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SYNOPSIS

We describe a new technique for automatic identification of stationary, linear systems with a single output. This class of models includes all linear, time-invariant, stochastic, difference equations driven by arbitrary inputs and having stationary, normal disturbances with rational spectra.

The parameters of the model are estimated by the method of maximum likelihood. A numerical algorithm for solving the likelihood equations is presented. The algorithm is essentially a modified Newton-Raphson algorithm, which takes advantage of the particular structure of the problem.

Conditions for consistency and asymptotic efficiency of the estimates are given for increasing sample length. It is shown that these properties are exclusively determined by the information matrix. An estimate of the latter is obtained without additional computations. The information matrix also yields an estimate of the accuracy of the estimates in each case.

The approach has been tested on artificially generated input/output data. It is also immediately applicable to power spectrum analysis of time series, having advantages over ordinary non-parametric methods in that it always gives a non-negative estimate without the problems of trend removal and of the choice of spectral windows turning up.

The basic idea can be extended to larger classes of systems. Also the identification is easily done recursively, which implies that the method is well suited for real time modelling.

Locator Terms for the IBM Subject Index

Identification, analyzing
Mathematics, models, synthesizing, automatic
Equations of state, dynamic, stochastic disturbances
Estimating parameters
Process control, computer

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table of contents

INTRODUCTION

Several powerful theoretical methods have been developed during the last decade for the synthesis of control systems. Many of these techniques, based on realistic assumptions, make it possible to include random disturbances, measurement errors, etc. The synthesis can easily be automatized using a digital computer, and can easily handle multivariable systems. The control algorithms obtained from these techniques also contain the class of algorithms, such as those obtained from three-term controllers (PID regulators), presently in use in the process industries. For example, the linear stochastic control theory (LSCT) leads to a regulator which can be interpreted as a multivariable, linear, time-varying network. This includes a set of interconnected three term controllers, as a special case.

It seems very attractive to apply linear stochastic control theory to obtain an algorithm which in turn can be put into a digital computer and used as a real time process controller. As the algorithm obtained by a number of interconnected PID regulators is included in the algorithm produced by the linear stochastic control theory, it is always possible to obtain a system which is equal to or better than what can be achieved by a collection of PID regulators.

To apply synthesis techniques such as those obtained from linear stochastic control theory, we need a mathematical model of the process and its environment. Such models might be constructed from first principles, as is generally the case with mechanical, electrical, and space technology systems, or the process might be identified directly from measurements of the relevant inputs and outputs.

When trying to identify an industrial process like paper, steel or cement-making from first principles, one meets the obstacle that such principles often do not exist, or at best are dubious. Knowledge is lacking about the processes, and at present, intuition acts as a substitute. Even if the structure of the equations governing the process can be found, the parameters of these

equations are often not known. Also, when a model of the process is found, the problem of describing the transducers and measuring equipment remains.

In this report we will describe a technique for numerical identification of a process using operating data. This technique attempts to represent the observed input/output relation as a single input, single output, linear dynamical system with stationary normal disturbances and measurement errors having rational spectra. Such a system can be described by a finite number of parameters, and the identification problem can then be regarded as a parameter estimation problem. The parameters are estimated using the method of maximum likelihood, and the essence of the method is a numerical algorithm for maximizing the likelihood function.

The identification algorithm requires samples of the input and the output and gives estimates of:

- The dynamics of the system including transducers and measuring equipment.
- The character (spectrum) of all disturbances affecting the output.
- The measurement errors, if they are uncorrelated.
- The accuracy of the estimates.

Let us exemplify with the dryer section of a paper machine in which the input steam pressure in a number of steam heated cylinders are manipulated to influence the moisture content of the dried paper. If the input steam pressure is recorded and the output moisture is measured, then it should be possible to obtain:

- The time constants of the heating cylinders.
- The spectrum and variance of the ingoing moisture + air moisture + any other disturbances.

With this information available it is possible to construct an optimal strategy for controlling output moisture through steam pressure, using

linear stochastic control theory. The strategy is optimal in the sense that it minimizes the output variance for all possible strategies. The numerical identification technique has been tested on artificially generated time series. The technique gives estimates which are efficient for long series. The estimates of the parameters will also converge to true values as the sample length increases.

There are two reasons for choosing our particular model:

- It is linear, which implies that it is possible to attack mathematically, with some generality.
- The linear stochastic control problem has previously been solved for this particular model, and computer programs for computation of the optimal control strategies are available [3].

In practice, the model should cover all stationary processes which are not greatly disturbed. Hence, by solving this identification problem, we are closing the chain data logging, mathematical model, control strategy. In this way, much of the otherwise tedious manual work can be done by computers.

The restrictive assumptions of our approach are as follows:

- The model is linear and invariant in time.
- The model has a single input and a single output.
- The disturbances at the output are considered normal, stationary, and with rational spectra.
- The process is sampled with a fixed sampling rate.

There are two fundamental aspects to regard when discussing the significance of these restrictions.

- Concerning the algorithm: to what extent are the restrictions essential for solving the problem?

- Concerning the process: to what extent are the restrictions essential for applying the algorithm to actual physical processes?

The assumptions of time invariance and linearity of the model, and of stationary disturbances with rational spectra imply a finite-dimensional parameter space, which is essential for the algorithm. These assumptions can be relaxed in several directions to give similar algorithms for other processes. Time-varying systems and nonlinear systems with a given structure that are characterized by a finite number of parameters as well as nonlinear systems with a given structure can be treated in the same way. The crucial restriction is the demand that the independent random component $e(t)$ driving the model be expressed as a function of the input record $\{u(t'), t' = 1, \ldots, t\}$, the output record $\{y(t'), t' = 1, \ldots, t\}$, and the parameters θ for every t. The condition of fixed sampling rate can be relaxed. Several inputs are easily treated. The assumption of a single output makes it possible to choose a canonical form for representing the process. The problem can be extended to multiple-output linear systems if a suitable canonical form can be found. The assumption of normality is essential, since it makes it simple to write down the likelihood function explicitly.

In theory, it is required that the process studied be linear and the disturbances gaussian. In practice, it should be possible to estimate a linearized model of a nonlinear process that operates sufficiently close to its equilibrium. If the equilibrium is slowly drifting, e.g. due to seasonal variations, the identification must naturally be repeated periodically. With regards to the assumption of normality, let it suffice to mention that the algorithm has been successfully used to identify simulated linear systems with rectangularly distributed disturbances.

The identification algorithm is intrinsically capable of estimating built-in transportation lags as multiples of the sampling interval. However, more accurate estimates can be obtained if we are able to incorporate the a priori knowledge (or anticipation) that a pure transportation lag is present. It is then advantageous to repeat the identification for systematically shifted

input data and compare the minimal losses (likelihood functions). In this way the process can generally be identified as a system of lower order than would be possible otherwise. The statistical aspects of this procedure have not yet been investigated.

The assumption of linearity and time invariance generally prevents the application of the identification algorithm to batch processes, since here the state of the process varies over a wide range, and most processes can be considered linear only in a small region. It should be possible to identify batch processes that are indeed linear over the whole range. However, in such cases the estimator is not efficient. A statement of the problem is given in section 2. It is of interest to observe that the formulation of the problem includes regression analysis, identification of linear noise-free input/output systems, determination of parameters in moving averages and autoregressive processes, and time series analysis as special cases. In section 2, we also give a description of the most general model of the desired structure as a descrete-time, stationary, dynamical system driven by a sequence of independent random variables. Alternative representations of the dynamical system are also discussed. Naturally all representations contain the same number of parameters and can be transformed from one to the other by changing parameters.

In sections 3 and 4, we derive an expression for the likelihood function and its partial derivatives. The derivations of sections 3 and 4 are essentially the same, the only difference is that in section 3 the difference equations representing the process are kept as an implicit relation, but in section 4, this relation is eliminated explicitly. In fact, the identification algorithms were derived independently by the authors starting from different viewpoints and with different mathematical means (difference equations and matrix algebra respectively). It gradually turned out that the two methods of solution were merely two aspects of the same method, indicated by the fact that the algorithms were similar and that exactly the same computational tricks could be utilized in both. Even the number of multiplications and the amount of necessary storage space coincided. The remaining difference is trivial and consists of the fact that the likelihood functions are minimized with respect to different parameters, the transformation

between the two sets of parameters being simple. However, both methods have been included in order to give additional insight into the problem. They represent two different viewpoints, each having advantages for certain aspects of the problem, and we hope that the reader will benefit from both. The authors certainly did.

The computation of the likelihood function and its derivatives can be interpreted as the solution of a set of difference equations all having the same homogeneous part but driven by inputs and observations in different ways.

In section 5, we give the conditions for consistency and asymptotic efficiency of the estimator for large samples. To show consistency, we must assume that the system is stable and that the z-transform representation of the disturbances have no poles and no seros on the unit circle.

The proofs of the results are collected in appendix A. They are extensions of the well-known results of Wald and Cramér on the consistency and asymptotic normality of the maximum likelihood estimate for independent samples from the same distribution. In our case the observations are dependent random variables. In order to extend Wald's and Cramér's proofs, it is necessary to show almost certain convergence of the likelihood function and the asymptotic normality of the partial derivatives of the likelihood function.

Consistency is shown to be determined exclusively by the so called information matrix, for which an estimate is also found. This estimate is obtained as a by-product of the identification algorithm. Thus, it will be possible to determine whether a computed estimate is actually consistent and efficient in each case. The information matrix also yields the covariances of the estimates so that estimates of the accuracy of the parameters are obtained in each case. It is also necessary to have some restrictions on the input signals. The essential factor is to guarantee that the system is properly excited by the inputs. A simple criterion for this is also given.

In section 6, we discuss the numerical algorithm for maximizing the likelihood function. Essentially the difficulty is to find the absolute maxi-

mum of the likelihood function. There is no guaranteed way of doing this. The algorithm used is a Newton-Raphson algorithm which will converge to a local maximum under suitable regularity conditions. It can, however, be shown that at least asymptotically there are not two local, absolute maxima. There might, however, still be several local maxima of lower magnitude. This must be investigated using different starting points for the Newton-Raphson technique.

If the spectrum of the disturbances is known, the likelihood function is convex and the algorithm converges in one step, (Markov estimates). The choice of starting values for the Newton-Raphson technique is also discussed. It is found that in many cases, the starting values can be chosen as zero.

Several computer programs have been developed: A typical IBM 1401 FORTRAN program is listed in section 6. This program has been used to test the identification algorithm and is practical (with respect to computing time) for first and second order systems having 7 and 11 parameters respectively, possibly third order systems having 14 parameters. Computing time is then about an hour for 100 pairs of observations. In section 7 we give some numerical examples for the identification of artificially generated processes, together with estimated accuracy. The true and estimated spectra of disturbances are shown by diagrams.

Our conclusions are given in section 8, where we also discuss the relation of our identification scheme to others e.g. the model reference technique. The application of our algorithm to time series analysis is also briefly covered in that section.

In section 9 we give some notes and references.

STATEMENT OF THE PROBLEM

A sequence of inputs $\{u(t),\ t = 1, 2, \ldots, N\}$ are given and a sequence of corresponding outputs $\{y(t),\ t = 1, 2, \ldots, N\}$ for a dynamical system have been observed. The problem is to find a representation of the observed input/output relation with the form

$$y(t) + \alpha_1 y(t-1) + \ldots + \alpha_n y(t-n) \tag{2.1}$$

$$= \beta_o u(t) + \beta_1 u(t-1) + \ldots + \beta_n u(t-n) + \lambda[e(t) + \gamma_1 e(t-1) + \ldots + \gamma_n e(t-n)] + \varkappa$$

where $\{e(t),\ t = \ldots, -1, 0, 1, \ldots\}$ is a sequence of independent normal random variables $N(0, 1)$. Introducing the translation operator z defined by

$$x(t+1) = z\,x(t) \tag{2.2}$$

(See e.g. Fröberg [27]) we find that the equation (2.1) can be written as

$$y(t) = \frac{\beta_o z^n + \beta_1 z^{n-1} + \beta_n}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n}\, u(t) + \lambda \frac{z^n + \gamma_1 z^{n-1} + \ldots + \gamma_n}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n}\, e(t) + k \tag{2.3}$$

where

$$k = \frac{\varkappa}{1 + \alpha_1 + \alpha_2 + \ldots + \alpha_n} \tag{2.4}$$

The rational functions of the right member of the equation have also interpretations as pulse transfer functions. See Ragazzini and Franklin [68 p. 66-69] and Zadeh and Desoer [89].

The reasons that disturbances have been included is that they are always present in a practical problem and that a description of their character is required to design a controller for the system.

There are many ways in which disturbances can enter the system, as measurement errors in input and output, as inputs which are not

measurable, as disturbances generated inside the system, etc. As we are only considering linear models, the disturbances can always be transformed so as to appear as disturbances entering at the output. The model structure given by (2.1) is equivalent to the assumption that the disturbances are stationary random processes whose spectral density functions are rational (functions of exp $i\omega$). The model structure of (2.1) is in fact the general representation of a finite-dimensional, completely controllable, completely observable, single input/single output system with arbitrary disturbances in terms of stationary, gaussian, random processes with rational spectral densities. To recognize this we can argue as follows. As the system is linear, we can use the principle of superposition, and consider the influence of inputs and disturbances separately. Assuming no disturbances the input/output relation of a finite dimensional, completely controllable and completely observable linear system can always be written as

$$y_1(t) = \frac{P_1(z)}{Q_1(z)} \; u(t) \tag{2.5}$$

where $P_1(z)$ and $Q_1(z)$ are polynomials in the translation operator z with no common factors. See Kalman [47]. Using linearity we can reduce all disturbances to an equivalent disturbance at the output. When calculating the spectral density function of this equivalent disturbance we use the well-known rules for transformations of spectral densities when a random signal is passed through a linear system [68]. Let $\varphi(\omega)$ be the spectral density of the equivalent disturbance. The original disturbances have rational spectra and the system is finite dimensional, the function $\varphi(\omega)$ will thus be rational in exp $i\omega$

Introduce the function $\Psi(z)$ defined by

$$\Psi(\exp i\omega) = \varphi(\omega) \tag{2.6}$$

The function $\Psi(z)$ can always be written as

$$\Psi(z) = \frac{P_2(z)}{Q_2(z)} \qquad \frac{P_2(z^{-1})}{Q_2(z^{-1})} \tag{2.7}$$

where $P_2(z)$ and $Q_2(z)$ are polynomials. See Doob [19 p. 501-502]. We assume that the polynomials $P_2(z)$ and $Q_2(z)$ have all zeros strictly inside the unit circle. Using the representation theorem the equivalent disturbance can be represented as

$$y_2(t) = \frac{P_2(z)}{Q_2(z)} \quad e(t) \tag{2.8}$$

where $\{e(t),\ t = \ldots, -1, 0, 1, \ldots\}$ is a sequence of independent normal $N(0, 1)$ random variables. Using superposition, adding the influence of the input, the disturbances, and a constant, we get

$$y(t) = y_1(t) + y_2(t) + k$$

$$y(t) = \frac{P_1(z)}{Q_1(z)}\, u(t) + \frac{P_2(z)}{Q_2(z)}\, e(t) + k$$

$$= \frac{P_1(z)\, Q_3(z)}{Q(z)}\, u(t) + \frac{P_2(z)\, Q_4(z)}{Q(z)}\, e(t) + k$$

$$= \frac{P_3(z)}{Q(z)}\, u(t) + \frac{P_4(z)}{Q(z)}\, e(t) + k \tag{2.9}$$

which is of the same form as (2.1).

In this expression $Q(z)$ is the least common denominator of $Q_1(z)$ and $Q_2(z)$

$$Q(z) = Q_1(z) \cdot Q_3(z) = Q_2(z) \cdot Q_4(z) \tag{2.10}$$

Notice, there is no loss in generality in assuming that the coefficient of $y(t)$ is one, and that λ is a common factor of all $e(t)$. The last statement follows from the fact that $P_2(z)$ of (2.8) can be multiplied with z without altering the spectral density function $\varphi(\omega)$. Also notice that we cannot assume that anyone of the β_i:s is nonzero without losing generality.

From the above discussion, it also follows that the dynamics represented by the polynomial $Q(z)$ is due partly to the system dynamics and partly to the representation of the disturbances. An investigation of common factors of $P_3(z)$, $P_4(z)$ and $Q(z)$ will separate one part from the other. An analysis

of the polynomials will thus give interesting information about possible structures for the system. For example, if $P_4(z)$ and $Q(z)$ have all zeros in common we can immediately conclude that a possible system structure is a noise free system with independent measurement errors.

The equation (2.1) contains 4n + 3 parameters: the coefficients α_1, α_2, ..., α_n, β_o, β_1, ..., β_n, γ_1, γ_2, ..., γ_n, λ, $\varkappa$ and n initial conditions. The identification problem can be stated as follows.

PROBLEM

Given observations of the inputs $\{u(t),\ t = 1, 2, \ldots, N\}$ and the outputs $\{y(t),\ t = 1, 2, \ldots, N\}$ find an estimate of the 4n + 3 parameters of (2.1).

Special cases of this problem have been solved previously; for n = 0 we have

$$y(t) = \varkappa + \beta_o u(t) + \lambda e(t)$$

that is a zero order model, a model without dynamics, or a regression model. From this point of view the model (2.1) can thus be regarded as a natural extension of a regression model to include dynamics i.e. a dynamic regression. The parameter n should actually be estimated. In practice we will do this by repeating the estimation for different values of n and observing the minimal values of the likelihood functions. As we start this procedure with n = 0, we will always first try a regression model. The problem of developing tests for the order n has not yet been considered.

Associated problems and special cases have been considered in literature.

1. $\alpha_1 = \ldots = \alpha_n = 0$, $\beta_1 = \ldots = \beta_n = 0$, $\gamma_1 = \ldots = \gamma_n = 0$ is the ordinary case of regression analysis [84].

2. $\beta_o = \ldots = \beta_n = 0$, $\gamma_1 = \ldots = \gamma_n = 0$ is the problem of identifying an autoregressive series, and has been considered by many authors, e.g.

Mann and Wald [59].

3. $\alpha_1 = \ldots = \alpha_n = 0$, $\beta_o = \ldots = \beta_n = 0$ is the problem of identifying a moving average, considered by Whittle [87], Walker [78], and Durbin [20], etc. Whittle derives the Maximum Likelihood equations for the large sample case ($N \to \infty$) and proposes a method to solve them. Walker reconsiders the problem and discards Whittle's method because of computational difficulties. He derives an iterative scheme for identifying the autocovariance function of the moving average.

4. $\beta_o = \ldots = \beta_n = 0$ is equivalent to parametric estimation of a rational power spectrum of a stationary stochastic process and has been considered by Durbin [22]. The case of non-parametric spectral estimation is treated in extensive literature, see e.g. Blackman and Tukey [15]. Durbin developes his method in three papers [20], [21], [22]. Its essence is an iterative method of computing approximate maximum likelihood estimates of the parameters of a rational spectrum of fixed order from the sample covariances. His results suggest that only the first few sample covariances (i.e. a fixed number k as $N \to \infty$) are needed to obtain efficient estimation. Thus, less data has to be stored ($k << N$) after the initial calculation of the sample covariances. However, his method cannot be immediately extended to include inputs ($\beta_i \neq 0$). Estimators based on sample covariances have also been designed by Åström [4] for the special case of 4) $\gamma_i = \alpha_i$, $i = 1, \ldots, n$, i.e. an autoregressive series with independent measurement errors, or equivalently, a rational spectrum consisting of an inverse trigonometric polynomial plus a constant term. This estimator is based on a quadratic loss function.

5. $\gamma_1 = \ldots = \gamma_n = 0$ is the case of no measurement errors and was studied by Kalman [46] and Bigelow and Ruge [13]. It leads to a least squares estimate.

6. $\alpha_i = \gamma_i$, $i = 1, \ldots, n$ corresponds to a noise-free process, whose output is corrupted by independent measurement errors, and has been studied by Levin [53]. His estimates are, however, not maximum likelihood estimates and not shown to be asymptotically efficient.

The general case has been studied by Galtieri [28] [29]. He derives the maximum likelihood equations and obtains Levin's and Kalman's estimates for a noise-free system as simplified cases. The complete maximum likelihood estimate is rejected due to the "intolerable" amount of computations involved. Instead he proposes a Bayesian approach, i.e. a conditional expectation of the parameters with respect to the observed output and an a priori parameter distribution (which is taken to be one for a stable system and zero for an unstable one). However, for a large number of parameters, multi-dimensional integrals with integrands depending on the input and output data have to be evaluated numerically.

A second method of Galtieri's is a recursive estimation method, where he derives relations between the a posteriori probability functions for the parameters, given an increasing number of data (on line identification). The probability distributions can then be used to obtain Baye's estimates of the parameters of every sampling instant. The same computational difficulties must arise, however.

General aspects and particular cases of model building (linear and non-linear), using empirical data, have been presented by many authors. A clear and simple presentation is given by Box [17]. General solutions to estimation problems are given by the Bayesian estimates, e.g. Maslov [60].

It should be pointed out that the problem stated does not include estimation of linear functional relations between the expected values of inputs and outputs, see e.g. Williams, [84 Chapter 11], Kendall and Stuart [48 Chapter 29], Madansky [58].

The results for those special cases as well as analyses of other simple examples make it clear that we cannot hope to obtain a closed-form solution of the problem. Instead we will consider the problem solved if we can find an efficient computational algorithm.

To solve the problem we will regard it as a statistical parameter-estimation problem. The method of maximum likelihood will be used. As the system equations are linear and the random variables normal, an explicit equation for the likelihood equation can be obtained. The essence of the method is then a good numerical technique to maximize the likelihood function.

Before proceeding with the identification problem we will discuss some alternative representations for the systems described by equation (2.1). These representations are useful when the results of the identification are to be applied and/or interpreted.

State Space Representation I.

After some calculations, we find the following equivalent representation of input/output relationship given by equation (2.1).

$$z(t+1) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & 0 & & 0 & -a_{n-1} \\ 0 & 1 & 0 & & 0 & -a_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & & 1 & -a_1 \end{bmatrix} z(t) + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ b_1 \end{bmatrix} u(t) + \begin{bmatrix} c_n \\ c_{n-1} \\ c_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ c_1 \end{bmatrix} e(t)$$

$$y(t) = z_n(t) + b_o u(t) + c_o e(t) + k \qquad (2.11)$$

where $z(t)$ is an n-dimensional state-vector and

$$\begin{cases} a_i = \alpha_i \\ b_i = \beta_i - \beta_o \cdot \alpha_i \\ c_i = \lambda(\gamma_i - \alpha_i) \\ b_o = \beta_o \\ c_o = \lambda \\ k = \dfrac{\varkappa}{1 + \alpha_1 + \dots + \alpha_n} \end{cases} \qquad (2.12)$$

This representation is useful when the results of the identification is to be used for the design of a control system because it is in a standard form suitable for many design schemes. See e.g. [3] [75]. A block diagram of the system is shown in Figure 1.

