Minimax Linear Regulator Problems for Positive Systems

with applications to multi-agent synchronization

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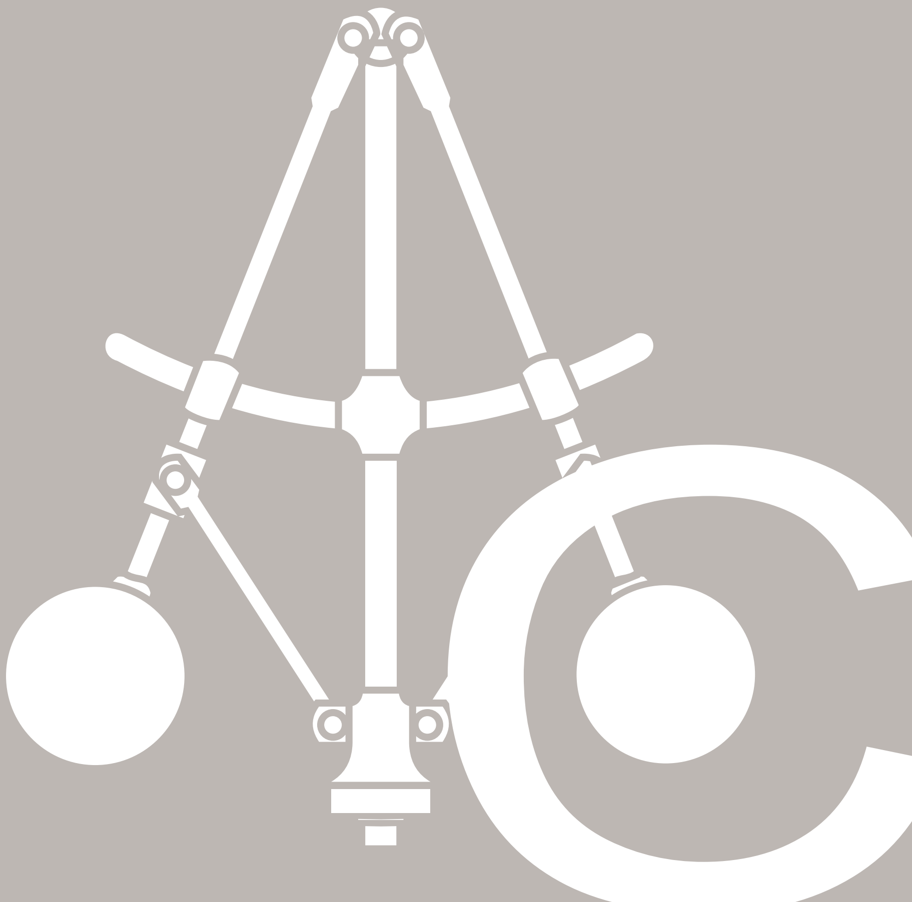

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Abstract

Exceptional are the instances where explicit solutions to optimal control problems are obtainable. Of particular interest are the explicit solutions derived for minimax problems, as they provide a framework for addressing challenges involving adversarial conditions and uncertainties. This thesis presents explicit solutions to a novel class of minimax optimal control problems for positive linear systems with linear costs, elementwise linear constraints in the control policy, and worst-case disturbances. We refer to this class of problems, in the absence of disturbances, as the linear regulator (LR) problem. Two types of worst-case disturbances are considered in this thesis: bounded by elementwise-linear constraints and unconstrained positive disturbances. Using dynamic programming theory, explicit solutions to the Bellman equation (in the discrete-time setting) and the Hamilton-Jacobi-Bellman equation (in the continuous-time setting) are derived for both finite and infinite horizons. For the infinite horizon case, a fixed-point method is proposed to compute the solution of the HJB equation. Furthermore, a necessary and sufficient condition for minimizing the l_1-induced gain of the system is derived and characterized by the disturbance penalty in the cost function of the minimax problem. This condition characterizes the solution of the l_1-induced gain minimization problem and demonstrates that, if a finite solution exists for the minimax problem under the presence of worst-case, unconstrained and positive disturbances, the solution to the minimax setting reduces to that of the LR problem in the absence of disturbances.

This thesis also analyzes the stabilizability and detectability properties of the LR problem. Similar to the Linear-Quadratic Regulator (LQR) problem, the LR problem is shown to facilitate the stabilization of positive systems. A linear programming formulation is introduced to compute the associated stabilizing controller, if one exists. The scalability and practical advantages of this theoretical framework for large-scale applications are demonstrated through its implementation in an optimal voltage control problem for a DC power network and in the management of a large-scale water network.

The second important contribution of this thesis is addressing positive synchronization on undirected graphs for homogeneous discrete and continuous-time positive systems. A static feedback protocol, derived from the Linear Regulator prob-

lem, is introduced. The stabilizing policy is derived by solving the linear programming formulation of the explicit solution to the LR problem under appropriate assumptions. Necessary and sufficient conditions are provided to ensure the positivity of each agent's trajectory for all nonnegative initial conditions. The effectiveness of this approach is illustrated through simulations on large regular graphs with varying nodal degrees.

Throughout the thesis, we demonstrate how the results can be applied to problems over networks with positive dynamics. Our results pave the way for robust networks that maintain stability and optimal performance despite adversarial conditions. By leveraging explicit solutions to minimax optimal control and multi-agent synchronization problems, this work provides a computationally efficient and scalable framework for controlling large-scale systems.

Acknowledgments

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Thank you to Mark Jeeninga for your co-supervision, mathematical enthusiasm, and rigor, as well as your empathy and ability to see the best in people. I look forward to finding and deriving the projects that will next converge to our shared synchronized trajectory!

A special thanks to Claus Führer, my master's thesis supervisor, for introducing me to the fascinating world of control theory and for helping me realize that pursuing a PhD was a natural and exciting next step for me.

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todo el cariño e ilusión (que la caracteriza), esta tesis también se la dedico a mi tía Gloria. Los demás sabéis quiénes sois y tendréis una dedicatoria en este manuscrito.

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Contents

Notation and symbols

Below is a list of frequently used symbols and notations that appear in this thesis.

$\mathbb{R}^n$	n-dimensional real space
$\mathbb{C}^n$	n-dimensional complex space
$\mathbb{Z}^n$	n-dimensional integer space
$\mathbb{R}^n_+$	nonnegative orthant
$\mathbb{R}^{n\times m}$	all $n \times m$ real matrices
$A \geq 0$	nonnegative matrix
$A > 0$	positive matrix
$A \gg 0$	strictly positive matrix
$\mathrm{diag}(a_1, a_2, \dots, a_n)$	diagonal $n \times n$ matrix with a_i its diagonal entries
$A^\top$	transpose of a matrix $A \in \mathbb{R}^{n\times n}$
A^{-1}	inverse of the matrix A
$\det(A)$	determinant of the matrix $A \in \mathbb{R}^{n\times n}$
a_{ij}	(i, j)-th entry of $A \in \mathbb{R}^{n\times n}$
I_n	identity matrix of dimension $n \times n$
$\sigma(A)$	spectrum of $A \in \mathbb{R}^{n\times m}$
$\alpha(A)$	spectral abscissa of $A \in \mathbb{R}^{n\times m}$
$\rho(A)$	spectral radius of $A \in \mathbb{R}^{n\times m}$
$\lambda_i(A)$	i-th eigenvalue of $A \in \mathbb{R}^{n\times n}$
$\mathrm{Re}(\lambda)$	real part of $\lambda \in \mathbb{C}$
$\lvert A\rvert$	entry wise absolute value of a matrix
$\mathbb{1}$	column vector with unit entries
$\mathbb{0}$	matrix or column vector with zero entries
$\otimes$	Kronecker product

$$\mathrm{sign}(x) = \begin{cases} \{+1\} & \text{if } x > 0 \\ \{-1\} & \text{if } x < 0 \\ [-1, +1] & \text{if } x = 0 \end{cases}$$ The set-valued signature of a scalar x.

1

Introduction

Control theory plays a crucial role in maintaining the stability and performance of dynamic systems, especially in the presence of disturbances and uncertainties. Disturbances, such as environmental variations, sensor noise, or unexpected inputs, can disrupt system behavior and reduce performance. Likewise, model uncertainties, such as inaccuracies in the system's mathematical representation or time-varying parameters, pose significant challenges to the reliability of controllers designed for ideal conditions. In this thesis, these issues are addressed by deriving explicit solutions to a novel minimax optimal control problem class, with optimal controllers that not only stabilize the system but also remain effective under worst-case disturbances. The challenges posed by disturbances and uncertainties are particularly pronounced in large-scale systems. Examples include power grids, transportation networks and sensor networks. For such systems, computational feasibility becomes a critical consideration. Solving high-dimensional optimal control problems or implementing complex controllers can become unreasonably expensive or impractical. Therefore, the existence of explicit solutions, which enable efficient computation and implementation even as the system scales, is considered invaluable, especially when working with this type of system dynamics.

An important feature of the dynamical systems studied in this thesis are the positive dynamics, a class of systems where all state variables remain non-negative under non-negative initial conditions and inputs. These systems naturally arise in applications such as population dynamics, epidemiology, chemical processes, and networked systems like traffic or communication networks. Their inherent non-negativity aligns with physical constraints in many real-world scenarios, making their study particularly relevant. Furthermore, positive systems exhibit special mathematical properties that simplify the analysis and design of control strategies. Throughout this thesis, it is demonstrated how these properties make them particularly suitable for deriving scalable and robust control solutions.

The final contribution of this thesis applies the theoretical framework developed to the multi-agent synchronization problem. This problem aims to achieve coordinated behavior among a group of agents, such as robots, drones, or computational nodes, by ensuring that they evolve according to a common dynamic behavior while exchanging information through a communication network. This problem is fundamental to the efficient operation of modern large-scale systems, including autonomous transportation, smart power grids, and decentralized networks, which are essential in today's interconnected society. A static feedback protocol design for large-scale positive multi-agent systems is proposed, based on the LR framework, including a condition for the positivity of all state trajectories, which is particularly relevant in applications where variables represent inherently positive quantities, such as population dynamics, epidemic spread, or resource allocation in decentralized networks.

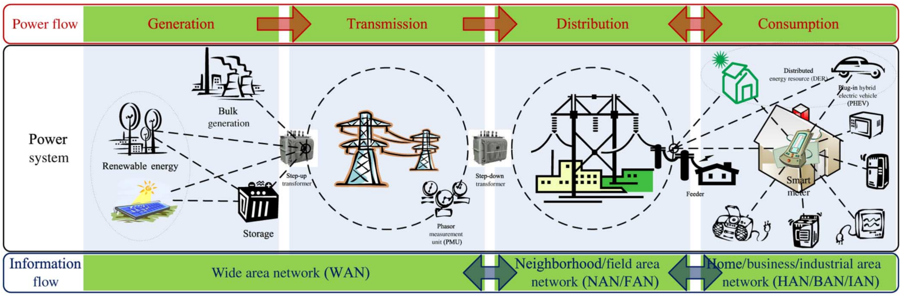
1.1 Minimax problems in control theory

Minimax optimal control problems are widespread across control theory and engineering disciplines, providing a framework for addressing challenges characterized by competitive dynamics and uncertainties. Their formulation often involves solving the Isaacs equation [Isaacs, 1965], a fundamental partial differential equation governing differential games and worst-case control strategies. These problems are particularly relevant in areas such as robust control, multi-agent systems and game theory, where it finds direct application in $\mathcal{H}_\infty$ robust control [Başar, 1984; Başar, 1989; Başar and Bernhard, 1989; Bernhard, 1991; Başar, 1991; Haurie and Zaccour., 2005]. Tackling such problems presents significant difficulties, particularly when we deal with large-scale systems, due to the computational complexity and dimensionality of the associated optimization problems.

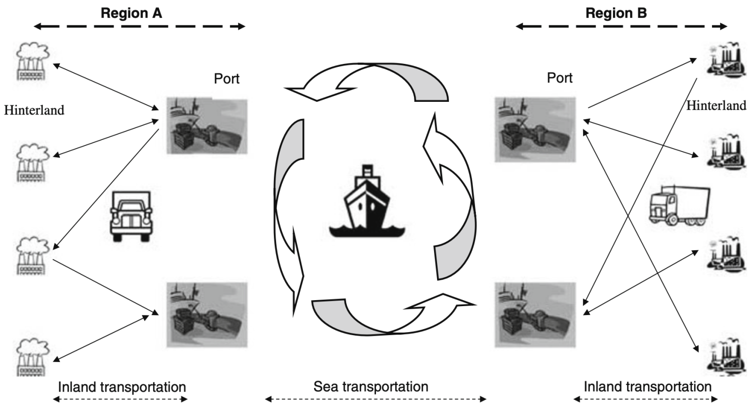
The study of optimal control problems with nonnegative cost dates back to [Strauch, 1966; Blackwell, 1965; Blackwell, 1967]. In particular, this work draws inspiration from the unified dynamic programming framework developed by D. Bertsekas [Bertsekas, 2005; Bertsekas, 2017]. Despite significant advancements in the theory of dynamic programming, obtaining exact solutions often remains challenging. A notable exception is the linear quadratic regulator problem, as presented in the pioneering work of Kalman [Kalman, 1960].

Linear quadratic regulator problem (LQR)

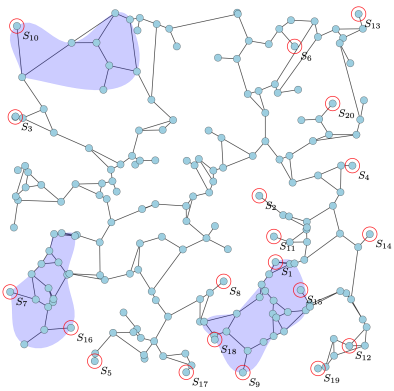
Before reviewing minimax problems from the literature, we will briefly revisit the classical problem of LQR. The linear quadratic regulator (LQR) is a foundational problem in control theory that aims to determine the optimal control policy for a linear dynamic system to minimize a quadratic cost function. This framework achieves stability and performance objectives by balancing state regulation with control effort, using penalties on both states and controls. The remainder of this section reviews the infinite-horizon setting of the LQR problem. For the finite-horizon scenario, we refer the reader to [Bertsekas, 2007, Ch. 3].

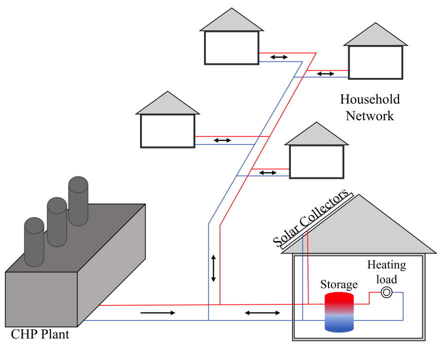


(a) Abstract picture of the smart grid



(b) Transportation network





(c) Large-scale generic water network **(d)** Sketch of a residential heating network

Figure 1.1 Examples of large-scale network systems.

(See List of Figures for image sources.)

The dynamics of the LQR problem are linear and can be described by

$$\begin{cases} x(t+1) = \tilde{A}x(t) + \tilde{B}u(t) & \text{Discrete-Time} \\ \dot{x}(t) = Ax(t) + Bu(t) & \text{Continuous-Time,} \end{cases} \tag{1.1}$$

where x is the n-dimensional vector of states and u is the m-dimensional control variable. The quadratic cost function is given by

$$\begin{cases} \sum_{t=0}^{\infty} g(x(t), u(t)) & \text{Discrete-Time} \\ \int_0^{\infty} g(x(t), u(t))dt & \text{Continuous-Time} \end{cases}$$

with

$$g(x,u) = x^\top Qx + u^\top Ru$$

where $Q \in \mathbb{R}^{n \times n}$ is the state weighting matrix and $R \in \mathbb{R}^{m \times m}$ is the control weighting matrix.

The optimal policy and cost of the LQR problem can be found using several techniques such us dynammic programming (DP) [Bellman, 1957], [Bertsekas, 2005] or Pontryagin's Maximum Principle[1]. The derivations for the solution to the novel class of problems that are presented in this thesis are mainly based on dynamic programming. The following derivations of the explicit solutions to the infinite-horizon LQR problem are further explained in Section 2.3, where the general dynamic programming (DP) theory is reviewed, and the Bellman equation is introduced in more detail.

In the infinite horizon case, the Bellman equation takes the following form

$$\begin{cases} \tilde{J}(x) = \min_u [g(x,u) + \tilde{J}(\tilde{f}(x,u))] & \text{Discrete-Time} \\ 0 = \min_u [g(x,u) + \frac{\partial J}{\partial x}^\top f(x,u)] & \text{Continuous-Time} \end{cases}$$

where $\tilde{f}(x,u) = \tilde{A}x + \tilde{B}u$, $f(x,u) = Ax + Bu$ respectively.

Using as candidate solutions $\tilde{J}(x) = x^\top \tilde{P}x$ and $J(x) = x^\top Px$, the Bellman equation for the LQR problem in infinite horizon results in the following algebraic equations

$$\tilde{P} = \tilde{A}^\top \tilde{P}\tilde{A} - \tilde{A}^\top \tilde{P}\tilde{B}\tilde{K} + Q \tag{1.2}$$

$$0 = PA + A^\top P - PBK + Q \tag{1.3}$$

and the optimal feedback laws $u(t) = \tilde{K}x(t)$ and $u(t) = Kx(t)$ where

$$\tilde{K} = (R + \tilde{B}^\top \tilde{P}\tilde{B})^{-1}\tilde{B}^\top \tilde{P}\tilde{A} \tag{1.4}$$

$$K = R^{-1}B^\top P.$$

[1] Recall that while the Bellman equations in DP theory are both necessary and sufficient conditions for optimality, the maximum principle is only a necessary condition for a controller to be optimal.

An important extension of the LQR problem arises when disturbances are introduced in the system dynamics, leading to a minimax linear quadratic regulator formulation. In this setup, the controller seeks to minimize the cost function, while the disturbances try to maximize it, resulting in a robust control strategy.

EXAMPLE 1—MINIMAX LQR IN INFITINE HORIZON

Assume that the dynamics of the LQR problem (1.1) are affected by the worst-case disturbance $\omega \in \mathbb{R}^l$

$$f(x,u,\omega) = \tilde{A}x + \tilde{B}u + \tilde{F}\omega.$$

Let the quadratic cost function (1.1) be

$$g(x,u,\omega) = g(x,u) - \gamma^2 \omega^\top \omega.$$

Then, substituting the value function $J(x) = x^\top P x$ in

$$\begin{aligned} J(x) &= \min_u \left[g(x,u,\omega) + J(f(x,u,\omega)) \right] \\ 0 &= \min_u \left[g(x,u,\omega) + \frac{\partial J}{\partial x}^\top f(x,u,\omega) \right] \end{aligned}$$

leads to the optimal policies $u = -\tilde{K}x$, $u = -Kx$ where feedback matrices are exactly (1.4) but where $\tilde{P}, P$ in (1.2) and (1.3) this time result from the equations

$$\begin{aligned} &\underbrace{\tilde{P} = \tilde{A}^\top \tilde{P}\tilde{A} - \tilde{A}^\top \tilde{P}\tilde{B}\tilde{K} + Q}_{(1.2)} - \frac{1}{\gamma^2}\tilde{A}^\top \tilde{P}\tilde{F}\tilde{F}^\top \tilde{P}\tilde{A} \\ &\underbrace{0 = A^\top P + PA - PBK + Q}_{(1.3)} - \frac{1}{\gamma^2} PFF^\top P. \end{aligned}$$ □

Minimax LQR problems can be categorized based on the nature of the disturbances and the uncertainty in system parameters. In this thesis, deterministic worst-case disturbances are studied. In these scenarios, the disturbance is modeled as an adversarial input that maximizes the system's cost. The control law is designed to counteract the worst-case effects of this disturbance, ensuring robust performance. This formulation of the LQR problem closely aligns with the principles of $\mathcal{H}_\infty$ control, where the controller seeks to minimize the $\mathcal{H}_\infty$ norm of the closed-loop transfer function to achieve robustness against bounded disturbances [Başar and Bernhard, 1991; Zhou et al., 1996].

Another prominent category involves stochastic minimax problems [Athans, 1971; Başar and Olsder, 1998], which incorporate probabilistic uncertainty. Here, the disturbances are modeled as random processes, and the objective is to minimize the expected worst-case cost. This approach often appears in robust control formulations blending ideas from $\mathcal{H}_\infty$ and Linear Quadratic Gaussian (LQG) control. A more generalized formulation considers parametric uncertainties in the system dynamics. This leads to a robust optimization problem where the control law must stabilize the system and achieve desired performance for all possible values of the uncertain parameters within a given set. Techniques such as linear matrix inequalities (LMIs) are commonly employed in these problems to compute robust stabilizing controllers [Boyd et al., 1994]. Hybrid approaches like mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control have been developed to address scenarios where both robustness (worst-case optimization) and efficiency (mean-square performance) are critical [Doyle et al., 1988]. These methods combine the benefits of robust control with traditional LQR formulations to achieve a balance between robustness and optimality. The probabilistic setting is beyond the scope of this work.

As previously mentioned, this thesis focuses on a novel class of deterministic minimax problems. This setup enables the design of robust controllers that ensure performance and stability, even in worst-case scenarios. Assume that the disturbance ω, which enters the system in Example 1, is unbounded. The output in Example 1 is defined as follows

$$z(t) = x(t)^\top Qx(t) + u(t)^\top Ru(t) - \gamma^2 w(t)^\top w(t).$$

In these scenarios, the LQR problem transforms into an $\mathcal{H}_\infty-$type control formulation, where the goal is to compute an optimal controller that minimizes the worst-case $l_\infty-$induced gain, ensuring robust performance against unbounded external perturbations.

DEFINITION 1
The $l_\infty-$induced gain from disturbance $\omega(t)$ to output $z(t)$ is the maximum energy gain from $\omega(t)$ to $z(t)$ and is defined as

$$\gamma_\infty = \sup_{\omega \neq 0} \frac{\|z(t)\|_\infty}{\|\omega(t)\|_\infty}.$$

It represents the worst-case amplification of disturbances ω through the system to the output z. □

1.2 Problem Formulation

Recent research [Rantzer, 2022; Li and Rantzer, 2024] proposes a very intriguing problem class in discrete-time for positive systems with nonnegative linear costs. This problem setting, which will be introduced next, is denoted throughout this work as *The Linear Regulator problem (LR)*. In both quadratic (LQR) and linear nonnegative cost settings (LR), the optimal cost is determined by an algebraic equation.

For large-scale systems, when the linear quadratic approach is chosen, the algebraic equation is the Riccati equation, and the number of unknown parameters grows quadratically with the state dimension. However, in the LR setting, the growth rate of the unknown parameters in the algebraic equation is linear with respect to the dimension of the state. This difference becomes particularly significant when the state dimension becomes large.

The Linear Regulator Problem (LR)

The system dynamics for the LR problem in [Rantzer, 2022] are linear and time-invariant and can be described by

$$x(t+1) = Ax(t) + Bu(t) \tag{1.5}$$

where $x \in \mathbb{R}^n_+$ is the $n-$dimensional vector of states and $u \in \mathbb{R}^m$ the $m-$dimensional control variable, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Furthermore, it is assumed that the control policy is subject to elementwise linear constraints, given by $|u| \leq Ex$ with $E \in \mathbb{R}^{m\times n}_+$. In addition, to guarantee the positivity of the closed-loop system's dynamics, it is required that $A \geq |B|E$.

The cost function of the LR problem is linear and is defined as

$$J = \sum_{t=0}^{\infty} \left[s^\top x(t) + r^\top u(t) \right]$$

where $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$ are such that $s \geq |E^\top r|$. This assumption guarantees that the cost $g(u,x) = s^\top x + r^\top u$ is nonnegative, and the problem is well-posed [Li and Rantzer, 2024]

$$f(x,u) \in X, \quad \forall x \in X, \quad u \in U(x).$$

Moreover, both s and r influence the detectability and stabilizability properties of the problem class. This is further analyzed in Chapter 4.

Similar to the LQR problem, the objective is to minimize the cost with respect to $\{u(t)\}_{t=0}^{\infty}$, resulting in the optimal controller

$$K \in \begin{bmatrix} \mathrm{sign}(r_1 + p^\top B_1)E_1 \\ \vdots \\ \mathrm{sign}(r_m + p^\top B_m)E_m \end{bmatrix}. \tag{1.6}$$

where p satisfies

$$p = s + A^\top p - E^\top |r + B^\top p|. \tag{1.7}$$

Equation (1.7) serves as the counterpart to the discrete algebraic Riccati equation in the LQR problem setting. Notably, in the LR framework, the number of unknown parameters corresponds to the dimension of the vector p whereas in the LQR framework, it corresponds to the dimension of the matrix P. Consequently, as the system dynamics scale, the number of unknown parameters in the LR setting grows linearly, offering a significant advantage in scalability.

The continuous-time formulation of this setting is derived in Chapter 3. However, the primary objective of this thesis is to extend the entire linear regulator problem class to encompass its worst-case minimax control framework in both discrete and continuous time. This expanded formulation is referred to as the Minimax Linear Regulator Problem throughout the document.

The Minimax Linear Regulator Problem

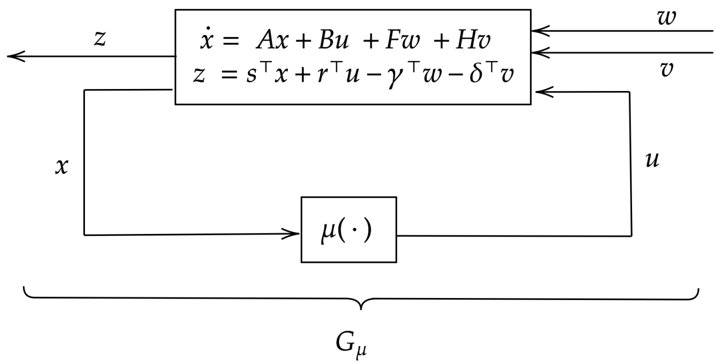


Figure 1.2 Block Diagram of the closed loop system dynamics.

The setup for the minimax linear regulator problem is represented in Figure 1.2 for the continuous-time scenario. An analogous representation can be obtained for the discrete-time setting. The diagram can be described as the continuous-time linear time invariant state-space model

$$G_\mu : \begin{cases} \dot{x} = Ax + B\mu(x) + Fw + Hv \\ z = s^\top x + r^\top \mu(x) - \boldsymbol{\gamma}^\top w - \boldsymbol{\delta}^\top v. \end{cases} \tag{1.8}$$

Denote

$$\begin{cases} \dot{x} & \text{Continuous-time case} \\ x(t+1) & \text{Discrete-time case} \end{cases}$$

where x is the n-dimensional vector of state variables, u the m-dimensional control variable, w the l-dimensional disturbance, v the c-dimensional disturbance, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times l}$, $H \in \mathbb{R}^{n\times c}$, μ is any, potentially nonlinear, control policy, z is a target output variable representing the system's performance, $s \in \mathbb{R}^n$ and $r \in \mathbb{R}^m$. Assuming zero initial conditions, the system dynamics (1.8) can be seen as an operator G_μ from the disturbance w to the output z.

This thesis considers two distinct types of deterministic disturbances:

1. *Disturbances bounded by elementwise linear constraints*:

$$|v(t)| \leq Gx(t), \quad G \in \mathbb{R}_+^{c\times n}$$

   which can be interpreted as model errors.

2. *Unconstrained nonnegative disturbances*:

$$w(t) \geq 0$$

   which can be understood as external disturbances that are inherently nonnegative.

The cost function of the minimax linear regulator problem is also linear.

$$J = \begin{cases} \sum\limits_{t=0}^{T} \left[s^\top x(t) + r^\top u(t) - \boldsymbol{\gamma}^\top w(t) - \delta^\top v(t)\right] & \text{Discrete-Time} \\ \int\limits_{0}^{T} \left[s^\top x(\tau) + r^\top u(\tau) - \boldsymbol{\gamma}^\top w(\tau) - \delta^\top v(\tau)\right] d\tau & \text{Continuous-Time} \end{cases}$$

where $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\boldsymbol{\gamma} \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^c$. The objective is to minimize the cost with respect to $u(t)$ while maximizing with respect to $w(t)$ and(or) $v(t)$.

Observe that, because we are dealing with positive systems, linear cost functions are particularly suitable, as variables such as flow, population, or inventory are inherently non-negative. In these scenarios, penalizing deviations directly with a linear term aligns naturally with the system's physical constraints and provides a straightforward interpretation of costs related to the use of resources or penalties for exceeding limits. In contrast, linear quadratic cost functions, which penalize the square of deviations and control efforts, emphasize dynamic trade-offs and smooth performance. Quadratic terms inherently penalize larger deviations more heavily than smaller ones, which may not always align with the characteristics of certain positive systems. For instance, in systems like water distribution networks or supply chains, the linear cost captures more accurately the proportional relationship between flow rates or inventory levels and the associated penalties or resource expenses.

While both cost functions have their advantages, the linear cost function can be particularly well-suited for positive systems due to its compatibility with the systems' natural properties. Additionally, its linear structure simplifies computation, enhancing scalability in large-scale settings.

Objective

This thesis focuses on extending the linear regulator framework to the minimax setting, addressing two distinct types of disturbances. Building on the discrete-time LR framework presented in [Rantzer, 2022], solutions for the continuous-time setting are derived and investigated. The contributions are motivated and illustrated by their computational efficiency and applicability to large-scale systems. In Chapter 3, dynamic programming is employed to derive closed-form solutions to the Bellman and Hamilton-Jacobi-Bellman equations within the multi-disturbance minimax linear regulator framework. Furthermore, the relationship between the system's l_1-induced norm and the existence of finite solutions to the optimal control problem is thoroughly examined (see Chapter 3). Subsequently, methods for solving the Bellman-HJB equation are explored. In the multi-disturbance setting, standard value iteration and a fixed-point method are proposed. For scenarios where the system is influenced solely by unconstrained disturbances, a linear programming formulation is established and discussed in Chapter 4. Finally, the application of this framework is demonstrated in the synchronization of positive multi-agent systems. A synchronization protocol is proposed for both continuous and discrete-time homogeneous agents operating within upper and lower-bounded families of undirected graphs (Chapter 5).

