Articles on Random Normal Matrix Theory

Cronvall, Joakim

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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JOAKIM CRONVALL

Lund University
Faculty of Science
Centre for Mathematical Sciences
Mathematics

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Joakim Cronvall

LUND UNIVERSITY

DOCTORAL THESIS
Thesis advisors: Docent Yacin Ameur
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Abstract
In this thesis, we investigate the random normal matrix model in which the eigenvalues correspond to the two-dimensional Coulomb gas at inverse temperature $\beta = 2$. The thesis consists of seven papers concerning different aspects of the random normal matrix model. The main topics of the thesis are asymptotics of the correlation kernel, fluctuations of linear statistics, asymptotics of the partition function and disk counting statistics. In particular, we investigate these topics in the setting of disconnected droplets, where interactions between the different components must be taken into account.

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Joakim Cronvall

LUND
UNIVERSITY

Mathematics
Centre for Mathematical Sciences
Box 118
SE-22100 LUND
Sweden

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Contents

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Joakim Cronvall
Lund, April 2025

Populärvetenskaplig sammanfattning

Inom matematisk fysik studeras fysikaliska fenomen med hjälp av matematiska modeller. Målet är att kunna förklara och förutsäga fysikaliska beteenden utifrån mer elementära principer.

Coulombgasen är en sådan modell. Den beskriver laddade partiklar som repellerar varandra, men som hålls samman av ett yttre elektriskt fält. Om antalet partiklar är mycket stort blir det matematiskt svårt att beskriva de enskilda partiklarnas beteende. Coulombgasen behandlar i stället partiklarna som ett enda stort system och ger en sannolikhetsteoretisk beskrivning av partikelsamlingen. Konfigurationer med låg energi tilldelas hög sannolikhet, medan konfigurationer med högre energi är mindre sannolika.

I denna avhandling behandlas en modell av slumpmässiga normala matriser, vars egenvärden motsvarar Coulombgasen i ett plan vid en viss temperatur. När antalet partiklar är mycket stort så dominerar konfigurationer vars energi är nära den minsta möjliga. Coulombgasen formerar sig då i en droppe i planet för att minimera energin.

Artiklarna i avhandlingen behandlar flera olika aspekter av Coulombgasen. Ett centralt ämne i avhandlingen är korrelationer mellan partiklar. Det är känt att korreltationen mellan partiklar på olika ställen inne i droppen är mycket liten. Nära droppens rand är däremot situationen en annan, även partiklar på olika delar av randen känner av varandra. I denna avhandling beskriver vi hur dessa randkorrelationer ser ut i olika situationer.

Formen på droppen som bildas beror på det elektriska fältet. I vissa fält så kan droppen bestå av flera olika separerade komponenter. Partiklarna på de olika komponenterna känner fortfarande av varandra och antalet partiklar i de olika komponenterna kan fluktuera. Beskrivningar av hur dessa fluktuationer ser ut är ett annat viktigt ämne i avhandlingen.

List of publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Szegő type asymptotics for the reproducing kernel in spaces of full-plane weighted polynomials**
Y. Ameur, J. Cronvall
Communications in Mathematical Physics, **398** (2023), 1291-1348.

II **The two-dimensional Coulomb gas: fluctuations through a spectral gap**
Y. Ameur, C. Charlier, J. Cronvall
arXiv:2210.13959

III **Free energy and fluctuations in the random normal matrix model with spectral gaps**
Y. Ameur, C. Charlier, J. Cronvall
arXiv:2312.13904

IV **On fluctuations of Coulomb systems and universality of the Heine distribution**
Y. Ameur, J. Cronvall
arXiv:2411.10288

V **Random normal matrices: eigenvalue correlations near a hard wall**
Y. Ameur, C. Charlier, J. Cronvall
Journal of Statistical Physics, **191** (2024), no. 98.

VI **Exponential moments for disk counting statistics at the hard edge of random normal matrices**
Y. Ameur, C. Charlier, J. Cronvall, J. Lenells
Journal of Spectral Theory, **13** (2023), no. 3, 841-902.

VII **Disk counting statistics near hard edges of random normal matrices: the multi-component regime**
Y. Ameur, C. Charlier, J. Cronvall, J. Lenells
Advances in Mathematics, **441** (2024), 109549.

Preface

1 Introduction and background

The topic of this thesis is the random normal matrix model or the Coulomb gas model in the determinantal case (i.e. at inverse temperature $\beta = 2$). Before discussing some of the results obtained in this thesis, we define the model and discuss some of its basic properties.

1.1 The Coulomb gas

The two-dimensional Coulomb gas is a model of n equally charged particles in an external field with logarithmic interactions in the complex plane $\mathbb{C}$. At the mathematical level, it is a probability measure $\mathbb{P}_n^\beta$ on the space of configurations $\mathbb{C}^n$. Given an external potential $Q : \mathbb{C} \to \mathbb{R} \cup \{+\infty\}$ we assign to each configuration of points $(z_1, \ldots, z_n) \in \mathbb{C}^n$ an energy H_n given by

$$H_n(z_1, \ldots, z_n) = \sum_{j \neq k} \log \frac{1}{|z_j - z_k|} + n \sum_{j=1}^{n} Q(z_j).$$

The energy consists of two terms, one is the Coulomb interaction between particles, the second is the interaction of the individual particles with the external field. The measure $\mathbb{P}_n^\beta$ is of Gibbs-type and defined by

$$d\mathbb{P}_n^\beta(z_1, \ldots, z_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{2} H_n(z_1 \ldots, z_n)} dA_n(z_1, \ldots, z_n), \tag{1}$$

where $\beta > 0$ is the inverse temperature and dA_n denotes n-fold area measure normalized so that the unit disk has area 1. The factor Z_n^β is a normalizing constant making $\mathbb{P}_n^\beta$ a probability measure. The convergence of the Z_n^β is ensured by assuming that Q has sufficient growth at infinity

$$\liminf_{|z| \to +\infty} \frac{Q(z)}{\log |z|^2} > 1.$$

The random ensemble of points obtained from $\mathbb{P}_n^\beta$ is what we refer to as the two-dimensional Coulomb gas in potential Q at inverse temperature β.

The general temperature case is of great interest and displays rich features. For example it can be seen that the Coulomb gas interpolates between Fekete point configurations for $\beta = +\infty$ [28, 30] and a Poisson point process for $\beta = 0$ [1].

This thesis is devoted to the case $\beta = 2$, for which the Coulomb gas is a random matrix model. This case is especially rich with connection to orthogonal polynomials, special functions, complex analysis and more. We will use the simplified notation $\mathbb{P}_n$ and Z_n when $\beta = 2$.

1.2 Random normal matrix model

As we will see, for $\beta = 2$ the Coulomb gas has a lot of structure. In fact the point process can viewed as eigenvalues of random normal matrices.

A normal matrix is a square matrix , M, over $\mathbb{C}$ commuting with its Hermitian adjoint, that is $MM^* = M^*M$. The collection of $n \times n$ normal matrices form a submanifold in $\mathbb{C}^{n\times n}$ and inherits a Riemannian volume form which we denote dM_n. We define a probability measure $\mathcal{P}_n$, on the space of normal matrices, by

$$d\mathcal{P}_n(M) = \frac{1}{\mathcal{Z}_n} e^{-n \operatorname{tr} Q(M)} dM_n(M), \tag{2}$$

where $\mathcal{Z}_n$ is a normalizing constant and $\operatorname{tr} Q(M)$, has the natural definition

$$\operatorname{tr} Q(M) = \sum_{\lambda \in \operatorname{Spec}(M)} Q(\lambda).$$

Here $\operatorname{Spec}(M)$ is the spectrum (the set of eigenvalues) of M. The entries of such a random matrix are of course not independent.

