathcal{L}(E'_+)(s)}{2 + \mathcal{L}(E'_+)(s)} \frac{\mathcal{L}(E'_-)(s)}{2 - \mathcal{L}(E'_-)(s)}, \\
&= \frac{2\alpha\Psi_X(s) 2\beta\Psi_Y(s)}{(2 - 2\alpha\Psi_X(s))(2 - 2\beta\Psi_Y(s))}, \\
&= \frac{\beta\Psi_X(s)}{1 - (1-\beta)\Psi_X(s)} \frac{\alpha\Psi_Y(s)}{1 - (1-\alpha)\Psi_Y(s)}.
\end{aligned}$$

The result follows since the equation above is the product of two equations, which are in the form of Equation (2). □

While Theorem 1 connects the distributions of the cycle length with functional properties of E_+ and E_-, it does not allow us to recover the distribution of T_+ and T_-. However, the next theorem allows us to recover the distribution of T_+ and T_- from the expected value functions of the non-stationary switch process.

Theorem 2. *Let $D(t)$, $t \in \mathbb{R}$ be a non-stationary switch process with the expected value functions E_+ and E_-. Then the following conditions are equivalent*

i) The functions $-E'_+(t)$ and $E'_-(t)$ are non-negative for $t > 0$,

ii) T_+ and T_- have the stochastic representations

$$T_+ \stackrel{d}{=} X + \sum_{k=1}^{\nu_\alpha - 1} Y_k, \qquad T_- \stackrel{d}{=} Y + \sum_{k=1}^{\nu_\beta - 1} X_k,$$

where ν_α and ν_β are independent and geometrically distributed with the parameters

$$\alpha = \frac{\mu_-}{\mu_+ + \mu_-},\ \beta = \frac{\mu_+}{\mu_+ + \mu_-},$$

and the densities of X_i and Y_i are given by

$$f_X(t) = -\frac{1}{2\alpha}E'_+(t),\ f_Y(t) = \frac{1}{2\beta}E'_-(t).$$

respectively. The sequences of X_i and Y_i are mutually independent sequences of iid random variables, which are also independent of ν_α and ν_β

Proof. $(i) \to (ii)$: First, we note that if a random variable, Z, has the stochastic representation $Z \stackrel{d}{=} X + \sum_{k=1}^{\nu_\alpha - 1} Y_k$, then

$$\begin{aligned}\Psi_Z(s) &= \mathsf{E}e^{s\left(X+\sum_{k=1}^{\nu_\alpha-1}Y_k\right)} = \Psi_X(s)\left(\mathsf{P}(\nu_\alpha = 1) + \sum_{n=2}^{\infty}\mathsf{E}e^{s\sum_{k=1}^{n-1}Y_k}\mathsf{P}(\nu_\alpha = n)\right)\\ &= \Psi_X(s)\sum_{n=1}^{\infty}\Psi_Y^{n-1}(s)(1-\alpha)^{n-1}\alpha = \frac{\alpha\Psi_X(s)}{1-(1-\alpha)\Psi_Y(s)}.\end{aligned}$$

By Proposition 3, Lemma 4, the property $\mathcal{L}(h')(s) = s\mathcal{L}(h) - h(0)$, and $\alpha = 1 - \beta$ we have

$$\Psi_+(s) = \frac{s\mathcal{L}(E_+)(s) - 1}{s\mathcal{L}(E_-)(s) - 1} = \frac{-\mathcal{L}(E'_+)(s)}{2 - \mathcal{L}(E'_-)(s)} = \frac{\alpha\Psi_X(s)}{1-\beta\Psi_Y(s)} = \frac{\alpha\Psi_X(s)}{1-(1-\alpha)\Psi_Y(s)}.$$

The identical argument for $\Psi_-(s)$ is omitted.
$(ii) \to (i)$: From Lemma 4 and Corollary 1 we have that

$$\begin{aligned}\mathcal{L}(E'_+)(s) &= -2\Psi_+(s)\frac{1-\Psi_-(s)}{1-\Psi_+(s)\Psi_-(s)}\\ -\frac{1}{2\alpha}\mathcal{L}(E'_+)(s) &= \frac{1}{\alpha}\frac{\alpha\Psi_X(s)}{1-\beta\Psi_Y(s)}\frac{1-\frac{\beta\Psi_Y(s)}{1-\alpha\Psi_X(s)}}{1-\frac{\alpha\Psi_X(s)}{1-\beta\Psi_Y(s)}\frac{\beta\Psi_Y(s)}{1-\alpha\Psi_X(s)}}\\ &= \Psi_X(s)\frac{1-\alpha\Psi_X(s)-\beta\Psi_Y(s)}{(1-\alpha\Psi_X(s))(1-\beta\Psi_Y(s))-\alpha\beta\Psi_X(s)\Psi_Y(s)}\\ &= \Psi_X(s)\frac{1-\alpha\Psi_X(s)-\beta\Psi_Y(s)}{1-\alpha\Psi_X(s)-\beta\Psi_Y(s)+\alpha\beta\Psi_X(s)\Psi_Y(s)-\alpha\beta\Psi_X(s)\Psi_Y(s)}\\ &= \Psi_X(s).\end{aligned}$$

Since $\Psi_X(s)$ is the Laplace transform of a density function, it follows that $(-2\alpha)^{-1}E'_+(t)$ will be non-negative and $E_+(t)$ will therefore be a monotone function. The identical argument for $E_-(t)$ is omitted for brevity. □

Theorem 2 extend the results of Bengtsson (2024), which showed that monotone expected values are equivalent to T_+ being 2−geometric divisible when $T_+ \stackrel{d}{=} T_-$. There is a straightforward interpretation for this case, with the geometric random variable being the number of switching attempts needed to switch, including the successful one. For the more general case when $T_+ \stackrel{d}{\neq} T_-$ there is also a natural interpretation of Theorem 2. Specifically, one can view the time X (or Y) as the random time until the first switching attempt is made. If this attempt is unsuccessful, ν_α (or ν_β) represents the number of failed switching attempts besides the first attempt, with the Y (or X) being the random time between the attempts.

We conclude this section with a corollary on the implication of Theorem 2 for the stationary switch process.

Corollary 2. *Let $D(t)$, $t \geq 0$, be a non-stationary switch process that satisfies the conditions of Theorem 2. Then the second derivative of the covariance function $R(t)$ of the stationary version $\tilde{D}(t)$, $t \in \mathbb{R}$ is of the form*

$$R''(t) = \frac{4}{\mu_+ + \mu_-} \left(\alpha f_X(t) + (1-\alpha) f_Y(t)\right), \tag{3}$$

and is proportional to the probability density function of the mixture $\xi X + (1-\xi)Y$. Where ξ is a Bernoulli variable with the parameter α, independent of X and Y.

From this corollary, it is clear that even if the switching time distribution satisfies the condition of Theorem 2, they can not be recovered from the covariance function alone. This follows from the inability to decompose the density into a mixture of X and Y such that X and Y are unique.

4. Examples

Asymmetry with a common divisor. One of the simplest ways of inducing asymmetry is to let T_+ and T_- be geometrically divisible with the same divisor $\tilde{T}$ but of different orders. The asymmetry is induced by modifying the single parameter, $\alpha \in (0,1)$ in the geometric summation, since $\beta = 1 - \alpha$. Hence, we have the following stochastic representation

$$T_+ \stackrel{d}{=} \sum_{k=1}^{\nu_\alpha} \tilde{T}_k, \quad T_- \stackrel{d}{=} \sum_{k=1}^{\nu_\beta} \tilde{T}_k.$$

Since this representation satisfies condition *ii)* in Theorem 2, the expected value functions E_+ and E_- will be monotone. Additionally, we have that $E_+/E_- = -\alpha/\beta$, meaning that they only differ by scaling. If the common divisor is exponential, this is similar to scaling T_+ and T_- with some constant, as demonstrated in the next example.

Asymmetry with a common divisor trough scaling. The asymmetry can be extended while maintaining the simplicity of having a common divisor, as in the previous example. The monotonicity property of the expected value functions E_+ and E_- is preserved by considering a scaling of the form

$$T_+ \stackrel{d}{=} b\tilde{T}_1 + a\sum_{k=2}^{\nu_\alpha} \tilde{T}_k, \quad T_- \stackrel{d}{=} a\tilde{T}_1 + b\sum_{k=2}^{\nu_\beta} \tilde{T}_k, \quad a, b > 0.$$

Since $\beta = 1 - \alpha$ this semi-parametric distribution has three numerical parameters $a > 0$, $b > 0$, $\alpha \in (0,1)$ and one functional parameter, which is the density f of $\tilde{T}$. We observe that $\mu_+ = (b + a(1/\alpha - 1))\,\mu$ and $\mu_- = (a + b(1/\beta - 1))\,\mu$, where $\mu = \mathsf{E}\tilde{T}$. Then the derivative of the expected value functions becomes

$$E'_+(t) = -2\frac{a + b(1/\beta - 1)}{a/\alpha + b/\beta} f(t/b)/b, \quad E'_-(t) = 2\frac{b + a(1/\alpha - 1)}{a/\alpha + b/\beta} f(t/a)/a.$$

Non-monotonic expected value functions. Consider a process with gamma-distributed switching times, with the scaling parameters $\theta_+ = 2$, $\theta_- = 1$, and the shape parameters $k_+ = 2$ and $k_- = 3$. The derivatives of the expected value functions can be evaluated via the Laplace transform using Corollary 1:

$$\mathcal{L}(E'_+)(s) = 2\,\frac{1-(1+s)^3}{(1+2s)^2(1+s)^3-1} = \frac{6+6s+2s^2}{7+19s+28s^2+16s^3+4s^4},$$
$$\mathcal{L}(E'_-)(s) = 2\,\frac{(1+2s)^2}{(1+2s)^2(1+s)^3-1} = \frac{8+8s}{7+19s+28s^2+16s^3+4s^4}.$$

The inverse Laplace transforms of these functions oscillate and provide an example where the conditions of Theorem 2 are not met. A direct consequence is that the covariance function $R(t)$, of the corresponding stationary process will also oscillate due to the relation in Proposition 4.

5. Conclusion

At the center of this paper are the characterizing functions of the two versions of the switch processes and their relationship to the switching time distributions. The main result shows that the monotonicity of the expected value functions is equivalent to the switching time distribution having a particular stochastic representation. This representation has a natural interpretation and allows us to recover the switching time distribution from the expected value functions. The recovery of the switching time distribution from the characterizing functions is one of the main motivations for studying these processes. We also show that the expected value functions should be the preferred characterization for this recovery. This is because using the covariance function of the stationary switch process leads to identifiability problems. These two observations have practical implications for the independent interval approximation framework, which approximates the excursions of Gaussian processes with a switch process. Applying these results to this framework is one of the two directions for future work. The other is to relax the independence assumptions between two switching time distributions.

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Appendix. Auxiliary results and proofs

In this Appendix, we have collected some of the proofs for the paper along with some elementary and supporting results. Before these results are presented, it should be noted that we use $*$ for convolutions of functions and $\star$ for the convolution of probability distributions. Namely, $F \star F$ denotes the distribution function of the sum of two independent random variables that follow the distribution F. Lastly, the proofs are presented under the assumptions of Section 2 unless stated otherwise. For convenience, the proofs and remarks of the appendix are grouped into three sections.

The Laplace transform and supporting results. The Laplace transform is heavily used within renewal theory and in this paper. We therefore provide a short introduction of it here. Recall that for a function $h(t)$, $t \geq 0$, its Laplace transform $\mathcal{L}(h)(s)$, defined through

$$\mathcal{L}(h)(s) = \int_0^\infty h(t)e^{-ts}\, dt,$$

for any s such that the integral is finite. Furthermore, we have that for a positive half-line distribution given by a cdf F; its Laplace transform Ψ is given by

$$\Psi(s) = \int_0^\infty e^{-ts}\, dF(t).$$

Additionally, there exists a well-known relation between the Laplace transform of the cumulative distribution function, F, and Ψ, namely

$$\mathcal{L}(F)(s) = \Psi(s)/s. \tag{4}$$

which follows from the fact that F is the convolution between the density f and the Heaviside step function. Lastly, we want to highlight the following property of the Laplace transform of derivatives, which sees extensive use in this paper:

$$\mathcal{L}(h')(s) = s\mathcal{L}(h)(s) - h(0).$$

With this introduction to the Laplace transform and the assumptions in place, we are now ready to define the non-stationary switch process.

