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Didenko, Victor; Helsing, Johan

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STABILITY OF THE NYSTRÖM METHOD FOR THE SHERMAN–LAURICELLA EQUATION*

VICTOR D. DIDENKO† AND JOHAN HELSING‡

Abstract. The stability of the Nyström method for the Sherman–Lauricella equation on piece-wise smooth closed simple contour Γ is studied. It is shown that in the space L_2 the method is stable if and only if certain operators associated with the corner points of Γ are invertible. If Γ does not have corner points, the method is always stable. Numerical experiments show the transformation of solutions when the unit circle is continuously transformed into the unit square, and then into various rhombuses. Examples also show an excellent convergence of the method.

1. Introduction. Let D be a domain in the complex plane $\mathbb{C}$ bounded by a simple closed piecewise smooth positively oriented contour Γ, and let $\mathfrak{M}_\Gamma := \{\tau_0, \tau_1, \ldots, \tau_{q-1}\}$ be the set of the corner points of Γ. Thus, at any point $\tau_j \in \mathfrak{M}_\Gamma$ the angle $\theta_j \in (0, 2\pi)$ between the right and left semitangents is not equal to π.

The boundary value problem

$$(1.1) \qquad \varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t), \quad t \in \Gamma,$$

where the bar denotes the complex conjugation and functions φ and ψ are analytic in the domain D, plays an important role in various applications. Thus the deflection of plates, the elastic equilibrium of solids, and the behavior of viscous flows with small Reynolds numbers can be modeled by (1.1) [16, 21, 23, 24, 25, 26]. More generally, a variety of boundary problems for the biharmonic equation can also be reduced to the boundary problem (1.1); see, for example, [7, Chapter 5]. On the other hand, the solution of problem (1.1) can be obtained from either of two famous integral equations, viz., from the Muskhelishvili or the Sherman–Lauricella equation. The first was suggested by Muskhelishvili [22] in the mid 1930s. The origin of the other is due to Lauricella [20]. He was the first to use the method of integral equations in elasticity. In 1940, Sherman [30] rewrote Lauricella's equation in a complex form and proposed a new simple way to derive it. More precisely, the unknown analytic functions φ and ψ in (1.1) are sought in the form

$$(1.2) \quad \varphi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\tau)}{\tau - z}\, d\tau, \quad z \in D,$$

$$(1.3) \quad \psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\omega(\tau)}}{\tau - z}\, d\tau + \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\tau)}{\tau - z}\, d\overline{\tau} - \frac{1}{2\pi i} \int_\Gamma \frac{\overline{\tau}\omega(\tau)}{(\tau - z)^2}\, d\tau, \quad z \in D,$$

†Department of Mathematics, Universiti Brunei Darussalam, Bandar Seri Begawan, BE1410 Brunei (diviol@gmail.com). This author was supported by the Universiti Brunei Darussalam under grant UBD/PNC2/2/RG/1(159).

‡Numerical Analysis, Centre for Mathematical Sciences, Lund University, Box 118, SE-221 00 Lund, Sweden (helsing@maths.lth.se). This author was supported by the Swedish Research Council under contract 621-2007-6234.

with an unknown function ω. If one substitutes the boundary values of functions (1.2) and (1.3) into (1.1), then ω satisfies the integral equation

$$(1.4) \qquad (A_\Gamma \omega)(t) \equiv \omega(t) + \frac{1}{2\pi i} \int_\Gamma \omega(\tau) \, d \ln\left(\frac{\tau - t}{\overline{\tau} - \overline{t}}\right)$$

$$- \frac{1}{2\pi i} \int_\Gamma \overline{\omega(\tau)} \, d\left(\frac{\tau - t}{\overline{\tau} - \overline{t}}\right) = f(t), \quad t \in \Gamma.$$

This equation is named after Sherman and Lauricella, and it is worth mentioning that (1.4) is not solvable in a closed analytic form. However, there is a vast literature describing various approximation methods employed to resolve (1.4). These methods mainly rely on Fourier series expansions, but under the circumstances the contour Γ must be a circle or a system of circles. For more involved geometries other approaches are needed. In particular, a very efficient tried-and-true method is the Nyström method. It was used to obtain approximate solutions of various problems in elasticity and fluid dynamics [5, 13, 14, 19]. Nevertheless, up to now there has been no rigorous proof of its stability even in the simplest case of smooth contour Γ. Such a situation is attributed to peculiarities of the operator A_Γ. To be more precise, let us first introduce some functional spaces. By $L_2(\Gamma)$ we denote the set of all Lebesgue measurable functions f on Γ such that

$$||f||_{L_2} = ||f||_2 := \left(\int_\Gamma |f(\tau)|^2 |d\tau|\right)^{1/2} < \infty$$

and let $W_2^1(\Gamma)$ refer to the closure of the set of all functions f with bounded derivatives in the norm

$$||f||_{W_2^1} := \left(\int_\Gamma |f(\tau)|^2 |d\tau| + \int_\Gamma |f'(\tau)|^2 |d\tau|\right)^{1/2}.$$

As usual, $C(\Gamma)$ is the space of the continuous functions on Γ with the supremum norm. It is well known that the operator A_Γ is not invertible in any of the spaces $C(\Gamma)$, $L_2(\Gamma)$, or $W_2^1(\Gamma)$. Therefore, in order to apply an approximation method, either the operator A_Γ or the space where it is considered should be corrected in a special way. In this paper we prefer to correct the operator A_Γ since such an operation allows us to not modify the approximation method used. Let us also point out another problem related to the numerical solution of the Sherman–Lauricella equation. If the contour Γ possesses corner points, the integral operators in the left-hand side of (1.4) are not compact in any of the functional spaces mentioned. This causes additional difficulties in the investigation of both the invertibility of the operator A_Γ and the stability of approximation methods for (1.4). In particular, the approaches used to study approximation methods for equations with compact operators (see, for example, [2, 3, 17]) are not working in the present situation. Instead, the theory of Mellin operators and a localization technique have to be employed.

Thus the present paper deals with the following issues. First, we study the Sherman–Lauricella equation on contours with corner points and show that on the space $L_2(\Gamma)$ the operator A_Γ is Fredholm and its index is always equal to zero. This result is needed in order to establish the invertibility of an operator $A_\Gamma + T_{SL}$, where T_{SL} is a specified compact operator. Moreover, for any function $f \in W_2^1(\Gamma)$ satisfying the condition

$$(1.5) \qquad\qquad \text{Re} \int_\Gamma f(\tau) \, \overline{d\tau} = 0,$$

the solution of the equation

$$(A_\Gamma + T_{SL})\omega = f \tag{1.6}$$

is simultaneously a solution of the original Sherman–Lauricella equation (1.4). Therefore, the modified Sherman–Lauricella equation (1.6) can be used to obtain approximate solutions of the original equation (1.4). In passing, note that (1.5) is a necessary condition for the solvability of the boundary value problem (1.1) [7, 21, 23, 26], so it cannot be avoided while studying the solvability of (1.4).

The remainder of the paper is devoted to approximating the solution of equation (1.6). For this we use a Nyström method based on composite Gauss–Legendre quadrature

$$\int_0^1 u(s)\,ds \approx \sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p u(s_{lp})/n, \tag{1.7}$$

where

$$s_{lp} = \frac{l + \varepsilon_p}{n}, \quad l = 0, 1, \ldots, n-1, \, p = 0, 1, \ldots, d-1, \tag{1.8}$$

and w_p and $0 < \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{d-1} < 1$ are the weights and the Gauss–Legendre points on the interval $[0, 1]$. The corresponding method is studied, and necessary and sufficient conditions of its stability are presented. It is shown that the Nyström method is stable if and only if certain operators $B_{\theta_j,\delta,\varepsilon}$ associated with the corner points $\tau_j \in \mathfrak{M}_\Gamma$ and with parameters of the quadrature formula are invertible. Note that the operators $B_{\theta_j,\delta,\varepsilon}$ belong to an algebra of Toeplitz operators. On the other hand, for the smooth contours the Nyström method is always stable. Finally, we consider a series of numerical examples that show an excellent convergence of the sequence of approximate solutions for smooth and piecewise smooth contours. Moreover, we monitor the transformation of the Sherman–Lauricella equation solutions when a smooth contour is continuously transformed into contours with angular points of various magnitude.

2. The invertibility of the auxiliary operators $A_\Gamma + T_{SL}$. Let $\gamma = \gamma(s)$ be a 1-periodic parametrization of Γ, and let $\tau_j \in \mathfrak{M}_\Gamma$ be the corner points of Γ. Without loss of generality, one can assume that $\tau_j = \gamma(j/q)$ for all $j = 0, 1, \ldots, q$. In addition, we assume that the function γ is two times continuously differentiable on each interval $(j/q, (j+1)/q)$ and

$$\left|\gamma'\left(\frac{j}{q}+0\right)\right| = \left|\gamma'\left(\frac{j}{q}-0\right)\right|, \quad j = 0, 1, \ldots, q-1.$$

Let $\mathfrak{L}_{add}(X)$ denote the set of all additive continuous operators on the Banach space X. Recall that an operator $A : X \mapsto X$ is called additive if $A(f+g) = Af + Ag$ for all $f, g \in X$. Consider now integral operators $L_\Gamma, K_\Gamma : L_2(\Gamma) \mapsto L_2(\Gamma)$ defined by

$$L_\Gamma\omega(t) := \frac{1}{2\pi i}\int_\Gamma \omega(\tau)\,d\ln\left(\frac{\tau-t}{\overline{\tau}-\overline{t}}\right), \quad K_\Gamma := \frac{1}{2\pi i}\int_\Gamma \omega(\tau)\,d\left(\frac{\overline{\tau}-\overline{t}}{\tau-t}\right), \tag{2.1}$$

and the operator of complex conjugation

$$M\omega(t) := \overline{\omega(t)}.$$

Then the operator A_Γ of (1.4) can be written as

$$A_\Gamma = I + L_\Gamma + MK_\Gamma.$$

With each angular point $\tau_j \in \mathfrak{M}_\Gamma$ we associate a curve $\Gamma_j = \Gamma_{\beta_j, \theta_j}$:

$$\Gamma_j = e^{i(\beta_j + \theta_j)}\mathbb{R} \cup e^{i\beta_j}\mathbb{R},$$

where $\beta_j \in [0, 2\pi)$ is the angle between the right semitangent and the real axis $\mathbb{R}$ and $\theta_j \in (0, 2\pi)$, $\theta \neq \pi$, is the angle between the right and left semitangents at the point τ_j.