It is a remarkable fact that eigenvalues of a matrix picked with respect to the measure $\mathcal{P}_n$ has the same distribution as a random sample of points from the measure $\mathbb{P}_n$. To see the connection we note that a normal matrix M, can be diagonalized as $M = UDU^*$, where D is diagonal with the eigenvalues of M as entries and U is a unitary matrix. The matrix U is unique up to right multiplication by a diagonal unitary matrix. The following factorization from [13] of the measure dM_n makes the connection to the Coulomb gas at $\beta = 2$ clear

$$dM_n = dU_n \prod_{j \neq k} |z_j - z_k| \, \pi^n \, dA_n(z_j), \tag{3}$$

where $\{z_j\}_{j=1}^n$ are the eigenvalues and dU_n is the Haar measure on the group of unitary matrices of size $n \times n$.

If we rewrite the measure in (2) using the factorization of the measure dM_n from (3) it becomes clear that the eigenvalues from $\mathcal{P}_n$ have the same distribution as a random sample from $\mathbb{P}_n$. Because of this equivalence we will, throughout the thesis, use the terms the random normal matrix model and the Coulomb gas model at $\beta = 2$ synonymously.

1.3 The correlation kernel

The eigenvalues of the random normal matrix model form what is known as a determinantal point process.

Since we are interested in a two-dimensional model, we restrict the discussion to point processes in the complex plane $\mathbb{C}$. A point process $\mathcal{X}$ on $\mathbb{C}$ is a random integer-valued Radon measure on $\mathbb{C}$. We call the point process $\mathcal{X}$ simple if it assigns (almost surely) at most measure one to singeltons.

The k-point correlation functions with respect to a reference measure μ are (if they exists) the functions $\rho_k : \mathbb{C}^k \to [0, \infty)$ for $k \geq 1$ such that for any finite collection of mutually disjoint sets $A_1, \ldots, A_n \subset \mathbb{C}$ the identity

$$\mathbb{E}\Big[\prod_{i=1}^{k} \mathcal{X}(A_i)\Big] = \int\limits_{\prod_{i=1}^{k} A_i} \rho_k(x_1, \ldots, x_k)\, d\mu(z_1) \cdots d\mu(z_k),$$

holds. We mention that the one-point function ρ_1 has the natural interpretation of the density of points with respect to the measure μ.

We say that a simple point process is a determinantal point process with reference measure μ if there exists a measurable function $\mathcal{K} : \mathbb{C}^2 \to \mathbb{C}$ such that

$$\rho_k(z_1, \ldots z_k) = \det(\mathcal{K}(z_i, z_j))_{i,j=1}^{k},$$

for every $k \geq 1$ and for all points $z_1, \ldots, z_k \in \mathbb{C}$. The function $\mathcal{K}$ is called a correlation kernel. We refer to the book [22] for more information about determinantal point processes.

The Coulomb gas (with $\beta = 2$) has precisely this structure. From the random configuration $\{z_j\}_{j=1}^{n}$ picked with respect to $\mathbb{P}_n$ we form a point process $\mathcal{X}_n$ by placing a point mass at each point in the configuration, that is

$$\mathcal{X}_n = \sum_{j=1}^{n} \delta_{z_j}.$$

It is not hard to see that the resulting point process is determinantal. Start by rewriting the density of the measure $\mathbb{P}_n$

$$e^{-H_n(z_1,\ldots,z_n)} = \prod_{j \neq k} |z_j - z_k| e^{-nQ(z_j)} = |\det(V_n(z_1, \ldots, z_n))|^2 \prod_{j=1}^{n} e^{-nQ(z_j)}, \quad (4)$$

where V_n is the Vandermonde matrix

$$V_n(z_1, \ldots, z_n) = (z_j^{k-1})_{j,k=1}^{n}.$$

The Vandermonde determinant can be rewritten in terms of orthonormal polynomials with respect to the measure $e^{-nQ(z)} dA(z)$. This will only change the determinant by a multiplicative constant. That is

$$\det V_n(z_1, \ldots, z_n) = c_n \cdot \det(p_{j-1,n}(z_k))_{j,k=1}^{n}, \quad (5)$$

where $p_{j,n}$ is the orthonormal polynomial of degree j obtained by applying the Gram-Schmidt process to the set $\{1, z, \dots, z^{n-1}\}$. Plugging equation (5) into (4) yields

$$e^{-H_n(z_1,\dots z_n)} = |c_n|^2 \cdot \det(K_n(z_j, z_k))_{j,k=1}^n,$$

where the function K_n is given by

$$K_n(z,w) = \sum_{j=0}^{n-1} p_{j,n}(z)\overline{p_{j,n}(w)} e^{-\frac{n}{2}(Q(z)+Q(w))}.$$

It now follows that the measure $\mathbb{P}_n$ can be written as

$$d\mathbb{P}_n(z_1,\dots,z_n) = \frac{1}{n!} \det(K_n(z_j,z_k))_{j,k=1}^n \, dA_n(z_1,\dots,z_n). \qquad (6)$$

(The factor $\frac{1}{n!}$ appears since the order of the z_j matters for $\mathbb{P}_n$ but not for the point process $\mathcal{X}_n$.)

The formula (6) shows that the n-point correlation function ρ_n can be written as a determinant of the function K_n. We can see that the same is in fact true for any k-point correlation function (with $k \le n$) simply by integrating out $(n-k)$ variables and using the orthonormality of the polynomials. Hence the Coulomb gas with $\beta = 2$ is a determinantal point process with correlation function K_n with respect to the reference measure dA.

The correlation kernel K_n also has a natural interpretation as a reproducing kernel. We denote the space of polynomials of degree less than n equipped with the inner product from $L^2(\mathbb{C}, e^{-nQ}dA)$ by Pol_n. This is a reproducing kernel Hilbert space whose kernel we denote by $k_n(z,w)$. That is, for any $p \in \mathrm{Pol}_n$ we have

$$p(z) = \int_{\mathbb{C}} p(w)\overline{k_n(w,z)} e^{-nQ(w)} dA(w).$$

The kernel k_n can be expressed in terms of the orthonormal basis $\{p_{j,n}\}_{j=0}^{n-1}$ of the space Pol_n

$$k_n(z,w) = \sum_{j=0}^{n-1} p_{j,n}(z)\overline{p_{j,n}(w)}.$$

It follows that the correlation kernel K_n is a weighted reproducing kernel in the sense that

$$K_n(z,w) = k_n(z,w) e^{-\frac{n}{2}(Q(z)+Q(w))}.$$

The kernel K_n is one of the main objects of study in this thesis. The strong link between random normal matrix theory and planar orthogonal polynomials will be exploited throughout the thesis.

1.4 Potential theory and equilibrium measure

The Coulomb gas, as a statistical physics model, becomes particularly interesting in the thermodynamic limit, that is when the number of points n tends to infinity. When the number of points becomes large, configurations close to the minimal value of the energy H_n start to dominate the behaviour of the system. At a macroscopic level we will see that the Coulomb gas minimizes the energy, H_n, in the following way.