Lemma 1. *Let G,H be distribution functions on $[0,\infty)$ with corresponding densities g, h. Then for each $t \geq 0$:*

$$(g * h)(t) \leq \min\left(\sup_{u>0} g(u) \cdot H(t),\ \sup_{u>0} h(u) \cdot G(t)\right).$$

Proof. Since g, h are densities on $[0,\infty)$, $g(t-x) = 0$ for $x > t$, we have

$$\begin{aligned}(g * h)(t) &= \int_{-\infty}^{\infty} g(t-x)h(x)dx = \int_0^t g(t-x)h(x)dx \\ &\leq \int_0^t \sup_{u>0} g(u)h(x)dx = \sup_{u>0} g(u)H(t).\end{aligned}$$

We obtain the minimum since convolution is commutative, and we can freely choose to bound g or f with its supremum. □

Corollary 3. *Let G be a distribution function on $[0,\infty)$, with density function g. Then for all $n, m \in \mathbb{N}$ we have*

$$g^{*(n+m)}(t) \leq \sup_{u>0} g(u)^{*n} G(t)^{m\star}.$$

Lemma 2. *Let G,H be distribution functions on $[0,\infty)$, then we have*

$$(H \star G)(t) \leq H(t)G(t)$$

and for $n \in \mathbb{N}$,

$$H^{\star n}(t) \leq H^n(t).$$

Proof. Since G and H are distribution functions and thus monotone on $[0,\infty)$ we have

$$(H \star G)(t) = \int_{-\infty}^{\infty} H(t-u)\, dG(u) = \int_0^t H(t-u)\, dG(u) \leq H(t) \int_0^t dG(u) = H(t)G(t).$$

We obtain the last equation from the associative property of convolutions and repeated application of the bound. □

Proofs related to the non-stationary switch process. After these supporting results, we account for the properties of the switch process $D(t)$ used across the paper. We start with the following result for the counting process $N(t)$ that counts the number of switches in $(0,t]$ for $t > 0$ and those with the negative count, in $(t,0]$ for $t < 0$. Since we have

$$D(t) = (-1)^{N(t)+(1-\delta)/2},$$

it is clear that the distributional properties of $N(t)$ together with the initial state δ uniquely define the distribution of the process $D(t)$. The distribution of $N(t)$ on the positive half-line and conditionally on $\delta = \delta_0$, $\delta_0 \in \{-1,1\}$ is the same as of $\{-N(-t) - 1, t \geq 0\}$ conditionally on $\delta = -\delta_0$, hence it is sufficient to consider only the positive half-line.

Lemma 3. *Assume that $\mathcal{T}_0 = (T_i^-, T_i^+)_{i\in\mathbb{N}}$ is a sequence of iid pairs of independent random variables, with the distribution function F_- and F_+, respectively. The one-dimensional marginal distributions of $N(t)$ given δ and for $t \geq 0$ are given by*

$$\mathsf{P}\big(N(t) = k \big| \delta\big) = \begin{cases} (F_+ \star F_-)^{l\star}(t) - (F_+ \star F_-)^{l\star} \star F_\delta(t); & k = 2l, \\ (F_+ \star F_-)^{l\star} \star F_\delta(t) - (F_+ \star F_-)^{l\star}(t); & k = 2l+1, \end{cases} \tag{5}$$

where $l \in \mathbb{N} \cup \{0\}$.

Proof. We consider only the case of conditioning on $\delta = 1$ as the opposite case can be obtained by the symmetry argument. We first note that for a positive t and a non-negative

integer l:

$$\begin{aligned}
\mathsf{P}\big(N(t)=0\big|\delta=1\big) &= \mathsf{P}(T_1^+ > t) = 1 - F_+(t),\\
\mathsf{P}\big(N(t)=1\big|\delta=1\big) &= \mathsf{P}(T_1^+ + T_1^- > t \geq T_1^+)\\
&= 1 - F_+ \star F_-(t) - (1 - F_+(t))\,,\\
&= F_+(t) - F_+ \star F_-(t),\\
\mathsf{P}\big(N(t)=2l\big|\delta=1\big) &= \mathsf{P}\left(\sum_{i=1}^{l+1} T_i^+ + \sum_{i=1}^{l} T_i^- > t \geq \sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l} T_i^-\right)\\
&= 1 - F_+^{(l+1)\star} \star F_-^{l\star}(t) - \left(1 - F_+^{l\star} \star F_-^{l\star}(t)\right)\\
&= F_+^{l\star} * F_-^{l\star}(t) - F_+^{(l+1)\star} \star F_-^{l\star}(t),\\
\mathsf{P}\big(N(t)=2l+1\big|\delta=1\big) &= \mathsf{P}\left(\sum_{i=1}^{l+1} T_i^+ + \sum_{i=1}^{l+1} T_i^- > t \geq \sum_{i=1}^{l+1} T_i^+ + \sum_{i=1}^{l} T_i^-\right)\\
&= F_+^{(l+1)\star} \star F_-^{l\star}(t) - F_+^{(l+1)\star} \star F_-^{(l+1)\star}(t).
\end{aligned}$$

□

With this lemma, we now present the proof of Proposition 1.

Proof of Proposition 1. Let us first consider the case of $t > 0$ and $\delta = 1$. Then by Equations (5), (4) and Lemma 3:

$$\begin{aligned}
P_+(t) &= \sum_{l=0}^{\infty} \mathsf{P}(N(t)=2l) = \left(\sum_{l=0}^{\infty} (F_+ \star F_-)^{l\star}\right)(t) - \left(\sum_{l=0}^{\infty} (F_+ * F_-)^{l\star}\right) \star F_+(t)\\
&= \mathcal{L}^{-1}\left(\frac{1-\Psi_+(s)}{s(1-\Psi_+(s)\Psi_-(s))}\right).
\end{aligned}$$

For the case of $\delta = -1$, we have

$$\begin{aligned}
P_-(t) &= \left(\sum_{l=0}^{\infty} (F_+ \star F_-)^{l\star}\right) \star F_-(t) - \left(\sum_{l=1}^{\infty} (F_+ \star F_-)^{l\star}\right)(t)\\
&= \mathcal{L}^{-1}\left(\frac{\Psi_-(s)\,(1-\Psi_+(s))}{s(1-\Psi_+(s)\Psi_-(s))}\right).
\end{aligned}$$

Moreover, for $\delta = 1$

$$\begin{aligned}
E_+(t) &= 2P_+(t) - 1\\
&= \mathcal{L}^{-1}\left(\frac{1-2\Psi_+(s)+\Psi_+(s)\Psi_-(s)}{s(1-\Psi_+(s)\Psi_-(s))}\right),
\end{aligned}$$

while for $\delta = -1$ we obtain

$$\begin{aligned}
E_-(t) &= \mathcal{L}^{-1}\left(\frac{2\Psi_-(s)(1-\Psi_+(s)) - 1 + \Psi_+(s)\Psi_-(s)}{s(1-\Psi_+(s)\Psi_-(s))}\right)\\
&= \mathcal{L}^{-1}\left(\frac{-1+2\Psi_-(s)-\Psi_+(s)\Psi_-(s)}{s(1-\Psi_+(s)\Psi_-(s))}\right).
\end{aligned}$$

□

The differentiability of E_+ and E_- relies on the differentiability of P_+ and P_- defined in Proposition 1. We provide the conditions for differentiability in the following lemma.

Lemma 4. *Assume that both distributions F_+ and F_- are absolutely continuous with respect to the Lebesgue measure and have finite expectations, μ_+ and μ_-, respectively. The functions P_+ and P_- satisfy the following conditions*

i) $\lim_{t\to 0^+} P_+(t) = 1$ *and* $\lim_{t\to 0^+} P_-(t) = 0$,

ii) $\lim_{t\to\infty} P_+(t) = \lim_{t\to\infty} P_-(t) = \dfrac{\mu_+}{\mu_+ + \mu_-}$,

iii) If there exists $L \in \mathbb{N}$ for which $\sup_{t>0}(f_+ * f_-)^{*L}(t) < \infty$, then P_+ and P_- are differentiable and

$$P_+ = \left(\mathbb{I}_{[0,\infty)} - F_+\right) \star \sum_{l=0}^{\infty} \left(F_+ \star F_-\right)^{l\star},$$

$$P_- = F_- + \left(F_- - \mathbb{I}_{[0,\infty)}\right) \star \sum_{l=1}^{\infty} \left(F_+ \star F_-\right)^{l\star},$$

$$P'_+ = \sum_{l=0}^{\infty} \left(f_+^{*l} * f_-^{*l} - f_+^{*(l+1)} * f_-^{l\star}\right),$$

$$P'_- = f_+ + \sum_{l=1}^{\infty} \left(f_+^{*l} * f_-^{*(l+1)} - f_+^{*l} * f_-^{l\star}\right).$$

where the convergence of all series is locally uniform on $(0,\infty)$.

Proof. We have seen in the proof of Proposition 1 that

$$\begin{aligned} P_+ &= \mathbb{I}_{[0,\infty)} - F_+ + \sum_{l=1}^{\infty} \left(F_+^{l\star} \star F_-^{l\star} - F_+^{(l+1)\star} \star F_-^{l\star}\right) \\ &= \left(\mathbb{I}_{[0,\infty)} - F_+\right) \star U, \end{aligned}$$

where U is the renewal measure defined by the renewal processes of the cycle length

$$U(A) = \delta_{\{0\}} + \sum_{l=1}^{\infty} \left(F_+ \star F_-\right)^{l\star}(A).$$

from Lemma 2 we have,

$$\lim_{t\to 0^+} \left(\mathbb{I}_{[0,\infty)} - F_+\right)(t) = 1,$$

$$\lim_{t\to 0^+} \left| \sum_{l=1}^{\infty} \left(F_+^{l\star} \star F_-^{l\star} - F_+^{(l+1)\star} \star F_-^{l\star}\right) \right| (t) \le \lim_{t\to 0^+} \frac{F_+ \star F_-(t)}{1 - F_+ \star F_-(t)} (1 + F_+(t)) = 0,$$

which establishes *i)* for P_+. The argument for P_- is similar.

The line of the argument for *ii)* utilizes the Key Renewal Theorem, see p.86 in Daley and Vere-Jones (2003), which states that for an integrable function g that vanishes on $(-\infty, 0)$:

$$\lim_{t\to\infty} g \star U(t) = \frac{1}{\mu_+ + \mu_-} \int_0^{\infty} g(u)\, du.$$

For the limit at the infinity for P_+, one applies it to $g = \mathbb{I}_{[0,\infty)} - F_+$ since $P_+ = g \star U$. The approach is analog for P_-

To prove the differentiability, we use the fact that P_+ and P_- can be expressed as a series of differentiable terms; thus, from Theorem 7.17, Rudin (1976), it is sufficient to show local uniform convergence of the partial sums that would allow the exchange summation and derivation. Thus for each $0 < t_0 < t_1$, we need to show that, for each $\epsilon > 0$, there exist an N such that

$$\Big| \sum_{l=N+1}^{\infty} \left(f_+^{*l} * f_-^{*l} - f_+^{*(l+1)} * f_-^{*l} \right)(t) \Big| < \epsilon$$

for all $t \in [t_0, t_1]$.