Let us now consider the operators $A_{\Gamma_j} : L_2(\Gamma_j) \mapsto L_2(\Gamma_j)$ associated with the operator A_Γ and defined by

$$(2.2) \qquad\qquad A_{\Gamma_j} = I + L_{\Gamma_j} + MK_{\Gamma_j}.$$

Note that L_{Γ_j} and K_{Γ_j} are defined by (2.1), but the contour Γ is replaced by Γ_j.

PROPOSITION 2.1. *If Γ satisfies the above conditions, then the Sherman–Lauricella operator $A_\Gamma : L_2(\Gamma) \mapsto L_2(\Gamma)$ is Fredholm if and only if each operator $A_{\Gamma_j} : L_2(\Gamma_j) \mapsto L_2(\Gamma_j)$ is invertible,*

Proof. It is easily seen that the operator $A_\Gamma : L_2(\Gamma) \mapsto L_2(\Gamma)$ belongs to the real algebra of additive operators $B_2(\Gamma)$ that was studied earlier in connection with the Muskhelishvili equation, so the result follows from the proof of Theorem 5.2.1 of [7]. □

Thus the invertibility of the operators $A_{\Gamma_j} : L_2(\Gamma_j) \mapsto L_2(\Gamma_j)$, $j = 0, 1, \ldots, q-1$, has to be investigated. Usually this is a very difficult task. However, in our case the operators A_{Γ_j} are isometrically isomorphic to Mellin operators, so their invertibility can be studied effectively.

Let $\mathbb{M}$ and $\mathbb{M}^{-1}$ be the direct and inverse Mellin transforms,

$$(\mathbb{M}f)(z) = \int_0^{+\infty} x^{1/2 - zi - 1}f(x)dx, \quad (\mathbb{M}^{-1}f)(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x^{zi - 1/2}f(z)dz.$$

Recall that the Mellin convolution operator $\mathcal{M}_0(b)$ with the symbol b is defined by

$$(2.3) \qquad \mathcal{M}_0(b)x(\sigma) = ((\mathbb{M}^{-1}b\mathbb{M})f)(\sigma) = \int_0^\infty \mathbf{k}\left(\frac{\sigma}{s}\right)x(s)\,\frac{ds}{s}, \quad \mathbf{k} = \mathbb{M}^{-1}(b).$$

THEOREM 2.2. *For any angles $\beta_j \in [0, 2\pi)$ and $\theta_j \in (0, 2\pi)$, the operator $A_{\Gamma_j} : L_2(\Gamma_j) \mapsto L_2(\Gamma_j)$ is invertible.*

Proof. For a given $\theta \in (0, 2\pi)$ consider the Mellin convolution operators

$$\mathcal{N}_\theta(f)(\sigma) = \frac{1}{2}\left(\frac{1}{2\pi i}\int_0^{+\infty} \frac{f(s)\,ds}{s - \sigma e^{i\theta}} - \frac{1}{2\pi i}\int_0^{+\infty} \frac{f(s)\,ds}{s - \sigma e^{i(2\pi - \theta)}}\right),$$

$$\mathcal{M}_\theta f(\sigma) := \frac{1}{\pi}\int_0^{+\infty} \left(\frac{\sigma}{s}\right)\frac{\sin\theta}{(1 - (\sigma/s)e^{i\theta})^2}\frac{f(s)}{s}\,ds.$$

Using the obvious relation $M^2 = I$, where I is the identity operator, rewrite (2.2) in the form

$$A_{\Gamma_j} = I + L_{\Gamma_j} + (MK_{\Gamma_j}M)M.$$

Let $L_2^2(\mathbb{R}^+)$ denote the Cartesian product of two copies of the space $L_2(\mathbb{R}^+)$ provided with the norm

$$\|(u_1, u_2)\| := (\|u_1\|_{L_2(\mathbb{R}^+)} + \|u_2\|_{L_2(\mathbb{R}^+)})^{1/2}, \quad (u_1, u_2) \in L_2^2(\mathbb{R}^+).$$

Consider a mapping $h : L_2(\Gamma_j) \mapsto L_2^2(\mathbb{R}^+)$ defined by

$$(2.4) \qquad\qquad h(f) = (h_1(f), h_2(f))^T,$$

where

$$(2.5) \qquad\qquad h_1(f)(s) = f(se^{i(\beta_j + \theta_j)}), \quad h_2(f)(s) = f(se^{i\beta_j}).$$

Then the mapping $A \mapsto hAh^{-1}$ is an isometric algebra homomorphism of $\mathfrak{L}_{add}(L_2(\Gamma_j))$ onto $\mathfrak{L}_{add}(L_2^2(\mathbb{R}^+))$, and it can be shown that the operators L_{Γ_j} and $MK_{\Gamma_j}M$ are isometrically isomorphic to the operators

$$\widetilde{L_j} = \begin{pmatrix} 0 & \mathcal{N}_{\theta_j} \\ \mathcal{N}_{\theta_j} & 0 \end{pmatrix} \in \mathcal{L}(L_2^2(\mathbb{R}^+)),$$

$$\widetilde{K_j} = \begin{pmatrix} 0 & e^{i2\beta_j}\mathcal{M}_{2\pi-\theta_j} \\ -e^{i2(\beta_j+\theta_j)}\mathcal{M}_{\theta_j} & 0 \end{pmatrix} \in \mathcal{L}(L_2^2(\mathbb{R}^+)),$$

respectively. Therefore, the operator A_{Γ_j} is isometrically isomorphic to the block Mellin operator with conjugation $A_{\theta_j} : L_2^2(\mathbb{R}^+) \mapsto L_2^2(\mathbb{R}^+)$,

$$A_{\theta_j} = \mathcal{A}_j + \mathcal{B}_j M,$$

where

$$\mathcal{A}_j = \begin{pmatrix} I & \mathcal{N}_{\theta_j} \\ \mathcal{N}_{\theta_j} & I \end{pmatrix}, \quad \mathcal{B}_j = \begin{pmatrix} 0 & e^{i2\beta_j}\mathcal{M}_{2\pi-\theta_j} \\ -e^{i2(\beta_j+\theta_j)}\mathcal{M}_{\theta_j} & 0 \end{pmatrix}.$$

Note that for the operator of complex conjugation we use the same notation even if it is considered on different spaces. A little thought shows that for any operator $\mathcal{D}$ belonging to the algebra of Mellin operators, the operator $M\mathcal{D}M$ belongs to the same algebra again. Therefore, transformation (1.21) and Lemma 1.4.6 of [7] imply that the operator A_{θ_j} is invertible if and only if the block Mellin operator

$$\widetilde{A}_{\theta_j} = \begin{pmatrix} \mathcal{A}_j & \mathcal{B}_j \\ M\mathcal{B}_j M & M\mathcal{A}_j M \end{pmatrix}$$

is also invertible. Simple computations show that the operator $\widetilde{A}_{\theta_j}$ has the form

$$\widetilde{A}_{\theta_j} = \begin{pmatrix} I & \mathcal{N}_{\theta_j} & 0 & e^{i2\beta_j}\mathcal{M}_{2\pi-\theta_j} \\ \mathcal{N}_{\theta_j} & I & -e^{i2(\beta_j+\theta_j)}\mathcal{M}_{\theta_j} & 0 \\ 0 & -e^{-i2\beta_j}\mathcal{M}_{2\pi-\theta_j} & I & -\mathcal{N}_{2\pi-\theta_j} \\ e^{-i2(\beta_j+\theta_j)}\mathcal{M}_{\theta_j} & 0 & -\mathcal{N}_{2\pi-\theta_j} & I \end{pmatrix}.$$

However, it is well known [9] that the block Mellin operator is invertible if and only if its symbol does not vanish. The Mellin symbols $\mathbf{n}_\theta$ and $\mathbf{m}_\theta$ of the operators $\mathcal{N}_\theta$ and

$\mathcal{M}_\theta$ are also known [7, pp. 248–249], viz.,

$$\mathbf{n}_\theta(z) = \frac{\sinh(\pi - \theta)z - \sinh(\theta - \pi)z}{2\sinh \pi z}, \quad z = x + \frac{i}{2}, \quad x \in \mathbb{R},$$

$$\mathbf{m}_\theta(z) = -e^{-i\theta} \frac{z\sin\theta}{\sinh \pi z} e^{-(\theta-\pi)z}, \quad z = x + \frac{i}{2}, \quad x \in \mathbb{R},$$

so the symbol of the operator $\widetilde{A}_{\theta_j}$ is

$$\text{symb}(\widetilde{A}_{\theta_j})(z) =$$

(2.6)

$$\begin{pmatrix} 1 & \mathbf{n}_{\theta_j}(z) & 0 & e^{i2\beta_j}\mathbf{m}_{2\pi-\theta_j}(z) \\ \mathbf{n}_{\theta_j}(z) & 1 & -e^{i2(\beta_j+\theta_j)}\mathbf{m}_{\theta_j}(z) & 0 \\ 0 & -e^{-i2\beta_j}\mathbf{m}_{2\pi-\theta_j}(z) & 1 & \mathbf{n}_{\theta_j}(z) \\ e^{-i2(\beta_j+\theta_j)}\mathbf{m}_{\theta_j}(z) & 0 & \mathbf{n}_{\theta_j}(z) & 1 \end{pmatrix}.$$

Expanding the determinant of the matrix (2.6) by the first two rows and simplifying the expression obtained, we arrive at the formula

$$\det\text{symb}(\widetilde{A}_{\theta_j})(z)$$

(2.7)
$$= \frac{(z^2\sin^2(2\pi - \theta_j) - \sinh((2\pi - \theta_j)z))(z^2\sin^2\theta_j - \sinh(\theta_j z))}{\sinh^4(\pi z)}.$$

Let us now show that $\det\text{symb}(\widetilde{A}_{\theta_j})$ does not vanish on the line $L_{1/2} := \{z \in \mathbb{C} : z = \mathbb{R} + i/2\}$. Consider a factor ψ_{θ_j} in the numerator of (2.7),

$$\psi_{\theta_j}(z) = y^2\sin^2\theta_j - \sinh^2(\theta_j z).$$

For $z \in L_{1/2}$, the imaginary part of $z\sin\theta_j$ is

$$\text{Im}\,(z\sin\theta_j) = \frac{1}{2}\sin\theta_j.$$

On the other hand,

$$\text{Im}\,(\sinh(\theta_j z)) = \cosh(\theta_j x)\sin\left(\frac{\theta_j}{2}\right).$$

Thus on the line $L_{1/2}$ the minimum

$$\min_{z\in L_{1/2}}|\sinh(\theta_j z)| = \sin\left(\frac{\theta_j}{2}\right) > \frac{1}{2}|\sin\theta_j| = |\text{Im}\,(z\sin\theta_j)|$$

(recall that $\theta_j \neq \pi$). Therefore, $\det\text{symb}(\widetilde{A}_{\theta_j})$ does not vanish on the line $L_{1/2}$, so the operator A_{Γ_j} is invertible for any corner point of Γ. $\square$

Returning to the operator A_Γ, one can establish the following result.