Consider the non-linear functional

$$I_Q[\mu] = \int_{\mathbb{C}^2} \log \frac{1}{|z-w|} d\mu(z) d\mu(w) + \int_{\mathbb{C}} Q(z) d\mu(z), \tag{7}$$

acting on the set of compactly supported Borel probability measures. The functional I_Q is called the weighted logarithmic energy and can be seen as a continuous version of the energy H_n. It is well known from potential theory that for $Q \in C^2$ satisfying the growth condition

$$\liminf_{z\to\infty} Q(z) - \log|z|^2 = +\infty,$$

the minimizer of I_Q is unique and takes the form

$$d\sigma(z) = 1_S(z)\Delta Q(z) dA(z),$$

where S is a compact set called the droplet and $\Delta = \frac{1}{4}(\partial_x^2 + \partial_y^2)$; see e.g. the book [31] on weighted potential theory on $\mathbb{C}$ and [34] for an extension to Riemann surfaces. The Coulomb gas tends to follow the equilibrium measure σ in the sense that the random measure

$$\frac{1}{n}\mathcal{X}_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j},$$

converges weakly in probability to σ; see e.g. [19].

1.5 The Ginibre ensemble

In 1965, Ginibre studied in [18] the ensemble corresponding to $Q(z) = |z|^2$. The corresponding ensemble is nowadays known as the Ginibre ensemble.

This ensemble is also related to a different type of matrix model with independent entries. Namely, consider an $n \times n$ matrix with independent entries all of which have a complex Gaussian distribution with mean 0 and variance $\frac{1}{n}$. The n random eigenvalues of such a matrix have the same distribution as the Coulomb gas with potential $Q(z) = |z|^2$ and $\beta = 2$.

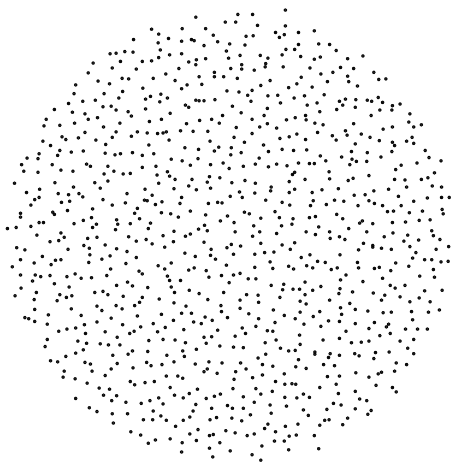

Figure 1: A sample from the Ginibre ensemble with $n = 1000$.

The radial symmetry of Q implies that the monomials $\{z^j\}_{j=0}^{n-1}$ are orthogonal in $L^2(\mathbb{C}, e^{-n|z|^2} dA)$. In fact the orthonormal polynomials simply becomes

$$p_{j,n}(z) = \sqrt{\frac{n^{j+1}}{j!}} z^j,$$

and the correlation kernel takes the form

$$K_n(z,w) = n \sum_{j=0}^{n-1} \frac{(nz\bar{w})^j}{j!} e^{-\frac{n}{2}(|z|^2+|w|^2)}.$$

The droplet in the Ginibre ensemble is the closed unit disk $S = \overline{\mathbb{D}}$ and the equilibrium measure is simply the uniform measure on $\overline{\mathbb{D}}$ since $\Delta|z|^2 = 1$.

The convergence to the uniform measure on the unit disk holds also for random entries that are not Gaussian. This is known as the circular law. It actually true in a very general setting. If the entries of an $n \times n$ matrix are assumed to be independent and identically distributed with mean 0 and variance $\frac{1}{n}$, then the spectral measure converges almost surely to the uniform measure on the unit disk. The theorem has a long history and we refer to [35] where the theorem was proven under the minimal assumptions as stated here. If the matrix entries are not Gaussian we lose the connection to random normal matrix theory.

We also mention that for the Ginibre ensemble there is an exact correspondence to non-interacting fermions in a plane with respect to a perpendicular magnetic field; see e.g. [17].

For a more comprehensive survey of the Ginibre ensemble and related models we refer to the recent book [9].

1.6 Rescaled kernel

We have seen that the Coulomb gas is completely determined by the correlation kernel. Studying the correlation kernel therefore naturally becomes central in the theory.

At the macroscopic level we have already seen that the density of the Coulomb gas tends to follow the equilibrium measure σ minimizing the weighted logarithmic energy (7). For the one-point function $\rho_1(z) = K_n(z,z)$ this implies that

$$\frac{1}{n} K_n(z,z) \to 1_S(z)\, \Delta Q(z), \quad \text{as } n \to \infty.$$

In order to see the microscopic picture we can rescale the kernel around a point $z_0 \in \mathbb{C}$. For a point z_0 in the interior of the droplet S such that $\Delta Q(z_0) > 0$, the limiting kernel is universal, i.e. independent of Q and z_0. It is given by

$$G(\zeta,\eta) = e^{\zeta\bar{\eta} - |\zeta|^2/2 - |\eta|^2/2},$$

and is the kernel of the infinite Ginibre ensemble found by Ginibre [18]. The universality result was proven in a slightly different context by Berman [6] and later for the random normal matrix model by Ameur, Hedenmalm and Makarov [3].

For a boundary point the limiting kernel is different and describes the transition from the interior to the exterior of the droplet. The kernel was discovered for the Ginibre ensemble by Forrester and Honner [15] and is given by

$$G(\zeta,\eta) \cdot \frac{1}{2}\,\mathrm{erfc}(\frac{\zeta+\bar{\eta}}{\sqrt{2}}),$$

where erfc is the complementary error function. Universality of this kernel has been given in different contexts in [5] using a rescaled Ward identity and in a more general setting in [21] using asymptotics of planar orthogonal polynomials.

We see that the transition from the droplet to the exterior of the droplet leads to a rapid Gaussian type decay in the particle density. What makes the boundary case complicated is that the kernel is not localized. The approach via orthogonal polynomials resolves this by considering the whole boundary at once.

1.7 Fluctuation around the equilibrium measure

We have seen that the average of the Coulomb gas converges to the equilibrium measure. An important question is to study the fluctuations around this equilibrium.

Given a function $f : \mathbb{C} \to \mathbb{R}$ we a associate a random variable $\text{trace}_n f$, also called a linear statistic, by

$$\text{trace}_n : f \mapsto \sum_{j=1}^{n} f(z_j),$$

where $\{z_j\}_{j=1}^n$ are taken at random with respect to the measure $d\mathbb{P}_n$. We assume that the function is smooth, say $f \in C_0^\infty$.

The large n behaviour of the linear statistic tends to follow the equilibrium measure in the sense that

$$\frac{1}{n}\mathbb{E}_n \, \text{trace}_n f \to \int_{\mathbb{C}} f(z) d\sigma(z),$$

as $n \to \infty$. Fluctuations around the equilibrium measure, $\text{fluct}_n f$, is defined by

$$\text{fluct}_n f = \text{trace}_n f - n \int_{\mathbb{C}} f(z) d\sigma(z),$$

that is by removing the leading order term.

We now mention some tools used to study fluctuations. The normalizing constant Z_n defined in (1) is called that partition function and is intimately connected with fluctuations of linear statistics. Consider a function f and the perturbed potential

$$Q_f(z) = Q(z) - \frac{s}{n} f(z).$$

Let $Z_{n,f}$ be the partition function with respect to Q_f and Z_n the partition function with respect to the unperturbed potential Q. The quotient of the two can be written as

$$\frac{Z_{n,f}}{Z_n} = \frac{1}{Z_n} \int e^{s \sum_{j=1}^{n} f(z_j)} e^{-H_n(z_1,\ldots,z_n)} dA_n = \mathbb{E}_n[e^{s \, \text{trace}_n f}],$$

where $\mathbb{E}_n$ is the expected value with respect to the measure $\mathbb{P}_n$. This shows that a good understanding of what effect a perturbation has on the partition function gives us information about the moment generating function of $\text{trace}_n f$.

