Radiation by an aperture antenna of arbitrary shape

Kristensson, Gerhard

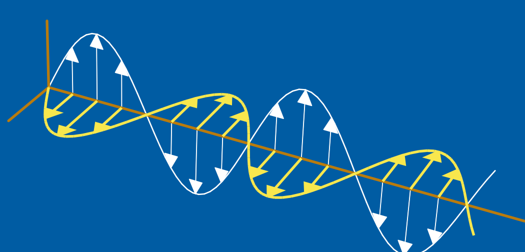
2000

CODEN:LUTEDX/(TEAT-7084)/1-29/(2000)

Radiation by an aperture antenna of arbitrary shape

Gerhard Kristensson

Electromagnetic Theory
Department of Electrical and Information Technology
Lund University
Sweden



Gerhard Kristensson

Department of Electrical and Information Technology
Electromagnetic Theory
Lund University
P.O. Box 118
SE-221 00 Lund
Sweden

Editor: Gerhard Kristensson

Abstract

This paper treats the radiation from a waveguide aperture in a perfectly conducting plane. The shape of the aperture is arbitrary. The radiated field into the half space, the reflected field in the waveguide, and the surface currents on the ground plane are calculated by means of a matching technique between the waveguide modes and the free space plane waves. A system of matrix equations determines the coupling between the radiation into the half space and the wave propagation in the waveguide. The accuracy of commonly used approximations of the aperture field is compared with the exact solution. Several numerical examples illustrate the method.

1 Introduction

Generic electromagnetic geometries have always played a great role in the understanding of complex electromagnetic scattering problems and in the design of applications. Examples are such generic geometries are scattering by a wedge (perfectly conducting or dielectric wedge), scattering by a cylinder, a sphere etc. Exactly soluble generic cases are also important in comparing the validity of various approximations that often are used to simplify the calculations.

The termination of a waveguide antenna in a perfectly conducting plane is often solved by approximating the field in the aperture by the exciting field, the use of magnetic currents, and the method of images, see *e.g.*, [2]. Other techniques, that apply to the two-dimensional case are presented in a recent paper [4]. The present paper solves the three-dimensional geometry exactly in the electromagnetic case, and we are therefore in a situation to find out how good the existing approximations are.

Provided the transverse electric field $\boldsymbol{E}_{xy}(\boldsymbol{r})$ in the aperture is known, the field in the far field zone in the half space is (the time convention $\exp\{-\mathrm{i}\omega t\}$ is used)

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{F}(\hat{\boldsymbol{r}})\frac{\mathrm{e}^{\mathrm{i}k_0 r}}{r}$$

The far-field amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}})$ can be expressed in terms of the aperture field $\boldsymbol{E}_{xy}(\boldsymbol{r})$ [2].

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \frac{\mathrm{i}k_0}{2\pi}\hat{\boldsymbol{r}} \times \left[\hat{\boldsymbol{z}} \times \iint_\Omega \boldsymbol{E}_{xy}(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}k_0\hat{\boldsymbol{r}}\cdot\boldsymbol{r}}\, \mathrm{d}x\, \mathrm{d}y\right], \quad \hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}} = \cos\theta > 0$$

where $\hat{\boldsymbol{z}}$ is the normal to the perfectly conducting plane, Ω is the aperture, $\hat{\boldsymbol{r}}$ is the direction of observation, see Figure 1, and $k_0 = \omega/c_0$ (c_0 is the speed of light in vacuum). The question of determining the aperture field in a very general setting is discussed and solved in this paper. We are then in a position of discussing the accuracy of commonly used approximations of the aperture field and comparing the results with the exact solution.

The fields in the waveguide and in the half space are analyzed in Sections 2 and 3, respectively, and in Section 4 the technique that matches the boundary conditions in the aperture and on the perfectly conducting plane is presented. Sections 5 and 6

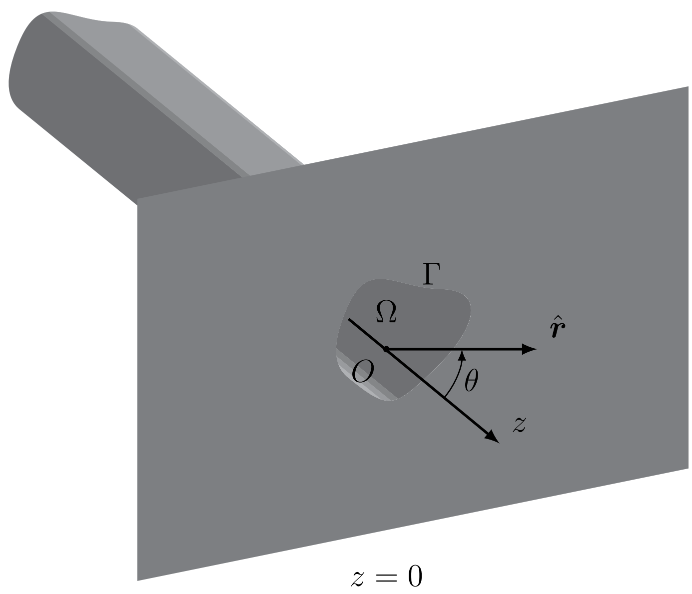


Figure 1: The geometry of the problem.

contain a proof of the power conservation of the problem and the expressions of the fields on the ground plane, respectively. The paper is completed by a presentation of explicit examples and numerical computations in Section 7 and a conclusion in Section 8.

2 Waveguide field

The perfectly conducting ground plane is located at $z = 0$. The sources in the domain $z < 0$ in the waveguide generate a mode $\ell_0\nu_0$, (ℓ is the collective mode index, and $\nu =$ TM, TE, or TEM[1]). The cross section of the waveguide is denoted Ω, see Figure 1, and it is arbitrary in shape and may consist of a central conducting part, *i.e.*, Ω is not a simply-connected region in the x-y-plane. The region inside the waveguide, $z < 0$ and $\boldsymbol{r}_\mathrm{c} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} \in \Omega$, is assumed to be vacuous, and the domain $z > 0$ is a half space, which also is assumed to be vacuum. Several generalizations of these assumptions are trivial to relax, and they are not pursued in this paper.

The main objective of this paper is to determine the reflected power in the waveguide, $z < 0$, the transmission power into the half space, $z > 0$, and to analyze the current induced on the ground plane, $z = 0$.

The sources generate an electric field in the waveguide, which is a mode denoted by $\boldsymbol{E}^+_{\ell_0\nu_0}(\boldsymbol{r})$. The corresponding magnetic field (mode) is denoted $\boldsymbol{H}^+_{\ell_0\nu_0}(\boldsymbol{r})$. We assume the frequency of this exciting wave is above cutoff for this mode, *i.e.*, $\omega/c_0 = k_0 > k_{\mathrm{t}\ell_0\nu_0}$, where $k_{\mathrm{t}\ell\nu}$ is the transverse wave number (the wave number of the cut-off frequency).

[1] TEM-modes are present only in the case where the waveguide structure supports such waves.

Since the waveguide is terminated at $z=0$, there is a reflected field in the region $z<0$. The electric and magnetic fields in the region $z<0$ are therefore [1, 3]

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}) = E_0\left(\boldsymbol{E}^+_{\ell_0\nu_0}(\boldsymbol{r}) + \sum\limits_{\ell,\nu} r_{\ell\nu}\boldsymbol{E}^-_{\ell\nu}(\boldsymbol{r})\right) \\ \boldsymbol{H}(\boldsymbol{r}) = E_0\left(\boldsymbol{H}^+_{\ell_0\nu_0}(\boldsymbol{r}) + \sum\limits_{\ell,\nu} r_{\ell\nu}\boldsymbol{H}^-_{\ell\nu}(\boldsymbol{r})\right) \end{cases} \quad z<0, \quad \boldsymbol{r}_\mathrm{c}\in\Omega \tag{2.1}$$

where $r_{\ell\nu}$ is the (relative) amplitude of the reflected $\ell\nu$-mode. The dimension of the complex constant E_0 is Vs.[2] The value of this constant is determined by the amount of power sent in the waveguide in the positive z-direction, which is

$$P_\mathrm{i} = \iint\limits_{\Omega} \hat{\boldsymbol{z}}\cdot <\boldsymbol{S}(t)> \,\mathrm{d}x\,\mathrm{d}y = \frac{|E_0|^2}{2\eta_0}\operatorname{Re} Y^E_{\ell_0\nu_0} = \frac{|E_0|^2}{2\eta_0} Y^E_{\ell_0\nu_0} \tag{2.2}$$

where we have introduced the notion of energy admittance $Y^E_{\ell\nu}$, see (4.15).

The notation of waveguide modes follows the established standards with a few modifications [1]. The TM-modes propagating in the $\pm z$-directions are

$$\begin{cases} \boldsymbol{E}^\pm_{\ell\nu}(\boldsymbol{r}) = \{\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \pm v_\ell(\boldsymbol{r}_\mathrm{c})\hat{\boldsymbol{z}}\}\,\mathrm{e}^{\pm\mathrm{i}k_{z\ell\nu}z} \\ \boldsymbol{H}^\pm_{\ell\nu}(\boldsymbol{r}) = \pm\boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\pm\mathrm{i}k_{z\ell\nu}z} \end{cases} \qquad \nu = \mathrm{TM}$$

where $k_{z\ell\nu} = \left(k_0^2 - k_{\mathrm{t}\ell\nu}^2\right)^{1/2}$, and $k_{\mathrm{t}\ell\nu}$ are the transverse wave number of the cut-off frequency. Similarly, for the TE-modes we have

$$\begin{cases} \boldsymbol{E}^\pm_{\ell\nu}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\pm\mathrm{i}k_{z\ell\nu}z} \\ \boldsymbol{H}^\pm_{\ell\nu}(\boldsymbol{r}) = \left\{\pm\boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) + \eta_0^{-1}w_\ell(\boldsymbol{r}_\mathrm{c})\hat{\boldsymbol{z}}\right\}\mathrm{e}^{\pm\mathrm{i}k_{z\ell\nu}z} \end{cases} \qquad \nu = \mathrm{TE}$$

In these expressions the index t denotes the transverse component of the field. The TEM-modes are defined by

$$\begin{cases} \boldsymbol{E}^\pm_{\ell\nu}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\pm\mathrm{i}kz} \\ \boldsymbol{H}^\pm_{\ell\nu}(\boldsymbol{r}) = \pm\boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\pm\mathrm{i}kz} \end{cases} \qquad \nu = \mathrm{TEM}$$

The dimension of $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})$ is 1/m. These expansion functions are only functions of the geometry of the cross section of the problem.

The relations between the functions w_ℓ, v_ℓ and the transverse components $\boldsymbol{E}_{\mathrm{t}\ell\nu}$ and $\boldsymbol{H}_{\mathrm{t}\ell\nu}$ are

$$\begin{cases} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = \dfrac{\mathrm{i}}{k_{\mathrm{t}\ell\nu}^2}\begin{cases} k_{z\ell\nu}\nabla_\mathrm{t} v_\ell(\boldsymbol{r}_\mathrm{c}), & \nu=\mathrm{TM} \\ -k_0\mathbf{J}\cdot\nabla_\mathrm{t} w_\ell(\boldsymbol{r}_\mathrm{c}), & \nu=\mathrm{TE}\end{cases} \\ \eta_0\mathbf{J}\cdot\boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = \dfrac{\mathrm{i}}{k_{\mathrm{t}\ell\nu}^2}\begin{cases} -k_0\nabla_\mathrm{t} v_\ell(\boldsymbol{r}_\mathrm{c}), & \nu=\mathrm{TM} \\ k_{z\ell\nu}\mathbf{J}\cdot\nabla_\mathrm{t} w_\ell(\boldsymbol{r}_\mathrm{c}), & \nu=\mathrm{TE}\end{cases} \end{cases} \tag{2.3}$$

[2] Remember that the Fourier transform of the electric field (dimension V/m) has dimension Vs/m.

where we have introduced the dyadic $\mathbf{J} = \hat{\boldsymbol{z}} \times \mathbf{I}_3$ that rotates a vector in the x-y-plane $\pi/2$ around the z-axis, and $\mathbf{I}_3$ is the identity dyadic in three dimensions. The TEM-modes have transverse components defined by

$$\begin{cases} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = -\nabla_\mathrm{t}\psi(\boldsymbol{r}_\mathrm{c}) \\ \eta_0 \mathbf{J}\cdot \boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = -\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = \nabla_\mathrm{t}\psi(\boldsymbol{r}_\mathrm{c}) \end{cases} \qquad \nu = \mathrm{TEM}$$

Note that we for convenience keep the index ℓ for the TEM-modes even if there are no dependence of the index ℓ.

