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Sei Zhen Khong and Anders Rantzer

Abstract— Stable positive linear time-invariant autonomous systems admit diagonal quadratic Lyapunov functions. Such a property is known to be useful in distributed and scalable control of positive systems. In this paper, it is established that the same holds for exponentially stable positive discrete-time and continuous-time linear time-varying systems.

Index Terms— linear time-varying systems, positive systems, Lyapunov functions, stability

I. Introduction

Monotone system models are ubiquitous in engineering and science. These models are characterized by dynamics that preserve an order relation in the state space [1], [2]. For linear models, the monotonicity property is equivalent to forward invariance of the nonnegative orthant and the class of systems is called positive systems [3], [4].

Besides their relevance to real-world modelling, monotone systems have gained an increasing attention over the last few decades owing to the fact that their order-preserving property can often be exploited to simplify analysis, synthesis and computations. For instance, [5] shows that stabilising static output feedback controllers for positive linear time-invariant (LTI) systems can be characterised using linear programming, by contrast to the more sophisticated semidefinite programming used for general linear systems [6]. Extensions along similar lines to input-output gain optimisation can be found in [7]. In particular, [7] introduces the use of integral linear constraints (ILCs) to characterise uncertainty and utilises linear programming to verify robust stability via a dissipativity approach. In [8], this is further generalised via an input-output approach to analyse robust stability of positive feedback systems, in a similar spirit to integral quadratic constraints based analysis [9].

The work [10] develops separator-based conditions for the stability of interconnected positive LTI systems. Robust stability of more general classes of cone-invariant LTI systems is characterised in [11]. In [12], it is shown that the input-output gain of positive systems can be evaluated using a diagonal quadratic storage function and this is exploiteid for $\mathbf{H}_\infty$ optimisation of decentralised controllers in terms of

linear matrix inequalities. The series of work [13], [14] introduces various methods for distributed analysis and control of positive systems. Positive linear systems are also known to demonstrate a high level of robustness against certain perturbations. For instance, it is shown in [15] that such systems are robustly stable against bounded time-varying delays. This is generalised to a class of monotone nonlinear systems in [16]. A generalisation of the class of monotone systems to differentially positive systems is introduced in [17], where the cone in which the state trajectory lies may vary over time.

It is well known that the stability of positive LTI systems can be characterised via the existence of a diagonal quadratic Lyapunov function [18], [14]. This is a form of sum-separable Lyapunov function, i.e. it is decomposable into a sum of functions, each of which is dependent on only one state variable. This also shows that the search for a Lyapunov function is amenable to distributed optimisation. Furthermore, diagonal quadratic Lyapunov functions supports the use of decentralised control laws and significantly facilitate scalable control of large-scale interconnected systems. This paper establishes the existence of diagonal quadratic Lyapunov functions for exponentially stable positive linear time-varying (LTV) systems, extending the features to this larger class of systems. Both discrete-time and continuous-time systems are considered. The results can be viewed as converse Lyapunov theorems [19] for positive LTV systems, with the additional requirement that the Lyapunov functions are quadratic and diagonal. An intermediate step to establishing the result for LTV systems involves doing so first for linear periodic systems. It is worth noting that the existence of max-separable Lyapunov functions has been established for monotone nonlinear systems on compact domains in [20].

The paper evolves along the following lines. Notation and preliminary material are introduced in the next section. The existence of diagonal quadratic Lyapunov functions is first established for positive linear periodically time-varying systems in Section III. This is then extended to positive LTV systems in Section IV. Some concluding remarks are provided in Section V.