1.3 Outline and contributions

The outline of this thesis and its key contributions can be summarized as follows:

Chapter 2: Preliminaries.
This chapter presents the technical preliminaries on positive systems, dynamic programming, graph theory, and the synchronization problem for multi-agent systems that are relevant to this thesis. No contributions of the author appear in this chapter.

Chapter 3: Minimax Linear Regulator for Positive Systems.
The first contribution of this thesis introduces a novel class of worst-case minimax optimal control problems. In [Rantzer, 2022], a discrete-time framework with linear cost, linear dynamics, and elementwise constraints on the control policy is introduced. This chapter extends the problem to a multi-disturbance worst-case minimax setting. Solutions for the extended problem are derived for both discrete and continuous-time settings, covering finite and infinite horizons. These solutions are obtained using dynamic programming, demonstrating that the optimal control policy, among all admissible policies, is linear. In the discrete-time setting, the policy is computed through standard value iteration, while in the continuous-time case, a fixed-point method is proposed to solve the Hamilton-Jacobi-Bellman equation. Additionally, the feedback matrix of the optimal controller inherits the sparsity structure of the constraint matrix in the problem formulation. This property allows for structural constraints in controller design and facilitates application to large-scale systems. An analysis of the l_1-induced gain of the system and its characterization through the disturbance penalty in the cost function is introduced. Examples on the scalability of this theoretical framework are provided through an optimal voltage control problem in a DC power network and a line-shaped water flow network.

The results in Chapter 3 have been published/submitted to:

- A. Gurpegui, E. Tegling, A. Rantzer. Minimax Linear Optimal Control of Positive Systems. *IEEE Control Systems Letters and American Control conference 2024 (joint submission),* Vol. 7, pp. 3920 - 3925, Dec 2023.
- A. Gurpegui, E. Tegling, A. Rantzer. A Minimax Optimal Controller for Positive Systems. Presented at *26th International Symposium on Mathematical Theory of Networks and Systems (Extended Abstract [Gurpegui et al., 2024]), Cambridge, UK, Aug 2024.*
- A. Gurpegui, M. Jeeninga, E. Tegling, A. Rantzer. Minimax Linear Optimal Control of Positive Systems. Submitted to *IEEE Transactions on Automatic control.*

linear regulator problems for positive systems

Chapter 4: Linear Programming Formulation of the LR Problem.
This chapter studies the stabilizability and detectability properties of the linear regulator (LR) problem and establishes a priori detectability conditions for the system. Additionally, a linear programming formulation is proposed to solve the Bellman equation for the LR problem in continuous time, under suitable assumptions. This formulation is also applicable to the minimax linear regulator setting with unconstrained disturbances, as the resulting solution, provided a finite solutions exist, coincides with the optimal setup in the absence of disturbances. The latter is a result presented in Chapter 3.

Chapter 4 is mainly based on

- A. Gurpegui, M. Jeeninga, E. Tegling, A. Rantzer. Minimax Linear Optimal Control of Positive Systems. Submitted to *IEEE Transactions on Automatic control.*

Chapter 5: Synchronization of Positive Multi-agent Systems.
Inspired by the ARE-based synchronization protocol presented in [Saberi et al., 2022], Chapter 5 introduces a static feedback protocol derived from the algebraic equation of the Linear Regulator problem. This protocol achieves positive synchronization in continuous and discrete-time homogeneous multi-agent systems operating on undirected graph families with appropriate eigenvalue bounds. Additionally, the chapter provides sufficient and necessary conditions to guarantee that each agent's trajectory remains positive for all nonnegative initial conditions.

The results in Chapter 5 have been submitted to:

- A. Gurpegui, M. Jeeninga, E. Tegling, A. Rantzer. A. Gurpegui, E. Tegling, A. Rantzer. Linear Regulator-Based Synchronization of Positive Multi-Agent Systems. Submitted to *the European Control Conference*, Thessaloniki, Greece, June 2025.

- A. Gurpegui, M. Jeeninga, E. Tegling, A. Rantzer. A. Gurpegui, E. Tegling, A. Rantzer. Linear Regulator Based Synchronization of Positive Multi-Agent Systems in Discrete-Time. Submitted to *the 10th IFAC Conference on Networked Systems*, Hong Kong, China, June 2025.

Chapter 6: Conclusions. In this chapter, we summarize the key findings of this thesis and discuss their implications, emphasizing the computational tractability of the proposed theoretical framework for large-scale positive systems and its applications to multi-agent system synchronization. Additionally, we examine the limitations of our results and suggest potential directions for future research.

2

Preliminaries

In this chapter, the mathematical background relevant to this thesis is reviewed and organized into sections that form the foundation for developing the contributions of this thesis. It is important to note that this chapter does not contain any original contributions by the author.

The first section focuses on positive system theory, beginning with the review of the Perron Frobenius theorem for nonnegative and strictly positive matrices. This theory forms the basis for understanding the structural properties of positive systems and their implications in optimal control. The second section addresses dynamic programming theory, presenting key results and extending them to minimax problems in both discrete and continuous time. These provide the basis for analyzing the minimax problem class and are explicitly used in the proof of the results of Chapter 3. Building on these foundations, the chapter reviews key concepts for the introduction of the LR-based synchronization protocol. It continues with a background section on dynamical systems over general graphs, providing the notation and necessary insights for describing dynamical systems in networked structures. This is followed by an introduction to the multi-agent system dynamics that are presented in this thesis. The chapter concludes by reviewing the state synchronization problem, presenting standard definitions and the theoretical results required to understand the derivation of the synchronization protocol introduced in Chapter 5.

2.1 Definitions for positive system theory

Some of the definitions and conditions of positivity for linear system presented later in the thesis require the notion of positivity of matrices and vectors.

DEFINITION 2—NONNEGATIVE MATRIX
A matrix $A \in \mathbb{R}^{n\times m}$ is nonnegative, $A \geq 0$, (equivalently $A \in \mathbb{R}^n_+$) if all its entries are nonnegative $a_{ij} \geq 0$. □

DEFINITION 3—POSITIVE MATRIX
A matrix that $A \in \mathbb{R}^{n\times m}$ is positive, $A > 0$, if all its entries are nonnegative and there exists at least one entry a_{ij} that is strictly positive $a_{ij} > 0$. □

DEFINITION 4—STRICTLY POSITIVE MATRIX
A matrix that $A \in R^{n\times m}$ is strictly positive, $A \gg 0$, if all its entries are strictly positive $a_{ij} > 0$. □

DEFINITION 5—POSITIVE (NEGATIVE) DEFINITE MATRIX
A matrix $A \in \mathbb{R}^{n\times m}$ is said to be positive(negative) definite $A \succ 0$ ($A \prec 0$) if $x^\top Ax > 0$ ($x^\top Ax < 0$) for all $x \in \mathbb{R}^n$ such that $x \neq 0$. □

DEFINITION 6—POSITIVE (NEGATIVE) SEMI-DEFINITE MATRIX
A matrix $A \in \mathbb{R}^{n\times m}$ is said to be positive(negative) semi-definite, $A \succeq 0$ ($A \preceq 0$) if $x^\top Ax \geq 0$ ($x^\top Ax \leq 0$) for all $x \in \mathbb{R}^n$ such that $x \neq 0$. □

Along these lines, we say that a matrix $A \in \mathbb{R}^{n\times m}$ is strictly greater than a matrix $B \in \mathbb{R}^{n\times m}$, denoted by $A \gg B$ if all the elements a_{ij} are greater than the corresponding elements of b_{ij}, i.e., $a_{ij} > b_{ij}$ for all i, j. If all the elements of A are greater than or equal to the corresponding elements of B, but at least one element of A is greater than the corresponding element in B, we say that A is greater than B denoted by $A > B$. Similarly, if all the elements of A are greater than or equal to the corresponding elements of B, then we say that A is greater than or equal to B, denoted by $A \geq B$, which is also satisfied when $A = B$. Analogous definitions and notations can be given also for $n-$dimensional vectors with $n \geq 0$. When dealing with scalars, note that strict positivity $a \gg 0$ coincides with positivity $a > 0$.

2.2 Positive systems

Positive dynamics have gained attention in control theory literature because of the many technological and physical phenomena that can be captured by positive dynamics. Classical books on the topic are [Berman, 1994] and [Luenberger, 1979]. In the latter, David Luenberger in 1979 devotes a chapter to positive dynamical systems, which is considered by many the initiation of "Positive System Theory." Significant research has been conducted to represent natural extensions of the class of positive systems, for instance positive systems with delays [Ebihara et al., 2017], positive switched systems [Blanchini et al., 2015a] and monotone systems [Smith, 1995]. Positive systems' theory has been useful to describe dynamical systems in a wide range of applications, such as, biology, ecology, physiology and pharmacology [Carson and Cobelli, 2001; Coxson and Shapiro., 1987; Haddad and Chellaboina., 2005; Haddad et al., 2010; Hernandez-Vargas et al., 2011; Jacquez, 1974], thermodynamics [Blanchini et al., 2015b; Haddad et al., 2010], epidemiology [Rami et al., 2013; Hernandez-Vargas and Middleton, 2013; Moreno et al., 2002], econometrics [Nieuwenhuis, 1986], filtering and charge routing networks or power systems [Benvenuti and Farina., 1996; Benvenuti and Farina, 2001; Benvenuti et al., 2001]. One of the main advantages of positive systems is that stability can be verified using linear Lyapunov functions [Blanchini and Giordano, 2014], [Rantzer, 2015], making this class of systems more tractable in a large scale setting because of their computational scalability [Rantzer and Valcher, 2021]. The purpose of this section is to establish a theoretical foundation that elucidates why positive system theory serves as a fundamental component of the theoretical framework presented in this thesis.

The Perron Frobenius theorem

Among the various properties of positive systems, the one associated with the dominant mode is particularly relevant, as it often enables a significant simplification of stability analysis. This property is expressed through a series of results known as the Perron-Frobenius theorems. The original work by Perron, in which these results for positive matrices were first introduced, can be found in [Perron, 1907]. The Perron-Frobenius theorems are presented here as stated in [Fiedler, 1986].

> **Perron Theorem for strictly positive matrices** *If A is a strictly positive square matrix, then $\rho(A)$ is a positive eigenvalue of A and there is only one linearly independent eigenvector belonging to the eigenvalue $\rho(A)$. Moreover, this eigenvector may be chosen to be positive [Fiedler, 1986, Lem. Perron].*

Frobenius extended Perron's results to nonnegative irreducible matrices in [Frobenius, 1912].

DEFINITION 7—REDUCIBLE AND IRREDUCIBLE MATRIX
A matrix $A \in \mathbb{R}^{n\times n}$ is reducible if there exists a permutation matrix P such that $PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where $A_{11},\ A_{22}$ are square matrices. A matrix $A \in \mathbb{R}^{n\times n}$ is irreducible if it is not reducible. □

> **Perron Frobenius Theorem for nonnegative matrices** *Let A be a nonnegative square matrix order $n > 0$. Then $\rho(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector of A belonging to this eigenvalue [Fiedler, 1986, Thm. 4.11].*
> *If A is irreducible, then $\rho(A)$ is a simple positive eigenvalue of A and there is a positive eigenvector belonging to the eigenvalue $\rho(A)$. No nonnegative eigenvector belongs to any other eigenvalue of A [Fiedler, 1986, Thm. 4.8].*

REMARK 1
The eigenvalue $\rho(A)$ is commonly known as the Perron root of A, while the corresponding eigenvector is referred to as the Perron vector of A. □

The remainder of the chapter is dedicated to reviewing the main results from positive systems theory presented in [Farina and Rinaldi, 2000; Kaczorek, 2002], which are relevant to the optimal control framework proposed in this thesis. The results for continuous-time positive systems are introduced first, followed by those for discrete-time positive systems.

Continuous-time positive systems

Consider the continuous-time linear system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, $u(t) \in \mathbb{R}^m$ the input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Linear systems have two definitions of positivity: external and internal. In this thesis, we primarily focus on internal positivity, which we will refer to simply as positivity. For context, we also review the notion of external positivity.

DEFINITION 8—CONTINUOUS-TIME EXTERNALLY POSITIVE LINEAR SYSTEM
The continuous-time linear system (2.1) is called externally positive if and only if for every nonnegative input $u(t) \in \mathbb{R}^m_+$ and initial state $x_0 = 0$ its output is nonnegative for all $t \geq 0$. □

It is important to emphasize that the positivity of a linear system depends on the basis chosen to represent inputs and outputs. In general, determining whether a basis exists that ensures a given linear system is positive is not straightforward. However, in most practical applications, the natural choice of basis also turns out to be the appropriate one.

LEMMA 1
A continuous-time linear system (2.1) is externally positive if and only if its impulse response is nonnegative [Kaczorek, 2002, Thm 2.1]. □

DEFINITION 9—METZLER MATRIX
A square matrix $A \in \mathbb{R}^{n\times n}$ is Metzler if $A + \alpha I \geq 0$ for some $\alpha \in \mathbb{R}$. □

Next, we formally introduce the definition of a continuous-time (internally) positive system, which is the notion of positivity for linear systems that is applied throughout our results in this thesis.

DEFINITION 10—CONTINUOUS-TIME POSITIVE LINEAR SYSTEM
The system (2.1), (2.2) is called (internally) positive system if and only if for any nonnegative initial state $x_0 \in \mathbb{R}^n_+$ and for every nonnegative input $u \in \mathbb{R}^m_+$, its state and output are nonnegative. □

Note that while the definition of positivity requires the output to be nonnegative for every nonnegative initial state, the definition of external positivity only requires the output to be nonnegative when $x_0 = 0$.

COROLLARY 2
Every continuous-time (internally) positive system is also externally positive. □

Observe that the reverse is not true. It is possible to determine formally whether a linear system is positive through the following results.

LEMMA 3
The continuous-time system (2.1), (2.2) is internally positive if and only if the matrix A is a Metzler, $B \geq 0$, $C \geq 0$, $D \geq 0$ [Farina and Rinaldi, 2000, Thm. 2]. □

EXAMPLE 2—[KACZOREK, 2002]
Given the circuit in Figure 2.1 with known resistances R_1, R_2, R_3, inductances L_1,

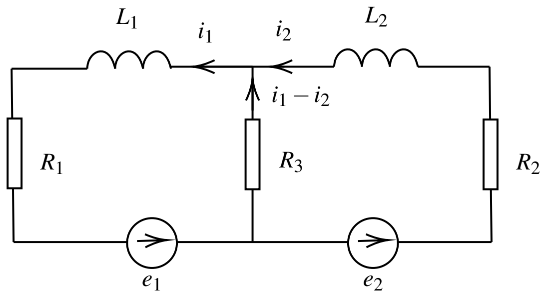


Figure 2.1

L_2 and source voltages $e_1 = e_1(t)$, $e_2 = e_2(t)$. The currents $i_1 = i_1(t)$, $i_2 = i_2(t)$ in the inductances are chosen as the state variables and the output is $y = y(t) = \begin{bmatrix} R_1 i_1 & R_2 i_2 \end{bmatrix}^\top$. Using the Kirchhoff law we obtain the equations

$$R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{d_{i_1}}{dt} = e_1$$
$$R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{d_{i_2}}{dt} = e_2.$$

The equations can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}; \; y = C \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

where

$$A = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}; \;\; B = \begin{pmatrix} \frac{1}{l_1} & 0 \\ 0 & \frac{1}{L_2} \end{pmatrix}; \;\; C = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

It follows that A is Metzler and B and C has nonnegative entries. Thus, the system is (internally) positive. □

Stability of continuous-time positive systems

Consider a continuous-time positive system described by

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{2.3}$$

where $A \in \mathbb{R}^{n\times n}$ is Metzler. The solution of (2.3) is

$$x(t) = e^{At}x_0. \tag{2.4}$$

DEFINITION 11
The positive system (2.3) is asymptotically stable if and only if the equation (2.4) satisfies that

$$\lim_{t\to\infty} x(t) = 0 \ \ \forall x_0 \in \mathbb{R}^n_+.$$ □

Next, necessary and sufficient conditions for the asymptotic stability of a continuous-time linear system are reviewed.

LEMMA 4
The positive system (2.3) is asymptotically stable if and only if all eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, such that $\det[I_n\lambda_i - A] = 0$ for all $i = 1, \ldots, n$ of the Metzler matrix A have negative real parts [Kaczorek, 2002, Thm. 2.7]. □

LEMMA 5
The positive system (2.3) is asymptotically stable if and only if all coefficients α_i $i = 0, 1, \ldots, n-1$ of the (characteristic) polynomial

$$\det[sI - A] = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

are positive i.e., $\alpha_i > 0$ [Kaczorek, 2002, Thm. 2.10]. □

LEMMA 6
The positive system (2.3) is unstable if at least one diagonal entry of the matrix A is positive [Kaczorek, 2002, Thm. 2.10]. □

DEFINITION 12—HURWITZ
A square matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz if all its eigenvalues have strictly negative real parts. In other words, $\mathrm{Re}(\lambda(A)) < 0$, for all $\lambda \in \mathrm{spec}(A)$. □

COROLLARY 7
A linear system of the form (2.3) is asymptotically stable if its state matrix $A \in \mathbb{R}^{n\times n}$ is Metzler and Hurwitz. □

The equilibrium points of asymptotically stable continuous-time systems are now characterized, as they allow for the derivation of additional stability properties for continuous-time positive systems.

DEFINITION 13
A vector x^* satisfying the condition $\dot{x} = 0$ is called the equilibrium point of the continuous-time system (2.3). □

Consider the continuous-time single-input system

$$\dot{x} = Ax + bu \tag{2.5}$$

with a constant positive input $\hat{u} > 0$. Let x^* be the equilibrium point of the system. Then (2.5) gives

$$Ax^* + b\hat{u} = 0. \tag{2.6}$$

By Lemma 4, if the positive system (2.5) is asymptotically stable then all eigenvalues of the system have negative real parts and $\det(A) \neq 0$. In this case the equilibrium point is given by

$$x^* = -A^{-1}b\hat{u}.$$

LEMMA 8
The equilibrium point x^* of an asymptotically stable positive system (2.5) for a constant input $\hat{u} > 0$ is positive, i.e. $x^* > 0$ if $b > 0$. Similarly, x^* is strictly positive if b is also strictly positive [Kaczorek, 2002, Thm. 2.11]. □

The properties of continuous-time positive systems that are relevant to this thesis, based on the theory reviewed in this subsection, are summarized in the following corollary.

COROLLARY 9— [RANTZER AND VALCHER, 2018B]
Let $A \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then the following properties are equivalent:

(i.) A is Hurwitz.

(ii.) There exists a vector $x \gg 0$ such that $Ax \ll 0$.

(iii.) There exists a vector $p \gg 0$ such that $p^\top A \ll 0$.

(iv.) There exists a diagonal matrix $P \succ 0$ such that $A^\top P + PA \prec 0$.

(v.) $-A^{-1}$ exists and has nonnegative entries. □

EXAMPLE 3
Consider the single input system system (2.5) with

$$A = \begin{bmatrix} -a & c \\ d & -b \end{bmatrix}; \qquad B = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}; \qquad \hat{u} = \gamma,$$

where $a, b, c, d, \beta_1, \beta_2, \gamma > 0$ and $ab - cd > 0$. Since

$$\det[sI - A] = s^2 + (a+b)s + ab - dc = 0,$$

by Lemma 5 the system is asymptotically stable. Moreover, using equation (2.2) it is possible to obtain

$$\begin{aligned} x^* = -A^{-1}b\hat{u} &= -\begin{bmatrix} -a & c \\ d & -b \end{bmatrix}^{-1} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \gamma = \frac{1}{ab-dc} \begin{bmatrix} b & c \\ d & a \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \gamma \\ &= \frac{1}{ab-dc} \begin{bmatrix} b\beta_1 + c\beta_2 \\ d\beta_1 + a\beta_2 \end{bmatrix} \gamma \gg 0. \end{aligned}$$ □

Discrete-time positive systems

Consider the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t) \tag{2.7}$$
$$y(t) = Cx(t) + Du(t) \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at the discrete time instant $t \in \mathbb{Z}_+$, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Analogously to the continuous-time setting two definitions of positive linear systems are introduced for the discrete-time setting.

DEFINITION 14—DISCRETE-TIME EXTERNALLY POSITIVE LINEAR SYSTEM
The discrete-time linear system in equation (2.7) is called externally positive if and only if for every nonnegative input sequence $u(t) \in \mathbb{R}^m_+$, $t \in \mathbb{Z}_+$ and initial state $x_0 = 0$ its output is nonnegative for all $t \in \mathbb{Z}_+$. □

LEMMA 10
The system (2.7) is externally positive if and only if its impulse response matrix is nonnegative [Kaczorek, 2002, Thm. 2.2]. □

Next, we introduce the definition of (internally) positive discrete-time systems.

DEFINITION 15—DISCRETE-TIME POSITIVE LINEAR SYSTEM
The system (2.7), (2.8) is called (internally) positive system if for every $x_0 \in \mathbb{R}^n_+$ and any nonnegative input sequence $u(t) \in \mathbb{R}^m_+$, its state and output are nonnegative, i.e., $x(t) \in \mathbb{R}^n_+$ $y(t) \in \mathbb{R}^p_+$ for all $t \in \mathbb{Z}_+$. □

LEMMA 11
The system (2.7), (2.8) is positive if and only if $A \geq 0$, $B \geq 0$, $C \geq 0$ and $D \geq 0$ [Kaczorek, 2002, Thm. 2.6]. □

It can be shown, analogous to the continuous-time case, that every discrete-time (internally) positive system is also externally positive.

Stability of discrete-time positive systems

Consider a discrete-time positive system described by the equation

$$x(t+1) = Ax(t), \ t \in \mathbb{Z}_+ \tag{2.9}$$

with $A \in \mathbb{R}_+^{n \times n}$ and $x(0) = x_0$. The solution of equation (2.9) has the form

$$x(t) = A^t x_0 \tag{2.10}$$

DEFINITION 16
The positive system in Equation (2.9) is called asymptotically stable if and only if the solution (2.10) satisfies the condition

$$\lim_{t \to \infty} x(t) = 0, \ \ \forall x_0 \in \mathbb{R}_+^n$$ □

Next, necessary and sufficient conditions for the asymptotic stability of a discrete-time system are given.

LEMMA 12
The positive system (2.9) is asymptotically stable if and only if all eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $A \in \mathbb{R}_+^n$, satisfying $\det(\lambda_i I_n - A)$ for $i = 1, \ldots, n$, have moduli less than 1, i.e. $|\lambda_i| < 1$ for $i = 1, \ldots, n$ [Kaczorek, 2002, Thm. 2012]. □

LEMMA 13
The positive system (2.9) is asymptotically stable if and only if all the coefficients δ_i of the (characteristic polynomial)

$$\det[zI_n - A + I] = z^n + \delta_{n-1} z^{n-1} + \cdots + \delta_1 z + \delta_0$$

are positive $\delta_i > 0$. □

LEMMA 14
The positive system (2.11) is unstable if at least one diagonal entry of the matrix A is greater than 1 [Kaczorek, 2002, Thm. 2.15]. □

DEFINITION 17—SCHUR MATRIX
A matrix $A \in \mathbb{R}^{n \times n}$ is called Schur if all its eigenvalues lie strictly inside the unit disk in the complex plane, i.e. $|\lambda| < 1$ for all eigenvalues λ of A. □

COROLLARY 15
If the state matrix A of a linear system of the form (2.9) is nonnegative and Schur, then the system is positive and asymptotically stable. □

Similarly to the continuous-time case, the equilibrium points of a linear system are also utilized to derive stability properties of discrete-time positive systems.

DEFINITION 18
A state vector x^* satisfying $x^* = Ax^*$ is called the equilibrium point of the discrete time system (2.9). □

Consider the single input discrete-time system

$$x(t+1) = Ax(t) + bu(t) \tag{2.11}$$

with constant input $\hat{u} > 0$. Let x^* be the equilibrium point of the system (2.11). Then

$$x^* = Ax^* + b\hat{u}. \tag{2.12}$$

By Lemma 12, if the positive system (2.11) is asymptotically stable, then all eigenvalues have moduli less than one. In this case, $\det[I - A] \neq 0$ and the equilibrium point of the system is

$$x^* = [I - A]^{-1} b\hat{u}. \tag{2.13}$$

LEMMA 16
The equilibrium vector x^* of the asymptotically stable positive system (2.11) with a constant input $\hat{u} > 0$ is positive if $b > 0$. Similarly, it is strictly positive, $x^* \gg 0$, if $b \gg 0$ [Kaczorek, 2002, Thm. 2.16]. □

Based on the theory reviewed in this subsection, the following corollary summarizes the key properties of discrete-time positive systems relevant to this thesis.

COROLLARY 17— [RANTZER AND VALCHER, 2018B]
Let $A \in \mathbb{R}^{n \times n}$ be a nonnegative matrix, the following properties are equivalent:

(i) A is Schur.

(ii) There exists $x \gg 0$ such that $Ax \ll x$.

(iii) There exists $p \gg 0$ such that $p^\top A \ll p^\top$.

(iv) There exists a diagonal $P \succ 0$ such that $A^\top PA - P \prec 0$.

(v) $(I_n - A)^{-1}$ exists and has nonnegative entries. □

EXAMPLE 4—[FARINA AND RINALDI, 2000]
Consider a set of firms $i = 1, \ldots, n$ in the stock market, and denote by b_i and a_{ij} the shares of firm i held by an outside shareholder and by firm j, $j = 1, \ldots, n$. This is

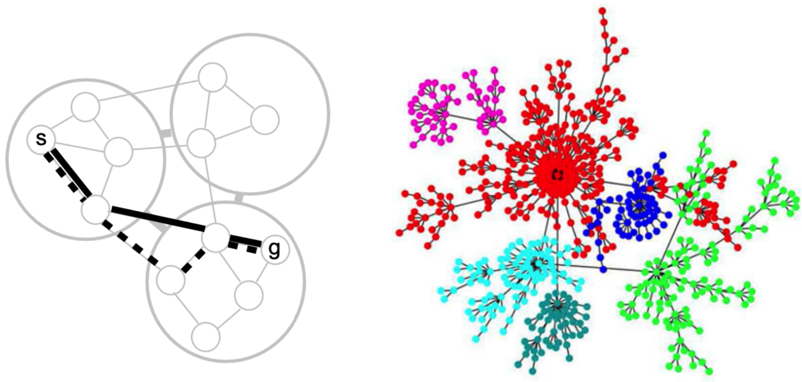
$$b_i + \sum_{j=1}^{n} a_{ij} = 1.$$

and the systems dynamics are given by $x(t+1) = Ax(t) + b$. Note that, because A is nonnegative and the sum of every row of A is $1 - b_i$, the Perron eigenvalue of A is smaller than one. Thus, A is asymptotically stable. Let x^* be the equilibrium vector of the system representing the influence level of the outside shareholders on each firm, i.e. $x_i^* = b_i + a_{i1}x_1^* + \cdots + a_{in}x_n^*$. It is direct that $x^* \gg 0$. □

2.3 Dynamic programming

In the 1950s, the RAND Corporation, with support from the U.S. Air Force, was interested on the multi-stage decision making problem [Åström and Kumar, 2014]. Richard Bellman, a mathematician attracted to this problem, pioneered the field of dynamic programming [Bellman, 1957]. Concurrently, Rufus Isaacs was researching dynamic continuous-time two-person zero-sum games. His work resulted in the Isaacs equation, a two-sided generalization of the Hamilton–Jacobi equation. This differential game theory was initially applied to military strategy problems [Zachrisson, 1964] and later became influential in robust control [Başar and Bernhard, 1991].

Dynamic programming was originally developed to address complex multi-stage decision processes and has since become a cornerstone in various fields, including control theory, operations research, artificial intelligence, and economics. In control theory, it plays a crucial role in solving optimal control problems [Bertsekas, 2005; Bertsekas, 2017; Bertsekas, 2023], where the objective is to determine a sequence of control inputs that optimize a given performance criterion over time. It has found wide applicability across diverse physical systems such as large-scale systems [Chen et al., 2023; Zhao et al., 2023; Jiang and Jiang, 2012] or robotics [Kang and McKay, 1986; Lee, 1995].



(a) **(b)**

Figure 2.2 Examples of dynammic programming for path planning (a) and clustering (b)

At its core, dynamic programming involves breaking down a large optimization problem into smaller, manageable subproblems, solving each subproblem only once, and storing the results to avoid redundant computations. This methodology is grounded in the *principle of optimality*, which was developed by R. Bellman and is central to dynamic programming theory.

The Principle of Optimality [Bellman, 1957]: *An optimal policy*[a] *has the property that whatever the initial state and initial decisions are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

[a] A decision-making rule that specifies the best action to take at each stage based on the current state.

This recursive property is fundamental to the efficiency and potential of dynamic programming, enabling it to address highly complex problems. Building on these foundational ideas, this section reviews auxiliary results that extend classical DP principles and adapt them to the deterministic minimax framework. Minimax dynamic programming introduces an adversarial perspective by embedding a worst-case optimization into the traditional DP formulation. This involves solving a two-layered problem: minimizing over control decisions while simultaneously maximizing over disturbances or adversarial inputs. The resulting Bellman equation reflects this nested optimization structure, incorporating the principle of optimality and the value function in a form suited to robust control.