COROLLARY 2.3. *If Γ is a simple closed piecewise smooth contour, then the operator $A_\Gamma : L_2(\Gamma) \mapsto L_2(\Gamma)$ is Fredholm and its index is always equal to zero.*

Proof. The first part of the above statement follows from Proposition 2.1 and Theorem 2.2. The proof of the equation

$$\kappa(A_\Gamma) = 0$$

can be done by standard homotopy arguments and is omitted here; cf. [10].   □

Now we can present the main result of this section. Let us consider an integral operator $T_{SL} : L_2(\Gamma) \mapsto L_2(\Gamma)$ defined by

$$(2.8) \qquad T_{SL}f(t) := \frac{1}{(\overline{t} - \overline{a})} \frac{1}{2\pi i} \int_\Gamma \left(\frac{\omega(\tau)}{(\tau - a)^2} d\tau + \frac{\overline{\omega(\tau)}}{(\overline{\tau} - \overline{a})^2} d\overline{\tau} \right),$$

where a is an arbitrary point in D.

THEOREM 2.4. *If Γ is a simple closed piecewise smooth contour, then operator*

$$(2.9) \qquad \widetilde{A}_\Gamma := A_\Gamma + T_{SL}$$

is invertible on the space $L_2(\Gamma)$. Moreover, if a function $f \in W_2^1(\Gamma)$ satisfies condition (1.5), then the solution of the equation

$$(2.10) \qquad \widetilde{A}_\Gamma \omega = f$$

belongs to the space $W_2^1(\Gamma)$ and is a solution of the original Sherman–Lauricella equation (1.4).

Proof. First we note that the operator $T_{SL} : L_2(\Gamma) \mapsto L_2(\Gamma)$ is compact. Therefore, on the space $L_2(\Gamma)$ the index of the operator $\kappa_{L_2}(\widetilde{A}_\Gamma)$ of the operator $\widetilde{A}_\Gamma$ is equal to the index of the operator A_Γ. Hence, by Corollary 2.3,

$$\kappa_{L_2}(\widetilde{A}_\Gamma) = 0.$$

On the other hand, on the Sobolev space $W_2^1(\Gamma)$, the index of the operator A_Γ is equal to zero as well [10]. Moreover, following the scheme proposed by Sherman [31] (see also [7, 21, 26]), one can prove two facts:

1. If the right-hand side f of (2.10) belongs to the Sobolev space $W_2^1(\Gamma)$ and satisfies (1.5), then for any solution ω of (2.10),

$$(2.11) \qquad T_{SL}\omega = 0.$$

2. The kernel $\ker_{W_2^1(\Gamma)}(\widetilde{A}_\Gamma)$ of the operator $\widetilde{A}_\Gamma$ on the Sobolev space $W_2^1(\Gamma)$ contains only the zero element; i.e.,

$$(2.12) \qquad \ker_{W_2^1(\Gamma)}(\widetilde{A}_\Gamma) = \{0\}.$$

Now (2.11) implies that in the Sobolev space $W_2^1(\Gamma)$ any solution ω of (2.10) is simultaneously a solution of the Sherman–Lauricella equation (1.4). On the other hand, since the indices of the operator $\widetilde{A}_\Gamma$ on both spaces $L_2(\Gamma)$ and $W_2^1(\Gamma)$ coincide and the Sobolev space $W_2^1(\Gamma)$ is dense in the space $L_2(\Gamma)$, the kernel dimensions of $\widetilde{A}_\Gamma$ on both spaces also coincide [11]:

$$\dim \ker_{L_2(\Gamma)}(\widetilde{A}(\Gamma)) = \dim \ker_{W_2^1(\Gamma)}(\widetilde{A}(\Gamma)).$$

Thus on the space $L_2(\Gamma)$ the kernel of the operator $\widetilde{A}_\Gamma$ is trivial. Since $\kappa_{L_2(\Gamma)} = 0$, the operator $\widetilde{A}_\Gamma : L_2(\Gamma) \mapsto L_2(\Gamma)$ is invertible.   □

Remark 2.5. There are other operators T that can be used to correct the Sherman–Lauricella operator in order to satisfy (2.10) and (2.12). Note that a proper choice of the operator T can improve conditioning and convergence of the algorithms used. Later on we present test results concerning the behavior of the condition numbers for different choices of the operator T.

3. The stability of the Nyström method. Let $\varphi^d = \varphi^d(s)$, $s \in \mathbb{R}$, be the uniform cardinal B-spline of degree d. Recall that

$$\varphi^0 = \chi_{[0,1)},$$

where $\chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$ and

$$\varphi^1(s) = \begin{cases} s & \text{if} \quad 0 \le s < 1, \\ 2 - s & \text{if} \quad 1 \le s \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

For other values $d \in \mathbb{N}$, the analytic expressions of the corresponding splines are also well known [29]. Note that the functions φ^d can also be obtained by the recursive formula

$$\varphi^d = (\varphi^0 * \varphi^{d-1})(s), \quad s \in \mathbb{R},$$

where $f * g$ is the convolution of f and g. Thus

$$\varphi^d(s) = \int_{\mathbb{R}} \chi_{[0,1)}(s - x)\, \varphi^{d-1}(x)\, dx.$$

For a fixed $d \in \mathbb{N}$, we set

$$\psi(s) := \varphi^d(s), \quad s \in \mathbb{R},$$

and note [6, Lemma 5.4.1] that

(3.1) $$\operatorname{supp}\{\psi\} \subset [0, d + 1],$$

where $\operatorname{supp}\{\psi\}$ refers to the support of the function ψ.

For any fixed positive integer n, one can define the functions

$$\psi_{kn}(s) := \psi(ns - k), \quad k \in \mathbb{Z}, \quad s \in \mathbb{R},$$

and this construction can be used to introduce spline spaces on the contour Γ. Consider the set Λ^d of all functions $\psi_{kn}, k \ge 0$, such that

$$\operatorname{supp}\{\psi_{kn}\} \cap [0, 1] \ne \emptyset.$$

It follows from (3.1) that $\psi_{kn} \in \Lambda^d$ if and only if $0 \le k \le n - 1$ and

$$\operatorname{supp}\{\psi_{kn}\} \subset \left[\frac{k}{n}, \frac{k + d + 1}{n} \right].$$

Thus if n is sufficiently large, one can consider the restriction $\psi_{kn}|_{[0,1+d/n)}$ of the function ψ_{kn} on the interval $[0, 1 + d/n)$ and extend it on the whole real line $\mathbb{R}$ in

a 1-periodic way. For simplicity, such extensions will still be denoted by the same symbol ψ_{kn}.

If $\gamma = \gamma(s)$, $s \in \mathbb{R}$, is a 1-periodic parametrization of the contour Γ, then for any $t \in \Gamma, t = \gamma(s)$, we set

$$\widetilde{\psi}_{kn}(t) := \psi_{kn}(s).$$

It is clear that if $(d+1)/n < 1$, then this definition is correct.

Let $S_n^d = S_n^d(\Gamma)$ denote the corresponding spline space on Γ; i.e., S_n^d is the set of all linear combinations of the functions $\widetilde{\psi}_{kn}$, $k = 0, 1, \ldots, n-1$.

Our goal now is to study operators associated with the Nyström method and the corner points of Γ. Recall that we assume that contour Γ has q corner points $\tau_0, \tau_1, \ldots, \tau_{q-1}$ and $\tau_j = \gamma(j/q)$, $j = 0, 1, \ldots, q-1$. In accordance with such parametrization, we choose $n = mq$ with an $m \in \mathbb{N}$. Thus any corner point of Γ is always a boundary point of an interval $(\gamma(k/n), \gamma(k+1)/n)$.

Let $0 < \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{d-1} < 1$ and $0 < \delta_0 < \delta_1 < \cdots < \delta_{d-1} < 1$ be real numbers. Consider two sets of points on the contour Γ,

$$\tau_{lp} = \gamma\left(\frac{l+\varepsilon_p}{n}\right), \quad t_{lp} = \gamma\left(\frac{l+\delta_p}{n}\right), \quad l = 0, 1, \ldots, n-1, \ p = 0, 1, \ldots, d-1,$$

and approximate the integral operator $\int_\Gamma k(t,\tau)x(\tau)\,d\tau$ according to the quadrature rule (1.7). Thus we set

$$\int_\Gamma k(t,\tau)x(\tau)\,d\tau = \int_0^1 k(\gamma(\sigma), \gamma(s))x(\gamma(s))\gamma'(s)\,ds$$

$$\approx K^{(\varepsilon,n)}x(t) = \sum_{l=0}^{n-1}\sum_{p=0}^{d-1} w_p k(t,\tau_{lp})x(\tau_{lp})\tau'_{lp}/n,$$

where $\tau'_{lp} = \gamma'(s_{lp})$ and s_{lp} are defined by (1.8). Note that if $k(t,\tau)$ is the kernel of the integral operator L_Γ or K_Γ, then for any point $\tau \in \Gamma \setminus \mathfrak{M}_\Gamma$, there is a finite limit $\lim_{t \to \tau} k(t,\tau)$ [8]. Therefore, for any $\tau \in \Gamma \setminus \mathfrak{M}_\Gamma$, one can define $k(\tau,\tau)$ by

$$k(\tau,\tau) := \lim_{t \to \tau} k(t,\tau).$$

Thus the expressions $k(t_{lp}, \tau_{lp})$ below are correctly defined even if $\varepsilon_p = \delta_p$.