Assume first that the support of $f_+ * f_-$ is the entire half-line $(0, \infty)$. For given t_0 and t_1, let $t \in [t_0, t_1]$, and $N > L$. We note that $(F_+ \star F_-)(t_1) < 1$. Then for every $\epsilon > 0$, by applying Corollary 3 and Lemma 2, we have

$$\begin{aligned}
&\Big| \sum_{l=N+1}^{\infty} \left(f_+^{*l} * f_-^{*l} - f_+^{*(l+1)} * f_-^{l\star} \right)(t) \Big| \\
&\leq (f_+ * f_-)^{*L} * \sum_{n=1}^{\infty} \left((f_+ * f_-)^{*(N-L+n)} + (f_+ * f_-)^{*(N-L+n)} * f_+ \right)(t) \\
&\leq \sup_{u>0} (f_+ * f_-)^{*L}(u) \cdot \sum_{n=1}^{\infty} \left((F_+ \star F_-)^{(N-L+n)\star} + (F_+ \star F_-)^{(N-L+n)\star} \star F_+ \right)(t) \\
&\leq \sup_{u>0} (f_+ * f_-)^{*L}(u) \cdot \sum_{n=1}^{\infty} (F_+ \star F_-)^{(N-L+n)}(t_1)\,(1 + F_+(t_1)) \\
&= \sup_{u>0} (f_+ * f_-)^{*L}(u) \cdot \frac{(F_+ \star F_-(t_1))^{(N-L+1)}}{1 - F_+ \star F_-(t_1)}\,(1 + F_+(t_1)),
\end{aligned}$$

where the last term can be made smaller than ϵ by taking N sufficiently large so that $(F_+ \star F_-(t_1))^{(N-L+1)}$ is small enough. This proves uniform convergence of P_+' and P_-' on compact subsets of $\mathbb{R}^+$. The proof can be extended for when the support of $f_+ * f_-$ is not the entire positive real line; this is technical and, therefore, omitted in this paper. However, the approach is to note that convolution extends the support. In essence, there is some k chosen large enough such that our t_1 falls within the support and $F_+ \star F_-^{k\star}(t_1) < 1$. Then, this is used as the size of the groupings of the subsequent terms in the series. □

Proof of Proposition 3. Form the observation that $\mathcal{L}(E_\pm)(s) = 2\mathcal{L}(P_\pm)(s) - 1/s$ we have the following system of equations

$$\begin{cases} \mathcal{L}(E_+)(s) + \frac{1}{s} &= 2\mathcal{L}(P_+)(s) \\ \mathcal{L}(E_-)(s) + \frac{1}{s} &= 2\mathcal{L}(P_-)(s). \end{cases}$$

By using the fact from Proposition 1 that $\mathcal{L}(P_-)(s)/\mathcal{L}(P_+)(s) = \Psi_-(s)$ and the first equation of the same proposition we obtain

$$\begin{cases} \mathcal{L}(E_+)(s) + \frac{1}{s} &= \frac{2}{s} \frac{1 - \Psi_+(s)}{1 - \Psi_+(s)\Psi_-(s)} \\ \frac{s\mathcal{L}(E_-)(s)+1}{s\mathcal{L}(E_+)(s)+1} &= \Psi_-(s). \end{cases}$$

Now, we can solve the top equation by substitution, and we have

$$\begin{aligned}
(s\mathcal{L}(E_+)(s) + 1) \left(1 - \Psi_+(s) \frac{s\mathcal{L}(E_-)(s) + 1}{s\mathcal{L}(E_+)(s) + 1} \right) &= 2(1 - \Psi_+(s)) \\
(s\mathcal{L}(E_+)(s) - 1) - \Psi_+(s)\,(s\mathcal{L}(E_-)(s) - 1) &= 0,
\end{aligned}$$

which leads to

$$\begin{cases} \frac{s\mathcal{L}(E_+)(s)-1}{s\mathcal{L}(E_-)(s)-1} & = \Psi_+(s) \\ \frac{s\mathcal{L}(E_-)(s)+1}{s\mathcal{L}(E_+)(s)+1} & = \Psi_-(s). \end{cases}$$

□

Proofs related to the stationary switch process. Before moving to the results on the stationary switch process, we need to define some well-known objects in renewal theory in the context of this paper. Define $S_L(t)$ and $S_R(t)$ as the instants of the most recent switch prior to t and the first switch after t, respectively. The residual time since the last switch is denoted by $B(t) = t - S_L(t)$, and the excess time is $A(t) = S_R(t) - t$.

Lemma 5. *The joint distribution of $A(t), B(t), N(t)$ is given through the following conditional probabilities for $u \in [0,t]$ and $v > 0$:*

$$\mathsf{P}\big(A(t) > v \big| B(t) = u, N(t) = k, \delta\big) = \begin{cases} \bar{F}_\delta(v+u)/\bar{F}_\delta(u); & k = 2l, \\ \bar{F}_{-\delta}(v+u)/\bar{F}_{-\delta}(u); & k = 2l+1, \end{cases}$$

$$\mathsf{P}\big(S_L(t) \le u \big| N(t) = k, \delta\big) = \begin{cases} F_\delta^{l\star} \star F_{-\delta}^{l\star}(u) \cdot \overline{\left(F_\delta^{l\star} \star F_{-\delta}^{l\star}\right)_u \star F_\delta}(t); & k = 2l; \\ F_\delta^{(l+1)\star} \star F_{-\delta}^{l\star}(u) \cdot \overline{\left(F_\delta^{(l+1)\star} \star F_{-\delta}^{l\star}\right)_u \star F_{-\delta}}(t); & k = 2l+1; \end{cases}$$

A probability measure represented by F which is restricted to $[0,u]$ is denoted by F_u i.e., $F_u(x) = F(x)/F(u)$ for $x \in [0,u]$ and one for $x > u$.

Further, the process $\Delta N(s) = N(s + S_R(t)) - N(S_R(t))$, $s > 0$, depends on the triplet $(S_L(t), S_R(t), N(t))$ only through $\delta_0 = (-1)^{N(t)+1}$. Hence, for a given $\delta = \delta_0$, it has the same distribution (as the entire process) as the process $N(s)$, $s > 0$.

Proof. For $B(t) = u, N(t) = k, \delta = 1$, we have $\sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l} T_i^- = t - u$, and $t - u + T_{l+1}^+ > t$, i.e. $T_{l+1}^+ > u$. Given these conditions, $A(t) > v$ is equivalent to $t-u+T_{l+1}^+ > t+v$, i.e. $T_{l+1}^+ > u + v$. Thus

$$\begin{aligned} \mathsf{P}\big(A(t) > v \big| B(t) = u, N(t) = k, \delta = 1\big) &= \begin{cases} \mathsf{P}(T_{l+1}^+ - u > v \big| T_{l+1}^+ > u) & ; k = 2l, \\ \mathsf{P}(T_{l+1}^- - u > v \big| T_{l+1}^- > u) & ; k = 2l+1, \end{cases} \\ &= \begin{cases} \bar{F}_+(u+v)/\bar{F}_+(u) & ; k = 2l, \\ \bar{F}_-(u+v)/\bar{F}_-(u) & ; k = 2l+1, \end{cases} \end{aligned}$$

The case of $\delta = -1$ is symmetric.

Further,

$$\begin{aligned}
\mathsf{P}\big(B(t) > u \big| N(t) = 2l, \delta = 1\big) &= \mathsf{P}\left(t - u > \sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l} T_i^- > t - T_i^+\right), \\
&= \int_0^{t-u} \bar{F}_+(t-p) dF_+^{l\star} \star F_-^{l\star}(p) \\
&= F_+^{l\star} \star F_-^{l\star}(u) \cdot \overline{\left(F_+^{l\star} \star F_-^{l\star}\right)_u \star F_+}(t), \\
\mathsf{P}\big(B(t) \le u \big| N(t) = 2l+1, \delta = 1\big) &= \mathsf{P}\left(\sum_{i=1}^{l+1} T_i^+ + \sum_{i=1}^{l} T_i^- \le u, \sum_{i=1}^{l+1} T_i^+ + \sum_{i=1}^{l+1} T_i^- > t\right), \\
&= \int_0^{u} \bar{F}_-(t-p) dF_+^{(l+1)\star} \star F_-^{l\star}(p) \\
&= F_+^{(l+1)\star} \star F_-^{l\star}(u) \cdot \overline{\left(F_+^{(l+1)\star} \star F_-^{l\star}\right)_u \star F_-}(t).
\end{aligned}$$

The second part of the result follows from the mutual independence of T_i^+'s and T_i^-'s. □

The following proof uses the key renewal theorem or, more precisely, one of its consequences, the limiting behavior of alternating processes.

The proof of Proposition 2. Let $D(t)$ be the regular switch process starting at time zero. Since the distribution of $N(s)$ is the same as $\Delta N(s)$ in Lemma 5, the stationary distribution can be obtained by finding the distributional limit of the triple $(A(t), B(t), D(t))$. The limit of the triple can be obtained from the Key Renewal Theorem.

Consider the renewal process defined via $X_n = T_n^+ + T_n^-$ and let S_n be the cumulative sum of X_n. By defining $Z_n = T_n^+$, we obtain an alternating process that is ON over $[S_n, S_n + T_{n+1}^+)$, $n \in \mathbb{N}$, and OFF, otherwise. Thus, the limiting distribution of probability that at time t the process is in the ON state is $\mathsf{E}Z_1/\mathsf{E}X_1$, see Theorem 3.4.4 in Ross (1996). This essentially proves the limiting distribution of $D(t)$.

To show the joint limiting distribution of $A(t)$ and $B(t)$ given $D(t) = 1$, we note that for an alternating process defined by $Z_n = T_n^+ \wedge (T_n^+ - b) \wedge a$ the system is ON if and only if $D(t) = 1, A(t) \le a, B(t) > b$. We note that $\mathsf{E}Z_n = \int_0^a \bar{F}_+(u+b)\, du$ and thus the limiting probability for $\mathsf{P}(\delta(t) = 1, A(t) \le a, B(t) > b)$ is equal to $\int_0^a \bar{F}_+(u+b)\, du/(\mu_+ + \mu_-)$, which demonstrates the distributional form of (δ, A, B), see Proposition 3.4.5, Ross (1996).

The expected value of the stationary switch process $\tilde{D}$ is given by $\mathsf{E}\delta = (\mu_+ - \mu_-)/(\mu_+ + \mu_-)$. To find $\mathsf{P}(\tilde{D}(t) = 1) | \tilde{D}(0) = 1)$, we follow the same path as in the proof of Proposition 1. If $\tilde{N}(t)$ denotes the number of switches between 0 and t for the stationary switch process, then its distribution at time t is given by

$$\begin{aligned}
&\mathsf{P}\Big(\tilde{N}(t) = k \Big| \tilde{D}(0) = \delta\Big) = \\
&\quad = \begin{cases}
1 - F_{A,\delta}(t); k = 0 \\
(F_+ \star F_-)^{(k/2-1)\star} \star F_{-\delta} \star F_{A,\delta}(t) - (F_+ \star F_-)^{(k/2)\star} \star F_{A,\delta}(t); & k > 0 \text{ is even}, \\
(F_+ \star F_-)^{(k-1)/2\star} \star F_{A,\delta}(t) - (F_+ \star F_-)^{(k-1)/2\star} \star F_{-\delta} \star F_{A,\delta}(t); & k \text{ is odd},
\end{cases}
\end{aligned}$$

where $F_{A,\delta}$ is the cdf of the initial delay given in Remark 1. The proof is a slight modification of the one in Theorem 3 in which one needs to account for the delay A distribution that is different from T_1^+, see Remark 1.

Since

$$\tilde{D}(t) = (-1)^{N_s(t)+(1-\delta)/2} \tag{6}$$

we can get, in the same way as in the proof of Proposition 1, that

$$\begin{aligned}
\tilde{P}_+(t) &= \mathsf{P}(\tilde{D}(t)=1|\tilde{D}(0)=1) = \sum_{l=0}^{\infty} \mathsf{P}(\tilde{N}(t)=2l\big|\tilde{D}(0)=1) \\
&= 1 - F_{A,+}(t) + \sum_{l=1}^{\infty} (F_+ \star F_-)^{(l-1)\star} \star F_- \star F_{A,+}(t) - \sum_{l=1}^{\infty} (F_+ \star F_-)^{l\star} \star F_{A,+}(t) \\
&= \mathcal{L}^{-1}\left(\frac{1}{s}\left(1 - \frac{1-\Psi_{A,+}(s)}{(1-\Psi_+(s)\Psi_-(s))}\right)\right)(t)
\end{aligned}$$

and

$$\tilde{P}_-(t) = \mathsf{P}(\tilde{D}(t)=1|\tilde{D}(0)=-1) = \mathcal{L}^{-1}\left(\frac{\Psi_{A,-}(s)\,(1-\Psi_+(s))}{s(1-\Psi_+(s)\Psi_-(s))}\right).$$

Then the final formula for the Laplace transform of $P_\delta(t)$ follows from the form of $\Psi_{A,\delta}(s)$ that is given in Remark 1.