Another important tool when studying fluctuations is the Ward identity. This important identity can be deduced as follows. Consider a test function $\psi \in C_0^\infty$. For $j = 1, \dots n$, integration by parts gives the identity

$$\mathbb{E}_n[\partial\psi(z_j)] = \mathbb{E}_n[\psi(z_j) \cdot \partial_j H_n(z_1, \dots, z_n)], \tag{8}$$

where ∂_j denotes differentiation with respect to the j-th variable. Define the three random variables I_n, II_n and III_n as follows:

$$\mathrm{I}_n[\psi] = \frac{1}{2} \sum_{j \neq k} \frac{\psi(z_j) - \psi(z_k)}{z_j - z_k},$$

$$\mathrm{II}_n[\psi] = n \sum_{j=1}^{n} \partial Q(z_j) \cdot \psi(z_j),$$

$$\mathrm{III}_n[\psi] = \sum_{j=1}^{n} \partial\psi(z_j).$$

A direct consequence of (8) is the following variant of the Ward identity

$$\mathbb{E}_n[\mathrm{I}_n[\psi] - \mathrm{II}_n[\psi] + \mathrm{III}_n[\psi]] = 0. \tag{9}$$

In physics, Ward identities are well known, see e.g; [37], and they play an important role in conformal field theories; see [25] for a mathematical treatment. The identity (9) was used in [4] to study limiting fluctuations of linear statistics. Under some regularity conditions and assuming that the droplet is connected the result from [4] shows that

$$\text{fluct}_n f \to N(e_f, v_f),$$

as $n \to \infty$, where $N(e_f, v_f)$ is a real Gaussian random variable. The expectation, e_f is given by

$$e_f = \int_S \Delta f + f \Delta L \, dA + \int_{\partial S} f \, \mathcal{N}(L^S) \, ds, \tag{10}$$

where $L(z) = \log \Delta Q(z)$ and $\mathcal{N}$ is the Neumann's jump operator (the definition can be found in [4] or Paper IV). L^S is the Poisson modification of L, that is the function equal to L on the droplet S and harmonic in the complement. The variance v_f is given by

$$v_f = \frac{1}{4} \int_{\mathbb{C}} |\nabla f^S(z)|^2 dA(z), \tag{11}$$

where f^S is the Poisson modification of f.

Note that the variance of $\text{fluct}_n f$ is bounded. The convergence of $\text{fluct}_n f$ is in this way different from the classical central limit theorem of independent random variables where the variance grows like n.

The Gaussian convergence for the Ginibre ensemble was proven before in [29] using the method of cumulants. The one-dimensional counterpart to the Ward identity was used by Johansson [23] to study fluctuations of random Hermitian matrices.

1.8 Hard edge and Balayage measure

From the Coulomb gas perspective it is natural to consider the point process restricted to a compact set $\mathcal{D} \subset \mathbb{C}$. This is equivalent to letting the weight Q be $+\infty$ outside of $\mathcal{D}$. For such a potential it even makes sense to consider the unweighted case $Q = 0$ on $\mathcal{D}$. In fact this is related to classical potential theory as the equilibrium measure in this case is the probability measure minimizing the logarithmic energy

$$I[\mu] = \int_{\mathcal{D}} \log \frac{1}{|z-w|} d\mu(z) d\mu(w). \tag{12}$$

It is well known that in this setting the extremizer is the harmonic measure rooted at infinity of the component of $\hat{\mathbb{C}} \setminus \mathcal{D}$ containing ∞. The orthogonal polynomials related to such a potential are also classical and asymptotics appeared first in Carleman's fundamental work [12]. The Coulomb gas confined to such a set has been studied in e.g. [2, 24].

The weighted hard edge case, when $Q \neq 0$, is much more complicated. In general not even the potential theory is well understood. However in some cases it is possible to give a good description. Consider a set $G \subset S$. We modify the potential such that

$$\tilde{Q}(z) = \begin{cases} Q(z), & z \in \mathbb{C} \setminus G \\ +\infty, & z \in G. \end{cases}$$

We refer to such a potential as a hard edge potential. Naturally the equilibrium measure changes. No mass can be placed in G. By definition, the logarithmic potential of a measure μ is

$$U^\mu(z) = \int \log \frac{1}{|z-w|} d\mu(w), \quad \text{for any } z \in \mathbb{C}.$$

This function can be seen as the potential energy created by a charge distribution μ. Assume that μ is supported in G. It is well known from potential theory [31] that there exists a measure μ' supported on the boundary ∂G with the same mass as μ such that $U^{\mu}(z) = U^{\mu'}(z)$ for $z \in \mathbb{C} \setminus G$. The measure μ' can be thought of as sweeping out the measure μ to the boundary, while keeping the logarithmic potential. We call μ' the balayage of μ onto ∂G. This generalizes the idea behind the harmonic measure. The balayage of a point mass at $z \in G$ onto ∂G is precisely the harmonic measure in G rooted at z.

It turns out that the balayage describes the equilibrium measure with respect to the hard edge potential $\tilde{Q}$. The equilibrium measure in potential Q restricted to G, that is $\nu := 1_G(z) \cdot \Delta Q(z) dA(z)$, is swept out to the boundary of G, while the rest of the equilibrium measure is unchanged. Thus the resulting equilibrium measure in potential $\tilde{Q}$ becomes

$$d\sigma = 1_{S \setminus G}\, d\sigma + 1_G\, d\sigma = 1_{S \setminus G}\, \Delta Q\, dA + d\nu'$$

where ν' is the balayage measure of ν. The boundary ∂G in this setting will be called a hard edge to distinguish it from the natural boundary of the droplet in a smooth potential which we will call a soft edge.

The equilibrium measure of (12) can also be thought of as the balayage measure of a pointmass at ∞ onto the boundary $\partial \mathcal{D}$. The idea of balayage goes back to Poincaré; see [31] for a detailed treatment.

2 Results

We now give a presentation of some of the results obtained in this thesis. We point out that the results in this section might be stated in a slightly different way than in the papers. Many results have been omitted. For the complete picture we refer the reader to the individual papers.

2.1 Kernel asymptotics for a soft edge

2.1.1 Long-range correlations in the simply connected setting

In a pioneering work, Forrester and Jancovici [16] study long-range correlations along the boundary of the droplet with respect to the elliptic Ginibre ensemble. This ensemble may be realised as the random normal matrix model with the quadratic potential

$$Q(z) = ax^2 + by^2, \quad \text{for } z = x + iy,$$

where $a, b > 0$. For this potential the droplet is an elliptic disk, and it is found in [16] that for two distinct points z, w on the boundary ellipse, the correlation kernel $K_n(z, w)$ is large, of order $\sqrt{n}$. This is in sharp contrast with long-range correlations in the bulk, which are exponentially small as $n \to \infty$. It is also conjectured in [16] that similar results should hold for a general potential Q.

In Paper I we verify the conjecture for a large class of potentials Q and find an asymptotic formula for the corresponding correlation kernel. We need some setup before stating the theorem.