In this section, we first revisit the discrete-time deterministic dynamic programming (DP) results from [Bertsekas, 2023, Sec. 1.2] and [Bertsekas, 2005, Ch. 1], along with their dynamic game formulation presented in [Başar and Bernhard, 1991, Ch. 2.2]. Next, we review the deterministic continuous-time DP framework from [Bertsekas, 2005, Ch. 3] and its corresponding continuous-time dynamic game formulation in [Başar and Bernhard, 1991, Ch. 2.3]. These results serve as the foundation for deriving closed-form solutions, in both discrete and continuous time, for a novel class of minimax optimal control problems, which are analyzed over finite and infinite horizons in Chapter 3 and represent the primary contribution of this thesis.

This methodology is characterized by two key components: a discrete-time dynamic system and a cost function defined over time. The dynamic system captures the evolution of certain variables, representing the system's state, as influenced by decisions made at discrete time steps. The system dynamics are

$$x(t+1) = f(x(t), u(t), \omega(t)) \tag{2.14}$$

where $x(t)$ is the state of the system and summarizes past information that is relevant for future optimization, $u(t)$ is the control or decision variable to be selected at time t, $\omega(t)$ is, in this case, a deterministic disturbance at time t, T is the number of times the control is applied, and f is a function that describes the system, in particular the mechanism by which the state is updated. In the discrete-time setting the cost function is additive in the sense that the cost incurred at time t, denoted by

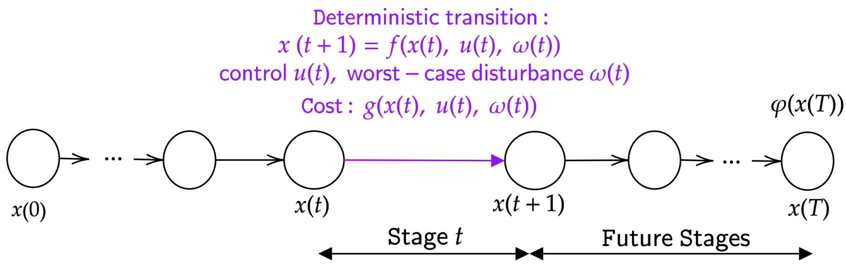


Figure 2.3 Deterministic T-stage optimal control problem. Starting from state $x(t)$ the next state under control $u(t)$ and disturbance $\omega(k)$ is generates according to $f(x(t), u(t), \omega(t))$, incurring a cost of $g(x(t), u(t), \omega(t))$.

$g(x(t), u(t), \omega(t))$ accumulates over time. The total cost is

$$\varphi(x(T)) + \sum_{t=0}^{T-1} g(x(t), u(t), \omega(t))$$

where $\varphi(x(T))$ is a terminal cost incurred at the end of the process. The optimal policy μ^* and disturbance ω^* are the ones that respectively minimize and maximize the cost; that is

$$J_{\mu^*,\omega^*}(x_0) = \inf_{\mu} \sup_{\omega} J_{\mu,\omega}(x_0)$$

where x_0 is the initial state, U is the set of all admissible policies and Ω the set of all admissible disturbances. Note that the optimal policy μ^* and the optimal disturbance ω^* are associated with a fixed initial state x_0. Hence, the optimal cost depends on x_0 and is denoted by $J^*(x_0)$, that is

$$J^*(x_0) = \inf_{\mu} \sup_{\omega} J_{\mu,\omega}(x_0). \tag{2.15}$$

It is helpful to consider J^* as a function that assigns the optimal cost $J^*(x_0)$ to each initial state x_0. This function is referred to as *the optimal cost function*. As previously mentioned, dynamic programming techniques are based on the principle of optimality, following the notation in [Bertsekas, 2023, Ch. 1.2] this principle can be mathematically expressed in our setting as follows.

The Principle of Optimality:
Let $\mu^* = \{\mu^*(x_0), \mu^*(x(1)), \ldots, \mu^*(x(T-1))\}$ be an optimal control sequence, and let $\{\omega^*(0), \ldots, \omega^*(T-1)\}$ be the sequence of worst-case disturbances, which, together with x_0 determines the corresponding state sequence $\{x^*(1), \ldots, x^*(T)\}$ via the system equation (2.14). Consider the subproblem whereby we start at $x^*(k)$ at time k and wish to minimize the cost-to-go from time k to time T,

$$\varphi(x(T)) + \sum_{t=T-k}^{T-1} g(x(t), \mu(x(t)), \omega^*(t))$$

over $\{\mu(x(t)), \mu(x(t+1)), \ldots, \mu(x(T-1))\}$ with $u(t) \in U(x(t))$. Then the truncated optimal control sequence $\{\mu^*(x(t)), \mu^*(x(t+1)), \ldots, \mu^*(x(T-1))\}$ is optimal for this subproblem.

Observe that if the truncated policy $\{\mu^*(x(t)), \mu^*(x(t+1)), \ldots, \mu^*(x(T-1))\}$ were not optimal as assumed, it would be possible to further reduce the cost by switching to an optimal policy for the subproblem starting at $x(t)$. This indicates that an optimal policy can be constructed step by step, beginning with the optimal policy for the final step and recursively extending it to construct an optimal policy for the entire problem.

EXAMPLE 5—DETERMINISTIC SCHEDULING [BERTSEKAS, 2005]
For a particular product, four operations denoted by A, B, C, and D, must be performed on a specific machine. It is assumed that operation B can only be performed after operation A is completed, and operation D can only follow operation B. The cost C_{AB} represents the cost of transitioning from operation A to B. Additionally, there is an initial cost S_A or S_C for starting with operation A or C, respectively. The total cost of a sequence is the sum of its setup costs. For instance, the cost of a sequence $ACDB$ is defined as

$$S_A + C_{AC} + C_{CD} + C_{DB}.$$

We can view this problem as a sequence of three decisions, namely the choice of the first three operations to be performed (the last operation is determined from the preceding three). The possible state transitions for this problem are illustrated in Figure 2.4. Because the problem is deterministic, each choice of control at a given state leads to a uniquely determined next state. For instance, from state AC choosing to perform operation D results in state ACD with a cost of C_{CD}. The optimal solution is the path that starts at the initial state, ends at a terminal state, and minimizes the total arc costs plus the terminal cost. □

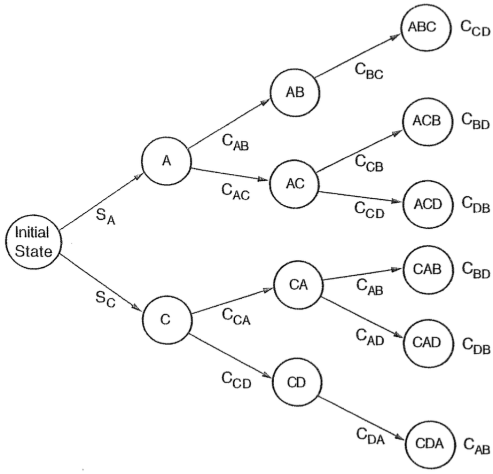


Figure 2.4 The transition graph for the deterministic scheduling problem in Example 5.

Discrete Time

In this subsection, we revisit the results from [Bertsekas, 2005, Ch. 1] on deterministic discrete-time optimal control. We review two key auxiliary results for the discrete-time worst-case minimax setting, considering both finite and infinite horizons. For the reader's convenience, we provide the proofs for the infinite-horizon case and refer to [Başar and Bernhard, 1991, Ch. 2.2] for the derivation of the finite-horizon scenario.

FINITE HORIZON

We define, a general discrete-time, minimax optimal control problem with continuous cost function and continuous constraints

$$\begin{aligned}
&\inf_{\mu} \sup_{\omega} \quad \left[\sum_{t=0}^{T-1} g(x(t), u(t), \omega(t)) \right] \\
&\text{subject to} \\
&\qquad x(t+1) = f(x(t), u(t), \omega(t)) \\
&\qquad x(t) \in X \subseteq \mathbb{R}^n; \quad x(0) = x_0 \; ; \quad u(t) = \mu(x(t)) \\
&\qquad u(t) \in U(x(t)) \subseteq \mathbb{R}^m; \quad \omega(t) \in \Omega(x(t)) \subseteq \mathbb{R}^l
\end{aligned} \tag{2.16}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}_+$ are continuous with respect to x, u and ω, x represents the vector of n-dimensional state variables, u is the m-dimensional control variable and ω is the l-dimensional disturbance variable. The objective is to minimize the worst-case cost over all possible control strategies.

LEMMA 18
Suppose

$$\max_{\omega}\left[g(x,u,\omega)\right] \geq 0$$

$\forall x \in X$, $\forall u \in U(x)$, $\omega \in \Omega(x)$. Then, the general optimal control problem (2.16) attains a finite value for every $x_0 \in \mathbb{R}^n_+$ if and only if the Bellman equation

$$\begin{aligned} J^*_k(x) &= \min_{u \in U(x)} \max_{\omega \in \Omega(x)} \left[g(x,u,\omega) + J^*_{k-1}(f(x,u,\omega))\right] \\ J^*_0(x) &= 0 \end{aligned} \tag{2.17}$$

has a finite solution $J^*_k(x)$, $\forall x \in X$, $k = 0, 1, \ldots, T$. Moreover, if this holds, then the minimal value of (2.16) is equal to $J^*_T(x(T))$. □

INFINITE HORIZON

Consider the infinite time-horizon variant of the general minimax optimal control problem (2.16).

$$\inf_{\mu} \sup_{\omega} \left[\sum_{t=0}^{\infty} g(x(t), u(t), \omega(t))\right] \tag{2.18}$$

subject to

$$\begin{aligned} &x(t+1) = f(x(t), u(t), w(t)), \\ &x(t) \in X \subseteq \mathbb{R}^n; \ \ x(0) = x_0 \ ; \ \ u(t) = \mu(x(t)) \\ &u(t) \in U(x(t)) \subseteq \mathbb{R}^m; \ \ \omega(t) \in \Omega(x(t)) \subseteq \mathbb{R}^l \end{aligned}$$

In the next lemma, we study an important instance of Lemma (18) where the time horizon of the minimax control problem (2.16) approaches infinity, $T \to \infty$.

LEMMA 19
Suppose

$$\max_{\omega}\left[g(x,u,\omega)\right] \geq 0$$

$\forall x \in X$, $\forall u \in U(x)$, $\omega \in \Omega(x)$. Then, the following statements are equivalent.

(i) The general optimal control problem in (2.18) attains a finite value for every $x_0 \in \mathbb{R}^n_+$.

(ii) The recursive sequence $\{J_k\}_{k=0}^{\infty}$ with $J_0 = 0$ and

$$J_k(x) = \min_{u} \max_{\omega}\left[g(x,u,\omega) + J_{k-1}(f(x,u,\omega))\right] \tag{2.19}$$

has a finite limit $\forall x \in X$.

(iii) The Bellman equation

$$J^*(x) = \min_u \max_\omega \left[g(x,u,\omega) + J^*(f(x,u,\omega))\right] \tag{2.20}$$

has nonnegative solution $J^*(x)$, $\forall x \in X$. □

Proof: Note that, by dynamic programming, the increasing monotone recursive sequence $\{J_k\}_{k=1}^{\infty}$ (see Proposition 20) defined in (2.19) also satisfies

$$J_k(x) = \inf_\mu \sup_\omega \sum_{t=0}^{k} \left[g(x(t),u(t),\omega(t))\right]. \tag{2.21}$$

To prove the equivalence we will prove implications $(i) \Longrightarrow (ii)$, $(ii) \Longrightarrow (iii)$, $(iii) \Longrightarrow (ii)$, $(iii) \Longrightarrow (i)$.

$(i) \Longrightarrow (ii)$ Assume (i). Then, in (2.21) when $k \longrightarrow \infty$ we get $J_k(x) < \alpha < \infty$ for all x, with α representing a finite value for (2.18). Hence, the recursive sequence (2.19) has a finite limit.

$(ii) \Longrightarrow (iii)$ Assume (ii). Taking the limit when $k \longrightarrow \infty$ on both sides of the equation (2.19) we get (iii).

$(iii) \Longrightarrow (ii)$ Assume (iii). We want to prove that $\lim_{k\longrightarrow\infty} J_k(x) < \infty$ for all x with $J_k(x)$ defined in (2.21). To achieve this, we use induction over $J_k(x) \leq J^*(x) < \infty$. It is clear that $0 = J_0(x) \leq J^*(x)$ with J^* nonnegative by definition. For the induction step, we assume that $J_k(x) \leq J^*(x)$. We want to prove that $J_{k+1}(x) \leq J^*(x)$. From the induction hypothesis, it is direct that

$$\begin{aligned} J_{k+1}(x) &= \min_u \max_\omega \left[g(x,u,\omega) + J_k(f(x,u,\omega))\right] \\ &\leq \min_u \max_\omega \left[g(x,u,\omega) + J^*(f(x,u,\omega))\right] \\ &= J^*(x). \end{aligned}$$

Thus, $J_{k+1}(x) \leq J^*(x)$ for all x, and $\lim_{k\longrightarrow\infty} J_k(x) < \infty$ for all k and for all x, as we wanted to prove.

$(iii) \Longrightarrow (i)$ Assume (iii). Define for all x

$$\mu^*(x) = \arg\min_u \max_\omega \left\{g(x,u,\omega) + J^*(f(x,u,\omega))\right\}$$

such that

$$J^*(x) = \min_u \max_\omega \left[g(x,u,\omega) + J^*(f(x,u,\omega))\right].$$

Indeed, from (2.21) and implication $(iii) \Longrightarrow (ii)$ it can be observed that

$$\begin{aligned} \max_\omega \sum_{t=0}^{k} \left[g(x,\mu^*(x),\omega)\right] &\leq \inf_\mu \max_\omega \sum_{t=0}^{k} \left[g(x(t),u(t),\omega(t))\right] \\ &= J_k(x) \leq J^*(x) < \infty. \end{aligned}$$

Hence,

$$\max_{\omega} \sum_{t=0}^{k} [g(x, \mu^*(x), \omega)] \leq J^*(x)$$

for all k and for all x. This proves (i). ■

PROPOSITION 20
Let

$$\max_{\omega} [g(x, u, \omega)] \geq 0$$

$\forall x \in X$ and $\forall u \in U$. Then, the recursive sequence $\{J_k(x)\}_{k=0}^{\infty}$ with $J_0(x) = 0$ and

$$J_k(x) = \min_{u} \max_{\omega} [g(x, u, \omega) + J_{k-1}(f(x, u, \omega))] \tag{2.22}$$

satisfies that $0 \leq J_0(x) \leq J_1(x) \leq J_2(x) \leq ...$ for all $x \in X$ and all $k \in \mathbb{N}$. □

Proof: To prove this we use induction over $J_k(x)$. From the proposition statement $J_0(x) = 0$ gives

$$\begin{aligned} J_1(x) &= \min_{u} \max_{w} [g(x, u, \omega) + J_0(f(x, u, \omega))] = \\ &= \min_{u} \max_{\omega} [g(x, u, \omega)] \geq 0 = J_0(x) \end{aligned}$$

for all x. For the induction step we assume that $J_k(x) \geq J_{k-1}(x)$. We then want to prove that $J_{k+1} \geq J_k$. From (2.19) and the induction hypothesis we have that

$$\begin{aligned} J_{k+1}(x) &= \min_{u \in U(x)} \max_{\omega \in \Omega(x)} [g(x, u, \omega) + J_k(f(x, u, \omega))] \\ &\geq \min_{u \in U(x)} \max_{\omega \in \Omega} [g(x, u, \omega) + J_{k-1}(f(x, u, \omega))] \\ &= J_k(x) \end{aligned}$$

Thus, $J_{k+1}(x) \leq J_k(x)$ for all k and for all x. ■

Continuous time

The deterministic continuous-time optimal control results from [Bertsekas, 2005, Ch. 3] are revisited in this subsection. Two auxiliary results for deriving solutions to the continuous-time worst-case minimax setting, presented in Chapter 3, are reviewed. These results are formulated for both finite and infinite horizons. Analogous to the discrete-time setting, the proofs for the infinite-horizon case are included for the reader's convenience, while [Başar and Bernhard, 1991, Ch. 2.2] is referenced for the derivation of the finite-horizon minimax results.

Consider the continuous-time dyamics:

$$\dot{x}(t) = f(x(t), u(t), \omega(t)) \quad 0 \leq t \leq T \tag{2.23}$$

where $x(t)$ represents the vector of state variables at time t, $u(t) \in U$ is the control variable vector at time t and $\omega(t) \in \Omega$ the disturbance at time t. $U(x)$ is the control constrained set, $\Omega(x)$ the disturbance set and $T \in \mathbb{R}_+$ is the terminal time or time horizon. The system (2.23) represents the n first-order differential equations

$$\frac{dx_i(t)}{dt} = f_i(x(t), u(t), \omega(t)), \quad i = 1, \cdots, n. \tag{2.24}$$

We view $\dot{x}(t)$, $x(t)$, $u(t)$ and $\omega(t)$ as column vectors, with ∇_t denoting the partial derivative with respect to t and ∇_x denoting the column vector of partial derivatives with respect to x of appropiate dimensions.

FINITE HORIZON

We define, a general continuous-time, minimax optimal control problem with continuous constraints and nonnegative final conditions $\phi(x(T))$ as

$$\begin{aligned} &\inf_{\mu} \max_{\omega} \quad \left[\phi(x(T)) + \int_0^T g(x(\tau), u(\tau), \omega(\tau))\, d\tau\right] \\ &\text{subject to} \\ &\qquad \dot{x} = f(x(t), u(t), \omega(t)) \\ &\qquad x(t) \in X \subseteq \mathbb{R}^n; \;\; x(0) = x_0\;; \;\; u(t) = \mu(x(t)) \\ &\qquad u(t) \in U(x(t)) \subseteq \mathbb{R}^m; \;\; \omega(t) \in \Omega(x(t)) \subseteq \mathbb{R}^l \end{aligned} \tag{2.25}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}_+$ are continuously differentiable functions with respect to x, and continuous with respect to u and ω. $\phi : X \to X$ is continuously differentiable with respect to x, x represents the vector of n-dimensional state variables, u is the m-dimensional control variable and ω is the l-dimensional disturbance variable. The objective is to minimize the worst-case cost over all possible control strategies.

LEMMA 21
Suppose

$$\max_{\omega} \left[g(x, u, \omega)\right] \geq 0$$

$\forall x \in X$, $\forall u \in U(x)$, $\omega \in \Omega(x)$. Then, the optimal control problem (2.25) has a finite value for every $x_0 \in \mathbb{R}^n_+$ if and only if the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE)

$$\begin{aligned} -\nabla_t J^*(t,x) &= \min_u \max_{\omega} \left[g(x,u,\omega) + \nabla_x J^*(t,x)^\top f(x,u,\omega)\right] \\ J^*(T,x) &= \phi(x(T)) \end{aligned} \tag{2.26}$$

has a solution $J^*(t,x)$, $\forall x \in X$, $t \in [0,T]$. Moreover, if (ii) holds, then the minimal value of (2.25) is equal to $J^*(0,x)$. □

INFINITE HORIZON

Consider the infinite time-horizon variant of the general minimax optimal control problem (2.25)

$$\begin{aligned} &\inf_{\mu} \max_{\omega} \int_0^{\infty} g(x(\tau),u(\tau),\omega(\tau))\, d\tau \\ &\text{subject to} \\ &\qquad \dot{x} = f(x(t),u(t),\omega(t)) \\ &\qquad x(t) \in X \subseteq \mathbb{R}^n; \quad u(t) = \mu(x(t)) \\ &\qquad u(t) \in U(x(t)) \subseteq \mathbb{R}^m; \quad \omega(t) \in \Omega(x(t)) \subseteq \mathbb{R}^l \end{aligned} \tag{2.27}$$

In the next lemma, we study an important instance of Lemma (21) where the time horizon of the minimax control problem (2.25), approaches infinity, $T \longrightarrow \infty$, and the HJB equation (2.26) takes a simpler form.

LEMMA 22
Suppose

$$\max_{\omega}\,[g(x,u,\omega)] \geq 0$$

$\forall x \in X$ and $\forall u \in U(x)$. Then, the following statements are equivalent.

(i) The general optimal control problem in (2.27) has a finite value for every $x_0 \in \mathbb{R}^n_+$.

(ii) The HJB differential equation

$$0 = \min_{u} \max_{\omega} \left[g(x,u,\omega) + \nabla_x J^*(x)^\top f(x,u,\omega) \right] \tag{2.28}$$

has a finite solution $J^*(x)$, for all $x \in X$. □

Proof: From Lemma (21) we know that the finite horizon minimax control problem (2.25) has a minimal value for all $x_0 \in \mathbb{R}^n_+$ if and only if $\forall x \in X$ and $t \in [0,T]$ the HJB equation (2.26)

$$\begin{aligned} -\nabla_t J^*(t,x) &= \min_{u} \max_{\omega} \left[g(x,u,\omega) + \nabla_x J^*(t,x)^\top f(x,u,\omega) \right] \\ J^*(T,x) &= \phi(x(T)) \end{aligned} \tag{2.29}$$

has a solution $J^*(t,x)$ of the form

$$J^*(t,x) = \inf_{\mu} \max_{\omega} \phi(x(T)) + \int_t^T g(x(\tau),u(\tau),\omega(\tau))\, d\tau. \tag{2.30}$$

$(i) \Longrightarrow (ii)$ Recall (2.30). Observe that as $T \longrightarrow \infty$ the dependency of the optimal cost on $t \in [0,T]$ in (2.30) vanishes. Thus, the value function of the infinite horizon problem (2.27) becomes

$$J^*(x) = \inf_{\mu} \max_{\omega} \int_{t_0}^{\infty} g(x(\tau),u(\tau),\omega(\tau))\, d\tau. \tag{2.31}$$

Note that (2.31) is independent on the initial time t_0, which can be observed from the fact that shifting the initial time leads to the same optimal u and ω with the same time shift.

Moreover, from Lemma (21), the HJB equation (2.29) has a solution $J^*(x)$ satisfying (2.31), $\forall x \in X$, $t \in [0,\infty]$. The value function (2.31) gives

$$\nabla_t J^*(x) = 0$$

which reduces the HJB equation (2.29) to (2.28). This proves (ii).

$(ii) \Longrightarrow (i)$ Suppose that the HJB differential equation (2.28) has a finite solution $\hat{J}(\hat{x})$ for all $x \in X$, that is, $\hat{J}(\hat{x})$ is a continuously differentiable function representing the optimal value function.

Let $\{\hat{u}(t) \mid t \in [0,\infty]\}$, be the control trajectory associated with $\hat{J}(\hat{x})$ and $\{\hat{x}(t) \mid t \in [0,\infty]\}$ the corresponding state trajectory. Define a disturbance trajectory $\{\omega^*(t) \mid t \in [0,\infty]\}$ such that

$$\int_0^{\infty} g(\hat{x}(\tau),\hat{u}(\tau),\omega^*(\tau))\ d\tau = \max_{\omega} \int_0^{\infty} g(\hat{x}(\tau),\hat{u}(\tau),\omega(\tau))\ d\tau.$$

Thus, $\{\omega^*(t) \mid t \in [0,\infty]\}$ is the worst-case disturbance for $\hat{J}(\hat{x})$.

From the HJB equation (2.28), we have that for all $t \in [0,\infty]$

$$0 \leq g(\hat{x}(t),\hat{u}(t),\omega^*(t)) + \nabla_x \hat{J}(\hat{x}(t))^\top f(\hat{x}(t),\hat{u}(t),\omega^*(t)).$$

Define $\dot{\hat{x}}(t) = f(\hat{x}(t),\hat{u}(t),\omega^*(t))$, $\hat{x}_0 = \hat{x}(0)$. The right hand side of (2.28) is equivalent to

$$g(\hat{x}(t),\hat{u}(t),\omega^*(t)) + \frac{d}{dt} J^*(\hat{x}(t)) \geq 0.$$

where $\frac{d}{dt} J^*(\hat{x}(t)) = \nabla_x J^*(\hat{x})^\top \dot{\hat{x}}(t)$. This is

$$g(\hat{x}(t),\hat{u}(t),\omega^*(t)) \geq -\frac{d}{dt} \hat{J}(\hat{x}(t)). \tag{2.32}$$

Now, we integrate both sides of (2.32) over time from 0 to some $T > 0$

$$\int_0^T g(\hat{x}(t), \hat{u}(t), \omega^*(t))\, d\tau \geq \hat{J}(\hat{x}(0)) - \hat{J}(\hat{x}(T)).$$

By assumption, there is no terminal cost and the value function is finite, therefore, the value function converges as $T \to \infty$, which implies that $\hat{J}(\hat{x}(T)) \to 0$ as $T \to 0$. Thus we can write

$$\int_0^\infty g(\hat{x}(t), \hat{u}(t), \omega^*(t))\, d\tau \geq \hat{J}(\hat{x}(0)). \tag{2.33}$$

Finally, let $J^*(x_0^*)$ be the resulting cost for the control and state trajectories $u^*(t)$ and $x^*(t)$ respectively, where the inequality (2.33) becomes an equality, i.e.:

$$J^*(x_0^*) = \int_0^\infty g(x^*(\tau), u^*(\tau), \omega^*(\tau))\, d\tau.$$

Note that the cost corresponding to $\{u^*(t) \mid t \in [0,T]\}$ is no larger than the cost corresponding to any other admissible control trajectory $\{u(t) \mid t \in [0,T]\}$, in particular $\{\hat{u}(t) \mid t \in [0,T]\}$. It follows that $\{u^*(t) \mid t \in [0,T]\}$ is optimal with

$$\mu(x) = \operatorname*{argmin}_u \max_\omega \left\{ g(x,u,\omega) + \nabla_x J^*(t,x)^\top f(x,u,\omega) \right\}$$

and

$$\hat{J}(\hat{x}_0) = J^*(x_0^*)$$

is the minimal value of (2.27) for all $x_0 \in \mathbb{R}_+^n$. Moreover, the preceding argument can be repeated with any initial time $t \in [0,\infty]$ and any initial state x. Thus,

$$\hat{J}(x) = J^*(x), \quad \text{for all } x \in X.$$

This proves $(ii) \implies (i)$. ∎

2.4 Dynamical systems over general graphs

Graph description

A graph $\mathcal{G}$ is also defined as a set $\mathcal{G} = \{\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}\}$, with a set of $N = |\mathcal{V}_\mathcal{G}|$ nodes and a set $\mathcal{E}_\mathcal{G} \subseteq \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$ of edges, each of which has an associated nonnegative weight w_{ij}. In this work, only undirected graphs are considered, hence, the edge $(i,j) \in \mathcal{E}_\mathcal{G}$ is bidirectional (i.e. it does not have a direction). Moreover, every graph $\mathcal{G}$ has a connected spanning tree. This means there exists a path from any node $i \in \mathcal{V}_\mathcal{G}$ to every other node $j \in \mathcal{V}_\mathcal{G} \setminus i$. The Laplacian matrix $\mathcal{L}(\mathcal{G})$ of $\mathcal{G}$, or simply $\mathcal{L}$ when $\mathcal{G}$ is clear from the context, is defined as

$$\mathcal{L}(\mathcal{G})_{ij} = \begin{cases} -w_{ij} & i \neq j \\ \sum_{k \in \mathcal{N}_i} w_{ik} & i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2.34}$$

Alternatively, in terms of the degree matrix D and the adjacency matrix A of the graph $\mathcal{G}$, the Laplacian matrix is given by $\mathcal{L} = D - A$.

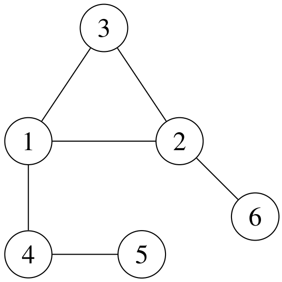


Figure 2.5 Example of an undirected and connected graph with $N = 6$ nodes.

EXAMPLE 6

Given the graph in Figure 2.5, observe that its degree, adjacency and Laplacian matrices are

$$D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}. \qquad \square$$

Note that the Laplacian matrix of undirected graphs is symmetric, thus, its eigenvalues are real. The set of connected graphs is denoted by $\mathbb{G}$ and is characterized by the following property.

LEMMA 23

A graph $\mathcal{G}$ is connected if and only if the associated Laplacian matrix $\mathcal{L}$ has a simple eigenvalue at the origin. Furthermore, in this case, the eigenvector associated with the eigenvalue at the origin is $\mathbb{1}$ and all other eigenvalues lie in the open right half-plane. $\square$

Denote by $\lambda_k(\mathcal{G})$, or λ_k if $\mathcal{G}$ is clear from the context, with $k = 1,\dots N$ the eigenvalues of $\mathcal{L}$. The eigenvalues are numbered so that $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. The eigenvalue λ_2 is known as the algebraic connectivity of $\mathcal{G}$.

DEFINITION 19—ALGEBRAIC CONNECTIVITY
The algebraic connectivity of a graph $\mathcal{G}$ is a measure of its connectivity based on the spectrum of its Laplacian matrix. It is defined as the second smallest eigenvalue $\lambda_2(\mathcal{G})$. If $\lambda_2(\mathcal{G}) > 0$ indicates that $\mathcal{G}$ is connected, whereas if $\mathcal{G} = 0$ indicates that the graph is disconnected. Larger values of λ_2 imply a more robust and well-connected graph. $\square$

In [Saberi et al., 2022, Ch. 3] the regularized Laplacian matrix of a graph is considered. This matrix, denoted as $\mathcal{D}$ is defined as

$$\mathcal{D} = I - (I+D)^{-1}\mathcal{L} \tag{2.35}$$

where D is diagonal with

$$D_{ii} = \sum_{k=1}^{N} w_{ik}$$

the degree matrix of the graph $\mathcal{G}$. The matrix $\mathcal{D}$ has eigenvalues μ_i, $i = 1,\dots N$, which are real because $\mathcal{D}$ is similar to the symmetric matrix

$$I - (I+D)^{-\frac{1}{2}}\mathcal{L}(I+D)^{-\frac{1}{2}}.$$

The matrix $\mathcal{D}$ is nonnegative and row stochastic, meaning each row of $\mathcal{D}$ sums to 1, i.e., $\mathcal{D}\mathbb{1} = \mathbb{1}$. From the above equality it is immediate that for $\lambda_1 = 0$, $\mathcal{D}\mathbb{1} = \mathbb{1}$, thus, $\mu_1 = 1$ is the largest eigenvalue of $\mathcal{D}$ with associated eigenvector $\mathbb{1}$.