The eigenfunctions $v_\ell(\boldsymbol{r}_\mathrm{c})$ an $w_\ell(\boldsymbol{r}_\mathrm{c})$ are normalized and orthogonal if integrated over the cross section Ω, *i.e.*,

$$\begin{cases} \displaystyle\iint_\Omega v_\ell(\boldsymbol{r}_\mathrm{c}) v_{\ell'}(\boldsymbol{r}_\mathrm{c}) \,\mathrm{d}x\,\mathrm{d}y = \delta_{\ell,\ell'} \\ \displaystyle\iint_\Omega w_\ell(\boldsymbol{r}_\mathrm{c}) w_{\ell'}(\boldsymbol{r}_\mathrm{c}) \,\mathrm{d}x\,\mathrm{d}y = \delta_{\ell,\ell'} \end{cases}$$

and the function ψ is normalized by

$$\iint_\Omega \nabla_\mathrm{t}\psi(\boldsymbol{r}_\mathrm{c}) \cdot \nabla_\mathrm{t}\psi(\boldsymbol{r}_\mathrm{c}) \,\mathrm{d}x\,\mathrm{d}y = 1$$

The dimensions of $v_\ell(\boldsymbol{r}_\mathrm{c})$ and $w_\ell(\boldsymbol{r}_\mathrm{c})$ are 1/m and ψ has no dimension. The eigenfunctions $v_\ell(\boldsymbol{r}_\mathrm{c})$ and $w_\ell(\boldsymbol{r}_\mathrm{c})$ satisfy

$$\begin{cases} \nabla_\mathrm{t}^2 v(\boldsymbol{r}_\mathrm{c}) + k_{\mathrm{t}\ell\nu}^2 v(\boldsymbol{r}_\mathrm{c}) = \dfrac{\partial^2 v(\boldsymbol{r}_\mathrm{c})}{\partial x^2} + \dfrac{\partial^2 v(\boldsymbol{r}_\mathrm{c})}{\partial y^2} + k_{\mathrm{t}\ell\nu}^2 v(\boldsymbol{r}_\mathrm{c}) = 0 \\ v(\boldsymbol{r}_\mathrm{c}) = 0 \qquad \boldsymbol{r}_\mathrm{c} \text{ on } \Gamma \end{cases} \qquad \text{(TM-case)}$$

where Γ denotes the boundary curve of the cross section Ω, and

$$\begin{cases} \nabla_\mathrm{t}^2 w(\boldsymbol{r}_\mathrm{c}) + k_{\mathrm{t}\ell\nu}^2 w(\boldsymbol{r}_\mathrm{c}) = \dfrac{\partial^2 w(\boldsymbol{r}_\mathrm{c})}{\partial x^2} + \dfrac{\partial^2 w(\boldsymbol{r}_\mathrm{c})}{\partial y^2} + k_{\mathrm{t}\ell\nu}^2 w(\boldsymbol{r}_\mathrm{c}) = 0 \\ \dfrac{\partial w}{\partial n}(\boldsymbol{r}_\mathrm{c}) = 0 \qquad \boldsymbol{r}_\mathrm{c} \text{ on } \Gamma \end{cases} \qquad \text{(TE-case)}$$

and for the TEM-case

$$\begin{cases} \nabla_\mathrm{t}^2 \psi(\boldsymbol{r}_\mathrm{c}) = 0 \\ \psi(\boldsymbol{r}_\mathrm{c}) = \text{constant} \qquad \boldsymbol{r}_\mathrm{c} \text{ on } \Gamma \end{cases} \qquad \text{(TEM-case)}$$

Moreover, the tangential field satisfy an orthogonal relation, which reads

$$\eta_0 \iint_\Omega \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \cdot \mathbf{J} \cdot \boldsymbol{H}_{\mathrm{t}\ell'\nu'}(\boldsymbol{r}_\mathrm{c}) \,\mathrm{d}x\,\mathrm{d}y = Y_{\ell\nu}\delta_{\ell,\ell'}\delta_{\nu,\nu'} \tag{2.4}$$

where the waveguide admittance is given by

$$Y_{\ell\nu} = \begin{cases} \dfrac{k_0 k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2} = \dfrac{k_0 \left(k_0^2 - k_{\mathrm{t}\ell\nu}^2\right)^{1/2}}{k_{\mathrm{t}\ell\nu}^2}, & \text{if } \nu = \text{TE, TM} \\ -1, & \text{if } \nu = \text{TEM} \end{cases} \tag{2.5}$$

3 Fields in the half space

A general expansion of the fields in the domain $z > 0$ is given as a plane wave expansion (we assume no sources in $z > 0$, *i.e.*, all waves are propagation in the $+z$-direction)

$$\begin{cases} \boldsymbol{E}(\boldsymbol{r}) = \dfrac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \boldsymbol{E}(\boldsymbol{k}_\mathrm{t}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c} + \mathrm{i}k_z z} \, \mathrm{d}k_x \, \mathrm{d}k_y \\ \boldsymbol{H}(\boldsymbol{r}) = \dfrac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \boldsymbol{H}(\boldsymbol{k}_\mathrm{t}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c} + \mathrm{i}k_z z} \, \mathrm{d}k_x \, \mathrm{d}k_y \end{cases} \qquad z > 0 \tag{3.1}$$

where the spectral components of the fields are

$$\begin{cases} \boldsymbol{E}(\boldsymbol{k}_\mathrm{t}) = \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) + E_z(\boldsymbol{k}_\mathrm{t})\hat{\boldsymbol{z}} \\ \boldsymbol{H}(\boldsymbol{k}_\mathrm{t}) = \boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t}) + H_z(\boldsymbol{k}_\mathrm{t})\hat{\boldsymbol{z}} \end{cases}$$

and the relation between the transverse components $\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})$ and $\boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t})$, and between the z-components and the transverse components are [5]

$$\begin{cases} \eta_0 \mathbf{J} \cdot \boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t}) = -\dfrac{k_z}{k_0} \left(\dfrac{\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}}{k_z^2} + \mathbf{I}_2 \right) \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) = -\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \\ E_z(\boldsymbol{k}_\mathrm{t}) = \dfrac{\eta_0 \boldsymbol{k}_\mathrm{t}}{k_0} \cdot \mathbf{J} \cdot \boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t}) = -\dfrac{1}{k_z} \boldsymbol{k}_\mathrm{t} \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \\ \eta_0 H_z(\boldsymbol{k}_\mathrm{t}) = -\dfrac{\boldsymbol{k}_\mathrm{t}}{k_0} \cdot \mathbf{J} \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \end{cases} \tag{3.2}$$

The dimension of $\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})$ is Vsm. Notice that all fields can be constructed from the field $\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})$. The longitudinal wave number k_z is

$$k_z = \left(k_0^2 - k_\mathrm{t}^2\right)^{1/2} = \begin{cases} \sqrt{k_0^2 - k_\mathrm{t}^2} \text{ for } k_\mathrm{t} < k_0 \\ \mathrm{i}\sqrt{k_\mathrm{t}^2 - k_0^2} \text{ for } k_\mathrm{t} > k_0 \end{cases}$$

and the dyadic $\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t})$ is defined as

$$\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) = \frac{k_z}{k_0} \left(\frac{\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}}{k_z^2} + \mathbf{I}_2 \right) = \frac{k_z}{k_0} \left(\frac{\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}}{k_0^2 - k_\mathrm{t}^2} + \mathbf{I}_2 \right)$$

This dyadic has the following important projections:

$$\begin{cases} \boldsymbol{k}_\mathrm{t} \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{k}_\mathrm{t} = \dfrac{k_\mathrm{t}^2 k_0}{k_z} \\ \boldsymbol{k}_\mathrm{t} \cdot \mathbf{J} \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \mathbf{J} \cdot \boldsymbol{k}_\mathrm{t} = -k_\mathrm{t}^2 \dfrac{k_z}{k_0} \end{cases} \tag{3.3}$$

The field at large distances from the aperture is

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{F}(\hat{\boldsymbol{r}}) \frac{\mathrm{e}^{\mathrm{i}k_0 r}}{r}$$

where the far-field amplitude $\boldsymbol{F}(\hat{\boldsymbol{z}})$ is [2]

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \frac{\mathrm{i}k_0}{2\pi}\hat{\boldsymbol{r}} \times \left[\hat{\boldsymbol{z}} \times \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t} = k_0 \sin\theta \hat{\boldsymbol{r}}_\mathrm{c})\right], \quad \cos\theta > 0 \tag{3.4}$$

The power radiated in the direction $\hat{\boldsymbol{r}}$ is

$$<\boldsymbol{S}(\hat{\boldsymbol{r}})> = \frac{|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2 \hat{\boldsymbol{r}}}{2\eta_0 r^2} \tag{3.5}$$

4 Mode matching

The boundary conditions at the interface $z = 0$ give the following condition for the transverse electric field, see (2.1) and (3.1):

$$E_0 \chi_\Omega(\boldsymbol{r}_\mathrm{c}) \left(\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{r}_\mathrm{c}) + \sum_{\ell,\nu} r_{\ell\nu} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \right) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}k_x \, \mathrm{d}k_y \quad \boldsymbol{r}_\mathrm{c} \in \mathbb{R}^2 \tag{4.1}$$

where $\chi_\Omega(\boldsymbol{r}_\mathrm{c})$ is the characteristic function of the aperture Ω, and for the transverse magnetic field we have

$$E_0 \left(\boldsymbol{H}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{r}_\mathrm{c}) - \sum_{\ell,\nu} r_{\ell\nu} \boldsymbol{H}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \right) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}k_x \, \mathrm{d}k_y \quad \boldsymbol{r}_\mathrm{c} \in \Omega \tag{4.2}$$

Take the Fourier transform in the $\boldsymbol{r}_\mathrm{c}$-variables of (4.1). With the use of the definition of the Fourier transform, the result is

$$E_0 \left(\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) + \sum_{\ell,\nu} r_{\ell\nu} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) \right) = \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \tag{4.3}$$

where $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})$ is the Fourier transform of the waveguide mode $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})$, *i.e.*,

$$\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) = \iint_{\mathbb{R}^2} \chi_\Omega(\boldsymbol{r}_\mathrm{c}) \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}x \, \mathrm{d}y = \iint_{\Omega} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}x \, \mathrm{d}y$$

with inverse

$$\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c}) = \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}k_x \, \mathrm{d}k_y$$

The Fourier transform of the TEM-mode is most conveniently computed directly from the field $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})$, but for the TM- and the TE-mode the Fourier transform can be related to the Fourier transform of the functions $v_\ell(\boldsymbol{r}_\mathrm{c})$ and $w_\ell(\boldsymbol{r}_\mathrm{c})$, respectively. In fact, from (2.3) and Stokes' analogous theorem $\iint_\Omega \hat{\boldsymbol{z}} \times \nabla\phi(\boldsymbol{r}_\mathrm{c})\,\mathrm{d}x\,\mathrm{d}y = \int_\Gamma \phi(\boldsymbol{r}_\mathrm{c})\,\mathrm{d}\boldsymbol{r}_\mathrm{c}$ we get (Γ is the boundary curve of the cross section Ω)

$$\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) = -\frac{1}{k_{\mathrm{t}\ell\nu}^2}\begin{cases} k_{z\ell\nu}\boldsymbol{k}_\mathrm{t} v_\ell(\boldsymbol{k}_\mathrm{t}), & \nu = \mathrm{TM} \\ -k_0 \mathbf{J}\cdot\boldsymbol{k}_\mathrm{t} w_\ell(\boldsymbol{k}_\mathrm{t}) + \mathrm{i}k_0 \int\limits_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}, & \nu = \mathrm{TE} \end{cases} \tag{4.4}$$

The dimension of $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})$ is m, and the Fourier transforms of the eigenfunctions $v_\ell(\boldsymbol{r}_\mathrm{c})$ and $w_\ell(\boldsymbol{r}_\mathrm{c})$ are

$$\begin{cases} v_\ell(\boldsymbol{k}_\mathrm{t}) = \iint\limits_{\mathbb{R}^2} \chi_\Omega(\boldsymbol{r}_\mathrm{c}) v_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} v_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}x\,\mathrm{d}y \\ w_\ell(\boldsymbol{k}_\mathrm{t}) = \iint\limits_{\mathbb{R}^2} \chi_\Omega(\boldsymbol{r}_\mathrm{c}) w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}x\,\mathrm{d}y \end{cases}$$

and $v_\ell^*(\boldsymbol{k}_\mathrm{t}) = v_\ell(-\boldsymbol{k}_\mathrm{t})$ and $w_\ell^*(\boldsymbol{k}_\mathrm{t}) = w_\ell(-\boldsymbol{k}_\mathrm{t})$ if the eigenfunctions $v_\ell(\boldsymbol{r}_\mathrm{c})$ and $w_\ell(\boldsymbol{r}_\mathrm{c})$ are chosen real-valued. The dimensions of $v_\ell(\boldsymbol{k}_\mathrm{t})$ and $w_\ell(\boldsymbol{k}_\mathrm{t})$ are m.

We now proceed with the second boundary condition in (4.2). Multiply the second integral in (4.2) with $\boldsymbol{E}_{\mathrm{t}\ell'\nu'}(\boldsymbol{r}_\mathrm{c})\cdot\mathbf{J}$, integrate over the cross section Ω and use (2.4). We get

$$\begin{aligned} &E_0 Y_{\ell'\nu'}\left(\delta_{\ell_0,\ell'}\delta_{\nu_0,\nu'} - r_{\ell'\nu'}\right) \\ &\quad= \frac{\eta_0}{4\pi^2}\iint\limits_{\Omega} \boldsymbol{E}_{\mathrm{t}\ell'\nu'}(\boldsymbol{r}_\mathrm{c})\cdot \iint\limits_{\mathbb{R}^2} \mathbf{J}\cdot\boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}x\,\mathrm{d}y \\ &\quad= \frac{\eta_0}{4\pi^2}\iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell'\nu'}(-\boldsymbol{k}_\mathrm{t})\cdot\mathbf{J}\cdot\boldsymbol{H}_{xy}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \\ &\quad= -\frac{1}{4\pi^2}\iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell'\nu'}(-\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \end{aligned}$$

where we also used (3.2). Solve for $r_{\ell\nu}$, and we get

$$r_{\ell\nu} = \delta_{\ell_0,\ell}\delta_{\nu_0,\nu} + \frac{1}{4\pi^2 Y_{\ell\nu}E_0}\iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell'\nu'}(-\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \tag{4.5}$$

which inserted in (4.3) becomes

$$\begin{aligned} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) &= 2E_0\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) \\ &\quad+ \sum_{\ell,\nu}\frac{1}{4\pi^2 Y_{\ell\nu}}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})\iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell'\nu'}(-\boldsymbol{k}'_\mathrm{t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}'_\mathrm{t})\cdot\boldsymbol{E}_{xy}(\boldsymbol{k}'_\mathrm{t})\,\mathrm{d}k'_x\,\mathrm{d}k'_y \end{aligned}$$

This equation can be written as an integral equation of the second kind in $\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})$.

$$\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) + \iint\limits_{\mathbb{R}^2} \mathbf{K}(\boldsymbol{k}_\mathrm{t},\boldsymbol{k}'_\mathrm{t})\cdot\boldsymbol{E}_{xy}(\boldsymbol{k}'_\mathrm{t})\,\mathrm{d}k'_x\,\mathrm{d}k'_y = 2E_0\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) \tag{4.6}$$

where the kernel $\mathbf{K}(\boldsymbol{k}_{\rm t},\boldsymbol{k}_{\rm t}')$ is given by

$$\mathbf{K}(\boldsymbol{k}_{\rm t},\boldsymbol{k}_{\rm t}') = -\sum_{\ell,\nu}\frac{1}{4\pi^2 Y_{\ell\nu}}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})\boldsymbol{E}_{{\rm t}\ell\nu}(-\boldsymbol{k}_{\rm t}')\cdot\boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}')$$

This is the basic integral equation that determines the aperture field $\boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t})$.