We consider a potential Q with droplet S. We assume that the component U of $\hat{\mathbb{C}} \setminus S$ containing ∞ is simply connected and that Q is real-analytic in a neighbourhood of ∂U. We also assume that the boundary ∂U is a real-analytic Jordan curve. It follows from Sakai's regularity theory [32] that these assumptions are not too restrictive.

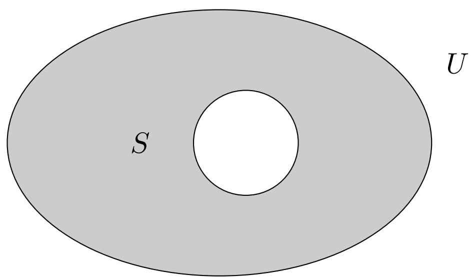

Figure 2: Example of the droplet S (grey) and the exterior component U.

Our main result from Paper I is expressed in terms of the reproducing kernel of a certain Hardy space. Consider the space $H^2_0(U)$ consisting of holomorphic functions in U vanishing at infinity with the following L^2-norm on the boundary

$$\|f\|^2 = \int\limits_{\partial U} |f(z)|^2 (\Delta Q(z))^{-1/2} ds(z),$$

where ds denotes the arclength measure. The reproducing kernel of the space $H^2_0(U)$ can be explicitly written as

$$S(z,w) = \frac{1}{2\pi} \frac{\sqrt{\phi'(z)}\overline{\sqrt{\phi'(w)}}}{\phi(z)\overline{\phi(w)} - 1} e^{\frac{1}{2}\mathcal{H}(z)} e^{\frac{1}{2}\overline{\mathcal{H}(w)}},$$

where $\phi(z) : U \to \mathbb{D}_e$ ($\mathbb{D}_e = \{z \in \mathbb{C} : |z| > 1\}$) is the conformal map normalized by $\phi(\infty) = \infty$ and $\phi'(\infty) > 0$, and $\mathcal{H}$ is a holomorphic function in U such that

$$\operatorname{Re} \mathcal{H}(z) = \frac{1}{2} \log \Delta Q(z), \quad z \in \partial U.$$

To make it unique we assume that $\operatorname{Im} \mathcal{H}(\infty) = 0$.

To state our result we need to define one more function $\mathcal{Q}$ as the holomorphic function in U with $\operatorname{Re} \mathcal{Q}$ equal to Q on ∂U and with $\operatorname{Im} \mathcal{Q}(\infty) = 0$.

Let $\delta_n = M\sqrt{\frac{\log\log n}{n}}$ where M is a constant depending on Q. By $N(U, \delta_n)$ we denote a δ_n-neighbourhood of U. We have the following theorem from Paper I.

Theorem 2.1 *Fix constants $c > 0$ and $0 < \beta < \frac{1}{4}$. For $z, w \in N(U, \delta_n)$ such that $|z - w| > c$ we have*

$$K_n(z,w) = \sqrt{2\pi n}\, S(z,w)\, (\phi(z)\overline{\phi(w)})^n\, e^{\frac{n}{2}\mathcal{Q}(z) + \frac{n}{2}\overline{\mathcal{Q}(w)}}\, e^{-\frac{n}{2}Q(z) - \frac{n}{2}Q(w)} \cdot (1 + \mathcal{O}(n^{-\beta})),$$

as $n \to \infty$.

Our proof uses the asymptotic formula for planar orthogonal polynomials of Hedenmalm and Wennman [20, 21] and a technique involving summation by parts, allowing us to obtain results extending across the boundary.

The kernel is of order $\sqrt{n}$ but only in a thin band of width $\mathcal{O}(n^{-1/2})$ along the boundary. This can be seen from the following expansion

$$2\log|\phi(z)| + \operatorname{Re} \mathcal{Q}(z) - Q(z) = -2\Delta Q(z_0)\ell^2 + \mathcal{O}(\ell^3),$$

where z_0 is the closest point from z on ∂U and ℓ is the distance in the normal direction of ∂U from z to z_0.

This gives a new manifestation of the boundary field that is seen when studying fluctuations of linear statistics. In fact in [14] Forrester demonstrates how the long-range correlations in the Ginibre ensemble can be used to prove that the fluctuations of linear statistics converges to a Gaussian field on the boundary (in addition to the independent Gaussian field in the bulk). As far as we are aware, a rigorous proof of the fluctuation theorem using the asymptotics of Theorem 2.1 has not been carried out, but it seems like an interesting direction for future research.

2.1.2 Long-range correlations in doubly connected setting

In Paper I we used the asymptotic formulas for orthogonal polynomials from [21] to study the long-range correlations along the boundary of a droplet. This method relies heavily on the unbounded component U of the complement of S being simply connected. For a multi-connected component of $\hat{\mathbb{C}} \setminus S$, the orthogonal polynomials close to the boundary are not known in general. In fact the only previously known asymptotic results for orthogonal polynomials in such a setting was obtained for certain lemniscate ensembles, see [7] and references therein.

In Paper II we study the long-range correlations in a doubly connected gap created by a radially symmetric smooth potential. To be precise the potential satisfies $Q(z) = Q(|z|)$ which implies that also the droplet is radially symmetric. We assume that there is a bounded doubly connected component in $\hat{\mathbb{C}} \setminus S$ which we denote by G. From the radial symmetry of Q it is clear that G is an annulus, i.e. there exists $0 < r_1 < r_2 < 0$ such that

$$G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

For a radial potential the orthonormal polynomials are nothing but normalized monomials. Although the structure of the polynomials is simple, the correlation kernel has a rich structure. Interestingly, when considering long-range correlations along the boundary, the asymptotics of the correlation kernel are again described in terms of the reproducing kernel for a Hardy space.

The space depends on a parameter x_n determined in the following way. Let $M = \sigma(\{|z| \leq r_1\})$, that is the mass of the inner component with respect to the equilibrium measure. Now let $x_n = \{Mn\} = Mn - \lfloor Mn \rfloor$ be the fractional part of Mn.

We now explain how long-range correlations along the boundary of G can be described in terms of reproducing kernels of a certain Hardy space. However, rather than a space of analytic functions, the appropriate space turns out to consist of multi-valued functions in the present case. Fortunately such spaces where already introduced by Widom [36] in his work on polynomials on systems of Jordan curves. Let us define these spaces.

Consider multi-valued analytic functions on G with single-valued absolute value. Assume a positive orientation of the boundary of G. The increment of the argument along the inner boundary component (or the outer) determines a class of multi-valued functions. We consider such multi-valued holomorphic functions of class x_n on G and let $H^2(G, x_n)$ be the Hardy space of such functions with norm

$$\|f\|^2 = \int_{\partial G} |f(z)|^2 (\Delta Q(z))^{-1/2} ds(z).$$

The space has a reproducing kernel $S_n(z,w)$. Note that $S_n(z,w)$ depends on x_n and therefore also on n.

We can now formulate one of the theorems from Paper II in the following way.

Theorem 2.2 *Let* $z = e^{i\theta_1}(r_j + \frac{t}{\sqrt{n\Delta Q(r_1)}})$ *and* $w = e^{i\theta_2}(r_k + \frac{s}{\sqrt{n\Delta Q(r_2)}})$ *with* $j,k \in \{1,2\}$ *and* $t,s,\theta_1,\theta_2 \in \mathbb{R}$*. Assume also that* $|z-w| \geq c$*, for some constant* $c > 0$*. Then we have the following asymptotics for* $K_n(z,w)$

$$K_n(z,w) = \sqrt{2\pi n}\, S_n(z,w)\, e^{-t^2-s^2} e^{iMn(\theta_1-\theta_2)} + \mathcal{O}(\log^5 n),$$

as $n \to \infty$*.*

In Paper II we did not make the connection to the Hardy space of multi-valued function, but the statements are equivalent.