II. Notation and preliminaries

Let $\mathbb{R}$ ($\mathbb{Z}$) and $\mathbb{R}_{\geq 0}$ ($\mathbb{Z}_{\geq 0}$) denote, respectively, the real and nonnegative real numbers (integers). Given a column vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm, i.e. $\|x\| := \sqrt{\sum_i^n x_i^2}$. Given a matrix $A \in \mathbb{R}^{m \times n}$, $A^T \in \mathbb{R}^{n \times m}$ denotes its transpose. $A_{i,j}$ denotes the (i,j) entry of A. $\|A\|$ denotes the matrix 2-norm, i.e. the largest singular value of A. I_n denotes the identity matrix of dimensions $n \times n$; the dimension n is often omitted when it is clear from the

context. Given an $A \in \mathbb{R}^{m \times n}$, the inequality $A > 0$ ($A \geq 0$) means that all entries of A are positive (nonnegative). An $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if all its off-diagonal entries are nonnegative. It is called *Hurwitz* (resp. *Schur*) if all its eigenvalues have strictly negative real parts (resp. lie strictly inside the unit circle). The spectral radius of A is denoted as $\mathrm{rad}(A)$. An $A \in \mathbb{R}^{n \times n}$ is said to be symmetric if $A = A^T$. A symmetric A is said to be positive (semi)definite if $(x^T A x \geq 0)$ $x^T A x > 0$ for all $x \in \mathbb{R}^n$; this is denoted as $(A \succeq 0)$ $A \succ 0$. Given two symmetric matrices A and B, the notation $A \succeq B$ is used to denote $A - B \succeq 0$. Similarly, given two matrices A and B, the notation $A \geq B$ is used to denote $A - B \geq 0$.

This paper is concerned with the class of discrete-time and continuous-time homogeneous linear time-varying (LTV) finite-dimensional state-space systems. The discrete-time systems in question are of the form:

$$x(k+1) = A(k)x(k); \quad x(0) = x_0 \in \mathbb{R}^n, \qquad (1)$$

where $A : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n}$ is a uniformly bounded discrete-time matrix-valued function. On the contrary, the continuous-time systems are of the form:

$$\dot{x}(t) = A(t)x(t); \quad x(0) = x_0 \in \mathbb{R}^n \qquad (2)$$

where $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ is a uniformly bounded continuous-time matrix-valued function. The unique existence of the solution $x(t)$, $t \geq t_0$ to (2) follows from certain properties on A. In the case where A is piecewise continuous and uniformly bounded, the unique existence of x follows from the fact that the right hand side of (2) is globally Lipschitz in the variable x [19, Thm. 3.2].

Starting from any initial state $x(m)$ at time $m \geq 0$, the solution of (1) is obtained as

$$x(k) = \Phi(k, m)x(m) \quad k \geq m,$$

where the discrete-time state transition matrix $\Phi(k, m)$ is given by

$$\Phi(k, m) = \begin{cases} I & k = m \\ A(k-1)A(k-2)\cdots A(m) & k > m. \end{cases} \qquad (3)$$

Note that $\Phi(k, m) = \Phi(k, p)\Phi(p, m)$ for all $k \geq p \geq m \geq 0$.

For the continuous-time system (2), the general solution starting at any initial state $x(s)$ at time $s \geq 0$ is of the form [21], [22]:

$$x(t) = \Phi(t, s)x(s), \qquad (4)$$

where Φ is the unique solution of the matrix differential equation

$$\frac{d\Phi(t, s)}{dt} = A(t)\Phi(t, s), \ t \geq s; \quad \Phi(s, s) = I.$$

In particular, $\Phi(t, s)$ is called the continuous-time state transition matrix and is invertible for all t, s with $\Phi(t, s)^{-1} = \Phi(s, t)$. Moreover, it satisfies $\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)$ for all $t, \tau, s \geq 0$. In the case where A is a constant matrix, i.e. (2)

is an LTI system, we have $\Phi(t, s) = e^{A(t-s)}$.