We now aim at reformulating this integral equation as an infinite system of linear equations. This is possible since the kernel $\mathbf{K}(\boldsymbol{k}_{\rm t},\boldsymbol{k}_{\rm t}')$ is separable. To simplify the notation, we introduce the pertinent normalization factor defined as, *cf.*, (2.5)

$$N_{\ell\nu} = \begin{cases} \dfrac{\sqrt{k_0\sqrt{k_0^2-k_{{\rm t}\ell\nu}^2}}}{k_{{\rm t}\ell\nu}}, & \text{if } \nu=\text{TM, TE, and } k_0 \geq k_{{\rm t}\ell\nu} \\ \dfrac{\sqrt{k_0\sqrt{k_{{\rm t}\ell\nu}^2-k_0^2}}}{k_{{\rm t}\ell\nu}}{\rm e}^{{\rm i}\pi/4}, & \text{if } \nu=\text{TM, TE, and } k_0 \leq k_{{\rm t}\ell\nu} \\ {\rm i}, & \text{if } \nu=\text{TEM} \end{cases}$$

Notice that $N_{\ell\nu}^2 = Y_{\ell\nu}$. Introduce a dimensionless scalar quantity $\alpha_{\ell\nu}$ (containing the unknown $\boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t})$) defined by

$$\alpha_{\ell\nu} = \frac{1}{4\pi^2 N_{\ell\nu}E_0}\iint_{\mathbb{R}^2}\boldsymbol{E}_{{\rm t}\ell\nu}(-\boldsymbol{k}_{\rm t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}_{\rm t})\cdot\boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t})\,{\rm d}k_x\,{\rm d}k_y$$

Equation (4.6) is then rewritten as

$$\boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t}) - \sum_{\ell,\nu}\alpha_{\ell\nu}\frac{E_0}{N_{\ell\nu}}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) = 2E_0\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) \tag{4.7}$$

Provided we can solve for the coefficients $\alpha_{\ell\nu}$, the solution $\boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t})$ can be found since the Fourier transformed functions $\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})$ are all defined by the geometry of the waveguide.

If we multiply the equation (4.7) with the appropriate quantity and integrate over the variable $\boldsymbol{k}_{\rm t}$, we get

$$\alpha_{\ell\nu} + \sum_{\ell',\nu'}A_{\ell\nu\ell'\nu'}\alpha_{\ell'\nu'} = \beta_{\ell\nu} \tag{4.8}$$

where

$$\begin{cases} \beta_{\ell\nu} = \dfrac{1}{2\pi^2 N_{\ell\nu}}\displaystyle\iint_{\mathbb{R}^2}\boldsymbol{E}_{{\rm t}\ell\nu}(-\boldsymbol{k}_{\rm t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}_{\rm t})\cdot\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t})\,{\rm d}k_x\,{\rm d}k_y \\ A_{\ell\nu\ell'\nu'} = -\dfrac{1}{4\pi^2 N_{\ell\nu}N_{\ell'\nu'}}\displaystyle\iint_{\mathbb{R}^2}\boldsymbol{E}_{{\rm t}\ell\nu}(-\boldsymbol{k}_{\rm t})\cdot\boldsymbol{\gamma}(\boldsymbol{k}_{\rm t})\cdot\boldsymbol{E}_{{\rm t}\ell'\nu'}(\boldsymbol{k}_{\rm t})\,{\rm d}k_x\,{\rm d}k_y \end{cases} \tag{4.9}$$

The expression in (4.8) is a set of linear equations for the unknown $\alpha_{\ell\nu}$, since the entries of the matrix $A_{\ell\nu\ell'\nu'}$ contain only known functions. Moreover, the entries of

$A_{\ell\nu\ell'\nu'}$ and $\beta_{\ell\nu}$ are dimensionless, and the matrix $A_{\ell\nu\ell'\nu'}$ is symmetric in the indices $\ell\nu$ and $\ell'\nu'$ since the dyadic $\boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t})$ is symmetric and even in $\boldsymbol{k}_\mathrm{t}$. Notice that

$$\beta_{\ell\nu} = -2N_{\ell_0\nu_0}A_{\ell\nu\ell_0\nu_0}$$

The reflection coefficients $r_{\ell\nu}$ in (4.5) can also be written in the coefficients $\alpha_{\ell\nu}$.

$$r_{\ell\nu} = \delta_{\ell_0,\ell}\delta_{\nu_0,\nu} + \frac{\alpha_{\ell\nu}}{N_{\ell\nu}}$$

To further simplify the matrix equation in (4.8), make a change in the variable $\alpha_{\ell\nu}$ and define a new quantity $a_{\ell\nu}$ as

$$a_{\ell\nu} = \delta_{\ell_0,\ell}\delta_{\nu_0,\nu}N_{\ell_0\nu_0} + \alpha_{\ell\nu}$$

Then $a_{\ell\nu}$ satisfies

$$a_{\ell\nu} + \sum_{\ell',\nu'} A_{\ell\nu\ell'\nu'}a_{\ell'\nu'} = b_{\ell\nu} \tag{4.10}$$

where

$$b_{\ell\nu} = N_{\ell_0\nu_0}\left(\delta_{\ell_0,\ell}\delta_{\nu_0,\nu} - A_{\ell\nu\ell_0\nu_0}\right) \tag{4.11}$$

Moreover, we have from (4.7)

$$\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) = E_0\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) + \sum_{\ell,\nu} a_{\ell\nu}\frac{E_0}{N_{\ell\nu}}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) \tag{4.12}$$

and

$$r_{\ell\nu} = \frac{a_{\ell\nu}}{N_{\ell\nu}} \tag{4.13}$$

The far-field amplitude is for $\cos\theta > 0$, see (3.4)

$$\boldsymbol{F}(\hat{\boldsymbol{r}}) = \frac{\mathrm{i}k_0E_0}{2\pi}\hat{\boldsymbol{r}}\times\left[\hat{\boldsymbol{z}}\times\left(\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t} = k_0\sin\theta\hat{\boldsymbol{r}}_\mathrm{c}) + \sum_{\ell,\nu}\frac{a_{\ell\nu}}{N_{\ell\nu}}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t} = k_0\sin\theta\hat{\boldsymbol{r}}_\mathrm{c})\right)\right]$$

and its absolute value is

$$\begin{aligned}|\boldsymbol{F}(\hat{\boldsymbol{r}})|^2 = \frac{|E_0|^2 k_0^2}{4\pi^2}\Bigg\{&\left|\hat{\boldsymbol{r}}_\mathrm{c}\cdot\left(\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) + \sum_{\ell,\nu}\frac{a_{\ell\nu}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})}{N_{\ell\nu}}\right)\right|^2 \\ &+ \cos^2\theta\left|\hat{\boldsymbol{\phi}}\cdot\left(\boldsymbol{E}_{\mathrm{t}\ell_0\nu_0}(\boldsymbol{k}_\mathrm{t}) + \sum_{\ell,\nu}\frac{a_{\ell\nu}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})}{N_{\ell\nu}}\right)\right|^2\Bigg\}_{\boldsymbol{k}_\mathrm{t}=k_0\sin\theta\hat{\boldsymbol{r}}_\mathrm{c}}\end{aligned}$$

4.1 The matrix entries

For the TM- and the TE-mode there is an alternative formulation of the matrix entries in $A_{\ell\nu\ell'\nu'}$. With the use of (4.4), we obtain

$$A_{\ell\mathrm{TM}\ell'\mathrm{TM}} = \frac{\sqrt{k_{z\ell\mathrm{TM}}}\sqrt{k_{z\ell'\mathrm{TM}}}}{4\pi^2 k_{\mathrm{t}\ell\mathrm{TM}}k_{\mathrm{t}\ell'\mathrm{TM}}}\iint_{\mathbb{R}^2}\frac{k_\mathrm{t}^2}{k_z}v_\ell(-\boldsymbol{k}_\mathrm{t})v_{\ell'}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y$$

$$A_{\ell\mathrm{TM}\ell'\mathrm{TE}} = \frac{\mathrm{i}k_0\sqrt{k_{z\ell\mathrm{TM}}}}{4\pi^2\sqrt{k_{z\ell'\mathrm{TE}}}k_{\mathrm{t}\ell\mathrm{TM}}k_{\mathrm{t}\ell'\mathrm{TE}}} \iint\limits_{\mathbb{R}^2} \frac{1}{k_z} v_\ell(-\boldsymbol{k}_\mathrm{t})\boldsymbol{k}_\mathrm{t} \cdot \int\limits_\Gamma w_{\ell'}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}\,\mathrm{d}k_x\,\mathrm{d}k_y$$

$$A_{\ell\mathrm{TE}\ell'\mathrm{TM}} = -\frac{\mathrm{i}k_0\sqrt{k_{z\ell'\mathrm{TM}}}}{4\pi^2\sqrt{k_{z\ell\mathrm{TE}}}k_{\mathrm{t}\ell\mathrm{TE}}k_{\mathrm{t}\ell'\mathrm{TM}}} \iint\limits_{\mathbb{R}^2} \frac{1}{k_z} v_{\ell'}(\boldsymbol{k}_\mathrm{t})\boldsymbol{k}_\mathrm{t} \cdot \int\limits_\Gamma w_{\ell}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}\,\mathrm{d}k_x\,\mathrm{d}k_y$$

and

$$\begin{aligned} A_{\ell\mathrm{TE}\ell'\mathrm{TE}} = \frac{1}{4\pi^2\sqrt{k_{z\ell\mathrm{TE}}}\sqrt{k_{z\ell'\mathrm{TE}}}k_{\mathrm{t}\ell\mathrm{TE}}k_{\mathrm{t}\ell'\mathrm{TE}}} \Bigg\{ &\iint\limits_{\mathbb{R}^2} k_\mathrm{t}^2 k_z w_\ell(-\boldsymbol{k}_\mathrm{t}) w_{\ell'}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \\ &+ \mathrm{i}\iint\limits_{\mathbb{R}^2} k_z w_\ell(-\boldsymbol{k}_\mathrm{t})\boldsymbol{k}_\mathrm{t}\cdot\mathbf{J}\cdot\int\limits_\Gamma w_{\ell'}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}\,\mathrm{d}k_x\,\mathrm{d}k_y \\ &- \mathrm{i}\iint\limits_{\mathbb{R}^2} k_z w_{\ell'}(\boldsymbol{k}_\mathrm{t})\boldsymbol{k}_\mathrm{t}\cdot\mathbf{J}\cdot\int\limits_\Gamma w_{\ell}(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}\,\mathrm{d}k_x\,\mathrm{d}k_y \\ &+ \iint\limits_{\mathbb{R}^2} k_z \int\limits_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c} \\ &\quad \cdot\left(\frac{\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}}{k_z^2} + \mathbf{I}_2\right)\cdot\int\limits_\Gamma w_{\ell'}(\boldsymbol{r}'_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}'_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}'_\mathrm{c}\,\mathrm{d}k_x\,\mathrm{d}k_y \Bigg\} \end{aligned}$$

4.2 Orthogonality relation

The waveguide expansion functions satisfy an orthogonality relation, *cf.*, (2.4). There exists also another orthogonality relation related to the energy integral, which is used in this section.

The use of the Parseval's identity gives

$$\begin{aligned} -Y_{\ell\nu}^E\delta_{\ell,\ell'}\delta_{\nu,\nu'} &= \eta_0 \iint\limits_\Omega \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})\cdot\mathbf{J}\cdot\boldsymbol{H}^*_{\mathrm{t}\ell'\nu'}(\boldsymbol{r}_\mathrm{c})\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{4\pi^2}\gamma_{\ell'\nu'} \iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{E}^*_{\mathrm{t}\ell'\nu'}(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \end{aligned} \tag{4.14}$$

where the energy admittance $Y_{\ell\nu}^E$ is very similar to the admittance defined in (2.5).

$$Y_{\ell\nu}^E = \begin{cases} \dfrac{k_0 k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2}, & \text{if } \nu = \mathrm{TM} \\ \dfrac{k_0 k_{z\ell\nu}^*}{k_{\mathrm{t}\ell\nu}^2}, & \text{if } \nu = \mathrm{TE} \\ 1, & \text{if } \nu = \mathrm{TEM} \end{cases} \tag{4.15}$$

We notice that only the TE part differs between $Y_{\ell\nu}^E$ and $Y_{\ell\nu}$ and that $\operatorname{Re} Y_{\ell\nu}^E = \operatorname{Re} Y_{\ell\nu}$ (for the TEM-mode there is a difference in sign). Moreover, the difference between the electric and the magnetic expansion functions $\gamma_{\ell\nu}$ is ($\eta_0\mathbf{J}\cdot\boldsymbol{H}_{\mathrm{t}\ell\nu}^*(\boldsymbol{r}_\mathrm{c}) = \gamma_{\ell\nu}\boldsymbol{E}_{\mathrm{t}\ell\nu}^*(\boldsymbol{r}_\mathrm{c})$)

$$\gamma_{\ell\nu} = \begin{cases} -\dfrac{k_0}{k_{z\ell\nu}^*}, & \text{if } \nu = \mathrm{TM} \\ -\dfrac{k_{z\ell\nu}^*}{k_0}, & \text{if } \nu = \mathrm{TE} \\ -1, & \text{if } \nu = \mathrm{TEM} \end{cases}$$

In the Fourier domain, we therefore have the following orthogonality relation

$$\iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})\cdot\boldsymbol{E}_{\mathrm{t}\ell'\nu'}^*(\boldsymbol{k}_\mathrm{t})\ \mathrm{d}k_x\ \mathrm{d}k_y = \gamma_{\ell\nu}^E\delta_{\ell,\ell'}\delta_{\nu,\nu'} \tag{4.16}$$

where

$$\gamma_{\ell\nu}^E = -4\pi^2 Y_{\ell\nu}^E/\gamma_{\ell\nu} = 4\pi^2\begin{cases} \dfrac{k_{z\ell\nu}^* k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2}, & \text{if } \nu = \mathrm{TM} \\ \dfrac{k_0^2}{k_{\mathrm{t}\ell\nu}^2}, & \text{if } \nu = \mathrm{TE} \\ 1, & \text{if } \nu = \mathrm{TEM} \end{cases}$$

5 Power flow and conservation

The purpose of this section is to prove the conservation of power, *i.e.*, the amount of power introduced into the system, (2.2), is either reflected in the waveguide, P_r, or radiated into the half-space, P_s. In this section we prove that $P_\mathrm{i} = P_\mathrm{r} + P_\mathrm{s}$.