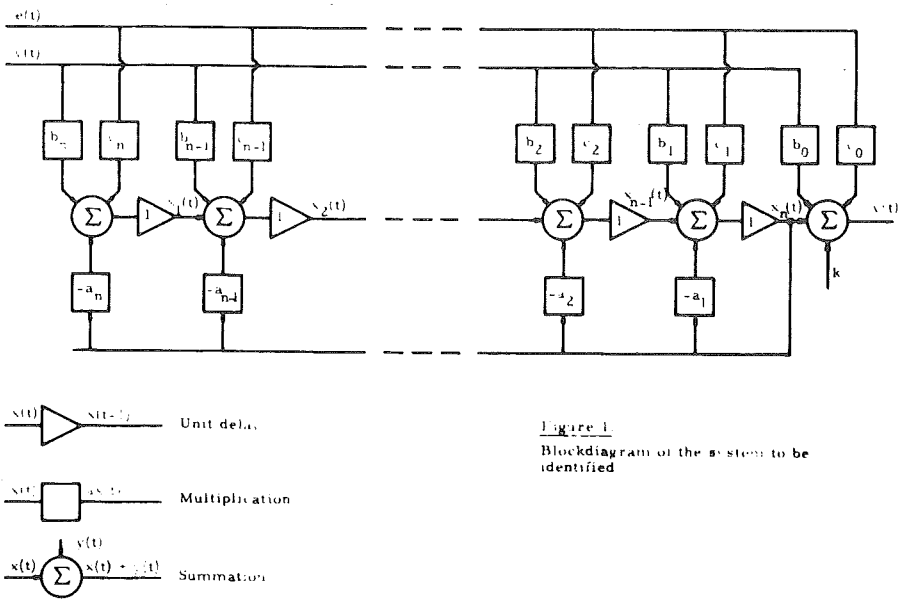


Figure 1
Blockdiagram of the system to be identified

In (2.10) there is linear dependence between the disturbances representing the measurement errors and the disturbances forcing the system. In certain situations it is known a priori that the measurement errors are independent of the other disturbances in the system. In such a case it is of interest to have a representation which reflects this a priori knowledge. When deriving such a representation, we will again use the principle of superposition, and consider the influence of the random disturbances $\{ e(t) \}$ on the output, separately. The spectral density of the disturbances is given by the function $\Psi(z)$ of equation (2.7). We rewrite $\Psi(z)$ as

$$\Psi(z) = d_o^2 + \frac{R(z)}{Q(z)} \cdot \frac{R(z^{-1})}{Q(z^{-1})} \tag{2.13}$$

where

$$\begin{cases} R(z) = d_1 z^{n-1} + d_2 z^{n-2} + \ldots + d_n \\ Q(z) = z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n \end{cases} \tag{2.14}$$

we then obtain the following representation of the input/output relation given by equation (2.1)

$$z(t+1) = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & 0 & & 0 & -a_{n-1} \\ 0 & 1 & 0 & & 0 & -a_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & & 1 & -a_1 \end{bmatrix} z(t) + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ b_1 \end{bmatrix} u(t) + \begin{bmatrix} d_n \\ d_{n-1} \\ d_{n-2} \\ \cdot \\ \cdot \\ \cdot \\ d_1 \end{bmatrix} e(t)$$

$$y(t) = z_n(t) + b_o u(t) + d_o v(t) + k \tag{2.15}$$

where z(t) is an n-dimensional state vector and $\{ e(t) \}$ and $\{ v(t) \}$ are sequences of independent normal random variables N(0, 1). The coefficients d_i are related to the coefficients (2.10) by the equations

$$\begin{cases} d_o^2 \alpha_n = \lambda^2 \cdot \gamma_n \\ d_o^2 (\alpha_{n-1} + \alpha_1 \alpha_n) + d_1 d_n = \lambda^2 (\gamma_{n-1} + \gamma_1 \gamma_n) \\ \cdot \\ \cdot \\ \cdot \\ d_o^2 (1 + \alpha_1^2 + \ldots + \alpha_n^2) + d_1^2 + d_2^2 + \ldots + d_n^2 = \lambda^2 (1 + \gamma_1^2 + \ldots + \gamma_n^2) \end{cases} \tag{2.16}$$

Even if the computation of all the components $d_1, d_2, \ldots, d_n$ involves the solution of algebraic equations, we observe that the computation of d_o^2, which represents the measurement error, is wieldy. Notice also that equation (2.16) does not always have a real solution.

MAXIMUM LIKELIHOOD ESTIMATE I

We will now carry out the calculation of the maximum likelihood estimate in detail. To do this we will find an expression for the probability density function of the observations $\{y(t),\ t = 1, 2, \ldots, N\}$ as function of the inputs $\{u(t),\ t = 1, 2, \ldots, N\}$ and the parameters. For this purpose we express the variables $e(t)$ of equation (2.1) as functions of the observations. Exploiting the symmetry of equation (2.1) and the state-space representation (2.14) we find the following relation

$$\begin{aligned} x(t+1) &= \Phi x(t) + \Gamma u(t) + \Delta[y(t) - k] \\ \epsilon(t) &= c_o e(t) = x_n(t) - b_o u(t) + y(t) - k \end{aligned} \tag{3.1}$$

where x in an n-dimensional state vector and

$$\Phi = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -\gamma_n \\ 1 & 0 & 0 & & 0 & -\gamma_{n-1} \\ 0 & 1 & 0 & & 0 & -\gamma_{n-2} \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & & 1 & -\gamma_1 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} -\beta_n & + \beta_o\gamma_n \\ -\beta_{n-1} & + \beta_o\gamma_{n-1} \\ \cdot & \\ \cdot & \\ \cdot & \\ -\beta_1 & + \beta_o\gamma_1 \end{bmatrix} \qquad \Delta = \begin{bmatrix} \alpha_n & - \gamma_n \\ \alpha_{n-1} & - \gamma_{n-1} \\ \cdot & \\ \cdot & \\ \cdot & \\ \alpha_1 & - \gamma_1 \end{bmatrix} \tag{3.2}$$

The variables ϵ are thus independent normal $N(0, c_o)$ random variables. Let L be the logarithm of the likelihood function and we get

$$L = \frac{1}{2c_o^2} \sum_{t=1}^{N} \epsilon^2(t) + N \log c_o + \frac{N}{2} \log 2\pi \tag{3.3}$$

Notice that x(t) of the dynamical system (3.1) can be interpreted physically as the maximum likelihood estimate of z(t), the state of the system (2.11), at time t given the observations y(1), y(2), . . . , y(N).

To find the maximum likelihood estimate of the 4n + 3 parameters, we will have to find the minimum of the function (3.3) where $\{\epsilon(t)\}$ is implicitly related to the observations $\{y(t)\}$ and $\{u(t)\}$ through equation (3.1). Instead of eliminating $\epsilon(t)$ we will retain (3.1) as an implicit relation. We notice that the likelihood function L has continuous partial derivatives of all orders for $c_o \neq 0$ and that the minimum is finite. The gradient of the function will thus vanish at the minimum. Notice, however, that the minimum is not necessarily unique. To simplify the notations we introduce the parameter vector θ whose components are defined by

$$\begin{cases} \theta_i = \gamma_i & i = 1, 2, \ldots, n \\ \theta_{n+i} = -\beta_i + \beta_o \gamma_i & i = 1, 2, \ldots, n \\ \theta_{2n+i} = \alpha_i - \gamma_i & i = 1, 2, \ldots, n \\ \theta_{3n+i} = x_i(1) & i = 1, 2, \ldots, n \\ \theta_{4n+1} = b_o \\ \theta_{4n+2} = k \\ \theta_{4n+3} = c_o = \gamma \end{cases} \tag{3.4}$$

There is nothing special in this choice of parameters. The following analysis can be carried out in exactly the same way for other choices.

The solution of the identification problem is thus reduced to that of finding the maximum of a function of 4n+3 parameters. There are a large number of techniques ranging from simple search techniques to elaborate gradient routines for solving this problem. The method to be described utilizes the structure of this particular problem. It is a gradient technique where fast convergence is obtained through the use of second partial derivatives, equivalently a Newton-Raphson method applied to the likelihood equations. See Kendall [48, sect. 18.21]. As with all procedures of this type, we

may have the difficulty of multiple maxima. We first notice that the sum $\Sigma \epsilon^2(t)$ only depends on the parameters $\theta_1, \ldots, \theta_{4n+2}$ and we can thus maximize the likelihood function separately with respect to these parameters. To do this we introduce the function $V(\theta)$ defined by

$$V(\theta) = \frac{1}{2} \sum_{t=1}^{N} \epsilon^2(t) \tag{3.5}$$

where θ from now on stands for the vector

$$\theta = \{\theta_i,\ i = 1, \ldots, 4n+2\} \tag{3.6}$$

When we have found the value $\hat{\theta}$ for which $V(\theta)$ has an absolute minimum, the estimate of θ_{4n+3} is obtained from

$$\hat{\theta}^2_{4n+3} = \hat{c}^2_o = \frac{1}{N} \min_{\theta} \sum_{t=1}^{N} \epsilon^2(t) \tag{3.7}$$

To find $\hat{\theta}$ we use the following Newton-Raphson algorithm

$$\theta^{k+1} = \theta^k - \left[V_{\theta\theta}(\theta^k)\right]^{-1} V_\theta(\theta^k) \tag{3.8}$$

Where V_θ denotes the gradient and $V_{\theta\theta}$ denotes the matrix of second partial derivatives of $V(\theta)$ i.e.

$$\{V_\theta\}_i = \frac{\partial V(\theta)}{\partial \theta_i} \qquad i = 1,\ 2, \ldots, 4n+2 \tag{3.9}$$

$$\{V_{\theta\theta}\}_{ij} = \frac{\partial^2 V(\theta)}{\partial \theta_i \partial \theta_j} \qquad i,\ j = 1,\ 2, \ldots, 4n+2 \tag{3.10}$$

Under suitable regularity conditions the algorithm (3.8) will converge to a value $\hat{\theta}$ for which the function $V(\theta)$ has a local minimum. One essential difficulty is that the function $V(\theta)$ may have several local minima. The choice of initial estimates for the algorithm is discussed in section 6
To use the algorithm (3.8) the partial derivatives of $V(\theta)$ must be evaluated. This is done as follows

$$\frac{\partial V}{\partial \theta_i} = \sum_{t=1}^{N} \epsilon(t) \frac{\partial \epsilon(t)}{\partial \theta_i} \qquad i = 1, \ldots, 4n+2 \tag{3.11}$$

$$\frac{\partial^2 V}{\partial \theta_i \partial \theta_j} = \sum_{t=1}^{N} \frac{\partial \epsilon(t)}{\partial \theta_i} \cdot \frac{\partial \epsilon(t)}{\partial \theta_j} + \sum_{t=1}^{N} \epsilon(t) \frac{\partial^2 \epsilon(t)}{\partial \theta_i \partial \theta_j} \quad i, j, = 1, 2, \ldots, 4n+2 \tag{3.12}$$

where the derivatives of $\epsilon(t)$ are obtained from (3.1) i.e.

$$\begin{aligned}
\frac{\partial \epsilon(t)}{\partial \theta_i} &= \frac{\partial x(t)}{\partial \theta_i} && i = 1, 2, \ldots, 4n \\
\frac{\partial \epsilon(t)}{\partial \theta_i} &= -u(t) && i = 4n+1 \\
\frac{\partial \epsilon(t)}{\partial \theta_i} &= -1 + \frac{\partial x(t)}{\partial \theta_i} && i = 4n+2 \qquad (3.13) \\
\frac{\partial^2 \epsilon(t)}{\partial \theta_i \partial \theta_j} &= \frac{\partial^2 x(t)}{\partial \theta_i \partial \theta_j} && i, j = 1, 2, \ldots, 4n \\
\frac{\partial^2 \epsilon(t)}{\partial \theta_i \partial \theta_j} &= 0 && \text{either of } i,j = 4n+1,\ 4n+2
\end{aligned}$$

and the derivatives of $x(t)$ are given by

$$\begin{aligned}
\frac{\partial x(t+1)}{\partial \theta_i} &= \Phi \frac{\partial x(t)}{\partial \theta_i} + \frac{\partial \Phi}{\partial \theta_i} x(t) + \frac{\partial \Gamma}{\partial \theta_i} u(t) + \frac{\partial \Delta}{\partial \theta_i} y(t) && i = 1, 2, \ldots, 4n \\
\frac{\partial x(1)}{\partial \theta_i} &= 0 && i = 1, 2, \ldots, 3n \\
\frac{\partial x_i(1)}{\partial \theta_j} &= \delta_{ij} && i, j = 3n+1, 3n+2, \ldots, 4n \qquad (3.14)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x(t+1)}{\partial \theta_i} &= \Phi \frac{\partial x(t)}{\partial \theta_i} - \Delta && i = 4n+2 \\
\frac{\partial x(1)}{\partial \theta_i} &= 0 && \qquad (3.15)
\end{aligned}$$

$$\frac{\partial^2 x(t+1)}{\partial\theta_i \partial\theta_j} = \Phi \frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} + 2 \frac{\partial\Phi}{\partial\theta_i} \cdot \frac{\partial x(t)}{\partial\theta_j}$$

$$\frac{\partial^2 x(1)}{\partial\theta_i \partial\theta_j} = 0 \qquad i, j = 1, 2, \ldots, n \tag{3.16}$$

and

$$\frac{\partial^2 x(t+1)}{\partial\theta_i \partial\theta_j} = \Phi \frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} + \frac{\partial\Phi}{\partial\theta_i} \cdot \frac{\partial x(t)}{\partial\theta_j}$$

$$\frac{\partial^2 x(1)}{\partial\theta_i \partial\theta_j} = 0 \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, 3n \end{matrix} \tag{3.17}$$

$$\frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} = 0 \qquad \text{either } i = 1, 2, \ldots n \quad j = 3n+1, 3n+2, \ldots, 4n$$
$$\text{or } i, j = n+1, \ldots, 4n$$

$$\frac{\partial^2 x(t+1)}{\partial\theta_i \partial\theta_j} = \Phi \frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} + \frac{\partial\Phi}{\partial\theta_i} \cdot \frac{\partial x(t)}{\partial\theta_j} \qquad i = 1, 2, \ldots, n \; ; \; j = 4n+2$$

$$\frac{\partial^2 x(1)}{\partial\theta_i \partial\theta_j} = 0 \tag{3.18}$$

$$\frac{\partial^2 x(t+1)}{\partial\theta_i \partial\theta_j} = \Phi \frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} - \frac{\partial\Delta}{\partial\theta_i} \qquad i = 2n+1, \ldots, 3n \; ; \; j = 4n+2$$

$$\frac{\partial^2 x(1)}{\partial\theta_i \partial\theta_j} = 0 \tag{3.19}$$

$$\frac{\partial^2 x(t)}{\partial\theta_i \partial\theta_j} = 0 \qquad i = n+1, \ldots, n \; ; \; j = 4n+2 \tag{3.20}$$

Notice that the logarithm of the likelihood function as given by equation (3.3) is an analytic function of $\theta_1, \theta_2, \ldots, \theta_{4n+3}$ if $\theta_{4n+3} \neq 0$. This function does not grow faster than N as N increases if the matrix Φ is stable. If the parameters $\theta_1, \ldots, \theta_n$ are constants, it follows from equations (3.1) and (3.3) that log L is a convex quadratic function of $\theta_{n+1}, \ldots, \theta_{4n+1}$. In this case, the Newton-Raphson algorithm (3.8) will converge in one step for all initial values. This situation corresponds physically to the identification of a linear model when the disturbance is

a known moving average of independent normal variables. In particular, if $\theta_1 = \theta_2 = \ldots = \theta_n = 0$ we get in this way the least squares estimate of the parameters $\theta_{n+1}, \theta_{n+2}, \ldots, \theta_{4n+1}$. Also notice that equations (3.1), (3.5), (3.11) - (3.15) for the function $V(\theta)$ and its derivatives can be interpreted as dynamical systems and that the matrices defining the dynamics of these systems are identical. This is illustrated in Figure 2 where we show a block diagram for the computation of function V and its first order derivatives.

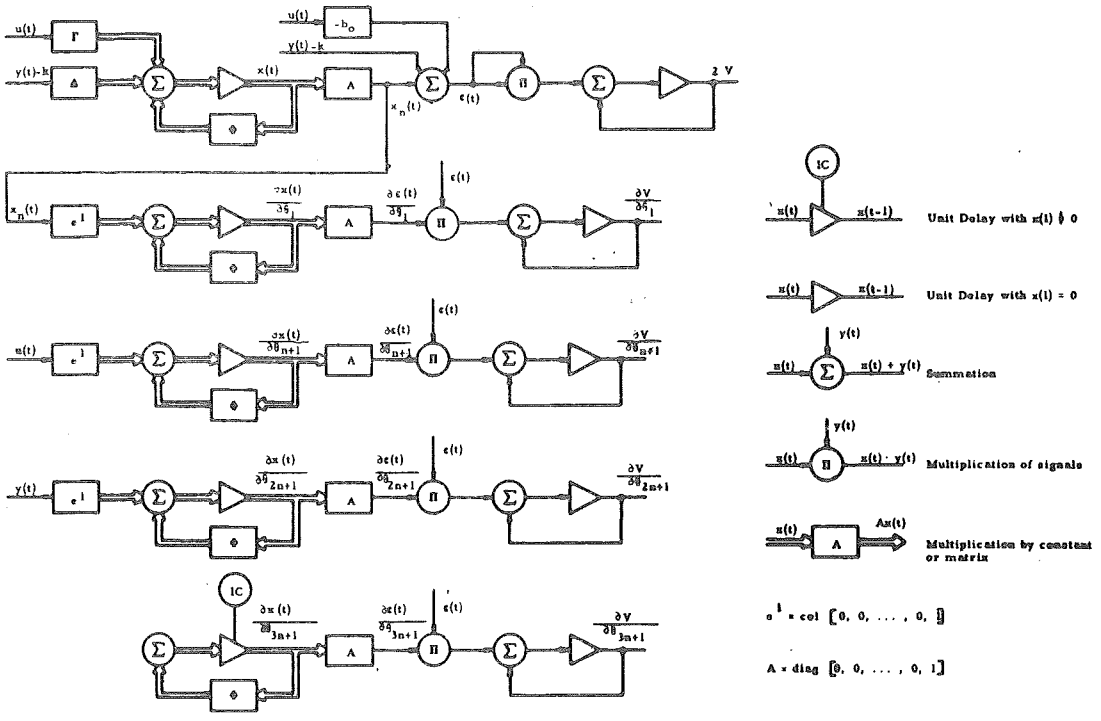


Figure 2.

Computational Considerations

We will now make a few observations which can be used to save on computing time.

- Computation of the partial derivatives V_θ.

To compute the gradient of $V(\theta)$ we must first compute the partial derivatives of $x_n(t)$ with respect to θ. See the equation (3.11). Furthermore, the form of the Φ matrix (3.2) implies that the derivatives of the component x_n with respect to θ_i cannot be computed without computing the derivatives of all the components. We also notice that for $i = 1, 2, \ldots, 3n$ only one of the terms of the right member of the equation (3.14) is non-vanishing and we have

$$\frac{\partial \Phi}{\partial \theta_i} = [0, 0, \ldots, e^i] \qquad i = 1, 2, \ldots, n \qquad (3.21)$$

$$\frac{\partial \Gamma}{\partial \theta_i} = e^{i-n} \qquad i = n+1, n+2, \ldots, 2n \qquad (3.22)$$

$$\frac{\partial \Delta}{\partial \theta_i} = e^{i-2n} \qquad i = 2n+1, 2n+2, \ldots, 3n \qquad (3.23)$$

where e^i is a vector whose components are given by

$$e^i_j = \delta_{j-n+i-1} \qquad (3.24)$$

Hence in order to evaluate the derivatives of $x_n(t)$ with respect to θ_i $i = 1, 2, \ldots, 3n$ we have to solve difference equations of the type

$$z^i(t+1) = \Phi z^i(t) + e^i v(t) \qquad (3.25)$$

with initial conditions

$$z(1) = 0$$

The solution of equation (3.21) is

$$z^i(t+1) = \sum_{s=1}^{t} \Phi^{t-s} e^i v(s)$$

$$= \sum_{s=1}^{t-1} \Phi^{t-1-s} \Phi e^i v(s) + e^i v(t) \qquad (3.26)$$

We observe that

$$z_n^i(t) = 0 \qquad t = 1, 2, \ldots, i \qquad (3.27)$$

The equations (3.26) and (3.27) imply

$$\Phi e^i = e^{i-1} \qquad i = 2, \ldots, n, n+2, \ldots, 2n, 2n+2, \ldots, 3n$$

Hence

$$z^i(t+1) = z^{i-1}(t) + e^i v(t)$$

and in particular

$$z_n^i(t+1) = z_n^{i+1}(t) \quad i = 2, \ldots, n, n+2, \ldots, 2n, 2n+2, \ldots, 3n \qquad (3.28)$$

Similarly we find that equation (3.14) for the derivatives of $x(t)$ with respect to $\theta_{3n+1}, \theta_{3n+2}, \ldots, \theta_{4n}$ can be written as

$$z^i(t+1) = \Phi z^i(t)$$
$$z^i(1) = e^i$$

and we can thus conclude that equation (3.28) holds also for $i = 3n+2, \ldots, 4n$. Summarizing we find

$$\frac{\partial x_n(t)}{\partial \theta_i} = \frac{\partial x_n(t-i+1)}{\partial \theta_1} \qquad i = 2, \ldots, n$$

$$\frac{\partial x_n(t)}{\partial \theta_i} = \frac{\partial x_n(t-i+n+1)}{\partial \theta_{n+1}} \qquad i = n+2, \ldots, 2n$$

$$\frac{\partial x_n(t)}{\partial \theta_i} = \frac{\partial x_n(t-i+2n+1)}{\partial \theta_{2n+1}} \qquad i = 2n+2, \ldots, 3n$$

$$\frac{\partial x_n(t)}{\partial \theta_i} = \frac{\partial x_n(t-i+3n+1)}{\partial \theta_{3n+1}} \qquad i = 3n+2, \ldots, 4n \qquad (3.29)$$

and we can thus conclude that in order to evaluate the partial derivatives of $\epsilon(t) = x_n(t)$ with respect to $\theta_1, \ldots, \theta_{4n}$ it suffices to calculate four derivatives.

- Computation of the second partial derivatives $V_{\theta\theta}$

To compute the second partial derivatives of $V(\theta)$ we must first compute the second partial derivatives of $x_n(t)$. See equation (3.12). By differentiating equation (3.29) and using equation (3.28) we find

$$\frac{\partial^2 x_n(t)}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 x_n(t-i-j+2)}{\partial \theta_1 \partial \theta_1} \qquad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n$$

$$\frac{\partial^2 x_n(t)}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 x_n(t-i-j+2+n)}{\partial \theta_1 \partial \theta_{n+1}} \qquad i = 1, 2, \ldots, n \quad j = n+1, n+2, \ldots, 2n$$

$$\frac{\partial^2 x_n(t)}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 x_n(t-i-j+2+2n)}{\partial \theta_1 \partial \theta_{2n+1}} \qquad i = 1, 2, \ldots, n \quad j = 2n+1, 2n+2, \ldots, 3n$$

$$\frac{\partial^2 x_n(t)}{\partial \theta_i \partial \theta_j} = \frac{\partial^2 x_n(t-i-j+2+3n)}{\partial \theta_1 \partial \theta_{3n+1}} \qquad i = 1, 2, \ldots, n \quad j = 3n+1, 3n+2, \ldots, 4n \qquad (3.30)$$

Hence in order to calculate the $3.5n^2 + 0.5n$ nonzero second partial derivatives with respect to $\theta_1, \ldots, \theta_{4n}$ it suffices to calculate four derivatives of $x_n(t)$

Finally we point out a possibility to save on time when computing the sums

$$\sum_{t=1}^{n} \frac{\partial x_n(t)}{\partial \theta_i} \cdot \frac{\partial x_n(t)}{\partial \theta_j} \qquad (3.31)$$

Instead of treating a general case we assume $1 \le i \le n$, $i \le j \le n$.

Introduce

$$s_{ij}(t) = \sum_{s=1}^{t} \frac{\partial x_n(s)}{\partial \theta_i} \cdot \frac{\partial x_n(s)}{\partial \theta_j} = \sum_{s=j+1}^{t} \frac{\partial x_n(s)}{\partial \theta_i} \cdot \frac{\partial x_n(s)}{\partial \theta_j} \tag{3.32}$$

The equation (3.29) implies

$$s_{ij}(t) = \sum_{s=j+1}^{t} \frac{\partial x_n(s-i+1)}{\partial \theta_1} \cdot \frac{\partial x_n(s-j+1)}{\partial \theta_1}$$

$$= \sum_{s=j-i+2}^{t-i+1} \frac{\partial x_n(s)}{\partial \theta_1} \cdot \frac{\partial x_n(s-j+1)}{\partial \theta_1} \tag{3.33}$$

and we can thus conclude that in order to obtain $s_{ij}(t)$ for i, j = 1, 2, . . . , n it suffices to calculate $s_{11}(t)$ Exploiting this idea, we find that in order to calculate the 2n(4n+1) sums (3.31) it suffices to calculate 16n-6 sums.