Definition 2.1: The discrete-time LTV system (1) is said to be (uniformly) exponentially stable if there exist $\gamma > 0$ and $0 < \lambda < 1$ such that

$$\|x(k)\| \leq \gamma \lambda^{(k-m)} \|x(m)\| \quad \forall k \geq m \geq 0, x(m) \in \mathbb{R}^n.$$

Equivalently,

$$\|\Phi(k, m)\| \leq \gamma \lambda^{(k-m)} \quad \forall k \geq m \geq 0.$$

Analogously, the continuous-time LTV system (2) is said to be (uniformly) exponentially stable if there exist $\gamma, \lambda > 0$ such that

$$\|x(t)\| \leq \gamma e^{-\lambda(t-s)} \|x(s)\| \quad \forall t \geq s \geq 0, x(s) \in \mathbb{R}^n.$$

Equivalently,

$$\|\Phi(t, s)\| \leq \gamma e^{-\lambda(t-s)} \quad \forall t \geq s \geq 0.$$

Lyapunov's direct method is commonly used to establish exponential stability of (1) or (2). In particular, exponential stability of (1) is equivalent to the existence of a $P : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n}$ such that with

$$V(x, k) := x(k)^T P(k) x(k), \qquad (5)$$

it holds that for all $k \geq 0$,

$$\begin{aligned} \eta \|x(k)\|^2 &\leq V(x, k) \leq \rho \|x(k)\|^2 \\ V(x, k+1) - V(x, k) &\leq -\nu \|x(k)\|^2, \end{aligned} \qquad (6)$$

where η, ρ, and ν are finite positive constants [22, Thm. 23.3]. Analogously, exponential stability of (2) is equivalent to the existence of a differentiable $P : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ such that with

$$V(x, t) := x(t)^T P(t) x(t), \qquad (7)$$

it holds that for all $t \geq 0$,

$$\begin{aligned} \eta \|x(t)\|^2 &\leq V(x, t) \leq \rho \|x(t)\|^2 \\ \dot{V}(x, t) = \frac{\partial V(x, t)}{\partial t} &\leq -\nu \|x(t)\|^2, \end{aligned} \qquad (8)$$

where η, ρ, and ν are finite positive constants [22, Thm. 7.4 and 7.8]. Note that the time-derivative of V along the trajectories of the LTV system is given by

$$\begin{aligned} \dot{V}(x, t) &= \frac{\partial}{\partial t} V(x, t) \\ &= x^T (A(t)^T P(t) + P(t)A(t) + \dot{P}(t))x. \end{aligned}$$

In the following, we will focus on positive linear systems. We show in Section IV that the converse Lyapunov result takes a stronger form for such systems. In particular, P can be taken to be a *diagonal* matrix-valued function over time, which is equivalent to saying that the Lyapunov function in

(5) or (7) is sum-separable:

$$V(x, k) = \sum_{i=1}^{n} P_i(k)x_i(k)^2 \quad \text{in discrete-time and}$$

$$V(x, t) = \sum_{i=1}^{n} P_i(t)x_i(t)^2 \quad \text{in continuous-time.}$$

To this end, the following definition and collection of preliminary results are in order.

Definition 2.2: The discrete-time LTV system (1) is said to be positive if $x(m) \in \mathbb{R}_{\geq 0}^{n}$ implies $x(k) \in \mathbb{R}_{\geq 0}^{n}$ for all $k \geq m \geq 0$. The continuous-time LTV system (2) is said to be positive if $x(s) \in \mathbb{R}_{\geq 0}^{n}$ implies $x(t) \in \mathbb{R}_{\geq 0}^{n}$ for all $t \geq s \geq 0$.

Phrased differently, a positive linear system leaves the nonnegative orthant $\mathbb{R}_{\geq 0}^{n}$ forward invariant. Observe that (1) is positive if, and only if, $A(k) \in \mathbb{R}_{\geq 0}^{n \times n}$, i.e. it is nonnegative for all $k \geq 0$. Moreover, it can be seen from (4) that (2) is positive if, and only if, $\Phi(t, s) \in \mathbb{R}_{\geq 0}^{n \times n}$, i.e. it is a nonnegative matrix, for all $t \geq s \geq 0$. Under the assumption that A is continuous, the positivity of a linear system is equivalent to $A(t)$ being Metzler for all $t \geq 0$ [1, Lem. VIII.1].

When A is a constant nonnegative matrix, it is known that A is Schur if, and only if, there exists a *diagonal $P \succ 0$* such that $A^T P A - P \prec 0$ [14, Prop. 2]. On the other hand, when A is a constant Metzler matrix, it is known that A is Hurwitz if, and only if, there exists a *diagonal $P \succ 0$* such that $A^T P + PA \prec 0$ [18, Thm. 6.2.3]. Thus, the main results of this paper can be seen as generalisations of these facts to LTV systems.