In this thesis we focus on families $\mathcal{F} \subseteq \mathbb{G}$ (in the continuous-time case) and $\mathcal{F} \subseteq \tilde{\mathbb{G}}$ (in the discrete-time case) of connected, undirected graphs, for which the eigenvalues of the associated Laplacian $\mathcal{L}$ (continuous-time case) and respectively the row stochastic matrix $\mathcal{D}$ (discrete-time case) satisfy upper and lower bounds.

DEFINITION 20
The set of undirected graphs for which the associated Laplacian matrix has nonzero eigenvalues λ_i, $i = 2,\dots N$ such that $a \leq \lambda_i \leq b$ is defined by

$$\mathbb{G}_{[a,b]} = \{\mathcal{G} \in \mathbb{G} \mid \lambda_i(\mathcal{G}) \in [a,b], \ \ \forall i > 1\}.$$ $\square$

In the continuous-time scenario, a is a lower bound for the algebraic connectivity of the graphs in $\mathbb{G}_{[a,b]}$. In the literature, the upper bound b has been be related to the number of agents and their degree of connectivity.

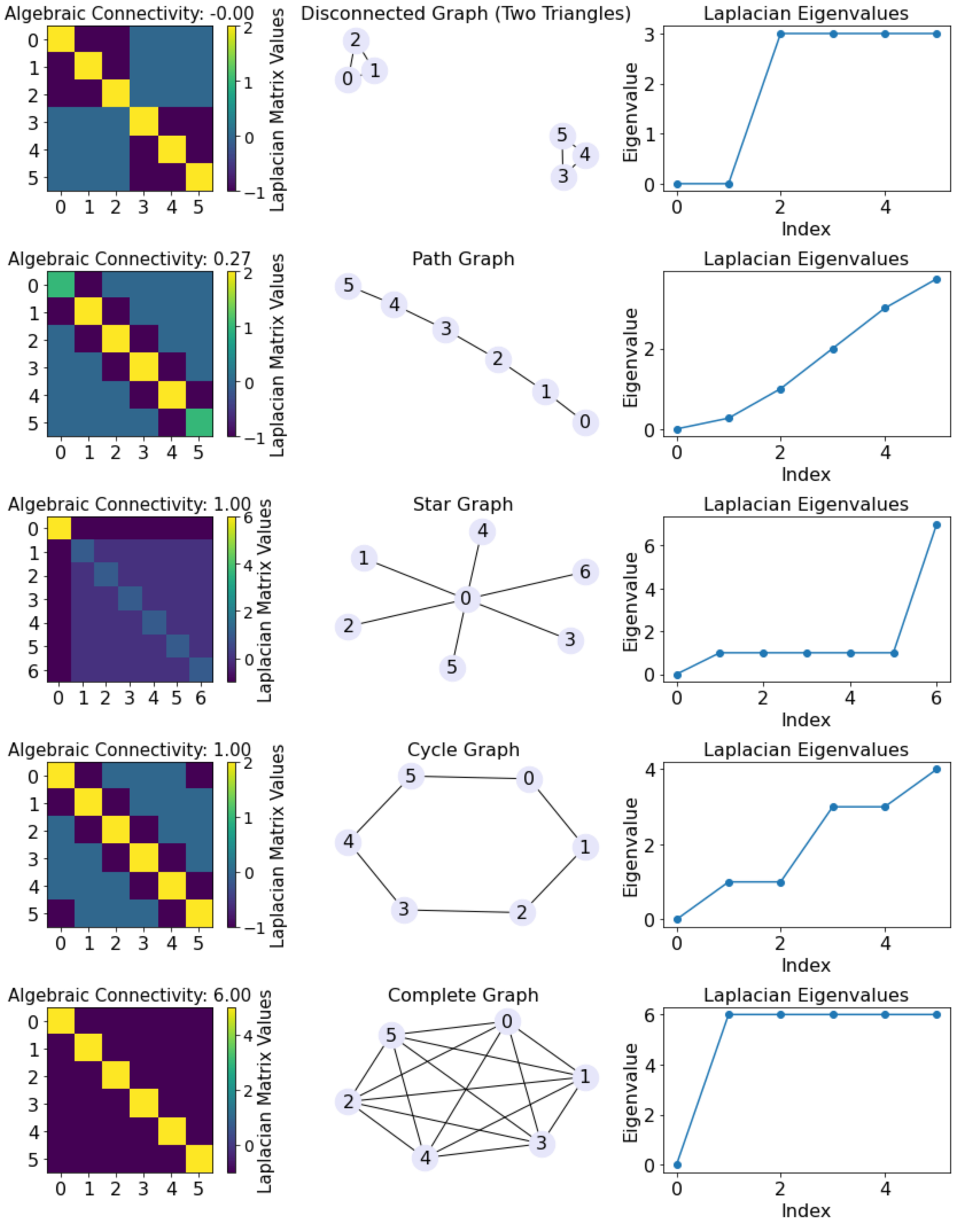


Figure 2.6 Visualization of Graph topologies, its algebraic connectivity, Laplacian matrix and spectrum.

DEFINITION 21

The set of undirected graphs for which the associated row-stochastic matrix $\mathcal{D}$ satisfies the property that all eigenvalues, except for $\mu_1 = 1$, have an absolute value smaller than d and greater than c is defined by

$$\tilde{\mathbb{G}}_{[c,d]} = \{\mathcal{G} \in \mathbb{G} \mid \mu_i(\mathcal{G}) \in [c,d],\ \forall i > 1\}.$$

where $c \in (-1,d]$, $d \in (0,1)$. □

Expressions for c and d has been studied in the literature—See *e.g.,* [Banerjee and Mehatari, 2016].

2.5 Multi-agent systems

Multi-agent systems (MAS) are systems composed of multiple interacting agents that work collaboratively or competitively to achieve individual or collective objectives. These agents are autonomous entities, such as robots, vehicles, software programs or sensors, capable of sensing their environment, making decisions, and taking actions based on their observations and interactions. In the remainder of this section, the definitions and notation follow those used in [Saberi et al., 2022].

Continuous-time

This thesis focuses on continuous-time multi-agent systems (MAS) composed by an arbitrary number of identical, linear time-invariant agents of the form

$$\dot{x}_i = Ax_i + Bu_i \tag{2.36}$$

don't we just need the cosed loop dyn to be positive?

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are, respectively, the state and input vectors of agent i, $A \in \mathbb{R}^{n\times n}$ is Metzler and $B \in \mathbb{R}^{n\times m}$.

It is assumed that the communication network provides each agent with a linear combination of its own state relative to the states of its neighboring agents. Each agent $i \in \{1,\dots,N\}$ in the network can access the relative information of the full states of its neighboring agents compared to its own state. Specifically, each agent has access to the quantity

we are using 1 to N instead of arbitrary number

$$\zeta_i = \sum_{j=1}^{N} w_{ij}(x_i - x_j) \tag{2.37}$$

where $w_{ij} \geq 0$, $w_{ii} = 0$ for $i, j \in \{1,\dots,N\}$. The network topology is described by an undirected graph $\mathcal{G} \in \mathbb{G}_{[\beta,\gamma]}$ where nodes represent the agents and edges correspond to the nonzero coefficients w_{ij}. In particular, $w_{ij} > 0$ indicates the presence of an edge from agent j to i with the edge weight equal to the magnitude of a_{ij}.

The communication in the continuous-time network is expressed using the Laplacian matrix $\mathcal{L}$, associated with this weighted graph $\mathcal{G}$. In particular, ζ_i can be written as

$$\zeta_i = \sum_{j=1}^{N} l_{ij} x_j \tag{2.38}$$

where l_{ij} are elements of $\mathcal{L}$.

Discrete-time

In this thesis discrete-time multi-agent systems (MAS) composed by an arbitrary number of identical, linear time-invariant agents are also studied. The dynamics of each agent $i = 1, \ldots, N$ are described by

$$x_i(t+1) = Ax_i(t) + Bu_i(t); \quad A \in \mathbb{R}_{+}^{n\times n}, B \in \mathbb{R}^{n\times m} \tag{2.39}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are, respectively, the state and input vectors of agent i.

Each agent communicates with its neighbors through a network described by an undirected graph $\mathcal{G} \in \tilde{\mathbb{G}}$ where nodes represent agents, and edges represent communication links. The edge weights $w_{ij} \geq 0$ determine the strength of the connection between agents i and j, with $w_{ii} = 0$. The adjacency matrix of the graph is denoted as $\mathcal{W} = [w_{ij}]$ and the associated Laplacian matrix is given by

$$\mathcal{L} = D - \mathcal{W}$$

Agents access the relative information of their neighbors through full-state measurements. Specifically, each agent i has access to the quantity

$$\zeta_i(k) = \frac{1}{1+\sum_{j=1}^{N} w_{ij}} \sum_{k\in\mathcal{N}_i} w_{ik}(y_i(t) - y_k(t)), \tag{2.40}$$

This information exchange can be expressed compactly in terms of the row-stochastic matrix $\mathcal{D}$ defined as (2.35). Rewriting $\zeta_i(k)$ we have

$$\zeta_i(k) = x_i(t) - \sum_{k\in\mathcal{N}_i} d_{ik}(x_i(t) - x_k(t)) \tag{2.41}$$

k instead of t

where d_{ij} are the elements of $\mathcal{D}$.

2.6 The state synchronization problem

A relevant problem in control theory is the design of protocols that lead to the synchronization of interconnected systems. Synchronization is a desired behavior in many dynamical systems associated with numerous applications [Ren and Beard, 2005; Tegling et al., 2023; Fabiny et al., 1993; Fax and Murray, 2004]. Note that the results and definitions in this section follow those in [Saberi et al., 2022].

A multi-agent system is homogeneous when all agents share identical dynamics. Synchronization in such systems is achieved when differences in the state of neighbouring agents asymptotically converge to zero.

DEFINITION 22—STATE SYNCHRONIZATION
Consider the continuous-time MAS described by (2.36) and (2.37) (the discrete time MAS described by (2.39) and (2.41)). The agents in the continuous-time (discrete-time) network achieve state synchronization if

$$\lim_{t\to\infty} [x_i(t) - x_j(t)] = 0, \qquad \forall i, j \in 1, \dots, N. \tag{2.42}$$

□

REMARK 2
The continuous-time agents in (2.36) (discrete-time agents (2.39)) are described by an LTI system, where $y(t) = x(t)$. This coupling, where the full state of the agents is used in the control law, is referred to in the literature as *full-state coupling*. □

The state synchronization problem Consider a continuous-time MAS described by (2.39), (2.41), (discrete-time MAS (2.36), (2.37)). Let $\mathcal{F}$ be a given family of graphs such that $\mathcal{F} \subseteq \mathbb{G}$[a] ($\mathcal{F} \subseteq \tilde{\mathbb{G}}$). The *state synchronization problem* with a set of network graphs $\mathcal{F}$ consists of finding, if possible, a linear static protocol of the form

$$u_i(t) = F\zeta_i(t) \tag{2.43}$$

for $i = 1, \dots, N$ such that for any graph $\mathcal{G} \in \mathcal{F}$ and all initial conditions of the agents, state synchronization is achieved. Furthermore, this problem is referred to as the *the positive consensus problem* if, for any selection of nonnegative initial conditions, the states of all agents remain nonnegative.

[a] Recall that $\mathbb{G}$, ($\tilde{\mathbb{G}}$) is the sets of connected, undirected graphs where nodes represents continuous-time (discrete time) agents and edges represent communication links.

Protocol design for continuous-time MAS

After implementing the linear static protocol (2.43), the MAS described by (2.36) and (2.37) follows from the dynamics

$$\dot{x}_i = Ax_i + BF\zeta_i$$

with $i = 1, \dots, N$. Then, the overall dynamics of the N agents can be written as

$$\dot{x} = (I_N \otimes A + \mathcal{L} \otimes BF)x. \tag{2.44}$$

It has been shown in [Mengran and Sandip, 2017], [Feketa et al., 2019], that the synchronization of the system (2.44) is equivalent to the asymptotic stability of the following $N-1$ subsystems

$$\dot{\tilde{\eta}}_i = (A+\lambda_i BF)\tilde{\eta}_i, \qquad i=2,\dots,N \tag{2.45}$$

where λ_i, $i=2,\dots,N$ are the nonzero eigenvalues of $\mathcal{L}$.

LEMMA 24
The MAS (2.36) and (2.37) achieves state synchronization if and only if the subsystems (2.45) are globally asymptotically stable [Saberi et al., 2022, Thm. 2.5]. □

The continuous-time methodology presented in Chapter 5 in this thesis follows the notation of [Saberi et al., 2022] and is presented in the next Remark.

REMARK 3
In the proof of Lemma 24 in [Saberi et al., 2022], the following coordinate transformation is defined

$$\eta := (T^{-1}\otimes I_n)x = (\eta_1 \dots \eta_N)$$

where $\eta_i \in \mathbb{R}^n$ and T is the matrix of eigenvectors of the Laplacian $\mathcal{L}$, satisfying $\mathcal{L} = T^{-1}JT$. In this context, note that the function η_1 satisfies

$$\dot{\eta}_1 = A\eta_1, \quad \eta_1(0) = (w\otimes I_n)x(0).$$

which can be shown by using the fact that 0 is a simple eigenvalue of the Laplacian. Here, w represents the first row of T^{-1}, i.e., the normalized eigenvector associated with the zero eigenvalue, which corresponds to $\mathbb{1}$, in the case of an undirected graph. Consequently, the proof of Lemma 24 shows that the synchronized trajectory of the network is given by,

$$x_s(t) = e^{At}\frac{1}{N}\sum_{i=1}^{N} x_i(0). \tag{2.46}$$

□

EXAMPLE 7—SYNCHRONIZATION OF CONTINUOUS-TIME MAS
Consider a system of N agents interacting over a regular graph where each agent has two neighbors. The system dynamics are

$$\dot{x}_i(t) = -\sum_{j\in\mathcal{N}_i}(x_i(t)-x_j(t)).$$

Let the interconnection network be a cycle graph represented by the Laplacian matrix

$$\mathcal{L}_{ij} = \begin{cases} 2, & \text{if } i=j \\ -1, & \text{if i,j neighbors} \\ 0, & \text{otherwise} \end{cases} \tag{2.47}$$

and the initial states of the agents satisfy that $x_i(0) \in [0,10]$ for $i = 1,\dots,N$ and are generated randomly. The continuous-time dynamics can be simulated as

$$\dot{x}(t) = -\mathcal{L}x(t) \qquad \square$$

where $x(t) \in \mathbb{R}^n$ is the vector of states of all agents at time t. Figure 2.7 shows that the states of a network of $N = 4, 50$ agents converge over time achieving synchronization. The synchronized state is given by (2.46).

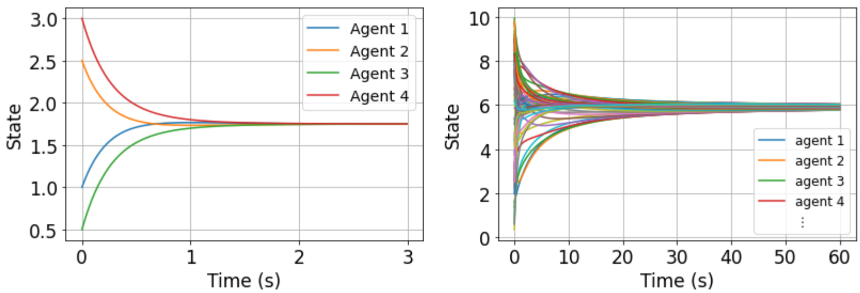


Figure 2.7 Convergence of the 4 and 50 continuous-time agent network system to the synchronized trajectory in Example 7.

Protocol design for discrete-time MAS

After implementing the linear static protocol (2.43), the MAS described by (2.39) and (2.41) evolves according to the dynamics

$$x_i(t+1) = Ax_i(t) + BF\zeta_i(t)$$

with $i = 1,\dots,N$. The overall dynamics of the N agents can be written compactly as

$$x(t+1) = (I_N \otimes A + (I - \mathcal{D}) \otimes BF)x(t). \qquad (2.48)$$

I_n

It has been shown in [Feketa et al., 2019] and [Mengran and Sandip, 2017], that the synchronization of the system (2.48) is equivalent to the asymptotic stability of the following $N-1$ subsystems

$$\tilde{\eta}_i(t+1) = (A + (1-\mu_i)BF)\tilde{\eta}_i(t), \quad i = 2,\dots,N \qquad (2.49)$$

where μ_i, $i = 2,\dots,N$ are the eigenvalues of $\mathcal{D}$ inside the unit disc.

LEMMA 25
The MAS described by (2.39) and (2.41) achieves state synchronization if and only if the subsystems (2.49) are globally asymptotically stable [Saberi et al., 2022, Thm. 3.4]. $\square$

The discrete-time methodology presented in Chapter 5 in this thesis follows the notation of [Saberi et al., 2022] and is presented in the next Remark.

REMARK 4
In the proof of Lemma 25 in [Saberi et al., 2022] the following coordinate transformation is defined

$$\eta := (T^{-1} \otimes I_n)x = (\eta_1 \dots \eta_N)$$

where $\eta_i \in \mathbb{R}^n$ and T is the matrix of eigenvectors of the row stochastic matrix $\mathcal{D}$, satisfying $\mathcal{D} = T^{-1} J_D T$. In this context, note that the function η_1 satisfies

$$\eta_1(t+1) = A\eta_1(t), \quad \eta_1(0) = (w \otimes I_n)x(0),$$

where w is the first row of T^{-1}, i.e., the normalized eigenvector associated with the eigenvalue 1 of $\mathcal{D}$. This implies that the synchronized trajectory is given by

$$x_s(t) = A^t \frac{1}{N} \sum_{i=1}^{N} x_i(0). \tag{2.50}$$

□

EXAMPLE 8—SYNCHRONIZATION OF A DISCRETE-TIME MAS
Consider again a system of N agents over the cycle graph represented by the Laplacian in (2.47) where each agent is connected to its two nearest neighbors and the initial states of the agents satisfy that $x_i(0) \in [0, 10]$ for $i = 1, \dots, 50$ and are generated randomly. The state of each agent evolves according to

$$x(t+1) = \mathcal{D}x(t),$$

where $x(t) \in \mathbb{R}^n$ is the vector os states of all agents at time step t and $\mathcal{D}$ is the row-stochastic matrix $\mathcal{D}$ governing the dynamics defined as $\mathcal{D} = I - (I + D)^{-1}\mathcal{L}$, with $D = \text{diag}(2, 2, \dots, 2)$. Thus, each agent $x_i(t)$ evolves according to

$$x_i(t+1) = x_i(t) + \frac{1}{3}\left(\sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))\right)$$

□

Figure 2.8 shows that the states of a network of 4 and 50 agents converge over time, achieving synchronization. The synchronized state is given by (2.50).

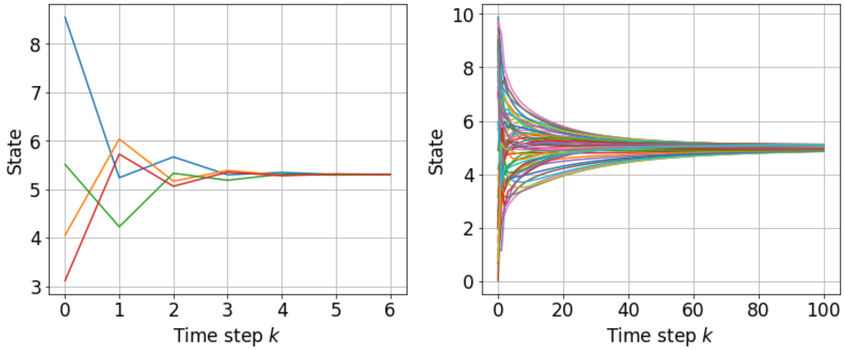


Figure 2.8 Convergence of the 4 and 50 discrete-time agent network system to the synchronized trajectory in Example 8.

3

Minimax linear regulator for positive systems

In this chapter, the theoretical framework and primary contributions of this thesis are introduced. The framework is formulated as a multi-disturbance, worst-case minimax optimal control problem, characterized by a linear cost function and positive linear dynamics. Unlike the Dynamic Programming section in Chapter 2, this chapter begins by presenting the continuous-time framework, addressing both finite and infinite horizon cases, as these are the settings in which the main contributions are made. Next, the discrete-time variant is also presented. This multi-disturbance approach accounts for two types of disturbances: elementwise bounded disturbances, which model situations where disturbances are proportional to the state of the system and are limited in magnitude, and unconstrained disturbances. Examples of the first include actuator constraints, such as motor capacity or battery power, and bounded external factors, such as load variations in power systems or temperature fluctuations. Additionally, unconstrained disturbances capture scenarios involving unrestricted, non-negative inputs, such as unbounded growth or external demands that can only increase. Throughout this chapter, explicit solutions to this theoretical framework are derived, revealing that the optimal control policy (among all possible policies) is linear. This policy can in turn be computed through standard value iterations. Moreover, the feedback matrix of the optimal controller inherits the sparsity structure from the constraint matrix of the problem statement. This permits structural controller constraints in the problem design and simplifies the application to large-scale systems. Additionally, a section on the l_1-induced gain of the continuous-time setting is presented, linking the existence of a finite solution to the minimax optimal control problem with the disturbance penalty in the cost function. To illustrate the methodology, two main examples are provided: a large-scale water distribution network and an optimal DC power network.

3.1 Continuous-time setting

Finite-horizon case

The optimal control problem in this section is formulated as a continuous-time minimax problem with nonnegative linear cost, positive linear dynamics, elementwise linear constraints on the control policy and the disturbance $v \in \mathbb{R}^c$, nonnegative unconstrained disturbance w and final conditions equal to zero,

$$\begin{aligned} &\inf_{\mu} \max_{w,v} \int_0^T \left[s^\top x(\tau) + r^\top u(\tau) - \boldsymbol{\gamma}^\top w(\tau) - \boldsymbol{\delta}^\top v(\tau) \right] d\tau \\ &\text{Subject to} \\ &\qquad \dot{x}(t) = Ax(t) + Bu(t) + Fw(t) + Hv(t) \\ &\qquad x(0) = x_0, \quad u(t) = \mu(x(t)), \\ &\qquad |u| \leq Ex, \quad w \geq 0, \quad |v| \leq Gx. \end{aligned} \tag{3.1}$$

Here, x represents the n-dimensional vector of state variables, u the m-dimensional control variable, w, v are the l-dimensional and c-dimensional disturbance, μ is any, potentially nonlinear, control policy, E prescribes the structure of the control law, G determines the linear dependency of the disturbance v and the state, and $T \in \mathbb{R}_+$ is the time horizon. The objective is to minimize the worst-case cost over all possible control strategies.

In the next theorem we use dynamic programming theory (Subsection 2.3) to derive explicit solutions to the Hamilton-Jacobi-Bellman equation of (3.1) and give necessary and sufficient conditions for the existence of finite solutions in finite time.

THEOREM 26
Let $A \in \mathbb{R}^{n\times n}$, $B = [B_1 \dots B_m] \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}_+^{n\times l}$, $H \in \mathbb{R}^{n\times c}$, $E = \left[E_1^\top \dots E_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$ such that $E_i^\top \neq \mathbb{0}$ for all i, $G \in \mathbb{R}_+^{c\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\boldsymbol{\gamma} \in \mathbb{R}_+^l$ and $\boldsymbol{\delta} \in \mathbb{R}^c$. Suppose that $A - |B|E$ is Metzler and

$$s \geq E^\top |r| - G^\top |\boldsymbol{\delta}|. \tag{3.2}$$

Then the following statements are equivalent:

(i) The optimal control problem (3.1) has a finite value for every $x_0 \in \mathbb{R}_+^n$;

(ii) The differential equation in $p(t) \in \mathbb{R}^n$,

$$\begin{aligned} &-\dot{p}(t) = s + A^\top p(t) - E^\top \left| r + B^\top p(t) \right| + G^\top \left| -\boldsymbol{\delta} + H^\top p(t) \right|, \\ &p(T) = 0, \end{aligned} \tag{3.3}$$

has a unique solution and

$$\boldsymbol{\gamma} \geq F^\top p(0). \tag{3.4}$$

Moreover, if the above conditions hold, the minimal value of the optimal control problem (3.1) is $p(0)^\top x_0$ and the optimal control policy is given by $u^*(t) = -K(t)x^*(t)$ with

$$K(t) \in \begin{bmatrix} \text{sign}(r_1 + p(t)^\top B_1)E_1 \\ \vdots \\ \text{sign}(r_m + p(t)^\top B_m)E_m \end{bmatrix}. \tag{3.5}$$

□

REMARK 5
The right-hand side of (3.5) is a set, since multiple controllers may exists that achieve the optimal cost. For any index i such that $r_i + p(t)_i^\top B_i = 0$ all controllers in the set

$$\mathcal{K} = \left\{ DE \mid D_{ii} \in [-1,1], D_{jj} \in \text{sign}(r_j + p(t)_j^\top B_j) \text{ for } j \neq i \right\}$$

correspond to same and unique solution $p(t)$ of the HJB ODE equation (3.3). □

REMARK 6
Condition (3.2) will be needed when applying Lemma 21 in Subsection 2.3 to the objective function in (3.1). □

REMARK 7
The condition $E_i^\top \neq \mathbb{0}$ for all $i = 1, \ldots, m$, is imposed to prevent ill-posed scenarios where certain control inputs u_i satisfy $|u_i| \leq E_i^\top x = 0$ leading to trivial control actions. □

REMARK 8
Even though dynamic programming is executed without a priori condition on linearity or sparsity, these properties result from the optimization criteria and constraints. In fact, the sparsity structure of the control gain $K(t)$ in (3.5) is inherited from the E matrix which can be determined by the problem designer. □

Proof: The proof of this theorem relies on Lemma 21 in the Appendix. To apply this Lemma, define

$$\begin{aligned} f(x,u,w) &:= Ax + Bu + \begin{bmatrix} F & H \end{bmatrix} \omega \\ g(x,u,\omega) &:= s^\top x + r^\top u - \begin{bmatrix} \gamma^\top & \delta^\top \end{bmatrix} \omega \\ \omega &:= \begin{bmatrix} w^\top & v^\top \end{bmatrix}^\top. \end{aligned} \tag{3.6}$$

It is direct that $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}_+$ are linear and continuously differentiable with respect to x and continuous with respect to u, w. Thus, it is clear that the minimax control problem (3.1) is a special case of the general minimax setting (2.25) in Section 2.3. Therefore, (i) in Theorem 26 is equivalent to (i) in Lemma 21 in the Appendix. Note that condition (3.2) gives

$$\begin{aligned}
&\max_{\substack{w\geq 0\\|v|\leq Ex}} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v\right] \\
&= s^\top x + r^\top u + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] + \max_{|v|\leq Gx}\left[-\delta^\top v\right] \\
&\geq \left(E^\top |r| - G^\top |\delta|\right)^\top x + r^\top u + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] + |\delta|^\top Gx \\
&\geq \left(E^\top |r|\right)^\top x - \left|r^\top\right| Ex + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] \\
&\geq \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] \geq 0
\end{aligned} \tag{3.7}$$

Thus,

$$\max_{w,v}\left[g(x,u,\omega)\right] = \max_{\substack{w\geq 0\\|v|\leq Gx}} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v\right] \geq 0$$

as required in Lemma 21. Next we verify that the differential equation (3.3) in (ii) is equivalent to (2.26) in Lemma 21. Recall the cost-to-go function (2.30). Define $J^*(t,x) = p^\top(t)x$,

$$\begin{aligned}
0 &= \min_u \max_\omega \left[g(x,u,\omega) + \nabla_t J^*(t,x) + \nabla_x J^*(t,x)^\top f(x,u,\omega)\right] \\
&= \min_u \max_{v,w} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v + \dot{p}(t)^\top x + p(t)^\top (Ax + Bu + Fw + Hv)\right] \\
&= \dot{p}(t)^\top x + (s^\top + p(t)^\top A)x + \min_{|u|\leq Ex}\left[(r^\top + p(t)^\top B)u\right] \\
&\qquad + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w + p(t)^\top Fw\right] + \max_{|v|\leq Gx}\left[-\delta^\top v + p(t)^\top Hv\right] \\
&= \dot{p}(t)^\top x + (s^\top + p(t)^\top A)x - \left|r^\top + p(t)^\top B\right| Ex \\
&\qquad + \max_{w\geq 0}\left[(-\boldsymbol{\gamma}^\top + p(t)^\top F)w\right] + \left|-\delta^\top + p(t)^\top H\right| Gx.
\end{aligned} \tag{3.8}$$

Given the linear nature of the optimization setting and the policy constraint design, the resulting HJB minimax equation becomes decoupled. Moreover, due to the linear nature of the value function (2.30) in this setting, the optimizing variables attain their optimal values on the boundary, i.e. $u_i \in \{-E_i x, E_i x\}$, $w_i \in \{0, \infty\}$, $v_i \in \{-G_i x, G_i x\}$ for all i. Thus, the differential equation (3.8) admits a finite solution if and only if (3.4) holds. Assuming (3.4) is satisfied, the HJB equation becomes

$$\dot{p}(t)^\top x + (s^\top + p(t)^\top A)x - \left|r^\top + p(t)^\top B\right| Ex + \left|-\delta^\top + p(t)^\top H\right| Gx = 0 \tag{3.9}$$

which is exactly (3.3). Since the right-hand side of (3.3) is Lipschitz, a solution exists and is unique. Hence, (i) and (ii) in Theorem 26 and in Lemma 21 in Subsection 2.3 are equivalent. Consequently, the proof of this theorem relies on Lemma 21.