MAXIMUM LIKELIHOOD ESTIMATE II

Consider the representation (2.1) of the process

$$\sum_{k=o}^{n} \alpha_k y(t-k) = \sum_{k=o}^{n} \beta_k u(t-k) + \lambda \sum_{k=o}^{n} \gamma_k e(t-k) + \varkappa, \qquad t = 1, \ldots, N \tag{4.1}$$

$$\alpha_o = \gamma_o = 1$$

Here $y(t)$ and $u(t)$ are unknown for $t \leq 0$

Make

$$y_o(t-n+1) = -\sum_{k=o}^{t} \alpha_k y(t-k) + \sum_{k=o}^{t} \beta_k u(t-k) + \lambda \sum_{k=o}^{t} \gamma_k e(t-k) + \varkappa,$$

$$t = 0, \ldots, n-1 \tag{4.2}$$

Then $y_o(t)$, $t = -n+1, \ldots, 0$ may be regarded as the n initial conditions required to determine the solution of the n^{th} order difference equation (4.1). It is easily seen that the representation (4.1) together with the initial conditions (4.2) contains exactly the required 4n+3 parameters, viz.

$$\begin{cases} \alpha_k, & k = 1, \ldots, n \\ \beta_k, & k = 0, \ldots, n \\ \gamma_k, & k = 1, \ldots, n \\ y_o(t), & t = -n+1, \ldots, 0 \\ \varkappa & \\ \lambda & \end{cases} \tag{4.3}$$

since $\{y(t),\ t = 1, \ldots, n\}$ can be uniquely determined from (4.1), (4.2) given the parameters and the sequences $\{u(t),\ t = 1, \ldots, n\}$ and $\{e(t),\ t = 1, \ldots, n\}$.

Geometric Interpretation

Regard the sequences of inputs, outputs, and disturbances as points u, y, and e in the N-dimensional euclidian space R^N, i.e. introduce the (row-) vectors

$y = \{y(j),\ j = 1, \ldots, N\}$
$u = \{u(j),\ j = 1, \ldots, N\}$
$e = \{e(j),\ j = 1, \ldots, N\}$

Then points y and u are observed, while e is a sample from an N-dimensional normal variable, $Ee = 0$, $Eee^T = I$.

Further, introduce the parameter-dependent matrices

$$\begin{cases} A = \sum_{k=o}^{n} \alpha_k I_k \\ B = \sum_{k=o}^{n} \beta_k I_k \\ C = \sum_{k=o}^{n} \gamma_k I_k \\ K = \varkappa I \\ Y_o = \sum_{k=o}^{n-1} y_o(k-n+1) I_k \end{cases}$$

Here

$I_k = \{\delta_{i-j-k},\ i, j = 1, \ldots, N\}$ = `shift matrix´

Notice that A, B, C, K, and $Y_o \in T$ = the class of left triangular, Toepliz [35] matrices. It is easy to verify the following rules

A, B $\in$ T $\Rightarrow$
i) $A^{-1} \in T$ if $\alpha_o \neq 0$
ii) AB = BA $\in$ T (i.e. A and B commute)

In fact, we can perform any finite number of elementary algebraic operations without ever leaving the class T.

Using matrix notations, a compact form of (4.1) is obtained

$$yA^T = uB^T + \lambda eC^T + iK^T + i_1 Y_o^T \qquad (4.5)$$

where

$i = \{1,\ j = 1, \ldots, N\}$ (a row vector with all components = unity)

$i_1 = \{\delta_{j-1},\ j = 1, \ldots, N\}$

Equation (4.5) can be regarded as a linear relation between non-singular transformations A, B, and C of the points y, u, and e respectively. The problem is to estimate these transformations, the vectors iK^T and $i_1 Y_o^T$, and the scalar λ, given y, u, and the distribution of e.

If the Maximum Likelihood estimate is chosen, we need the distribution of y as a function of u and the parameters [83].

But this is readily obtained from (4.5). Since e is normal (0, I), then y is normal and

$$\begin{cases} Ey = (uB^T + iK^T + i_1 Y_o^T)\, A^{T^{-1}} \\ Cov\,(y,y) = \lambda^2 A^{-1} C C^T \quad A^{T^{-1}} \end{cases}$$

so that the logarithm of the likelihood function is (except for a constant)

$$L = -N \text{ Log } \lambda - \frac{1}{2\lambda^2} \epsilon C^{T^{-1}} C^{-1} \epsilon^T \qquad (4.6)$$

since det A = det C = 1.

Here

$$\epsilon = yA^T - uB^T - iK^T - i_1 Y_o^T = \lambda eC^T \qquad (4.7)$$

= the disturbances in the output.

The Maximum Likelihood Equations

Include all parameters except λ in the common vector θ and make

$$V(\theta) = \frac{1}{2}\ \epsilon C^{T^{-1}}\ C^{-1}\ \epsilon^T \tag{4.8}$$

The M. L. estimate $\hat{\theta}$ of θ is then obtained from

$$V(\hat{\theta}) = \min_{\theta} V(\theta) \tag{4.9}$$

and the M. L. estimate of λ from

$$\hat{\lambda}^2 = \frac{2}{N} V(\hat{\theta}) \tag{4.10}$$

by differentiating (4.6) with respect to λ.

It is now easy to find the derivatives of $V(\theta)$ by straight forward differentiation of (4.7). Notice that I_r and C^{-1} commute. The results are

$$V = \frac{1}{2}\epsilon^1\ \epsilon^{1T}$$

$$\begin{cases} \dfrac{\partial V}{\partial \alpha_r} = \epsilon^1 I_r y^{1T} & r = 1, \ldots, n \\ \dfrac{\partial V}{\partial \beta_r} = -\epsilon^1 I_r u^{1T} & r = 0, \ldots, n \\ \dfrac{\partial V}{\partial \gamma_r} = -\epsilon^1 I_r \epsilon^{2T} & r = 1, \ldots, n \\ \dfrac{\partial V}{\partial \varkappa} = -\epsilon^1 i^{1T} & \\ \dfrac{\partial V}{\partial y_o(r+1-n)} = -\epsilon^1 I_r i_1^{1T} & r = 0, \ldots, n-1 \end{cases} \tag{4.11}$$

where

$$\epsilon^\nu = \epsilon C^{T^{-\nu}},\quad y^\nu = yC^{T^{-\nu}},\quad u^\nu = uC^{T^{-\nu}},\quad i^\nu = iC^{T^{-\nu}},\quad i_1^\nu = i_1^\nu C^{T^{-\nu}}$$

The M. L. estimate is obtained as the zero $\hat{\theta}$ of the first derivatives (4.11) minimizing $V(\theta)$.

Solution of the Maximum Likelihood Equations

The estimation problem is essentially a problem of solving the Maximum Likelihood equations.

Notice that where C is known, the remaining equations can be solved exactly for A, B, K, and Y_o, in which case the solution takes the form of weighted Least Squares (= Markov) estimates. This is not true if C is unknown. It is known [53] that if C is not correct, the Markov estimate may even be asymptotically biased ($N \to \infty$).

There are several hill climbing techniques which can be applied in minimizing $V(\theta)$. Since $V(\theta)$ is given in closed form it is advantageous to use one that utilizes the knowledge of the structure. A Newton-Raphson technique will be used [45].

The second derivatives of $V(\theta)$ are required. Since the parameters $\{\alpha_i\}$, $\{\beta_i\}$, $\varkappa$, and $\{y_o(i + 1 - n)\}$ enter the vector ϵ in essentially the same way (4.5), there are only three different second derivatives that need to be considered.

Let f_r, g_s be any two of the parameters $\{\alpha_i\}$, $\{\beta_i\}$, $\varkappa$, or $\{y_o(i + 1 - n)\}$. Let ξ, η be the corresponding vectors -y, u, i, i_1, and N_ξ, N_η the corresponding appropriate sets of subscripts r, s. Thus a correspondence is obtained

f_r	—	ξ	—	N_ξ
g_r	—	η	—	N_η
α_r	—	-y	—	$\{1, \ldots, n\}$
β_r	—	u	—	$\{0, \ldots, n\}$
$\varkappa$	—	i	—	0
$y_o(r+1-n)$	—	i_1	—	$\{0, \ldots, n-1\}$

Also let

$$\gamma_r \quad \text{—} \quad \epsilon \quad \text{—} \quad N_\epsilon = \{1, \ldots, n\}$$

Define as before

$$\xi^\nu = \xi C^{T^{-\nu}}, \quad \eta^\nu = \eta C^{T^{-\nu}}$$

We have for $\xi, \eta = -y, u, i, i_1$

$$V = \frac{1}{2} \epsilon^1 \epsilon^{1T}$$

$$\begin{cases} \dfrac{\partial V}{\partial f_r} = -\epsilon^1 I_r \xi^{1T} & r \in N_\xi \\ \dfrac{\partial V}{\partial \gamma_r} = -\epsilon^1 I_r \epsilon^{2T} & r \in N_\epsilon \\ \dfrac{\partial^2 V}{\partial f_r \partial g_s} = \eta^1 I_s^T I_r \xi^{1T} & r \in N_\xi, \ s \in N_\eta \\ \dfrac{\partial^2 V}{\partial f_r \partial \gamma_s} = \epsilon^1 I_{r+s} \xi^{2T} + \epsilon^2 I_s^T I_r \xi^{1T} & r \in N_\xi, \ s \in N_\epsilon \\ \dfrac{\partial^2 V}{\partial \gamma_r \partial \gamma_s} = 2\epsilon^1 I_{r+s} \epsilon^{3T} + \epsilon^2 I_s^T I_r \epsilon^{2T} & r, s \in N_\epsilon \end{cases} \tag{4.12}$$

Let V_θ be the column vector of first derivatives and $V_{\theta\theta}$ the matrix of second derivatives according to (4.12). Then the Newton-Raphson algorithm is

$$\hat{\theta}^{k+1} = \hat{\theta}^k - V_{\theta\theta}^{-1}(\hat{\theta}^k) \, V_\theta(\hat{\theta}^k) \tag{4.13}$$

If the sequence $\{\hat{\theta}^k\}$ converges, then $\hat{\theta} = \lim_{k\to\infty} \hat{\theta}^k$ is a root of $V_\theta(\hat{\theta}) = 0$.

Concerning the initial estimate $\hat{\theta}^o$ see section 6.

Finally, notice that so far nothing is assumed about the stability of the process, i.e. the stability of the difference equation $\sum_{k=o}^{n} \alpha_k \, X_{t-k} = 0$

Computational considerations

Each cycle of the Newton-Raphson procedure (4.13) requires essentially the following computations. See (4.12)

1) vectors ξ^ν and ϵ^ν, viz.

$$y,\ y^1,\ y^2,\ u,\ u^1,\ u^2,\ i,\ i^1,\ i^2,\ i_l,\ i_l^1,\ i_l^2,\ \epsilon,\ \epsilon^1,\ \epsilon^2,\ \epsilon^3$$

2) inner products $\eta^\mu I_s^T I_r \xi^\nu$, viz.

<u>function:</u> $\epsilon^1 \epsilon^{1T}$

<u>1st derivatives:</u> $\epsilon^1 I_r y^{1T},\ \epsilon^1 I_r u^{1T},\ \epsilon^1 I_r \epsilon^{2T},\ \epsilon^1 i^{1T},\ \epsilon^1 I_r i_l^{1T}$

<u>2nd derivatives:</u> $y^1 I_s^T I_r y^{1T},\ u^1 I_s^T I_r y^{1T},\ u^1 I_s^T I_r u^{1T},$

$\epsilon^2 I_s^T I_r y^{1T},\ \epsilon^1 I_{r+s} y^{2T},\ \epsilon^2 I_s^T I_r u^{1T},\ \epsilon^1 I_{r+s} u^{2T},$

$\epsilon^2 I_s^T I_r \epsilon^{2T},\ \epsilon^1 I_{r+s} \epsilon^{3T},\ i^1 I_r y^{1T},\ i^1 I_r u^{1T},$

$i^1 I_r \epsilon^{2T},\ i^2 I_r \epsilon^{1T},\ i^1 i^{1T},$

$i_l^1 I_s^T I_r y^{1T},\ i_l^1 I_s^T I_r u^{1T},\ i_l^1 I_s^T I_r \epsilon^{2T},\ i_l^2 I_r \epsilon^{1T},$

$i_l^1 I_s^T i^{1T},\ i_l^1 I_s^T I_r i_l^{1T}$

3) the solution $\hat\theta^{k+1} - \hat\theta^k$ of the linear system of equations

$$V_{\theta\theta}(\hat\theta^k)(\hat\theta^{k+1} - \hat\theta^k) = V_\theta(\hat\theta^k)$$

It is advantageous to solve it by inverting $V_{\theta\theta}$, since $V_{\theta\theta}^{-1}$ is needed for estimating the covariance of $\hat\theta$. See section 5.

Now y and u are given, and ϵ is easily computed from (4.7)

$$\begin{cases} \epsilon(t) = \sum_{k=0}^{n} \alpha_k y(t-k) - \sum_{k=0}^{n} \beta_k u(t-k) - \varkappa - \sum_{k=0}^{n-1} y_o(k+1-n)\, \delta_{t-1-k} \\ \\ y(t) = u(t) = 0 \quad t \le 0 \end{cases} \tag{4.14}$$

The vectors ξ^ν (which will be called state-vectors) satisfy $\xi^\nu = \xi^{\nu-1} C^{T^{-1}}$ so that ξ^ν is obtained from $\xi^{\nu-1}$ through the linear equation $\xi^\nu C^T = \xi^{\nu-1}$

(4.15)

But this equation is solved easily, since C is triangular.
The solution is

$$\begin{cases} \xi^\nu(t) = -\sum_{k=1}^{n} \gamma_k \xi^\nu(t-k) + \xi^{\nu-1}(t), \quad t = 1, \ldots, N \\ \\ \xi^\nu(t) = 0 \quad t \le 0 \end{cases} \tag{4.16}$$

The components of the state vectors $\xi^\nu(j)$ can then be interpreted as the solution of a linear difference equation (4.16) (which is stable according to postulate) with zero initial conditions and driven by $\xi^{\nu-1}(j)$. Or equivalently as the output of a linear filter, defined by $\{\gamma_i\}$ with the input $\xi^{\nu-1}(t)$. We need the outputs of this filter for the following inputs

u = the observed input $\quad \nu = 1, 2$
y = the observed output $\quad \nu = 1, 2$
ϵ = the calculated error $\quad \nu = 1, 2, 3$
i = a unit step $\quad \nu = 1, 2$
i_1 = a unit pulse $\quad \nu = 1, 2$

The inner products are formed between vectors $I_r \xi^\nu$, which are the state vectors shifted a certain number of steps r (always $\le 2n$). They may be interpreted as various cross-correlations between the filter outputs ξ^ν.

There are $O(n^2)$ different such inner products, corresponding to the various second derivatives, each of them a sum of N products. However, there exist numerical dependences between the products which can be utilized in reducing the number of necessary multiplications in the following way.

Let ξ, η stand for any two of $-y$, u, ϵ, i, i_1 and define the scalar function

$$\psi_r^{\nu\mu}(\xi\ \eta,\ t) = \eta^{\mu} J_t I_r \xi^{\nu T} = \sum_{k=1}^{t} \eta^{\mu}(k)\ \xi^{\nu}(k-r) \qquad (r \geq 0)$$

$$\psi_r^{\nu\mu}(\xi,\ \eta,\ t) = \psi_{-r}^{\mu\nu}(\eta,\ \xi,\ t) \qquad (r < 0) \qquad (4.17)$$

Here

$$J_t = \{(\delta_{i-j},\ i = 1, \ldots, t;\ 0,\ i = t+1, \ldots, N),\ j = 1, \ldots, N\}$$

= 'truncate matrix'.

We have then, since $I_s^T I_r = J_{N-s} I_{r-s} \qquad (r \geq s)$,

$$\eta^{\mu} I_s^T I_r \xi^{\nu T} = \psi_{r-s}^{\nu\mu}(\xi,\ \eta,\ N-s), \qquad (r \geq s) \qquad (4.18)$$

Since $\psi_r^{\nu\mu}(\xi\ \eta,\ t)$ are computed sequentially for $t = 1, \ldots, N$, $\psi_{r-s}^{\nu\mu}(\xi,\ \eta,\ N-s)$ for $s = 0, \ldots, n$ is computed with the same effort as $\psi_r^{\nu\mu}(\xi\ \eta,\ N)$, and only $NO(n)$ multiplications are required. (The number of storage locations are however still $O(n^2)$.)

Of special interest is the case where N is a large number, while n is moderate. It is then desirable to perform the computations in such a way that the storage space required does not increase with N (except for the external memory necessary to store the input/output u, y).

This is accomplished if the computations are organized in the following way. For each $t = 1, \ldots, N$

- $\epsilon(t)$ is computed according to (4.14). $2n + 2$ storage locations are required to store $y(t-k)$, $u(t-k)$, $k = 0, \ldots, n$
- The values $\xi^{\nu}(t)$ are computed from (4.16). Only a moderate (= bounded, $N \to \infty$) number of storage locations are necessary (i.e. for the various $\xi^{\nu}(t-k)$, $k = 0, \ldots, n$)

- The cross products $\xi^{\nu}(t-r)\,\eta^{\mu}(t)$ according to (4.12) are computed and added to $\Psi_r^{\nu\mu}(\xi,\ \eta,\ t-1)$ according to (4.17)

- The state variables $\xi^{\nu}(t-k+1)$ and the accumulated inner products $\Psi_r^{\nu\mu}(\xi,\ \eta,\ t-k+1)$ are moved to the locations for $\xi^{\nu}(t-k)$ and $\Psi_r^{\nu\mu}(\xi,\ \eta,\ t-k)$ respectively.

The inversion of $V_{\theta\theta}$ is done by standard techniques [27]. No caution is required here as to computing time and storage space.

Finally, notice that if N is large

$$\eta^{\mu} I_s^T I_r \xi^{\nu^T} \sim \Psi_{r-s}^{\nu\mu}(\xi,\ \eta,\ N)$$

the relative error is of the order N^{-1}. This follows, as the difference equation

$\sum_{k=0}^{n} \gamma_k X_{t-k} = 0$ being stable implies $C^{-1} \underset{N\to\infty}{=} O(1)$.

The Asymptotic Number of Multiplications

Of crucial importance for the computing time necessary to execute the identification algorithm, when N is large, is the number of multiplications necessary to perform one step of the Newton-Raphson algorithm, times the number of such steps required to obtain an approximate root with prescribed accuracy. The number of multiplications will now be derived. What matters is the number of multiplications inside the main loop $t = 1, \ldots, N$ (which is gone through only once per step), because the number of operations outside is constant when N increases.

For each $t = 1, \ldots, N$ the following table is obtained. The zeros hold asymptotically for large t, since the responses to the step and pulse inputs i and i_1 can be computed to any degree of accuracy with a bounded number of multiplications.

Quantity to be computed	Range of sub or Superscripts	Formula	Number of multiplications
$y^{\nu}(t)$	$\nu = 1, 2$	(4.16)	2n
$u^{\nu}(t)$	$\nu = 1, 2$	(4.16)	2n
$i^{\nu}(t)$	$\nu = 1, 2$	(4.16)	0
$i_1^{\nu}(t)$	$\nu = 1, 2$	(4.16)	0
$\epsilon(t)$		(4.14)	2n+1
$\epsilon^{\nu}(t)$	$\nu = 1, 2, 3$	(4.16)	3n
$\Psi_o^{11}(\epsilon, \epsilon, t)$		(4.17)	1
$\Psi_r^{11}(-y, \epsilon, t)$	$r = 1, \ldots, n$	(4.17)	n
$\Psi_r^{11}(u, \epsilon, t)$	$r = 0, \ldots, n$	(4.17)	n+1
$\Psi_r^{21}(\epsilon, \epsilon, t)$	$r = 1, \ldots, n$	(4.17)	n
$\Psi_o^{11}(i, \epsilon, t)$		(4.17)	0
$\Psi_r^{11}(i_1, \epsilon, t)$	$r = 0, \ldots, n-1$	(4.17)	0
$\Psi_r^{11}(-y, -y, t)$	$r = 0, \ldots, n-1$	(4.17)	n
$\Psi_r^{11}(-y, u, t)$	$r = -n, \ldots, n-1$	(4.17)	2n
$\Psi_r^{11}(u, u, t)$	$r = 0, \ldots, n$	(4.17)	n+1
$\Psi_r^{12}(-y, \epsilon, t)$	$r = -n+1, \ldots, n-1$	(4.17)	2n-1
$\Psi_r^{21}(-y, \epsilon, t)$	$r = 2, \ldots, 2n$	(4.17)	2n-1
$\Psi_r^{12}(u, \epsilon, t)$	$r = -n+1, \ldots, n$	(4.17)	2n
$\Psi_r^{21}(u, \epsilon, t)$	$r = 1, \ldots, 2n$	(4.17)	2n
$\Psi_r^{22}(\epsilon, \epsilon, t)$	$r = 0, \ldots, n-1$	(4.17)	n
$\Psi_r^{31}(\epsilon, \epsilon, t)$	$r = 2, \ldots, 2n$	(4.17)	2n-1
$\Psi_r^{11}(-y, i, t)$	$r = 1, \ldots, n$	(4.17)	0
$\Psi_r^{11}(u, i, t)$	$r = 0, \ldots, n$	(4.17)	0
$\Psi_r^{21}(\epsilon, i, t)$	$r = 1, \ldots, n$	(4.17)	0
$\Psi_r^{12}(\epsilon, i, t)$	$r = 1, \ldots, n$	(4.17)	0
$\Psi_o^{11}(i, i, t)$		(4.17)	0
$\Psi_r^{11}(-y, i_1, t)$	$r = -n+2, \ldots, n$	(4.17)	0
$\Psi_r^{11}(u, i_1, t)$	$r = -n+1, \ldots, n$	(4.17)	0
$\Psi_r^{21}(\epsilon, i_1, t)$	$r = -n+2, \ldots, n$	(4.17)	0
$\Psi_r^{12}(\epsilon, i_1, t)$	$r = 1, \ldots, 2n-1$	(4.17)	0
$\Psi_r^{11}(i, i_1, t)$	$r = -n+1, \ldots, 0$	(4.17)	0
$\Psi_r^{11}(i_1, i_1, t)$	$r = 0, \ldots, n-1$	(4.17)	0

The components of V_θ and $V_{\theta\theta}$ are obtained from the quantities $\Psi_r^{\nu\mu}(\xi, \eta, N-s)$, $s = 0, \ldots, n$, according to (4.18) and (4.12), by a fixed number of multiplications.

Summing the left column of the table, we obtain the asymptotic number of multiplications necessary to compute one step of the Newton-Raphson procedure

$$\sim N(27n + 1)$$

This is very convenient, since it implies that doubling the order of the model just doubles the computing time needed to identify it.

The number of multiplications per parameter is

$$N(27n + 1) / (3n + 3) \to 9N, \quad n \to \infty$$

CONSISTENCY AND ASYMPTOTIC NORMALITY OF THE ESTIMATES

In this section we analyze the properties of the maximum likelihood estimate. In particular, we are interested in the behavior of the estimate ($\hat{\theta}$) as the number of observations (N) increases. The standard results for the maximum likelihood estimate of parameters from independent equally distributed samples do not apply directly.

We first analyze the consistency of the estimate. Before giving any general results we will consider a few special cases which illustrate the case when the estimate is not consistent.

Example 1

Let the system be described by

$$x(t+1) = ax(t)$$

$$y(t) = x(t) + e(t)$$

where $\{e(t)\}$ are independent normal $(0,\sigma)$ random variables, a is a given number $-1 < a < 1$ and $y(1), \ldots, y(N)$ are the observed outputs. Let the initial condition $x(0) = \theta$ be the parameter which is to be estimated. The maximum likelihood estimate of θ is given by

$$\hat{\theta} = \frac{1 - a^2}{a^2 - a^{2N+2}} \sum_{t=1}^{N} a^t y(t)$$

This estimate is unbiased and has the variance

$$\text{Var } \hat{\theta} = \frac{\sigma^2}{a^2} \quad \frac{1 - a^2}{1 - a^{2N+2}}$$

As $N \to \infty$ the variance converges to $\sigma^2(1 - a^{-2})$, and the estimate of $\theta = x(0)$ thus is not consistent. If in this case the parameter a also should be estimated, it can be shown that this estimate is not consistent.