III. LINEAR PERIODIC SYSTEMS

This section establishes the existence of a diagonal quadratic Lyapunov function for positive linear periodically time-varying systems that are exponentially stable. This is then generalised to general LTV systems in the succeeding section. Both discrete-time and continuous-time systems are considered.

A. Discrete-time systems

Consider the following discrete-time linear periodic system with period p:

$$x(k+1) = A(k)x(k); \quad x(0) = x_0; \quad A(k+p) = A(k), \tag{9}$$

where $A : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n}$ is uniformly bounded. The monodromy matrix at time k, defined as

$$\Psi(k) := \Phi(k+p, k),$$

relates the value of the state at a given time k to the value after one period at $k + p$:

$$x(k+p) = \Psi(k)x(k).$$

It plays a major role in the stability analysis of (9). In particular, (9) is exponentially stable if, and only if, $\Psi(k)$ is Schur for all $k \geq 0$; see [23], [24]. Note that by the

definition of the state-transition matrix in (3), the eigenvalues of $\Psi(k)$ are independent of the time tag k and are the same as those of $A(p-1)A(p-2)\cdots A(0)$; see also [24, Section 3.1.1]. In what follows, it is convenient to employ the cyclic reformulation of a linear periodic system; see [24, Section 6.3]. Specifically, define

$$\hat{A} := \begin{bmatrix} 0 & 0 & \cdots & 0 & A(p-1) \\ A(0) & 0 & \cdots & 0 & 0 \\ 0 & A(1) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A(p-2) & 0 \end{bmatrix}. \tag{10}$$

Note that

$$\hat{A}^p = \begin{bmatrix} \Psi(0) & 0 & \cdots & 0 \\ 0 & \Psi(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \Psi(p-1) \end{bmatrix}, \tag{11}$$

and hence the eigenvalues of $\hat{A}$ are the pth-roots of those of $\Psi(k)$.

Theorem 3.1: A positive linear p-periodic discrete-time system of the form (9) is exponentially stable if, and only if, there exists a *diagonal P* such that $P(k+p) = P(k)$,

$$\eta I \preccurlyeq P(k) \preccurlyeq \rho I$$
$$A(k)^T P(k+1)A(k) - P(k) \preccurlyeq -\nu I$$

for all $k \geq 0$. In other words, (6) holds with respect to the V defined in (5).

Proof: Sufficiency follows from the standard Lyapunov argument described in Section II. For necessity, note that exponential stability of (9) is equivalent to $\Psi(k)$ being Schur for all $k \geq 0$. It follows from (11) that $\hat{A} \in \mathbb{R}^{np \times np}$ defined in (10) is Schur. By the positivity hypothesis, $\hat{A}$ is also a nonnegative matrix. Therefore, there exists of a *diagonal $D \succ 0$* such that $\hat{A}^T D \hat{A} - D \prec 0$. Let the diagonal blocks of D be denoted by $[D]_k \in \mathbb{R}^{n \times n}$, where $k = 1, \ldots, p$. By defining

$$P(k) := [D]_{k+1}, \quad k = 0, 1, \ldots, p-1,$$

it can be seen that $A(k)^T P(k+1)A(k) - P(k) \prec 0$ for $k = 0, 1, \ldots, p-1$. The claim of the theorem then follows from extending the definition of P p-periodically. ∎