5.1 Power flow in the waveguide

The power flow in the waveguide is (use (2.1) and (4.14))

$$\begin{aligned}\iint\limits_{\Omega}\hat{\boldsymbol{z}}\cdot <\boldsymbol{S}(t)>\ \mathrm{d}x\ \mathrm{d}y &= -\frac{1}{2\eta_0}\operatorname{Re}\iint\limits_{\Omega}\boldsymbol{E}(\boldsymbol{r})\cdot\mathbf{J}\cdot\boldsymbol{H}^*(\boldsymbol{r})\ \mathrm{d}x\ \mathrm{d}y \\ &= \frac{|E_0|^2}{2\eta_0}\left\{\operatorname{Re}Y_{\ell_0\nu_0}^E \mathrm{e}^{-2\operatorname{Im}k_{z\ell_0\nu_0}z} - \sum_{\ell,\nu}\operatorname{Re}Y_{\ell\nu}^E\left|r_{\ell\nu}\right|^2\mathrm{e}^{2\operatorname{Im}k_{z\ell\nu}z}\right\} \\ &= \frac{|E_0|^2}{2\eta_0}\left\{Y_{\ell_0\nu_0}^E - \sum_{\ell<k_0,\nu}Y_{\ell\nu}^E\left|r_{\ell\nu}\right|^2\right\} = \frac{|E_0|^2}{2\eta_0}\left\{Y_{\ell_0\nu_0} - \sum_{\ell<k_0,\nu}Y_{\ell\nu}\left|r_{\ell\nu}\right|^2\right\} = P_\mathrm{i} - P_\mathrm{r}\end{aligned}$$

since it is assumed that $k_0 > k_{\mathrm{t}\ell_0\nu_0}$, and where $\ell < k_0$ in the summation means that the summation in ℓ is only over those modes for which $k_{\mathrm{t}\ell\nu} < k_0$ (propagating modes), *i.e.*, $Y_{\ell\nu}$ real. The input power is given by (2.2)

$$P_\mathrm{i} = \iint\limits_{\Omega}\hat{\boldsymbol{z}}\cdot <\boldsymbol{S}(t)>\ \mathrm{d}x\ \mathrm{d}y = \frac{Y_{\ell_0\nu_0}^E\left|E_0\right|^2}{2\eta_0}$$

and the reflected power back into the waveguide is

$$P_{\rm r} = \frac{|E_0|^2}{2\eta_0} \sum_{\ell<k_0,\nu} Y^E_{\ell\nu}\, |r_{\ell\nu}|^2$$

Inserting (4.13) gives

$$P_{\rm r} = \frac{|E_0|^2}{2\eta_0} \sum_{\ell<k_0,\nu} |a_{\ell\nu}|^2$$

5.2 Power flow in the half space

The radiated power into the half space $z > 0$ is (see also Appendix B and (B.1))

$$P_{\rm s} = \frac{1}{8\pi^2\eta_0} \operatorname{Re} \iint\limits_{|\boldsymbol{k}_{\rm t}|\le k_0} \boldsymbol{E}_{xy}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}^*_{xy}(\boldsymbol{k}_{\rm t})\ {\rm d}k_x\ {\rm d}k_y$$

Using (4.12) we get

$$\begin{aligned}P_{\rm s} = \frac{|E_0|^2}{8\pi^2\eta_0} \operatorname{Re} \iint\limits_{|\boldsymbol{k}_{\rm t}|\le k_0} &\left(\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})}{N_{\ell\nu}}\right)\\ &\cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \left(\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})}{N_{\ell\nu}}\right)^* {\rm d}k_x\ {\rm d}k_y\end{aligned}$$

5.3 Power balance

We now prove that $P_{\rm s} = P_{\rm i} - P_{\rm r}$. We expand the domain of integration (see Appendix B) and get

$$\begin{aligned}P_{\rm s} = \frac{|E_0|^2}{8\pi^2\eta_0} \operatorname{Re} \iint\limits_{\mathbb{R}^2} &\left(\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})}{N_{\ell\nu}}\right)\\ &\cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \left(\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}\boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t})}{N_{\ell\nu}}\right)^* {\rm d}k_x\ {\rm d}k_y\end{aligned}$$

or

$$\begin{aligned}P_{\rm s} =&\frac{|E_0|^2}{8\pi^2\eta_0} \operatorname{Re} \iint\limits_{\mathbb{R}^2} \Bigg\{ \boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}^*_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t})\\ &+ \sum_{\ell,\nu} \frac{a^*_{\ell\nu}}{N^*_{\ell\nu}} \boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}^*_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}}{N_{\ell\nu}} \boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}^*_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t})\\ &+ \sum_{\ell,\nu} \sum_{\ell',\nu'} \frac{a_{\ell\nu}a^*_{\ell'\nu'}}{N_{\ell\nu}N^*_{\ell'\nu'}} \boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}^*_{{\rm t}\ell'\nu'}(\boldsymbol{k}_{\rm t}) \Bigg\}\ {\rm d}k_x\ {\rm d}k_y\end{aligned}$$

Use the fact that

$$Y_{\ell\nu}\boldsymbol{E}_{\mathrm{t}\ell\nu}^*(\boldsymbol{k}_\mathrm{t}) = -\left(Y_{\ell\nu}^E\right)^* \boldsymbol{E}_{\mathrm{t}\ell\nu}(-\boldsymbol{k}_\mathrm{t})$$

and the definition of the matrix $A_{\ell\nu\ell'\nu'}$, see (4.9). We get

$$\begin{aligned}P_\mathrm{s} = \frac{|E_0|^2}{2\eta_0}\,\mathrm{Re}\Big\{ &\frac{\left(Y_{\ell_0\nu_0}^E\right)^*}{Y_{\ell_0\nu_0}}N_{\ell_0\nu_0}^2 A_{\ell_0\nu_0\ell_0\nu_0} + N_{\ell_0\nu_0}\sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* A_{\ell\nu\ell_0\nu_0} \\ &+ N_{\ell_0\nu_0}\frac{\left(Y_{\ell_0\nu_0}^E\right)^*}{Y_{\ell_0\nu_0}}\sum_{\ell,\nu} a_{\ell\nu}A_{\ell_0\nu_0\ell\nu} + \sum_{\ell,\nu}\sum_{\ell',\nu'}\frac{\left(Y_{\ell'\nu'}^E\right)^* N_{\ell'\nu'}}{Y_{\ell'\nu'}N_{\ell'\nu'}^*}a_{\ell\nu}a_{\ell'\nu'}^* A_{\ell'\nu'\ell\nu}\Big\}\end{aligned}$$

Since $Y_{\ell_0\nu_0}^E = Y_{\ell_0\nu_0}$ is real and

$$\frac{Y_{\ell\nu}^E N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*} = \begin{cases} 1, & k_0 \geq k_{\mathrm{t}\ell\nu} \text{ and the TEM-mode} \\ \pm\,\mathrm{i}, & k_0 \leq k_{\mathrm{t}\ell\nu}\end{cases}$$

where the upper (lower) sign holds for the TM (TE)-mode, we get

$$\begin{aligned}P_\mathrm{s} = \frac{|E_0|^2}{2\eta_0}\,\mathrm{Re}\Big\{ &N_{\ell_0\nu_0}^2 A_{\ell_0\nu_0\ell_0\nu_0} + N_{\ell_0\nu_0}\sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* A_{\ell\nu\ell_0\nu_0} \\ &+ N_{\ell_0\nu_0}\left(b_{\ell_0\nu_0} - a_{\ell_0\nu_0}\right) + \sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^*\left(b_{\ell\nu} - a_{\ell\nu}\right)\Big\}\end{aligned}$$

We have here also applied (4.10). Using (4.11), we have

$$\begin{aligned}&\mathrm{Re}\left\{N_{\ell_0\nu_0}\sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* A_{\ell\nu\ell_0\nu_0} - N_{\ell_0\nu_0}a_{\ell_0\nu_0} + \sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* b_{\ell\nu}\right\} \\ =&\mathrm{Re}\left\{\sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^*\left(N_{\ell_0\nu_0}\delta_{\ell_0,\ell}\delta_{\nu_0,\nu} - b_{\ell\nu}\right) - N_{\ell_0\nu_0}a_{\ell_0\nu_0} + \sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* b_{\ell\nu}\right\} \\ =&0\end{aligned}$$

since

$$\frac{\left(Y_{\ell_0\nu_0}^E\right)^* N_{\ell_0\nu_0}}{Y_{\ell_0\nu_0}N_{\ell_0\nu_0}^*} = 1$$

We finally get (use (4.11))

$$\begin{aligned}P_\mathrm{s} =&\frac{|E_0|^2}{2\eta_0}\,\mathrm{Re}\Big\{N_{\ell_0\nu_0}^2 - \sum_{\ell,\nu}\frac{\left(Y_{\ell\nu}^E\right)^* N_{\ell\nu}}{Y_{\ell\nu}N_{\ell\nu}^*}a_{\ell\nu}^* a_{\ell\nu}\Big\} \\ =&\frac{|E_0|^2}{2\eta_0}\,\mathrm{Re}\Big\{N_{\ell_0\nu_0}^2 - \sum_{\ell<k_0,\nu}|a_{\ell\nu}|^2\Big\} = P_\mathrm{i} - P_\mathrm{r}\end{aligned}$$

and power flow is conserved.

6 Currents on the ground plane

The currents on the ground plane, $z = 0$, are determined by, see *e.g.*, (3.1)

$$\boldsymbol{J}(\boldsymbol{r}_{\rm c}) = \hat{\boldsymbol{z}} \times \boldsymbol{H}(\boldsymbol{r}_{\rm c}) = \mathbf{J} \cdot \boldsymbol{H}(\boldsymbol{r}_{\rm c}) = \frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \mathbf{J} \cdot \boldsymbol{H}(\boldsymbol{k}_{\rm t}) e^{i\boldsymbol{k}_{\rm t}\cdot\boldsymbol{r}_{\rm c}} \, dk_x \, dk_y$$

which, by (3.2) and (4.12), becomes

$$\eta_0 \boldsymbol{J}(\boldsymbol{r}_{\rm c}) = -\frac{E_0}{4\pi^2} \iint\limits_{\mathbb{R}^2} \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \left(\boldsymbol{E}_{{\rm t}\ell_0\nu_0}(\boldsymbol{k}_{\rm t}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}}{N_{\ell\nu}} \boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) \right) e^{i\boldsymbol{k}_{\rm t}\cdot\boldsymbol{r}_{\rm c}} \, dk_x \, dk_y$$

We introduce the notation

$$\boldsymbol{F}_{\ell\nu}(\boldsymbol{r}_{\rm c}) = \frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \boldsymbol{\gamma}(\boldsymbol{k}_{\rm t}) \cdot \boldsymbol{E}_{{\rm t}\ell\nu}(\boldsymbol{k}_{\rm t}) e^{i\boldsymbol{k}_{\rm t}\cdot\boldsymbol{r}_{\rm c}} \, dk_x \, dk_y \tag{6.1}$$

These vectors are known and can be computed for a given geometry of the cross section of the waveguide. The current on the ground plane can be written as

$$\eta_0 \boldsymbol{J}(\boldsymbol{r}_{\rm c}) = -E_0 \left(\boldsymbol{F}_{\ell_0\nu_0}(\boldsymbol{r}_{\rm c}) + \sum_{\ell,\nu} \frac{a_{\ell\nu}}{N_{\ell\nu}} \boldsymbol{F}_{\ell\nu}(\boldsymbol{r}_{\rm c}) \right)$$

7 Explicit examples

In this section we analyze a few explicit examples in detail and compare with often used approximations.

7.1 Rectangular waveguide

The eigenmodes and eigenvalues for a waveguide with rectangular cross section, sides a and b in the x- and the y-directions, respectively, see Figure 2, are given in Table 1. The collective mode index $\ell = \{m, n\}$ in this case. The Fourier transforms of the eigenmodes in Table 1 are

$$\begin{cases} v_\ell(\boldsymbol{k}_{\rm t}) = 2\sqrt{ab} \dfrac{m\pi \left(1 - (-1)^m e^{-ik_x a}\right)}{m^2\pi^2 - k_x^2 a^2} \dfrac{n\pi \left(1 - (-1)^n e^{-ik_y b}\right)}{n^2\pi^2 - k_y^2 b^2} \\ \qquad m, n = 1, 2, 3, \ldots \\ w_\ell(\boldsymbol{k}_{\rm t}) = -\sqrt{\epsilon_m \epsilon_n} \sqrt{ab} \dfrac{k_x a \left(1 - (-1)^m e^{-ik_x a}\right)}{m^2\pi^2 - k_x^2 a^2} \dfrac{k_y b \left(1 - (-1)^n e^{-ik_y b}\right)}{n^2\pi^2 - k_y^2 b^2} \\ \qquad m, n = 0, 1, 2, 3, \ldots, (m, n) \neq (0, 0) \end{cases}$$

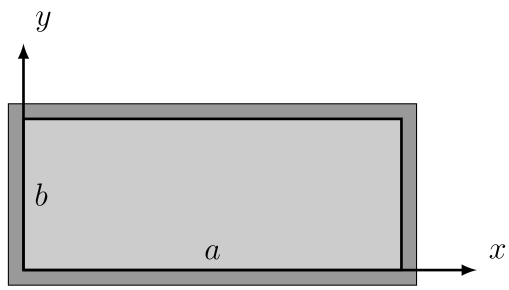


Figure 2: Geometry of the rectangular waveguide.