A similar behaviour as in the simply connected case appears here, where there are lone-range correlations along the boundary in a band of width $\mathcal{O}(n^{-1/2})$. Again this shows that there is a boundary field. As we will see later, this field is not (purely) Gaussian.

A special case of long-range correlations for a disconnected droplet has also been studied for the lemniscate ensemble [11]. It would be interesting to see if the asymtotics of the correlation kernel takes the same form as in Theorem 2.2.

2.1.3 One-point function

The one-point function $K_n(z,z)$ measures the density of eigenvalues at z. We are interested in the behaviour near a boundary point of the gap G. Since G is not simply connected, even the leading order behaviour of the one point function was unknown, but we verify that it has the expected error-function behaviour. What is more interesting is the subleading term which includes geometric information about the gap and is in this sense non-local. Before stating the theorem we define the Jacobi theta function $\theta(z;\tau)$ by

$$\theta(z;\tau) := \sum_{\ell=-\infty}^{+\infty} e^{2\pi i \ell z} e^{\pi i \ell^2 \tau},$$

for $z \in \mathbb{C}$ and $\tau \in i(0,+\infty)$.

A result from Paper II states the following:

Theorem 2.3 *Let $G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ be a gap in the droplet S, consider $\alpha \in [0, 2\pi)$ fixed and put*

$$z = e^{i\alpha}\left(r_1 + \frac{t}{\sqrt{2n\Delta Q(r_1)}}\right).$$

As $n \to +\infty$, the one-point function $K_n(z,z)$ has the asymptotic expansion

$$\begin{aligned} K_n(z,z) = n\Delta Q(r_1)\frac{\operatorname{erfc}(t)}{2} + \frac{\sqrt{n\Delta Q(r_1)}}{\sqrt{2\pi}\, r_1} e^{-t^2}\Bigg[\frac{1}{6}(t^2-2) \\ + r_1 \frac{\partial_{\mathrm{n}}\Delta Q(r_1)}{\Delta Q(r_1)}\left(\frac{1}{2}\sqrt{\pi} t \operatorname{erfc}(t) e^{t^2} - \frac{1}{12}(2t^2+5)\right) \\ + \frac{\log\frac{\Delta Q(r_2)}{\Delta Q(r_1)}}{4\log(r_2/r_1)} + \frac{1}{2\log(r_2/r_1)} (\log\theta)'\left(Mn + \frac{\log\frac{\Delta Q(r_2)}{\Delta Q(r_1)}}{4\log(r_2/r_1)}; \frac{\pi i}{\log(r_2/r_1)}\right)\Bigg] + \mathcal{O}(\log^4 n). \end{aligned}$$

Here $\partial_n \Delta Q(r_1)$ denotes the derivative of ΔQ in the normal direction of ∂G at r_1 pointing into G.

The theta function is periodic, $\theta(z+1,\tau) = \theta(z,\tau)$, and we see that the asymptotics only depend on the fractional part of Mn, that is x. In the same way as in Theorem 2.2 we see a pseudo-periodic behaviour of the kernel.

The subleading term is not known for a general potential Q. At the outer boundary it was conjectured by Lee and Riser [27] that the subleading term is universal and related to the curvature of the droplet. However as we observe in Paper II the conjecture must be modified for potentials where ΔQ is non-constant in a neighbourhood of the boundary.

2.2 Fluctuations

The fluctuation result from [4] requires the droplet to be connected. The proof uses a decomposition of the test function f in the exterior component U of the droplet. Since U is assumed to be simply connected, the function f can be written as

$$f = f_0 + \operatorname{Re} g, \tag{13}$$

where f_0 is zero on the boundary of the droplet and g is an analytic function in $\hat{\mathbb{C}} \setminus S$. The decomposition is combined with a limiting form of Ward identity. If an exterior component of the droplet is multi-connected, that is to say the droplet is disconnected, this method fails. The decomposition in (13) does not work since harmonic functions are no longer the real part of analytic functions.

2.2.1 Disconnected droplet

In Paper II, some first fluctuation results for a disconnected droplet are given. Here the potential as well as the test function are assumed to be radially symmetric. In Paper III, the fluctuation result in the radial setting follows from considering the partition function in a perturbed potential. Paper IV generalizes the fluctuation results in Paper II and Paper III for disconnected droplets in two ways. Firstly, the class of potentials is generalized, allowing for some annular type gaps that are not radially symmetric. Secondly, the class of test function includes all smooth and bounded functions.

The setting of Paper IV is more general, but we shall, for simplicity, assume here that the potential is radially symmetric and that the droplet consists of two components separated by an annulus $G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$. Let $f \in C_b^\infty$ be a smooth and bounded test function. We assume for again for simplicity that f is supported in a neighbourhood of the gap G, so that we do not need to consider any other components of the complement of S other than G.

The proof again builds on a decomposition of the test function. However the doubly connected gap G forces us to include one more term in the decomposition. We write

$$f = f_0 + \operatorname{Re} g + c\,\omega,$$

where as before f_0 is zero on the boundary of G and g is an analytic function in G. The new term ω is a smooth function equal to 0 on a neighbourhood of the inner component and 1 on a neighbourhood of the outer component of the droplet.

The fluctuations of the function ω measure the fluctuations of points between the two components. As it turns out these fluctuations are not asymptotically Gaussian. Instead they have a discrete normal distribution. That is an integer-valued random variable Y with probability mass function

$$\mathbb{P}(Y = j) = \frac{1}{C}\theta^j q^{\frac{1}{2}j(j-1)},$$

for every $j \in \mathbb{Z}$, where $\theta > 0$ and $0 < q < 1$ are parameters and C is a normalizing constant.

The fluctuation result from Paper IV in the setting of a radial potential becomes the following:

Theorem 2.4 *Let $f \in C^\infty$ and let $f = f_1 + \lambda\omega(z)$ where f_1 is of the form $f_0 + \operatorname{Re} g$. Then the cumulant generating function, $F_n(t)$, of $\mathrm{fluct}_n f$ satisfies*

$$F_n(t) = F_X(t) + F_{\lambda Y_n}(t) + o(1),$$

as $n \to \infty$. Here X is a normal random variable with expected value e_{f_1} and variance v_{f_1}, given by (10) and (11). Y_n is a discrete normal random variable with parameters $q = (\frac{r_1}{r_2})^2$ and $\theta_n = (\frac{r_1}{r_2})^{1+2x_n}\sqrt{\frac{\Delta Q(r_1)}{\Delta Q(r_2)}}$.

The discrete normal variable Y_n depends on n through the parameter x_n, similarly to what we have seen before concerning the kernel asymptotics. Similar behaviour has been seen for one-dimensional models; see [8].

2.3 Partition function

The normalizing constant Z_n is called the partition function. The partition function is a central object in statistical mechanics. For the Coulomb gas, it incorporates a lot of important information.

In [37], Zabrodin and Wiegman made predictions about the asymptotic expansion of the free energy, $\log Z_n$, for general β down to the constant order. The predictions were based on computations in the setting of radial potential.