B. Continuous-time systems

Consider the following continuous-time linear periodic system with period h:

$$\dot{x}(t) = A(t)x(t); \quad x(0) = x_0; \quad A(t+h) = A(t), \tag{12}$$

where $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ is piecewise continuous and uniformly bounded. The monodromy matrix at time t, defined as

$$\Psi(t) := \Phi(t+h, t),$$

plays an important role in determining the stability of (12); see [23], [24]. In particular, $\Psi(t)$ relates the value of the

state at a given time t to the value after one period at $t + h$:

$$x(t + h) = \Psi(t)x(t).$$

Therefore, the sampled state $x_t(k) := x(t + kh)$ is governed by the shift-invariant discrete-time equation

$$x_t(k + 1) = \Psi(t)x_t(k).$$

It follows that the periodic system (12) is exponentially stable if, and only if, $\Psi(t)$ is Schur for all $t \in [0, 0 + h]$. One may also examine the stability of (12) via the Lyapunov's direct method. To be specific, it is known that (12) is exponentially stable if, and only if, there exists differentiable P such that $P(t + h) = P(t)$ and for all $t \geq 0$ with respect to the quadratic Lyapunov function defined in (7) [24]. The next theorem shows that P can be chosen to be *diagonal* when the system in question is positive.

Theorem 3.2: A positive linear h-periodic continuous-time system of the form (12) is exponentially stable if, and only if, there exists a differentiable *diagonal* P such that $P(t + h) = P(t)$ and with

$$V(x, t) := x^T P(t)x(t),$$

it holds that

$$\eta\|x(t)\|^2 \leq V(x, t) \leq \rho\|x(t)\|^2$$
$$\dot{V}(x, t) = \frac{\partial}{\partial t}V(x, t) \leq -\nu\|x(t)\|^2$$

for all $t \geq 0$ and some $\eta, \rho, \nu > 0$.

Proof: Sufficiency follows from the Lyapunov's direct method described above. For necessity, note that $\Psi(0)$ is Schur. If $\Psi(0)$ is not irreducible, apply small perturbations on A so that it is. Then by the Perron-Frobenius theorem [18, Thm. 2.1.4] there exist $y > 0$ and $z > 0$ such that $\Psi(0)^T y = \lambda y$ and $\Psi(0)z = \lambda z$, where $\lambda = \text{rad}(\Psi(0)) < 1$. Define $y(t) = \lambda^{t/h}\Phi(h, t)^T y$ and $z(t) = \lambda^{-t/h}\Phi(t, 0)z$. Then $y(t) > 0$ and $z(t) > 0$ for all t, because $\Phi(h, t)$ and $\Phi(t, 0)$ are invertible and nonnegative. It follows that $P(t) = \text{diag}(y_1(t)/z_1(t), \ldots, y_n(t)/z_n(t))$ defines the desired quadratic Lyapunov function. To see this, observe that P is h-periodic by construction and

$$\dot{y}(t) = \frac{\log(\lambda)}{h}\lambda^{t/h}\Phi(h, t)^T y(t) - \lambda^{t/h}A(t)^T\Phi(h, t)^T y(t)$$
$$\dot{z}(t) = -\frac{\log(\lambda)}{h}\Phi(t, t)z(t) + \lambda^{-t/h}A(t)\Phi(t, 0)z(t).$$

Consequently,

$$(\dot{P}(t) + A(t)^T P(t) + P(t)A(t))z(t)$$
$$= \dot{y}(t) - P(t)\dot{z}(t) + A(t)^T y(t) + P(t)A(t)z(t)$$
$$= \frac{\log(\lambda)}{h}(y(t) + P(t)z(t))$$
$$= \frac{2\log(\lambda)}{h}y(t) < 0.$$

By [18, Thm. 6.2.3] (see also [14, Prop. 1]), it follows that the Metzler matrix $\dot{P}(t) + A(t)^T P(t) + P(t)A(t) \prec 0$ for all $t \in [0, h]$, which remains true for sufficiently small

perturbations on A. This concludes the proof. ∎

IV. LINEAR TIME-VARYING SYSTEMS

The main results of the paper are presented in this section. The key idea of the proofs involves extending the existence of diagonal quadratic Lyapunov functions of linear periodic systems established in the preceding section to time-varying systems on successive time intervals.