	Eigenmodes $v_\ell(\boldsymbol{r}_\mathrm{c}), w_\ell(\boldsymbol{r}_\mathrm{c})$	**Eigenvalues** $k_{\mathrm{t}\ell\nu}$
TM	$v_\ell(\boldsymbol{r}_\mathrm{c}) = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$	$\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$
TE	$w_\ell(\boldsymbol{r}_\mathrm{c}) = \sqrt{\frac{\epsilon_m\epsilon_n}{ab}} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$	$\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$

Table 1: Table over the normalized eigenmodes in a rectangular waveguide, see Figure 2. The integer m and n assume the values $m, n = 0, 1, 2, 3, \ldots$, with the exception that m and n are nonzero for the TM-modes, and that m and n are **both** not zero for the TE-modes ($\epsilon_m = 2 - \delta_{m,0}$).

or in a form that more clearly shows the behavior at $k_x = \pm m\pi$ and $k_y = \pm n\pi$

$$\begin{cases} v_\ell(\boldsymbol{k}_\mathrm{t}) = -\frac{\sqrt{ab}}{2} \mathrm{e}^{-\mathrm{i}(k_x a - m\pi)/2 - \mathrm{i}(k_y b - n\pi)/2} \\ \qquad \times \left(\mathrm{sinc}\frac{k_x a - m\pi}{2} - (-1)^m \mathrm{sinc}\frac{k_x a + m\pi}{2}\right) \\ \qquad \times \left(\mathrm{sinc}\frac{k_y b - n\pi}{2} - (-1)^n \mathrm{sinc}\frac{k_y b + n\pi}{2}\right) \\ \qquad m, n = 1, 2, 3, \ldots \\ w_\ell(\boldsymbol{k}_\mathrm{t}) = \frac{\sqrt{\epsilon_m\epsilon_n}\sqrt{ab}}{4} \mathrm{e}^{-\mathrm{i}(k_x a - m\pi)/2 - \mathrm{i}(k_y b - n\pi)/2} \\ \qquad \times \left(\mathrm{sinc}\frac{k_x a - m\pi}{2} + (-1)^m \mathrm{sinc}\frac{k_x a + m\pi}{2}\right) \\ \qquad \times \left(\mathrm{sinc}\frac{k_y b - n\pi}{2} + (-1)^n \mathrm{sinc}\frac{k_y b + n\pi}{2}\right) \\ \qquad m, n = 0, 1, 2, 3, \ldots, (m, n) \neq (0, 0) \end{cases}$$

The general expressions of the Fourier transform of the eigenfunctions are for

$\nu = \mathrm{TM}$, see (4.4)

$$\begin{aligned}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) =& -\frac{k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2}\boldsymbol{k}_\mathrm{t} v_\ell(\boldsymbol{k}_\mathrm{t})\\ =& -2\boldsymbol{k}_\mathrm{t}\sqrt{ab}\frac{\sqrt{k_0^2-\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}}{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\frac{m\pi}{m^2\pi^2-k_x^2a^2}\left(1-(-1)^m\mathrm{e}^{-\mathrm{i}k_xa}\right)\\ &\cdot\frac{n\pi}{n^2\pi^2-k_y^2b^2}\left(1-(-1)^n\mathrm{e}^{-\mathrm{i}k_yb}\right)\\ & m,n=1,2,3,\ldots\end{aligned}$$

and for $\nu = \mathrm{TE}$

$$\begin{aligned}\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) =& \frac{k_0}{k_{\mathrm{t}\ell\nu}^2}\mathbf{J}\cdot\boldsymbol{k}_\mathrm{t} w_\ell(\boldsymbol{k}_\mathrm{t}) - \mathrm{i}\frac{k_0}{k_{\mathrm{t}\ell\nu}^2}\int\limits_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c}\\ =& -\mathbf{J}\cdot\boldsymbol{k}_\mathrm{t}\sqrt{\epsilon_m\epsilon_n}\sqrt{ab}\frac{k_0}{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\frac{k_xa}{m^2\pi^2-k_x^2a^2}\left(1-(-1)^m\mathrm{e}^{-\mathrm{i}k_xa}\right)\\ &\cdot\frac{k_yb}{n^2\pi^2-k_y^2b^2}\left(1-(-1)^n\mathrm{e}^{-\mathrm{i}k_yb}\right)\\ &+\sqrt{\epsilon_m\epsilon_n}\frac{k_0\left(1-(-1)^n\right)}{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sqrt{\frac{a}{b}}\hat{\boldsymbol{x}}\frac{k_xa}{m^2\pi^2-k_x^2a^2}\left(1-(-1)^m\mathrm{e}^{-\mathrm{i}k_xa}\right)\\ &-\sqrt{\epsilon_m\epsilon_n}\frac{k_0\left(1-(-1)^m\right)}{\pi^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sqrt{\frac{b}{a}}\hat{\boldsymbol{y}}\frac{k_yb}{n^2\pi^2-k_y^2b^2}\left(1-(-1)^n\mathrm{e}^{-\mathrm{i}k_yb}\right)\\ & m,n=0,1,2,3,\ldots,\quad (m,n)\neq(0,0)\end{aligned}$$

The rectangular waveguide is not pursued further due to the fact that the matrix entries of $A_{\ell\nu\ell'\nu'}$ in (4.9) lead to integrals that involve extensive numerical integration. Instead, the circular waveguide is more suited for numerical computations. This is done in the following subsection.

7.2 Circular waveguide

The next example is the circular waveguide. The eigenmodes and eigenvalues for the circular waveguide with radius a, see Figure 3, are given in Table 2. The collective mode index $\ell = \{m, n, \sigma\}$ in this case. The mode with the lowest cutoff frequency is TE_{11}, *i.e.*,

$$\begin{aligned}\boldsymbol{E}_{\mathrm{t}11\sigma\mathrm{TE}}(\boldsymbol{r}_\mathrm{c}) =& -\frac{\mathrm{i}k_0a}{\eta_{11}}\mathbf{J}\cdot\nabla_\mathrm{t}\frac{\sqrt{2}J_1(\eta_{11}r_\mathrm{c}/a)}{\sqrt{\pi\,(\eta_{11}^2-1)}J_1(\eta_{11})}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\\ =&\frac{\mathrm{i}k_0a}{\eta_{11}}\hat{\boldsymbol{r}}_\mathrm{c}\frac{\sqrt{2}J_1(\eta_{11}r_\mathrm{c}/a)}{r_\mathrm{c}\sqrt{\pi\,(\eta_{11}^2-1)}J_1(\eta_{11})}\begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}\\ &-\mathrm{i}k_0\hat{\boldsymbol{\phi}}\frac{\sqrt{2}J_1'(\eta_{11}r_\mathrm{c}/a)}{\sqrt{\pi\,(\eta_{11}^2-1)}J_1(\eta_{11})}\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\end{aligned}$$

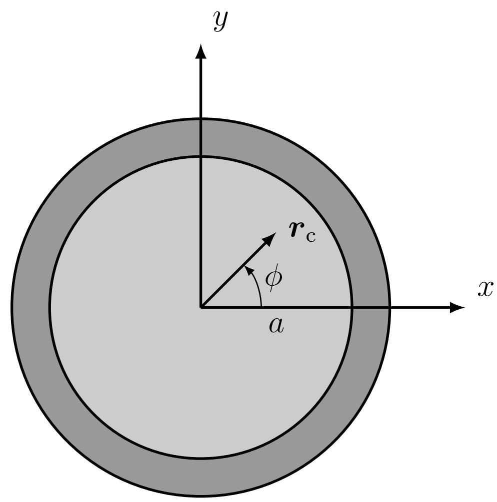


Figure 3: Geometry of the circular waveguide.

where $\eta_{11} \approx 1.841$. For the index $\sigma = \mathrm{o}$ the field is oriented along the $\hat{\boldsymbol{x}}$-axis. In fact, we have

$$\boldsymbol{E}_{\mathrm{t}11\mathrm{oTE}}(\boldsymbol{r}_\mathrm{c} = \boldsymbol{0}) = \frac{\mathrm{i}k_0}{\sqrt{2\pi\,(\eta_{11}^2 - 1)}J_1(\eta_{11})}\hat{\boldsymbol{x}}$$

The Fourier transforms of the eigenmodes are, see (A.1), (A.5), and (A.6)

$$\begin{cases} v_\ell(\boldsymbol{k}_\mathrm{t}) = 2\pi(-\mathrm{i})^m \dfrac{\sqrt{\epsilon_m}}{\sqrt{\pi}} \dfrac{\xi_{mn} a J_m(k_\mathrm{t}a)}{k_\mathrm{t}^2 a^2 - \xi_{mn}^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \\ w_\ell(\boldsymbol{k}_\mathrm{t}) = \dfrac{2\pi(-\mathrm{i})^m \sqrt{\epsilon_m}\eta_{mn} a}{\sqrt{\pi\,(\eta_{mn}^2 - m^2)}} \dfrac{k_\mathrm{t} a J_m'(k_\mathrm{t}a)}{\eta_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \end{cases}$$

where the indices are $m = 0, 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, and $\sigma = \mathrm{e}, \mathrm{o}$. Moreover, we have, see (A.3)

$$\int_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\,\mathrm{d}\boldsymbol{r}_\mathrm{c} = 2\pi(-\mathrm{i})^{m-1} \frac{\sqrt{\epsilon_m}\eta_{mn}}{\sqrt{\pi\,(\eta_{mn}^2 - m^2)}a} \mathbf{J} \cdot \nabla_{\boldsymbol{k}_\mathrm{t}} J_m(k_\mathrm{t}a) \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix}$$

The general expressions of the Fourier transform of the transverse components $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})$ are for $\nu = \mathrm{TM}$, see (4.4)

$$\begin{aligned} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) &= -\frac{k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2}\boldsymbol{k}_\mathrm{t} v_\ell(\boldsymbol{k}_\mathrm{t}) \\ &= -2\pi(-\mathrm{i})^m a \frac{\sqrt{k_0^2 a^2 - \xi_{mn}^2}}{\xi_{mn}} \frac{\sqrt{\epsilon_m}}{\sqrt{\pi}} \frac{\boldsymbol{k}_\mathrm{t} a J_m(k_\mathrm{t}a)}{k_\mathrm{t}^2 a^2 - \xi_{mn}^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \\ & m = 0, 1, 2, 3, \ldots, n = 1, 2, 3, \ldots, \sigma = \mathrm{e}, \mathrm{o} \end{aligned}$$

	Eigenmodes $v_\ell(\boldsymbol{r}_\mathrm{c}), w_\ell(\boldsymbol{r}_\mathrm{c})$	Eigenvalues $k_{\mathrm{t}\ell\nu}$
TM	$v_\ell(\boldsymbol{r}_\mathrm{c}) = \dfrac{\sqrt{\epsilon_m} J_m(\xi_{mn} r_\mathrm{c}/a)}{\sqrt{\pi} a J'_m(\xi_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$	$\dfrac{\xi_{mn}}{a}$
TE	$w_\ell(\boldsymbol{r}_\mathrm{c}) = \dfrac{\sqrt{\epsilon_m}\eta_{mn} J_m(\eta_{mn} r_\mathrm{c}/a)}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)} a J_m(\eta_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$	$\dfrac{\eta_{mn}}{a}$

Table 2: Table over the normalized eigenmodes in a circular waveguide, see Figure 3. The constant ξ_{mn} is the nth positive zero to $J_m(x)$ and the constant η_{mn} is the nth positive zero to $J'_m(x)$, *i.e.*, $J_m(\xi_{mn}) = 0$ and $J'_m(\eta_{mn}) = 0$, $m = 0, 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$, $\epsilon_m = 2 - \delta_{m,0}$, and $\sigma = \mathrm{e}, \mathrm{o}$. Notice that $\xi_{mn} > m$ and $\eta_{mn} > m$, $n = 1, 2, 3, \ldots$, see Ref. 6.

and for $\nu = \mathrm{TE}$

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) =& \frac{k_0}{k_{\mathrm{t}\ell\nu}^2} \mathbf{J} \cdot \boldsymbol{k}_\mathrm{t} w_\ell(\boldsymbol{k}_\mathrm{t}) - \mathrm{i} \frac{k_0}{k_{\mathrm{t}\ell\nu}^2} \int_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}\boldsymbol{r}_\mathrm{c} \\
=& a \frac{k_0 a}{\eta_{mn}} \mathbf{J} \cdot \boldsymbol{k}_\mathrm{t} a \frac{2\pi(-\mathrm{i})^m \sqrt{\epsilon_m}}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)}} \frac{k_\mathrm{t} a J'_m(k_\mathrm{t} a)}{\eta_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \\
&+ a \frac{k_0 a}{\eta_{mn}} \frac{2\pi(-\mathrm{i})^m \sqrt{\epsilon_m}}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)}} \\
&\cdot \left(\frac{\mathbf{J} \cdot \boldsymbol{k}_\mathrm{t} a}{k_\mathrm{t} a} J'_m(k_\mathrm{t} a) \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} - \frac{\boldsymbol{k}_\mathrm{t} a}{k_\mathrm{t} a} \frac{m J_m(k_\mathrm{t} a)}{k_\mathrm{t} a} \begin{pmatrix} -\sin m\alpha \\ \cos m\alpha \end{pmatrix} \right) \\
=& a \frac{k_0 a}{\eta_{mn}} \frac{2\pi(-\mathrm{i})^m \sqrt{\epsilon_m}}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)}} \left\{ \frac{\mathbf{J} \cdot \boldsymbol{k}_\mathrm{t} a}{k_\mathrm{t} a} \frac{\eta_{mn}^2 J'_m(k_\mathrm{t} a)}{\eta_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \right. \\
&\left. - \frac{\boldsymbol{k}_\mathrm{t} a}{k_\mathrm{t} a} \frac{m J_m(k_\mathrm{t} a)}{k_\mathrm{t} a} \begin{pmatrix} -\sin m\alpha \\ \cos m\alpha \end{pmatrix} \right\} \\
& m = 0, 1, 2, 3, \ldots, n = 1, 2, 3, \ldots, \sigma = \mathrm{e}, \mathrm{o}
\end{aligned}$$

From these expressions we get

$$\boldsymbol{E}_{\mathrm{t}\ell\nu}(-\boldsymbol{k}_\mathrm{t}) = (-1)^{m+1} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t})$$

We also have

$$\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t} = \mathbf{0}) = \begin{cases} \mathbf{0}, & \nu = \mathrm{TM} \\ \mathrm{i} \dfrac{k_0 a^2}{\eta_{mn}} \dfrac{\delta_{m,1}\sqrt{2\pi}}{\sqrt{\eta_{mn}^2 - 1}} \begin{pmatrix} -\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{x}} \end{pmatrix}, & \nu = \mathrm{TE} \end{cases}$$