The results of the example can be generalized to cover the general case of section 2. The conclusion is that if the system defined by equation (2.1) is stable, the estimate of the initial state is never consistent. Notice, however, that this is not a serious matter because for a stable system the contribution of the initial conditions to the likelihood function will be negligible, as $N \to \infty$.

We will now give another example which shows that the estimates of the coefficients b_i cannot be consistent for all possible choices of the input u.

Example 2

Consider the system

$$x(t+1) = ax(t) + bu(t)$$

$$y(t) = x(t) + e(t)$$

where $x(0) = 0$, and a is a given number $-1 < a < 1$ and

$$u(t) = \begin{cases} 1 & t = 0 \\ 0 & t = 1, \ldots, N \end{cases}$$

Iterating the equation once we find

$$x(t+1) = ax(t)$$

$$x(1) = b \cdot u(1)$$

The situation is then the same as the one discussed in Example 1 and we can immediately conclude that the estimate of the parameter b is not consistent. Roughly speaking, this example shows that in order to obtain consistent estimates of the parameters b, the input u must excite the system persistently.

We will finally give a third example which illustrates another difficulty.

Example 3

Consider the system

$$x_1(t+1) = -0.06\, x_2(t) + 0.3\, u(t)$$

$$x_2(t+1) = x_1(t) - 0.5\, x_2(t) + u(t)$$

$$y(t) = x_1(t) + e(t)$$

Introducing new state-variables z related to the x variables by

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & -1 \\ -10 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we find that the input/output relation can be written as

$$z_1(t+1) = -0.2\, z_1(t) + 0.1\, u(t)$$

$$z_2(t+1) = -0.3\, z_2(t)$$

$$y(t) = 10\, z_1(t) + 5\, z_2(t) + e(t)$$

From this representation it is clear that regarding the parameter -0.3 entering the equation for $z_2(t)$ we have the same situation as when estimating the parameter θ in example 1. The conclusion is thus that this parameter cannot be estimated consistently. Back-tracking, we find that the parameter -0.3 enters both the coefficients a_1 and a_2 of the original representation and the conclusion is thus that it is not possible to estimate the parameters $a_1 = 0.5$ and $a_2 = 0.06$ consistently. On the other hand, it is clear from the representation with z as state variable that the input/output relation can be represented by a first order system. The parameters of this system can be estimated consistently. The reason for this is that the states x_1 and x_2 are not controllable from the output u.

Consistency

We will now proceed to give conditions for the consistency of the estimator. We first introduce some notations: θ denotes the vector of parameters defined by equation (3.4) or any other vector of 4n+3 parameters completely defining the process, the vector of true parameters is denoted by θ_o, y denotes the vector of observed outputs defined by equation (2.1) or (4.5), and E_o denotes mathematical expectation with respect to the distribution of y when the parameters θ have their true values θ_o. The logarithm of the likelihood function is denoted by $L(y; \theta)$. Both $L(y; \theta)$ and the maximum likelihood estimate $\hat{\theta}$ will depend on the number of observed outputs N. To indicate this we introduce the notations $\hat{\theta}^N$ and $L^N(y; \theta)$.

We will prove consistency by modifying the elegant proof of Wald [76] for independent samples. The possibility of doing such extensions was pointed out by Wald [76] [77].

The present case differs from Wald's case in two respects. The samples are not independent and there are inputs u. Naturally some restrictions must be placed on the inputs. We will generally assume that the following regularity condition is satisfied.

ASSUMPTION A

Assume that the input u is such that u(t) and u(t) u(t + t) are Cersaro summable i.e. the following limits

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t + t)$$

exist for all t = 1, 2, ...

Before stating the main theorem we will give some sub-results which are of interest in themselves. These results concern the convergence of the logarithm of the likelihood function $L^N(y; \theta)$.

LEMMA 1

Let R be a region in 4n+3 dimensional Euclidian space defined by

$R = \{\theta \mid \lambda > 0$ and all zeros of the polynomials

$$z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$$

and $z^n + \gamma_1 z^{n-1} + \ldots + \gamma_n$ have magnitudes strictly less than one.$\}$

Further let the input satisfy assumption A. Then

$$\lim_{N\to\infty} \frac{1}{N} L^N(y; \theta) = \lim_{N\to\infty} \frac{1}{N} E_o L^N(y; \theta) = L(\theta; \theta_o) \qquad (5.1)$$

with probability one if $\theta \in R$ and $\theta_o \in R$.

This is essentially an ergodic theorem. It gives the asymptotic properties of the likelihood function. The proof is given in the appendix where we also show

LEMMA 2

Let the input satisfy assumption A and let $R' \subset R$ be a closed set. Then $L(\theta; \theta_o)$ is an analytic function of θ in R' and we have

$$\lim_{N\to\infty} \frac{1}{N} \frac{\partial}{\partial\theta_i} L^N(y; \theta) = \lim_{N\to\infty} \frac{1}{N} \frac{\partial}{\partial\theta_i} E_o L^N(y; \theta) = \frac{\partial}{\partial\theta_i} L(\theta; \theta_o) \qquad (5.2)$$

with probability one. The above equation also holds for higher derivatives.

We further have

THEOREM 1

Let S_o be a set in 4n+3 dimensional Euclidian space defined by

$$S_o = \{ \theta \mid L(\theta, \theta_o) = L(\theta_o, \theta_o) \} \tag{5.3}$$

Assume that the input satisfies the assumption A, and that for all sufficiently large N, $\hat{\theta}^N \in R'$ where $R' \subset R$ is a closed set. Then

$$\| \hat{\theta}^N - \mathcal{P} \hat{\theta}^N \| \to 0 \tag{5.4}$$

with probability one, where $\mathcal{P}\theta$ is the projection of θ on $S_o \cap R'$ i.e. the nearest point $\in S_o \cap R'$.

The proof of this theorem is also given in the appendix. Notice in particular that the proof depends critically on the fact that $\hat{\theta}^N$ is chosen so that the likelihood function has an absolute maximum. This is very difficult to guarantee in practice. Our algorithm will always yield a local maximum but not necessarily a global maximum. However, if two local maxima of equal magnitude exist, one can choose either one and in the limit $\hat{\theta}^N$ will still belong to the set $S_o \cap R'$.

Apparently $\theta_o \in S_o$, furthermore, if S_o only contains one point the theorem implies that the maximum likelihood estimate is consistent. This implies that the maximum of the likelihood function is unique for sufficiently large N if S_o is a point, even if the likelihood function has several maxima of equal magnitude for finite N.

In the appendix we also show that under the assumption A the following holds:

$$\| \frac{1}{N} L^N_{\theta\theta}(y, \hat{\theta}^N) - L_{\theta\theta}(\hat{\theta}^N; \theta_o) \| \to 0 \tag{5.5}$$

..th p ..bility one. This means that the quantity $L^N_{\theta\theta}(y, \hat{\theta}^N)$ which is computed in the numerical algorithm is an almost sure estimate of the information matrix $I^N(\theta) = N L_{\theta\theta}(\theta;\theta_o)$ for large values of N. It is thus not necessary to compute the information matrix separately.

We further have the following result

THEOREM 2

Let θ be defined by (4.3), let $\Lambda^N(y;\hat{\theta}^N)$ be the diagonal matrix of eigenvalues of $\frac{1}{N} L^N_{\theta\theta}(y;\hat{\theta}^N)$ and let $P^N(y;\hat{\theta}^N)$ be a matrix of corresponding orthongonal eigenvectors.

Then

$$\lim_{N\to\infty} \| \Lambda^N(y;\hat{\theta}^N)\, P^{N^T}(y;\hat{\theta}^N)\, \hat{\theta}^N - \Lambda^N(y;\hat{\theta}^N)\, P^{N^T}(y;\hat{\theta}^N)\, \theta_o \| = 0 \qquad (5.6)$$

with probability one.

The proof is given in the appendix.

The theorem is a kind of consistency theorem for certain linear transformations of $\hat{\theta}^N$. Even if some or all components of $\hat{\theta}^N$ are inconsistent, the theorem gives the linear combinations that are consistent. The fact that those combinations vary with N is due to the fact that the consistency property is expressed in terms which are observable (computable).

Theorem 2 implies in particular that if $\frac{1}{N} L^N_{\theta\theta}(y;\hat{\theta}^N)$ converges then the quality $\frac{1}{N} L^N_{\theta\theta}(y;\hat{\theta}^N)\,\hat{\theta}^N$ is strongly consistent.

We will now consider the set S_o introduced in Theorem 1. A complete characterization of this set is given in Lemma 3 and Lemma 4 of the appendix. The essential result is that the set S_o can be represented as $S_o = S_o' \cap R'$ where S_o' is a set which is linear in the 3n+3 parameters $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta_o, \beta_1, \ldots, \beta_n, \varkappa$ and λ. In particular, we have found that the set S_o will always contain the n-dimensional subspace spanned by the parameters associated with the initial conditions i.e. these parameters can never be consistently estimated. A simple demonstration of this was given in Example 1. This situation is not disturbing

because in the practical application we are only interested in the 3n+3 parameters which describe the system. The parameters associated with the initial conditions were only introduced for the purpose of writing the likelihood function explicitly. This will motivate the following concept.

DEFINITION

A system is said to be completely identifiable if the maximum likelihood estimates of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta_o, \beta_1, \ldots, \beta_n, \kappa$ and λ (or an equivalent set of 3n+3 parameters which completely defines the system) are all consistent.

We will finally give a convenient condition for a system to be identifiable. To do so we have to impose some additional restrictions on the input signal u. This is done as follows

DEFINITION

An input signal u is said to be persistently exciting of order m if the limits

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \tag{5.7}$$

$$r_u(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t+t) \qquad t = 0, 1, 2, \ldots \tag{5.8}$$

exist, and if the matrix

$$\begin{bmatrix} r_u(0) & r_u(1) & \ldots & r_u(m) \\ r_u(1) & r_u(0) & & r_u(m-1) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ r_u(m) & r_u(m-1) & & r_u(0) \end{bmatrix} \tag{5.9}$$

is positive definite.

We can now state the main result

THEOREM 3

The system (2.1) is completely identifiable if the input u is persistently exciting of order $2n$, and the system (2.1) is completely controllable either from u or e.

The theorem is proven in the appendix. Both conditions are easy to verify. In fact, when stating the problem in section 2, we actually made assumptions which imply that the system (2.1) is completely controllable from u or e. In practice, however, we never know the properties of the system to be identified and the condition on controllability may be violated. This occurs e.g. when we try to identify a model of higher order than the original system. Compare Example 3. However, whenever the system is not identifiable the estimate of the information matrix becomes singular for large values of N (Theorem 2).

Asymptotic Normality

We have the following result

THEOREM 4

Let θ denote the (3n+3) dimensional vector of parameters where those corresponding to the initial states have been deleted. Assume that S_o contains only the point θ_o so that $\hat{\theta}^N$ is consistent. Then the stochastic variable $L_{\theta\theta}(\theta_o, \theta_o)\sqrt{N}(\hat{\theta}^N - \theta_o)$ is asymptotically normal $(L', -L_{\theta\theta})$. If in addition $L_{\theta\theta}(\theta_o, \theta_o)$ is non-singular, then $\hat{\theta}^N$ is asymptotically normal $(\theta_o, -\frac{1}{N}L_{\theta\theta}^{-1})$ and asymptotically efficient.

This theorem is analogous to one of the standard theorems for maximum likelihood estimates with independent samples. The proof is given in the appendix.

The Information Matrix

Fisher's information matrix $I^N(\theta)$ is defined by

$$I^N(\theta) = -E_o L^N_{\theta\theta}(y;\theta_o) = E_o L^N_\theta(y;\theta_o)\,[L^N_\theta(y;\theta_o)]^T \qquad (5.10)$$

This matrix yields valuable information about the estimation procedure. We have previously seen that Theorem 2 implies that an analysis of the rank of the information matrix reveals which components of θ that can be consistently estimated. In particular, Theorem 3 implies that if the submatrix $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta_o, \beta_1, \ldots, \beta_n, \kappa$ and λ is positive definite then all system parameters are consistently estimated. Theorem 4 then tells that the estimate of the system parameters is asymptotically normal with a covariance matrix that is easily obtained from the information matrix.

We have the following asymptotic expression for the value of the information matrix at the true parameters θ_o

$$I^N(\theta_o) = -N L_{\theta\theta}(\theta_o, \theta_o) + o(N)$$

See Lemma 2. In the computational procedure we are actually evaluating

$$L^N_{\theta\theta}(y;\hat{\theta}^N)$$

But according to Lemma 5, this quantity converges with probability one to the information matrix as N increases. Hence, with the numerical procedure we will actually obtain an almost sure estimate of the value of the information matrix at $\theta = \hat{\theta}^N$

We will now evaluate the information matrix directly. We have (3.3)

$$-L^N(y;\theta) = \frac{1}{2c_o^2} \sum_{t=1}^{N} \varepsilon^2(t) + N \log c_o - \frac{N}{2} \log 2\pi$$

where $\varepsilon(t)$ is related to the input u and the observation y by (3.1). In the equation (3.1) the quantities Φ, Γ, Δ, b_o and κ are functions of the parameters θ as given by (3.2) and (3.4). The output y is generated by the

dynamical system (2.11) which we write in the compact form

$$z(t+1) = F_o z(t) + G_o u(t) + H_o e(t)$$

$$y(t) = z_n(t) + b_{oo} u(t) + c_o e(t) + k_o \qquad (5.11)$$

where F_o, G_o, H_o, b_{oo}, c_{oo} and k_o are functions of the true parameters θ_o.

$$F = \begin{bmatrix} 0 & 0 & 0 & & 0 & -a_n \\ 1 & 0 & 0 & & 0 & -a_{n-1} \\ 0 & 1 & 0 & & 0 & -a_{n-2} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & & 1 & -a_1 \end{bmatrix}$$

$$G = \begin{bmatrix} b_n \\ b_{n-1} \\ . \\ . \\ . \\ b_1 \end{bmatrix} \quad , \quad H = \begin{bmatrix} c_n \\ c_{n-1} \\ . \\ . \\ . \\ c_1 \end{bmatrix} \qquad (5.12)$$

Differentiating (3.3) we get

$$\frac{\partial L^N(\theta)}{\partial c_o} = \frac{1}{c_o^3} \sum_{t=1}^{N} \varepsilon^2(t) + \frac{N}{c_o}$$

$$\frac{\partial^2 L^N(\theta)}{\partial c_o \partial \theta_i} = \frac{2}{c_o^3} \sum_{t=1}^{N} \varepsilon(t) \frac{\partial \varepsilon(t)}{\partial \theta_i} \qquad i = 1, 2, \ldots, 4n+2$$

$$\frac{\partial^2 L^N(\theta)}{\partial c_o^2} = \frac{3}{c_o^4} \sum_{t=1}^{N} \varepsilon^2(t) - \frac{N}{c_o^2}$$

But $\epsilon(t)$ and $\frac{\partial \epsilon(t)}{\partial \theta_i}$ are independent if $\theta = \theta_o$ see (3.13) - (3.15). Hence

$$E_o \frac{\partial^2 L^N(\theta_o)}{\partial c_o \partial \theta_i} = 0 \qquad i = 1, 2, \ldots, 4n+2$$

$$E_o \frac{\partial^2 L^N(\theta_o)}{\partial c_o^2} = \frac{2N}{c_o^2}$$

The information matrix thus has the form

$$I^N(\theta) = \left[\begin{array}{c|c} c_o^{-2} E_o V_{\theta\theta} & 0 \\ \hline 0 & 2Nc_o^{-2} \end{array}\right] \qquad (5.13)$$

We will now evaluate $E_o V^N_{\theta\theta}$. To do this we will proceed in the same way as was done in section 3. One could equally well proceed along the lines of section 4. Using the fact that for $\theta = \theta_o$ the random variables $\epsilon(t)$ and $\frac{\partial^2 \epsilon(t)}{\partial \theta_i \partial \theta_j}$ are independent and taking mathematical expectation of the equation (3.12) we get

$$I^N_{ij}(\theta) = E_o \frac{\partial^2 V(\theta)}{\partial \theta_i \partial \theta_j} = \sum_{t=1}^{N} E_o \frac{\partial \epsilon(t)}{\partial \theta_i} \cdot \frac{\partial \epsilon(t)}{\partial \theta_j}$$

$$= \sum_{t=1}^{N} E_o \frac{\partial \epsilon(t)}{\partial \theta_i} + \sum_{t=1}^{N} \operatorname{cov}_o \left(\frac{\partial \epsilon(t)}{\partial \theta_i}, \frac{\partial \epsilon(t)}{\partial \theta_j} \right) \qquad i, j = 1, 2, \ldots, 4n+2 \quad (5.14)$$

To evaluate the information matrix, we thus have to compute the first two moments of $\frac{\partial \epsilon(t)}{\partial \theta_i}$. In section 3, we found that these derivatives $\frac{\partial \epsilon(t)}{\partial \theta_i}$ could be conveniently represented as outputs of dynamical systems driven by the inputs $u(t)$ and the observations $y(t)$. This representation is now conveniently used when evaluating the moments of the derivatives. It is convenient to consider the two terms of (5.14) separately.

We will first calculate the mean of $\frac{\partial \epsilon(t)}{\partial \theta_i}$. We will consider four cases

Case 1

Let $i = 1, 2, \ldots, n, 2n+1, 2n+2, \ldots, 3n$. We then have (3.13)

$$\frac{\partial \epsilon(t)}{\partial \theta_i} = \frac{\partial x_n(t)}{\partial \theta_i}$$

The derivatives of x_n are given by (3.14). Taking mathematical expectation and observing that $x(t) = -z(t)$ if $\theta = \theta_o$ we get

$$\begin{bmatrix} \frac{\partial \bar{x}(t+1)}{\partial \theta_i} \\ \frac{\partial \bar{x}(t+1)}{\partial \theta_{2n+i}} \\ \bar{z}(t+1 \end{bmatrix} = \begin{bmatrix} \Phi & 0 & E^i \\ 0 & \Phi & E^i \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{x}(t)}{\partial \theta_i} \\ \frac{\partial \bar{x}(t)}{\partial \theta_{2n+i}} \\ \bar{z}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ G \end{bmatrix} u(t)$$

$$\begin{bmatrix} \frac{\partial \bar{x}(1)}{\partial \theta_i} \\ \frac{\partial \bar{x}(1)}{\partial \theta_{2n+i}} \\ \bar{z}(1) \end{bmatrix} = 0 \qquad (5.15)$$

where $\bar{x}$ stands for the mathematical expectation of x. Notice that due to (3.29) it is sufficient to solve the equation for $i = 1$.

The equation (5.15) implies that

$$\frac{\partial \bar{x}(t)}{\partial \theta_i} - \frac{\partial \bar{x}(t)}{\partial \theta_{2n+i}} = 0 \text{ , for all } t$$

This is a consequence of the fact that all the components in the sub-space $\frac{\partial \bar{x}}{\partial \theta_i} - \frac{\partial \bar{x}}{\partial \theta_{2n+i}}$ of the dynamical system (5.15) are not controllable from the input $u(t)$. This is, of course, very natural, as the coefficients

$\theta_i - \theta_{2n+i} = \alpha_i$ do not reflect the transmission of the input u(t) to the output y(t).

Case 2

Now consider i = n+1, n+2, . . . , 2n. We have then $\frac{\partial \epsilon(t)}{\partial \theta_i} = \frac{\partial x_n(t)}{\partial \theta_i}$

and the derivatives of x_n are given by (3.14). Specializing this equation, we get

$$\frac{\partial x(t+1)}{\partial \theta_i} = \Phi \frac{\partial x(t)}{\partial \theta_i} + \Gamma u(t) \qquad i = n+1, n+2, \ldots, 2n$$

$$\frac{\partial x(1)}{\partial \theta_i} = 0$$

This equation does not contain any stochastic elements.

Case 3

Now consider i = 3n+1, 3n+2, . . . , 4n. We find by (3.14) that

$$\frac{\partial \epsilon(t)}{\partial \theta_i} = \frac{\partial x_n(t)}{\partial \theta_i}$$

The derivatives of x_n are given by (3.14) i.e.

$$\frac{\partial x(t+1)}{\partial \theta_i} = \Phi \frac{\partial x(t)}{\partial \theta_i}$$

$$\frac{\partial x_j(1)}{\partial \theta_i} = \delta_{ij}$$

This equation does not contain any stochastic elements.

Case 4

Now consider i = 4n+1 and i = 4n+2. We have (3.13)

$$\frac{\partial \epsilon(t)}{\partial \theta_{4n+1}} = -u(t)$$

$$\frac{\partial \epsilon(t)}{\partial \theta_{4n+2}} = \frac{\partial x_n(t)}{\partial \theta_{4n+2}} - 1$$

and the derivatives of $x_n(t)$ with respect to θ_{4n+2} are given by

$$\frac{\partial x(t+1)}{\partial \theta_{4n+2}} = \Phi \frac{\partial x(t)}{\partial \theta_{4n+2}} - \Delta$$

$$\frac{\partial x(1)}{\partial \theta_{4n+2}} = 0$$

These equations do not contain any stochastic elements.

Now we will compute the last term of the equation (5.14). It follows from the previous analysis that only terms with i, j = 1, 2, . . . , n, 2n+1, 2n+2, . . . , 3n are nonvanishing.

Introduce

$$\tilde{x} = x - Ex = x - \bar{x}$$

We get

$$\begin{bmatrix} \frac{\partial \tilde{x}(t+1)}{\partial \theta_i} \\ \frac{\partial \tilde{x}(t+1)}{\partial \theta_{2n+i}} \\ \tilde{z}(t+1) \end{bmatrix} = \begin{bmatrix} \Phi & 0 & E^i \\ 0 & \Phi & E^i \\ 0 & 0 & F \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{x}(t)}{\partial \theta_i} \\ \frac{\partial \tilde{x}(t)}{\partial \theta_{2n+i}} \\ \tilde{z}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_o e^i \\ H \end{bmatrix} e(t)$$

$$\begin{bmatrix} \frac{\partial \tilde{x}(1)}{\partial \theta_i} \\ \frac{\partial \tilde{x}(1)}{\partial \theta_{2n+i}} \\ \tilde{z}(1) \end{bmatrix} = 0$$

Now let R denote the covariance matrix of the solution of this equation i.e.

$$R = E \begin{bmatrix} \frac{\partial \tilde{x}}{\partial \theta_i} \\ \frac{\partial \tilde{x}}{\partial \theta_{n+i}} \\ \tilde{z} \end{bmatrix} \left[\frac{\partial \tilde{x}}{\partial \theta_i} , \frac{\partial \tilde{x}}{\partial \theta_{n+i}} , \tilde{z} \right]$$

we get the following equation for R

$$R(t+1) = \Xi R(t) \Xi^T + R_o$$

$$R(1) = 0$$

where

$$\Xi = \begin{bmatrix} \Phi & 0 & E^i \\ 0 & \Phi & E^i \\ 0 & 0 & \Psi \end{bmatrix} \quad R_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e^i (e^i)^T c_o^2 & c_o e^i D^T \\ 0 & D(e^i)^T c_o & DD^T \end{bmatrix}$$

Again we notice that it is only necessary to consider the solution for i = 1, the solutions for other values of i are then obtained from (3.30).