A. Discrete-time systems

Theorem 4.1: A linear positive homogeneous discrete-time system

$$x(k + 1) = A(k)x(k); \quad x(0) = x_0,$$

where $A : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n}$ uniformly bounded for all $k \geq 0$, is exponentially stable if, and only if, there exists a *diagonal* P such that

$$\eta I \preccurlyeq P(t) \preccurlyeq \rho I$$
$$A(k)^T P(k + 1)A(k) - P(k) \preccurlyeq -\nu I$$

for all $k \geq 0$. In other words, (6) holds with respect to the diagonal Lyapunov function V defined in (5).

Proof: Sufficiency follows from the Lyapunov's direct method described in Section II. Necessity is established below. Since the system is exponentially stable, there exist $\gamma > 0$ and $0 < \lambda < 1$ such that

$$\|\Phi(k, m)\| \leq \gamma\lambda^{(k-m)} \quad \forall k \geq m \geq 0. \tag{13}$$

Let p_1 be an even number such that $\|\Phi(p_1, 0)\| < 1$, which in turn implies that $\Phi(p_1, 0)$ is Schur. Define

$$A_1(k + ip_1) := A(k) \quad \forall k = 0, 1, \ldots, p_1 - 1, i \in \mathbb{Z}_{\geq 0}$$

and consider the following exponentially stable linear periodic system:

$$x_{p_1}(k + 1) = A_1(k)x_{p_1}(k); \quad x_{p_1}(0) = x(0).$$

By Theorem 4.1, there exists a *diagonal* P_1 such that $\eta_1 I \preccurlyeq P_1(t) \preccurlyeq \rho_1 I$ and $A_1(k)^T P_1(k+1)A_1(k) - P_1(k) \preccurlyeq -\nu_1 I$ for all $k \geq 0$ and some $\eta_1, \rho_1, \nu_1 > 0$. Note that $A(k) = A_1(k)$ and $x(k) = x_{p_1}(k)$ for $k = 0, 1, \ldots, p_1 - 1$, and hence with $P(t) := P_1(t)$ for $k = 0, 1, \ldots, p_1 - 1$, it holds that $\eta_1 I \preccurlyeq P(t) \preccurlyeq \rho_1 I$ and $A(k)^T P(k+1)A(k) - P(k) \preccurlyeq -\nu_1 I$ for $k = 0, 1, \ldots, p_1 - 1$.

Now let p_2 be an even number such that $\|\Phi(p_2, p_1/2)\| < 1$. Such a p_2 exists by (13). By repeating the same steps as above, $P(k)$ can be extended to $k = p_1, \ldots, p_2 - 1$ so that $\eta_2 I \preccurlyeq P(t) \preccurlyeq \rho_2 I$ and $A(k)^T P(k+1)A(k) - P(k) \preccurlyeq -\nu_2 I$ for $k = 0, 1, \ldots, p_2 - 1$ for some $\eta_2, \rho_2, \nu_2 > 0$. Repeated applications of the arguments above while noting that A is uniformly bounded then yields a *diagonal* P such that $\eta I \preccurlyeq P(t) \preccurlyeq \rho I$ and $A(k)^T P(k+1)A(k) - P(k) \preccurlyeq -\nu I$ for all $k \in \mathbb{Z}_{\geq 0}$ and some $\eta, \rho, \nu > 0$. ∎

B. Continuous-time systems

Theorem 4.2: A linear positive homogeneous continuous-time system

$$\dot{x}(t) = A(t)x(t); \quad x(0) = x_0, \tag{14}$$

where $A : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ is piecewise continuous and uniformly bounded for all $t \geq 0$, is exponentially stable if, and only if, there exists a Lyapunov function

$$V(x,t) = x^T P(t)x(t)$$

with differentiable *diagonal P* such that

$$\eta\|x(t)\|^2 \leq V(x,t) \leq \rho\|x(t)\|^2$$
$$\dot{V}(x,t) = \frac{\partial}{\partial t}V(x,t) \leq -\nu\|x(t)\|^2$$

for all $t \geq 0$ and some $\eta, \rho, \nu > 0$.