Finally, for the covariance function, we note that

$$\begin{aligned}
R(t) &= \mathsf{E}(\tilde{D}(0)\tilde{D}(t)) - \frac{(\mu_+-\mu_-)^2}{(\mu_++\mu_-)^2} \\
&= \left(2\tilde{P}_+(t)-1\right)\frac{\mu_+}{\mu_++\mu_-} - \left(2\tilde{P}_-(t)-1\right)\frac{\mu_-}{\mu_++\mu_-} - \frac{(\mu_+-\mu_-)^2}{(\mu_++\mu_-)^2} \\
&= \frac{2}{\mu_++\mu_-}\left(\tilde{P}_+(t)\mu_+ - \tilde{P}_-(t)\mu_- + \mu_+\frac{\mu_--\mu_+}{\mu_++\mu_-}\right)
\end{aligned}$$

The Laplace transform of R can now be easily obtained by combining all the above results. □

Proof of Proposition 4. The first equations we obtain from Proposition 2 and 1

$$\begin{aligned}
\mathcal{L}(\tilde{P}'_+)(s) &= s\mathcal{L}(\tilde{P}_+)(s) - \mathsf{P}(X_+(0)=1|X_+(0)=1), \\
&= 1 - \frac{1}{\mu_+ s}\frac{(1-\Psi_+(s))\,(1-\Psi_-(s))}{1-\Psi_+(s)\Psi_-(s)} - 1, \\
&= \frac{1}{\mu_+ s}\frac{(1-\Psi_+(s))\,(1-\Psi_-(s))}{1-\Psi_+(s)\Psi_-(s)}, \\
&= \frac{1}{\mu_+}\left(\frac{1-\Psi_+(s)}{s(1-\Psi_+(s)\Psi_-(s))} - \frac{\Psi_-(s)(1-\Psi_+(s))}{s(1-\Psi_+(s)\Psi_-(s))}\right), \\
&= -\left(\frac{\mathcal{L}(P_+)(s)-\mathcal{L}(P_-)(s)}{\mu_+}\right).
\end{aligned}$$

The proof is very similar for $\mathcal{L}(\tilde{P}'_+)(s)$ and observing that from the construction of $\tilde{P}'_+$ if follows that $\mathsf{P}(X_-(0)=1|X_-(0)=-1)=0$. The second equation follows from substituting in the first and that $E_+(t) = 2\mathsf{P}(X_+(t)=1)-1$. □

IV

SLEPIAN MODEL BASED INDEPENDENT INTERVAL APPROXIMATION FOR THE LEVEL EXCURSION DISTRIBUTIONS

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Abstract. The independent interval approximation of the excursion time distributions for Gaussian processes has been used in physics and engineering. A new but related approach matches the expected value of the clipped Slepian to the expected value of a non-stationary binary stochastic process. This approach is extended to non-zero crossings and provides a probabilistic foundation for the validity of the approximations for a large class of processes. Both the above and below distributions are approximated. While the Slepian-based method was shown to be equivalent to the classical IIA for the zero-level, this is not the case for non-zero excursions.

1. Introduction

Excursion distribution has received considerable attention from the mid-1930s until today. This is not surprising, considering the importance of characterizing the distribution of excursions in many applications. For example, the length of a drought or a heatwave are two problems where characterizing the exceedance distributions is important. Here, classical methods in extreme value theory, such as peaks over threshold or block maxima, fail since it is the length of the excursion and not the maxima that is of interest.

The formulation of the excursion distribution is both simple and intuitive, but this hides the true complexity of the problem. The problem of fully characterizing the excursion distribution from characteristics of the underlying process is still unsolved despite considerable efforts. The study of excursion distributions originates in the study of the number of real roots of a random polynomial. It was perhaps Littlewood and Offord (1938) who first formulated the question mathematically, and early work was done by Kac (1943), Littlewood and Offord (1939), and Erdös and Offord (1956). From this early work Rice (1944) studied the expected number of roots of a stochastic process, which led to the famous Rice formula on the expected number of crossing in a unit of time for an ergodic stationary process. The study of the number of zeros would evolve into investigating the length between zeros, and hence, the first investigation of excursion times began. A

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discussion of the development is incomplete without mentioning the results of McFadden (1956).

McFadden (1956) investigated the length of axis crossings and first proposed the notion of clipping a stochastic process. If $X(t)$ is the process of interest, the clipped process $Z(t)$ can be constructed by computing $Z(t) = sgn(X(t) - u)$. While $Z(t)$ will be less rich in terms of information, all the information regarding the excursion behavior of $X(t)$ will be contained in $Z(t)$.

Due to the lack of results connecting the characteristics of the process of interest to the excursion distribution, approximation methods have been developed. A method commonly used in statistical physics is the so-called independent interval approximation (IIA). The basis for this is to approximate the clipped process with a stationary binary process taking values one and minus one, where the time spent in the one and minus one states are independent. The extensive survey paper by Bray et al. (2013) is a good overview of the use of the IIA in statistical physics. However, Longuet-Higgins (1962) showed that this approximation method is never valid for Gaussian processes. Nevertheless, the IIA has been further developed, and in particular Sire (2007) extended the IIA framework to non-zero levels. There are, however, questions about the mathematical validity of the IIA framework due to its lack of criterion on the characteristics used for the approximation.

The IIA has also been used to derive the so-called persistency coefficient. This coefficient describes the tail behavior of the excursion distribution and is used to characterize various systems in statistical physics. However, deriving the persistency coefficient is a difficult problem, and therefore, they are only known for a handful of processes. For example, Poplavskyi and Schehr (2018) derived the persistency coefficient for zero-level crossings of the two-dimensional diffusion process.

Recently, another version of the IIA framework was proposed by Bengtsson and Podgórski (2024b). This version is based on clipping the Slepian process and matching it to a non-stationary switch process. However, this was only done for zero-level crossings, and this approach was shown to be equivalent to the ordinary IIA. In this paper, the Slepian-based IIA framework is extended to non-zero-level excursions. Conditions are also provided for the mathematical validity of this approach. These conditions are based on recent results on the properties of the switch process by Bengtsson and Podgórski (2024a).

The paper is organized as follows. Section 2 and 3 introduce the Slepian model and the Switch processes and thus serve as preliminaries for subsequent sections. In Section 4, properties of the clipped processes are derived, which are then used when matching characteristics. The latter is done in Section 5, and in this section, we demonstrate the difficulty of using the Clipped process and stationary switch process for non-zero level IIA. In Section 6, the Slepian-based IIA is applied to approximate the excursion distribution of Gaussian diffusion processes in two dimensions. Persistency coefficients are approximated for this process, and the results are compared to results based on simulating trajectories of the process.

2. Slepian process

Recall that the Slepian model at level u corresponding to a smooth stationary Gaussian process $X(t)$ with the twice continuously differentiable normalized covariance function $r(t)$ is given through

$$X_u(t) = u \cdot r(t) - R \cdot r'(t)/\sqrt{-r''(0)} + \Delta(t), \tag{1}$$

where $r(t)$ is the covariance of X, R is a standard Rayleigh variable, i.e. with density $f_R(s) = se^{-s^2/2}$ independent of a (non-stationary) Gaussian process Δ with covariance

$$r_\Delta(t,s) = r(t-s) - r(t)r(s) + r'(t)r'(s)/r''(0).$$

This process describes the statistical behavior of the original process X at the instants of an up-crossing of the level u, i.e., the process behaves how a trajectory of a stationary Gaussian process is seen when observed at the u-level up-crossing instant.

The Slepian process decomposes the behavior at the $u-$up crossing into three distinct parts. The first is fully deterministic and only depends on the level u: $r_u(t) = u\ r(t)$. The second is the deterministic function $-r'(t)/\sqrt{r''(0)}$, scaled by a standard Rayleigh random variable R. The last component is a non-stationary Gaussian stochastic residual process

$$\Delta(t) = X(t) - X(0) \cdot r(t) - X'(0) \cdot r'(t)/r''(0),$$

independent of R and u. In Figure 1, we present samples from these three components for Gaussian diffusion processes together with samples from the Slepian models combined from these components.

We also note that due to the symmetry of Gaussian processes, the Slepian model $\tilde{X}_u$ of the process X at a u-down-crossing is given by

$$\tilde{X}_u(t) \stackrel{d}{=} -X_{-u}(t), \tag{2}$$

where the relation denotes equality of the distributions of the two processes. Throughout this paper, unless otherwise stated, we will assume that all covariance functions are normalized such that $r(0) = 1$. With the Slepian process introduced, we will move to the binary processes, which are used in the independent interval approximation method.

3. Stationary and non-stationary switch processes

The key idea of the Slepian-based IIA is to match the expected value of the clipped Slepian process to the expected value of a switch process. This section provides a short account of the necessary results for this matching and the results that later give the probabilistic motivation for why the Slepian-based IIA is valid. For an overview of the switch process, see Bengtsson (2024) and Bengtsson and Podgórski (2024a). The switch process is defined by interlacing intervals of random length such that after the end of each random length, the process switches either from one to minus one or from minus one to one. The process is defined on the positive half line starting at the origin, and the initial

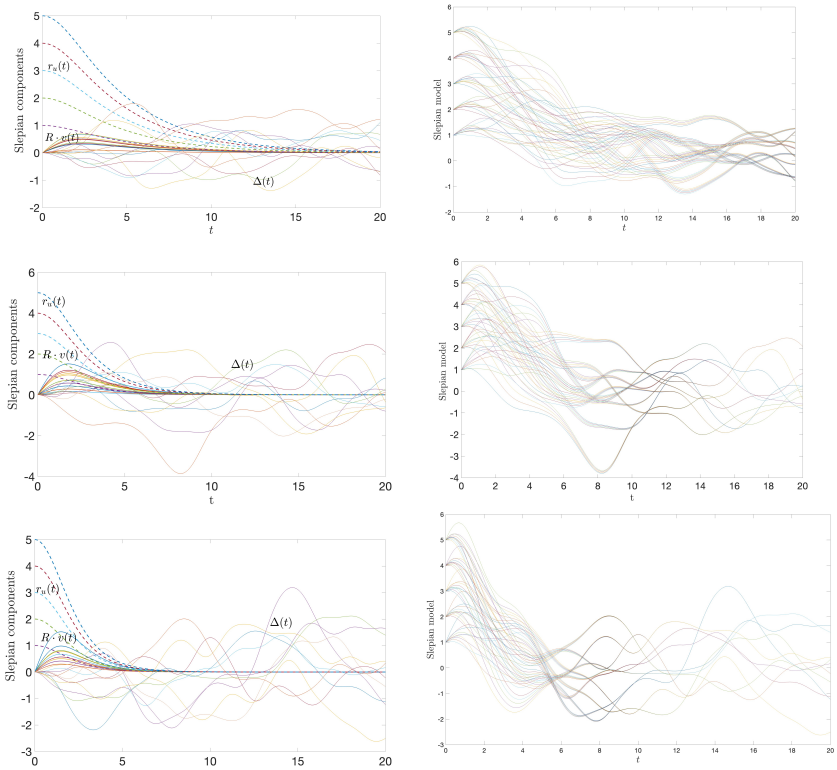

FIGURE 1. The Slepian model for three Gaussian diffusions: 1-dimensional *(top)*, 2-dimensional *(middle)*, 3-dimensional *(bottom)*. *Right:* Three components of the decomposition; *Left:* Components put together to create ten samples from the Slepian model at five different levels.

value is determined by a Bernoulli random variable, such that the process starts at one with some probability p_0 and at minus one with $1 - p_0$. The Bernoulli random variable is independent of the intervals, which are denoted by T_i^-, and T_i^+, for $i \in \mathbb{N}$ for the time spent in state minus one and one, respectively. Additionally, it's assumed that T_i^+ and T_i^- sequences of independent identically distributed (iid) random variables which are also mutually independent.