Lastly, an expression for the optimal control policy is given

$$\begin{aligned}\mu(x) &= \arg\min_{u \le Ex} \left[g(x,u,w) + \left(\nabla_x p(t)^\top x \right) (f(x,u,w)) \right] \\ &= \arg\min_{u \le Ex} \sum_{i=1}^{m} \left[\left(r_i^\top + p(t)^\top B_i \right) u_i \right].\end{aligned}$$

Because for all $i = 1, \ldots, m$ the inequality $|u| \le Ex$ restricts u_i to the interval $[-E_i x, E_i x]$, the minimum is attained when $(r_i + p(t)^\top B_i)$ and u_i have opposite signs. If $(r_i + p(t)^\top B_i) = 0$, then any $u_i \in [-E_i x, E_i x]$ is admissible. Thus, $u_i \in -\text{sign}(r_i + p(t)^\top B_i) E_i x$. This proves the formula for $K(t)$ in (3.5). ■

Although the solution to the ODE (3.3) is unique, the same is not necessarily true for the control policy (3.5) that achieves the optimal cost. At any time instant for which there occurs a switch in the sign of $r_i + p(t)^\top B_i$, any choice $u_i \in [-E_i x, E_i x]$ would render the control policy optimal. Naturally, this can be neglected if such behavior occurs on a set of zero measure. It should be noted, however, that $(r_i + p(t)^\top B_i) = 0$ could also hold on a set with positive measure, meaning that multiple choices of the controller achieve the optimal cost. This is illustrated in the next example. Nevertheless, consistently choosing either $u_i = -E_i x$ or $u_i = E_i x$ when $(r_i + p(t)^\top B_i) = 0$ yields a bang-bang controller and is sufficient for optimality.

EXAMPLE 9
Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}; E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F, G, H = 0; \;\; s = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top; \;\; r = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top.$$

The solution to the Bellman equation (3.3) is

$$p(t) = (1 - e^{-t}) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top. \tag{3.10}$$

Then

$$r^\top + p(t)^\top B = \begin{bmatrix} 0 & 2(1 - e^{-t}) \end{bmatrix}$$

and both K_1 and K_2 given by

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E; \;\; K_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} E$$

achieve the optimal cost (3.10) and are stabilizing, with

$$A - BK_1 = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \; A - BK_2 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad \square$$

Infinite-horizon case

Consider the optimal control problem specified in (3.1). In this section, we present its infinite horizon variant.

$$\begin{aligned} &\inf_{\mu} \sup_{w,v} \int_0^{\infty} \left[s^\top x(\tau) + r^\top u(\tau) - \boldsymbol{\gamma}^\top w(\tau) - \delta^\top v(\tau) \right] d\tau \\ &\text{Subject to} \\ &\qquad \dot{x}(t) = Ax(t) + Bu(t) + Fw(t) + Hv(t) \qquad (3.11) \\ &\qquad x(0) = x_0, \;\; u(t) = \mu(x(t)), \\ &\qquad |u| \le Ex, \;\; w \ge 0, \;\; |v| \le Gx. \end{aligned}$$

THEOREM 27

Let $A \in \mathbb{R}^{n\times n}$, $B = [B_1, \ldots, B_m] \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}_+^{n\times l}$, $H \in \mathbb{R}^{n\times c}$, $E = \left[E_1^\top, \ldots, E_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$ such that $E_i^\top \neq \mathbb{0}$ for all i, $G \in \mathbb{R}_+^{c\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\boldsymbol{\gamma} \in \mathbb{R}_+^l$ and $\delta \in \mathbb{R}^c$. Suppose that $A - |B|E$ is Metzler and

$$s \ge E^\top |r| - G^\top |\delta|. \qquad (3.12)$$

Then the following statements are equivalent:

(i) The optimal control problem (3.11), has a finite value for every $x_0 \in \mathbb{R}_+^n$;

(ii) There exists $p \in \mathbb{R}_+^n$ such that

$$A^\top p = E^\top \left| r + B^\top p \right| - G^\top \left| -\delta + H^\top p \right| - s \qquad (3.13)$$

and

$$\boldsymbol{\gamma} \ge F^\top p \qquad (3.14)$$

Moreover, if (i)–(ii) are satisfied, then the infinite horizon minimax control problem (3.11) has minimum value $p^\top x_0$ and the control law $u(t) = -Kx(t)$ is optimal when

$$K \in \begin{bmatrix} \operatorname{sign}(r_1 + p^\top B_1)E_1 \\ \vdots \\ \operatorname{sign}(r_m + p^\top B_m)E_m \end{bmatrix}. \qquad (3.15)$$

$\square$

Note that, Remark 6, 7 and 8 also apply to the infinite horizon case.

Analogous to the finite time horizon case, it may happen that the control law that achieves the optimal cost is not unique. Taking $T \to \infty$ in (9) provides an example of such a situation.

Proof: The proof of this Theorem relies on Lemma 22 in Subsection 2.3. Analogous to Theorem 26, define (3.6). Then the minimax optimal control problem (3.11) is a special case of the general minimax setting (2.27) in Section 2.3. Therefore, (i) in Theorem 27 is equivalent to (i) in Lemma 22 in Section 2.3. Additionally, condition (3.2) gives

$$\max_{w,v}\left[g(x,u,\omega)\right] = \max_{\substack{w\geq 0\\ |v|\leq Gx}} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v\right] \geq 0.$$

as required in Lemma 22. Next, we verify that equation (3.13) in (ii) is equivalent to (2.28) in Lemma 22. Define $J^*(x) = p^\top x$. Equation (2.28) in Lemma 22 in the Appendix implies that

$$\begin{aligned}
0 &= \min_u \max_w \left[g(x,u,w) + \nabla_x J^*(x) f(x,u,w)\right] \qquad (3.16)\\
&= \min_u \max_{w,v} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v + p^\top (Ax + Bu + Fw + Hv)\right]\\
&= \left(s^\top + p^\top A\right)x + \min_{|u|\leq Ex}\left[(r^\top + p^\top B)u\right]\\
&\qquad + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w + p^\top F w\right] + \max_{|v|\leq Gx}\left[-\delta^\top v + p^\top H v\right]\\
&= \left(s^\top + p^\top A\right)x - \left|r^\top + p^\top B\right| Ex\\
&\qquad + \max_{w\geq 0}\left[(-\boldsymbol{\gamma}^\top + p^\top F)w\right] + \left|-\delta^\top + p^\top H\right| Gx.
\end{aligned}$$

By applying condition (3.14) we obtain

$$\left(s^\top + p^\top A\right)x - \left|r^\top + p^\top B\right| Ex + \left|-\delta^\top + p^\top H\right| Gx = 0$$

which is equation (3.13) in (ii). Hence, (i), (ii) in Theorem 27 and in Lemma 22 are equivalent. As a consequence, the proof of equivalence between (i), (ii) in Theorem 27 follows from the proof of Lemma 22.

Finally, the formula for K in (3.15) is derived analogously to the formula for $K(t)$ in the finite horizon case (3.5). ■

The following theorem proposes an iterative fixed point method to solve the HJB equation (3.13). It relies on an iterative method for solving the Bellman equation of the discrete-time version of the optimal control problem (3.11), which appears in the following subsection and was introduced in Section 2.3.

THEOREM 28—CONTINUOUS-TIME VALUE ITERATION
Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{n\times c}$, $E \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\delta \in \mathbb{R}^c$ and $h \in \mathbb{R}$. Assume $s \geq E^\top |r| - G^\top |\delta|$, $A - |B|E$ is Metzler and $A - |B|E + hI_n \geq 0$ with $h > 0$. Define $\hat{A} = \frac{1}{h}A + I_n$, $\hat{B} = \frac{1}{h}B$, $\hat{H} = \frac{1}{h}H$, $\hat{E} = E$, $\hat{G} = G$, $\hat{s} = \frac{1}{h}s$, $\hat{r} = \frac{1}{h}r$ and $\hat{\delta} = \frac{1}{h}\delta$. Then, the recursive sequence $\{p_k\}_{k=0}^\infty$ with $p_0 = 0$ and

$$p_{k+1} = \hat{s} + \hat{A}^\top p_k - \hat{E}^\top \left|\hat{B}^\top p_k + \hat{r}\right| + \hat{G}^\top \left|\hat{H}^\top p_k - \hat{\delta}\right| \tag{3.17}$$

has a finite limit if and only if there exists $p \in \mathbb{R}^n$ such that (3.13), in which case $p_k \to p$. □

Proof: From Theorem 1 in [Gurpegui et al., 2023] it is direct that (3.17) has a finite limit if and only if there exists a $p \in \mathbb{R}_+^n$ such that

$$p = \hat{s} + \hat{A}^\top p - \hat{E}^\top \left|\hat{B}^\top p + \hat{r}\right| + \hat{G}^\top \left|\hat{H}^\top p - \hat{\delta}\right|. \tag{3.18}$$

and in this case, $p_k \to p$. By definition of A, B, H, E, G, s, r and δ, adding $-p$ to both sides of (3.18) and multiplying by h yields

$$\begin{aligned}
0 &= \hat{s} + (\hat{A} - I_n)^\top p - \hat{E}^\top \left|\hat{B}^\top p + \hat{r}\right| + \hat{G}^\top \left|\hat{H}^\top p - \hat{\delta}\right| \Rightarrow \\
0 &= h\left(\hat{s} + (\hat{A} - I_n)^\top p - \hat{E}^\top \left|\hat{B}^\top p + \hat{r}\right| + \hat{G}^\top \left|\hat{H}^\top p - \hat{\delta}\right|\right) \\
&= s + A^\top p - E^\top \left|B^\top p + r\right| + G^\top \left|H^\top p - \delta\right|.
\end{aligned}$$

Thus, p solves (3.18) if and only if p solves (3.13). ■

3.2 l_1-induced gain analysis

The l_1-induced gain of a positive system plays an important role in robust stability analysis against dynamical and parametric uncertainties [Rantzer and Valcher, 2018a; Briat, 2013; Ebihara et al., 2011]. Past studies on switched positive systems [Zappavigna et al., 2010b; Zappavigna et al., 2010a], show how the l_1-induced gain can also be employed as a performance index to be minimized.

Recall that the l_1-induced gain of a system is the maximum ratio of the l_1 norm of the system's output to the l_1 norm of the control and disturbance input. Thus, it measures the maximum amplification of the input disturbances and control signals to the system's output.

Formally, we define the l_1-induced gain of the system

$$\dot{x}(t) = Ax(t) + Bu(t) + Fw(t) + Hv(t)$$

with respect to a disturbance ω as

$$\|G_{\mu,\omega}\|_{1-\text{ind}} = \sup_{\omega \neq 0} \frac{\|z\|_1}{\|\omega\|_1} = \sup_{\omega \neq 0} \frac{\int_0^\infty \left[s^\top x(\tau) + r^\top u(\tau)\right]\, d\tau}{\int_0^\infty \mathbb{1}^\top \omega(\tau)\, d\tau}. \tag{3.19}$$

In this section, the l_1-induced gain of the closed loop system

$$G_\mu : \begin{cases} \dot{x}(t) & = Ax(t) + B\mu(x(t)) + Fw(t) + Hv(t) \\ z(t) & = s^\top x(t) + r^\top \mu(x(t)) - \boldsymbol{\gamma}^\top w - \delta^\top v \end{cases}$$

with respect to the disturbance $w \geq 0$ and the worst-case disturbance of (3.11) are given. The same results are obtained for the infinite time setting in [Blanchini et al., 2023, Sec. 6]. However, this section extends that analysis by characterizing the l_1-induced gain of the system with respect to the disturbance penalty in the cost function. In addition, we give tight bounds on the disturbance penalty to ensure that the optimal control problem achieves a finite cost, which motivates the conditions (3.4) and (3.14) in the context of the l_1-induced gain of the disturbance.

Assume in Theorem 26 and Theorem 27 that $G = H^\top = 0$, $w \neq 0$ and denote $\boldsymbol{\gamma} = \gamma \mathbb{1}$. Let the l_1-induced gain of the system in Figure 1.2 with respect to the disturbance w be bounded by a parameter γ^*,

$$\|G_{\mu,w}\|_{1-\text{ind}} = \sup_{w \geq 0} \frac{\|z\|_1}{\|w\|_1} \leq \gamma^*. \tag{3.20}$$

It is immediate from (3.19) that (3.20) gives

$$\sup_{w \geq 0} \int_0^T \left[s^\top x(\tau) + r^\top u(\tau) - \gamma^* \mathbb{1}^\top w(\tau)\right]\, d\tau \leq 0.$$

Therefore, given a fixed time horizon $T \in [0, \infty]$

$$\begin{aligned} &\arg\max_{w \geq 0} \left[\int_0^T \left[s^\top x(\tau) + r^\top u(\tau) - \gamma \mathbb{1}^\top w(\tau)\right]\, d\tau\right] \\ &= \begin{cases} 0 & \gamma \geq \gamma^* \\ \infty & \gamma < \gamma^* \end{cases} \end{aligned} \tag{3.21}$$

holds. Thus, the cost function (3.21) has a finite value when $\gamma \geq \gamma^*$. From Theorem (26) and Theorem (27) we know that there exists a finite solution to the optimal control problem (3.1) and (3.11) if and only if the conditions in (3.4) and (3.14) hold, respectively. Thus, the minimum l_1-induced gain of the system in Figure 1.2 with $G = H^\top = 0$ is

$$\boldsymbol{\gamma}^* = \boldsymbol{\gamma} = \begin{cases} F^\top p(0), & \text{if } T < \infty \\ F^\top p & \text{if } T = \infty. \end{cases} \tag{3.22}$$

The worst-case disturbance can be found in [Blanchini et al., 2023].

inequality with equality in at least one entry

3.3 Line-shaped water-flow network

Consider a river system with a downstream flow represented by β and the dissipation capacity α. The river flow is segmented into n sections, where the state x reflects the volume of water in each section, with initial volume one. Dams positioned at these sections act as control points u modulating the downstream flow. In this context, disturbances v arise due to leakage effects, where water is dissipated more than is accounted for, resulting in extraneous flow towards downstream reservoirs. These leakages intensify as the disturbances propagate downriver, influencing the water volumes of subsequent sections. Additionally, we denote w as an unconstrained positive disturbance in the form of rain. The system dynamics of

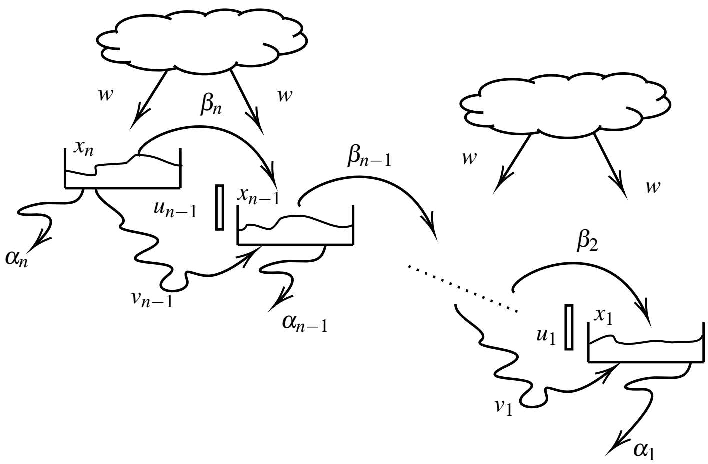


Figure 3.1 Scalable Water-flow diagram.

the water-flow diagram in Figure 3.1 describe the water-flow of a river with different stations represented by the nodes x_i, $i = 1, \dots, n$. The state-space model of the system dynamics in Figure 3.1 can be described by

$$\begin{aligned}
\dot{x}_1 &= -(\alpha_1)x_1 + \beta_2 x_2 - u_1 + v_1, \\
\dot{x}_i &= -(\alpha_i + \beta_i)x_i + \beta_{i+1}x_{i+1} - u_i + u_{i-1} + v_i - v_{i-1} \\
\dot{x}_n &= -(\alpha_n + \beta_n)x_n + u_{n-1} - v_{n-1}
\end{aligned} \tag{3.23}$$

$i = 2, \dots, n-1$. Assume that the network is homogeneous, i.e. $\alpha_i = \alpha$ for all $i = 1, \dots, n$, $\beta_i = \beta$ for all $i = 2, \dots, n$. Then the system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

and initial state $x(0) = x_0$ are

$$A = \begin{bmatrix} -\alpha & \beta & \cdots & 0 \\ 0 & -(\alpha+\beta) & \cdots & 0 \\ \vdots & \vdots & \ddots & \beta \\ 0 & 0 & 0 & -(\alpha+\beta) \end{bmatrix}; \; B = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 1 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 0 & 0 \; \cdots & 0 & 1 \end{bmatrix};$$

$$H = -B; \qquad x_0 = \mathbb{1}. \tag{3.24}$$

Leakages: load disturbances

Initially, assume there is no additional rain input, i.e. $w = 0$. The control and disturbance actions in this network system are constrained by $|u| \leq Ex$, $|v| \leq Gx$ with $E, G \geq 0$. Here, G represents the slope of the water flow network, where a higher altitude corresponds to a greater leakage disturbance. The choice of parameters E, s, r, δ are designed to satisfy the assumptions (3.27), (3.28):

$$E = \begin{bmatrix} 0 & \zeta_u & 0 & \cdots & 0 \\ 0 & 0 & \zeta_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \zeta_u \end{bmatrix}; \; G = \frac{1}{n} \begin{bmatrix} \zeta_v & 0 & \dots & 0 & 0 \\ 0 & 2\zeta_v & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (n-1)\zeta_v & 0 \end{bmatrix};$$

$$s = \rho_s \begin{bmatrix} 1 \; 0 \; \dots \; 0 \; 0 \end{bmatrix}; \qquad r = \rho_u \begin{bmatrix} 2/n \; \dots \; n-1/n \; 0 \end{bmatrix}^\top; \tag{3.25}$$

and $\delta = \rho_v \mathbb{1}$. Here ζ_u and ζ_v scale, respectively, the control and disturbance capacity. ρ_s, ρ_u, ρ_v are scaling parameters for the penalties on state, control, and disturbance terms in the cost function. The sparsity structure of E relies on the Metzler condition, ensuring a viable setup under the given constraints.

The optimal control problem setup (3.11) of this example is

$$\begin{aligned} &\min_{\mu} \sup_{v} \int_0^T \left[s^\top x(\tau) + r^\top u(\tau) - \delta^\top v(\tau) \right] d\tau \\ &\text{Subject to} \\ &\qquad \dot{x}(t) = Ax(t) + Bu(t) + Hv(t) \\ &\qquad x(0) = x_0, \;\; u(t) = \mu(x(t)), \\ &\qquad |u| \leq Ex, \;\; |v| \leq Gx. \end{aligned} \tag{3.26}$$

The choice of ζ_u, ζ_v, ρ_s, ρ_u, and ρ_v depends on following conditions:

$$A - |B|E = \begin{bmatrix} -\alpha & \beta - \zeta_u & \cdots & 0 \\ 0 & -(\alpha+\beta) - \zeta_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta - \zeta_u \\ 0 & 0 & \cdots & -(\alpha+\beta) - \zeta_u \end{bmatrix} \tag{3.27}$$

has to be Metzler. Hence, it is necessary that $\beta \geq \zeta_u$. Moreover,

$$s - \left(E^\top |r| - G^\top |\delta|\right) = \begin{bmatrix} \rho_s + \frac{1}{n}\zeta_v\rho_v \\ 0 - \frac{2}{n}\left(\zeta_u\rho_u - \zeta_v\rho_v\right) \\ \vdots \\ 0 - \frac{n-1}{n}\left(\zeta_u\rho_u - \zeta_v\rho_v\right) \\ 0 \end{bmatrix} \geq 0. \tag{3.28}$$

Therefore, for all $\rho_s \geq 0$ it is sufficient that, $\zeta_u\rho_u - \zeta_v\rho_v \leq 0$.

Set $\alpha = 3$, $\beta = 10$ and assume the river flow has $n = 100$ sections. Note that, under these parameters, the state matrix has a Perron root of $\lambda_p = -3$.

Now, consider $\zeta_v = 3$. In the presence of leakages, the Perron root shifts to $\lambda_p \approx 0.19$ causing the system dynamics to become unstable.

Simulations. In Figure 3.2, the optimal cost evolution shows that, despite disturbances causing an increase in the cost and making the system unstable, the controller derived from our minimax framework successfully stabilizes and reduces the cost to the level observed in the absence of disturbances. This demonstrates that, under pertinent assumptions and proper tuning of the different optimization variables, the controller's effectiveness allows the system to maintain performance comparable to the disturbance-free scenario. Figure 3.3 complements this by showing the state trajectories for the open-loop system in the absence of disturbances, the open-loop system with disturbances, and the closed-loop system with control.

Finally, Figure 3.4 illustrates the behavior of the total optimal cost when $T \to \infty$ and as the network size increases. As the number of nodes grows, the cost in the presence of disturbances rises significantly, indicating how disturbances escalate the cost in larger networks. In contrast, when the controller derived from the minimax framework is introduced, it effectively stabilizes the system and significantly reduces the cost. These simulation results demonstrate that, with appropriate parameter tuning, the derived optimal policy from our novel problem class effectively compensates for disturbances entering locally at each node of the system, even when their effects cascade through the line-networked structure. Moreover, exploring alternative network topologies appears to be a promising direction for designing optimal network structures within the context of this theoretical framework.

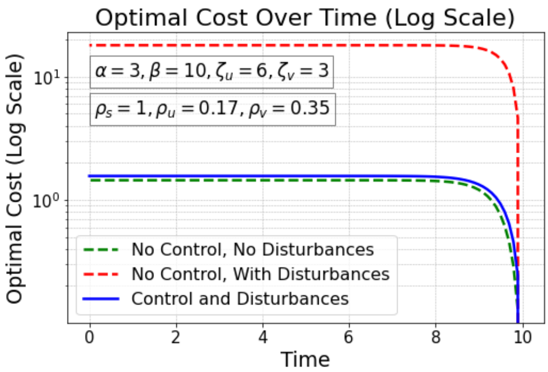


Figure 3.2 Optimal cost $p(t)^\top x_0$ evolution over time $t \in [0, T]$ with $T = 10$.

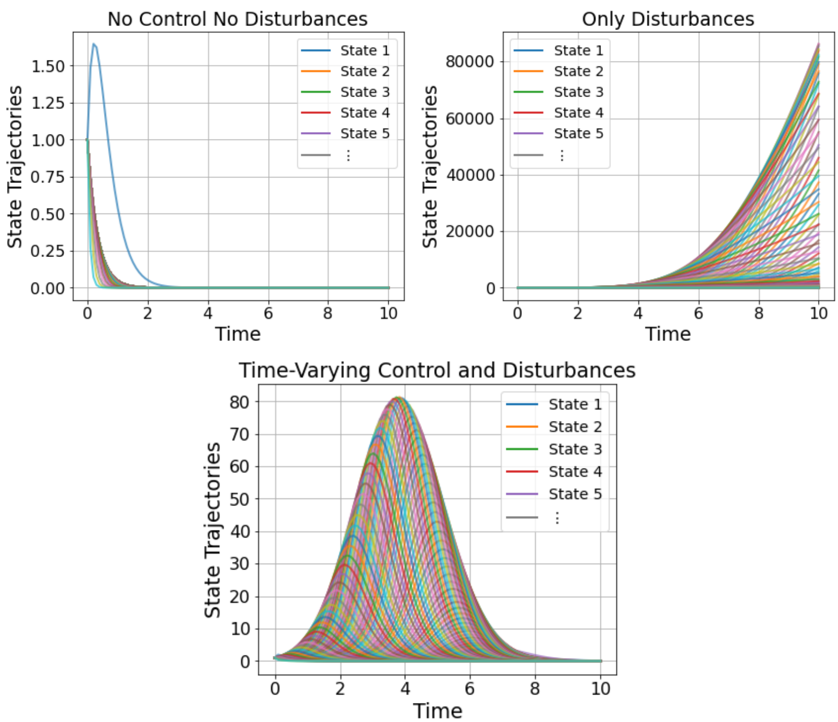


Figure 3.3 Trajectories $x(t) = e^{\tilde{A}t}x_0$ over time $t \in [0, T]$, with $T = 10$ of (a) the open-loop system $\tilde{A} = A$, (b) the open loop system in the presence of disturbances $\tilde{A} = A + |H|G$, (c) the closed loop system in presence of disturbance and control $\tilde{A} = A + |H|G - BK$.

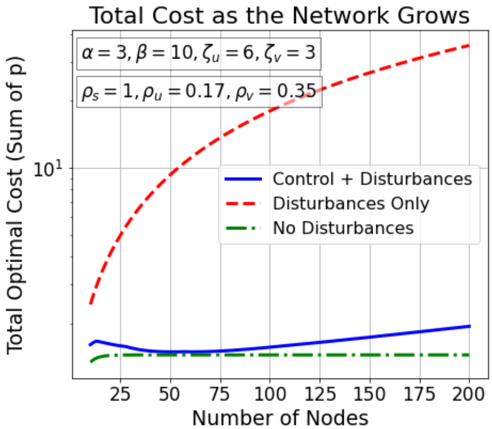


Figure 3.4 Logarithmic growth of the optimal cost $p^\top x_0$ when $T \to \infty$, as the water-flow network is scaled from $n = 2$ to 200 sections.

Rain: positive unconstrained disturbance.

Assume that it starts raining. Let the rain be interpreted as a worst-case positive, unconstrained disturbance $w \geq 0$ affecting all nodes homogeneously, i.e. $F = \mathbb{1}$. Suppose this disturbance persists over a time horizon of $T = 24$. An upper bound on the amount of rain that the feedback controller, derived from the minimax framework, can compensate for is studied. To establish this bound, condition (3.4) from Theorem 26 is applied. This condition provides a measure of the system's capacity to handle the disturbance, with $\boldsymbol{\gamma} \geq F^\top p(0)$ representing the minimum l_1-induced gain required to counteract the worst-case rain disturbance and maintain system stability. The extended optimal control problem becomes

$$\begin{aligned}
&\min_{\mu} \sup_{v} \int_0^{24} \left[s^\top x(\tau) + r^\top u(\tau) - \delta^\top v(\tau) - \boldsymbol{\gamma}^\top w \right] d\tau \\
&\text{Subject to} \\
&\qquad \dot{x}(t) = Ax(t) + Bu(t) + Hv(t) + \mathbb{1}w \qquad (3.29) \\
&\qquad x(0) = x_0, \quad u(t) = \mu(x(t)), \\
&\qquad |u| \leq Ex, \quad |v| \leq Gx, \quad w \geq 0.
\end{aligned}$$

Assuming the same parameter choice as before, it is obtained that $F^\top p(0) \approx 1.56$. Therefore, if the optimal control problem (3.26) has a finite solution $p^\top(0)x_0$ and $\gamma \geq 1.56$ then the extended optimal control problem (3.29) also has a finite solution, and its solution coincides with that of (3.26). This result establishes that the controller derived for the original problem is robust enough to handle the additional rain disturbance, provided that the gain condition is satisfied.

3.4 Discrete-time setting

Finite-horizon case

The optimal control problem in this section is formulated as a discrete-time minimax problem with nonnegative linear cost, positive linear dynamics, elementwise linear constraints on the control policy and the disturbance $v \in \mathbb{R}^c$, nonnegative unconstrained disturbance w and initial conditions equal to zero,

$$\min_{\mu} \max_{w,v} \sum_{t=0}^{T-1} \left[s^\top x(t) + r^\top u(t) - \boldsymbol{\gamma}^\top w(t) - \delta^\top v(t) \right]$$

$$\begin{aligned}
&\text{Subject to} \\
&\quad x(t+1) = Ax(t) + Bu(t) + Fw(t) + Hv(t) \\
&\quad x(0) = x_0, \quad u(t) = \mu(x(t)), \\
&\quad |u| \leq Ex, \quad w \geq 0, \quad |v| \leq Gx.
\end{aligned} \tag{3.30}$$

Here, x represents the n-dimensional vector of state variables, u the m-dimensional control variable, w, v are the l-dimensional and c-dimensional disturbance, μ is any, potentially nonlinear, control policy, E prescribes the structure of the control law, G determines the linear dependency of the disturbance v and the state, and $T \in \mathbb{R}_+$ is the time horizon. The objective is to minimize the worst-case cost over all possible control strategies.

In the next theorem we use dynamic programming theory (Subsection 2.3) to derive explicit solutions to the Bellman equation of the minimax setup (3.30) and give necessary and sufficient conditions for the existence of finite solutions in finite time.

THEOREM 29
Let $A \in \mathbb{R}^{n\times n}$, $B = \left[B_1^\top, \ldots, B_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$, $F \in \mathbb{R}^{n\times l}$, $H \in \mathbb{R}^{n\times c}$, $E = \left[E_1^\top, \ldots, E_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$, such that $E_i^\top \neq \mathbb{0}$ for all i, $G \in \mathbb{R}_+^{c\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^c$. Suppose that

$$A \geqslant |B| E \tag{3.31}$$

$$s \geqslant E^\top |r| - G^\top |\delta| . \tag{3.32}$$

Then the following statements are equivalent:

(i) The optimal control problem (3.30), has a finite value for every $x_0 \in \mathbb{R}_+^n$.