From now on, let R denote the solution associated with i = 1. Then

$$\text{cov} \left(\frac{\partial \epsilon(t)}{\partial \theta_1} , \frac{\partial \epsilon(t)}{\partial \theta_1} \right) = R_{n,n}(t)$$

$$\text{cov} \left(\frac{\partial \epsilon(t)}{\partial \theta_1} , \frac{\partial \epsilon(t)}{\partial \theta_{2n+1}} \right) = R_{n,2n}(t)$$

$$\text{cov} \left(\frac{\partial \epsilon(t)}{\partial \theta_{2n+1}} , \frac{\partial \epsilon(t)}{\partial \theta_{2n+1}} \right) = R_{2n,2n}(t)$$

To obtain the results for other values of i and j we use (3.29) and obtain

$$\operatorname{cov}\left(\frac{\partial \epsilon(t)}{\partial \theta_{i+1}}, \frac{\partial \epsilon(t)}{\partial \theta_{2n+j+1}}\right) = \{\Xi^{|i-j|} R(t-i-j)\}_{n,n}$$

$$\operatorname{cov}\left(\frac{\partial \epsilon(t)}{\partial \theta_{i+1}}, \frac{\partial \epsilon(t)}{\partial \theta_{2n+j+1}}\right) = \{\Xi^{|i-j|} R(t-i-j)\}_{n,2n}$$

$$\operatorname{cov}\left(\frac{\partial \epsilon(t)}{\partial \theta_{i+1}}, \frac{\partial \epsilon(t)}{\partial \theta_{2n+j+1}}\right) = \{\Xi^{|i-j|} R(t-i-j)\}_{2n,2n}$$

Thus having completed the computations, we will summarize and discuss the results. The information matrix can be written as the sum of two terms. One term has the components

$$\sum_{t=1}^{N} E_o \frac{\partial \epsilon(t)}{\partial \theta_i} E_o \frac{\partial \epsilon(t)}{\partial \theta_i}$$

which only depend on the input $u(t)$. The other term has the components

$$\sum_{t=1}^{N} \operatorname{cov}\left(\frac{\partial \epsilon(t)}{\partial \theta_i}, \frac{\partial \epsilon(t)}{\partial \theta_j}\right)$$

Neither term is positive definite. The properties of $I^N(\theta)$ as N tends to infinity are of particular interest because we have

$$L_{\theta\theta}(\theta_o, \theta_o) = \lim_{N\to\infty} \frac{1}{N} I^N(\theta_o)$$

where $L_{\theta\theta}$ is the matrix of Theorems 2 and 4.

In particular, we find that

$$\{I_{ij}^N(\theta);\ i,j = 3n+1,\ 3n+2, \ldots, 4n\}$$

is bounded. It then follows from Theorem 2 that the estimate of the corresponding parameters i.e. the initial conditions are not consistent. Similarly, we find that a sufficient condition for the estimates of $\theta_1, \ldots, \theta_n$ and $\theta_{2n+1}, \ldots, \theta_{3n}$ to be consistent is that the difference equation

$$z(t+1) = F\, z(t) + He(t)$$

is controllable.

For the other parameters we find that two types of conditions have to be imposed if the estimates should be consistent. The difference equations generating the partial derivatives must be controllable, and the input u(t) must be such that the system is persistently excited (Theorem 3). For example, a step input is not sufficient. In such a case, we find that $\frac{\partial \varepsilon(t)}{\partial \theta_i}$ $i = n+1, \ldots, 2n$ and $\frac{\partial \varepsilon(t)}{\partial \theta_{4n+2}}$ will converge to constants. The matrix

$$\{\frac{1}{N} I^N_{ij}(\theta), \quad i,j = n+1, n+2, \ldots, 2n, 4n+2\}$$

the n converges to a matrix of rank 1 and this implies that only a linear combination of the parameters θ_{n+1}, $\theta_{n+2}, \ldots, \theta_{2n}$ and θ_{4n+2} can be estimated (Theorem 2). This is very natural because the effect of a step disturbance cannot be separated from an unknown DC level in the output if N is large.

NUMERICAL ALGORITHM

In sections 3 and 4 the identification problem was reduced to one of minimizing the function $V(\theta)$ (maximizing the likelihood function). We will now discuss in detail how this can be done and some of the difficulties that may occur. In this chapter, the formulas and notations are adapted to the presentation in chapter 3, but the discussion is valid more generally.

The algorithm developed in chapters 3 and 4 is essentially a method of computing the functions V_θ and $V_{\theta\theta}$ for an arbitrary argument θ. The Newton-Raphson algorithm [27] [71] developed in section 3 essentially attempts to find a parameter value θ such that $V_\theta = 0$. When trying to find the maximum likelihood estimate using this algorithm, two difficulties may arise.

- The function $V(\theta)$ may have several stationary points.
- The algorithm may not converge.

An enlightening discussion of the difficulties associated with multiple maxima of the likelihood function, for the case of independent samples from a common distribution, is given by Huzurbazar [37]. In general there is very little that can be done about this case except to find all the local minima of $V(\theta)$. In practice, we let it suffice to choose a few different starting values. For the case of independent samples Huzurbazar has also proven that at least asymptotically there cannot be two absolute minima with the same magnitude.

A class of algorithms (gradient routines) can be written as

$$\theta^{k+1} = \theta^k - A(\theta^k)\, V_\theta(\theta^k) \qquad (6.1)$$

By suitable choices of matrix $A(\theta)$ we can obtain algorithms both for finding local minima of $V(\theta)$ and for finding its stationary points. A general discussion of gradient routines is found in [82].

Obviously, it is possible to substitute θ by any differentiable function of θ that is unique in both directions, and define new derivatives with respect to the new parameters. Since the structures in chapters 3 and 4 are slightly different (different parameters are chosen), the convergence properties need not necessarily be the same. The question, whether there is an "optimal" structure such that the convergence properties are particularly advantageous, has not been considered.

A few special cases of the function $A(\theta)$ are well known:

Steepest descent: $A(\theta) = kI$

Newton-Raphson method: $A(\theta) = V_{\theta\theta}^{-1}(\theta)$

Method of scoring: $A(\theta) = I^{-1}(\theta)$ ($I(\theta)$ = information matrix)

Markov estimate: $A(\theta) = [MV_{\theta\theta}M^T + I - MM^T]^{-1}M$

where $M = \{m_{ij}\}$ and

$$m_{ij} = \begin{cases} 0 & i = 1, \ldots, n \\ \delta_{i-j} & i = n+1, \ldots, 4n+2 \end{cases}$$

In the last case, the first n components (the noise parameters) are not changed from the starting values. This corresponds to the best estimate with a known noise spectrum. Specifically, if the noise parameters are made equal to zero, the Kalman estimate of the remaining parameters is obtained. It should be noted that for Markov and Kalman estimates, the procedure converges in one step from any starting point, but the noise components can never be estimated. A discussion of the first three cases and a few others is given by Kale [44], [45].

Some numerical experiments have been performed in order to obtain guidelines for the choice of suitable techniques. The method of steepest descent always converges to a local minimum for k small enough.

However, experiments have shown that the convergence is generally intolerably slow, even if k is chosen optimal for each case. This agrees with Kale's experience. The Newton-Raphson procedure converges very rapidly if the starting point is chosen near the root, otherwise generally not at all.

Between these two extremes a variety of modifications are feasible. The following alternatives have been considered:

- An approximate second derivative matrix $V^{*}_{\theta\theta}(\theta)$ with the property of being positive definite: $A(\theta) = V^{*-1}_{\theta\theta}(\theta)$.

This prevents the procedure from converging to any local saddle point or local maximum.

A matrix with the desired properties is obtained from

$$\frac{\partial^2 V^*}{\partial\theta_i \partial\theta_j} = \sum_{t=1}^{N} \frac{\partial\varepsilon(t)}{\partial\theta_i} \frac{\partial\varepsilon(t)}{\partial\theta_j} \qquad i, j = 1, \ldots, 4n+2 \qquad (6.2)$$

- A factor α to reduce the length of the steps when far from the root:

$$A(\theta) = \alpha V^{*-1}_{\theta\theta}(\theta)$$

- Steepest descent with computed local curvature.

$$A(\theta) = \frac{V_\theta^T V_\theta}{V_\theta^T V_{\theta\theta} V_\theta} I$$

- Various step control schemes, e.g. the step $A(\theta) V_\theta(\theta)$ halved if some test on next approximation gives an improbable result.

The matrix of (6.2), referred to as "approximative second derivative matrix", is a good approximation to the actual $V_{\theta\theta}$ at the true parameter value for large values of N, since it converges to $V_{\theta\theta}$ with probability one. Furthermore, it is always at least non-negative definite,

and it is obtained with less computation than the exact second derivatives. The alternative is closely related to the method of "scoring" $A(\theta) = I^{-1}(\theta)$ but the latter requires the additional computations involved in the calculation of the information matrix $I(\theta)$. (Compare section 5)

For the standard algorithm the following procedure is chosen:

1. Put the starting point $\theta^o = 0$ and

   $$A(\theta) = [MV_{\theta\theta}(\theta^o)\, M^T + I - MM^T]^{-1} M$$

   This yields a Kalman estimate for θ_i, $\quad i = n+1, \ldots, 4n+2$

2. Put $A(\theta) = V^{*-1}_{\theta\theta}(\theta)$, where $V^*_{\theta\theta}(\theta)$ is the "approximate second derivative matrix" and repeat the calculations

   $$\theta^{k+1} = \theta^k - A(\theta^k)\, V_\theta(\theta^k) \qquad k = 1, 2, \ldots$$

   until the computed step length is smaller than some preassigned value.

3. Put $A(\theta) = V^{-1}_{\theta\theta}(\theta)$ and repeat calculations until the required accuracy is reached.

Our experience has shown that this algorithm will generally converge for systems of low order. In certain cases, particularly if the matrix Φ has eigenvalues close to the unit circle, we have sometimes had difficulties (see Section 7). In these cases we have almost always obtained convergence by using some of the alternatives mentioned previously. This requires however a "man in the loop". This should not be a serious disadvantage for off line identification. In on line identification, the problem should also not be serious, since we have then got a good starting value. However, since little experience is available, we drop the subject here. (Schrödinger [72] p. 76)

The state vector x and its derivatives with respect to the parameters $\theta_1, \theta_{n+1}, \theta_{2n+1}, \theta_{3n+1}, \theta_{4n+2}$ are obtained from the difference equations

of the form $\xi(t+1) = \Phi\,\xi(t) + \gamma$

In the program the derivatives are represented by a matrix

$$x = \frac{\partial^2 x}{\partial\theta_1^2},\ \frac{\partial^2 x}{\partial\theta_1\partial\theta_{n+1}},\ \frac{\partial^2 x}{\partial\theta_1\partial\theta_{2n+1}},\ \frac{\partial^2 x}{\partial\theta_1\partial\theta_{4n+2}},\ \frac{\partial^2 x}{\partial\theta_1\partial\theta_{3n+1}},\ \frac{\partial^2 x}{\partial\theta_{2n+1}\partial\theta_{4n+2}},$$

$$\frac{\partial x}{\partial\theta_1},\ \frac{\partial x}{\partial\theta_{n+1}},\ \frac{\partial x}{\partial\theta_{2n+1}},\ \frac{\partial x}{\partial\theta_{4n+2}},\ \frac{\partial x}{\partial\theta_{3n+1}},\ x$$

The derivatives of the n:th component of x are then taken together in vectors in the following manner:

$$F1(i) = \frac{\partial x_n}{\partial\theta_i}$$

$$F2(i) = \frac{\partial x_n}{\partial\theta_{n+i}}$$

$$F3(i) = \frac{\partial x_n}{\partial\theta_{2n+i}}$$

$$F4(i) = \frac{\partial x_n}{\partial\theta_{3n+i}}$$

$$S1(i) = \frac{\partial^2 x_n}{\partial\theta_1\partial\theta_i} \qquad i = 1, 2, \ldots, n$$

$$S2(i) = \frac{\partial^2 x_n}{\partial\theta_1\partial\theta_{n+i}}$$

$$S3(i) = \frac{\partial^2 x_n}{\partial\theta_1\partial\theta_{2n+i}}$$

$$S4(i) = \frac{\partial^2 x_n}{\partial\theta_1\partial\theta_{3n+i}}$$

$$S5(i) = \frac{\partial^2 x_n}{\partial\theta_i\partial\theta_{4n+2}}$$

$$S6(i) = \frac{\partial^2 x_n}{\partial\theta_{2n+i}\partial\theta_{4n+2}}$$

Loss function:

$V1 = 2 \cdot V(\theta)$

First order derivatives of loss function with respect to the parameters are denoted by VZ(i).

Second order partial derivatives of loss function are denoted by VZZ(I, 3).

The intermediate sums of squares of $\epsilon(t)$ have the notations

$$VAA(i) = SUM1(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_i} \frac{\partial \epsilon}{\partial \theta_i}$$

$$VAB(i) = SUM2(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_1} \frac{\partial \epsilon}{\partial \theta_{n+i}}$$

$$VAC(i) = SUM4(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_1} \frac{\partial \epsilon}{\partial \theta_{2n+i}}$$

$$VAD(i) = SUM7(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_1} \frac{\partial \epsilon}{\partial \theta_{3n+i}}$$

$$VAE = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_i} \frac{\partial \epsilon}{\partial \theta_{4n+1}}$$

$$VAF = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_1} \frac{\partial \epsilon}{\partial \theta_{4n+2}}$$

$$VBA(i) = SUM11(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_i}$$

$$VBB(i) = SUM3(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_{n+i}}$$

$$VBC(i) = SUM5(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_{2n+i}}$$

$$VBD(i) = SUM8(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_{3n+i}}$$

$$VBE = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+1}}$$

$$VBF = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+2}}$$

$$VCA(i) = SUM12(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{i}}$$

$$VCB(i) = SUM13(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{n+i}}$$

$$VCC(i) = SUM6(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{2n+i}}$$

$$VCD(i) = SUM9(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{3n+i}}$$

$$VCE = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+1}}$$

$$VCF = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+2}}$$

$$VDA(i) = SUM14(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{3n+1}} \frac{\partial \epsilon}{\partial \theta_{i}}$$

$$VDB(i) = SUM15(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{3n+1}} \frac{\partial \epsilon}{\partial \theta_{n+i}}$$

$$VDC(i) = SUM16(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{2n+i}}$$

$$VDD(i) = SUM10(i) = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{3n+1}} \frac{\partial \epsilon}{\partial \theta_{3n+i}}$$

$$VDE = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+1}}$$

$$VDF = \sum_{1}^{t} \frac{\partial \epsilon}{\partial \theta_{2n+1}} \frac{\partial \epsilon}{\partial \theta_{4n+2}}$$

The terms $\Sigma \epsilon \frac{\partial^2 \epsilon}{\partial \theta_i \partial \theta_j}$ if desired are summed up by the variables SM

$$SM1(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial \theta_i \partial \theta_1}$$

$$SM2(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial\theta_i \partial\theta_{n+1}}$$

$$SM3(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial\theta_i \partial\theta_{2n+1}}$$

$$SM4(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial\theta_i \partial\theta_{3n+1}}$$

$$SM5(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial\theta_i \partial\theta_{4n+2}}$$

$$SM6(i) = \sum_{1}^{t} \epsilon \frac{\partial^2 \epsilon}{\partial\theta_{2n+i} \partial\theta_{4n+2}}$$

The matrix of second order derivatives VZZ can be formed from these variables.

```
STMNT       FORTRAN STATEMENT

C           IDENTIFICATION OF LINEAR SINGLE INPUT SINGLE OUTPUT SYSTEM WITH
C           FORCING NOISE AND MEASUREMENT ERRORS
C
C           INPUT DATA
C            1 TWO TITLE CARDS
C            2 N      NUMBER OF VALUES OF THE INPUT SERIES
C              K6     FIRST ORDER OF MODEL
C              KMAX   MAXIMUM ORDER OF MODEL
C              DELTA  STOP VARIABLE FOR ITERATION OF INITIAL STATE DERIVATIVES
C              ALFA   CONVERGATION FACTOR
C              LEAST = 1 IF BEGIN RUN WITH LEAST SQUARE ESTIMAT
C            3 Y  MEASUREMENT
C              U  INPUT SERIES
C            4 INITIAL ESTIMAT OF MODEL PARAMETERS
C
C           SWITCH
C            C ON  EXACT SECOND ORDER DERIVATIVE MATRIX
C            D ON  NEW CONVERGENCE FACTOR IS READ IN
C            G ON  ITERATION IS INTERRUPTED AFTER COMPLETING LAST STEP
C
C           TAPE
C            3  INPUT 3
C            4  OUTPUT FOR PLOTTER
C            5  OUTPUT
C
      0   DIMENSION  A(2),B(2),C(2),D(2),X(2,12),F1(2),F2(2),F3(2),F4(2),
      1   S1(3),S2(3),S3(3),S4(3),SM1(3),SM2(3),SM3(3),SM4(3),SUM1(2),SUM2(2
      2   ),SUM3(2),SUM4(2),SUM5(2),SUM6(2),SUM7(2),SUM8(2),SUM9(2),SUM10(2)
      3   ,SUM11(2),SUM12(2),SUM13(2),SUM14(2),SUM15(2),SJM16(2),VZ(10),
      4   VZZ(10,11),G2(10,11),VEC(92),S5(2),S6(2),SM5(2),SM6(2)
          EQUIVALENCE (G2,VZZ)
      0   EQUIVALENCE (VEC(1),F1),(VEC(3),F2),(VEC(5),F3),(VEC(7),F4),
      1   (VEC(9),SUM1),(VEC(11),SUM2),(VEC(13),SUM3),(VEC(15),SUM4),
      2   (VEC(17),SUM5),(VEC(19),SUM6),(VEC(21),SUM7),(VEC(23),SUM8),
      3   (VEC(25),SUM9),(VEC(27),SUM10),(VEC(29),SUM11),(VEC(31),SUM12),
      4   (VEC(33),SUM13),(VEC(35),SUM14),(VEC(37),SUM15),(VEC(39),SUM16),
      5   (VEC(41),S1),(VEC(44),S2),(VEC(47),S3),(VEC(50),S4),(VEC(53),SM1),
      6   (VEC(56),SM2),(VEC(59),SM3),(VEC(62),SM4),(VEC(65),VZ)
      7   ,(VEC(75),VAE),(VEC(76),VBE),(VEC(77),VCE),(VEC(78),VAF),(VEC(79),
      8   VBF),(VEC(80),VCF),(VEC(81),VEE),(VEC(82),VEF),(VEC(83),VFF),
      9   (VEC(84),S5),(VEC(86),S6),(VEC(88),SM5),(VEC(90),SM6),(VEC(92),XF)
C
C           INPUT
1           READ 601,N,K6,KMAX,DELTA ,ALFA,LEAST
601         FORMAT (3I5,2E10.5,I5)
602         FORMAT(5E16.8)
C
C           PRINT INPUT
            PRINT 800
800         FORMAT(1H1)
802         FORMAT(1X5E16.8)
C
C           CHOOSE K
```

```
STMNT       FORTRAN STATEMENT

100         K = K6
            K4 = K-1
            K6 = K+1
            KK = K*K
            KK4 = KK-1
            KKK = KK*K
            KKK6 = KKK+1
            KKKK = KKK*K
            KKKK6 = KKKK+1
            KKKK7 = KKKK+2
            KKKK8 = KKKK+3
            NK = N-K
C
C           READ INITIAL GUESS OF PARAMETERS
            READ 602,(A(I),I = 1,K)
            READ 602,(B(I),I = 1,K)
            READ 602,(C(I),I = 1,K)
            READ 602,(D(I),I = 1,K)
            READ 602,E,F
            PRINT 800
            PRINT 802,(A(I),I = 1,K)
            PRINT 802,(B(I),I = 1,K)
            PRINT 802,(C(I),I = 1,K)
            PRINT 802,(D(I),I = 1,K)
            PRINT 802,E,F
          0 WRITE TAPE 5,(A(I),I = 1,K),(B(I),I = 1,K),(C(I),I = 1,K),(D(I),
          1 I = 1,K),E,F
C
C           INITIATION
200         REWIND 3
            DO 220 I = 1,K
            DO 215 J = 1,11
215         X(I,J) = 0.
            L = K6-I
220         X(I,12) = D(L)
            X(K,11) = 1.
            DO 230 I = 1,92
230         VEC(I) = 0.
            F4(1) = 1.
            S10(1) = 1.
            READ TAPE 3,U,Y
            Y = Y-F
            S = D(1)-E*U+Y
            V1 = S*S
            VZ(KKK6) = S
            VZ(KKKK6) = -S*U
            VZ(KKKK7) = -S
            VDE = -U
            VDF = -1.
            VEE = U*U
            VEF = U
            VFF = 1.
            KFI = 0
```

```
STMNT       FORTRAN STATEMENT

            I1 = 1
            IF (SENSE SWITCH 2) 300,250
250         I1 = 7
C
C           INTEGRATION
300         DO 570 LOOP = 2,N
C
C           STATE EQUATION
            DO 330 I = I1,12
            S = X(K,I)
            IF (K4) 31000,31000,305
305         DO 310 J = 1,K4
            L = K-J
310         X(L+1,I) = X(L,I)-A(J)*S
31000       X(1,I) = -A(K)*S
            IF (I-11) 3100,330,322
3100        GO TO (311,312,312,314,314,316,317,318,319,320),I
311         X(K,1) = X(K,1)-X(K,7)-X(K,7)
            GO TO 330
312         X(K,I) = X(K,I)-X(K,I+6)
            GO TO 330
314         X(K,I) = X(K,I)-X(K,I+6)
            IF (KFI) 315,330,330
315         I = 5
            GO TO 330
316         X(K,6) = X(K,6)-1.
            GO TO 330
317         X(K,7) = X(K,7)-X(K,12)
            GO TO 330
318         X(K,8) = X(K,8)+U
            GO TO 330
319         X(K,9) = X(K,9)+Y
            GO TO 330
320         IF (KFI) 321,330,330
321         I = 11
            GO TO 330
322         DO 325 J = 1,K
            L = K6-J
            X(J,10) = X(J,10)-C(L)
325         X(J,12) = X(J,12)+B(L)*U+C(L)*Y
330         CONTINUE
            XF = X(K,10)-1.
C
C           EXAMINE LAST POWER OF FI
            IF (KFI) 370,340,340
340         S = 0.
            DO 345 I = 1,K
345         S = S+ABSF(X(I,11))
            IF (S-DELTA ) 350,350,360
350         KFI = KFI+1
            IF (KFI-K) 370,370,355
355         KFI = -1
            GO TO 370
```

```
STMNT       FORTRAN STATEMENT

360         KFI = 0
C
C           TRANSLATION OF PARTIAL DERIVATIVES OF STATE
370         IF (K4) 376,376,371
371         DO 375 I = 1,K4
            J = K-I
            J6 = J+1
            F1(J6) = F1(J)
            F2(J6) = F2(J)
            F3(J6) = F3(J)
            IF (KFI) 375,374,374
374         F4(J6) = F4(J)
375         CONTINUE
376         F1(1) = X(K,7)
            F2(1) = X(K,8)
            F3(1) = X(K,9)
            F4(1) = X(K,11)
C
            IF (SENSE SWITCH 2) 380,390
380         DO 385 I = 2,KK4
            J = KK-I
            J6 = J+1
            S1(J6) = S1(J)
            S2(J6) = S2(J)
            S3(J6) = S3(J)
            IF (KFI) 385,384,384
384         S4(J6) = S4(J)
385         CONTINUE
            DO 388 I = 1,K4
            J = K-I
            J6 = J+1
            S5(J6) = S5(J)
388         S6(J6) = S6(J)
            S1(1) = X(K,1)
            S2(1) = X(K,2)
            S3(1) = X(K,3)
            S4(1) = X(K,5)
            S5(1) = X(K,4)
            S6(1) = X(K,6)
C
C           LOSS FUNCTION
390         READ TAPE 3,U,Y
            Y. = Y-F
            S = X(K,12)-E*U+Y
            V1 = V1+S*S
C
C           FIRST ORDER PARTIAL DERIVATIVES OF LOSS
            DO 420 I = 1,K
            J = I+K
            L = I+KK
            M = I+KKK
            VZ(I) = VZ(I)+S*F1(I)
            VZ(J) = VZ(J)+S*F2(I)
```

```
STMNT       FORTRAN STATEMENT

            VZ(L) = VZ(L)+S*F3(I)
            IF (KFI) 420,415,415
415         VZ(M) = VZ(M)+S*F4(I)
420         CONTINUE
            VZ(KKKK6) = VZ(KKKK6)-S*U
            VZ(KKKK7) = VZ(KKKK7)+S*XF
C
C           SUM PRODUCT OF EPSILON AND SECOND ORDER PARTIAL DERIVATIVES
            IF (SENSE SWITCH 2) 425,450
425         DO 435 I = 1,KK4
            SM1(I) = SM1(I)+S*S1(I)
            SM2(I) = SM2(I)+S*S2(I)
            SM3(I) = SM3(I)+S*S3(I)
            IF (KFI) 435,430,430
430         SM4(I) = SM4(I)+S*S4(I)
435         CONTINUE
            DO 440 I = 1,K
            SM5(I) = SM5(I)+S*S5(I)
440         SM6(I) = SM6(I)+S*S6(I)
C
C           SUM PRODUCT OF FIRST ORDER DERIVATIVES OF STATE
450         S = F1(1)
            T = F2(1)
            SS = F3(1)
            TT = F4(1)
            DO 470 I = 1,K
            SUM1(I) = SUM1(I)+S*F1(I)
            SUM2(I) = SUM2(I)+S*F2(I)
            SUM3(I) = SUM3(I)+T*F2(I)
            SUM4(I) = SUM4(I)+S*F3(I)
            SUM5(I) = SUM5(I)+T*F3(I)
            SUM6(I) = SUM6(I)+SS*F3(I)
            IF (KFI) 470,460,460
460         SUM7(I) = SUM7(I)+S*F4(I)
            SUM8(I) = SUM8(I)+T*F4(I)
            SUM9(I) = SUM9(I)+SS*F4(I)
            SUM10(I) = SUM10(I)+TT*F4(I)
470         CONTINUE
C
            IF (K4) 494,494,480
480         DO 490 I = 2,K
            SUM11(I) = SUM11(I)+F1(I)*T
            SUM12(I) = SUM12(I)+F1(I)*SS
            SUM13(I) = SUM13(I)+F2(I)*SS
            IF (KFI) 496,485,485
485         SUM14(I) = SUM14(I)+F1(I)*TT
            SUM15(I) = SUM15(I)+F2(I)*TT
            SUM16(I) = SUM16(I)+F3(I)*TT
490         CONTINUE
494         IF (KFI) 496,495,495
495         VDE = VDE-F4(1)*U
            VDF = VDF+F4(1)*XF
496         VAE = VAE-F1(1)*U
```

```
STMNT       FORTRAN STA E^C^NT

            VZZ(L2,KKKK7)   = VCF
            VZZ(L3,KKKK7) = VDF
560         IF (LOOP-N) 570,8110,8110
8110        PRINT 802,(VZ(I),I = 1,KKKK7)
            PRINT 802,V1
570         CONTINUE
C
C           ADD SUM OF EPS AND SECOND ORDER DERIVATIVES OF STATE TO VZZ
            IF (SENSE SWITCH 2) 575,600
575         DO 595 I = 1,K
            DO 590 J = 1,K
            L = I*J-1
            IF (I-J) 580,580,585
580         VZZ(I,J) = VZZ(I,J)+SM1(L)
585         M = J+K
            VZZ(I,M) = VZZ(I,M)+SM2(L)
            M = J+KK
            VZZ(I,M) = VZZ(I,M)+SM3(L)
            M = J+KKK
590         VZZ(I,M) = VZZ(I,M)+SM4(L)
            J = I+KK
            VZZ(I,KKKK7) = VZZ(I,KKKK7)+SM5(I)
595         VZZ(J,KKKK7) = VZZ(J,KKKK7)+SM6(I)
C
C           SOLVE NEWTON RAPHSON EQUATION
600         VZZ(KKKK6,KKKK6) = VEE
            VZZ(KKKK6,KKKK7) = VEF
            VZZ(KKKK7,KKKK7) = VFF
            IF (SENSE SWITCH 3) 6005,6010
6005        READ 602,ALFA
6010        S = ALFA
            IF (LEAST) 604,604,601
601         DO 603 I = 1,K
            I6 = I+1
            DO 602 J = I6,KKKK7
602         VZZ(I,J) = 0.
            VZZ(I,I) = 1.
603         VZ(I) = 0.
            S = 1.
            LEAST = 0
604         DO 610 I = 1,KKKK7
            DO 605 J = I,KKKK7
605         VZZ(J,I) = VZZ(I,J)
610         VZZ(I,KKKK8) = VZ(I)*S
            DO 1855 I = 1,KKKK7
1855        PRINT 802,(VZZ(I,J),J = 1,KKKK7)
            WRITE TAPE 5,ALFA,V1,(VZ(I),I = 1,KKKK7)
            WRITE TAPE 5,VZZ
            ALFA = 1.-(1.-ALFA)**2
C
C           JORDAN CALCULATION
            DO 9150 I = 1,KKKK7
            T = 1./G2(I,I)
```

```
STMNT       FORTRAN STATEMENT

            DO 9120 J = 1,KKKK8
9120        G2(I,J) = G2(I,J)*T
            I6 = I+1
            DO 9150 J =  1,KKKK7
            IF (J-I) 9130,9150,9130
9130        T = G2(J,I)
            G2(J,I) = 0.
            DO 9140 L = I6,KKKK8
9140        G2(J,L) = G2(J,L)-T*G2(I,L)
9150        CONTINUE
C
            DO 720 I = 1,K
            A(I) = A(I)-VZZ(I,KKKK8)
            J = I+K
            B(I) = B(I)-VZZ(J,KKKK8)
            J = I+KK
            C(I) = C(I)-VZZ(J,KKKK8)
            J = I+KKK
720         D(I) = D(I)-VZZ(J,KKKK8)
            E = E-VZZ(KKKK6,KKKK8)
            F = F-VZZ(KKKK7,KKKK8)
C
            PRINT 802,(A(I),I = 1,K)
            PRINT 802,(B(I),I = 1,K)
            PRINT 802,(C(I),I = 1,K)
            PRINT 802,(D(I),I = 1,K)
            PRINT 802,E,F
          0 WRITE TAPE 5,(A(I),I = 1,K),(B(I),I = 1,K),(C(I),I = 1,K),(D(I),
          1 I = 1,K),E,F
C
C           EXAMINE THE DIFFERENCE IN PARAMETERS
            IF (SENSE SWITCH 6) 660,200
660         IF (K-KMAX) 100,1,1
C
            END
```