The resulting plus-minus process is denoted by $D(t)$, $t > 0$. However, $D(t)$ can be extended to the negative real line by attaching another realization at zero that starts from $-\delta$ in the negative direction. In Figure 2, we illustrate a trajectory of such a process, which starts at minus one. Before presenting some elementary results needed for this paper, we need to establish the notation used for the Laplace transform since this is the main mathematical tool used. Let $\Psi(s)$ be the Laplace transform of a probability

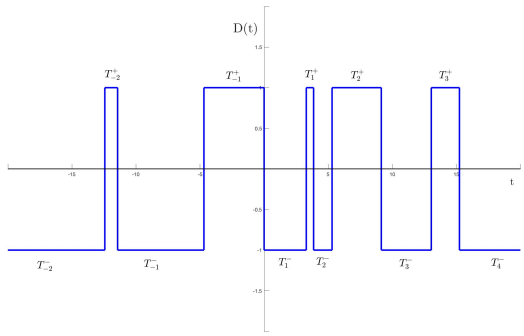

FIGURE 2. The definition of the switch process attached to the origin, with the initial state $\delta = -1$.

distribution defined by the cdf F with support on the positive half line. The Laplace transforms of T_i^+, $i \in \mathbb{N}$ are denoted by $\Psi_+(s)$ and $\Psi_-(s)$.

In this work, we utilize Proposition 1 from Bengtsson and Podgórski (2024a) on the Laplace transform of the expected value function of the switch process.

Proposition 1. *Let $D(t)$ be a non-stationary switch process then the Laplace transform of $P_\delta(t) = \mathsf{P}\Big(D(t) = 1\Big|\delta\Big)$, $t > 0$, is given by*

$$\mathcal{L}(P_\delta)(s) = \frac{1 - \Psi_+(s)}{s(1 - \Psi_+(s)\Psi_-(s))} \cdot \begin{cases} 1 & ;\delta = 1, \\ \Psi_-(s) & ;\delta = -1, \end{cases}$$

Additionally, the Laplace transform of the expected value function $E_\delta(t) = \mathsf{E}(D(t)\big|\delta)$, $t > 0$ is given by

$$\mathcal{L}(E_\delta)(s) = \frac{\Psi_-(s) - \Psi_+(s) + \delta(1 - \Psi_-(s))(1 - \Psi_+(s))}{s(1 - \Psi_+(s)\Psi_-(s))}.$$

Focusing on the expected value functions to characterize this process is natural since $D(t)$ is not stationary. The reason for this lack of stationery is the attachment of the switch at zero.

In the ordinary IIA, a stationary switch process is used to approximate the clipped process, and matching is done through the covariance function. A stationary switch process can be constructed from the non-stationary version by delaying the process forward and backward around zero. Hence, the origin will be placed in some interval $[-B, A]$, and two independent non-stationary switch processes are attached at each end of this interval. The value of the process on this interval is chosen by a binary random variable δ, that takes the value one with probability $\mu_+/(\mu_+ + \mu_-)$ and minus one with probability $\mu_-/(\mu_+ + \mu_-)$. The main question is now how to find the distributions of A and B such that the delay results in a stationary process. The answer follows from well-known results

in renewal theory see, for example, Cox (1962), and the densities, conditional on δ, are given by

$$f_{A|\delta}(t) = f_{B|\delta}(t) = \frac{1 - F_\delta(t)}{\mu_\delta}.$$

Where F_δ is the cdf of T_+ or T_- depending on if δ takes the value one or minus one, respectively. Throughout the rest of the paper, we will denote this stationary switch process by $\tilde{D}(t)$, $t \in \mathbb{R}$. The characteristic associated with the stationary switch process is the covariance function. We therefore present a shortened version of Proposition 2 from Bengtsson and Podgórski (2024a), which gives us the covariance function of $\tilde{D}(t)$ in the Laplace domain.

Proposition 2. *Let $\tilde{D}(t)$ be a stationary switch process constructed from the interval distributions T_i^+ and T_i^- for $i \in \mathbb{N}$ with the expected values μ_+ and μ_- respectively. Define $\tilde{P}_\delta(t) = P(\tilde{D}(t) = 1 | \tilde{D}(0) = \delta)$, then the process $\tilde{D}(t)$ is uniquely characterized by $\tilde{P}_\delta(t)$ which have the following form in the Laplace domain*

$$\mathcal{L}(\tilde{P}_\delta)(s) = \frac{1}{s} \begin{cases} 1 - \dfrac{1}{\mu_+ s} \dfrac{\big(1 - \Psi_+(s)\big)\big(1 - \Psi_-(s)\big)}{1 - \Psi_+(s)\Psi_-(s)} & ; \delta = 1, \\ \dfrac{1}{\mu_- s} \dfrac{\big(1 - \Psi_+(s)\big)\big(1 - \Psi_-(s)\big)}{1 - \Psi_+(s)\Psi_-(s)} & ; \delta = -1. \end{cases}$$

The covariance of $\tilde{D}(t)$, $R(t) = \mathsf{Cov}(\tilde{D}(u), \tilde{D}(t+u))$ is given by

$$R(t) = \frac{2}{\mu_+ + \mu_-} \left(\tilde{P}_+(t)\mu_+ - \tilde{P}_-(t)\mu_- + \mu_+ \frac{\mu_- - \mu_+}{\mu_+ + \mu_-} \right). \tag{3}$$

and its Laplace transform is given by

$$\mathcal{L}(R)(s) = \frac{4}{s\,(\mu_+ + \mu_-)} \left(\frac{\mu_+ \mu_-}{\mu_+ + \mu_-} - \frac{1}{s} \frac{(1 - \Psi_+(s))(1 - \Psi_-(s))}{1 - \Psi_-(s)\Psi_+(s)} \right). \tag{4}$$

We now have a collection of characteristics in the Laplace domain, $\Psi_+(s)$, $\Psi_-(s)$, $\mathcal{L}(E_+)(s)$, $\mathcal{L}(E_-)(s)$ and $\mathcal{L}(R)(s)$. The main goal now is to retrieve $\Psi_+(s)$ and $\Psi_-(s)$ from either the covariance or the expected value functions. It was shown in Bengtsson and Podgórski (2024a) that $\Psi_+(s)$ and $\Psi_-(s)$ are obtainable from the derivatives of the expected value function, and this results in the following expression

$$\Psi_+(s) = \frac{\mathcal{L}(E'_+)(s)}{\mathcal{L}(E'_-)(s) - 2}, \quad \Psi_-(s) = \frac{\mathcal{L}(E'_-)(s)}{\mathcal{L}(E'_+)(s) + 2}. \tag{5}$$

However, if one wants to use the above relations to match the expected value functions of the clipped Slepian process with the crossing distributions, one needs to show that the above functions are completely monotone so that they indeed correspond to the Laplace transform of probability distributions. This is not easy to demonstrate directly from the definition. However, Theorem 1 in Bengtsson and Podgórski (2024a) gives sufficient conditions on the derivatives of E_+ and E_+, which can be used to solve this problem.

Theorem 1. *Let $D(t)$ be a non-stationary switch process with the expected value functions $E_+(t)$ and $E_-(t)$. Then the following conditions are equivalent*

i) Functions $-E'_+(t)$ and $E'_-(t)$ are non-negative,

ii) T_+ and T_- have the stochastic representations

$$T_+ \stackrel{d}{=} X + \sum_{k=1}^{\nu_\alpha - 1} Y_k, \qquad T_- \stackrel{d}{=} Y + \sum_{k=1}^{\nu_\beta - 1} X_k,$$

where the sums are considered zero whenever summing over the empty set. Additionally, X_i's and Y_i's mutually independent and iid, with the densities

$$f_X(t) = -\frac{1}{2\alpha} E'_+(t) \qquad f_Y(t) = \frac{1}{2\beta} E'_-(t),$$

respectively. ν_α and ν_β are geometrically distributed with the parameters

$$\alpha = \frac{\mu_-}{\mu_+ + \mu_-} \qquad \beta = \frac{\mu_+}{\mu_+ + \mu_-},$$

which also serves as the normalizing coefficients of $-E'_+$ and E'_-.

It should be mentioned that the monotonicity of the expected value functions is often easy to verify in practice.

4. Clipping Slepian and Gaussian processes

The length of u-level excursions of a smooth stationary process $X(t)$ forms a sequence of random variables. Additionally, if these excursions are marked with whether they fall below or above the level u, the process exhibits a structure similar to a switch process. However, the intervals for this process will not be independent, which is assumed for the switch process. We denote these interval lengths with $\bar{T}_i^+$ and $\bar{T}_i^-$, respectively, $i \in \mathbb{Z}$. More formally, we define the clipped process at level u by

$$\tilde{D}_u(t) = sgn(X(t) - u).$$

It is clear that $\tilde{D}_u(t)$ will inherit the stationarity property and contain information about the u-level excursions of $X(t)$ both above and below the level. In Figure 3, the idea of clipping a process is illustrated for a Gaussian process clipped at $u = 0.5$. In the ordinary IIA approach, the covariance of the process $X(t)$, $r(t)$, is matched to the covariance of the clipped process. For the simplest case, the process is a zero mean Gaussian process clipped at $u = 0$; we obtain the well-known and explicit arcsin formula

$$R_0(t) = \frac{2}{\pi} \arcsin r(t).$$

For the non-zero level, greater care is needed. Let F_X be the cdf of $X(0)$ and $\bar{F}_X = 1 - F_X$. Assuming continuity of the distribution of $X(t)$, the covariance of the clipped stationary

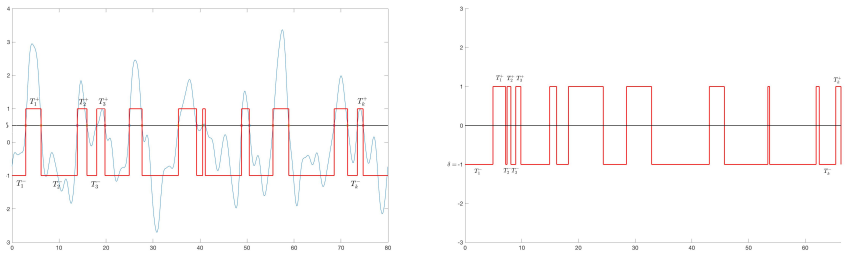

FIGURE 3. The excursion intervals of a process $X(t)$ together with the corresponding clipped process (left) and a sample of the switch process under the IIA (right) with exponential distributions having the means matched those in the top graph.

process can be written as follows

$$\begin{aligned} R_u(t) &= \mathsf{P}(X(t) > u | X(0) > u)\bar{F}_X(u) + \mathsf{P}(X(t) \le u | X(0) \le u)F_X(u) \\ &\quad - \mathsf{P}(X(t) \le u | X(0) > u)\bar{F}_X(u)\mathsf{P}(X(t) > u | X(0) \le u)F_X(u) - \big(1 - 2F_X(u)\big)^2 \\ &= 2\left(\mathsf{P}(X(t) > u | X(0) > u)\bar{F}_X(u) + \mathsf{P}(X(t) \le u | X(0) \le u)F_X(u) + 2F_X(u)\bar{F}_X(u)\right). \end{aligned}$$

If we assume that the process $X(t)$, $t \in \mathbb{R}$ is distributionally symmetric in value, i.e. it has the same distribution as $-X(t)$, $t \in \mathbb{R}$, then for $A_u(t) = \mathsf{P}(X(t) \le u, X(0) \le u)$ we have

$$R_u(t) = 2\left(A_{-u}(t) + A_u(t) + 2F_X(u)F_X(-u)\right).$$

For a zero mean Gaussian process $X(t)$ with the covariance $r(t)$ and if we set $\tilde{u} = u/\sqrt{r(0)}$ and $\rho_t = r(t)/r(0)$:

$$\begin{aligned} A_u(t) &= \mathsf{P}(\rho_t Z + \sqrt{1-\rho_t^2}Y < \tilde{u}, Z < \tilde{u}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tilde{u}} e^{-\frac{z^2}{2}} \Phi\left(\frac{\tilde{u} - \rho_t z}{\sqrt{1-\rho_t^2}}\right) dz = B(\tilde{u}, \rho_t), \end{aligned}$$

where Z and Y are independent standard normal variables, and the last equation serves as a definition of B. Consequently, we obtain the covariance of the clipped processes in terms of the covariance of the underlying Gaussian process

$$R_u(t) = 2\left(B\left(\frac{u}{r(0)}, \frac{r(t)}{r(0)}\right) + B\left(\frac{-u}{r(0)}, \frac{r(t)}{r(0)}\right) + \Phi\left(\frac{u}{r(0)}\right)\Phi\left(\frac{-u}{r(0)}\right)\right). \tag{6}$$

Since the clipped process of a stationary process is stationary, it is natural to characterize the clipped process by its covariance function.