(ii) The algebraic equation in $p_k \in \mathbb{R}_+^n$,

$$p_k = s + A^\top p_{k-1} - E^\top \left| r + B^\top p_{k-1} \right| + G^\top \left| -\delta + H^\top p_{k-1} \right| \tag{3.33}$$

$$p_0 = 0,$$

has a unique solution and

$$\boldsymbol{\gamma} \geq F^\top p_T. \tag{3.34}$$

Moreover, if the above conditions holds, the minimal value of the optimal control problem (3.30) is $p_T^\top x_0$ and the optimal control policy is given by $u^*(t) = -K_k x(k)$, is optimal when

$$K_k \in \begin{bmatrix} \text{sign}(r_1 + p_{k-1}^\top B_1)E_1 \\ \vdots \\ \text{sign}(r_m + p_{k-1}^\top B_m)E_m \end{bmatrix}. \tag{3.35}$$

□

REMARK 9
The right-hand side of (3.5) is a set, since multiple controllers may exists that achieve the optimal cost. For any index i such that $r_i + [p_k]_j^\top B_i = 0$ all controllers in the set

$$\mathcal{K} = \left\{ DE \mid D_{ii} \in [-1,1], D_{jj} \in \text{sign}(r_j + [p_{k-1}]_j^\top B_j) \text{ for } j \neq i \right\}$$

correspond to same and unique solution p_k of the Bellman equation (3.33). □

REMARK 10
Condition (3.34) will be needed when applying Lemma 18 in section 2.3 to the objective function in (3.30). □

REMARK 11
The condition $E_i^\top \neq \mathbb{0}$ for all $i = 1, \ldots, m$, is imposed to prevent ill-posed scenarios where certain control inputs u_i satisfy $|u_i| \leq E_i^\top x = 0$ leading to trivial control actions. □

The following remark concerning the sparsity of the optimal controller still holds for the discrete-time setting.

REMARK 12
As in the continuous-time scenario, dynamic programming is conducted without explicitly enforcing linearity or sparsity; however, these properties naturally emerge as a result of the optimization criteria and constraints. Furthermore, the sparsity structure of the control gain K_k in (3.35) is determined by the E matrix, which is specified by the problem designer. □

Proof: The general nonlinear minimax optimal control problem (2.16) presented in the subsection 2.3 reduces to our problem set up (3.41) if

$$\begin{aligned} f(x,u,w) &:= Ax + Bu + \begin{bmatrix} F & H \end{bmatrix} \omega \\ g(x,u,w) &:= s^\top x + r^\top u - \begin{bmatrix} \gamma^\top & \delta^\top \end{bmatrix} \omega \\ \omega &:= \begin{bmatrix} w^\top & v^\top \end{bmatrix}^\top. \end{aligned} \tag{3.36}$$

It is direct that $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \longrightarrow \mathbb{R}_+$ is linear. Thus, the minimax control problem (3.30) is a special case of the general minimax setting (2.16) in section 2.3. Observe that, condition (3.43) gives

$$\begin{aligned}
&\max_{\substack{w\geq 0\\ |v|\leq Ex}} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \boldsymbol{\delta}^\top v\right] \\
&= s^\top x + r^\top u + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] + \max_{|v|\leq Gx}\left[-\boldsymbol{\delta}^\top v\right] \\
&\geq \left(E^\top |r| - G^\top |\delta|\right)^\top x + r^\top u + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] + |\delta|^\top Gx \\
&\geq \left(E^\top |r|\right)^\top x - \left|r^\top\right| Ex + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] \\
&\geq \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w\right] \geq 0
\end{aligned} \tag{3.37}$$

Therefore,

$$\max_{w,v}\left[g(x,u,\omega)\right] = \max_{\substack{w\geq 0\\ |v|\leq Gx}}\left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \boldsymbol{\delta}^\top v\right] \geq 0$$

as required in Lemma 18. Recall the cost-to-go function (2.21). Define $J_0(x) = p_0$ and $J_k(x) = p_k^\top x$. It follows from the Bellman equation (2.17) in Lemma 18 that

$$\begin{aligned}
J_k^*(x(t)) &= \min_u \max_\omega \left[g(x(t),u(t),\omega(t)) + J_{k-1}^*(f(x(t),u(t),\omega(t)))\right] \qquad (3.38)\\
&= \min_u \max_{v,w}\left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \boldsymbol{\delta}^\top v + p_{k-1}^\top (Ax + Bu + Fw + Hv)\right] \\
&= s^\top x + p_{k-1}^\top Ax + \min_{|u|\leq Ex}\left[r^\top u + p_{k-1}^\top Bu\right] \\
&\qquad + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w + p_{k-1}^\top Fw\right] + \max_{|v|\leq Gx}\left[-\delta^\top v + p_{k+1}^\top Hv\right] \\
&= s^\top x + p_{k-1}^\top Ax - \left|r + B^\top p_{k-1}\right|^\top Ex \\
&\qquad + \max_{w\geq 0}\left[-\boldsymbol{\gamma}^\top w + p_{k-1}^\top Fw\right] + \left|-\delta + H^\top p_{k-1}\right|^\top Gx.
\end{aligned}$$

Analogous to the continuous-time setting, because of the linear nature of the optimization setting and the policy constraint design, the resulting Bellman minimax equation becomes decoupled. Moreover, due to the linear nature of the value function (2.21), the optimizing variables attain their optimal values on the boundary, i.e. $u_i \in \{-E_i x, E_i x\}$, $w_i \in \{0,\infty\}$, $v_i \in \{-G_i x, G_i x\}$ for all i. Applying (3.34), the Bellman equation becomes

$$s^\top x + p_{k-1}^\top Ax - \left|r + B^\top p_{k-1}\right|^\top Ex + \left|-\delta + H^\top p_{k-1}\right|^\top Gx = p_k^\top \tag{3.39}$$

which is exactly (3.33). Hence, (i) and (ii) in Theorem 29 and Lemma 18 are equivalent. Consequently, the proof of this theorem relies on Lemma 18. Lastly, an expression for the optimal control policy $u(t) = \mu(x(t))$ is given by

$$\begin{aligned}\mu(x) &= \arg\min_{u \leq Ex} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v + p_{k-1}^\top (Ax + Bu + Fw + Hv) \right] \\ &= \arg\min_{u \leq Ex} \sum_{i=1}^{m} \left[\left(r_i^\top + p_{k-1}^\top B_i \right) u_i \right].\end{aligned}$$

Because for all $i = 1, \ldots, m$ the inequality $|u| \leq Ex$ restricts u_i to the interval $[-E_i x, E_i x]$, the minimum is attained when $(r_i + p_{k-1}^\top B_i)$ and u_i have opposite signs. If $(r_i + p_{k-1}^\top B_i) = 0$, then any $u_i \in [-E_i x, E_i x]$ is admissible. Thus, $u_i \in -\text{sign}(r_i + p_{k-1}^\top B_i) E_i x$. This proves the formula for K_k in (5.20). ■

While the solution to the Bellman equation (3.3) is unique, the corresponding control policy (5.20) that achieves the optimal cost may vary. Specifically, when $(r_i + p_{k-1}^\top B_i) = 0$, multiple control actions can lead to the same optimal cost. This is illustrated in the next example. Nonetheless, consistently selecting either $u_i = -E_i x$ or $u_i = E_i x$ whenever $(r_i + p_{k-1}^\top B_i) = 0$ ensures a bang-bang control strategy, which is sufficient to achieve optimality.

EXAMPLE 10
Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}; B = \frac{1}{4} \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 3 \end{bmatrix}; E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$F, H, G = 0; \;\; s = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^\top; \;\; r = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top.$$

The solution to the Bellman equation (3.18) for a general time step k can be written as

$$p = [1 \ 1 \ 1]^\top. \tag{3.40}$$

where $\alpha_k \geq 0$ is the value of the first and second coordinates, and $\beta_k \geq 0$ alternates between 1 (odd k) and 0 (even k). Then

$$r^\top + p^\top B = \begin{bmatrix} 0 & 3/4 \end{bmatrix}$$

and both K_1 and K_2 given by

$$K_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} E; \;\; K_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} E$$

achieve the optimal cost (3.40) and are stabilizing, with

$$A - BK_1 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}; \quad A - BK_2 = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}. \qquad \square$$

Infinite-horizon case

Consider the optimal control problem specified in (3.30). In this section, we present its infinite horizon variant.

$$\inf_{\mu} \max_{w} \sum_{t=0}^{\infty} \left[s^\top x(t) + r^\top u(t) - \boldsymbol{\gamma}^\top w(t) - \delta^\top v(t) \right] \tag{3.41}$$

subject to

$$\begin{aligned} & x(t+1) = Ax(t) + Bu(t) + Fw(t) + Hv(t), \\ & x(0) = x_0; \quad u(t) = \mu(x(t)) \\ & |u| \leq Ex\ ; w \geq 0; \ |v| \leq Gx \end{aligned}$$

THEOREM 30
Let $A \in \mathbb{R}^{n\times n}$, $B = \left[B_1^\top, \ldots, B_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$, $F \in \mathbb{R}^{n\times l}$, $H \in \mathbb{R}^{n\times c}$, $E = \left[E_1^\top, \ldots, E_m^\top\right]^\top \in \mathbb{R}_+^{m\times n}$ such that $E_i^\top \neq \mathbb{0}$ for all i, $G \in \mathbb{R}_+^{c\times n}$, $s \in \mathbb{R}^n$, $r \in \mathbb{R}^m$, $\gamma \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^c$. Suppose that

$$A \geqslant |B| E \tag{3.42}$$

$$s \geqslant E^\top |r| - G^\top |\delta|. \tag{3.43}$$

Then the following statements are equivalent:

(i) The optimal control problem (3.41), has a finite value for every $x_0 \in \mathbb{R}_+^n$.

(ii) The recursive sequence $\{p_k\}_{k=0}^{\infty}$ with $p_0 = 0$ and

$$p_k = s + A^\top p_{k-1} - E^\top \left| r + B^\top p_{k-1} \right| + G^\top \left| -\delta + H^\top p_{k-1} \right| \tag{3.44}$$

has a finite limit.

(iii) There exists $p \in \mathbb{R}_+^n$ such that

$$p = s + A^\top p - E^\top \left| r + B^\top p \right| + G^\top \left| -\delta + H^\top p \right|. \tag{3.45}$$

and

$$\boldsymbol{\gamma} \geq F^\top p. \tag{3.46}$$

If (iii) is true then (3.41) has the minimal, finite, nonnegative value $p^T x_0$, with p being the limit of the recursive sequence $\{p_k\}_{k=0}^{\infty}$ in (ii). Moreover, the control law $u(t) = -Kx(t)$, is optimal when

$$K \in \begin{bmatrix} \mathrm{sign}(r_1 + p^\top B_1)E_1 \\ \vdots \\ \mathrm{sign}(r_m + p^\top B_m)E_m \end{bmatrix}. \tag{3.47}$$

□

Analogous to the finite time horizon case, it may happen that the control law that achieves the optimal cost is not unique. Taking $T \to \infty$ in Example 10 provides an example of such a situation.

Proof: The proof of this Theorem relies on Lemma 19 in Section 3.4. Analogous to Theorem 29 define (3.36). Then the minimax optimal control problem (3.41) is a special case of the general minimax setting (2.18) in Section 3.4. Additionally, condition (3.43) gives

$$\max_{w,v} [g(x,u,\omega)] = \max_{\substack{w \geq 0 \\ |v| \leq Gx}} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v \right] \geq 0$$

as required in Lemma 19 in the Section 2.3. Now, it is clear that (i) in Theorem 30 is equivalent to (i) in Lemma 19 in the Section 2.3. Next, we will verify that the recursive sequence in (ii) is equivalent to (ii) in Lemma 19. To prove this we use induction over $p_k^\top x = J_k(x)$. By definition, it is direct that $p_0^\top x = 0 = J_0(x)$ for all x. For the induction step we assume that $p_k^\top x = J_k(x)$. Now we want to prove that $p_{k+1}^\top x = J_{k+1}(x)$. From (2.19) and the induction hypothesis we have that

$$\begin{aligned}
J_{k+1}(x) &= \min_u \max_\omega [g(x,u,w) + J_k(f(x,u,\omega))] \\
&= \min_u \max_{w,v} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v + p_k^\top (Ax + Bu + Fw + Hv)) \right] \\
&= s^\top x + p_k^\top Ax + \min_{|u| \leq Ex} \left[r^\top u + p_k^\top Bu \right] \\
&\qquad + \max_{w \geq 0} \left[-\boldsymbol{\gamma}^\top w + p_k^\top Fw \right] + \max_{|v| \leq Gx} \left[-\delta^\top v + p_k^\top Hv \right] \\
&= s^\top x + p_k^\top Ax - \left| r + B^\top p_k \right|^\top Ex \\
&\qquad + \max_{w \geq 0} \left[-\boldsymbol{\gamma}^\top w + p_k^\top Fw \right] + \left| -\delta + H^\top p_k \right|^\top Gx.
\end{aligned}$$

Because the sequence in (ii) has a finite limit, the Bellman equation becomes

$$s^\top x + p_k^\top Ax - \left| r + B^\top p_k \right|^\top Ex + \left| -\delta + H^\top p_k \right|^\top Gx = p_{k+1}^\top x.$$

Therefore, $p_k^\top x = J_k(x)$ for all $k \in \mathbb{N}$ and all $x \in \mathbb{R}^n_+$, and $p^\top x = J^*(x)$ for all x. Hence, (ii) and (iii) both in Theorem 30 and Lemma 19 are equivalent. Furthermore, note that under homogeneous constraints the linearity of J_k is preserved during value iteration.

Because (i), (ii) and (iii) in this theorem and in Lemma 19 are equivalent, the proof of equivalence between (i), (ii) and (iii) in Theorem 30 follows from the proof of Lemma 19 in Section 2.3.

To finish this proof it is just left to give an expression for the optimal control policy $u(t) = \mu(x(t))$,

$$\begin{aligned}\mu(x) &= \arg\min_{|u|\le Ex} \left[s^\top x + r^\top u - \boldsymbol{\gamma}^\top w - \delta^\top v + p^\top (Ax + Bu + Fw + Hv)\right] \\ &= \arg\min_{|u|\le Ex} \sum_{i=1}^{m} \left[\left(r_i + p^\top B_i\right) u_i\right].\end{aligned}$$

Finally, since for all $i = 1 \dots m$ the inequality $|u| \le Ex$ restricts u_i to the interval $[-E_i x, E_i x]$, the minimum is attained when $(r_i + p^\top B_i)$ and u_i have opposite signs. Thus, $u_i = -\text{sign}(r_i + p^\top B_i)E_i$ for all $i = 1 \dots m$. ■

3.5 Optimal voltage control in a DC power network

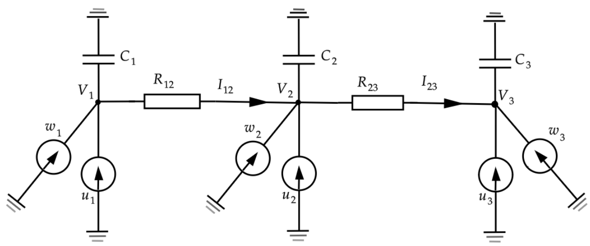


Figure 3.5 Example of a DC network of 3 terminals (buses) and 5 lines. The controls u_i are used to control the voltage when the system is subjected to the disturbances w_i.

The optimal control problem (3.41) admits sparsity constraints on the controller, making it particularly useful for problems defined over network graphs. Here, we consider a simple example of voltage control in a DC (i.e., direct current) power network. Here, the nodes represent voltage source converters with positive voltage dynamics, interconnected through resistive lines. The model can, for example, capture an envisioned multi-terminal high-voltage DC network, whose design aims to transmit power over long distances while maintaining low losses [Van Hertem and Ghandhari, 2010; Andreasson et al., 2017] or a simplified DC distribution network [Karlsson and Svensson, 2003]. The (continuous) voltage dynamics at the DC bus i (node i) is given by

$$\begin{aligned} C_i\dot{V}_i(t) &= -\sum_{j=1}^{n} I_{ij} + u_i(t) + v_i(t) \\ &= -\sum_{j=1}^{n} \frac{1}{R_{ij}}(V_i(t) - V_j(t)) + u_i(t) + v_i(t), \end{aligned} \tag{3.48}$$

for all $i = 1, 2, \ldots, n$, where u_i denotes the controlled injected current, R_{ij} the resistance of transmission line (i, j) (with $R_{i,j} = \infty$ if there exists no line connecting nodes i and j), and C_i is the total capacitance at bus i.[1] We have also included the disturbance current w_i, arising from variations in local generation and load. Defining the vector $V = [V_1(t), ..., V_n(t)]^\top$, u and v analogously, and $C = \text{diag}([C_1, ...C_n])$, we may write (3.48) on vector form as

$$C\dot{V}(t) = -\mathcal{L}_R V(t) + u(t) + v(t). \tag{3.49}$$

Here, $\mathcal{L}_R$ is the weighted Laplacian matrix of the graph representing the transmission lines, whose edge weights are given by the conductances $\frac{1}{R_{ij}}$, i.e.,

$$[\mathcal{L}_R]_{i,j} = \begin{cases} -\frac{1}{R_{i,j}} & \text{if } \; i \neq j \\ \sum_{j=1}^{n} \frac{1}{R_{i,j}} & \text{if } \; i = j \end{cases}.$$

Note that $-\mathcal{L}_R$ is Metzler, and the system (3.49) thus positive.
The dynamics in 3.49 can be discretized as

$$C(V(\tau h + h) - V(\tau h))/h = -\mathcal{L}_{\mathcal{R}} V(\tau h) + u(\tau h) + v(\tau h).$$

Setting $t = \tau h$ and re-defining the state $x(t) = V(\tau h)$ gives the discrete-time dynamics

$$x(t+1) = \left[I - hC^{-1}\mathcal{L}_R\right] x(t) + hC^{-1}u(t) + hC^{-1}v(t). \tag{3.50}$$

Now, we formulate the optimal control problem (3.41) for the dymamics (3.50)

$$\inf_{\mu} \max_{v} \sum_{t=0}^{\infty} \left[s^\top x(t) + r^\top u(t) - \gamma^\top v(t) \right] \tag{3.51}$$

Subject to

$$\begin{aligned} & x(t+1) = \left[I - hC^{-1}\mathcal{L}_R\right] x(t) + hC^{-1}u(t) + hC^{-1}v(t) \\ & u(t) = \mu(x(t)) \; ; \;\; x(0) = x_0 \\ & |u| \leq Ex \; ; \; |v| \leq Gx \end{aligned}$$

[1] Any line capacitances can for the purpose of this example be absorbed in to the buses.

Identifying A and B, condition (3.42) reads

$$\left[I - hC^{-1} \cdot \mathcal{L}_R\right] \geqslant hC^{-1}E + hC^{-1}G. \tag{3.52}$$

Clearly, the right hand side of the inequality must inherit the zero pattern of $\mathcal{L}_R$, i.e. its sparsity pattern. In other words, the disturbances and control signal must be compatible with the physical network structure and depend only on connected nodes. E, or G can, however be more sparse than $\mathcal{L}_R$. Furthermore, (3.52) reveals that the diagonals of the left hand side must satisfy

$$(e_{ii} + g_{ii}) + \sum_{j=1}^{N} \frac{1}{R_{ij}} \leq \frac{C_i}{h}$$

This can always be satisfied by making h sufficiently small. However, the off-diagonals reveal conditions on e_{ij}, g_{ij} that depend on the line resistances R_{ij} in a manner best illustrated by (3.53).

In Fig. 3.5 a 3-terminal DC power network system is introduced. For this network, the condition (3.52) reads

$$\begin{bmatrix} 1 - \sum_{j=1}^{3} \frac{h}{R_{1,j} \cdot C_1} & \frac{h}{R_{1,2} \cdot C_1} & 0 \\ \frac{h}{R_{2,1} \cdot C_2} & 1 - \sum_{j=1}^{3} \frac{h}{R_{2,j} \cdot C_2} & \frac{h}{R_{2,3} \cdot C_2} \\ 0 & \frac{h}{R_{3,2} \cdot C_3} & 1 - \sum_{j=1}^{3} \frac{h}{R_{3,j} \cdot C_3} \end{bmatrix}$$
$$\geqslant \begin{bmatrix} \frac{h}{C_1}(e_{1,1} + g_{1,1}) & \frac{h}{C_1}(e_{1,2} + g_{1,2}) & 0 \\ \frac{h}{C_2}(e_{2,1} + g_{2,1}) & \frac{h}{C_2}(e_{2,2} + g_{2,2}) & \frac{h}{C_2}(e_{2,3} + g_{2,3}) \\ 0 & \frac{h}{C_3}(e_{3,2} + g_{3,2}) & \frac{h}{C_3}(e_{3,3} + g_{3,3}) \end{bmatrix}. \tag{3.53}$$

This element-wise matrix inequality shows necessary constraints on the elements of E and G.

In parallel, condition (3.43) means that E and G need to satisfy

$$s \geqslant E^\top |r| - G^\top |\gamma|.$$

Here, the structure of E determines the states available to the local current controllers and G the manner in which disturbances enter the system.

Particularly, in our 3 terminal DC power network we need the problem design to satisfy

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \geqslant \begin{bmatrix} e_{1,1} & e_{2,1} & 0 \\ e_{1,2} & e_{2,2} & e_{3,2} \\ 0 & e_{2,3} & e_{3,3} \end{bmatrix} \left| \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \right| - \begin{bmatrix} g_{1,1} & g_{2,1} & 0 \\ g_{1,2} & g_{2,2} & g_{3,2} \\ 0 & g_{2,3} & g_{3,3} \end{bmatrix} \left| \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} \right|.$$

In this example, the resulting optimal controller (3.47) can take 8 different configurations depending on the sign of each element of the parameter r. If r is positive, because in this problem $B = hC^{-1}$ is positive, all the signs of the rows in K are positive so that $u(t) = -Ex(t)$ becomes optimal. However, if r is not positive, it is possible to use this parameter to modify the signs of the rows in the resulting control action.

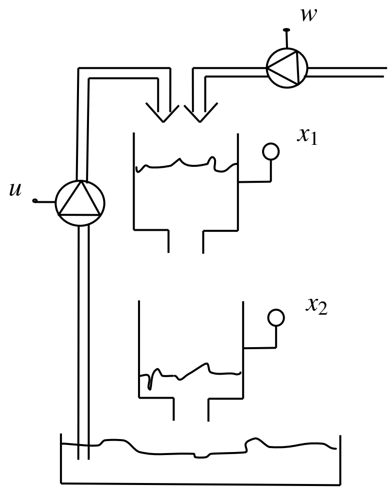


Figure 3.6 Double Tank process with disturbance w.

Double-tank Process

Consider the discretized double-tank process dynamics from [Hansson and Boyd, 1998], represented in Figure 3.6, in the presence of disturbances, which are treated similarly to the control action. The system dynamics are described by:

$$x(k+1) = Ax(k) + Bu(k) + Fw(k)$$

where

$$A = \begin{bmatrix} 0.9648 & 0 \\ 0.0345 & 0.9648 \end{bmatrix}; \quad B = F = \begin{bmatrix} 0.0971 \\ 0.0017 \end{bmatrix}.$$

Let $\gamma \geq F^\top p$, $E = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $s = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ and $r = 0.2$. The optimal control problem follows the problem setting in (3.41) with $G, H = \mathbb{0}$ and $\delta = \mathbb{0}$.

First we check that conditions (3.42) and (3.43) hold

$$A = \begin{bmatrix} 0.9648 & 0 \\ 0.0345 & 0.9648 \end{bmatrix} \geq \begin{bmatrix} 0.0971 & 0 \\ 0.0017 & 0 \end{bmatrix} = |B|E$$

$$s = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 0 \end{bmatrix} 0.2 = E^\top r.$$

Now, solving the Bellman equation (3.45) we obtain

$$p = \begin{bmatrix} 13.09 \\ 28.41 \end{bmatrix}. \tag{3.54}$$

Finally, from Theorem 30 we know that the problem (3.41) has a solution if and only if $\gamma \geq F^\top p \approx 1.32$. For this value of γ the optimal solution is (3.54) and its respective feedback controller matrix is $K = E$.

3.6 Concluding summary of Chapter 3

This Chapter provides an extensive framework for minimax problems for positive systems that considers elementwise-bounded and unconstrained worst-case disturbances. Dynamic programming is applied without imposing predefined constraints on linearity or sparsity; instead, these properties arise naturally from the optimization criteria and constraints. This approach differs from methods in which the control design explicitly dictates specific structures. In both finite and infinite horizon settings, the minimization and the two distinct maximization problems are decoupled. This property allows the explicit solution of the Bellman equation (for the discrete-time case) and the Hamilton-Jacobi-Bellman (HJB) equation (for the continuous-time case) to be extended to the minimax framework, resulting in a closed-form solution. In the finite-horizon setting, the Bellman equation is represented as a difference equation, while the HJB equation takes the form of an ordinary differential equation (ODE). In contrast, in the infinite-horizon case, both equations simplify to an algebraic equation which grows linearly with the state dimension. A fixed-point method for computing the solution to the algebraic HJB equation in the presence of elementwise bounded disturbances is provided in Theorem 28.

This novel class of optimal control problems is shown to enable efficient scaling to large dynamical systems. This efficiency stems partly from the sparsity of the optimal policy, which is a direct result of the constraints imposed by the problem statement and the optimization process. In addition, the l_1-induced gain of the system with respect to unbounded disturbance was formulated, and a tight bound for the finiteness of the cost under this disturbance was presented. The methodology is illustrated through an example involving a DC power network and a large-scale water management network, highlighting its scalability and practical relevance.

4

Linear programming formulation of the LR problem

Building on recent work in discrete time [Rantzer, 2022; Li and Rantzer, 2024], this chapter introduces a linear programming formulation for the minimax linear regulator problem in the infinite-horizon setting in continuous-time (see (3.11) in Chapter (3)). This formulation assumes that the system is influenced exclusively by positive, unconstrained disturbances. Firstly, the stabilizability and detectability properties of the minimax linear regulator problem (3.11) are examined. Subsequently, a linear programming formulation of the Hamilton-Jacobi-Bellman (HJB) equation is developed, and the equivalence of its solutions is thoroughly investigated.

Assume for the entire chapter that $H = G^\top = 0$, meaning that the system is only subject to worst-case unconstrained disturbances $w \geq 0$. Furthermore, under the context of Theorem 27, also assume the condition $\boldsymbol{\gamma} \geq F^\top p$. Then, it follows that the solution p o the minimization problem (4.1) coincides with the solution to the minimax problem (3.11).

$$
\begin{aligned}
&\inf_{\mu} \int_0^\infty \left[s^\top x(\tau) + r^\top u(\tau) \right] d\tau \\
&\text{Subject to} \\
&\qquad \dot{x}(t) = Ax(t) + Bu(t) \\
&\qquad x(0) = x_0, \;\; u(t) = \mu(x(t)), \\
&\qquad |u| \leq Ex.
\end{aligned}
\tag{4.1}
$$

In this context, the minimal value solution is $p^\top x_0$, where p is obtained by solving the algebraic equation

$$0 = s + A^\top p - E^\top \left| r + B^\top p \right|. \tag{4.2}$$

Moreover, if Theorem 27 holds, the optimal value of the optimal control problem (4.1) is $p^\top x_0$ and the optimal policy is given by $u^* = -Kx(t)$, satisfies (3.15).

4.1 E-Stabilizability and detectability conditions

In the context of (4.1), two relevant definitions are presented. A pair (A, B) is *E-stabilizable* if there exists a feedback law $u = -Kx$ with $|u| \leq Ex$ such that $A - BK$ is Hurwitz. The system is *detectable* if all unobservable states are stable. Next, a detectability characterization for positive systems is presented.

PROPOSITION 31—PROPOSITION 3 [DAMM AND ETHINGTON, 1969]
Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^n$. Consider the autonomous system

$$\dot{x} = Ax, \qquad y = Cx$$

where A is a Metzler matrix and $C \geq 0$. The pair (C, A) is detectable if and only if $Cv > 0$ for any nonnegative eigenvector v corresponding to a nonnegative eigenvalue λ of A. □

THEOREM 32
Suppose that the Bellman equation (4.2) has a finite solution p. Let K satisfy (3.15). If the pair $(s^\top - r^\top K, A - BK)$ is detectable, then $u(t) = -Kx(t)$ stabilizes the system. □

Proof: Since p solves the Bellman equation, the controller K achieves the optimal cost. The optimal cost in closed loop equals $J^*(x_0) = \int_0^t (s^\top - r^\top K)x(\tau)d\tau$ and is optimal. It satisfies $J^*(x_0) = p^\top x_0$ and is bounded. Therefore, $(s^\top - r^\top K)x(\tau) \to 0$ as $\tau \to \infty$. If the pair $(s^\top - r^\top K, A - BK)$ is detectable, this implies that the closed-loop matrix $A - BK$ is Hurwitz. ■

Detectability of the pair $(s^\top - r^\top K, A - BK)$ can be verified through Proposition 31, since the closed-loop system is again a positive system and $s^\top - r^\top K$ is nonnegative thanks to (3.12). The next corollary gives a simple but restrictive *a priori* condition that implies observability for all of the controllers that can be produced by Theorem 27. A more sophisticated condition is introduced in Theorem 35.

COROLLARY 33
Suppose $s - E^\top |r| \gg 0$, then $(s^\top - r^\top K, A - BK)$ is detectable for any K satisfying $|K| \leq E$. Moreover, if (i)–(ii) in Theorem 27 hold with $G = H^\top = 0$ and K that satisfies (3.5), then $u(t) = -Kx(t)$ stabilizes the system. □

Proof: Since $s - K^\top r \geq s - E^\top |r| \gg 0$ we have $(s - K^\top r)^\top v > 0$ for every nonzero nonnegative eigenvector v of $A - BK$. Hence $(s^\top - r^\top K, A - BK)$ is detectable by Proposition 31. The remainder follows from Theorem 32. ■

In the next theorem, we demonstrate that the detectability of $(s^\top - K^\top r, A - BK)$ is guaranteed if every nonzero state incurs a positive cost regardless of the policy applied.