NUMERICAL EXAMPLES

The identification scheme has been tested on a number of artificially generated input/output records. In this section we give nine examples of first order systems generated from either of the equations

$$\begin{cases} x(t+1) = -a_1\, x(t) + b_1\, u(t) + d_1\, e(t) \\ y(t) = x(t) + b_o\, u(t) + d_o\, v(t) + k \end{cases} \tag{7.1}$$

or

$$y(t) + \alpha_1\, y(t-1) = \beta_o\, u(t) + \beta_1\, u(t-1) + \lambda\,[e(t) + \gamma_1\, e(t-1)] + \varkappa \tag{7.2}$$

The tenth example is a second order system with no input, taken from reference [5].

The numbers u(t), e(t), and v(t) were generated as suitable scaled sums of twelve rectangular (0, 1) pseudo-random numbers obtained from a modified Fibonacci series. The estimated representation is obtained on the form (ex 1-9)

$$\begin{cases} x(t+1) = -a_1\, x(t) + b_1\, u(t) + c_1\, e(t) \\ y(t) \quad\;\; = x(t) + b_o\, u(t) + c_o\, e(t) + k \end{cases} \tag{7.3}$$

The parameters a_1, b_1, c_1, b_o, k, c_o corresponding to the generating relation, are given in each case. The estimated information matrix is given in order to check the correspondence with the actual estimation error. The initial value is not estimated.

All disturbances are substituted by equivalent disturbances in the output. The power spectra of the equivalent disturbances have been computed. The spectral density functions $\varphi(\omega)$ have been computed from the formula

$$\varphi(2\pi f) = \lambda^2 \cdot \left| \frac{1 + \sum_{k=1}^{n} \gamma_k e^{i2\pi kf}}{1 + \sum_{k=1}^{n} a_k e^{i2\pi kf}} \right|^2$$

In all cases the following standard procedure was used. The initial values were chosen as zero and a Kalman estimate was computed. The approximative second derivatives were then used in a few steps and the exact second derivatives in the last iterations were used. If this procedure failed to converge other methods were tried. This is discussed in detail in the examples.

Example 1

Generating equation

$$\begin{cases} x(t) = 0.95\, x(t-1) + u(t-1) \\ y(t) = x(t) + v(t) \end{cases} \qquad t = 1, \ldots, 300 \qquad (7.4)$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	-0.95	1	0	0	0	1
Estimated	-0.948	1.034	0.012	0.081	-0.067	1.047

Estimated covariances

0.000006	0.000051	0.000014	0.000000	-0.000078	0.000000
0.000051	0.000959	0.000120	0.000074	0.000043	0.000000
0.000014	0.000120	0.000431	-0.000027	-0.000011	0.000021
0.000000	0.000074	-0.000027	0.003873	0.000313	0.000000
-0.000078	0.000043	-0.000011	0.000313	0.007912	0.000000
0.000000	0.000000	0.000021	0.000000	0.000000	0.001828

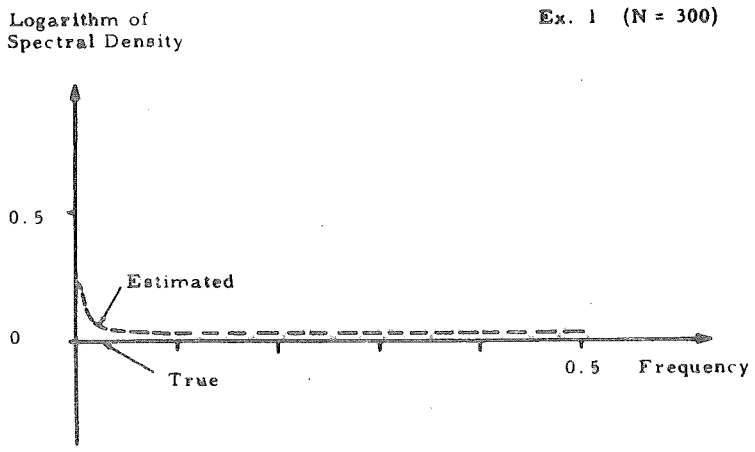


Figure 3
True and estimated power spectral densities of example 1.

Comment

The model (7.4) represents a very slow, first order system observed with independent errors. The process could not be identified for N = 100, the reason being that the equivalent parameter γ_1 in the representation (7.2) is close to the boundary of the region R of the validity of consistency theorem 1 of section 5. The parameter which minimizes the loss function, falls just outside that region and the equations generating the likelihood function are then unstable. That the absolute minimum is found in an unstable region is hence due to the relatively short sample. When the sample length was increased to N = 300 no numerical difficulties occurred.

Example 2

Generating equation

$$\begin{cases} x(t) = 0.95\, x(t-1) + e(t-1) \\ y(t) = x(t) + v(t) \end{cases} \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	-0.95	0	0.9214	0	0	1.5964
Estimated	-0.866	0.269	0.892	0.290	0.397	1.563

Estimated covariances

0.004929	-0.003054	0.001808	-0.001304	-0.018401	0.000000
-0.003054	0.039480	-0.000533	0.018488	0.074621	0.000000
0.001808	-0.000533	0.027220	0.001765	-0.001911	0.006975
-0.001304	0.018488	0.001765	0.029278	0.039690	0.000000
-0.018401	0.074621	-0.001911	0.039690	0.863380	0.000000
0.000000	0.000000	0.006975	0.000000	0.000000	0.012215

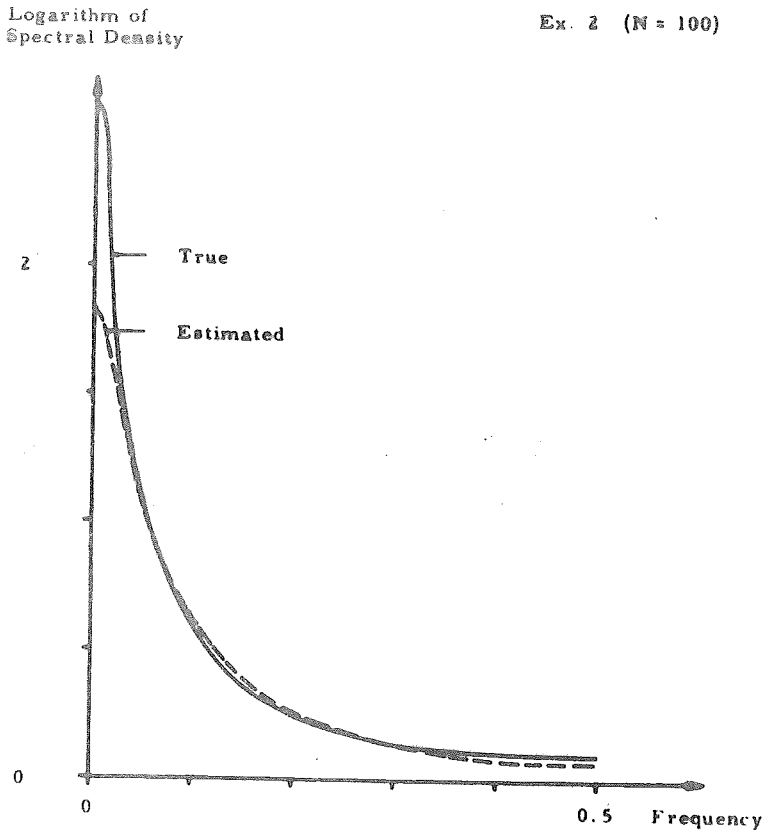


Figure 4

True and estimated power spectral densities.

Comment

The same system as in example 1, but where the input has not been observed, so that the output has nothing to do with the input u. No numerical difficulties occurred, since parameter γ_1 equals 0.372, which is well inside the region R = (-1, 1). The example is equivalent to a pure spectral analysis.

Example 3

Generating equation

$$y(t) = 0.95\, y(t-1) + u(t-1) + e(t) - 0.5\, e(t-1) \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	-0.95	1	0.45	0	0	1
Estimated	-0.929	0.947	0.480	0.033	-0.100	0.941

Estimated covariances

0.000664	-0.000109	0.000345	-0.000442	-0.020029	0.000000
-0.000109	0.012591	-0.000941	0.006253	0.042707	0.000000
0.000354	-0.000941	0.008556	-0.001889	-0.015363	0.002258
-0.000442	0.006253	-0.001889	0.011125	0.034995	0.000000
-0.020029	0.042707	-0.015363	0.034995	1.335370	0.004426

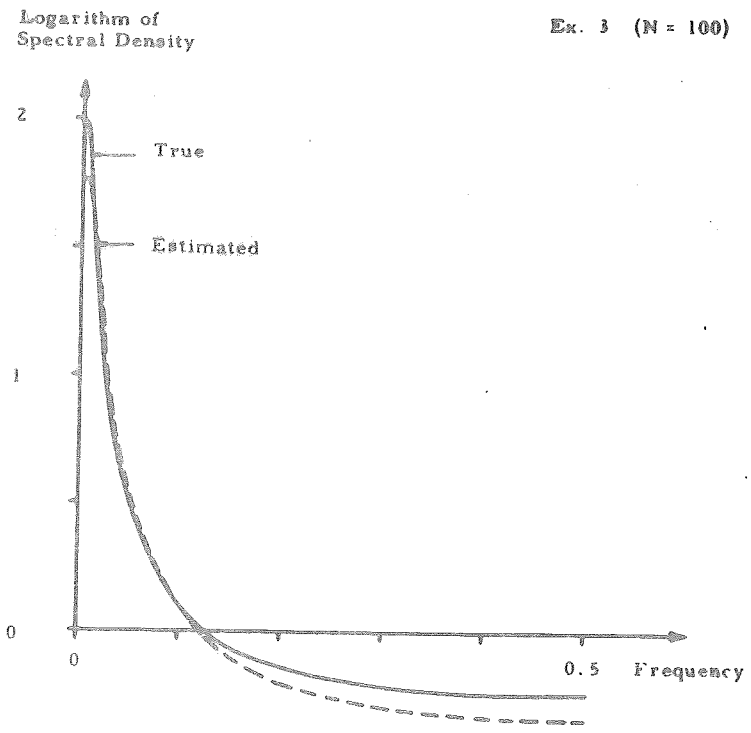


Figure 5
True and estimated power spectral densities of example 3.

Comment

The system is equivalent to a slow (= high rate sampled) first order system with white input noise and independent measurement errors. No numerical difficulties ($\gamma_1 = -0.5$).

Example 4

Generating equation

$$\begin{cases} x(t) = 0.95\, x(t-1) + e(t-1) \\ y(t) = x(t) + u(t) + 1 + v(t) \end{cases} \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	-0.95	0	0.9214	1	1	1.5964
Estimated	-0.873	0.267	0.897	1.289	0.536	1.564

Estimated covariances

0.004584	-0.003222	0.001871	-0.001401	-0.022835	0.000000
-0.003222	0.039964	-0.000575	0.018795	0.082869	0.000000
0.001871	-0.000575	0.027395	0.001679	-0.004083	0.007013
-0.001401	0.018795	0.001679	0.029409	0.043645	0.000000
-0.022835	0.082869	-0.004083	0.043645	0.990694	0.000000
0.000000	0.000000	0.007013	0.000000	0.000000	0.012234

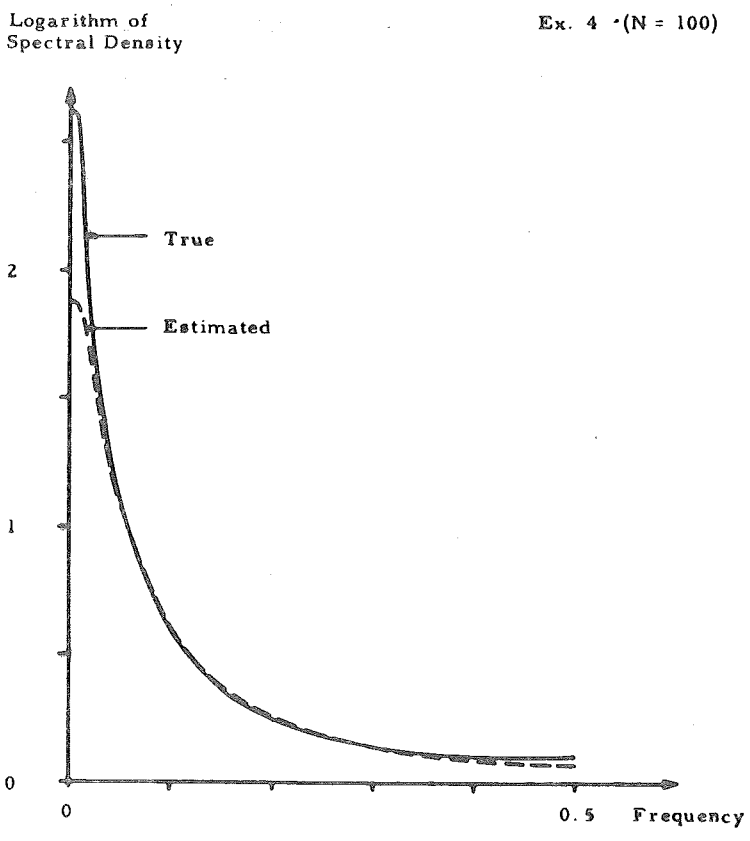


Figure 6

True and estimated power spectral densities of example 4.

Comment

The process can be regarded as an ordinary regression model with colored disturbances. No numerical difficulties.

Example 5

Generating equation

$$y(t) = u(t) - u(t-1) + e(t) - e(t-1) + 1 \qquad t = 1, \ldots, 100$$

Comment

The output is a difference of the function $u(t) + e(t) + t$, and constitutes a case for which the consistency of the identification algorithm has not been proven ($\gamma_1 = -1$), so that the parameter θ_o lies on the boundary of R. The numerical troubles that occured were of the same kind as those found in example 1. In this case the true power spectrum is zero at the origin.

Example 6

Generating equation

$$\begin{cases} x(t) = -0.5\, x(t-1) + u(t-1) \\ y(t) = x(t) + v(t) \end{cases} \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	0.5	1	0	0	0	1
Estimated	0.624	0.868	0.010	0.023	-0.059	0.982

Estimated covariances

0.005275	-0.003612	0.002288	-0.001166	-0.001137	0.000000
-0.003612	0.009763	-0.001613	0.000610	0.001705	0.000000
0.002288	-0.001613	0.009356	-0.000472	-0.000583	0.000051
-0.001166	0.000610	-0.000472	0.011608	0.002601	0.000000
-0.001137	0.001705	-0.000583	0.002601	0.010654	0.000000
0.000000	0.000000	0.000051	0.000000	0.000000	0.004823

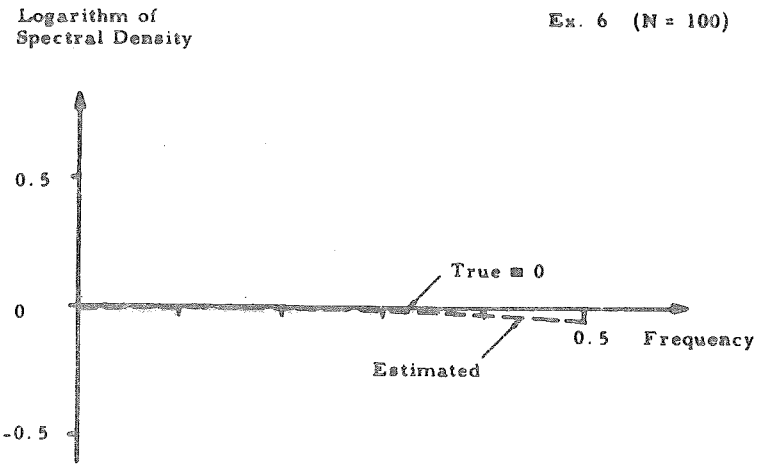


Figure 7
True and estimated power spectral densities of example 6.

Comment

A first order oscillating system observed with independent measurement errors. No numerical difficulties.

Example 7

Generating equation

$$\begin{cases} x(t) = -0.5\, x(t-1) + e(t-1) \\ y(t) = x(t) + v(t) \end{cases} \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	0.5	0	-0.3878	0	0	1.4604
Estimated	0.663	-0.122	-0.328	0.314	0.083	1.469

Estimated covariances

0.025971	0.003512	0.000381	0.001803	0.001120	0.000000
0.003512	0.022655	0.001202	-0.005540	0.001798	0.000000
0.000381	0.001202	0.016009	0.000494	0.000394	-0.002410
0.001803	-0.005540	0.000494	0.025453	0.004662	0.000000
0.001120	0.001798	0.000394	0.004662	0.017474	0.000000
0.000000	0.000000	-0.002410	0.000000	0.000000	0.010796

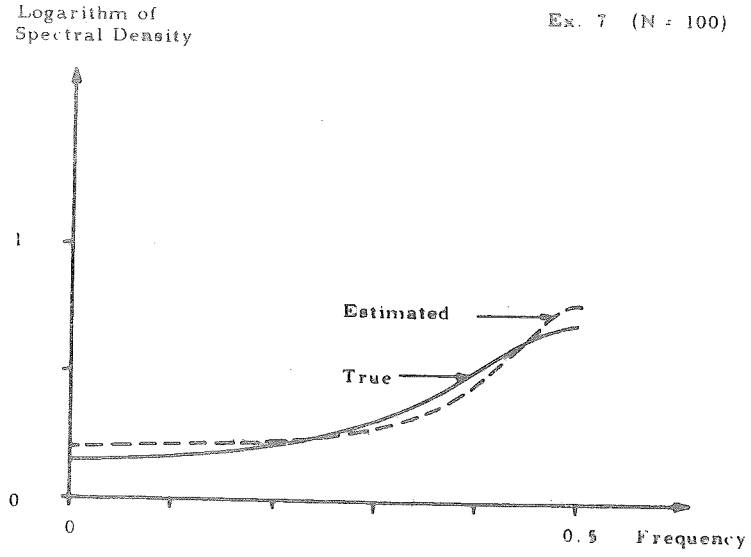


Figure 8

True and estimated power spectral densities of example 7.

Comment

The same system as in example 6 but the input u is not observed (spectral analysis). Some numerical difficulties occurred. The estimates were oscillating. However, when switching to the complete second derivative matrix $V_{\theta\theta}$ convergence was easily obtained. No simple explanation has been found, just a reminder that convergence is not guaranteed under all circumstances.