Let $Q_n^\delta : L_\infty(\Gamma) \mapsto S_n^d(\Gamma)$ denote the interpolation projection on the subspace $S_n^d(\Gamma)$ such that

$$Q_n^\delta x(t_{lp}) = x(t_{lp}), \quad l = 0, 1, \ldots, n-1, \ p = 0, 1, \ldots, d-1$$

for all x from the set $\mathbf{R}(\Gamma)$ of the Riemann integrable functions. In passing, note that the sequence of interpolation projections $(Q_n^\delta)_{n\in\mathbb{N}} : \mathbf{R}(\Gamma) \mapsto L_2(\Gamma)$ strongly converges to the corresponding embedding operator [27],

$$(3.2) \qquad\qquad \lim_{n\to\infty} \|Q_n^\delta f - f\|_{L_2(\Gamma)} = 0, \quad f \in \mathbf{R}(\Gamma).$$

Let $P_n : L_2(\Gamma) \mapsto S_n^d$ be the orthogonal projection onto the spline space S_n^d. The Nyström method determines an approximate solution ω_n of the Sherman–Lauricella equation

$$A_\Gamma\omega \equiv \omega + L_\Gamma\omega + MK_\Gamma\omega + T_{SL}\omega = f$$

by the equation

$$Q_n^\delta A_\Gamma^{\varepsilon,n} P_n \omega_n \equiv Q_n^\delta P_n \omega_n + Q_n^\delta L_\Gamma^{(\varepsilon,n)} P_n \omega_n + Q_n^\delta M K_\Gamma^{(\varepsilon,n)} P_n \omega_n + Q_n^\delta T_{SL}^{(\varepsilon,n)} P_n \omega_n$$

(3.3)
$$= Q_n^\delta f, \quad \omega_n \in S_n^d(\Gamma), \quad n \in \mathbb{N}.$$

Operator equation (3.3) is equivalent to the following system of algebraic equations:

$$\omega(t_{kr}) + \frac{1}{2\pi i} \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p \omega(\tau_{lp}) \left(\frac{\tau'_{lp}}{\tau_{lp} - t_{kr}} - \frac{\overline{\tau'}_{lp}}{\overline{\tau}_{lp} - \overline{t}_{kr}} \right) \frac{1}{n}$$

$$- \frac{1}{2\pi i} \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p \overline{\omega(\tau_{lp})} \left(\frac{1}{\overline{\tau}_{lp} - \overline{t}_{kr}} \frac{\tau'_{lp}}{n} - \frac{\tau_{lp} - t_{kr}}{(\overline{\tau}_{lp} - \overline{t}_{kr})^2} \frac{\overline{\tau'}_{lp}}{n} \right)$$

(3.4)
$$+ \frac{1}{(\overline{t}_{kr} - \overline{a})} \frac{1}{2\pi i} \sum_{l=0}^{n-1} \sum_{p=0}^{d-1} w_p \left(\frac{\omega_{lp}}{(\tau_{lp} - a)^2} \frac{\tau'_{lp}}{n} + \frac{\overline{\omega(\tau_{lp})}}{(\overline{\tau}_{lp} - \overline{a})^2} \frac{\overline{\tau'}_{lp}}{n} \right) = f(t_{kr}),$$

$$k = 0, 1, \ldots, n-1, \, r = 0, 1, \ldots, d-1.$$

Let $(A_n)_{n \in \mathbb{N}}$ be a bounded sequence of bounded additive operators $A_n : S_n^d \mapsto S_n^d$. The set $\mathcal{T}$ of all such sequences (A_n), equipped with componentwise operations of addition, multiplication, involution, and multiplication by real scalars, and the norm

$$||(A_n)|| = \sup_{n \in \mathbb{N}} ||A_n||,$$

becomes a real C^*-algebra; cf. [7] for more details.

DEFINITION 3.1. *The sequence* $(A_n) \in \mathcal{T}$ *is called stable if there is an* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$ *the operators* $A_n P_n : S_n^d \mapsto S_n^d$ *are invertible and the norms* $||(A_n P_n)^{-1} P_n||_{n \geq n_0}$ *are uniformly bounded.*

As was observed in [18], the stability of an approximation method (A_n) is equivalent to the invertibility of the coset $(A_n) + \mathcal{G}$ in the quotient algebra $\mathcal{T}/\mathcal{G}$, where $\mathcal{G}$ is an ideal of the algebra $\mathcal{T}$ consisting of all sequences of bounded additive operators uniformly convergent to zero, i.e.,

$$\mathcal{G} := \{(C_n) \in \mathcal{A} : \lim_{n \to \infty} ||G_n|| = 0\}.$$

Thus the study of the stability of approximation methods can be reduced to the investigation of the invertibility in the algebra $\mathcal{T}/\mathcal{G}$. However, in many cases the algebra $\mathcal{T}/\mathcal{G}$ is too large for the invertibility problem to be effectively treated. To tackle this problem, Silbermann [32] proposed considering a subalgebra $\mathcal{A}$ of the algebra $\mathcal{T}$ and an extension $\mathcal{J} \supseteq \mathcal{G}$ of the ideal $\mathcal{G}$ such that the invertibility of the corresponding coset $(A_n) + \mathcal{G}$ in $\mathcal{T}/\mathcal{G}$ can be studied via the invertibility of the coset $(A_n) + \mathcal{J}$ in $\mathcal{A}/\mathcal{J}$. This approach facilitated the use of powerful Banach algebra techniques in approximation methods. A version of the corresponding results of [32] is used below.

Let $\mathcal{A} \subset \mathcal{T}$ denote the set of all sequences (A_n) for which there is an operator $A \in \mathfrak{L}_{add}(L_2(\Gamma))$ such that the strong limits

$$s - \lim_{n \to \infty} A_n P_n = A, \quad s - \lim_{n \to \infty} (A_n)^* P_n = A^*$$

exist.

Let us recall the following result (see, for example, [7, Proposition 1.6.4]).

THEOREM 3.2.   *Let $\mathcal{K}_{add}(L_2(\Gamma))$ denote the ideal of all compact operators in $\mathfrak{L}_{add}(L_2(\Gamma))$. Then the collection of the sequences*

$$\mathcal{J} := \{(J_n) : J_n = P_n K P_n + C_n, \quad K \in \mathcal{K}_{add}(L_2(\Gamma)), \quad (C_n) \in \mathcal{G}\}$$

forms a closed two-sided ideal in $\mathcal{A}$. Moreover, the mapping $W : \mathcal{A} \mapsto \mathfrak{L}_{add}(L_2(\Gamma))$, $(A_n) \mapsto s-\lim A_n P_n$ is a Banach algebra homomorphism, and the sequence $(A_n) \in \mathcal{A}$ is stable if and only if the operator $W(A_n)$ is invertible in $\mathfrak{L}_{add}(L_2(\Gamma))$ and the coset $(A_n) + \mathcal{J}$ is invertible in $\mathcal{A}/\mathcal{J}$.

This result can be used to study the Nyström method. It follows from (3.2) that the sequence of approximation operators $(A_n)_{n \in \mathbb{N}}$ corresponding to this method strongly converges to the operator $\widetilde{A}_\Gamma$. A similar limit relation takes place for the sequence of adjoint operators. However, our operators are not linear over the field of complex numbers, so for the adjoint operators one has to use the definition from [7, section 1.2]. By Theorem 2.4, the operator $\widetilde{A}_\Gamma$ is invertible in $\mathfrak{L}_{add}(L_2(\Gamma))$. Therefore, we need only establish the invertibility of the corresponding coset $(A_n)^0 := (A_n) + \mathcal{J}$ in the quotient algebra $\mathcal{A}/\mathcal{J}$. This can be done by using a localizing principle. Recall that localizing principles connect the invertibility of an element from a given Banach or C^*-algebra with the invertibility of its local representatives [1, 12]. Notice that in the neighborhoods of nonangular points $\tau \in \Gamma, \tau \notin \mathfrak{M}_\Gamma$, the integral operators of (1.4) behave like compact operators [8], so for such points $\tau \in \Gamma$ our approximation sequence $(A_n)_{n \in \mathbb{N}}$ is locally equivalent to the sequence of projections $(P_n)_{n \in \mathbb{N}}$ which is a stable sequence. Thus we need only identify and study local representatives of the Nyström method for the angular points $\tau_j \in \Gamma$.

With each corner point τ_j we associate a model approximation method for the operator A_{Γ_j} of (2.2) considered on the space $L_2(\Gamma_j)$. More precisely, let us approximate the integral $\int_{\Gamma_j} u(\tau)\, d\tau$ by the quadrature formula

$$\int_{\Gamma_j} u(\tau)\, d\tau$$

(3.5)

$$\approx \sum_{l=-\infty}^{-1} \sum_{p=0}^{d-1} w_p u\left(\frac{l+\varepsilon_p}{n} e^{i(\beta_j+\theta_j)}\right) \frac{e^{i(\beta_j+\theta_j)}}{n} + \sum_{l=0}^{\infty} \sum_{p=0}^{d-1} w_p u\left(\frac{l+\varepsilon_p}{n} e^{i\beta_j}\right) \frac{e^{i\beta_j}}{n},$$

where $\varepsilon_p,\ p = 0, 1, \ldots, d-1$, are as above. We also need spline spaces on the contours Γ_j and $\mathbb{R}^+$. Thus let $S_n^{\beta_j, \theta_j}$ be the smallest subspace of $L_2(\Gamma_j)$ that contains all functions

(3.6)
$$\widetilde{\varphi}_{kn}(t) := \begin{cases} \begin{cases} \psi_{kn}(s) & \text{if} \quad t = e^{i\beta_j} s, \\ 0 & \text{otherwise,} \end{cases} & k \geq 0, \\[2mm] \begin{cases} \psi_{k-d,n}(s) & \text{if} \quad t = e^{i(\beta_j+\theta_j)} s, \\ 0 & \text{otherwise,} \end{cases} & k < 0. \end{cases}$$

The spline space $S_n(\mathbb{R}^+)$ is constructed analogously, but in this case we set $\beta_j = 0$ in (3.6) and take the functions $\widetilde{\varphi}_{kn}$ with $k \geq 0$ only. Let $\widetilde{P}_n$ and $\widehat{P}_n$ denote, respectively, the orthogonal projections from $L_2(\Gamma_j)$ onto $S_n^{\beta_j, \theta_j}$ and from $L_2(\mathbb{R}^+)$ onto $S_n(\mathbb{R}^+)$.