THEOREM 34
Assume conditions (i) and (ii) in Theorem 27 hold with $G = H^\top = 0$. Let K satisfy (3.15). If there exists a solution $p \in \mathbb{R}^n_+$ to (4.2) such that $p \gg 0$ and $\alpha(A - BK) \neq 0$, then the pair $(s^\top - r^\top K, A - BK)$ is detectable and $u(t) = -Kx(t)$ is a stabilizing policy.

Moreover, if $\alpha(A - BK) = 0$ then the pair $(s^\top - r^\top K, A - BK)$ is not detectable.□

Proof: Suppose there exists a vector $p \in \mathbb{R}^n_+$ solving (4.2) with $p > 0$. Note that

$$K^\top (B^\top p + r) = E^\top |B^\top p + r|.$$

Substituting this into (4.2), it follows that

$$-(A - BK)^\top p = s - K^\top r. \tag{4.3}$$

If $A - BK$ is Hurwitz, then all unobservable states of $A - BK$ are asymptotically stable. Thus, the pair $(s^\top - r^\top K, A - BK)$ is detectable and $u(t) = -Kx(t)$ is a stabilizing policy.

Suppose $A - BK$ is not Hurwitz. Since $A - |B|E$ is Metzler, $A - BK$ is also Metzler. Therefore, there exists a nonnegative eigenvector, specifically the Perron eigenvector v_p of $A - BK$, corresponding to the largest eigenvalue $\lambda_\text{p} = \alpha(A - BK) \neq 0$. This eigenvector satisfies $\lambda_\text{p} v_\text{p} = (A - BK)v_\text{p}$, thus $v_\text{p} \neq 0$. The right-hand side of (4.3) can be rewritten as

$$p^\top (-(A - BK))v_\text{p} = p^\top (-\lambda_\text{p})v_\text{p}. \tag{4.4}$$

Since $p \gg 0$, $v_\text{p} \geq 0$ and $\lambda_\text{p} > 0$ it follows that $p = p^\top(-\lambda_\text{p})v_\text{p} \ll 0$. This contradicts the left-hand side of (4.3), $(s - K^\top r)^\top v_\text{p} \geq 0$. Therefore, if $p \gg 0$ and $\lambda_\text{p} \neq 0$ the inequality cannot hold. Consequently, if there exists a solution $p \gg 0$ to (4.2) and $\alpha(A - BK) \neq 0$ the pair $(s^\top - r^\top K, A - BK)$ is detectable and $u(t) = -Kx(t)$ is a stabilizing policy.

Finally, if $\alpha(A - BK) = 0$, the Perron root is $\lambda_\text{p} = 0$ with $v_\text{p} \geq 0$. Therefore, (4.4) gives $p^\top(-\lambda_\text{p})v_\text{p} = (s - K^\top r)^\top v_\text{p} = 0$. Thus, $(s^\top - r^\top K, A - BK)$ is not detectable by Proposition 31. ■

Recall the observability condition in [Li and Rantzer, 2024, Assumption 4.1].

$$(s^\top - |r^\top|E) \sum_{i=0}^{n-1} (A - |B|E)^i \gg 0. \tag{4.5}$$

Condition (4.5) is based on the sum of the observability matrix in the discrete-time framework. In continuous time, the observability matrix involves the integral $\int_0^\infty e^{(A-|B|E)t}dt$. This integral does not converge for the modes λ of $A-|B|E$ satisfying $\text{Re}(\lambda) \geq 0$. Thus, the observability condition

$$\int_0^\infty e^{(A-|B|E)^\top}(s-E^\top|r|)dt \gg 0$$

ensures that no unstable mode $v \neq 0$ associated with $\text{Re}(\lambda) \geq 0$ is unobservable in continuous-time. Inspired by (4.5), the next theorem proposes an *a priori* detectability condition for the continuous-time setting.

THEOREM 35
Assume (i)–(ii) in Theorem 26 and Theorem (27) hold with $G = H^\top = 0$. Let $h \in \mathbb{R}$ such that $A-|B|E+hI_n \geq 0$ and $h>0$. Suppose also that

$$(s^\top - |r^\top|E)\sum_{i=0}^{n-1}\left(\frac{1}{h}(A-|B|E)+I\right)^i \gg 0. \tag{4.6}$$

Then, $(s^\top - r^\top K, A-BK)$ is detectable for any K satisfying $|K| \leq E$. □

Proof: Suppose (4.6) holds. Let $M = A-BK$, $N = s-K^\top r$, $\hat{M} = A-|B|E$ and $\hat{N} = s-E^\top|r|$. Note that $M \geq \hat{M}$ and $N \geq \hat{N}$. The ODE (3.3) can be rewritten as $-\dot{p}(t) = Mp(t)+N$ with $p(T)=0$. Define $w(t) = p(T-t)$ and suppose $T>0$, then $w(0)=0$ and $w(t)$ for $t \in [0,T]$ satisfies

$$w(t) = \int_0^t e^{M^\top \tau} N d\tau. \tag{4.7}$$

It is possible to approximate the integral by

$$w(t) = \int_0^t e^{M^\top \tau} N d\tau = \lim_{h\to\infty} \frac{1}{h}\sum_{i=0}^{\lfloor \frac{t}{h} \rfloor -1}(\frac{1}{h}M^\top + I)^i N \tag{4.8}$$

Because $M+hI_n \geq \hat{M}+hI_n \geq 0$ and $N \geq \hat{N} \geq 0$, the summation in (4.8) can be lower bounded by

$$\lim_{h\to\infty}\frac{1}{h}\sum_{i=0}^{n-1}\left(\frac{1}{h}\hat{M}^\top + I\right)^i \hat{N} \gg 0$$

which is positive by assumption. For a nonnegative vector $v \neq 0$ corresponding to an eigenvalue $\lambda \geq 0$ of M it holds that

$$\begin{aligned} v^\top w(t) &= v^\top \lim_{h\to\infty}\frac{1}{h}\sum_{i=0}^{\lfloor \frac{t}{h} \rfloor -1}(\frac{1}{h}M^\top + I)^i N \\ &= \lim_{h\to\infty}\frac{1}{h}\sum_{i=0}^{\lfloor \frac{t}{h} \rfloor -1}(\frac{\lambda}{h}+1)^i v^\top N = (\int_0^t e^{\lambda\tau}d\tau)v^\top N. \end{aligned}$$

Clearly $\int_0^t e^{\lambda\tau} d\tau > 0$. At the same time

$$\begin{aligned} v^\top w(t) &= v^\top \lim_{h\to\infty} \frac{1}{h} \sum_{i=0}^{\lfloor \frac{t}{h} \rfloor - 1} (\frac{1}{h} M^\top + I)^i N \\ &\geq v^\top \lim_{h\to\infty} \frac{1}{h} \sum_{i=0}^{n-1} (\frac{1}{h} \hat{M}^\top + I)^i \hat{N} > 0. \end{aligned}$$

Thus, $v^\top N = v^\top (s - K^\top r) > 0$ and $(s^\top - r^\top K, A - BK)$ is detectable by Proposition 31. ■

EXAMPLE 11—DETECTABILITY OF THE LINE-SHAPED WATER NETWORK
Recall the Line-Shaped water network in Section 3.3. Let $\tilde{A} = A + |H|G$ and $\tilde{s} = s^\top + |\delta^\top|G$. The detectability of the pair $(\tilde{s} - r^\top K, \tilde{A} - BK)$ is established via Theorem 35. However, the sufficient condition provided in Corollary 33 is overly restrictive for this example, as (3.28) is not strictly positive. Since $\lambda_p \approx 0.19$ it suffices to choose $h = 20$ to verify the inequality

$$\frac{1}{h}(\tilde{s}^\top - |r^\top|E) \sum_{i=0}^{n-1} \left(\frac{1}{h}(\tilde{A} - |B|E) + I \right)^i \gg 0.$$

Therefore, the pair $(\tilde{s} - r^\top K, \tilde{A} - BK)$ is detectable for any K satisfying $|K| \leq E$. □

4.2 Linear programming

It is possible to obtain a solution to the Bellman equation (4.2) through the linear program (LP)

$$\begin{aligned} &\text{Maximize} \;\; \mathbb{1}^\top p \text{ over } p \in \mathbb{R}^n_+, \; \zeta \in \mathbb{R}^m_+ \\ &\text{Subject to} \;\; A^\top p \geq E^\top \zeta - s \\ &\qquad\qquad -\zeta \leq r + B^\top p \leq \zeta. \end{aligned} \tag{4.9}$$

Lemma 36 characterizes the boundedness of (4.9) by considering the dual linear program.

LEMMA 36
The following are equivalent:

(i) The primal linear program (4.9) has a bounded solution;

(ii) The dual linear program

$$\begin{aligned} &\text{Minimize} \quad s^\top x + r^\top u \;\; \text{with } x \in \mathbb{R}^n_+ \\ &\text{Subject to} \quad Ax + Bu \leq -\mathbb{1} \\ &\qquad\qquad -Ex \leq u \leq Ex \end{aligned} \tag{4.10}$$

is feasible;

(iii) There exists a $D \in \mathbb{R}^{m\times m}$ satisfying $-I \le D \le I$ such that $A - BDE$ is Hurwitz. □

Proof: $(i) \iff (ii)$: This follows from the weak duality of the linear programs, and the fact the the primal problem is feasible. Indeed, taking $p = 0$ and $\zeta = |r|$ renders the primal linear program feasible since $s \ge E^\top |r|$.

$(ii) \implies (iii)$: For any feasible (x,u) let D such that $-I \le D \le I$ and $u = -DEx$. It follows that $(A-BDE)x \le -\mathbb{1} < \mathbb{0}$. The matrix $A - BDE$ is Metzler since $A - |B|E$ is Metzler and $-I \le D \le I$. Since $x \ge 0$, it follows by item 1 and 18 of [Fiedler, 1986, Thm. 5.1] that $A - BDE$ is Hurwitz.

$(iii) \implies (ii)$: By item 2 and 18 of [Fiedler, 1986, Thm. 5.1] there exists a $v \gg \mathbb{0}$ such that $(A - BDE)v \ll 0$. Let $\alpha = \min_i |(A-BDE)_i|$ and take $x = \frac{1}{\alpha}v$ and $u = -DEx$, then $Ax + Bu = (A - BDE)x \le -\mathbb{1}$, $-Ex \le u \le Ex$ and $x \ge \mathbb{0}$. ■

THEOREM 37
If the linear program has a bounded value, then the optimizer p solves the Bellman equation (4.2). □

Proof: Let p and ζ optimize (4.9), then $A^\top p = E^\top \zeta - s$ and $\zeta = |r + B^\top p|$, and thus (3.13) with $G = H^\top = 0$ is satisfied. ■

Note that the existence of a bounded solution to the LP (4.9) is not necessary for the existence of solution to the HJB equation.

EXAMPLE 12
Let

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix};\quad E^\top = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$s = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top;\quad r = 0.$$

The third state is unstable and not detectable (nor stabilizable). The solution to the Bellman equation (3.3) is

$$p = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top. \tag{4.11}$$

However, the third entry of the p vector in the linear program is unbounded. □

Moreover, a system that is not detectable might still be stabilizable, but the Bellman equation might fail to identify the stabilizing solution.

EXAMPLE 13
Let

$$A = \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \ B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \ E^\top = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$s = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}^\top; \ r = 0.$$

The Perron Frobenius eigenvalue of A is $\lambda_p = 0.5$ indicating that the first state is unstable. Furthermore, the detectability condition fails since

$$(s^\top - |r^\top|E)v_p = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top = 0$$

where v_p is the eigenvector associated with λ_p. This implies that the first state is not detectable. However, the system is stabilizable: Choosing $K = E$ yields

$$A - BK = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ \lambda_{\mathrm{p}} = -0.5.$$

In this context, the solution to the Bellman equation (4.2) is

$$p = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}^\top \tag{4.12}$$

with $K = \operatorname{sign}(0)E$, i.e. any $K \in [-E, E]$ is minimizing. However, $K = -E$ does not give a stable closed loop, so not all minimizing solutions are stabilizing. □

Lemma 36 implies that the linear program (4.9) converges only if (A, B) stabilizable by a feedback $u = -Kx$ with $|u| \leq Ex$. Therefore, a solution to the Bellman equation (4.2) is an optimizer to (4.9) as long as the closed-loop system is detectable.

Next, a partial converse to Theorem 37 is presented:

THEOREM 38
If p solves the Bellman equation (3.13), K satisfies (3.15) and K is such that

$$\left(s - K^\top r, A - BK\right) \tag{4.13}$$

is detectable, then p maximizes (4.9), with $\zeta = |r + B^\top p|$. □

Proof: For the primal linear program to be bounded we require by Lemma 36 that there exists a feedback law $u(t) = -Kx(t)$ with $K = DE$ and $|D| \leq I$ that stabilizes the system. By Theorem 32, it follows that any K that satisfies (3.15) stabilizes the system. Taking p in (4.9) as the solution to (3.13) and $\zeta = |r + B^\top p|$ satisfies constraints of (4.9). If p would not be optimal, then applying the iteration (28) to p

would result in a $\tilde{p}$ for which $\mathbb{1}^\top p \leq \mathbb{1}^\top \tilde{p}$ [Rantzer, 2022]. However, since p solves the Bellman equation we have $p = \tilde{p}$, and thus p is optimal. ■

From the proof of Lemma 36 it is clear that the dual linear program (4.10) always generates a stabilizing controller. We use this fact to show that the primal linear program (4.9) also generates at least one stabilizing controller.

THEOREM 39
If the primal linear program (4.9) has a bounded solution p, then all controllers $u(t) = -Kx(t)$ satisfying

$$K \in \operatorname{diag}(\operatorname{sign}(B^\top p + r))E$$

achieve the optimal cost and at least one stabilizes the system, i.e., the resulting closed loop matrix $A - BK$ is Hurwitz. Moreover, if $(s^\top - r^\top K, A - BK)$ is detectable, then K always stabilizes the system. □

Proof: The primal and dual linear programs (4.9) and (4.10) achieve the same cost due to strong duality (*e.g.*, see [Dantzig and Thapa, 2003, Ch. 2]). Let p^*, ζ^*, x^* and u^* be such as the optimal cost of both programs is achieved. That is, $s^\top x^* + r^\top u^* = \mathbb{1}^\top p^*$. Thanks to optimality of the constraints we have

$$\begin{aligned} |r + B^\top p^*| &= \zeta^*, \quad E^\top \zeta^* = s + A^\top p^*, \\ Ax^* + Bu^* &= -\mathbb{1}, \quad |u| = Ex^*. \end{aligned}$$

Let $D = \operatorname{diag}(\operatorname{sign}(u^*))$ such that $u^* = DEx^*$. If $x_i^* = 0$, note that $((A + BDE)x^*)_i \geq 0$ since all off-diagonal entries of $A + BDE$ are nonnegative. This violates $(A + BDE)x^* = -\mathbb{1}$ and therefore $x^* > 0$. By assumption, E is nonnegative and has no all-zero rows, and so $Ex^* > \mathbb{0}$. The established equalities yield

$$\begin{aligned} |r + B^\top p^*|^\top Ex^* &= \zeta^{*\top} Ex^* = s^\top x^* + p^{*\top} Ax^* \\ &= s^\top x^* - p^{*\top}\mathbb{1} - p^{*\top} Bu^* = -r^\top u^* - p^{*\top} Bu^* \\ &= -(r + B^\top p^*)^\top u^* = -(r + B^\top p^*)^\top DEx^*. \end{aligned}$$

It follows that equality holds if $D \in -\operatorname{diag}(\operatorname{sign}(r + B^\top p^*))$. The controller is not necessarily unique if there exists an index i such that $(r + B^\top p^*)_i = 0$. At least one of such D satisfies $D = \operatorname{diag}(\operatorname{sign}(u^*))$ and therefore stabilizes the system. The optimal cost $J^*(x_0) = \int_0^t (s^\top - r^\top K)x(\tau)\mathrm{d}\tau$ satisfies $J^*(x_0) = p^{*\top} x_0$ and is bounded. Therefore, $(s^\top - r^\top K)x(\tau) \to 0$ as $\tau \to \infty$. If the pair $(s^\top - r^\top K, A - BK)$ is detectable, this implies that $A - BK$ is Hurwitz. ■

4.3 Concluding summary of Chapter 4

This chapter extends previous results in discrete time presented in [Rantzer, 2022] and [Li and Rantzer, 2024], focusing on the continuous-time version of the novel class of problems introduced in those works. A linear programming formulation is proposed to solve the HJB equation for the linear regulator problem and the minimax linear regulator problem in the presence of unconstrained disturbances $w \geq 0$ under appropriate assumptions.

The stabilizability and detectability properties of the system dynamics are analyzed and linked to the existence and equivalence of solutions of the HJB equation derived in the previous chapter, and the linear programming formulation derived in this one. Additionally, sufficient detectability conditions are established to ensure the uniqueness of a stabilizing policy when the HJB equation admits a finite solution and it is demonstrated that the proposed linear programming approach always identifies a stabilizing policy, provided one exists.

5

Synchronization of positive multi-agent systems

Synchronization is a critical behavior in many dynamical systems and has broad applications across various domains [Ren and Beard, 2005; Tegling et al., 2023; Fabiny et al., 1993; Fax and Murray, 2004]. A fundamental challenge in control theory is designing protocols that achieve synchronization in interconnected systems. In recent research, a widely adopted approach to addressing the synchronization problem involves the design of stabilizing controllers for the closed-loop dynamics of individual agents. These controllers are then extended to develop a synchronization-achieving controller for the overall system. A detailed overview of this approach is provided in [Saberi et al., 2022]. Central to stabilizing the local system is solving the algebraic Riccati equation for the agent dynamics. Notably, the resulting controller ensures that increasing the input signal maintains stability in the local system, a critical requirement for synchronization.

Research on positive systems has also expanded significantly to the study positive consensus [Valcher and Zorzan, 2016; Valcher and Zorzan, 2017]. For example, [Valcher and Misra, 2014] addresses the positive consensus problem using a static output feedback approach for single-input, single-output positive agents, with the controller gain matrix encoding the network's connectivity structure.

This chapter builds on these ideas by proposing a synchronization controller for systems with positive homogeneous dynamics. The approach relies on solving the Linear Regulator problem in Chapter 3, which is analogous to the algebraic Riccati equation in the Linear Quadratic Regulator framework. However, the Linear Regulator problem offers significant computational advantages: its algebraic equation can be solved using linear programming, and the number of unknowns grows linearly with the state dimension, in contrast to the quadratic scaling in the Riccati equation. Moreover, the sparsity structure of the resulting optimal control policy can be chosen by the problem designer. Necessary and sufficient conditions ensuring the positivity of each agent's trajectory for all nonnegative initial conditions are also provided. Numerical simulations on large regular graphs with different nodal degrees illustrate the proposed results and demonstrate its effectiveness.

5.1 The LR-based method in continuous-time

In [Saberi et al., 2022, Prot. 2.1], a protocol design method based on an algebraic Riccati equation (ARE) is presented. For this protocol, it is shown that full-state coupling is always solvable for a family in $\mathbb{G}_{[\beta,\infty)}$[1], where no upper bound is required for the nonzero eigenvalues of $\mathcal{L}$.

ARE-based protocol for continuous-time MAS

Consider a MAS described by (2.36) and (2.37). We consider the protocol

$$u_i = \rho F \zeta_i, \tag{5.1}$$

where $\rho \geq 1$ and $F = -B^\top P$ with $P > 0$ being the unique solution of the continuous-time algebraic Riccati equation

$$A^\top P + PA - 2\beta PBB^\top P + Q = 0 \tag{5.2}$$

where $Q > 0$ and β is a lower bound for the real part of the nonzero eigenvalues of all Laplacian matrices associated with a graph in the set of connected graphs $\mathbb{G}_{[\beta,\infty)}$.

Recall the Linear Regulator set up (4.1).

$$\begin{aligned}
&\inf_{\mu} \int_0^\infty \left[s^\top x(\tau) \right] d\tau \\
&\text{Subject to} \\
&\qquad \dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t),\; x(0) = x_0 \\
&\qquad u(t) = \mu(x(t)),\; |u| \leq \tilde{E}x.
\end{aligned} \tag{5.3}$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $\tilde{E} \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}_+^n$ such that $s \gg 0$, and $\tilde{A} - |\tilde{B}|\tilde{E}$ is Metzler. For $F, H, G = 0$ it follows from Theorem (27) that (5.3) has a finite solution for every $x_0 \in \mathbb{R}_+^n$ if and only if there exists a nonnegative vector $p \in \mathbb{R}_+^n$ such that

$$\tilde{A}^\top p = \tilde{E}^\top |\tilde{B}^\top p| - s. \tag{5.4}$$

[1] Recall Definition 20

If either are satisfied, then (5.3) has the minimal value $p^\top x_0$. Moreover, the control law $u(t) = -Kx(t)$ is optimal when

$$K := \text{diag}\left(\text{sign}(\tilde{B}^\top p)\right)\tilde{E}. \tag{5.5}$$

REMARK 13
To verify the E-stabilizability of the pair (A,B) with A Metzler, it is shown in Lemma 36 that it is necessary and sufficient to verify the feasibility of

$$Ax + Bu \leq -\mathbb{1}, \quad -Ex \leq u \leq Ex.$$

This feasibility problem can be verified by any linear program solver. □

Note that by construction $s > 0$. Assuming also that the pair $(\tilde{A},\tilde{B})$ is $\tilde{E}$-stabilizable, by Corollary 33 and Theorem 35 in Chapter 4 a vector $p \geq 0$ solves (5.4) if and only if p maximizes linear program

$$\begin{aligned} \text{Maximize } & \mathbb{1}^\top p \text{ over } p \in \mathbb{R}^n_+,\ \zeta \in \mathbb{R}^m_+ \\ \text{Subject to } & \tilde{A}^\top p \leq \tilde{E}^\top \zeta - s \\ & -\zeta \leq \tilde{B}^\top p \leq \zeta. \end{aligned} \tag{5.6}$$

Inspired by the ARE-based protocol (5.1.1) and the Continuous-time Linear regulator protocol framework this section presents solution to two problems.

PROBLEM 1—SYNCHRONIZATION PROBLEM
Design a linear feedback controller of the form (5.1) that solves the State Synchronization Problem in Section 2.6 and satisfies the constraint

$$|u_i| \leq E\,|\zeta_i|\,. \tag{5.7}$$

□

PROBLEM 2—POSITIVE SYNCHRONIZATION PROBLEM
Find necessary and sufficient conditions for our protocol, which ensures that the positivity of each agent's trajectory is preserved for all nonnegative initial conditions. □

REMARK 14
For B nonnegative we always have that $K = E$, implying that the internal positivity of each agent ensures the positivity of the interconnected system. □

LR-based protocol for continuous-time MAS

Consider the MAS described by (2.36) and (2.37) with $A \in \mathbb{R}^{n\times n}$ Metzler and $B \in \mathbb{R}^{n\times m}$. Let $E \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}_+^n$ and $\mathcal{L}$ be a Laplacian matrix associated with a graph $\mathcal{G} \in \mathbb{G}_{[\beta,\gamma]}$ with N agents. Suppose

$$A - \gamma|B|E \tag{5.8}$$

is Metzler and $\rho \geq \frac{1}{\beta}$. The LR-based protocol is given by

$$u_i = -\rho K \zeta_i, \tag{5.9}$$

where K follows from (5.5) with $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{E} = \frac{1}{\rho}E$, $F = H = G = 0$ and $s > 0$.

THEOREM 40
Consider a graph family $\mathcal{F} \subseteq \mathbb{G}_{[\beta,\gamma]}$ and the MAS described by (2.36) and (2.37). If the pair (A,B) is E-stabilizable then the protocol (5.9) solves the state synchronization problem for any undirected graph $\mathcal{G} \in \mathcal{F}$. Moreover, the synchronized trajectory is given by (2.46), and each u_i satisfies the bound (5.7). □

Proof: Let $\mathcal{G}$ be any graph in $\mathcal{F}$. The overall dynamics of the N agents with the protocol (5.1.1) can be written as

$$\dot{x} = (I_N \otimes A - \rho \mathcal{L}(\mathcal{G}) \otimes BK)x. \tag{5.10}$$

By Lemma 24, the synchronization of the system (5.10) is equivalent to the asymptotic stability of

$$\dot{\tilde{\eta}}_i = (A - \lambda_i \rho BK)\tilde{\eta}_i, \qquad i = 2,\dots,N \tag{5.11}$$

where $\lambda_i = \lambda_i(\mathcal{G})$, $i = 2,\dots,N$ are positive. We prove that $A - \lambda_i\rho BK$ is Hurwitz for all i.

Let $i = 2,\dots,N$ and $\alpha_i = \lambda_i\rho$. Recall that $\lambda_i \geq \beta \geq \frac{1}{\rho}$, and thus $\alpha_i \geq 1$. By assumption, $\gamma \geq \lambda_i$ and $A - \gamma\rho\,|B|\,\tilde{E}$ is Metzler and since $BK \leq |B|\tilde{E}$, $A - \alpha_i BK$ is also Metzler. Observe from (5.5) that K satisfies

$$K^\top(B^\top p) = \tilde{E}^\top \left|B^\top p\right| \geq 0,$$

where $p \geq 0$ solves (5.4). It follows from (5.4) that

$$\begin{aligned}(A - \alpha_i BK)^\top p &= A^\top p - \alpha_i K^\top B^\top p \\ &= \tilde{E}^\top \left|B^\top p\right| - s - \alpha_i \tilde{E}^\top \left|B^\top p\right| (1-\alpha_i)E^\top \left|B^\top p\right| - s \leq -s < 0.\end{aligned}$$

Therefore, by [Fiedler, 1986, Thm. 5.1] it is implied that $A - \alpha_i BK$ is Hurwitz as we wanted to prove. ■

Positive systems often appear as dynamical systems for which the states are representing (physical) quantities that cannot become negative. In some contexts it is therefore desired that the trajectories remain in the nonnegative orthant. The following theorem characterizes when this is the case.

THEOREM 41
Consider a graph family $\mathcal{F} \subseteq \mathbb{G}_{[\beta,\gamma]}$ and the MAS described by (2.36) and (2.37). Suppose the pair (A,B) is E-stabilizable and consider the protocol (5.9). The trajectories of the MAS remain nonnegative for all nonnegative initial conditions if and only if BK is nonnegative. □

Proof: The dynamics (2.36) of each agent i is rewritten as

$$\dot{x}_i = \hat{A}x_i + \hat{B}\hat{u}_i \tag{5.12}$$

where $\hat{A} = A - \rho BK \sum_{j=1}^{N} a_{ij}$, $\hat{B} = \rho BK$ and $\hat{u}_i = \sum_{j=1}^{N} a_{ij}x_j$. Note that $\gamma \geq \sum_{j=1}^{N} a_{ij}$ and $|B|E \geq BK$, hence, $\hat{A}$ is Metzler.

($\Longrightarrow$): Suppose that the matrix BK has, at least, one negative element, then there exists $\hat{B}_{p,q} < 0$ for some $p,q \in \mathbb{N}$. Let $x_i(0)_p = 0$ and $x_j(0)_q$ be sufficiently large for some $q \neq p$. Then $\hat{u}_i(0)_q \gg 0$ and from (5.12) it follows that $\dot{x}_i(0)_p < 0$. Thus, x_i leaves the nonnegative orthant.

($\Longleftarrow$): Because $\hat{A}$ is Metzler and $\hat{B} \geq 0$ the system is internally positive with respect to $\hat{u}$ [Rantzer and Valcher, 2018a]. Thus, the trajectories of (5.12) remain nonnegative. ■

Recall that the LR-based consensus protocol requires that the gain parameter ρ satisfies $\rho \geq \frac{1}{\lambda_2}$. Violation of this bound may destabilize the systems (5.11) for some i, resulting in dissensus. At the same time we require in the proof of Theorem 40 that the state matrix $A - \lambda_i \rho BK$ in (5.11) is Metzler, in order to apply the Linear Regulator theory. For this to hold, we need $\lambda_i \rho \leq \alpha$, where

$$\alpha = \operatorname{argmax}_{\tau \geq 0} \left\{ A - \tau |B| \tilde{E} \text{ is Metzler} \right\}, \tilde{E} = \frac{1}{\rho} E. \tag{5.13}$$

For any graph family $\mathcal{F} \subseteq \mathbb{G}_{[\beta,\gamma]}$ we have $\lambda_i(\mathcal{G}) \in [\beta, \gamma]$ for any $\mathcal{G} \in \mathcal{F}$, implying the bounds

$$\frac{1}{\lambda_i} \leq \frac{1}{\beta} \leq \rho \leq \frac{\alpha}{\gamma} \leq \frac{\alpha}{\lambda_i}.$$

We therefore conclude that, in order to apply the LR-based protocol, we require that

$$\alpha \geq \frac{\gamma}{\beta}. \tag{5.14}$$

Note that $\frac{\gamma}{\beta} \geq 1$, and thus $\alpha \geq 1$ is required as well. Note also that the left-hand side of (5.14) depends fully on the local agent dynamics via (5.13), whereas the right-hand side depends on the class $\mathbb{G}_{[\beta,\gamma]}$ that contains the graph family of interest. It follows that the system dynamics dictate for which graph families the Protocol 5.1.1 can reach consensus. Although the matrix $\tilde{E}$ could be scaled down such that α is increased, doing so might violate the E-stabilizability of (A,B).

Numerical simulations

Consider the MAS (2.36) described by

$$A = \begin{bmatrix} -2.21 & 2.40 \\ 0.43 & -0.44 \end{bmatrix}; \quad B = \begin{bmatrix} 0.27 \\ 0 \end{bmatrix}; \quad E = \begin{bmatrix} 0.06 \\ 0.6 \end{bmatrix}^\top$$

and composed by 150 agents. Consider also a connected undirected graph $\mathcal{G}$ in the family of regular graphs of degree $d = 4,7$ for $\mathcal{F}_R$. The matrix A is unstable $\sigma(A) = \{-2.63, 0.03\}$.