Recently a rigorous study of the partition function in a radial and strictly subharmonic potential for $\beta = 2$ was carried out by Byun, Kang and Seo [10]. Although not matching perfectly, the predictions from [37] where not too far off. What is shown in [10] is that the partition function has an asymptotic expansion

$$\log Z_n = C_0 n^2 + C_1 n \log n + C_2 n + C_3 \log n + C_4 + o(1),$$

as $n \to \infty$, where all constants are given explicitly. The expansion down to C_2 is treated for general β by Leblé and Serfaty [26]. The constants C_1 and C_3 are independent of Q and of lesser interest. The first constant, the leading order term, is equal to $-I_Q[\sigma]$ that is minus the weighted energy of the equilibrium measure. This should not come as a surprise since the main contribution to the partition function should come from configurations for which the energy H_n is close to minimal. The constant C_1 is related to the entropy of the equilibrium measure. It is given by

$$C_2 = \frac{\log(2\pi)}{2} - 1 - \frac{1}{2}\int_{\mathbb{C}} \log(\Delta Q(z))\, d\sigma(z).$$

The constant order term C_4 is perhaps the most interesting one. It was conjectured in [37] that C_4 is related to spectral determinants via Polyakov-Alvarez formula.

In [10], the authors assume that the potential is globally subharmonic and radially symmetric. These assumptions restrict the possible geometry of the droplet: it will be wither a disk or an annulus. In Paper III we remove the global subharmonicity assumption and consider radially symmetric potentials that are allowed to be superharmonic in certain portions of the plane. This means that the droplet may be disconnected with one or many annular spectral gaps.

Most of the expansion that we obtain is the same as in [10], but the constant order term changes. In fact there appears q-factorials related to the discrete normal appearing in the fluctuation theorems.those from the fluctuation theorems from Paper II, III and IV. Using free-energy expansions we can re-prove some results on fluctuations and we also obtain some new results.

2.4 Kernel asymptotics for a hard edge

In Paper V we study the correlation kernel near a hard edge. We start with the Mittag-Leffler potential

$$Q(z) = |z|^{2b} - \frac{2\alpha}{n} \log |z|, \tag{14}$$

where $b > 0$ and $\alpha > -1$. The droplet with respect to this potential is a disk, namely $S = \{|z| \leq b^{-1/2}\}$. Then we modify the potential Q by letting it be $+\infty$ in an annulus $G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ inside the droplet. The new potential is then

$$\tilde{Q}(z) = \begin{cases} Q(z), & z \notin G \\ +\infty, & z \in G. \end{cases} \tag{15}$$

We have seen that the modification of the potential has a dramatic effect on the equilibrium measure. The measure in G is swept out to the boundary ∂G. The radial symmetry means that the equilibrium measure with respect to the potential $\tilde{Q}$ takes the form

$$d\sigma(z) = 1_{S\setminus G}(z)\, \Delta Q(z) dA(z) + \nu_1 \frac{ds_1(z)}{2\pi} + \nu_2 \frac{ds_2(z)}{2\pi},$$

where ν_1 and ν_2 are positive constants and ds_1 and ds_2 are the arclength measures on the two boundary components of G.

2.4.1 Long-range correlation

In Paper V we study long-range correlations along the hard edge, similar to what was done in Paper I and II for soft edges. The result is surprisingly similar to the soft edge case, however they take place on different scales. In the soft edge the boundary regime is of order $\mathcal{O}(n^{-1/2})$. For the hard edge system the points that are pushed to the boundary form a hard edge regime of order $\mathcal{O}(n^{-1})$. The long-range correlations are again described by the reproducing kernel $S_n(z,w)$, of a Hardy space $H^2(G,x_n)$ of multi-valued analytic functions in G of class x_n with L^2-norm

$$\|f\|^2 = \int\limits_{|z|=r_1} |f(z)|^2 \frac{1}{\nu_1}\, ds_1(z) + \int\limits_{|z|=r_2} |f(z)|^2 \frac{1}{\nu_2}\, ds_2(z).$$

The parameter x_n is determined in a similar way as in the soft edge case. It is given by

$$x_n = \{n\sigma(|z| \le r_1) - \alpha\} = n\sigma(|z| \le r_1) - \alpha - \lfloor n\sigma(|z| \le r_1) - \alpha \rfloor.$$

We can now reformulate a result from Paper V in the following way.

Theorem 2.5 *Let* $z = e^{i\theta_1}(r_1 - \frac{t}{\nu_1 n})$ *and* $w = e^{i\theta_2}(r_2 + \frac{s}{\nu_2 n})$ *where* $s, t \ge 0$, *then*

$$K_n(z,w) = 2\pi n\, S_n(z,w)\, e^{-t-s} e^{i(n\sigma(|z|\le r_1)-\alpha)(\theta_1-\theta_2)} + o(n),$$

as $n \to \infty$.

Notice that the kernel is exponentially decaying in s and t, that is the correlations are decaying as the points moves in the normal direction to the boundary into the droplet. This is different from the soft edge case where we saw in Theorems 2.1 and 2.2 that the decay was Gaussian.

The radial symmetry of the potential makes it possible to find good asymptotics of the orthogonal polynomials and through them the kernel. In the case of the Mittag-Leffler ensemble they are even expressed in terms of special functions, such as the incomplete gamma function. Whether similar very precise results such as Theorem 2.5 can be obtained for more general potentials is not clear.

2.4.2 One-point function

While the balayage measure is only supported on the boundary of the gap, for finite n, the hard edge contribution is living on a band of width $\mathcal{O}(n^{-1})$ around the boundary. The local limit at a hard edge boundary point was studied by Seo in [33] for a general radially symmetric subharmonic potential. The kernel obtained by Seo was

$$K(z,w) = \int\limits_0^1 te^{-t(z+\bar{w})}dt \cdot 1_{\mathbb{H}_+}(z) \cdot 1_{\mathbb{H}_+}(w), \tag{16}$$

where $\mathbb{H}_+$ is the right half plane. To prove universality of this kernel beyond the radial setting is an interesting open question.

In Paper V, besides long-range correlations, we perform a detailed analysis of the one-point function at the hard edge. We consider again the potential from (15) and rescale around a point on the hard edge. That is we define the point z by $z = e^{i\alpha}(r_1 - \frac{t}{\nu_1 n})$, with $t \geq 0$. Then we obtain an asymptotic expansion

$$K_n(z, z) = C_1 n^2 + C_2\, n \log n + C_3^{(n)} n + C_4 \sqrt{n} + \mathcal{O}(n^{2/5}),$$

as $n \to \infty$. We give explicit formulas for the terms C_j. They are all constant in n except for $C_3^{(n)}$ which is given in terms of a Jacobi theta function depending on n through the parameter x_n.

2.5 Disk counting statistics

In Papers VI and VII we study what is known as disk counting statistics. Both papers consider the Mittag-Leffler potential Q from (14) with hard edges. In Paper VI the potential is modified to be $+\infty$ on the set $\{z : |z| > \rho\}$ creating one boundary with a hard wall. In Paper VII we study the potential $\tilde{Q}$ from (15).

A disk counting statistics is a random variable $N(R)$ equal to the number of points in a disk of radius r centred at the origin. We observe that $N(R)$ can be written as

$$N(r) = \mathrm{trace}_n\, 1_{\mathbb{D}_R},$$

where $\mathbb{D}_R$ is a disk of radius R centred at the origin. We can in this way think of $N(r)$ as a very non-smooth linear statistic.