7.2.1 Matrix entries of $A_{\ell\nu\ell'\nu'}$

The matrix entries of $A_{\ell\nu\ell'\nu'}$ are needed to compute the interaction between the ground plane and the waveguide modes. From (4.9) we have

$$\begin{aligned} A_{\ell\mathrm{TM}\ell'\mathrm{TM}} =& 2\left(1-\delta_{m,0}\delta_{\sigma,\mathrm{o}}\right)\delta_{m,m'}\delta_{\sigma,\sigma'}\sqrt{\sqrt{\kappa^2-\xi_{mn}^2}\sqrt{\kappa^2-\xi_{mn'}^2}} \\ &\cdot \int_0^\infty \frac{x^2\left(J_m(x)\right)^2}{\left(x^2-\xi_{mn}^2\right)\left(x^2-\xi_{mn'}^2\right)}\frac{x\ \mathrm{d}x}{\sqrt{\kappa^2-x^2}} \end{aligned}$$

where we have introduced the dimensionless frequency parameter $\kappa = k_0 a$. To continue, we have

$$\begin{aligned} A_{\ell\mathrm{TM}\ell'\mathrm{TE}} =& \delta_{m,m'}\left(\delta_{\sigma,\mathrm{e}}\delta_{\sigma',\mathrm{o}}-\delta_{\sigma,\mathrm{o}}\delta_{\sigma',\mathrm{e}}\right)\frac{2m\kappa\sqrt{\sqrt{\kappa^2-\xi_{mn}^2}}}{\sqrt{\sqrt{\kappa^2-\eta_{mn'}^2}}}\frac{1}{\sqrt{(\eta_{mn'}^2-m^2)}} \\ &\cdot \int_0^\infty \frac{\left(J_m(x)\right)^2}{x^2-\xi_{mn}^2}\frac{x\ \mathrm{d}x}{\sqrt{\kappa^2-x^2}} \end{aligned}$$

$$\begin{aligned} A_{\ell\mathrm{TE}\ell'\mathrm{TM}} =& \delta_{m,m'}\left(\delta_{\sigma,\mathrm{o}}\delta_{\sigma',\mathrm{e}}-\delta_{\sigma,\mathrm{e}}\delta_{\sigma',\mathrm{o}}\right)\frac{2m\kappa\sqrt{\sqrt{\kappa^2-\xi_{mn'}^2}}}{\sqrt{\sqrt{\kappa^2-\eta_{mn}^2}}}\frac{1}{\sqrt{\eta_{mn}^2-m^2}} \\ &\cdot \int_0^\infty \frac{\left(J_m(x)\right)^2}{x^2-\xi_{mn'}^2}\frac{x\ \mathrm{d}x}{\sqrt{\kappa^2-x^2}} \end{aligned}$$

and

$$\begin{aligned} A_{\ell\mathrm{TE}\ell'\mathrm{TE}} =& \delta_{m,m'}\delta_{\sigma,\sigma'}\frac{2\kappa^2}{\sqrt{\sqrt{\kappa^2-\eta_{mn}^2}\sqrt{\kappa^2-\eta_{mn'}^2}}}\frac{1}{\sqrt{(\eta_{mn'}^2-m^2)}}\frac{1}{\sqrt{(\eta_{mn}^2-m^2)}} \\ &\cdot \left\{\left(1-\delta_{m,0}\delta_{\sigma,\mathrm{o}}\right)\frac{\eta_{mn}^2\eta_{mn'}^2}{\kappa^2}\int_0^\infty \frac{\left(J_m'(x)\right)^2\sqrt{\kappa^2-x^2}}{\left(\eta_{mn}^2-x^2\right)\left(\eta_{mn'}^2-x^2\right)}x\ \mathrm{d}x \right. \\ &\left. + m^2\int_0^\infty \frac{\left(J_m(x)\right)^2}{x^2}\frac{x\ \mathrm{d}x}{\sqrt{\kappa^2-x^2}}\right\} \end{aligned}$$

7.2.2 Orthogonality relations

From (4.16) we can obtain integral identities for the different modes. These integral identities seem not know in the literature and for this reason we give the explicit expressions of these identities in this subsection. For the TM-modes the following integral identity is obtained ($m = 0, 1, 2, 3, \ldots,\ n, n' = 1, 2, 3, \ldots$)

$$2\int_0^\infty \frac{\left(J_m(x)\right)^2}{\left(x^2-\xi_{mn}^2\right)\left(x^2-\xi_{mn'}^2\right)}x^3\ \mathrm{d}x = \delta_{n,n'}$$

and for the TM-modes we get ($m = 0, 1, 2, 3, \ldots$, $n, n' = 1, 2, 3, \ldots$)

$$\frac{2}{\sqrt{(\eta_{mn}^2 - m^2)(\eta_{mn'}^2 - m^2)}} \int_0^\infty \left\{ \frac{\eta_{mn}^2 \eta_{mn'}^2 \left(J_m'(x)\right)^2}{(\eta_{mn}^2 - x^2)(\eta_{mn'}^2 - x^2)} + \frac{m^2 \left(J_m(x)\right)^2}{x^2} \right\} x \, \mathrm{d}x = \delta_{n,n'}$$

or

$$2\eta_{mn}^2 \eta_{mn'}^2 \int_0^\infty \frac{\left(J_m'(x)\right)^2}{(\eta_{mn}^2 - x^2)(\eta_{mn'}^2 - x^2)} x \, \mathrm{d}x = \sqrt{(\eta_{mn}^2 - m^2)(\eta_{mn'}^2 - m^2)}\delta_{n,n'} - m$$

For the cross-mode terms we get ($m = 0, 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$)

$$\int_0^\infty \frac{\left(J_m(x)\right)^2}{x^2 - \xi_{mn}^2} x \, \mathrm{d}x = 0$$

7.2.3 The current on the ground plane

The expansion functions of the current on the ground plane are given by (6.1). For the circular waveguide they are

$$\boldsymbol{F}_{\ell\mathrm{TM}}(\boldsymbol{r}_\mathrm{c}) = \mathrm{i}\kappa \frac{\sqrt{\kappa^2 - \xi_{mn}^2}}{\xi_{mn}} \frac{\sqrt{\epsilon_m}}{\sqrt{\pi}} \nabla \int_0^\infty \frac{J_m(x) J_m(x r_\mathrm{c}/a)}{x^2 - \xi_{mn}^2} \frac{x \, \mathrm{d}x}{\sqrt{\kappa^2 - x^2}} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$$

and

$$\begin{aligned} \boldsymbol{F}_{\ell\mathrm{TE}}(\boldsymbol{r}_\mathrm{c}) = -\mathrm{i} & \frac{1}{\eta_{mn}\sqrt{\eta_{mn}^2 - m^2}} \frac{\sqrt{\epsilon_m}}{\sqrt{\pi}} \\ & \times \left\{ \eta_{mn}^2 \mathbf{J} \cdot \nabla \int_0^\infty \frac{J_m'(x) J_m(x r_\mathrm{c}/a)}{\eta_{mn}^2 - x^2} \sqrt{\kappa^2 - x^2} \, \mathrm{d}x \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} \right. \\ & \left. - \kappa^2 m \nabla \int_0^\infty \frac{J_m(x) J_m(x r_\mathrm{c}/a)}{x} \frac{\mathrm{d}x}{\sqrt{\kappa^2 - x^2}} \begin{pmatrix} -\sin m\phi \\ \cos m\phi \end{pmatrix} \right\} \end{aligned}$$

7.2.4 Numerical examples

We illustrate the analysis presented in this paper with a series of numerical examples for the circular waveguide. The incident wave in the waveguide is assumed to be a TE_{11}-mode. In Figure 4, the normalized intensity at the normalized frequency $\kappa = k_0 a = 1.01\eta_{11} \approx 1.86$ is depicted as a function of the observation angle θ. This frequency is just above the lowest cutoff frequency $f = c_0\eta_{11}/2\pi a$. The approximate solutions, when the aperture field is approximated by the incident mode, are illuatrated with red lines. As expected, the agreement in the forward direction is very good. Both curves are normalized with their forward intensities, which are 0.036 (exact) and 0.23 (approximate), respectively. The directivities[3] of the aperture antenna are 4.09 dB (exact) and 3.80 dB (approximate), respectively.

[3]The directivity is related to radiation in a half space.

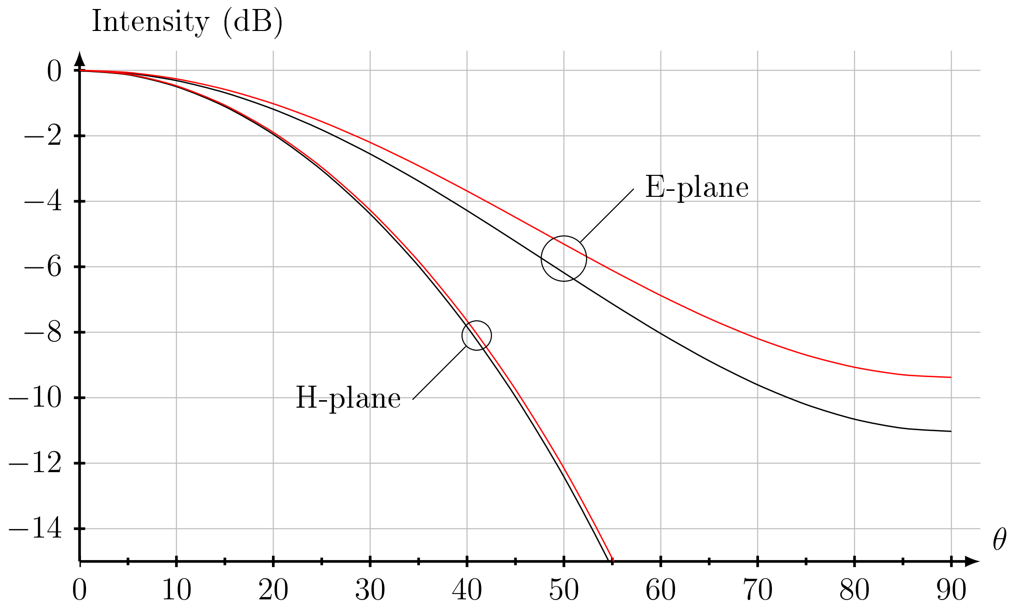


Figure 4: The normalized intensity in a dB-scale as a function of the observation angle θ at the normalized frequency $\kappa = k_0 a = 1.01\eta_{11} \approx 1.86$ (E-plane and H-plane). The black lines are the exact solutions, while the red lines are the approximate solutions (see text), when the aperture field is identical to the incident mode.

In Figure 5, the normalized intensity at a higher frequency, $\kappa = k_0 a = 1.5\xi_{11} \approx 5.75$, is shown. Both curves are normalized with their forward intensities, which are 21.1 (exact) and 21.4 (approximate), respectively. The directivities of the aperture antenna are 11.6 dB (exact) and 11.6 dB (approximate), respectively. Both radiation pattern and directivity are well predicted by the approximate solution. The reason for this is found in Figure 6 which shows the reflected power in the waveguide. This reflected power is negligible except for frequencies close to the cutoff frequency $f = c_0\eta_{11}/2\pi a$. Notice the jump discontinuities at the location of the cutoff frequencies. Finally, we give an illustration of the surface currents on the ground plane at the normalized frequency $\kappa = k_0 a = 1.5\xi_{11} \approx 5.75$ in Figure 7. The polarization ellipses at several locations on the ground plane are depicted. Dots mark the center of the ellipse and the current at a synchronized time ($t = 0$). This figure clearly illustrates the confinement of the current to the aperture.

7.3 Coaxial waveguide

In this final example we address the coaxial waveguide. The waveguide has an outer radius a and an inner radius b, see Figure 8. This geometry is slightly more complex than the previous ones, but very important from an application point of view.

The collective mode index $\ell = \{m, n, \sigma\}$ is the same as for the circular waveguide in Section 7.2. The radial eigenfunctions, $Z_{mn}(r_c)$ and $Y_{mn}(r_c)$, and the normaliza-

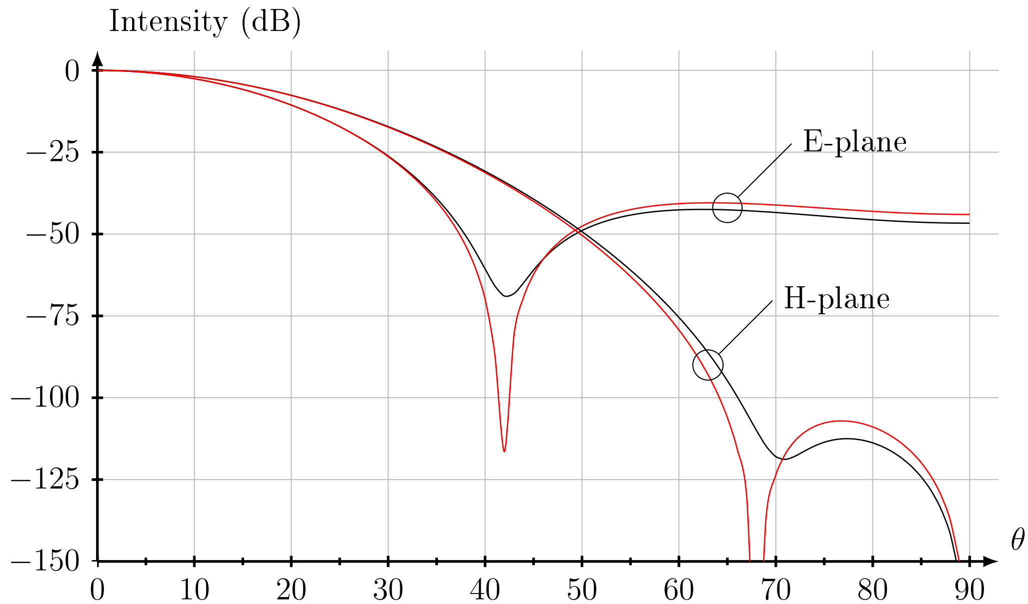


Figure 5: The normalized intensity in a dB-scale as a function of the observation angle θ at the normalized frequency $\kappa = k_0 a = 1.5\xi_{11} \approx 5.75$ (E-plane and H-plane). The black lines are the exact solutions, while the red lines are the approximate solutions when the aperture field is identical to the incident mode.

tion constants, C_{mn} and D_{mn}, are defined by ($m = 0, 1, 2, \ldots$, and $n = 1, 2, 3, \ldots$)

$$\begin{cases} Z_{mn}(r_c) = J_m(\zeta_{mn} r_c/a) N_m(\zeta_{mn}) - J_m(\zeta_{mn}) N_m(\zeta_{mn} r_c/a) \\ C_{mn}^{-2} = \dfrac{2\pi}{\epsilon_m} \displaystyle\int_b^a (Z_{mn}(r_c))^2 \, r_c \, dr_c = \dfrac{\pi}{\epsilon_m} \left\{ b^2 \left(Z'_{mn}(b)\right)^2 - a^2 \left(Z'_{mn}(a)\right)^2 \right\} \end{cases}$$
$$\begin{cases} Y_{mn}(r_c) = J_m(\gamma_{mn} r_c/a) N'_m(\gamma_{mn}) - J'_m(\gamma_{mn}) N_m(\gamma_{mn} r_c/a) \\ D_{mn}^{-2} = \dfrac{2\pi}{\epsilon_m} \displaystyle\int_b^a (Y_{mn}(r_c))^2 \, r_c \, dr_c \\ \qquad = \dfrac{\pi a^2}{\epsilon_m \gamma_{mn}^2} \left\{ \left(\gamma_{mn} b^2/a^2 - m^2\right) (Y_{mn}(b))^2 - \left(\gamma_{mn} - m^2\right) (Y_{mn}(a))^2 \right\} \end{cases} \tag{7.1}$$

and the eigenvalues, ζ_{mn} and γ_{mn}, are determined by the positive roots ($m = 0, 1, 2, \ldots$, and $n = 1, 2, 3, \ldots$) of the transcendental equations

$$\begin{cases} Z_{mn}(b) = J_m(\zeta_{mn} b/a) N_m(\zeta_{mn}) - J_m(\zeta_{mn}) N_m(\zeta_{mn} b/a) = 0 \\ Y'_{mn}(b) = J'_m(\gamma_{mn} b/a) N'_m(\gamma_{mn}) - J'_m(\gamma_{mn}) N'_m(\gamma_{mn} b/a) = 0 \end{cases} \tag{7.2}$$

where the Bessel function and the Neumann function of order m are denoted $J_m(z)$ and $N_m(z)$, respectively, and $\epsilon_m = 2 - \delta_{m,0}$. Notice that $Z_{mn}(a) = 0$ and $Y'_{mn}(a) = 0$.