Let the eigenvalues of every $\mathcal{G} \subseteq \mathcal{F}_R$ be upper bounded by $\gamma = 13$ and lower bounded by $\beta = 1$ such that $\rho = 1/\beta = 1$. To solve the state synchronization problem, the LR-based Protocol 5.1.1 is implemented. Consider $s = \mathbb{1} > 0$, and $\tilde{E} = \frac{1}{\rho}E = \begin{bmatrix} 0.06 & 0.6 \end{bmatrix}$ such that $A - |B|\tilde{E}$ is Metzler. The linear program (4.9) is maximized by a vector p^*, which satisfies the algebraic equation (5.4) resulting in $K = E$.

Observe that, $A - \gamma|B|E$ with $\tilde{E} = \frac{1}{\rho}E$ is Metzler and $BK = BE \geq 0$. Hence, from Theorem 41 the trajectories of the MAS remain nonnegative for all initial conditions. The initial condition is arbitrarily chosen to be nonnegative.

Figure 5.1 illustrates the evolution of the first and second states of each of the 150 agents in an interconnected system, where each agent is connected to 4 and 5
7 neighbors, respectively. In Figure 5.2 the state synchronization of the agents is illustrated with a 2D plot of their trajectories. As expected, the trajectories do not leave the positive orthant. Moreover, the synchronization is achieved faster as the nodal degree is increased. The distance from the trajectories to the synchronized trajectory is represented in Figure (5.3).

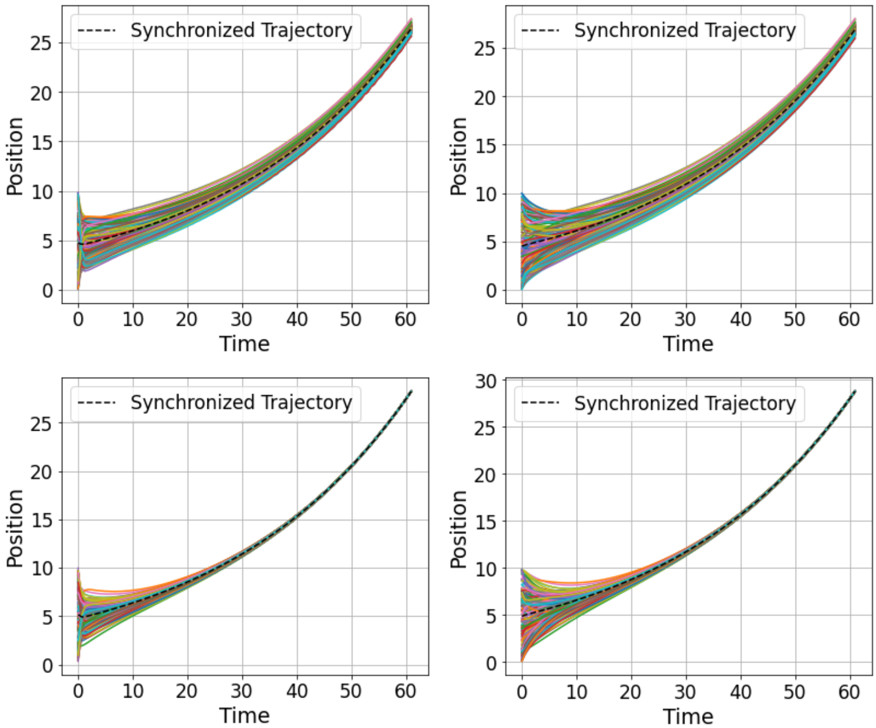


Figure 5.1 Evolution over time of the first (left panels) and the second (right panels) state of each agent $i = 1, \ldots, 150$ synchronizing over 5-regular graphs (upper panels) and 7-regular graphs.

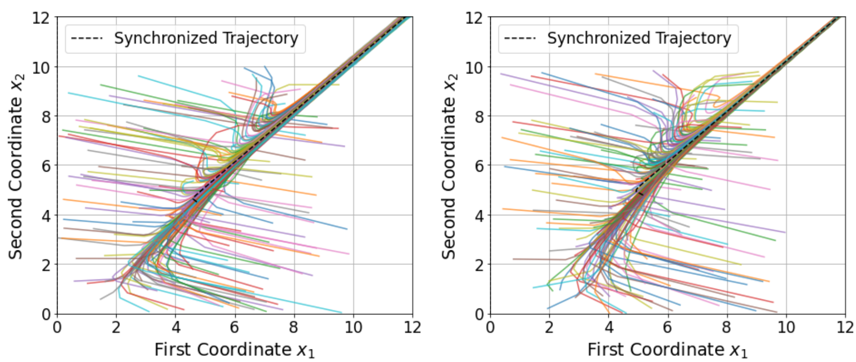


Figure 5.2 Trajectories of agents synchronizing over a 5-regular graph (left panel) and 7-regular graph.

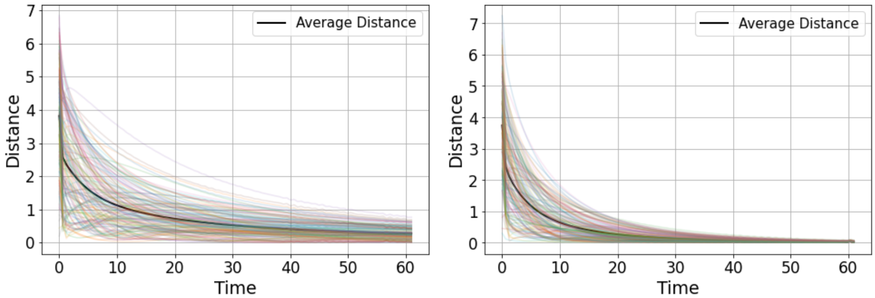


Figure 5.3 Euclidean distance to the synchronized trajectory $x_s(t)$ i.e. $\|x_i(t) - x_s(t)\|$ for $i = 1, \ldots 150$ over 5-regular graphs (left panel) and 7-regular graphs.

5.2 The LR-based method in discrete-time

Analogous to the continuous-time case, in [Saberi et al., 2022, Prot. 3.5], a protocol design method based on the discrete-time algebraic Riccati equation is introduced. This method demonstrates that full-state coupling synchronization is always achievable for a family of graphs in $\tilde{\mathbb{G}}_{(-\infty,\beta]}$[2], where $\beta \in (0,1)$ is fixed.

ARE-based protocol for discrete-time MAS

Consider a MAS described by (2.39) and (2.41). The protocol is given by

$$u_i = F_\delta \zeta_i \tag{5.15}$$

where

$$F_\delta = -(1-\beta^2)^{-1}(B^\top P_\delta B + I)^{-1} B^\top P_\delta A,$$

with $P_\delta > 0$ determined as the unique solution of

$$P_\delta = A^\top P_\delta A + \delta I - A^\top P_\delta B (B^\top P_\delta B + I)^{-1} B^\top P_\delta A, \tag{5.16}$$

for some $\delta > 0$ satisfying

$$\beta^2 B^\top P_\delta B < (1-\beta^2) I. \tag{5.17}$$

[2] Recall Definition 21

Consider the discrete-time version of the Linear Regulator set up (4.1).

$$
\begin{aligned}
&\inf_{\mu} \sum_{t=0}^{\infty} \left[s^\top x(t) \right] \\
&\text{Subject to} \\
&\qquad x(t+1) = \tilde{A}x(t) + \tilde{B}u(t),\ x(0) = x_0 \\
&\qquad u(t) = \mu(x(t)),\ |u| \leq \tilde{E}x.
\end{aligned}
\tag{5.18}
$$

where $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $\tilde{E} \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}_+^n$ such that $s \gg 0$ and $\tilde{A} - |\tilde{B}|\tilde{E} \geq 0$. For $F, H, G = 0$ it follows from Theorem (30) that the optimal control problem (5.18) has a finite solution for every $x_0 \in \mathbb{R}_+^n$ if and only if there exists a nonnegative vector $p \in \mathbb{R}_+^n$ such that

$$p = s + \tilde{A}^\top p - \tilde{E}^\top |\tilde{B}^\top p|. \tag{5.19}$$

If the above conditions are met, the minimal value equals $p^\top x_0$. Moreover, the optimal control law $u(t) = -Kx(t)$ is

$$K := \operatorname{diag}\big(\operatorname{sign}(\tilde{B}^\top p)\big)\tilde{E}. \tag{5.20}$$

REMARK 15
Analogous to the continuous-time result in Lemma 36, verifying E-stabilizability of (A, B) with $A \geq 0$, reduces to solving

$$(A - I)x + Bu \leq -\mathbb{1}, \quad -Ex \leq u \leq Ex.$$

which can be efficiently checked using a linear programming solver. □

Note that by construction $s > 0$. Assuming also that the pair $(\tilde{A}, \tilde{B})$ is $\tilde{E}$-stabilizable, by Corollary 33 and Theorem 35 in Chapter 4 a vector $p \geq 0$ solves (5.4) if and only if p maximizes linear program

$$
\begin{aligned}
&\text{Maximize } \ \mathbb{1}^\top p \text{ over } p \in \mathbb{R}_+^n,\ \zeta \in \mathbb{R}_+^m \\
&\text{Subject to } \ p \leq s + \tilde{A}^\top p - \tilde{E}^\top \zeta \\
&\qquad\qquad\quad -\zeta \leq \tilde{B}^\top p \leq \zeta.
\end{aligned}
\tag{5.21}
$$

Inspired by the ARE-based protocol (5.2.1) and the Discrete-time Linear regulator protocol framework this section presents solution to Problem 1 and Problem 2 for Discrete-Time MAS.

LR-based protocol for discrete-time MAS

Consider the MAS described by (2.39) and (2.41) with $A \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $E \in \mathbb{R}_+^{m\times n}$, $s \in \mathbb{R}_+^n$ such that $s > 0$ remove and $\mathcal{D}$ be the row stochastic matrix associated with a graph $\mathcal{G} \in \tilde{\mathbb{G}}_{[\gamma,\beta]}$ with N agents. Suppose

$$A - (1-\gamma)|B|E \geq 0 \tag{5.22}$$

$\gamma \in (-1,1)$. The LR-based protocol is given by

$$u_i = -\rho K \zeta_i, \tag{5.23}$$

where $\rho \geq \frac{1}{1-\beta}$, $\beta \in (0,1)$ and K follows from (5.20) with $\tilde{A} = A$, $\tilde{B} = B$, $\tilde{E} = \frac{1}{\rho}E$ and $s > 0$.

THEOREM 42
Consider a family of graphs $\mathcal{F} \subseteq \tilde{\mathbb{G}}_{[\gamma,\beta]}$ and the MAS described by (2.39) and (2.41). If the pair (A,B) is E-stabilizable then the protocol (5.23) solves the state synchronization problem for any undirected graph $\mathcal{G} \in \mathcal{F}$. Moreover, the synchronized trajectory is given by (2.46), and each u_i satisfies the bound (5.7). □

2.50

Proof: Let $\mathcal{G}$ be any graph in $\mathcal{F}$. The dynamics of the N agents under the protocol (5.1.1) can be expressed as

$$x(t+1) = (I_N \otimes A - (I - \mathcal{D}(\mathcal{G})) \otimes BK)x(t). \tag{5.24}$$

By Lemma 24, the synchronization of the system (5.24) is equivalent to the asymptotic stability of the following $N-1$ subsystems

$$\tilde{\eta}_i(t+1) = (A - (1-\mu_i)\rho BK)\tilde{\eta}_i(t), \tag{5.25}$$

where $\mu_i = \mu_i(\mathcal{G})$, $i = 2,\dots,N$ are the eigenvalues inside the unit disc for the row stochastic matrix $\mathcal{D}$ associated with $\mathcal{G} \in \tilde{\mathbb{G}}_{[\gamma,\beta]}$. We prove that $A-(1-\mu_i)\rho BK$ is Schur stable for all i.

Let $\delta_i = (1-\mu_i)\rho$ for all $i = 2,\dots,N$. Recall that $\mu_i \leq \beta \leq 1-\frac{1}{\rho}$, thus $\delta_i \geq 1$. By assumption, $\gamma \leq \mu_i$ and $A-(1-\gamma)\rho|B|\tilde{E} \geq 0$. Since $BK \leq |B|\tilde{E}$, then $A-\delta_i|B|E \geq 0$. Observe from (5.20) that K satisfies

$$K^\top(B^\top p) = \tilde{E}^\top \left|B^\top p\right| \geq 0,$$

where $p \geq 0$ solves (5.19). Using that $s > 0$, $\rho \geq \frac{1}{1-\beta}$, $\gamma \in (-1, \beta)$, $\beta \in (0,1)$, it follows from (5.19) that

$$\begin{aligned}(A-(1-\mu_i)\rho BK)^\top p &= A^\top p - \delta_i K^\top (B^\top p)\\ &\leq A^\top p - \delta_i \tilde{E}^\top \left|B^\top p\right|\\ &\leq A^\top p - \tilde{E}^\top \left|B^\top p\right|\\ &= p - s < p.\end{aligned}$$

Therefore, by Proposition 17 in Section 2.2, $A - \delta_i BK$ is Schur stable for all $i = 1, \ldots, N$ with $\gamma \leq \mu_i \leq \beta$. ■

The following theorem provides an analogous characterization to the one in Theorem 41 for the trajectories of a discrete-time MAS.

THEOREM 43
Consider a graph family $\mathcal{F} \subseteq \tilde{\mathbb{G}}_{[\gamma,\beta]}$ and the MAS described by (2.39) and (2.41). Suppose the pair (A,B) is E-stabilizable and consider the protocol (5.23). Suppose also that

$$-1 < \gamma \leq \frac{1}{1+\sum_{j=1}^{N} w_{ij}} \quad \forall i = 1, \ldots N.$$

The trajectories of the MAS remain nonnegative for all nonnegative initial conditions if and only if BK is nonnegative. □

Proof: The dynamics (2.39) of each agent i is rewritten as

$$x_i(t+1) = \hat{A}x_i(t) + \hat{B}\hat{u}_i(t) \tag{5.26}$$

where $\hat{A} = A - \rho BK \frac{\sum_{j=1}^{N} w_{ij}}{1+\sum_{j=1}^{N} w_{ij}}$, $\hat{B} = \rho BK$ and $\hat{u}_i = \sum_{j=1}^{N} \frac{w_{ij}}{1+\sum_{k=1}^{N} w_{ik}} x_j$. Note that, by assumption $A - |B|E \geq 0$ and $|B|E \geq \rho BK$, hence $\hat{A} \geq 0$.

($\Longrightarrow$): Suppose that the matrix BK has, at least, one negative element, then there exists $\hat{B}_{p,q} < 0$ for some $p, q \in \mathbb{N}$. Let $x_i(0)_p = 0$ and $x_j(0)_q$ be sufficiently large for some $q \neq p$. Then $\hat{u}_i(0)_q \gg 0$ and from (5.26) it follows that $x_i(1)_p < 0$. Thus, x_i leaves the nonnegative orthant.

($\Longleftarrow$): Because $\hat{A}, \hat{B} \geq 0$ the system is internally positive with respect to $\hat{u}$ [Farina and Rinaldi, 2000, Ch. 2]. Thus, the trajectories of (5.26) remain nonnegative. ■

Numerical simulations

Consider the MAS (2.39) composed by 150 agents and described by

$$A = \begin{bmatrix} 0.4 & 0.8 \\ 0.4 & 0.7 \end{bmatrix}; \; B = \begin{bmatrix} -0.6 & 0.002 & -0.2 \\ -0.4 & 0.005 & 0.03 \end{bmatrix}; \; E = \begin{bmatrix} 0.07 & 0.2 \\ 0.3 & 0.3 \\ 0.3 & 0.01 \end{bmatrix}$$

with randomly generated initial conditions in the interval $[0,1]$. Consider also a connected undirected graph $\mathcal{G}$ in the family of regular graphs of degree $d = 7, 20$ denoted by $\mathcal{F}_R \subseteq \tilde{\mathbb{G}}_{[\gamma,\beta]}$. The matrix A is unstable with spectrum $\sigma(A) = \{-0.035235,\ 1.135235\}$.

Let the eigenvalues of every row stochastic matrix $\mathcal{D}$ associated with $\mathcal{G} \subseteq \mathcal{F}_R$ be upper bounded by $\beta = 0.25$ such that $\rho \geq 1/(1-\beta)$, in particular $\rho = 3.83$. To solve the state synchronization problem, the LR-based Protocol 5.2.1 is implemented. Consider $s = \mathbb{1} > 0$, and

$$\tilde{E} = \frac{1}{\rho} E = \begin{bmatrix} 0.02 & 0.05 \\ 0.07 & 0.07 \\ 0.07 & 0.003 \end{bmatrix}$$

such that $A - |B|\tilde{E}$ is nonnegative. The linear program (5.21) is maximized by a vector $p^* = [70.71,\ \ 128.27]$, which satisfies the algebraic equation (5.19) and results in

$$K = \begin{bmatrix} -0.07 & -0.2 \\ 0.3 & 0.3 \\ -0.3 & -0.01 \end{bmatrix}.$$

Observe that, $A - |B|E$ with $\tilde{E} = \frac{1}{\rho}E$ is nonnegative and

$$BK \approx \begin{bmatrix} 0.1 & 0.1 \\ 0.02 & 0.08 \end{bmatrix} \geq 0.$$

Hence, from Theorem 43 the trajectories of the MAS remain nonnegative for all initial conditions.

Figure 5.4 shows the evolution of the first and second states for each of the 150 agents in an interconnected system, where each agent is connected to 7 and 20 neighbors, respectively. It is clear from the figure that synchronization is achieved more rapidly as the nodal degree increases. In Figure 5.5, the state synchronization of the agents is depicted using a 2D plot of their trajectories. The distance from the trajectories to the synchronized trajectory is represented in Figure (5.6).

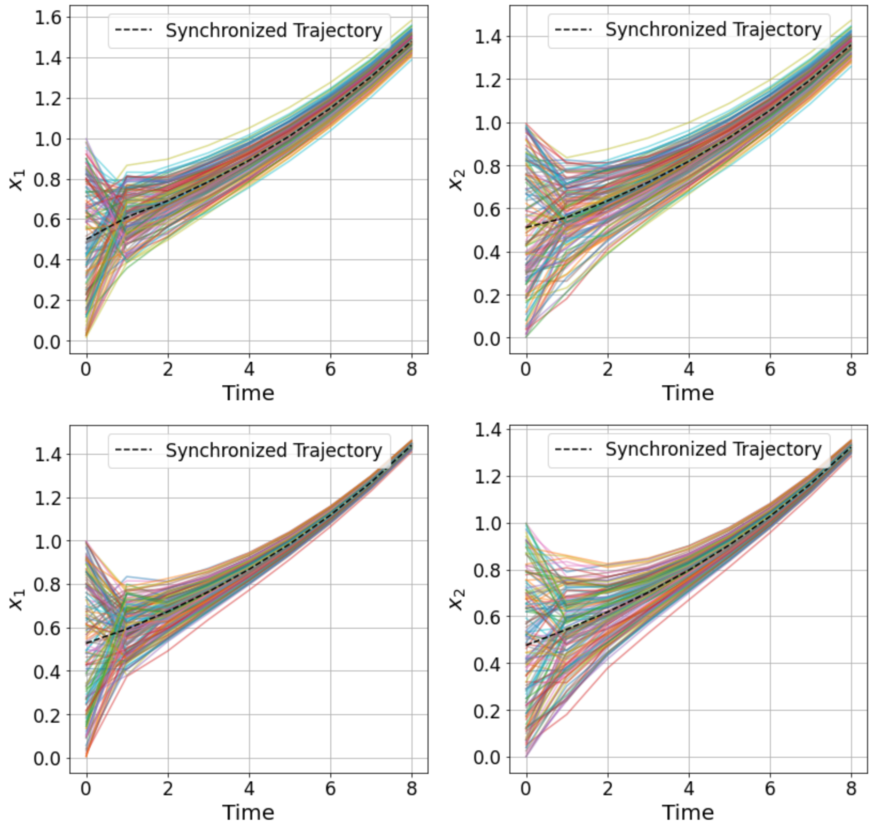


Figure 5.4 Evolution over time of the first (left panels) and the second (right panels) state of each agent $i = 1, \ldots, 150$ synchronizing over 7-regular graphs (upper panels) and 20-regular graphs.

5.3 Concluding summary of Chapter 5

In this Chapter we have introduced the linear-regulator based synchronization protocol for multi-agent systems with positive dynamics. The protocol was designed to ensure synchronization for all graphs within a specified family, assuming known upper and lower bounds on their eigenvalues. The approach serves as a positive systems analogue to the well-established LQR-based consensus protocol in [Saberi et al., 2022]. Building on previous results for the linear regulator problem in Chapters 3 and 4, it has been demonstrated that the proposed protocol ensures stability and provides an a priori proportional bound on the control input for each agent. Additionally, a condition guaranteeing the positivity of all state trajectories was established.

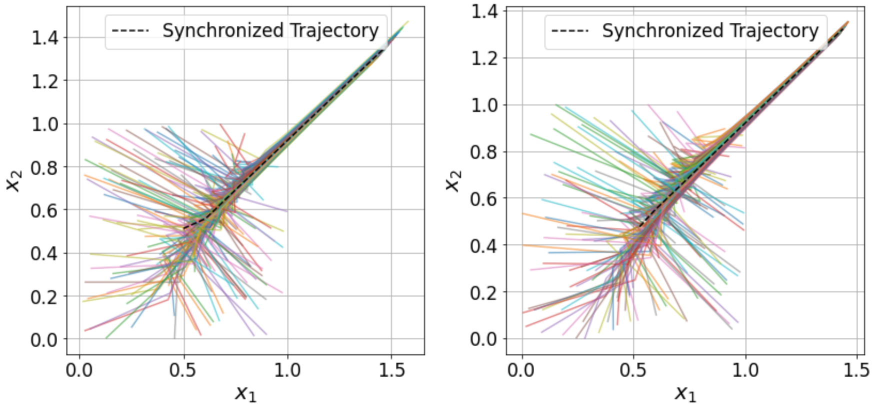


Figure 5.5 Trajectories of agents synchronizing over 4-regular graphs (left panel) and 20-regular graphs.

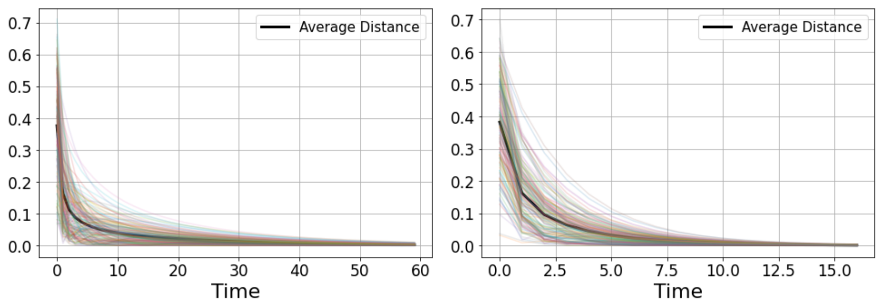


Figure 5.6 Euclidean distance to the synchronized trajectory $x_s(t)$ i.e. $\|x_i(t) - x_s(t)\|$ for $i = 1, \ldots 150$ over 7-regular graphs (left panel) and 20-regular graphs.

6

Conclusions and directions for future work

Explicit solutions to optimal control problems are rarely found or computed efficiently, especially for systems subject to disturbances or those of large scale. This thesis derives explicit solutions for a class of minimax optimal control problems with positive system dynamics (Chapter 3). Additionally, it characterizes the continuous-time, linear program formulation of the Linear Regulator problem, introduced in the discrete-time work [Rantzer, 2022], through detectability and stabilizability analysis (Chapter 4). Furthermore, this class of problems has been demonstrated to serve as a foundation for developing synchronization protocols in positive multi-agent systems (Chapter 5). We conclude the thesis by summarizing key findings related to these systems and outlining potential directions for future research.

6.1 Main conclusions

There are several optimal control problem classes. A well-studied class is the linear quadratic regulator. Using methodologies such as dynamic programming or Pontryagin's Maximum Principle one can derive a control law for a given system such that a certain optimality criterion is achieved. In this thesis a novel class of minimax control problems for positive systems is presented. Motivated by applications to large-scale systems, the computational tractability of explicit solutions for this class of problems is investigated. In particular, their linear program formulation is examined under the presence of unbounded, worst-case disturbances. Another interesting class of disturbances, those that are bounded with homogeneous constraints, is introduced, for which explicit solutions can also be found. Not only are large-scale systems utilized to enhance the advantages of this framework, but continuous and discrete-time multi-agent system dynamics are also shown to achieve consensus under the LR-based protocol introduced in the last contribution of this thesis. The topics are further elaborated below.

Minimax linear regulator framework

In Chapter 3, we demonstrate that the linear nature of the optimization framework, enables explicit solutions with parameters that scale linearly with the state dimension, making this approach particularly effective for large-scale systems. Notably, dynamic programming is performed without imposing a priori constraints on linearity or sparsity. Instead, these properties emerge naturally from the optimization criteria and constraints, ensuring that no nonlinear or nonsparse controller can achieve a lower cost. These explicit solutions rely on the positivity of the system dynamics, an essential feature of this problem class. For the infinite-horizon case, we derive explicit solutions using standard value iteration and a fixed-point iteration method for the continuous-time setting. Additionally, a necessary and sufficient condition on the disturbance penalty in the cost function is derived, addressing the l_1-induced gain minimization problem and guaranteeing the existence of solutions under unconstrained disturbances.

To illustrate the practical impact of these findings, two main examples are analyzed. Firstly, a continuous-time water-flow network subjected to disturbances from leakages and rainfall achieves system costs that closely approach those of a disturbance-free scenario. This demonstrates the efficiency of the optimal static feedback, provided appropriate tuning parameters are selected. Secondly, in the discrete-time setting, an optimal DC power network subject to disturbances from local generation and load variations highlights the diverse sparsity structure that the system designer can choose for the constraints on the control variable. These constraints determine the sparsity structure of the optimal control policy. Additionally, a discretized double-tank system illustrates how the sufficient and necessary condition for the disturbance penalty applies in the discrete-time setting.

Overall, this chapter introduces a robust and scalable optimal control framework that leverages the advantages of positive system dynamics, offering a novel and efficient approach to large-scale dynamical systems.

Sparsity of the optimal control law

The minimax regulator problems presented in this thesis and the linear regulator problem share the same optimal control policy structure. In Chapter 3, we demonstrate that this policy is static in the infinite horizon case and a bang-bang controller in the finite horizon case, for both continuous and discrete-time settings. A significant contribution of this problem class is that, unlike in the LQR framework, where the feedback matrix K is generally dense, the sparsity of the controller in our approach arises naturally from the constraints on the control variable. This characteristic is particularly advantageous for large-scale systems, where dense matrices pose computational challenges. Additionally, in applications such as distributed systems, it is often beneficial to enforce sparsity in K. Our method not only guarantees optimality but also offers the flexibility to define the sparsity structure of the controller. Ongoing research aims to establish precise guarantees on the level of sparsity that can be achieved while preserving the existence of optimal solutions.

Linear Program Formulation of the LR problem

To further highlight the strengths of our theoretical framework for large-scale systems, we demonstrate in Chapter 4 that, in the infinite-horizon minimax setting with positive unconstrained disturbances, the Hamilton-Jacobi-Bellman (HJB) equation can be reformulated as a linear program. Through the detectability and stabilizability analysis conducted in this chapter, we establish conditions guaranteeing the equivalence between the solutions of the HJB equation and its linear programming formulation. This linear programming formulation is subsequently employed to derive the static feedback protocol presented in Chapter 5. Taking advantage of the computational tractability of linear programming, this framework proves particularly effective when the system dynamics become large.

LR synchronization protocol

In Chapter 5, a synchronization protocol for positive systems based on the linear regulator static feedback matrix is introduced. This method highlights another compelling application of the theoretical framework developed in this thesis. The proposed approach inherits several advantages from the linear regulator setting. For instance, it allows for deriving the optimal policy through a linear programming formulation and facilitates the design of sparse optimal policies. This sparsity emerges directly from the constraints imposed on the control variable and subsequently propagates through the interconnected system.

6.2 Directions for future work

Many of the results presented in this thesis also provide interesting openings for further research. For example, ongoing research aims to characterize the family of E matrices that result in finite solutions under the assumption that the system (A, B) is positive and E-stabilizable. A related question is determining the most efficient sparsity structure of the E matrix, particularly as this sparsity is inherited by the optimal controller, which has important implications for computational efficiency and scalability.

Additionally, considering the high computational cost of value iteration, the exploration of alternatives such as policy iteration [Bertsekas, 2017], offers a compelling direction for future work, particularly in computing the solution to the multi-disturbance algebraic HJB and Bellman equation, since a linear program has not yet been formulated.

Another promising direction is characterizing how network topology impacts the existence of finite solutions. Building on the results in Chapters 3 and 4 to explore the output feedback setting, investigating how MPC can benefit from the LR framework, or incorporating stochastic disturbances and nonlinearities into the system dynamics also present exciting possibilities for future work. Along these lines, modeling the approximation errors arising from linearizing a system or the system uncertainties under our disturbance class representation introduces an intriguing topic for further research.

In the context of the LR-based protocol, potential extensions include deriving an LR-based protocol for directed graphs, addressing systems subjected to disturbances, and deriving a priori conditions to guarantee positive synchronization. Another interesting research direction is to further analyze and characterize the eigenvalue bounds of graph families that facilitate synchronization, as well as to elucidate the role of the sparsity structure in the optimal feedback matrix for the scalability of the interconnected system. These directions highlight the potential for further research and generalization of the results in this thesis.

E matrix

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