However, in the IIA approach, one has to match two characteristics corresponding to the distributions of the excursion above and below the level u. For this purpose, the Slepian process attached at u-crossing (up or down) constitutes a much better alternative

than the clipped stationary Gaussian process. The reason for this is that we do have two Slepian processes at a crossing of u. One is for the u-upcrossing and the other for the u-downcrossing. In our approach, we clip two Slepian processes given in (1) and (2)

$$D_u^+(t) = \operatorname{sgn}(X_u(t) - u) \tag{7}$$

$$D_u^-(t) = \operatorname{sgn}(\tilde{X}_u(t) - u). \tag{8}$$

We note that by (2), $D_u^- \stackrel{d}{=} -D_{-u}^+$.

In the next proposition, the expected value of the clipped process is presented. Later, this function will be used to obtain the approximated excursion distribution by matching it with a non-stationary switch process. Hence, this is an important equation for the Slepian-based IIA.

Proposition 3. *Let $D_u^+(t)$ be the clipped Slepian process at an u-level up crossing of a Gaussian process, $u \in \mathbb{R}$, with the covariance function $r(t)$ twice continuously differentiable. Then $E_u^+(t) = \mathsf{E}(D_u^+(t))$ is given by*

$$E_u^+(t) = 1 - 2\Phi\left(u\frac{1-r(t)}{\sqrt{1-r^2(t)+\frac{r'^2(t)}{r''(0)}}}\right) - 2\frac{1}{\sqrt{-r''(0)}}\frac{r'(t)}{\sqrt{1-r^2(t)}}\exp\left(-\frac{u^2}{2}\frac{1-r(t)}{1+r(t)}\right)$$
$$\times\,\Phi\left(\frac{-u}{\sqrt{-r''(0)}}\sqrt{\frac{1-r(t)}{1+r(t)}}\frac{r'(t)}{\sqrt{1-r^2(t)+\frac{r'^2(t)}{r''(0)}}}\right).$$

Let $E_u^-(t) = \mathsf{E}(D_u^-(t))$, then

$$E_u^-(t) = -E_{-u}^+(t).$$

Moreover,

$$\lim_{t\to 0^+} E_u^+(t) = 1, \quad \lim_{t\to 0^+} E_u^-(t) = -1,$$
$$\lim_{t\to\infty} E_u^+(t) = \lim_{t\to\infty} E_u^-(t) = 1 - 2\Phi(u).$$

The proof requires some technical lemmas, and these, together with the proof, have been relegated to the Appendix. While the expression of the clipped Slepian process might appear to be intimidating, it simplifies significantly for the zero level. For $u = 0$, it reduces to

$$E_0^+(t) = -\frac{1}{\sqrt{-r''(0)}}\frac{r'(t)}{\sqrt{1-r(t)^2}}.$$

In the next section, the characteristics of this section, such as the covariance of the clipped process and expected value functions of the clipped Slepian process, will be used to obtain approximations of the excursion distributions.

5. IIA for a non-zero crossing level

Intuitively, the IIA simply means that the characteristics of a clipped process are matched to the characteristics of a switch process, either the stationary one as in the ordinary IIA case or the non-stationary one in the Slepian-based IIA. Both approaches imply that the dependency between the $\bar{T}_i^+$'s and $\bar{T}_i^-$'s of the clipped process is presumed to be negligible so that independence can, for practical purposes, be assumed. After matching the characteristics, the relation between the characteristics and the switching times of the switch processes is utilized to obtain the switching time distributions, which serve as an approximation of the excursion time distributions. We start with an overview and treatment of the ordinary IIA and then introduce the Slepian-based IIA, which extends to non-zero levels of the work done in Bengtsson and Podgórski (2024b).

5.1. **The stationary IIA.** Initially, when zero-level crossings were considered, the stationary versions of the processes were matched through their covariance functions, see Bray et al. (2013). It was shown in Bengtsson and Podgórski (2024a) that the Slepian-based IIA coincides with the ordinary IIA for zero-level crossings, i.e. that the two approaches are equivalent. However, the non-zero crossing case is more complicated and has not been extensively studied. Using characteristics of a clipped process and matching it to characteristics of a stationary switch process was introduced in Sire (2007) and further investigated in Sire (2008). Here, we present a more rigorous introduction of the IIA at the non-zero crossings for the stationary case, which is needed to identify the gaps in the methodology that will be later addressed through the Slepian-based approach.

Following the previous sections, we have two stationary processes, the switch stationary one $\tilde{D}(t)$, $t \in \mathbb{R}$, and the clipped stationary Gaussian one $\tilde{D}_u(t) = \operatorname{sgn}(X(t)-u)$, $t \in \mathbb{R}$. It is clear that the stationary switch process $\tilde{D}(t)$ is simpler in structure and can be utilized for approximate analysis of $\tilde{D}_u(t)$ if the two are matched in one or the other way. In the more straightforward, zero-level/symmetric case, the two processes can be matched by their covariance structure, i.e., we call $\tilde{D}(t)$ the independent intervals approximation of $\tilde{D}_0(t)$ if the covariances of the two are matched, i.e.

$$R(t) = R_0(t).$$

For the case of $u = 0$, it simply means that the (identical) distribution of T_i^+ and T_i^- is given by the cdf F, the following relation have to be satisfied

$$\mathcal{L}\left(\frac{2}{\pi}\arcsin\rho_t\right)(s) = \frac{1}{s}\left(1 - \frac{2}{s\mu}\frac{1-\Psi(s)}{1+\Psi(s)}\right).$$

Through this relation, the distribution given by Ψ matches the covariance of the original process. Conversely, if the distribution of the excursion times is specified, then, through the above relation, we can obtain the covariance of the original process. This one-to-one correspondence (if such one exists) formally defines the IIA for zero-level crossings. Of course, it is not obvious that the relation leads to a valid distribution, which has been discussed in Bengtsson and Podgórski (2024b). In this case, one needs also match also μ,

which can be done by the reciprocal of the crossing intensity given by the Rice formula

$$\lambda_0 = \frac{\mathsf{E}\left(|X'(0)|\right)}{\sqrt{2\pi r(0)}} = \frac{1}{\pi}\sqrt{\frac{-r''(0)}{r(0)}},$$

so that the matching equality is

$$\mu = \frac{1}{\lambda_0} = \pi\sqrt{\frac{r(0)}{-r''(0)}},$$

We note that for the asymmetric case, i.e., when $u \neq 0$, there are two distributions to be matched, F_+ for the excursions above the u-level and F_- corresponding to these below the u-level. Thus, if we consider matching the covariance of the switch process and the clipped process, one needs one more relation to solve for both distributions. For that, different strategies could be taken.

Let us consider the matching through the covariance functions. First, the covariance R_u of the clipped process is given in (6). The relation between the covariance function of the stationary switch process and the interval distributions is given by Proposition 2. By computing the derivative of (3) and using and (5) we obtain the following

$$\mathcal{L}(R'')(s) = \frac{4}{\mu_+ + \mu_-} \cdot \frac{\Psi_+(s) + \Psi_-(s) - 2\Psi_+(s)\Psi_-(s)}{1 - \Psi_+(s)\Psi_-(s)}.$$

As it can be seen from this relation, the roles of Ψ_+ and Ψ_- cannot be distinguished solely based on the covariance function R of the stationary switch process and thus from the covariance R_u of the clipped process that matches R. There is a need for an additional relation.

Before discussing this, let us point out how μ_+ and μ_- can be matched. First, we have $\mu_+ + \mu_-$ can be matched by the reciprocal of the half of the u-level crossing intensity λ_u which is given by the Rice formula

$$\lambda_u = \frac{\mathsf{E}\left(|X'(0)|\right)}{\sqrt{2\pi r(0)}}\, e^{-u^2/(2r(0))} = \frac{1}{\pi}\sqrt{\frac{-r''(0)}{r(0)}}\, e^{-u^2/(2r(0))}$$

so

$$\mu_+ + \mu_- = 2\pi\sqrt{\frac{r(0)}{-r''(0)}} e^{u^2/(2r(0))}. \tag{9}$$

It is easy to argue that the ratio of averages of spending above and below the level u must be equal to the ratio of the probabilities of $X(0)$ being above and below u, yielding the second matching equation

$$\frac{\mu_+}{\mu_-} = \frac{\Phi\left(-u^2/(2r'(0))\right)}{1 - \Phi\left(-u^2/(2r'(0))\right)}, \tag{10}$$

where Φ is the cdf of a standard normal distribution.

In Sire (2007), where non-zero level crossing IIA has been attempted for the first time, and two matching characteristics have been proposed

$$R_{<u}(t) = \mathsf{E}((u - X(t))^+(u - X(0))^+), \tag{11}$$

$$N_{<u}(t) = \mathsf{E}\left(N\left(\{0 < s \le t; X(s) = u\}\right)\mathbb{I}_{X(0)<u}\right), \tag{12}$$

where x^+ is a positive part of a real number x, $\mathbb{I}_A$ is an indicator function of a set A, and $N(\cdot)$ is counting measure of a set.

These two characteristics of the clipped process parallel equivalent characteristics of the switch process. Let us introduce the switch counting process $\tilde{N}(t)$, $t > 0$ that counts switches in $(0, t]$ for the stationary switch process $\tilde{D}$. Clearly,

$$\tilde{D}(t) = (-1)^{\tilde{N}(t)+(1-\delta)/2},\ t \ge 0.$$

The quantities corresponding to (11) and (12) are, respectively,

$$A_<(t) = \mathsf{E}\left(\frac{1-\tilde{D}(t)}{2}\frac{1-\tilde{D}(0)}{2}\right),$$

$$N_<(t) = \mathsf{E}\left(\tilde{N}(t)|\delta = -1\right).$$

We have

$$A_<(t) = \frac{R(t)}{4} + \frac{\mu_-^2}{(\mu_+ + \mu_-)^2}.$$

Thus, by utilizing Theorem 2 and Proposition 4 given in the Appendix, we obtain the following Laplace transform relations

$$\mathcal{L}(A_<)(s) = \frac{1}{s(\mu_- + \mu_+)}\left(\mu_- - \frac{1}{s}\frac{(1-\Psi_+(s))(1-\Psi_-(s))}{1-\Psi_-(s)\Psi_+(s)}\right), \tag{13}$$

$$\mathcal{L}(N_<)(s) = \frac{1}{s^2\mu_-}\frac{\left(1+\Psi_+(s)\right)\left(1-\Psi_-(s)\right)}{1-\Psi_+(s)\Psi_-(s)}. \tag{14}$$

Given the characteristics $A_<$ and $N_<$, these two equations allow the identification of both Ψ_+ and Ψ_-.

Remark 1. We note that the formula for the Laplace transform of $N_<$ coincides with that in Sire (2007) and Sire (2008), and the transform of A is also matching presented there if one accounts matching for μ_+ and μ_- given in (9) and (10).

In the stationary IIA approach, which is presented above, one challenge is to evaluate explicitly $N_{<u}$ for the clipped process. Moreover, it is not apparent that matching through (13) and (14) leads to valid distribution functions given through Ψ_+ and Ψ_-.