Example 8

Generating equation

$$y(t) = -0.5\, y(t-1) + u(t-1) + e(t) - 0.5\, e(t-1) \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_o	k	c_o
True	0.5	1	-1	0	0	1
Estimated	0.610	0.903	-0.904	0.047	-0.008	0.934

Estimated covariances

0.005049	0.000630	-0.001166	0.000071	-0.000271	0.000000
0.000630	0.018352	0.001788	-0.009888	0.000236	0.000000
-0.001166	0.001788	0.013692	-0.002361	-0.000272	-0.004223
0.000071	-0.009888	-0.002361	0.010864	0.001038	0.000000
-0.000271	0.000236	-0.000272	0.001038	0.001692	0.000000
0.00000	0.000000	-0.004223	0.000000	0.000000	0.004362

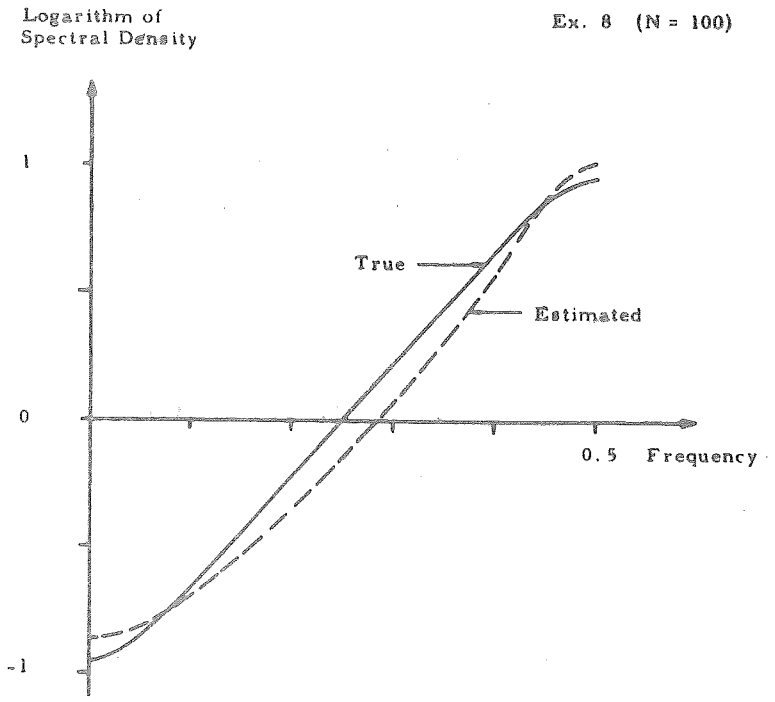


Figure 9
True and estimated power spectral densities of example 8.

Comment

The process does not have a representation (7.1) with independent measurement errors since the equivalent parameters in this representation are then complex. Physically, the conclusion can be drawn that the measurement errors are colored. No numerical difficulties.

Example 9

Generating equation

$$\begin{cases} x(t) = -0.5\, x(t-1) + e(t) \\ y(t) = x(t) + u(t) + 1 + v(t) \end{cases} \qquad t = 1, \ldots, 100$$

Parameters

	a_1	b_1	c_1	b_0	k	c_0
True	0.5	0	-0.3878	1	1	1.4604
Estimated	0.666	-0.126	-0.329	1.318	1.237	1.468

Estimated covariances

0.021056	0.002194	0.000049	0.001803	0.000680	0.000000
0.002194	0.022326	0.001164	-0.005607	0.001698	0.000000
0.000049	0.001164	0.015974	0.000454	0.000353	-0.002413
0.001803	-0.005607	0.000454	0.025426	0.004630	0.000000
0.000680	0.001698	0.000353	0.004630	0.017398	0.000000
0.000000	0.000000	-0.002413	0.000000	0.000000	0.010772

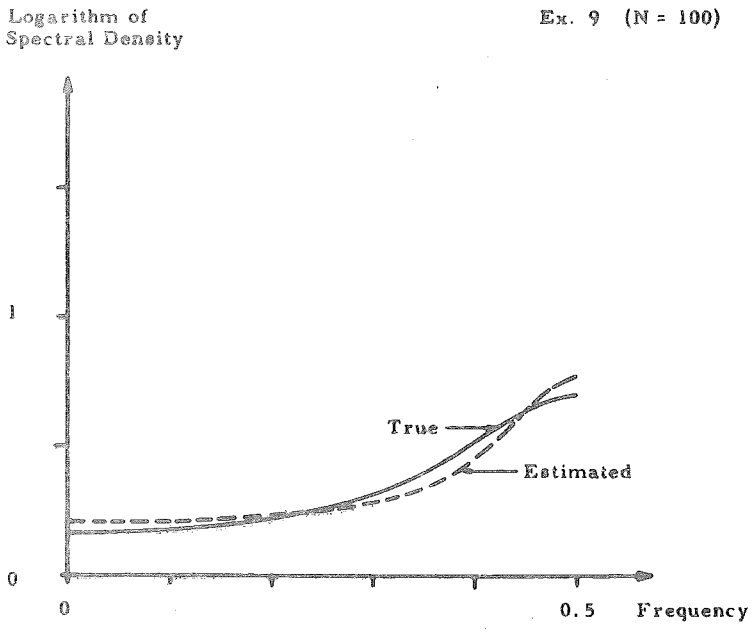


Figure 10

True and estimated power spectral densities of example 9.

Comment

A regression relation as in example 4, but with different "coloring" of the residuals. No numerical difficulties.

Example 10 (two series)

Generating equation

$$y(t) - 1.5\, y(t-1) + 0.7\, y(t-2) = e(t) - e(t-1) + 0.2\, e(t-2) \qquad t = 1, \ldots, 500$$

Parameters

	α_1	α_2	γ_1	γ_2	λ
True	-1.5	0.7	-1	0.2	1
Estimated	-1.524	0.689	-1.028	0.168	0.978
Estimated	-1.537	0.763	-1.034	0.246	1.020

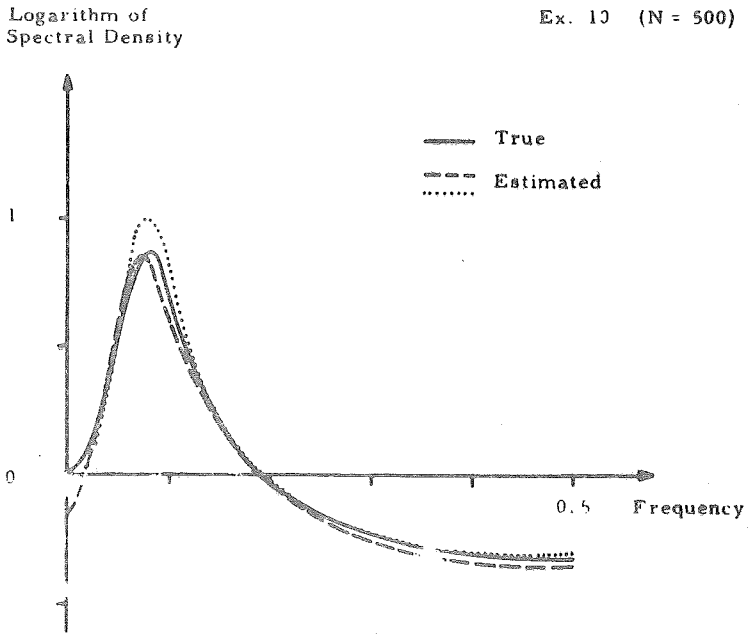


Figure 11
True and estimated spectral power densities of example 10.

Comment

This is an example of the application to power spectral analysis, where the spectrum is band pass. Two examples are given. No numerical difficulties. An earlier version of the program listed has been used.

COMMENTS AND CONCLUSIONS

In this section we give some of the reasoning that has led us to the chosen approach, briefly cover come alternative possibilities, and compare our approach with the model reference technique. Finally, we give some extensions and other applications of our technique.

Since random distrubances are a primary concern, the statistical approach is more or less given. Having established this, an appropriate approach is given by Wald's decision theory [52] [60]. However, when specialized to the present case (non-sequential estimation) the theory requires the postulating of certain important properties.

The first problem then presenting itself is that of a structure or framework for the model (Wald speaks about a parametric family of distributions). We adopt here the philosophy that every structure that is possible to handle and flexible enough to describe observations with the required accuracy is a good or "true" structure. Hence, the linear structure is postulated.

Even within the linear structure there are several possible alternatives. The structure of the system is equivalent to the description of the input/output relation to be used. Under controllability and observability assumptions [47] the input/output relation is conveniently described by the impulse response (= weighing function) or the transfer function of the system. Another description of the system which is frequently used, is a differential or difference equation model with a certain structure. In our approach we are also identifying the disturbances. Here again we have several possibilities. The random disturbances can be characterized by power spectra, covariance functions, or Karhunen Loève representations, [34] to mention a few examples. As stated in the introduction, our motivation for solving the identification problem is that we want to design a control system using the powerful tool of linear, stochastic control theory. This immediately leads us to a representation of the input/output relation as a difference equation and a Karhunen Loève representation of the disturbance. Such a model is the starting point for the linear, stochastic

control theory. Apart from this, there are a few other points which are worth noticing, namely, storage requirements and implementation.

If the system is represented by a weighting function, we need in principle an infinite number of parameters, while a difference equation representation only requires the storing of a finite number of quantities. If the weighting function of a dynamical system is known, there is still a considerable problem in implementing a system having such a weighting function; a representation by a difference equation can be implemented immediately, using either digital or analog equipment.

Anyway, once a linear, single-output, difference-equation model is postulated the problem of structure is essentially solved for there are canonical structures for such systems.

Two structural problems still remain: the choice of order of the system and the choice of representation for the random disturbances. The problem of order has not been considered. The choice of representation of the disturbance essentially boils down to choosing either of the models (2.11) or (2.15). The model (2.11) was chosen in preference to (2.15), because for that model the equations could be solved explicitly for e(t). This is of profound importance for the construction of the algorithm. Using the equivalent model (2.15) we have two sources for the random disturbances. The logarithm of the likelihood function becomes

$$-L(y;\theta) = \frac{1}{2c_o^2}\sum_{t=1}^{N} e^2(t) + \frac{1}{2d_o^2}\sum_{t=1}^{N} v^2(t) - N \log d_o c_o$$

In this case we cannot express e(t) and v(t) explicitly in y by solving the equation (2.15) but we most keep (2.15) as an implicit equation. Analysing the details we find that it leads to a two point boundary value problem for evaluating the function $V(\theta)$ and its partial derivatives.

When the structure is given, the next problem is to define a loss function, which is equivalent to stating the purpose of the identification, that is what the model is to be used for. We get around this by choosing the

maximum likelihood estimator which is invariant for reasonable changes of parameters. One set of parameters may be substituted by another set that is a function of the former set without affecting the output of the resulting model. The maximum likelihood estimator is thus very convenient since it generates a kind of general purpose model. The choice is justified by the fact that it generates estimates having desirable large-sample properties (section 5).

Alternative Approaches

Even if we have decided to identify the process as a linear, stochastic, difference-equation by the methods of mathematical statistics, there are still a number of alternatives to choose from. First we may choose between working on the time-axis or in the frequency plane, and we may adjust the model to sample covariances or to the sample itself. This gives immediately four alternatives.

- Fourier transforms and time samples: compute the Fourier transforms of input and output, and adjust the model transfer function to the ratio. This idea was discarded, since numerical Fourier transformation is generally a cumbersome operation. It might however be worthwhile to investigate it further, since simplifying approximations may be feasible.

- Power and cross-spectrum identification: compute the sample correlations and cross-correlations, and the corresponding spectra, and adjust the model power and cross-spectra to the data. Again the Fourier transformation gives rise to numerical difficulties (and to tedious computations), because the sample autocorrelation function has to be truncated, in order not to obtain a number of multiplications which increases quadratically with the amount of data.

- Autocorrelation and cross-correlation identification: compute the sample correlation and cross-correlation functions and adjust the model correlation function to the data. This method has the disadvantage that the exact sample distribution is complicated and that the sample

autocorrelation function has to be truncated. Furthermore, one has to make sure that the adjustment procedure leads to a noise autocorrelation which is indeed an autocorrelation function. This problem must always arise, since a truncated autocorrelation function is generally not an autocorrelation function (i.e. non-negative definite). For large samples, the sample correlation distributions can be approximated by normal ones, and then the approach may lead to an efficient estimator [22]. If a least squares loss function is used, the problem of deriving the distribution vanishes, but some loss of efficiency should be expected [4]. If these difficulties can be overcome, the approach may be feasible. It has been tried with some success by Durbin [22] for the case of a stationary time series. With a known noise spectrum (usually white), the approach is equivalent to the well known method of cross-correlation analysis [65]. However, it is not immediately clear how Durbin's method should be combined with the cross-correlation technique.

- Identification of time functions: adjust the solution of the difference equation for the model to the observed output, according to the distribution of the observations. This approach requires no preliminary treatment of the samples, and has the advantage that no approximations have to be made to compute the sample distribution. It is used in [53] and in this report.

Comparison with Model Reference Techniques

It is instructive to consider our solution to the identification problem in the framework usually used when discussing process identification. The identification problem is often characterized by three elements

- the class of input signals
- the structure of the system
- the criterion

The restriction we imposed on the input was that the input signal should be persistently exciting. See definition of section 5.

Such inputs might be obtained during normal operation of the process, that is, if there are scheduled quality changes which are frequent enough to permit a satisfactory identification (Theorem 3). Otherwise, they must be introduced during an experimental phase. Notice, however, that it is important that the inputs are generated externally and are not the results of feedback within the system.

A possible alternative would be to introduce persistent disturbances during normal operation that are small enough to be tolerable yet large enough to yield a satisfactory on-line identification.

The choice of model structure and criterion have already been discussed. The criterion is given implicitly when the problem is formulated as a statistical parameter estimation problem and we decide to use the method of maximum likelihood. We recall that this leads to the minimization of the "loss function" $V(\theta)$. If we so desire, our procedure can also be interpreted as follows: consider, for example, (3.1) as the model structure to be identified, and let the function $V(\theta)$ given by (3.5) be the criterion for the identification. Looked upon in this way, the development of section 3 can be regarded as a model reference adaption method for system (3.1). Equation (3.1) is actually an algorithm for a digital computer, but it can be equally well implemented by analog means. Consider in particular the case $\gamma_1 = \alpha_1$, $\gamma_2 = \alpha_2, \ldots, \gamma_n = \alpha_n$ i.e. the only disturbance in the actual system is measurement errors. Equation (3.1) is then identical to the actual system model (2.11), apart from a trivial change in sign of the state variable. Thus, our approach also gives a probabilistic interpretation of the conventional model reference technique. Such a representation is of interest since the results given in section 5 can be used to obtain estimates of the accuracy of the parameter estimates in conventional model reference techniques. Notice, however, that this is restricted to the situation where measurement errors are the only disturbances.

It is also of interest to compare the algorithm developed in sections 3 and 4 with the algorithms currently used in model reference techniques. Blandhol [16] only evaluates the function $V(\theta)$. Judging from our experience it is very difficult to get a reasonable convergence rate by probing techniques using only the values of $V(\theta)$. Blandhol also confirms this. The gradient $V_\theta(\theta)$ is evaluated in some model reference techniques that are implemented in adaptive systems e.g. [64] [69]. In these cases the parameter adjustment routine is chosen as

$$\theta^{k+1} = \theta^k - \alpha V_\theta(\theta^k)$$

Notice that a more effective algorithm is obtained with very little extra computational effort, using the approximate second partial derivatives. We conclude then that it appears worthwhile to consider this modification in model reference adaptive systems currently in use. By including this feature we would also obtain an estimate of the information matrix and thus also of the accuracy of the estimated parameters.

Other Applications of the Technique

The identification algorithm is primarily intended to generate stochastic models of stationary industrial processes for automatic design of a strategy yielding optimal control of the process [3]. The problem as stated in section 2 also includes that of time series analysis, a typical application being the design of optimal filters for estimation and prediction. The identification program can thus also be used for parametric estimation of power spectra of given stationary time series. [5 section 6]

The method then has the advantage over orthodox methods in that it avoids the problem of choosing spectral or lag windows [15]. Also, the spectral estimates will always be non-negative. Furthermore, the ever present problem of prior trend removal is obviated. A process containing a trend component is really not quite stationary and hence the ordinary power spectrum, strictly speaking, does not exist for such processes. With out method, trends are estimated as such, and the spectrum of the

remaining stationary part is interpreted as the spectrum of the time series.

Possible Extensions

The problem can be extended to several inputs immediately. Both the algorithm and the convergence proofs generalizes immediately.

The algorithm can also be extended directly to non-linear and/or time variable systems with a single output and a known structure. Consider for example, the following system:

$$x(t + 1) = g(x(t),\ u(t),\ t)$$

where g (x, u, t) is a function which contains some unknown parameters. Let u be the input(s) of the system and let the output y be given by

$$y(t) = x_n(t) + z_m(t) + c_o e(t)$$

where the vector z(t) is given by

$$z(t + 1) = Fz(t) + Ge(t)$$

The system described by these equations is an arbitrary non-linear system with a single output, with a random disturbance in the output that is stationary and has spectrum of order 2m. The problem is to identify the unknown parameters of the function g(x, u, t), the constants c_o and κ, and the elements of the matrices F and G. This identification problem can be solved immediately using the technique described in the report. To obtain the likelihood function we first write e(t) as a function of the inputs and the observations. We get

$$x(t + 1) = g(x\ (t),\ u(t),\ t)$$

$$z(t + 1) = Fz(t) + \frac{1}{c_o} G\left[y(t) - x_n(t) - z^{\circ}_m(t)\right]$$

$$\epsilon(t) = c_o e(t) = y(t) - x_m(t) - z_m(t)$$

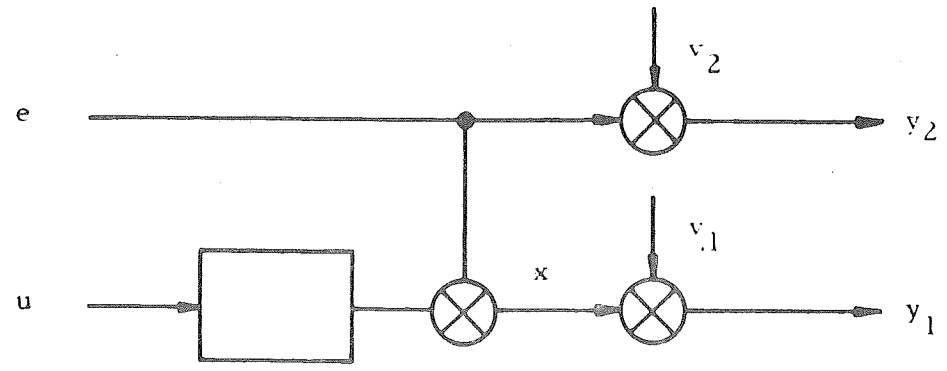
and the logarithm of the likelihood function becomes

$$-L(y;\theta) = \frac{1}{2c_o^2} \sum_{t=1}^{N} \varepsilon^2(t) + N \log c_o + \frac{N}{2} \log 2\pi$$

We can now proceed in exactly the same way as was done in section 3 to obtain an algorithm to maximize $L(y;\theta)$.

Regarding the extension to multiple outputs, the crucial problem is to find a suitable canonical form.

It is evident that if several outputs are to be controlled, we can repeat the identification on each one of them. In some cases, the multiple output character of the process is however essential. If some additional quantity is measured besides the output, in order to obtain additional information, it is not correct to regard this quantity as an input, unless the measurement is perfect [58]. Neither is it always possible to regard it as an output of a second single-output process, since the output of this model need not necessarily have anything to do with the output of the first model. The following situation will illustrate this case:



It originates in an attempt to increase the efficiency of the control of x through u by measuring the disturbances e.

The following modification of the linear identification problem is sometimes relevant. Consider a general linear process which is closed by a feedback loop from the output to the input where we wish to identify the process. It is evident that this is not possible without additional information. However, if the characteristics of the regulator are known, only minor modifications of the open loop algorithm are required to identify the process, provided it is identifiable at all.

The results can also be generalized in a different direction. So far, we have assumed that the estimate should be calculated from a complete record of inputs and observations. Such a situation is referred to as off-line estimation [61]. In certain applications, particularly in connection with adaptive control, the problem is different because the inputs and outputs are obtained recursively in time. This situation is referred to as on-line identification. In our procedure, the derivatives are computed sequentially and it is therefore very easy to also compute the estimate recursively.

NOTES AND REFERENCES

Literature on the identification problem is overwhelming. This problem and related ones are treated in literature on control systems and in literature on statistics. The included list of references is a selection and by no means complete. Out of the statistical literature, papers on general problems of statistical inference, parametric estimation [18], [48], [63], [83], spectral analysis [52] and time series analysis [9], [33], [70] are of interest for the identification problem. When the latter is formulated as a parameter estimation problem, Wald's decision theory provides a suitable framework [52]. The application of decision theory to the identification problem is discussed in [28], [29], [60].

Other papers from statistical literature that are of interest for the identification problem fall into one of three classes.

- Estimation of the coefficients in linear function under various hypothesis concerning the independent and dependent variables [17], [57], [58].

- Estimation of parameters in Markov processes [14], [33]. Of more general Markov processes the cases of a finite number of states and discrete time [2], [9] and continuous time [1] have been treated.

- Estimation of parameters in stochastic differential or difference equations. Special cases of difference equation are the autoregressive [4], [21], [87] and moving average series [20], [78] and combination thereof [22].

In control theory, the identification problem is of interest for its own sake, that is, for describing the object to be controlled [88], [89] and also because it is of fundamental interest for the design of adaptive systems [10], [65]. A recent survey of part of the control literature on the identification problem is given in [23]. In the book [65] there is a survey of the identification problem with particular emphasis on applications in adaptive control systems. In [61] there is also an annotated bibliography. Many additional papers are found in the proceedings of recent congresses [40], [41]. Only the case of

"normal operating records" is considered i.e. the input is given, and no special "test signals" must be applied.

In control theory, the process is regarded as an input/output system and a mathematical relation (model) is sought whose output approximates the measured relation for the given inputs according to a chosen loss function. Depending on the particular representation that is chosen, there are many alternative identification schemes that have been investigated.

- Fouries analysis [65], which is in effect a mathematical method of solving a linear integral equation for the transfer function of a linear deterministic model.

- Cross correlation or cross spectrum analysis [25], [66], [80], [65], [79], [36] which may be regarded as the statistical analog of the above method.

- Various model reference methods [8], [16], [73], [85], where a parametric class of models is postulated and the loss function is minimized by adjusting the parameters. This can be done either by hill climbing methods [82] or manually on analog or digital computers. This is the conventional method used in mathematical model building.

- Parameter estimation in stochastic difference equations [28], [53], [55], where statistical methods are applied to estimate the parameters (Maximum likelihood or least squares).

- Estimation of impulse response [53] [54], where model output is supposed to be a moving average of the input plus random errors, and the coefficients are determined by minimizing the risk (= ensemble average of loss) over the known distribution of random errors.

- Utilizing orthogonal functions [51], [30], [31], [49], [81] which is in effect an efficient way of minimizing the loss function. The observed input and output are developed in series or orthogonal functions, whose

coefficients are then non-interacting regarded as parameters in the loss function.

- Development of the plant functional in a functional space [6], [7], [38], [39].

- The application of quasilinearization to model building is found in [11], [12], [43], [67].

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APPENDIX A - PROOFS OF THEOREMS IN SECTION 5

In this appendix we have collected the proofs of the theorems of section 5. To simplify the reading we have also restated the theorems themselves.

LEMMA 1

Let R be a region in 4n+3 dimensional Euclidian space defined by
$R = \{\theta \mid \lambda > 0$, and all zeros of the polynomials $z^n + \alpha_1 z^{n+1} + \ldots + \alpha_n$ and $z^n + \gamma_1 z^{n-1} + \ldots + \gamma_n$ have magnitudes strictly less than one$\}$.

Assume that the inputs u(t) and all cross-products u(t) u(t + ν) are Cesàro summable, i.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \text{ and } \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t+\nu) \quad \exists \qquad \nu = 0, 1, 2, \ldots$$

Then

$$\lim_{N\to\infty} \frac{1}{N} L^N(y;\theta) = \lim_{N\to\infty} \frac{1}{N} E_o L^N(y;\theta) = L(\theta, \theta_o)$$

with probability one if $\theta \in R$ and $\theta_o \in R$.

Proof

Before giving a formal proof we will make some observations.