The Fourier transforms of the eigenmodes are presented in Table 3, see (A.1),

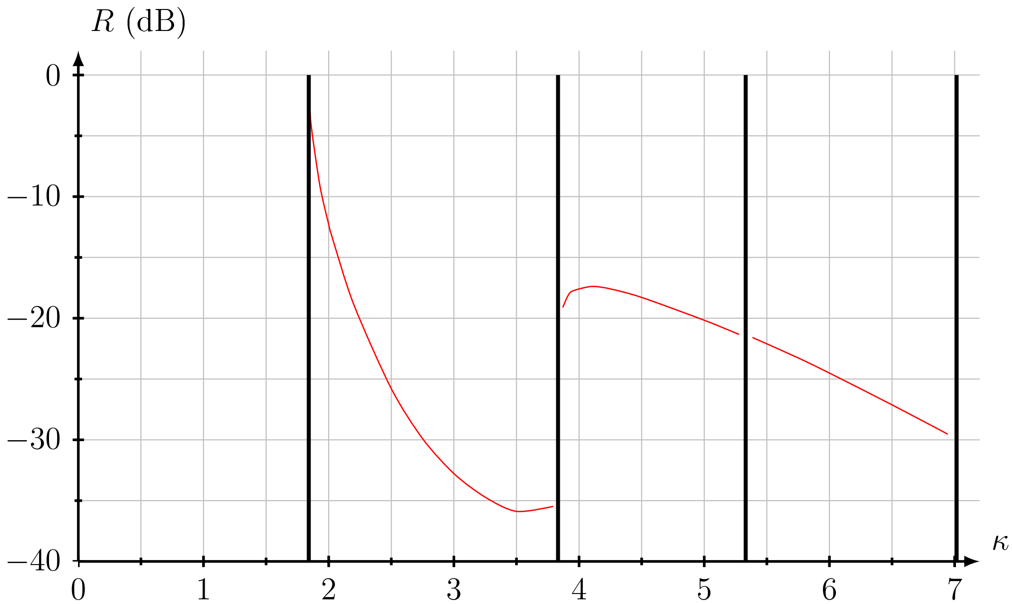


Figure 6: The reflected power in a dB-scale in the waveguide as a function of the normalized frequency κ. The vertical lines give the location of the cutoff frequencies in the circular waveguide.

(A.4)

$$\begin{cases} v_\ell(\boldsymbol{k}_\mathrm{t}) = -2\pi(-\mathrm{i})^m C_{mn} a\zeta_{mn} \dfrac{aZ'_{mn}(a)J_m(k_\mathrm{t}a) - bZ'_{mn}(b)J_m(k_\mathrm{t}b)}{\zeta_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \\ w_\ell(\boldsymbol{k}_\mathrm{t}) = 2\pi(-\mathrm{i})^m D_{mn} a^2 \dfrac{Y_{mn}(a)k_\mathrm{t}aJ'_m(k_\mathrm{t}a) - Y_{mn}(b)k_\mathrm{t}bJ'_m(k_\mathrm{t}b)}{\gamma_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \end{cases}$$

where the indices are $m = 0, 1, 2, 3, \dots$, $n = 1, 2, 3, \dots$, and $\sigma = \mathrm{e}$, o. We also have from (A.3)

$$\begin{aligned} \int\limits_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c}) \mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}} \, \mathrm{d}\boldsymbol{r}_\mathrm{c} =& 2\pi(-\mathrm{i})^{m-1} D_{mn} \\ & \times \mathbf{J} \cdot \nabla_{\boldsymbol{k}_\mathrm{t}} \left\{ Y_{mn}(a) J_m(k_\mathrm{t}a) - Y_{mn}(b) J_m(k_\mathrm{t}b) \right\} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \end{aligned}$$

The general expressions of the Fourier transform of the transverse components of the electric field $\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{r}_\mathrm{c})$ are for $\nu = \mathrm{TM}$, see (4.4)

$$\begin{aligned} \boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) = -\frac{k_{z\ell\nu}}{k_{\mathrm{t}\ell\nu}^2} \boldsymbol{k}_\mathrm{t} v_\ell(\boldsymbol{k}_\mathrm{t}) = 2\pi(-\mathrm{i})^m C_{mn} a^2 \frac{\sqrt{k_0^2 a^2 - \zeta_{mn}^2}}{\zeta_{mn}} \\ \times \boldsymbol{k}_\mathrm{t} \frac{aZ'_{mn}(a)J_m(k_\mathrm{t}a) - bZ'_{mn}(b)J_m(k_\mathrm{t}b)}{\zeta_{mn}^2 - k_\mathrm{t}^2 a^2} \begin{pmatrix} \cos m\alpha \\ \sin m\alpha \end{pmatrix} \\ m = 0, 1, 2, 3, \dots, n = 1, 2, 3, \dots, \sigma = \mathrm{e}, \mathrm{o} \end{aligned}$$

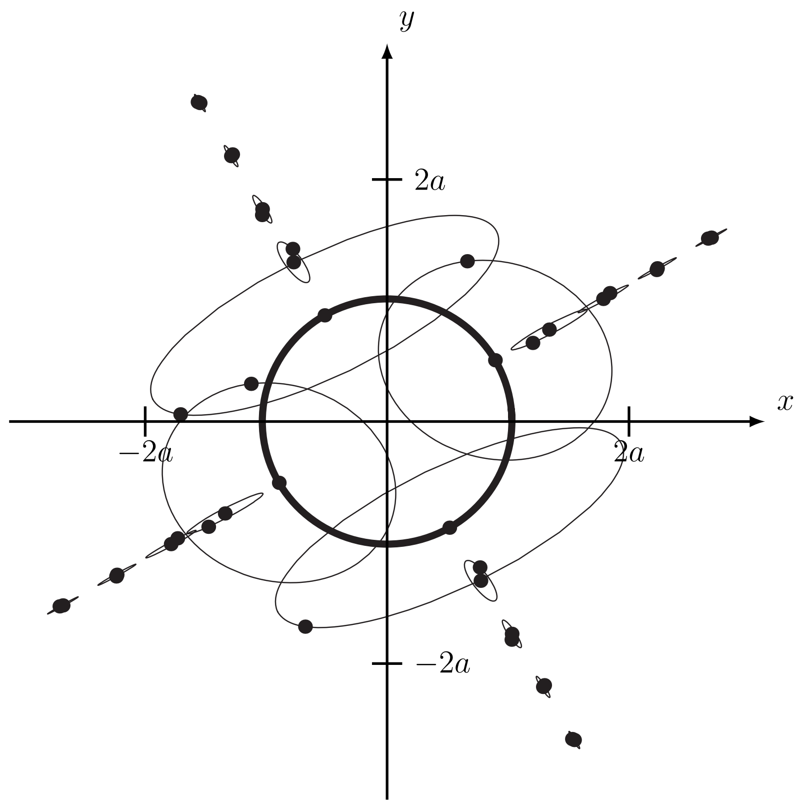


Figure 7: The polarization ellipse of the surface current on the ground plane at the normalized frequency $\kappa = k_0 a = 1.5\xi_{11} \approx 5.75$. The dots at the center of each ellipse locate the points at which the currents are computed and the dots on the ellipse give the currents at a synchronized time. The thick circle locates the circular aperture $r_\mathrm{c} = a$.

and for $\nu = \mathrm{TE}$

$$\begin{aligned}
\boldsymbol{E}_{\mathrm{t}\ell\nu}(\boldsymbol{k}_\mathrm{t}) =& \frac{k_0}{k_{\mathrm{t}\ell\nu}^2}\mathbf{J}\cdot\boldsymbol{k}_\mathrm{t} w_\ell(\boldsymbol{k}_\mathrm{t}) - \mathrm{i}\frac{k_0}{k_{\mathrm{t}\ell\nu}^2}\int\limits_\Gamma w_\ell(\boldsymbol{r}_\mathrm{c})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_\mathrm{t}\cdot\boldsymbol{r}_\mathrm{c}}\;\mathrm{d}\boldsymbol{r}_\mathrm{c}\\
=& 2\pi(-\mathrm{i})^m D_{mn}\frac{k_0 a^2}{\gamma_{mn}^2}\\
&\times\left\{\mathbf{J}\cdot\boldsymbol{k}_\mathrm{t} a^2\frac{Y_{mn}(a)k_\mathrm{t} a J_m'(k_\mathrm{t} a) - Y_{mn}(b)k_\mathrm{t} b J_m'(k_\mathrm{t} b)}{\gamma_{mn}^2 - k_\mathrm{t}^2 a^2}\begin{pmatrix}\cos m\alpha\\ \sin m\alpha\end{pmatrix}\right.\\
&\left.\quad + \mathbf{J}\cdot\nabla_{\boldsymbol{k}_\mathrm{t}}\left\{Y_{mn}(a)J_m(k_\mathrm{t} a) - Y_{mn}(b)J_m(k_\mathrm{t} b)\right\}\begin{pmatrix}\cos m\alpha\\ \sin m\alpha\end{pmatrix}\right\}\\
& m = 0,1,2,3,\ldots, n = 1,2,3,\ldots, \sigma = \mathrm{e},\ \mathrm{o}
\end{aligned}$$

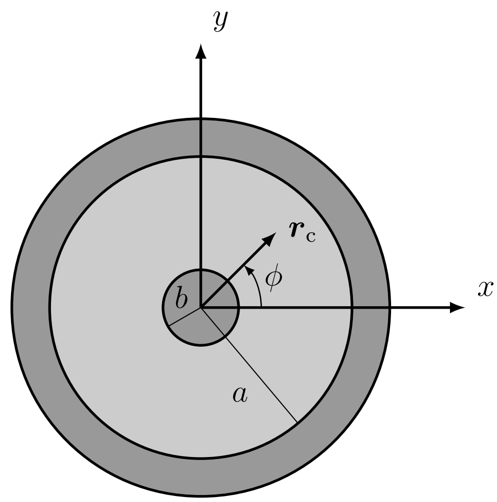


Figure 8: Geometry of the coaxial waveguide.

Similarly, the TEM-mode implies with the use of (A.2)

$$\begin{aligned}\boldsymbol{E}_{\mathrm{tTEM}}(\boldsymbol{k}_{\mathrm{t}}) &= -\int_0^{2\pi} d\phi \int_b^a \frac{\hat{\boldsymbol{r}}_{\mathrm{c}}}{2\pi r_{\mathrm{c}} \ln a/b} \mathrm{e}^{-\mathrm{i}k_{\mathrm{t}} r_{\mathrm{c}} \cos(\phi-\alpha)} r_{\mathrm{c}} \, \mathrm{d}r_{\mathrm{c}} \\ &= -\mathrm{i}\frac{\hat{\boldsymbol{k}}_t}{\ln a/b} \int_b^a J_0'(k_{\mathrm{t}} r_{\mathrm{c}}) \, \mathrm{d}r_{\mathrm{c}} = -\mathrm{i}\frac{\hat{\boldsymbol{k}}_t}{k_{\mathrm{t}} \ln a/b} \left(J_0(k_{\mathrm{t}} a) - J_0(k_{\mathrm{t}} b)\right)\end{aligned}$$

8 Conclusions

In this paper, we have presented a method for computing the radiation into a half space from a waveguide aperture in a perfectly conducting ground plane. The underlying integral equation is rewritten as a matrix equation, which is easily solved numerically by truncation. From the solution of the matrix equation, it is then straightforward to calculate the reflected power in the waveguide and the power radiated into the half space as well as the surface currents on the ground plane. Several numerical computations illustrate the performance of the method. It is possible to extend the present method to several apertures in a perfectly conducting ground plane.

Appendix A Integrals

A.1 Circular waveguide integrals

The following integrals are useful:

$$\int_0^{2\pi} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix} \begin{pmatrix} \cos m'\phi \\ \sin m'\phi \end{pmatrix} \mathrm{d}\phi = \frac{2\pi}{\epsilon_m} \left(1 - \delta_{m,0}\delta_{\sigma,\mathrm{o}}\right) \delta_{m,m'} \delta_{\sigma,\sigma'}$$
$$m, m' = 0, 1, 2, \ldots, \text{ and } \sigma, \sigma' = \mathrm{e}, \mathrm{o}$$

	Eigenmodes	Eigenvalues $k_{\mathrm{t}\ell\nu}$
TM	$v_\ell(\boldsymbol{r}_\mathrm{c}) = C_{mn} Z_{mn}(r_\mathrm{c}) \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}$	$\dfrac{\zeta_{mn}}{a}$
TE	$w_\ell(\boldsymbol{r}_\mathrm{c}) = D_{mn} Y_{mn}(r_\mathrm{c}) \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix}$	$\dfrac{\gamma_{mn}}{a}$
TEM	$\nabla_\mathrm{t}\psi(\boldsymbol{r}_\mathrm{c}) = \dfrac{\hat{\boldsymbol{r}}_\mathrm{c}}{2\pi r_\mathrm{c} \ln a/b}$	0

Table 3: Table over the normalized eigenmodes in a coaxial waveguide, see Figure 8. The functions, $Z_{mn}(r_\mathrm{c})$ and $Y_{mn}(r_\mathrm{c})$, the eigenvalues, ζ_{mn} and γ_{mn}, and the normalization constants, C_{mn} and D_{mn}, are defined in (7.1) and (7.2), respectively. The indices m, n, and σ take the values $m = 0, 1, 2, \ldots$, $n = 1, 2, 3, \ldots$, and $\sigma = \mathrm{e}, \mathrm{o}$. Note that for $m = 0$ only $\sigma = \mathrm{e}$ exists.