Let $\mathbf{R}_2(\Gamma_j)$ refer to the set of functions on Γ_j which are Riemann integrable on each finite part of Γ_j and satisfy the condition

$$\|f\|_{\mathbf{R}_2(\Gamma_j)} = \|f\|_{L_2(\Gamma_j)} + \left(\sum_{k=0}^{\infty} \sup_{t \in e^{i(\beta_j + \theta_j)}[k,k+1]} |f(t)|^2\right)^{1/2}$$
$$+ \left(\sum_{k=0}^{\infty} \sup_{t \in e^{i\beta_j}[k,k+1]} |f(t)|^2\right)^{1/2} < \infty.$$

Consider now the integral equation

$$(3.7) \qquad\qquad A_{\Gamma_j} x = f, \quad f \in \mathbf{R}_2(\Gamma_j).$$

Replacing x by an element $x_n \in S_n^{\beta_j,\theta_j}$ and applying quadrature formula (3.5) to the integrals in the left-hand side of (3.7), and using the interpolation projections $\widetilde{Q}_n^{\delta} : \mathbf{R}_2(\Gamma_j) \mapsto S_n^{\beta_j,\theta_j}$ defined by

$$\widetilde{Q}_n^{\delta} x(t_{lp}) = x(t_{lp}), \quad l \in \mathbb{Z}, \quad p = 0, 1, \ldots, d-1;$$

$$t_{lp} = \begin{cases} \dfrac{l + \delta_p}{n} e^{i(\beta_j + \theta_j)} & \text{if} \quad l \in \mathbb{Z}^-, \\ \dfrac{l + \delta_p}{n} e^{i\beta_j} & \text{if} \quad l \in 0 \cup \mathbb{Z}^+, \end{cases}$$

we approximate the operator A_{Γ_j} by the sequence of operators $(\widetilde{Q}_n^{\delta} A_{\Gamma_j}^{(\varepsilon,n)} \widetilde{P}_n)_{n\in\mathbb{N}}$, where $A_{\Gamma_j}^{(\varepsilon,n)}$ are the approximations of the operator A_{Γ_j} such that the identity operator is replaced by the projection operator $\widetilde{P}_n$, whereas the integrals are replaced by quadratures (3.5) with respect to variable τ. Let us point out that the equation

$$\widetilde{Q}_n^{\delta} A_{\Gamma_j}^{(\varepsilon,n)} \widetilde{P}_n x_n = \widetilde{Q}_n^{\delta} f, \quad x_n \in S_n^{\beta_j,\theta_j}, \quad n \in \mathbb{N},$$

is equivalent to an infinite system of algebraic equations. The structure of this system is similar to the structure of the system (3.4), but the expressions τ'_{lj} should be replaced by $e^{i\beta_j}$ or $e^{i(\beta_j+\theta_j)}$ depending on which part of the contour Γ_j is considered. Let us study the operator sequence $(\widetilde{Q}_n^{\delta} A_{\Gamma_j}^{(\varepsilon,n)} \widetilde{P}_n)_{n\in\mathbb{N}}$. We consider approximations of the equations

$$(3.8) \qquad\qquad \mathcal{M}_0(b) x = g, \quad g \in \mathbf{R}_2(\mathbb{R}^+),$$

where $\mathcal{M}_0(b)$ is the Mellin convolution operator (2.3) with the symbol b. Using a quadrature formula similar to (3.5) but only for the integrals on the positive semi-axis $\mathbb{R}^+$, we approximate the operator $\mathcal{M}_0(b)$ by the sequence of operators $(\widehat{Q}_n^{\delta} \mathcal{M}_0^{(\varepsilon,n)}(b) \widehat{P}_n)_{n\in\mathbb{N}}$. The interpolation operators $\widehat{Q}_n^{\delta}$ are similar to the operators $\widetilde{Q}_n^{\delta}$, but are defined on the positive semi-axis $\mathbb{R}^+$. The system of arising algebraic equations for (3.8) is given below (see (3.9)) and will be studied later.

LEMMA 3.3. *The sequence* $(\widetilde{Q}_n^{\delta} A_{\Gamma_j}^{(\varepsilon,n)} \widetilde{P}_n)_{n\in\mathbb{N}}$ *is stable if and only if the sequence*

$(\widehat{A}_{\theta_j}^{(\delta,\varepsilon,n)}\widehat{P}_n),$

$$\widehat{A}_{\theta_j}^{(\delta,\varepsilon,n)}\widehat{P}_n = \mathcal{A}_j^{(\delta,\varepsilon,n)} + \mathcal{B}_j^{(\delta,\varepsilon,n)}M,$$

$$\mathcal{A}_j^{(\delta,\varepsilon,n)} = \begin{pmatrix} \widehat{P}_n & \widehat{Q}_n^\delta \mathcal{M}_0^{(\varepsilon,n)}(b_j^1)\widehat{P}_n \\ \widehat{Q}_n^{1-\delta}\mathcal{M}_0^{(1-\varepsilon,n)}(b_j^{(1)})\widehat{P}_n & \widehat{P}_n \end{pmatrix},$$

$$\mathcal{B}_j^{(\delta,\varepsilon,n)} = \begin{pmatrix} 0 & \widehat{Q}_n^\delta \mathcal{M}_0^{(\varepsilon,n)}(b_j^{(2)})\widehat{P}_n \\ \widehat{Q}_n^{1-\delta}\mathcal{M}_0^{(1-\varepsilon,n)}(b_j^{(3)})\widehat{P}_n & 0 \end{pmatrix},$$

where

$$b_j^{(1)}(z) = \mathbf{n}_{\theta_j}(z), \quad b_j^{(2)}(z) = e^{i2\beta_j}\mathbf{m}_{2\pi-\theta_j}(z), \quad b_j^{(3)}(z) = -e^{i2(\beta_j+\theta_j)}\mathbf{m}_{\theta_j}(z),$$

is stable.

Proof. Let $h : L_2(\Gamma_j) \mapsto L_2^2(\mathbb{R}^+)$ be the mapping defined by (2.4)–(2.5). It is easy to check that

$$h\widetilde{Q}_n^\delta h^{-1} = \mathrm{diag}(\widehat{Q}_n^\delta, \widehat{Q}_n^{1-\delta}), \quad h\widetilde{P}_n h^{-1} = \mathrm{diag}(\widehat{P}_n, \widehat{P}_n), \quad n \in \mathbb{N},$$

where $\mathrm{diag}(A,B)$ denotes the diagonal matrix operator with entries $A, B \in \mathfrak{L}_{add}(L_2(\mathbb{R}^+))$. In the case where $A = B$, such diagonal operators are denoted simply by A. Thus we have

$$h(\widetilde{Q}_n^\delta A_{\Gamma_j}^{(\varepsilon,n)}\widetilde{P}_n)h^{-1} = (h\widetilde{Q}_n^\delta h^{-1})(hA_{\Gamma_j}^{(\varepsilon,n)}h^{-1})(h\widetilde{P}_n h^{-1}),$$

and the calculation of the expression $hA_{\Gamma_j}^{(\varepsilon,n)}h^{-1}$ completes the proof.   □

Let l_2 be the space of sequences $(\xi_l)_{l=0}^\infty$ of complex numbers $\xi_l, l = 0, 1, \dots$ such that $||(\xi_l)||_{l_2} = \left(\sum_{l=0}^\infty |\xi_l|^2\right)^{1/2} < \infty$, and let l_2^d refer to d copies of l_2. Further, by $E_n : l_2 \mapsto S_n(\mathbb{R}^+)$ and $E_{-n} : S_n(\mathbb{R}^+) \mapsto l_2$ we denote the operators defined by

$$E_n\left((\xi_l)_{l=0}^\infty\right) = \sum_{l=0}^\infty \xi_l\widetilde{\varphi}_{ln}(t), \quad E_{-n}\left(\sum_{l=0}^\infty \xi_l\widetilde{\varphi}_{ln}(t)\right) = ((\xi_l)_{l=0}^\infty).$$

It is known [4] that the operators E_n and E_{-n} are bounded and there is a constant C such that $||E_n||||E_{-n}|| \le C$ for all $n \in \mathbb{N}$.

COROLLARY 3.4.   *The sequence* $(\widetilde{Q}_n^\delta A_{\Gamma_j}^{(\varepsilon,n)}\widetilde{P}_n)_{n\in\mathbb{N}}$ *is stable if and only if the operator* $B_{\theta_j,\delta,\varepsilon} := E_{-1}\widehat{A}_{\theta_j}^{(\delta,\varepsilon,1)}\widehat{P}_1 E_1$ *is invertible.*

Proof. Consider the system of algebraic equations

(3.9)

$$\sum_{p=0}^{d-1} w_p \sum_{l=0}^\infty \mathbf{k}\left(\frac{k+\delta_r}{l+\varepsilon_p}\right)\frac{x_{pln}}{l+\varepsilon_p} = g\left(\frac{k+\delta_r}{n}\right), \quad r = 0, 1, \dots, d-1, \quad k = 0, 1, \dots,$$

where $x_{pln} := x((l+\varepsilon_p)/n)$, $p = 0, 1, \dots, d-1$, $l = 0, 1, \dots$, arising in the above approximation of (3.8).

If $(A_{rp}(b))_{r,p=0}^{d-1}$ refers to the matrix of the system (3.9), with $A_{rp} : l_2 \mapsto l_2$ for all $r, p = 0, 1, \dots, d-1$, then $E_{-n}\widehat{Q}_n^\delta \mathcal{M}_0^{(\varepsilon,n)}(b)\widehat{P}_n E_n = (A_{rp}(b))_{r,p=0}^{d-1}$. However, the

operators $E_{-n}\widehat{Q}_n^\delta \mathcal{M}_0^{(\varepsilon,n)}(b)\widehat{P}_n E_n$ do not depend on n, and neither do the entries of the operators $E_{-n}\widehat{A}_{\theta_j}^{(\delta,\varepsilon,n)}\widehat{P}_n E_n$. Hence the sequence $E_{-n}\widetilde{Q}_n^\delta A_{\Gamma_j}^{(\varepsilon,n)}\widetilde{P}_n E_n$ is stable if and only if any member of the constant sequence $E_{-n}\widehat{A}_{\theta_j}^{(\delta,\varepsilon,n)}\widehat{P}_n E_n$, say $E_{-1}\widehat{A}_{\theta_j}^{(\delta,\varepsilon,1)}\widehat{P}_1 E_1$, is invertible.   ∎

If the contour Γ and its parametrization γ satisfy the above condition, then the stability of the Nyström method (3.3) for the operator $R_\Gamma + T_{SL}$ depends on the behavior of the operators A_{τ_j} associated with the corner points $\tau_j, j = 0, 1, \ldots, q-1$. More precisely, the following result holds.