Consider now a collection of disk counting statistics $\{N(R_j)\}_{j=1}^m$. This is a collection of dependent random variables. We form the joint moment generating function

$$\mathbb{E}\Big[\prod_{j=1}^{m} e^{u_j N(R_j)}\Big], \tag{17}$$

with $u_1, \dots u_m \in \mathbb{R}$. The radii R_j are rescaled around the hard edges. We study the asymptotics of (17) as $n \to \infty$ and obtain asymptotic expansions. As a consequence we obtain several limiting theorems for the distribution of the disk counting statistics.

For example, consider the potential $\tilde{Q}$ with a gap $G = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ separating two components of the support of the equilibrium measure. If we let $m = 1$ and $R_1 = r_1$, the random variable $N(r_1)$ counts the number of points in the inner component of the equilibrium measure. The fluctuations of $N(r_1)$ therefore measures the fluctuations of points between the two components. A corollary in Paper VII shows that this is again described by a discrete normal random variable.

Remarkably the Weierstrass $\wp$-function appears in the formula for the covariance of $N(r_j)$ and $N(r_k)$ with the radii appropriately rescaled at the hard edge.

References

[1] Akemann, G., Byun, S.-S., *The High Temperature Crossover for General 2D Coulomb Gases*, J. Stat. Phys., **175** (2019), 1043-1065.

[2] Akemann, G., Kieburg, M., Nagaro, T., Parra, I., *Families of two-dimensional Coulomb gases on an ellipse: correlation functions and universality*, J. Phys. A: Math. Theor. **53** (2020), 075201.

[3] Ameur, Y., Hedenmalm, H., Makarov, N., *Fluctuations of eigenvalues of random normal matrices*, Duke Math. J. **159** (2011), no. 1, 31-81.

[4] Ameur, Y., Hedenmalm, H., Makarov, N., *Random normal matrices and Ward identities*, Ann. Probab. **43** (2015), 1157–1201.

[5] Ameur, Y., Kang, N.-G., Makarov, N., *Rescaling Ward identities in the random normal matrix model*, Constr. Approx. **50** (2019), 63–127.

[6] Berman, R., *Determinantal point processes and fermions on complex manifolds: Bulk universality*, Conference Proceedings on Algebraic and Analytic Microlocal Analysis (Northwestern, 2015), M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch, eds. Springer.

[7] Bertola, M., Elias Rebelo, J.G., Grava, T., *Painleve IV critical asymptotics for orthogonal polynomials in the complex plane*, SIGMA Symmetry Integrability Geom. Methods Appl. **14** (2014), Paper No. 091.

[8] Borot, G., Guionnet, A., *Asymptotic expansion of β matrix models in the multi-cut regime*, Forum Math.Sigma **12** (2024) e13.

[9] Byun, S.-S., Forrester, P.J., *Progress on the study of the Ginibre ensembles*, KIAS Springer Series in Mathematics **3** (2025).

[10] Byun, S.-S., Kang, N.-G., Seo, S.-M., *Partition functions of determinantal and Pfaffian Coulomb gases with radially symmetric potentials*, Commun. Math. Phys. **401** (2023), 16271663.

[11] Byun, S-S., Yang, M., *Determinantal Coulomb Gas Ensembles with a Class of Discrete Rotational Symmetric Potentials*, SIAM J. Math. Anal. **55** (2023), no. 6, 6867-6897.

[12] Carleman, T., *Uber die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen.* Ark. Mat. Astron. Fys. **17** (1923), 1-30.

[13] Chau, L.-L., Zaboronsky, O., *On the structure of correlation functions in the normal matrix model*, Comm. Math. Phys. **196** (1998), no. 1, 203–247.

[14] Forrester, P.J., *Fluctuation formula for complex random matrices*, J. Phys. A: Math. Gen., **32** (1999), L159.

[15] Forrester, P.J., Honner, G., *Exact statistical properties of the zeros of complex random polynomials*, J. Phys. A. **41** (1999), 375003.

[16] Forrester, P.J., Jancovici, B., *Two-dimensional one-component plasma in a quadrupolar field*, International Journal of Modern Physics A **11** (1996) no. 5.

[17] Garcia-Garcia, A. M., Nishigaki, S.M., Verbaarschot,J. J. M., *Critical statistics for non-Hermitian matrices*, Phys.Rev. E. **66** (2002), 016132.

[18] Ginibre, J., *Statistical ensembles of complex, quaternion, and real matrices*, J. Math. Phys. **6** (1965), 440.

[19] Hedenmalm, H., Makarov, N., Coulomb gas ensembles and Laplacian growth, Proc. London. Math. Soc. **106** (2013), 859–907.

[20] Hedenmalm, H., Wennman, A., *Off-spectral analysis of Bergman kernels*, Comm. Math. Phys. **373** (2020), 1049-1083.

[21] Hedenmalm, H., Wennman, A., *Planar orthogonal polynomials and boundary universality in the random normal matrix model*, Acta Math. **227** (2021), 309-406.

[22] Hough, J. Ben, Krishnapur, M., Peres, Y., Virág, B., *Zeros of Gaussian analytic functions and determinantal point processes*, University Lecture Series 51, AMS 2009.

[23] Johansson, K., *On fluctuations of eigenvalues of random Hermitian matrices*, Duke Math. J. **91** (1998), 151–204.

[24] Johansson, K., Viklund, F., *Coulomb gas and the Grunsky operator on a Jordan domain with corners*, arXiv:2309.00308 (2023).

[25] Kang, N.-G., Kang, Makarov, N. *Gaussian free field and conformal field theory*, Astérisque **353** (2013), viii+136 pp.

[26] Leblé, T., Serfaty, S., *Fluctuations of two dimensional coulomb gases*, Geom. Funct. Anal., 28 (2018), 443–508.

[27] Lee, S.-Y., Riser, R., *Fine asymptotic behaviour of random normal matrices: ellipse case*, J. Math. Phys. **57** (2016), 023302.

[28] Marceca, F., Romero, J.-L., *Improved discrepancy for the planar Coulomb gas at low temperatures*, to appear in J. Anal. Math (arxiv: 2212.14821).

[29] Rider, B., Virág, B., *The noise in the circular law and the Gaussian free field*, Int. Math. Res. Not. IMRN 2007, no. 2, Art. ID rnm006, 33 pp.

[30] Saff, E.B., Kuijlaars, A.B.J, *Distributing many points on a sphere*, The Mathematical Intelligencer **19** (1997), 5–11.

[31] Saff, E. B., Totik, V., *Logarithmic potentials with external fields*, Springer 1997.

[32] Sakai, M., *Regularity of a boundary having a Schwarz function*, Acta Math. **166** (1991), 263–297.

[33] Seo, S.-M., *Edge behavior of two-dimensional Coulomb gases near a hard wall*, Ann. Henri Poincaré **23** (2021), 2247–2275.

[34] Skinner, B., *Logarithmic potential theory on Riemann surfaces.* ProQuest LLC, Ann Arbor, MI, 2015.

[35] Tao, T., Vu, V., Krishnapur, M., *Random matrices: Universality of ESDs and the circular law*, Ann. Probab. **38** (2010), no. 5, 2023-2065.

[36] Widom, H. *Extremal polynomials associated with a system of curves in the complex plane*, Adv. Math. **3** (1969) 127–232.

[37] Wiegmann, P., Zabrodin, A., *Large N expansion for the 2D Dyson gas*, J. Phys. A **39** (2006), no. 28, 89338964.

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