Proof: Sufficiency follows from the Lyapunov's direct method described in Section II. For necessity, note that by the exponential stability hypothesis, there exist $\gamma, \lambda > 0$ such that

$$\|\Phi(t,s)\| \leq \gamma e^{-\lambda(t-\tau)} \quad \forall t \geq s \geq 0. \tag{15}$$

This implies that there exists an $h > 0$ such that $\|\Phi(h,0)\| < 1$, which in turn implies that $\Phi(h,0)$ is Schur. Define

$$A_1(t+kh) := A(t) \quad \forall t \in [0, 0+h), \, k = 0, 1, 2, \dots$$

and consider the following linear periodic system:

$$\dot{x}_h(t) = A_1(t)x_h(t).$$

By Theorem 3.2, there exists a differentiable *diagonal P_1* such that with $V_1(x_h, t) = x_h^T P_1(t)x_h(t)$, it holds that $\eta_1\|x_h(t)\|^2 \leq V_1(x_h, t) \leq \rho_1\|x_h(t)\|^2$ and $\dot{V}_1(x_h, t) \leq -\nu_1\|x_h(t)\|^2$ for $t \geq 0$ and some $\eta_1, \rho_1, \nu_1 > 0$. Note that $A(t) = A_1(t)$ and $x(t) = x_h(t)$ for $t \in [0, h)$, and hence with $P(t) := P_1(t)$ and $V(x,t) := x^T P(t)x(t)$ for $t \in [0, h_1)$, $\eta_1\|x(t)\|^2 \leq V(x,t) \leq \rho_1\|x(t)\|^2$ and $\dot{V}(x,t) \leq -\nu_1\|x(t)\|^2$ for $t \in [0, h)$.

Now by applying inequality (15) again, it follows that $\|\Phi(2h, h/2)\| < 1$. By repeating the arguments above, $P(t)$ can be extended to $t \in [h, 2h)$ so that $\eta_2\|x(t)\|^2 \leq V(x,t) \leq \rho_2\|x(t)\|^2$ and $\dot{V}(x,t) \leq -\nu_2\|x(t)\|^2$ for $t \in [0, h)$ and some $\eta_2, \rho_2, \nu_2 > 0$.

Repeating the lines of arguments above while noting that A, and hence Ψ, is uniformly bounded then results in a differentiable *diagonal P* satisfying $\eta\|x(t)\|^2 \leq V(x,t) \leq \rho\|x(t)\|^2$ and $\dot{V}(x,t) \leq -\nu\|x(t)\|^2$ for all $t \geq 0$ and some $\eta, \rho, \nu > 0$, where $V(x,t) := x^T P(t)x(t)$. This completes the proof. ∎

The results in this section are particularly useful for scalable analysis of large-scale positive systems. Consider the scenario where a diagonal quadratic Lyapunov function has been found for an exponentially stable positive system (14) with $A(t) \in \mathbb{R}^{n \times n}$. Suppose that an additional agent x_{n+1} joins the system, and the new state-space equation with $\hat{x} := [x^T, x_{n+1}]^T$ is given by $\dot{\hat{x}}(t) = \hat{A}(t)\hat{x}(t)$, where $\hat{A}(t)$ is Metzler, the top left $n \times n$ entries of $\hat{A}(t)$ are equal to those

of $A(t)$ and $\hat{A}_{i,(n+1)} \equiv \hat{A}_{(n+1),i} \equiv 0$ for $i = 1, 2, \dots n - 1$. It thus follows that in the search for a diagonal quadratic Lyapunov function for verifying the stability of the appended system, the first $n - 1$ functions in the sum can be chosen to be the same as those for the original system. Efforts can then be concentrated on the last two states/agents x_n and x_{n+1}.

V. Conclusions

This paper establishes the existence of diagonal quadratic Lyapunov functions for uniformly exponentially stable finite-dimensional time-varying linear positive systems in both discrete and continuous times. Despite being non-constructive, the results imply that the search for a Lyapunov function for such classes of systems can be performed in a distributed manner. Future work may involve extending the work to an input-output setting via quadratic storage functions and synthesising decentralised optimal $\mathbf{H}_\infty$ controllers for these systems. Existence of sum-separable Lyapunov functions for nonlinear systems will also be investigated.

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