THEOREM 3.5. *Let $n = qm$, $m \in \mathbb{N}$. The Nyström method for the operator $R_\Gamma + T_{SL} : L_2(\Gamma) \mapsto L_2(\Gamma)$ is stable if and only if all the operators $B_{\theta_j,\delta,\varepsilon}$, $j = 0, 1, \ldots, q-1$ are invertible. Moreover, if the operators $B_{\theta_j,\delta,\varepsilon}$, $j = 0, 1, \ldots, m-1$, are invertible, and if a function f belongs to the Sobolev space $W_2^1(\Gamma)$ and satisfies the condition (1.5), then the approximate solutions $\omega_n = \omega_n(t)$ obtained by the Nyström method (3.3) converge to the solution of the Sherman–Lauricella equation (1.4).*

Proof. Let $\mathcal{A}$ and $\mathcal{J}$ be the real C^*-algebra and its ideal introduced in Theorem 3.2. By $\mathcal{C}$ we denote the smallest closed C^*-algebra that contains the sequences $(P_n L_\Gamma P_n)$, $(P_n M K_\Gamma M P_n)$, $(P_n f P_n)$, $f \in C(\Gamma)$, and the ideal $\mathcal{J}$. Then $\mathcal{C}/\mathcal{J}$ is a C^*-subalgebra of $\mathcal{A}/\mathcal{J}$; hence $\mathcal{C}/\mathcal{J}$ is inverse closed in $\mathcal{A}/\mathcal{J}$ [7, Corollary 1.4.11]. Therefore, the coset $(A_\Gamma^{(\delta,n)} P_n) + \mathcal{J} \in \mathcal{C}/\mathcal{J}$ is invertible in $\mathcal{A}/\mathcal{J}$ if and only if it is invertible in $\mathcal{C}/\mathcal{J}$. However, the invertibility of this coset in $\mathcal{C}/\mathcal{J}$ can be proved by the localizing principle [7, Theorem 1.9.5(a)].

Thus the set

$$\mathcal{B} := \{(P_n f P_n) + \mathcal{J} : f \in C_\mathbb{R}(\Gamma)\},$$

where $C_\mathbb{R}(\Gamma)$ refer to the set of all real-valued continuous functions on Γ. It is known [27] that for any $f \in C(\Gamma)$ the operators $F_n : L_2(\Gamma) \mapsto L_2(\Gamma)$, $F_n := f P_n - P_n f$ are bounded and

$$\lim_{n \to \infty} ||F_n|| = 0.$$

This implies that $\mathcal{B}$ is a commutative subalgebra in the center of $\mathcal{C}$. Moreover, the space of maximal ideals of $\mathcal{B}$ is homeomorphic to Γ, and the maximal ideal $\mathcal{N}_\tau \subset \mathcal{B}$ associated with the point $\tau \in \Gamma$ is

$$\mathcal{N}_\tau := \{(P_n f P_n) + \mathcal{J} : f(\tau) = 0\}.$$

Let $\mathcal{J}_{\mathcal{N}_\tau}$ be the smallest closed ideal of the algebra $\mathcal{C}$ containing $\mathcal{N}_\tau$. According to [7, Theorem 1.9.5(a)], the coset $(A_\Gamma^{(\delta,n)} P_n) + \mathcal{J}$ is invertible in $\mathcal{C}/\mathcal{J}$ if and only if the coset $(A_\Gamma^{(\delta,n)} P_n)_\tau^\circ := ((A_\Gamma^{(\delta,n)} P_n) + \mathcal{J}) + \mathcal{J}_{\mathcal{N}_\tau}$ is invertible in the quotient-algebra $(\mathcal{C}/\mathcal{J})/\mathcal{J}_{\mathcal{N}_\tau}$ for each $\tau \in \Gamma$. However, as was already mentioned for $\tau \in \Gamma \setminus \mathfrak{M}_\Gamma$, a local representative of $(A_\Gamma^{(\delta,n)} P_n)_\tau^\circ$ is just $(P_n)_\tau^0$, which is obviously invertible. On the other hand, for any corner point $\tau \in \mathfrak{M}_\Gamma$ a local representative of the corresponding coset is the operator $B_{\theta_j,\delta,\varepsilon}$. Therefore, in this case the coset $(A_\Gamma^{(\delta,n)} P_n)_\tau^\circ$ is invertible if and only if the operator $B_{\theta_j,\delta,\varepsilon}$ is. Thus the conditions of Theorem 3.5 provide the invertibility of the coset $(A_\Gamma^{(\delta,n)} P_n) + \mathcal{J}$ in the quotient-algebra $\mathcal{C}/\mathcal{J}$, and that implies its invertibility in $\mathcal{A}/\mathcal{J}$. Recalling now that on the space $L_2(\Gamma)$ the operator $A_\Gamma + T_{SL}$ is also invertible and applying Theorem 3.2, we obtain the result.   ∎

Theorem 3.2 raises an interesting and important question about the invertibility of the auxiliary operators $B_{\theta_j,\delta,\varepsilon}$. It can be shown that entries of the corresponding

operator matrices belong to a famous algebra of Toeplitz operators $\mathfrak{T}_2$. This allows us to study conditions when the model operators $B_{\theta_j,\delta,\varepsilon}$ are Fredholm. However, at the moment there are no effective theoretical tools to verify the invertibility of the operators from the algebra $\mathfrak{T}_2$.

Remark 3.6. If an approximation method (A_n) is stable, then the error $||\omega - \omega_n||$ can be estimated as follows [27, Proposition 1.23(c)]:

$$||\omega - \omega_n|| \leq \inf_{v \in S_n^d(\Gamma)} (||\omega - v|| + ||A_n^{-1}|| \cdot ||A_n v - Q_n^\delta f||).$$

Thus to evaluate the error of the method under consideration one can use known results of the approximation theory. Note that examples below demonstrate an excellent numerical convergence of the method.

4. Numerical examples. The numerical examples for the Sherman–Lauricella integral equation are performed in the MATLAB environment (version 7.6) and executed on an ordinary workstation equipped with an IntelCore2 Duo E8400 CPU at 3.00 GHz and 4GB of memory. The integral equation is discretized with a Nyström scheme based on composite Gauss–Legendre quadrature (1.7) with $d = 15$. The GMRES iterative solver [28] with a low-threshold stagnation avoiding technique [15] is used for the linear systems, and the stopping criterion threshold is set to $10 \cdot \epsilon_{\text{mach}}$, where ϵ_{mach} is machine epsilon. We use the Parton and Perlin [26] choice T_{SL} of (2.8), except in the last numerical example, where we investigate the properties of other choices of T_{SL}. The point a is placed at the origin.
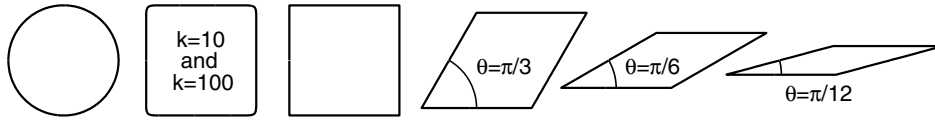

FIG. 4.1. *From unit circle, via superellipse and square, to rhombuses with decreasing opening angles θ. These curves Γ are used in the numerical examples.*

Note that in applications, right-hand sides f are not always continuous. Therefore, it is important to study the applicability of approximation methods in this case as well. Following the above scheme, one can replace the space $W_2^1(\Gamma)$ by the space $W_2^1(\Gamma, c_1, c_2, \ldots, c_q)$.[1] Functions from $W_2^1(\Gamma, c_1, c_2, \ldots, c_q)$ can be discontinuous at corner points, and examples with the so-called discontinuous square function show that the Nyström method picks up the smallest details of the solution. Moreover, they also show a gradual transformation of the solution during continuous deformation of the unit circle into the series of contours both with and without corner points.

4.1. Curves Γ and right-hand sides $f(z)$. All curves Γ (see Figure 4.1) are parameterized by a function $t = \gamma(s)$ with $-0.5 < s \leq 0.5$ and with the origin at the center. The superellipse (Lamé curve) has Cartesian equation

$$|x|^k + |y|^k = 1,$$

and we consider examples with $k = 10$ and $k = 100$. The square and the rhombuses have arc length equal to four. The parametrization is chosen such that the corners of

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[1]The definition of $W_2^1(\Gamma, c_1, c_2, \ldots, c_q)$ can be found in [10].

the square and the rhombuses and the points where the superellipse has the highest curvature (rounded corners) occur at parameter values $s_1 = -0.375$ (lower left corner), $s_2 = -0.125$ (lower right corner), $s_3 = 0.125$ (upper right corner), and $s_4 = 0.375$ (upper left corner). On the unit circle $\gamma(s_1) = \exp(-i3\pi/4)$, $\gamma(s_2) = \exp(-i\pi/4)$, $\gamma(s_3) = \exp(i\pi/4)$, and $\gamma(s_4) = \exp(i3\pi/4)$.

The numerical examples make use of two different right-hand sides. The first right-hand side is

$$(4.1) \qquad\qquad f_1(t) = |t| .$$

The second right-hand side is

$$(4.2) \qquad\qquad f_2(t) = f_0(t) - \frac{it}{2A}\mathrm{Re}\int_\Gamma f_0(\tau)\, d\bar{\tau} ,$$

where A is the area enclosed by Γ and $f_0(t)$ is the step function

$$f_0(t(s)) = \begin{cases} 0, & -0.5 \quad < s \le -0.375 , \\ -1, & -0.375 < s \le -0.125 , \\ 0, & -0.125 < s \le 0.125 , \\ 1, & 0.125 < s \le 0.375 , \\ 0, & 0.375 < s \le 0.5 . \end{cases}$$

Let us call this function f_2 *the discontinuous square function*. The denomination reflects the fact that if one goes around the above unit square, then the image of the pass under the mapping f_2 is the same unit square but taken not continuously.