5.2. **The Slepian-based IIA.** The core idea of the Slepian-based IIA is to use the Slepian process and Theorem 3 to obtain approximated excursion distributions as the

distributions $\hat{T}_+$ and $\hat{T}_-$ obtained through (5) as follows

$$\Psi_{\hat{T}_+}(s) = \frac{\mathcal{L}(E_u^{+\prime})(s)}{\mathcal{L}(E_u^{-\prime})(s) - 2} \qquad \Psi_{\hat{T}_-}(s) = \frac{\mathcal{L}(E_u^{-\prime})(s)}{\mathcal{L}(E_u^{+\prime})(s) + 2}. \tag{15}$$

For the equations to correspond to valid probability distributions, they need to be completely monotone. To verify it for a given pair of $E_u^{+\prime}$ and $E_u^{-\prime}$ is difficult at best. This is where the utility of Theorem 1 becomes apparent. Since it provides easy-to-check conditions on $E_u^{+\prime}$ and $E_u^{-\prime}$ such that they correspond to valid probability distributions, the next theorem formulates this in the context of the clipped Slepian process.

Theorem 2. *Let X be a stationary Gaussian process with the twice continuous differentiable covariance function $r(t)$ $t \in \mathbb{R}$. If for a given level u the functions E_u^+ and E_u^- from Proposition 3 are monotone, then the approximated excursion distributions above and below the level u in the Slepian based IIA has the stochastic representation*

$$\hat{T}_+ \stackrel{d}{=} X + \sum_{k=1}^{\nu_\alpha - 1} Y_k, \qquad \hat{T}_- \stackrel{d}{=} Y + \sum_{k=1}^{\nu_\beta - 1} X_k,$$

where sums over the empty set are zero when the sum is empty. ν_α and ν_β are geometrically distributed with parameters

$$\alpha = \Phi(u) \qquad \beta = 1 - \Phi(u)$$

respectively, and the variables X_i's and Y_i's have densities

$$f_X(t) = -\frac{1}{2\alpha} E_u^{+\prime}(t) \qquad f_Y(t) = \frac{1}{2\beta} E_u^{-\prime}(t).$$

The sequences X_i and Y_i are iid and independent of ν_α and ν_β which are also mutually independent.

As previously mentioned, for the zero level case, it was shown in Bengtsson and Podgórski (2024b) that the ordinary IIA and the Slepian-based IIA coincide. This does not seem to be true for the non-zero level crossings since $A_{<u}$ and $N_{<u}$ of the clipped process do not seem related to E_u^+ and E_u^- in the same way as $A_<$ and $N_<$ are related to E_+ and E_- for the switch process. We advocate for the Slepian-based IIA given through (15) since E_u^+ and E_u^- explicitly given in Theorem 3, while $A_{<u}$, that can be obtained from (6), involves integration formulas and obtaining a formula for $N_{<u}$ is a nontrivial open problem, yet to be solved.

6. Non-zero level persistency coefficients for Gaussian diffusion

This section presents an example of the IIA for non-zero level crossing for the Gaussian diffusion process. We focus on the 2−dimensional diffusion process for two reasons. The first reason is that it is one of the few processes for which the persistency coefficient is analytically known for the zero level, see Poplavskyi and Schehr (2018). The non-zero excursion has also been studied by, among others Sire (2008) using the ordinary IIA. The

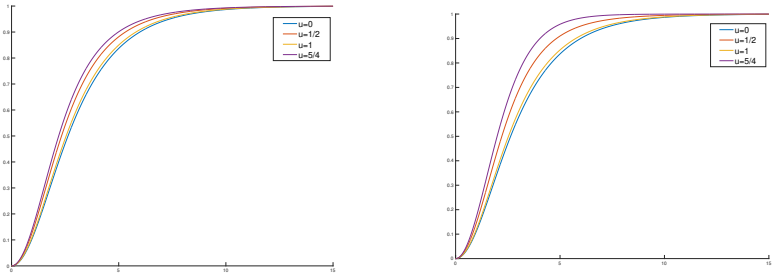

FIGURE 4. The expected values of the Slepian model based on the diffusion covariance, for an u up-crossing *(left)* and an u down-crossing *(right)*, for $u = 0, 1/2, 1, 5/4$.

second reason is its wide use in statistical physics, where the diffusion covariances were used in Majumdar et al. (1996), Bray et al. (2013), and Wong et al. (2001).

In particular, Wong et al. (2001) studied the case where a diffusing field starts from a random initial configuration. This model is theoretically attractive due to the linear form of the diffusion equation in any dimension and experimentally because of its natural physical interpretation in various non-equilibrium systems. For these reasons, the model constitutes a mathematically convenient and physically important benchmark for any methodology aiming at a persistency approximation and assessment.

For the sake of this presentation, it is sufficient to know that the covariance of the d-dimensional diffusion is given by $\operatorname{sech}^{d/2}(t/2)$, $d \in \mathbb{N}$ and that for $d = 2$ we have an explicit spectrum $\operatorname{sech}(\pi\omega)$. While simulating trajectories might seem adequate, it becomes impractical for higher levels of u where these crossing events occur less frequently, necessitating the simulation of very long trajectories, which is computationally costly.

The first step in approximating the persistency coefficients in the Slepian-based IIA framework is to verify that the monotonicity conditions in Theorem 2 are satisfied. This is verified by inserting the covariance function and its derivatives of the two-dimensional diffusion processes into the equations of Theorem 3. Following this, the expected value functions can be turned into cumulative distribution functions (CDFs).

Figure 4 illustrates the normalized expected value functions of the Slepian process for various levels of u. They correspond to the CDFs of the geometric divisors in the stochastic representation outlined in Theorem 1 for multiple levels of u. It is evident from the graphs that the conditions of Theorem 2 are satisfied. However, it is important to note that E_- loses its monotonicity properties for $u > 5/4$. Hence, we focus on the crossing levels for which the monotonicity conditions in the previous theorem are met. To estimate persistence, we utilize samples from the approximated excursion distribution. These samples are obtained by numerical inversion sampling from the CDFs of the normalized expected value functions depicted in Figure 4. The samples are then generated from the approximate excursion time distribution using the stochastic representation in Theorem 2.

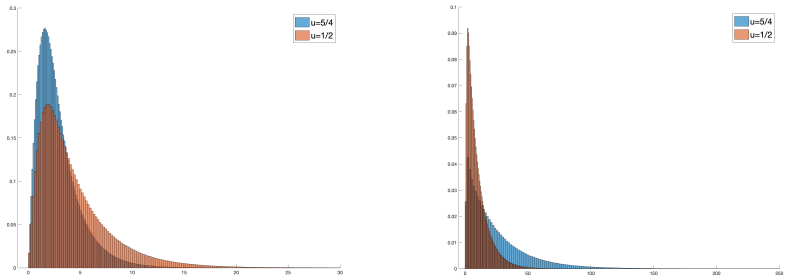

FIGURE 5. Normalized histograms of the approximated excursion distribution for the Slepian model based on the diffusion covariance, for an u up-crossing *(left)* and an u down-crossing *(right)*, for $u = 1/2, 5/4$.

However, the problem of estimating the persistency coefficients, θ_+ and θ_-, from these generated samples remains.

Suppose that the tails of $P(T_+ \geqslant t)$ and $P(T_- \geqslant t)$ are of an exponential form, i.e., $P(T_+ \geqslant t) \approx C \cdot e^{-\theta_+ t}$, for some large t, then the natural logarithm of $P(T_+ \geqslant t)$ will be on the linear form $ln(C) - \theta_+ t$. Hence, the persistency coefficients can be approximated via ordinary least squares (OLS) on the empirical survival function. While the notion of utilizing the empirical survival function for tail estimation is not new, having been introduced by Kratz and Resnick (1996) and Schultze and Steinebach (1996), it was employed in Bengtsson and Podgórski (2024b) for estimating persistence coefficients from generated samples. We obtain 10^6 samples from the approximated excursion time distribution to estimate the persistence coefficients. Only the values below one-half of the survival function are utilized for the OLS approximation of $\theta_\pm$, as only the tail is of interest. This approach is repeated to obtain 10 estimates, and approximate 95% confidence intervals are computed and presented in Table 1.

TABLE 1. Slepian-based IIA persistency coefficient estimate

Crossing level, u	Estimate, θ_+	Confidence interval	Estimate, θ_-	Confidence interval
0	0.1862	$\pm$0.000217	0.1861	$\pm$0.000195
1/2	0.2893	$\pm$0.000347	0.1105	$\pm$0.000131
1	0.4225	$\pm$0.000462	0.0591	$\pm$0.000061
5/4	0.5001	$\pm$0.000574	0.0411	$\pm$0.000044

For comparison, trajectories of the 2−dimensional Gaussian diffusion process were simulated using the WAFO-toolbox (WAFO-group, 2017). A total of 10^5 trajectories were simulated, each with a length of 10^6. The persistence coefficient was estimated in the same manner as before, using OLS on the extracted crossing intervals. The estimates are presented in Table 2 along with approximate 95% confidence intervals.

It should be noted that Slepian-based estimates of persistency are remarkably close to the estimates based on simulating trajectories. Except for the zero level crossings for

TABLE 2. Simulation estimates of persistency, based on trajectories

Crossing level, u	Estimate, θ_+	Confidence interval	Estimate, θ_-	Confidence interval
0	0.1885	$\pm$0.000164	0.1883	$\pm$0.000165
1/2	0.2932	$\pm$0.000271	0.1117	$\pm$0.000104
1	0.4295	$\pm$0.000482	0.0598	$\pm$0.000067
5/4	0.5101	$\pm$0.000661	0.0417	$\pm$0.000054

which the true persistency coefficient was shown to be 0.1875 by Poplavskyi and Schehr (2018). For the sake of this example, it can be stated that the Slepian-based IIA provides a good approximation of persistency for Gaussian diffusion processes.

The Slepian IIA not only estimates persistency coefficients but also provides information about the approximated excursion time distribution. In Figure 5, normalized histograms are presented such that they correspond to the density of the excursion distribution. From Figure 5, the exponential nature of the tail is evident, and the long tail of the down crossing for higher levels is also clear. It also shows how shifting the level from 1/2 to 5/4 alters the tail behavior of the distribution. Since an arbitrary number of samples can be generated, any characteristics of interest, such as the mean and moments, are approximately obtainable from these samples. This illustrates the usefulness of the Slepian-based IIA.

7. Conclusion

Using the Slepian-based IIA to approximate the excursion time, distributions have been extended to non-zero crossing levels. By utilizing the crossing behavior of the process of interest and matching the clipped version of this process to a non-stationary switch process, probabilistic valid approximations are obtained. An application to the Gaussian diffusion process is presented, and the results are in line with those obtained from simulations. There are several natural directions for future work on this topic. Perhaps the most natural step is to investigate the asymptotics of the IIA at high crossing levels.

8. Acknowledgement

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Appendix: Proofs and auxiliary results

A lemma was needed to derive the expectation of the clipped Slepian process, which is presented below. Recall that the Rayleigh distribution with the parameter σ_r^2 is given by the density $f_R(s) = se^{-s/\sigma_r^2}/\sigma_r^2$, and we shortly write $R \sim Rayleigh(\sigma_r)$ when a random variable R has this distribution.

Lemma 1. *Let $R \sim Rayleigh(\sigma_r)$ and $W \sim \mathcal{N}(\mu, \sigma_w^2)$ be independent of each other, then*

$$\Xi(\tilde{\mu}, \tilde{\sigma}) \stackrel{def}{=} \mathsf{P}(R < W) = \Phi(\tilde{\mu}) - \frac{\tilde{\sigma}}{\sqrt{1+\tilde{\sigma}^2}} \Phi\left(\tilde{\mu}\frac{\tilde{\sigma}}{\sqrt{1+\tilde{\sigma}^2}}\right) \exp\left(-\frac{\tilde{\mu}^2}{2(1+\tilde{\sigma}^2)}\right),$$

where $(\tilde{\mu}, \tilde{\sigma}) = (\mu, \sigma_r)/\sigma_w$.