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k_\mathrm{t} r_\mathrm{c}\cos(\phi-\alpha)} \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} \,\mathrm{d}\phi = 2\pi \mathrm{i}^m J_m(k_\mathrm{t} r_\mathrm{c}) \begin{pmatrix}\cos m\alpha \\ \sin m\alpha\end{pmatrix} \tag{A.1}$$

Useful are also

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k_\mathrm{t} r_\mathrm{c}\cos(\phi-\alpha)} \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} \hat{\boldsymbol{r}}_\mathrm{c} \,\mathrm{d}\phi &= \frac{2\pi \mathrm{i}^{m-1}}{r_\mathrm{c}} \nabla_{\boldsymbol{k}_\mathrm{t}} J_m(k_\mathrm{t} r_\mathrm{c}) \begin{pmatrix}\cos m\alpha \\ \sin m\alpha\end{pmatrix} \\ &= 2\pi \mathrm{i}^{m-1} \left(\hat{\boldsymbol{k}}_t J_m'(k_\mathrm{t} r_\mathrm{c}) \begin{pmatrix}\cos m\alpha \\ \sin m\alpha\end{pmatrix} + \hat{\boldsymbol{\alpha}} \frac{m J_m(k_\mathrm{t} r_\mathrm{c})}{k_\mathrm{t} r_\mathrm{c}} \begin{pmatrix}-\sin m\alpha \\ \cos m\alpha\end{pmatrix} \right)\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}k_\mathrm{t} r_\mathrm{c}\cos(\phi-\alpha)} \begin{pmatrix}\cos m\phi \\ \sin m\phi\end{pmatrix} \hat{\boldsymbol{\phi}} \,\mathrm{d}\phi &= \frac{2\pi \mathrm{i}^{m-1}}{r_\mathrm{c}} \mathbf{J} \cdot \nabla_{\boldsymbol{k}_\mathrm{t}} J_m(k_\mathrm{t} r_\mathrm{c}) \begin{pmatrix}\cos m\alpha \\ \sin m\alpha\end{pmatrix} \\ &= 2\pi \mathrm{i}^{m-1} \left(\hat{\boldsymbol{\alpha}} J_m'(k_\mathrm{t} r_\mathrm{c}) \begin{pmatrix}\cos m\alpha \\ \sin m\alpha\end{pmatrix} - \hat{\boldsymbol{k}}_t \frac{m J_m(k_\mathrm{t} r_\mathrm{c})}{k_\mathrm{t} r_\mathrm{c}} \begin{pmatrix}-\sin m\alpha \\ \cos m\alpha\end{pmatrix} \right)\end{aligned} \tag{A.3}$$

$$\int Z_m(\alpha x) Y_m(\beta x) x \,\mathrm{d}x = \frac{\beta x Z_m(\alpha x) Y_{m-1}(\beta x) - \alpha x Z_{m-1}(\alpha x) Y_m(\beta x)}{\alpha^2 - \beta^2} \tag{A.4}$$

where $Z_m(z)$ and $Y_m(z)$ are two arbitrary solutions of the Bessel's differential equation. Two special cases are

$$\int_0^a J_m(\xi_{mn} x/a) J_m(\beta x) x \,\mathrm{d}x = \begin{cases} a^2 \dfrac{\xi_{mn} J_m'(\xi_{mn}) J_m(\beta a)}{\beta^2 a^2 - \xi_{mn}^2}, & \beta \neq \xi_{mn}/a \\ \dfrac{a^2}{2} \left(J_{m+1}(\xi_{mn})\right)^2, & \beta = \xi_{mn}/a \end{cases} \tag{A.5}$$

and

$$\int_0^a J_m(\eta_{mn} x/a) J_m(\beta x) x \,\mathrm{d}x = \begin{cases} a^2 \dfrac{\beta a J_m(\eta_{mn}) J_m'(\beta a)}{\eta_{mn}^2 - \beta^2 a^2}, & \beta \neq \eta_{mn}/a \\ \dfrac{a^2}{2} \left(J_m(\eta_{mn})\right)^2 \left(1 - \dfrac{m^2}{\eta_{mn}^2}\right), & \beta = \eta_{mn}/a \end{cases} \tag{A.6}$$

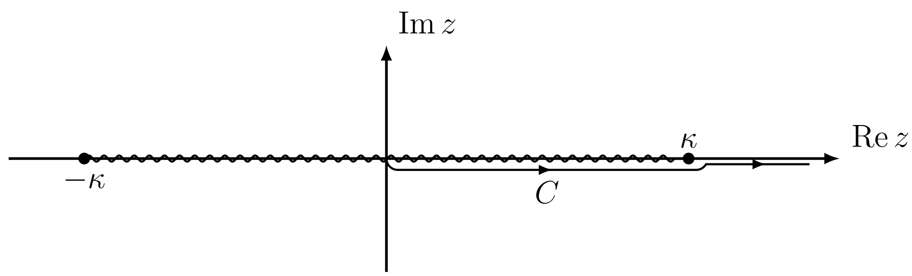


Figure 9: The integration contour C in the complex z-plane.

The entries in the matrix $A_{\ell\nu\ell'\nu'}$ contain integrals that have to be computed numerically. The following change of independent variable illuminates the square root singularity that is present in these integrals. To this end, assume $f(x)$ is a smooth function for $x \in \mathbb{R}$. For real α and β, we have

$$\begin{aligned}\int_0^\infty f(x)\frac{x\,\mathrm{d}x}{(\alpha^2 - x^2)^{1/2}} &= \int_0^\alpha f(x)\frac{x\,\mathrm{d}x}{\sqrt{\alpha^2 - x^2}} - \mathrm{i}\int_\alpha^\infty f(x)\frac{x\,\mathrm{d}x}{\sqrt{x^2-\alpha^2}} \\ &= \int_0^\alpha f(\sqrt{\alpha^2 - t^2})\,\mathrm{d}t - \mathrm{i}\int_0^\infty f(\sqrt{\alpha^2 + t^2})\,\mathrm{d}t\end{aligned}$$

A.2 Change of contour

To illustrate the effect of contour deformation, we consider the integral ($\xi_n = \xi_{1n}$, $n = 1, 2, 3, \ldots$)

$$\begin{aligned}I_1(\kappa, n, n') &= \int_C \frac{z^2\left(J_1(z)\right)^2}{(z^2-\xi_n^2)(z^2-\xi_{n'}^2)}\frac{z\,\mathrm{d}z}{(\kappa^2 - z^2)^{1/2}} \\ &= \int_C \frac{z^2 J_1(z)\left(H_1^{(1)}(z) + H_1^{(2)}(z)\right)}{2(z^2-\xi_n^2)(z^2-\xi_{n'}^2)}\frac{z\,\mathrm{d}z}{(\kappa^2 - z^2)^{1/2}}\end{aligned}$$

This integral enters in the matrix entries of $A_{\ell\mathrm{TM}\ell'\mathrm{TM}}$ for the circular waveguide. The contour C is along the real axis in the complex z-plane, and is located below the branch line between the points $z = \pm\kappa$, see Figure 9.

Deform the contour C in the upper (lower) half plane for the integral containing $H_1^{(1)}(z)$ ($H_1^{(2)}(z)$) and use $I_1(z) = -\mathrm{i}J_1(\mathrm{i}z)$, and $K_1(x) = -\pi/2H_1^{(1)}(\mathrm{i}x) = -\pi/2H_1^{(2)}(-\mathrm{i}x)$ for $x \geq 0$. We get

$$\begin{aligned}I_1(\kappa, n, n') =& \frac{2\mathrm{i}}{\pi}\int_0^\infty \frac{x^2 I_1(x)K_1(x)}{(x^2+\xi_n^2)(x^2+\xi_{n'}^2)}\frac{x\,\mathrm{d}x}{\sqrt{\kappa^2+x^2}} \\ &+ \int_0^\kappa \frac{x^2 J_1(x)H_1^{(1)}(x)}{(x^2-\xi_n^2)(x^2-\xi_{n'}^2)}\frac{x\,\mathrm{d}x}{\sqrt{\kappa^2-x^2}} - \delta_{nn'}\frac{\pi}{4}J_1'(\xi_n)Y_1(\xi_n)\frac{\xi_n}{\sqrt{\kappa^2-\xi_n^2}}\end{aligned}$$

Appendix B Power flow in the half space

The total power flow (average over one period) that passes the interface $z = z_1$ is

$$P_\mathrm{s} = \iint\limits_{\Omega} \hat{\boldsymbol{z}} \cdot <\boldsymbol{S}(t)> \,\mathrm{d}x\,\mathrm{d}y = -\frac{1}{2}\,\mathrm{Re} \iint\limits_{\Omega} \boldsymbol{E}_{xy}(\boldsymbol{r}) \cdot \left(\mathbf{J} \cdot \boldsymbol{H}_{xy}^*(\boldsymbol{r})\right)\big|_{z=z_1} \,\mathrm{d}x\,\mathrm{d}y$$

By the use of the Parseval's identity (or use (3.1)) we get

$$\begin{aligned} P_\mathrm{s} &= -\frac{1}{8\pi^2}\,\mathrm{Re} \iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}, z_1) \cdot \mathbf{J} \cdot \boldsymbol{H}_{xy}^*(\boldsymbol{k}_\mathrm{t}, z_1)\,\mathrm{d}k_x\,\mathrm{d}k_y \\ &= \frac{1}{8\pi^2 \eta_0}\,\mathrm{Re} \iint\limits_{\mathbb{R}^2} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})\mathrm{e}^{\mathrm{i}k_z z_1} \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t})^* \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t})\mathrm{e}^{-\mathrm{i}k_z^* z_1}\,\mathrm{d}k_x\,\mathrm{d}k_y \end{aligned}$$

Divide the domain of integration into $|\boldsymbol{k}_\mathrm{t}| \leq k_0$ and $|\boldsymbol{k}_\mathrm{t}| > k_0$.

$$P_\mathrm{s} = \frac{1}{8\pi^2 \eta_0}\,\mathrm{Re} \iint\limits_{|\boldsymbol{k}_\mathrm{t}| \leq k_0} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y + I$$

where

$$\begin{aligned} I = \frac{1}{8\pi^2 \eta_0}\,\mathrm{Re} \iint\limits_{|\boldsymbol{k}_\mathrm{t}| > k_0} &\frac{\mathrm{i}\sqrt{k_\mathrm{t}^2 - k_0^2}}{k_0} \mathrm{e}^{-2\sqrt{k_\mathrm{t}^2 - k_0^2} z_1} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \\ &\cdot \left(\frac{\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}}{k_0^2 - k_\mathrm{t}^2} + \mathbf{I}_2\right) \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y = 0 \end{aligned}$$

since $\boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \cdot (\boldsymbol{k}_\mathrm{t}\boldsymbol{k}_\mathrm{t}/k_z^2 + \mathbf{I}_2) \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t})$ is a real quantity. Finally we have

$$P_\mathrm{s} = \frac{1}{8\pi^2 \eta_0}\,\mathrm{Re} \iint\limits_{|\boldsymbol{k}_\mathrm{t}| \leq k_0} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t})\,\mathrm{d}k_x\,\mathrm{d}k_y \tag{B.1}$$

which is independent of the plane, $z = z_1$, at which the integral is evaluated on.

Finally, we prove that this expression, (B.1), of the radiated power is identical to the radiated power over a half sphere of the far-field amplitude $\boldsymbol{F}(\hat{\boldsymbol{r}})$. The radiated power into the half space $z > 0$ is, see (3.4) and (3.5)

$$\begin{aligned} P_\mathrm{s} &= \iint\limits_{\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} > 0} \hat{\boldsymbol{r}} \cdot <\boldsymbol{S}(t)> \,\mathrm{d}S \\ &= \frac{k_0^2}{8\pi^2 \eta_0} \int_0^{2\pi} \int_0^{\pi/2} \left|\hat{\boldsymbol{r}} \times \left[\mathbf{J} \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t} = k_0 \sin\theta \hat{\boldsymbol{r}}_\mathrm{c})\right]\right|^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \end{aligned}$$

This integral is of the form

$$I = \int_0^{2\pi} \int_0^{\pi/2} f(k_0 \sin\theta \hat{\boldsymbol{r}}_\mathrm{c}) \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

which we transform into ($\boldsymbol{k}_\mathrm{t} = k_0 \sin\theta \hat{\boldsymbol{r}}_\mathrm{c}$)

$$I = \iint\limits_{|\boldsymbol{k}_\mathrm{t}|\leq k_0} f(\boldsymbol{k}_\mathrm{t}) \frac{\mathrm{d}k_x \,\mathrm{d}k_y}{k_0\sqrt{k_0^2 - k_\mathrm{t}^2}}$$

We can now write the radiated power into the half space $z > 0$ as

$$P_\mathrm{s} = \frac{k_0^2}{8\pi^2\eta_0} \iint\limits_{|\boldsymbol{k}_\mathrm{t}|\leq k_0} \left|\hat{\boldsymbol{k}}_t \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})\right|^2 + \left|\frac{\sqrt{k_0^2 - k_\mathrm{t}^2}}{k_0} \hat{\boldsymbol{k}}_t \cdot \mathbf{J} \cdot \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t})\right|^2 \frac{\mathrm{d}k_x \,\mathrm{d}k_y}{k_0\sqrt{k_0^2 - k_\mathrm{t}^2}}$$

where we used the decomposition

$$|\hat{\boldsymbol{r}} \times [\mathbf{J} \cdot \boldsymbol{E}_{xy}]|^2 = |\boldsymbol{E}_{xy}|^2 - \sin^2\theta \left|\hat{\boldsymbol{\phi}} \cdot \boldsymbol{E}_{xy}\right|^2 = |\hat{\boldsymbol{r}}_\mathrm{c} \cdot \boldsymbol{E}_{xy}|^2 + \cos^2\theta \left|\hat{\boldsymbol{\phi}} \cdot \boldsymbol{E}_{xy}\right|^2$$

The use of (3.3) implies

$$P_\mathrm{s} = \frac{1}{8\pi^2\eta_0} \operatorname{Re} \iint\limits_{|\boldsymbol{k}_\mathrm{t}|\leq k_0} \boldsymbol{E}_{xy}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{\gamma}(\boldsymbol{k}_\mathrm{t}) \cdot \boldsymbol{E}_{xy}^*(\boldsymbol{k}_\mathrm{t}) \,\mathrm{d}k_x \,\mathrm{d}k_y$$

which is identical to the expression above.

References

[1] R. E. Collin. *Field Theory of Guided Waves.* IEEE Press, New York, NY, second edition, 1991.

[2] R. E. Collin. *Antennas and Radiowave Propagation.* McGraw-Hill, New York, NY, 1985.

[3] D. S. Jones. *Methods in Electromagnetic Wave Propagation.* IEEE Press, Piscataway, NJ, second edition, 1994.

[4] G. A. Kriegsmann. The flanged waveguide antenna: Discrete reciprocity and conservation. *Wave Motion*, **29**(1), 81–95, 1999.

[5] S. Rikte, G. Kristensson, and M. Andersson. Propagation in bianisotropic media—reflection and transmission. Technical Report LUTEDX/(TEAT-7067)/1–31/(1998), Lund Institute of Technology, Department of Electromagnetic Theory, P.O. Box 118, S-221 00 Lund, Sweden, 1998.

[6] G. N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, second edition, 1966.