Both functions (4.1) and (4.2) satisfy condition (1.5) for all curves in the examples.

Results from the computations, in graphical form, are presented in Figures 4.2, 4.3, and 4.4, where the left images show real and imaginary parts of the approximate solution ω_n as a function of parameter s, whereas the right images represent the corresponding graph of ω_n in the complex plane. Note that the right-hand side $f_1 \in W_2^1$, so the corresponding solutions are continuous functions for all contours under consideration (cf. Figure 4.2). However, especially intriguing is the change of the solution for the equations with the discontinuous square function $f_2 \in W_2^1(\Gamma, \gamma(s_1), \gamma(s_2), \gamma(s_3), \gamma(s_4))$ as a right-hand side of (1.4). If the contour Γ is subject to transformations shown in Figure 4.1, the changes of the solutions are presented in Figures 4.3 and 4.4. In particular, if the opening angle of the rhombus diminishes from $\pi/2$ to 0, the four solution branches split into two pairs with the elements of the corresponding pairs drifting to each other. This probably reflects the fact that two angles of the rhombus tend to π, and it looks like a disappearance of the corresponding corner points (cf. the behavior of the graphs at the points $s_3 = -0.125$ and $s_4 = 0.375$ in the left images of Figure 4.4). On the other hand, for the lower left and the upper right vertices, the angle tends to 0. The corresponding points on the rhombuses mimic cusps, and such points cause additional computational difficulties. Let us also point out that the corresponding parts of the solutions are experiencing a remarkable growth.

Table 4.1 shows numerical convergence with increased mesh refinement.

4.2. Different choices of T_{SL}. In the literature there are several choices for the operator T_{SL}. The Parton and Perlin [26] choice of T_{SL} is (2.8), and the Sherman choice of T_{SL} [21, section 56] is

$$(4.3) \qquad T_{SL}\omega(t) = \left(\frac{1}{t-a} - \frac{1}{\bar{t}-\bar{a}} + \frac{t-a}{(\bar{t}-\bar{a})^2}\right)\frac{1}{\pi i}\mathrm{Re}\int_\Gamma \frac{\omega(\tau)\, d\tau}{(\tau-a)^2} .$$
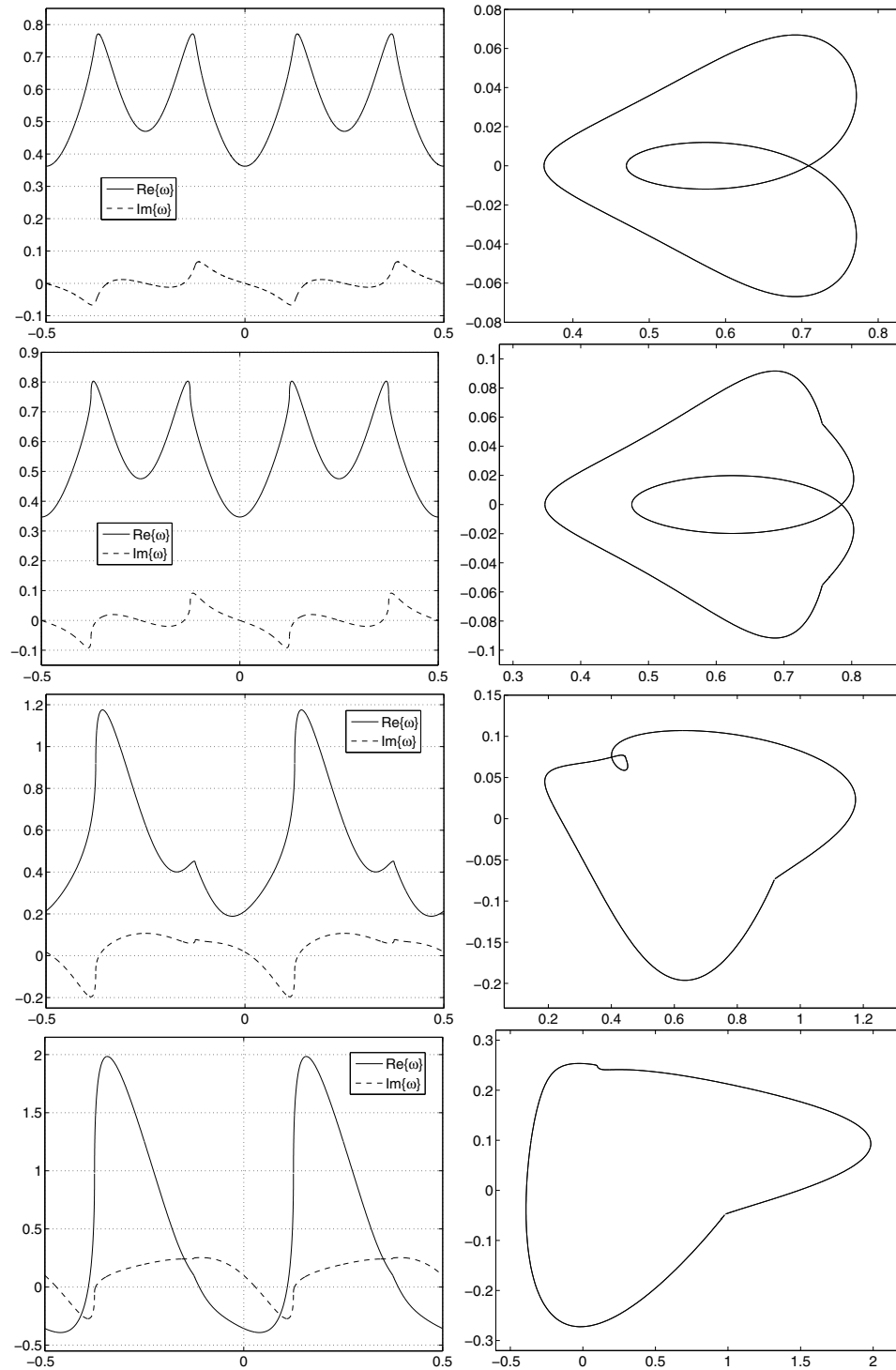
FIG. 4.2. *Solution ω of (1.6) with the right-hand side $f = f_1$. Left column: real and imaginary parts of ω as a function of parameter s. Right column: ω in the complex plane. First row: superellipse with $k = 10$. Second row: square. Third row: rhombus with $\theta = \pi/3$. Fourth row: rhombus with $\theta = \pi/6$.*
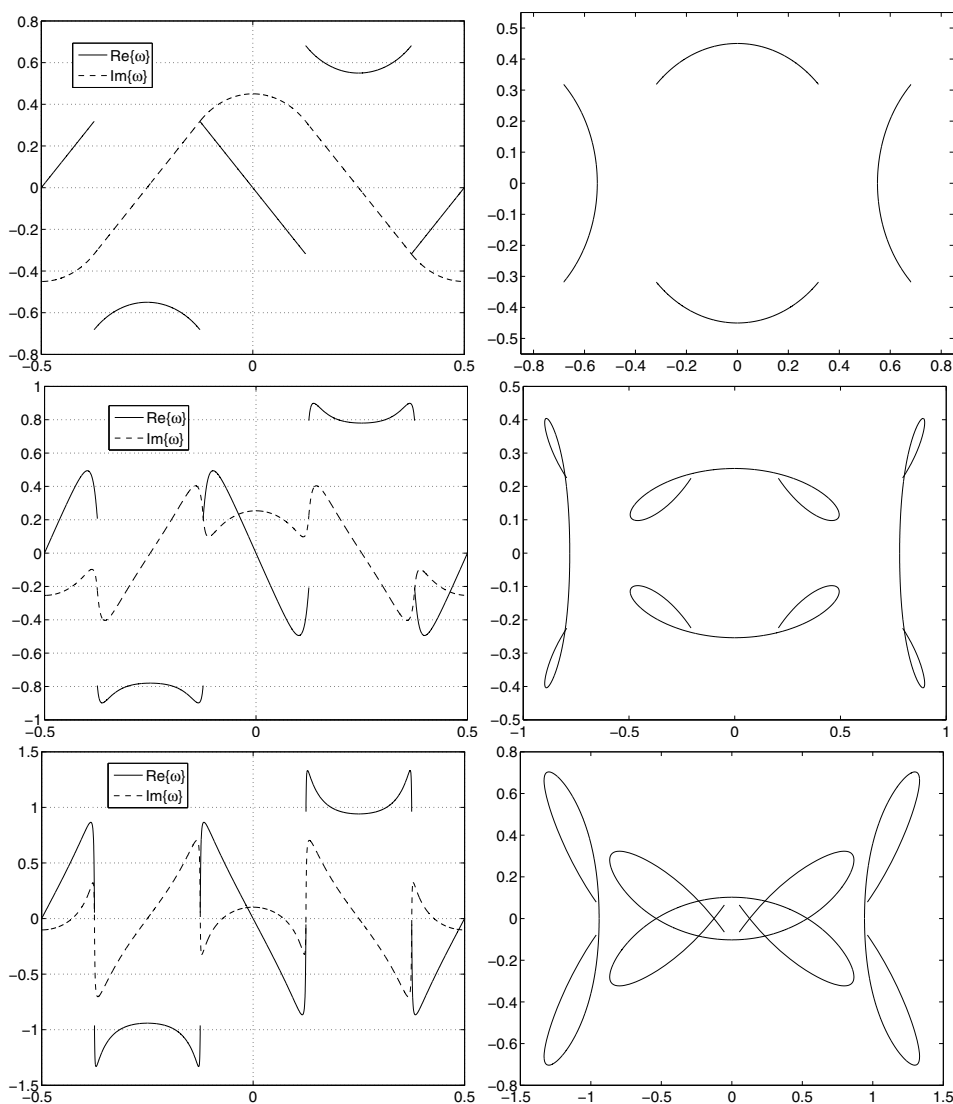
FIG. 4.3. *Same as Figure* 4.2 *but with* $f = f_2$. *First row: circle. Second row: superellipse with* $k = 10$. *Third row: superellipse with* $k = 100$.

In equation (26) of [14] an operator T_{SL} is defined by

$$(4.4) \qquad T_{SL}\omega(t) = \frac{in_t}{2S}\mathrm{Re}\int_\Gamma\left(\omega(\tau) + \frac{1}{\pi i}\int_\Gamma\frac{\omega(z)\,dz}{z - \tau}\right)d\bar\tau\,,$$

where S is the arc length of Γ and n_t is the outward unit normal at t on Γ.