Proof. Without loss of generality, we can assume that $\sigma_r = 1$. Under the stated conditions and F_R being the cdf of R:

$$\begin{aligned}\mathsf{P}(R < W) &= \mathsf{E}(F_R(W)) = \mathsf{E}\left(\mathbb{I}_{W>0}(1 - e^{-W^2/2})\right) \\ &= P(W > 0) - \frac{1}{\sqrt{2\pi\sigma_w^2}} \int_0^\infty e^{-\frac{w^2}{2} - \frac{(w-\mu)^2}{2\sigma_w^2}} dw.\end{aligned}$$

We have

$$\begin{aligned}\frac{1}{\sqrt{2\pi\sigma_w^2}} \int_0^\infty e^{-\frac{(w-\mu)^2 + \sigma_w^2 w^2}{2\sigma_w^2}} dw &= \frac{1}{\sqrt{2\pi\sigma_w^2}} \int_0^\infty e^{-\frac{w^2 - 2w\mu/(1+\sigma_w^2) + \mu^2/(1+\sigma_w^2)}{2\sigma_w^2/(1+\sigma_w^2)}} dw \\ &= \frac{1}{\sqrt{2\pi\sigma_w^2}} \int_0^\infty e^{-\frac{\left(w - \mu/(1+\sigma_w^2)\right)^2 + \mu^2\left(1/(1+\sigma_w^2) - 1/(1+\sigma_w^2)^2\right)}{2\sigma_w^2/(1+\sigma_w^2)}} dw \\ &= \frac{1}{\sqrt{1+\sigma_w^2}} e^{-\frac{\mu^2}{2(1+\sigma_w^2)}} \int_0^\infty \frac{1}{\sqrt{2\pi\frac{\sigma_w^2}{1+\sigma_w^2}}} e^{-\frac{\left(w - \mu/(1+\sigma_w^2)\right)^2}{2\sigma_w^2/(1+\sigma_w^2)}} dw \\ &= \frac{1}{\sqrt{1+\sigma_w^2}} e^{-\frac{\mu^2}{2(1+\sigma_w^2)}} \Phi\left(\frac{\mu}{\sigma_w}/\sqrt{1+\sigma_w^2}\right),\end{aligned}$$

which yields the result. □

Corollary 1. *If in the above result $\mu = 0$, then*

$$\mathsf{P}(R < W) = \frac{1}{2}\left(1 - \sqrt{\frac{\sigma_r^2}{\sigma_r^2 + \sigma_w^2}}\right). \tag{16}$$

Proof of Theorem 3. Let us define a normal variable

$$W = \frac{\sqrt{-r''(0)}}{r'(t)}\left(\Delta(t) + u\frac{r(t) - r(0)}{r(0)}\right). \tag{17}$$

The mean and variance of this variable are

$$\mu = u\sqrt{-r''(0)}\frac{r(t) - r(0)}{r(0)r'(t)},\ \sigma_w^2 = \frac{r''(0)}{r(0)}\frac{r^2(t) - r^2(0)}{r'^2(t)} - 1. \tag{18}$$

Using Lemma 1, we get

$$E_u^+(t) = 2\mathsf{P}(X_u > u) - 1 = 2\mathsf{P}(W > R) - 1 = 2\Xi(\mu/\sigma_w, 1/\sigma_w) - 1.$$

After some algebra, we get

$$\begin{aligned} E_u^+(t) = 2\Phi &\left(\frac{u}{\sqrt{r(0)}} \frac{r(t) - r(0)}{\sqrt{r^2(0) - r^2(t) + \frac{r(0)}{r''(0)} r'^2(t)}} \right) - 1 \\ &- 2\sqrt{-\frac{r(0)}{r''(0)}} \frac{r'(t)}{\sqrt{r^2(0) - r^2(t)}} \exp\left(-\frac{u^2}{2r(0)} \frac{r(0) - r(t)}{r(0) + r(t)} \right) \\ &\times \Phi\left(\frac{-u}{\sqrt{-r''(0)}} \sqrt{\frac{r(0) - r(t)}{r(0) + r(t)}} \frac{r'(t)}{\sqrt{r^2(0) - r^2(t) + \frac{r(0)}{r''(0)} r'^2(t)}} \right). \end{aligned}$$

□

Next, we evaluate the conditional expectation of a clipped Slepian process ((7)) for a given value of the Rayleigh variable R.

Lemma 2. *Let $u > 0$ and D_u^+ be defined as in* (7)*. Then for each $s > 0$:*

$$\mathsf{E}(D_u^+(t)|R = s) = 1 - 2\Phi\left(\frac{u\frac{r(0)-r(t)}{r(0)} - s\frac{r'(t)}{\sqrt{-r''(0)}}}{\sqrt{\frac{r^2(0)-r^2(t)}{r(0)} + \frac{r'^2(t)}{r''(0)}}} \right).$$

Proof. Since conditionally, on R, the Slepian process is Gaussian, the derivation is straightforward

$$\begin{aligned} E(Y_u(t)|R = s) =& E\big(I(X_0(t) > u) - I(X_0(t) < u)|R = s\big) \\ =& P(ur(t) - Rr'(t) + \Delta(t) > u|R = s) \\ & - P(ur(t) - Rr'(t) + \Delta(t) < u|R = s) \\ =& 1 - 2P(ur(t) - sr'(t) + \Delta(t) < u|R = s) \\ =& 1 - 2P\left(Z < -\frac{u(r(t) - 1) - sr'(t)}{\sqrt{r(0) - r(t)^2 - r'(t)^2}} \right) \\ =& 1 - 2\Phi\bigg(-\frac{u(r(t) - 1) - sr'(t)}{\sqrt{r(0) - r(t)^2 - r'(t)^2}} \bigg), \end{aligned}$$

where Z in the above is, as usual, a standard normal variable. □

In the discussion of the IIA, we used the following result to determine the distribution of the number of switches.

Lemma 3. *Let $\tilde{N}(t)$, $t \geq 0$, be the number of switches in $[0,t]$ of a stationary switch process $\tilde{D}$. Then*

$$\mathsf{P}\Big(\tilde{N}(t) = k \Big| \tilde{D}(0) = \delta\Big) =$$

$$= \begin{cases} 1 - F_{A|\delta}(t); k = 0 \\ (F_+ \star F_-)^{(k/2-1)\star} \star F_{-\delta} \star F_{A|\delta}(t) - (F_+ \star F_-)^{(k/2)\star} \star F_{A|\delta}(t); & k > 0 \text{ is even}, \\ (F_+ \star F_-)^{(k-1)/2\star} \star F_{A|\delta}(t) - (F_+ \star F_-)^{(k-1)/2\star} \star F_{-\delta} \star F_{A|\delta}(t); & k \text{ is odd}, \end{cases}$$

where $\star$ is used to denote the convolution of probability distributions (i.e., the distribution function of the sum of random variables), $F_{A|\delta}$ is the cdf of the initial delay conditionally on δ.

Proof. We consider only the case of conditioning on $\delta = 1$ as the opposite case can be obtained by the symmetry argument. We first note that for a positive t and a non-negative integer l:

$$\begin{aligned} \mathsf{P}\Big(\tilde{N}(t) = 0 \Big| \delta = 1\Big) &= \mathsf{P}\big(A > t | \delta = 1\big) = 1 - F_{A|\delta=1}(t), \\ \mathsf{P}\Big(\tilde{N}(t) = 1 \Big| \delta = 1\Big) &= \mathsf{P}(A + T_1^- > t \geq A | \delta = 1) \\ &= \mathsf{P}(A + T_1^- > t | \delta = 1) - \mathsf{P}(A + T_1^- > t, A > t | \delta = 1) \\ &= 1 - F_{A|\delta=1} \star F_-(t) - \big(1 - F_{A|\delta=1}(t)\big), \\ \mathsf{P}\Big(\tilde{N}(t) = 2l \Big| \delta = 1\Big) &= \mathsf{P}\left(A + \sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l} T_i^- > t \geq A + \sum_{i=1}^{l-1} T_i^+ + \sum_{i=1}^{l} T_i^- \Big| \delta = 1\right) \\ &= 1 - F_+^{l\star} \star F_-^{l\star} \star F_{A|\delta=1}(t) - \Big(1 - F_+^{(l-1)\star} \star F_-^{l\star} \star F_{A|\delta=1}(t)\Big) \\ &= F_+^{(l-1)\star} * F_-^{l\star} \star F_{A|\delta=1}(t) - F_+^{l\star} \star F_-^{l\star} \star F_{A|\delta=1}(t), \\ \mathsf{P}\Big(\tilde{N}(t) = 2l+1 \Big| \delta = 1\Big) &= \mathsf{P}\left(A + \sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l+1} T_i^- > t \geq A + \sum_{i=1}^{l} T_i^+ + \sum_{i=1}^{l} T_i^- \right) \\ &= F_+^{l\star} \star F_-^{l\star} \star F_{A|\delta=1}(t) - F_+^{l\star} \star F_-^{(l+1)\star} \star F_{A|\delta=1}(t). \end{aligned}$$

□

For the next proposition, we need the Laplace transform of $f_{A|\delta}$ and $F_{A|\delta}$.

$$\Psi_{A|\delta}(s) = \mathcal{L}(f_{A|\delta})(s) = \frac{1 - s\mathcal{L}(F_\delta)(s)}{s\mu_\delta} = \frac{1 - \Psi_\delta(s)}{s\mu_\delta}$$

$$\mathcal{L}(F_{A|\delta})(s) = \frac{\mathcal{L}(f_{A|\delta})(s)}{s} = \frac{1 - \Psi_\delta(s)}{s^2\mu_\delta}$$

Proposition 4. *For a stationary switch process, the average number of switches in* $[0,t]$, $N_<(t) = \mathsf{E}(\tilde{N}(t)|\delta=-1)$, $N_>(t) = \mathsf{E}(\tilde{N}(t)|\delta=1)$ *have Laplace transforms*

$$\mathcal{L}(N_>)(s) = \frac{1}{s^2\mu_+}\frac{\left(1+\Psi_-(s)\right)\left(1-\Psi_+(s)\right)}{1-\Psi_+(s)\Psi_-(s)},$$
$$\mathcal{L}(N_<)(s) = \frac{1}{s^2\mu_-}\frac{\left(1+\Psi_+(s)\right)\left(1-\Psi_-(s)\right)}{1-\Psi_+(s)\Psi_-(s)}.$$

Proof. The result follows from

$$\begin{aligned}
\mathcal{L}(N_>)(s) &= \sum_{k=1}^{\infty} k\cdot\mathcal{L}\left(\mathsf{P}\left(\tilde{N}(\cdot)=k|\delta=1\right)\right)(s)\\
&= \sum_{l=0}^{\infty}(2l+1)\left(\mathcal{L}\left(\left(F_+\star F_-\right)^{l\star}\star F_{A|\delta=1}\right)(s)-\mathcal{L}\left(\left(F_+\star F_-\right)^{l\star}\star F_-\star F_{A|\delta=1}\right)(s)\right)\\
&+\sum_{l=1}^{\infty}2l\left(\mathcal{L}\left(\left(F_+\star F_-\right)^{(l-1)\star}\star F_-\star F_{A|\delta=1}\right)(s)-\mathcal{L}\left(\left(F_+\star F_-\right)^{l\star}\star F_{A|\delta=1}\right)(s)\right)\\
&= \frac{\Psi_{A|\delta=1}}{s}\left(\left(1-\Psi_-\right)\sum_{l=0}^{\infty}(2l+1)\left(\Psi_+\Psi_-\right)^l+\left(1-\Psi_+\right)\sum_{l=1}^{\infty}2l\left(\Psi_+\Psi_-\right)^{(l-1)}\Psi_-\right)\\
&= \frac{1-\Psi_+}{s^2\mu_+}\left(\frac{1-\Psi_-}{1-\Psi_+\Psi_-}+2\Psi_-\sum_{l=1}^{\infty}l\left(\Psi_+\Psi_-\right)^{l-1}\left(\left(1-\Psi_-\right)\Psi_++1-\Psi_+\right)\right)\\
&= \frac{1-\Psi_+}{s^2\mu_+}\left(\frac{1-\Psi_-}{1-\Psi_+\Psi_-}+\frac{2\Psi_-\left(1-\Psi_-\Psi_+\right)}{\left(1-\Psi_+\Psi_-\right)^2}\right) = \frac{1-\Psi_+(s)}{s^2\mu_+}\frac{1+\Psi_-(s)}{1-\Psi_+(s)\Psi_-(s)}.
\end{aligned}$$

□

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