Yet another option is the simple choice of T_{SL} given by

$$(4.5) \qquad T_{SL}\omega(t) = \frac{it}{2A}\mathrm{Re}\int_\Gamma\omega(\tau)\,d\bar\tau\,,$$

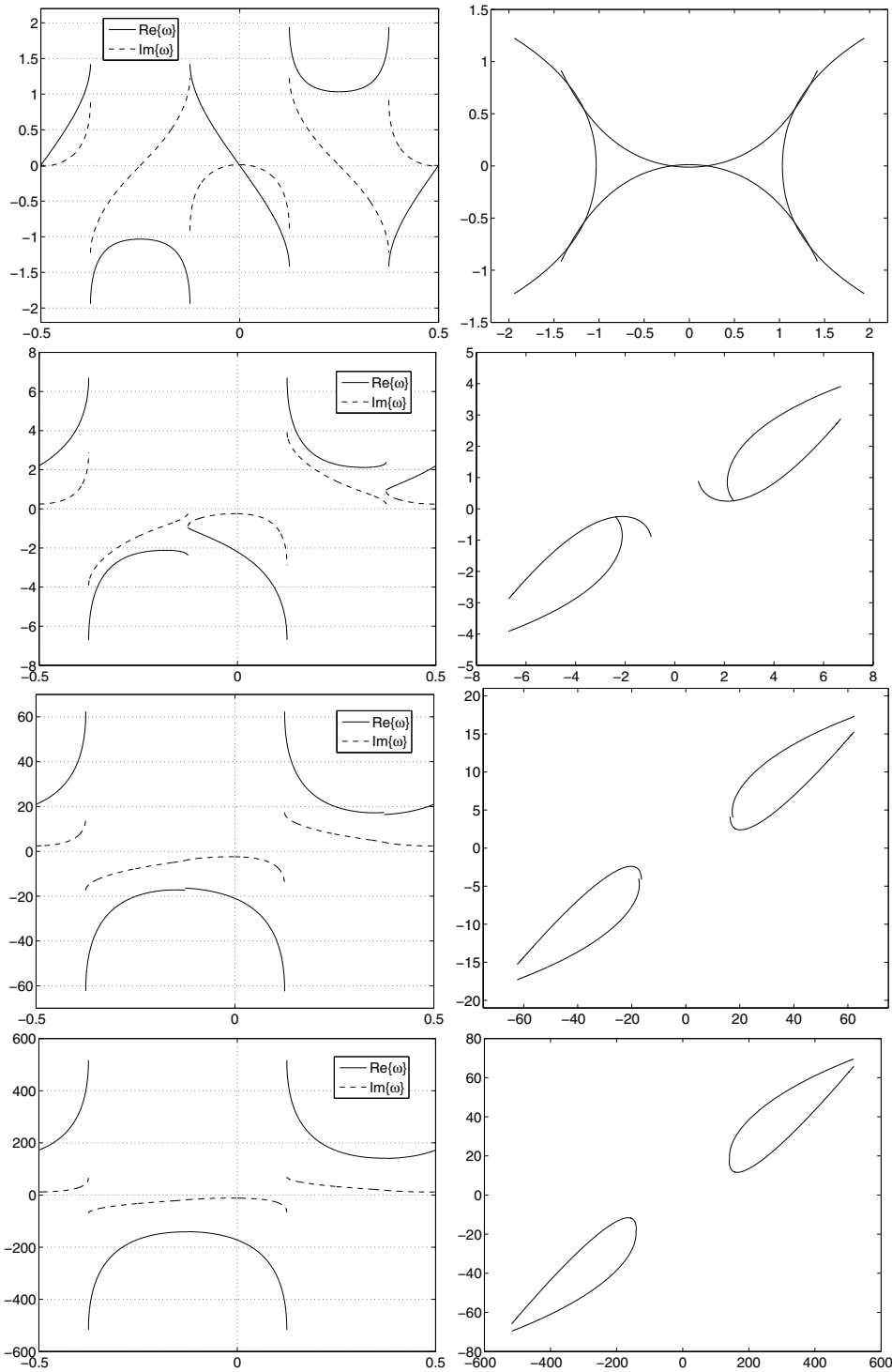where A is the area enclosed by Γ.

FIG. 4.4. *Continuation of Figure 4.3 with $f = f_2$. First row: square. Second row: rhombus with $\theta = \pi/3$. Third row: rhombus with $\theta = \pi/6$. Fourth row: rhombus with $\theta = \pi/12$.*

TABLE 4.1

Convergence of the relative error $||\omega_n - \omega_{2n}||_2/||\omega_{2n}||_2$. Each entry in the table compares solutions computed on two meshes. The first mesh has n quadrature subintervals and gives ω_n. The second mesh is constructed by dividing all subintervals of the first mesh into two. It has $2n$ subintervals and gives ω_{2n}. There are 16 quadrature points on each subinterval. The following abbreviations are used: C is the unit circle, E is the superellipse with $k = 10$, S is the square, and R_1, R_2, and R_3 are the rhombuses with opening angles $\theta = \pi/3$, $\theta = \pi/6$, and $\theta = \pi/12$, respectively. Uniform meshes are used for the circle and the superellipse. Adaptively refined meshes are used for the square and the rhombuses.

n	C, f_1	C, f_2	n	E, f_1	E, f_2
8	$2 \cdot 10^{-16}$	$2 \cdot 10^{-11}$	40	$4 \cdot 10^{-9}$	$3 \cdot 10^{-9}$
16	$6 \cdot 10^{-15}$	$2 \cdot 10^{-14}$	80	$2 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
32	$1 \cdot 10^{-14}$	$9 \cdot 10^{-15}$	160	$5 \cdot 10^{-14}$	$4 \cdot 10^{-14}$

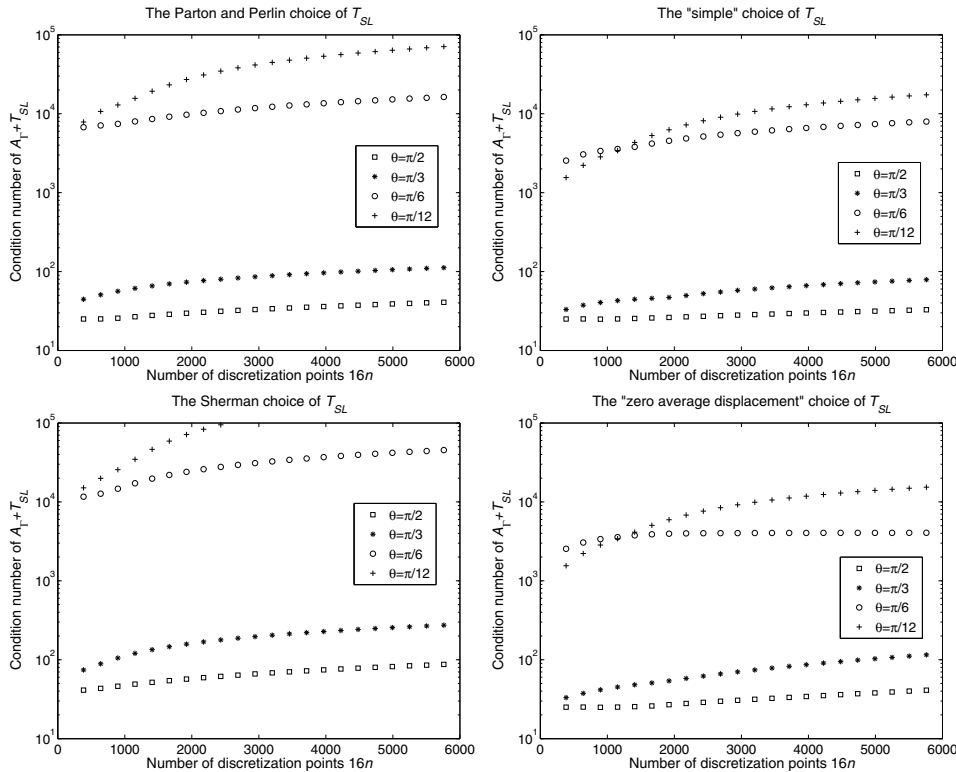| n	S, f_1	S, f_2	R_1, f_1	R_1, f_2	R_2, f_1	R_2, f_2	R_3, f_1	R_3, f_2
104	$5 \cdot 10^{-3}$	$1 \cdot 10^{-2}$	$3 \cdot 10^{-3}$	$2 \cdot 10^{-2}$	$5 \cdot 10^{-2}$	$2 \cdot 10^{-1}$	$1 \cdot 10^{-2}$	$9 \cdot 10^{-2}$
184	$9 \cdot 10^{-4}$	$3 \cdot 10^{-3}$	$6 \cdot 10^{-4}$	$3 \cdot 10^{-3}$	$4 \cdot 10^{-3}$	$6 \cdot 10^{-2}$	$7 \cdot 10^{-4}$	$2 \cdot 10^{-2}$
344	$3 \cdot 10^{-5}$	$9 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	$1 \cdot 10^{-4}$	$7 \cdot 10^{-5}$	$3 \cdot 10^{-3}$	$1 \cdot 10^{-5}$	$7 \cdot 10^{-4}$

FIG. 4.5. *Condition number in 2-norm of discretized operator $A_\Gamma + T_{SL}$ as a function of increased mesh refinement for rhombuses with different opening angles θ. The angle $\theta = \pi/2$ corresponds to the square. Four choices of T_{SL} are compared. Upper left: the Parton and Perlin choice of T_{SL} (2.8). Upper right: the "simple" choice of T_{SL} (4.5). Lower left: the Sherman choice of T_{SL} (4.3). Lower right: the "zero average displacement" choice of T_{SL} (4.4).*

Note that each of the operators (4.3), (4.4), (4.5) satisfies condition (2.11), so the solution of the corresponding corrected equation (2.9) is simultaneously a solution of the original Sherman–Lauricella equation (1.4).

Figure 4.5 compares the behavior of the condition numbers of the system matrix for different choices of T_{SL}. It seems that for the values θ close to $\pi/2$ the choice of the additional operator T_{SL} does not have a strong influence on the behavior of condition numbers. On the other hand, this becomes more important if some angles are small, and the operator T_{SL} proposed by Sherman seems to be the least attractive for computations.

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