A result similar to that of the lemma is also needed when proving the consistency of the ML estimator for the case of independent samples. In that case, the result is deduced from the strong law of large numbers. In this case we have however dependent variables, and the strong law of large numbers cannot be applied directly. The lemma is essentially an ergodic theorem and it can be proven via Birkhoff's individual ergodic theorem [19]. In the particular case, we can, however, obtain the desired result by less powerful tools.

We have

$$L^N(y;\theta) = -N \log \lambda - \frac{1}{2\lambda^2}\, \epsilon^1\, \epsilon^{1T}$$

where $\epsilon^1(t)$ are independent only for $\theta = \theta_o$. However, it is possible to make a linear transformation of the stochastic variables $\epsilon^1(t)$ such that $L^N(y;\theta)$ can be expressed as a sum of independent variables, not equally distributed, for all θ and Kolmogoroff's form of the strong law of large numbers, can be applied.

To simplify the calculations, we use the compact notations of section 4. We have from (4.6), 4.7), and (4.8)

$$L^N(y;\theta) = -N \log \lambda - \frac{1}{\lambda^2} V(\theta)$$

$$V(\theta) = \frac{1}{2}\, \epsilon\, C^{T^{-1}}\, C^{-1}\, \epsilon^T$$

$$\epsilon = yA^T - uB^T - iK^T - i_1 Y_o^T$$

$$\epsilon_o = yA_o^T - uB_o^T - iK_o^T - i_1 Y_{oo}^T = \lambda_o e C_o^T \qquad \text{(A1.1)}$$

where the zero subscripts denote the true parameter values. The convergence of $\frac{1}{N} L^N(y;\theta)$ is equivalent to the convergence of $\frac{1}{N} V(\theta)$ (since $\lambda \neq 0$). We have

$$\frac{1}{N} V(\theta) = \frac{1}{2N} (eG^T + u'')\ \ (Ge^T + u''^T)$$

where

$$u'' = u_o' A_o^{T^{-1}} A^T C^{T^{-1}}\ \ u' C^{T^{-1}}$$

$$u' = uB^T + iK^T + i_1 Y_o^T$$

$$u_o' = uB_o^T + iK_o^T + i_1 Y_{oo}^T$$

$$G^T = \lambda_o C_o^T A_o^{T^{-1}} A^T C^{T^{-1}} \qquad \text{(A1.2)}$$

But

$$\frac{1}{N} e G^T G e^T = \frac{1}{N} e M \Lambda M^T e^T = \frac{1}{N} \sum_{i=1}^{N} (v_i^2 - 1) \lambda_i + \frac{1}{N} \sum_{i=1}^{N} \lambda_i$$

where Λ is the diagonal matrix of (non-negative) eigenvalues of $G^T G$, and M is orthogonal.

Since M is orthogonal, $v = eM$ has uncorrelated components v_i, and since e is normal, v_i are independent (0, 1). The variables $(v_i^2 - 1) \lambda_i$ are then independent, and

$$E(v_i^2 - 1) \lambda_i = 0$$

$$D^2(v_i^2 - 1) \lambda_i = 3 \lambda_i^2$$

Then Kolmogoroff's criterion of the strong law of large numbers [24, p243] implies almost certain convergence of $\frac{1}{N} e G^T G e^T$ if $\Sigma \lambda_i^2 / i^2$ converges. Since the partial sums are non-decreasing, it is sufficient that they are bounded.

But

$$\sum_{i=1}^{N} \frac{\lambda_i^2}{i^2} \leq \sum_{i=1}^{N} \frac{1}{i^2} \max \lambda_i^2 \leq \text{const} \| G^T G \|^2 \leq \text{const} \| G^T \|^2 \| G \|^2$$

$$= \text{const} \| G \|^4 \leq \text{const} \| C_o \|^4 \| A_o^{-1} \|^4 \| A \|^4 \| C^{-1} \|^4$$

where the maximum norm is chosen $\| \{a_{ij}\} \| \equiv \max_i \sum_j |a_{ij}|$

(Notice that $\| G^T \| = \| G \|$ when G is Toeplitz.)

It is easily seen that $\| C_o \|$ and $\| A \|$ are always ≤ constant and a sufficient criterion is thus that $\| A_o^{-1} \|$ and $\| C^{-1} \|$ are bounded.

The linear term

$$\frac{1}{N} e G^T u''^T = \frac{1}{N} \sum_{i=1}^{N} e_i u_i'''$$

where

$$u''' = u'' G$$

converges by the same criterion to its mean value (which is zero) if

$$\sum_{i=1}^{N} \frac{1}{i^2} u_i'''^2$$

is bounded. But

$$\sum_{i=1}^{N} \frac{1}{i^2} u_i'''^2 \leq \text{const } \| u''' \|^2 \leq \text{const } \| G \|^2 \| u'' \|^2 \leq \text{const } \| A_o^{-1} \|^2 \| C^{-1} \|^2 \| u'' \|^2$$

(max. norm).

After elementary calculations using (A1.2) we get

$$\| u'' \| = \| u \ (B_o^T A_o^{T^{-1}} A^T - B^T) C^{T^{-1}} + [(i K_o^T + i_1 Y_{oo}^T) A_o^{T^{-1}} A^T$$

$$-(i K^T + i_1 Y_o^T)] C^{T^{-1}} \|$$

$$\leq \text{const } \| u \| [\| A_o^{-1} \| + \text{const }] \| C^{-1} \| + \text{const } [\| A_o^{-1} \| + \text{const }] \| C^{-1} \|$$

which is bounded if $\| A_o^{-1} \|$, $\| C^{-1} \|$, and $\| u \|$ are all bounded (max. norm).

Now C^{-1} is left triangular and Toeplitz ($C^{-1} \in T$). Its first column is by definition given by $C^{-1} i_1^T$, or by the arguments of section 4 as the solution of the difference equation

$$\sum_{k=1}^{n} x(t-k) \gamma_k = \delta_{t-1} .$$

If the vector γ generates a strictly stable difference equation, then the sum $\Sigma \mid x(t) \mid$ converges, so that $\| C^{-1} \|$ is bounded. The same argument applies to $\| A_o^{-1} \|$

Finally consider the constant term

$$\frac{1}{N} u'' u''^T = \frac{1}{N} \| uD^T + iF^T + i_1 H^T \|^2 \qquad \text{(Euclidian norm)}$$

where

$$D^T = (B_o^T A_o^{T^{-1}} A^T - B^T) C^{T^{-1}}$$

$$F^T = (K_o^T A_o^T A^T - K^T) C^{T^{-1}}$$

$$H^T = (Y_{oo}^T A_o^T A^T - Y_o^T) C^{T^{-1}}$$

We have immediately, if u and C^{-1} are bounded, that D, F, and H converge, and

$$\lim_{N\to\infty} \frac{1}{N} \| u'' \|^2 = \lim_{N\to\infty} \frac{1}{N} u D^T D u^T + 2 \lim_{N\to\infty} \frac{1}{N} u D^T F i^T + \text{constant.}$$

The quadratic term is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i,j,k=1}^{N} u(i)\, d_{j-i}\, d_{j-k}\, u(k) = \sum_{\nu=-\infty}^{\infty} \sum_{\ell=0}^{\infty} d_\ell\, d_{\ell+\nu} \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-\nu} u(t)\, u(t+\nu)$$

and the limit exists if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t+\nu) \qquad \exists \quad , \qquad \nu = 0, \pm 1, \pm 2, \ldots$$

The linear term

$$\sum_{i=0}^{\infty} d_i \sum_{j=0}^{\infty} f_j \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)$$

exists if the average input

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \qquad \exists$$

It has thus been shown that

$$\lim_{N\to\infty} \frac{1}{N} L^N(y;\theta) = \lim_{N\to\infty} \frac{1}{N} E_o L^N(y;\theta)$$

with probability one for all parameters θ such that

i) the difference equation generated by γ is stable

and provided that

ii) the difference equation generated by α_o is stable

iii) the input time averages

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t) \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-\nu} u(t)\,u(t+\nu)$$

exist for all integers ν.

But the difference equations are stable if the corresponding characteristic equations

$$\begin{cases} z^n + \gamma_1 z^{n-1} + \ldots + \gamma_n = 0 \\ \vdots \\ z^n + \alpha_{1o} z^{n-1} + \ldots + \alpha_{no} = 0 \end{cases}$$

have zeros of magnitudes strictly less than one. A forteriori the lemma holds for $\theta \in R$, $\theta_o \in R$.

Q. E. D.

In the following theorems and lemmas the regularity properties of lemma 1 on the input u(t) are supposed to be satisfied.

LEMMA 2

The function $L(\theta, \theta_o)$ is an analytic function of θ in a closed set $R' \subset R$ and we have

$$\lim_{N\to\infty} \frac{1}{N} \frac{\partial}{\partial\theta_i} L^N(y;\theta) = \frac{\partial}{\partial\theta_i} \lim_{N\to\infty} \frac{1}{N} E_o L^N(y;\theta) = \frac{\partial}{\partial\theta_i} L(\theta, \theta_o)$$

with probability one. The above equation also holds for higher derivatives.

Proof

First assume that θ is a scalar. The function $L^N(y;\theta)$ is infinitely differentiable in R'. Hence, by analytic continuation we can define an analytic function of a complex variable θ. As N increases the function $L^N(y;\theta)$ is monotone, but it does not increase faster than N. Hence, $\frac{1}{N} L^N(y;\theta)$ are all bounded and converges uniformly if $\theta \in R'$. According to a well-known theorem about analytic functions, Titchmarsch [74, pp 95-96], the limit is then an analytic function, and the lemma holds.

Now if θ is a vector the same discussion holds, the crucial step is the theorem about convergence of analytic function. The proof of this theorem utilizes only Cauchy's integral theorem which holds also for function of several complex variables, Goursat [32 pp 279-281].

THEOREM 1

Let S_o be a set in 4n+3 dimensional Euclidian space defined by

$$S_o = \{ \theta \mid L(\theta, \theta_o) = L(\theta_o, \theta_o) \}$$

Assume that for all sufficiently large N, $\hat{\theta}^N \in R'$, where $R' \subset R$ is a closed set.

Then

$$\| \hat{\theta}^N - P\hat{\theta}^N \| \to 0$$

with probability one, where $P\theta$ is the projection of θ and $S_o \cap R'$ = the nearest point $\in S_o \cap R'$.

Proof

The proof is essentially that given by Wald [76] for the case of independent samples. The possibility of the generalization has been pointed out by Wald. We will essentially follow Kendall's [48] exposition of Wald's proof. Let $\ell^N(y;\theta)$ denote the likelihood function and $L^N(y;\theta)$ the logarithm of $\ell^N(y;\theta)$. The ML estimate $\hat{\theta}^N$ is determined in such a way that

$$\ell^N(y;\hat{\theta}^N) \geq \ell^N(y;\theta)$$

for every θ.

If there are several $\hat{\theta}^N$ that give the same maximum value, we choose either one according to some arbitrary rule. Let θ_o be the true parameter value.

Let $\theta \notin S_o$. Lemma 1 then implies that

$$\lim_{N\to\infty} \log \ell^N(y;\theta) = L(\theta,\theta_o) \neq L(\theta_o,\theta_o) = \lim_{N\to\infty} \log \ell^N(y;\theta_o)$$

with probability one.

Hence to every $\epsilon > 0$ there is an $N_o(\epsilon)$ such that

$$\left| \frac{1}{N} \log \ell^N(y;\theta) - L(\theta,\theta_o) \right| < \epsilon$$

$$\left| \frac{1}{N} \log \ell^N(y;\theta_o) - L(\theta_o,\theta_o) \right| < \epsilon$$

for every $N > N_o(\epsilon)$

Put

$$\epsilon = \frac{1}{2} \left| -L(\theta,\theta_o) + L(\theta_o,\theta_o) \right|$$

Then

$$\left| \frac{1}{N} \log \ell^N(y;\theta) - \frac{1}{N} \log \ell^N(y;\theta_o) - L(\theta,\theta_o) + L(\theta_o,\theta_o) \right|$$

$$\leq \left| \frac{1}{N} \log \ell^N(y;\theta) - L(\theta,\theta_o) \right| + \left| \frac{1}{N} \log \ell^N(y,\theta_o) - L(\theta_o,\theta_o) \right| < 2\epsilon$$

$$= \left| L(\theta,\theta_o) - L(\theta_o,\theta_o) \right|$$

Hence

$$\left| \frac{1}{N} \log \ell^N(y;\theta) - \frac{1}{N} \log \ell^N(y;\theta_o) \right| \neq 0$$

or

$$\ell^N(y;\theta) \neq \ell^N(y;\theta_o) \qquad \text{for every } N > N_o(\epsilon)$$

with probability one.

Introduce the symbol E_o for the operation of taking mathematical expectation with respect to the distribution of y when $\theta = \theta_o$. Using the inequality between the geometric and the arithmetic means we get

$$E_o \log \frac{\ell^N(y;\theta^*)}{\ell^N(y;\theta_o)} < \log E_o \frac{\ell^N(y;\theta^*)}{\ell^N(y;\theta^o)}$$

if $\theta^* \notin S_o$. We further have

$$E_o \frac{\ell^N(y;\theta^*)}{\ell^N(y;\theta_o)} = \int \frac{\ell^N(y;\theta^*)}{\ell^N(y;\theta_o)} \ell^N(y;\theta_o)\, dy = 1$$

Hence

$$E_o \log \frac{\ell^N(y;\theta^*)}{\ell^N(y;\theta_o)} < 0$$

and S_o is given by

$$\log \frac{\lambda^2}{\lambda_o^2} + \frac{\lambda_o^2}{\lambda^2} 2 \frac{V(\theta, \theta_o)}{\lambda_o^2} = 1 \qquad \text{(A2.2)}$$

But the function $\log \frac{\lambda^2}{\lambda_o^2} + \frac{\lambda_o^2}{\lambda^2} 2 \frac{V(\theta, \theta_o)}{\lambda_o^2}$ has minimum with respect to λ

$$= \log 2 \frac{V(\theta, \theta_o)}{\lambda_o^2} + 1 \geq 1 \quad \text{for} \quad \frac{\lambda^2}{\lambda_o^2} = 2 \frac{V(\theta, \theta_o)}{\lambda_o^2}$$

Hence the equality (A2.2) can be satisfied only if

$$2 \frac{V(\theta, \theta_o)}{\lambda_o^2} = 1 \quad \text{and} \quad \lambda = \lambda_o$$

Because of the inequality (A2.1) this implies the following two conditions on the parameters θ, θ_o.

$$\begin{cases} \frac{1}{\lambda_o^2} \lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}(G^T G) = \lim_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} g_{i-j}^2 = 1 \\ \lim_{N\to\infty} \frac{1}{N} \|u''\|^2 = 0 \end{cases} \qquad \text{(A2.3)}$$

if $\theta \in S_o$.

But the first condition implies that $g_{i-j} = \delta_{i-j}$, since $g_o = 1$, and hence

$$\frac{1}{\lambda^2} G = C^{-1} A A_o^{-1} C_o = I$$

Hence $S_o = R \cap S_o^1 \cap S_o^2$ where S_o^1 is given by

$$\theta \in S_o^1 \Leftrightarrow A C_o = A_o C \text{ and } \lambda = \lambda_o \qquad \text{(A2.4)}$$

Notice that S_o^1 is independent of the input u.

The second condition $\theta \in S_o^2$ is from (A1. 2) and (A2. 4)

$$\lim_{N\to\infty} \frac{1}{N} \| (u B_o^T + i K_o^T + i_1 Y_{oo}^T) C_o^{T^{-1}} - (u B^T + i K^T + i_1 Y_o^T) C^{T^{-1}} \|^2 = 0$$

But, since $\lim_{N\to\infty} \frac{1}{N} u'' u''^T = 0$ then $\lim_{N\to\infty} \frac{1}{N} u'' C^T C u''^T = 0$ if C^{-1} is bounded, that is, if $\theta \in R$.

Hence for $\theta \in S_o$

$$\lim_{N\to\infty} \frac{1}{N} \| (u B_o^T + i K_o^T + i_1 Y_{oo}^T) C^T - (u B^T + i K^T + i_1 Y_o^T) C_o^T \|^2 = 0$$

or $\theta \in S_o^2 \Leftrightarrow$

$$\lim_{N\to\infty} \frac{1}{N} \| u D^T + i F^T \|^2 = 0 \qquad \text{(A2. 5)}$$

where

$$\begin{cases} D^T = B_o^T C^T - B^T C_o^T \\ F^T = K_o^T C^T - K C_o^T \end{cases}$$

Since S_o^1 and S_o^2 are defined by (A2. 4) and (A2. 5) even for $\theta \notin R$ and are obviously linear in A, C, and B, K, C respectively, we can conclude that $S_o = R \cap S_o'$ where $S_o' = S_o^1 \cap S_o^2$ is a linear set.

Q. E. D.

LEMMA 4

Assume that the system (2. 1) is completely controllable either from u(t) or e(t) and that the matrix function of inputs

$$R_u = \begin{bmatrix} r_u(0) & r_u(1) & \cdots & r_u(2n) \\ r_u(1) & r_u(0) & \cdots & r_u(2n-1) \\ \cdots & \cdots & \cdots & \cdots \\ r_u(2n) & r_u(2n-1) & \cdots & r_u(0) \end{bmatrix}$$

is positive definite

where

$$r_u(\nu) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-\nu} [\, u(t) - \bar{u} \,] \, [\, u(t+\nu) - \bar{u} \,]$$

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)$$

Then S_o contains only the set $\{ \theta \mid A = A_o,\ B = B_o,\ C = C_o,\ K = K_o \}$

Proof

We have from (A2.4) and A2.5) the characterization of S'_o :

$\theta \in S'_o \Leftrightarrow$

$$\begin{cases} A C_o = A_o C & \text{(A2.6)} \\ \lim\limits_{N\to\infty} \frac{1}{N} \left\| u (B_o^T C^T - B^T C_o^T) + i (K_o^T C^T - K^T C_o^T) \right\|^2 = 0 & \text{(A2.7)} \end{cases}$$

Put

$d = \{ d_i,\ i = 0, \ldots, 2n \}$ where $D = B_o C - B C_o = \sum_{r=0}^{2n} d_r I_r$

$U = \{ u(j-i+1),\ i = 1, \ldots, 2n,\ j = 1, \ldots N \}$ $u(t) = 0,\ t \leq 0$

$\bar{f} = i F i_1^T$ where $F = K_o C - K C_o$

Then (A2.7) becomes

$$d^T \lim_{N\to\infty} \frac{1}{N} UU^T d + 2d^T \lim_{N\to\infty} U i^T \bar{f} + \bar{f}^2 = 0 \qquad \text{(A2.8)}$$

The matrix of this quadratic form is

$$\begin{bmatrix} \lim\limits_{N\to\infty} \frac{1}{N} UU^T & \lim\limits_{N\to\infty} \frac{1}{N} U i^r \\ \lim\limits_{N\to\infty} \frac{1}{N} i U^T & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_u(0) + \bar{u}^2 & r_u(1) + \bar{u}^2 & \cdots & r_u(2n) + \bar{u}^2 & \bar{u} \\ r_u(1) + \bar{u}^2 & r_u(0) + \bar{u}^2 & \cdots & r_u(2n-1) + \bar{u}^2 & \bar{u} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r_u(2n) + \bar{u}^2 & r_u(2n-1) + u^2 & \cdots & r_u(0) + \bar{u}^2 & u \\ \bar{u} & \bar{u} & \cdots & \bar{u} & 1 \end{bmatrix}$$

But this is positive definite if, and only if R_u is positive definite because of the well-known identity

$$\det \begin{bmatrix} R_1 & R_2 \\ R_2^T & R_3 \end{bmatrix} = \det R_3 \cdot \det \left[R_1 - R_2 R_3^{-1} R_2^T \right]$$

Then (A2.8) has the single solution $d = 0$, $\bar{f} = 0$ and S_o' is characterized by the equations

$$\begin{cases} A C_o = A_o C \\ B C_o = B_o C \\ \nu \sum\limits_{i=1}^{n} \gamma_{io} = \nu_o \sum\limits_{i=1}^{n} \gamma_i \end{cases} \qquad \text{(A2.9)}$$

We will now investigate the implications of the equations (A2. 9).

Let $P_A(z)$ be the generating function of the Toeplitz matrix A i. e.

$$P(z) = \alpha_o z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$$

In this particular case we have

$$\alpha_o = \gamma_o = 1$$

The above equations can now be written as the following relations for the generating functions.

$$P_C(z)\, P_{A_o}(z) = P_{C_o}(z)\, P_A(z) \qquad \text{(A2. 10)}$$

$$P_C(z)\, P_{B_o}(z) = P_{C_o}(z)\, P_B(z) \qquad \text{(A2. 11)}$$

Equation (A2. 11) follows from the fact that we have assumed the order of the model to be known. Now the polynomial $P_{C_o}(z)$ may have factors in common with $P_{A_o}(z)$ and $P_{B_o}(z)$ say $P_{C_o'}(z)\, P_{C_o''}(z)$ but that

$$P_{C_o}(z) = P_{C_o'}(z)\, P_{C_o''}(z)$$

and

$$P_{A_o}(z) = P_{C_o'}(z)\, P_{A_o''}(z)$$

$$P_{B_o}(z) = P_{C_o''}(z)\, P_{B_o'}(z)$$

This means physically that every state of the system is controllable either from the input u or from the disturbance e. If the condition is not satisfied, there is at least one state of the system that is either not controllable from the input or from the disturbance.

Consider the equation (A2. 10). Dividing both members with the common factor $P_{C_o'}(z)$ we get

$$P_C(z)\, P_{A_o''}(z) = P_{C_o'''}(z)\, P_A(z)$$

Now let $(z - \lambda)^{\nu}$ be a factor of $P_{C_o'''}(z)$. The equation implies that this also is a factor of the polynomial of the right member. As it is not a factor of $P_{A_o''}(z)$ it must be a factor of $P_C(z)$.

Applying the same arguement to equation (A2. 11) we find that $P_{C_o'}(z)$ is a factor of $P_C(z)$. Combining this with the previous conclusion we find that $P_{C_o'}(z)\, P_{C_o'}(z) = P_{C_o}(z)$ is a factor of $P_C(z)$. But $P_C(z)$ and $P_{C_o}(z)$ are of the same degree.

Hence

$$P_C(z) = P_{C_o}(z)$$

and it now follows from (A2. 10) and A2. 11) that

$$P_A(z) = P_{A_o}(z) \quad \text{and}$$

$$P_B(z) = P_{B_o}(z)$$

The last equation of (A2. 9) then also implies that $\varkappa = \varkappa_o$.

Hence controllability implies $A = A_o$, $B = B_o$, $C = C_o$, $K = K_o$.

Q. E. D.

LEMMA 5

Assume that

$$\frac{1}{N} L_{\theta\theta}^{N}(y;\theta) \to L_{\theta\theta}(\theta, \theta_o) \qquad \text{(lemma 2)}$$

uniformly in $\theta \in R'$ and with probability one, where $L_{\theta\theta}(\theta, \theta_o)$ is continuous for $\theta \in R'$.

Further assume that

$$\| \hat{\theta}^N - P\hat{\theta} \| \to 0 \qquad \text{(theorem 1)}$$

with probability one, where $P\theta$ is the projection on $S_o \cap R'$.

Then

$$\| \frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N) - L_{\theta\theta}(P\hat{\theta}^N, \theta_o) \| \to 0$$

with probability one.

Comment

The lemma obviates the necessity of computing the information matrix $I(\hat{\theta}^N)$.

The assumptions follow from lemma 2 and theorem 1 respectively.

Proof

We have on a y-set A_1 with probability measure one, where

$y = \{y(t), t = 1, 2, \ldots\}$

that

$$\| \frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^{N'}) - L_{\theta\theta}(\hat{\theta}^{N'}, \theta_o) \| < \epsilon_1 \quad \text{for every } N, N' < N_1(\epsilon_1, y)$$

Since the convergence is uniform, N is independent of $\hat{\theta}^{N'}$. Hence the inequality is also true if $N' = N$ (since it holds for every $\theta \in R'$).

Furthermore in a set A_2 with probability measure one

$$\| \hat{\theta}^N - P\hat{\theta}^N \| < \epsilon_2 \quad \text{for every } N > N_2(\epsilon_2, y)$$

Because of the continuity of $L_{\theta\theta}(\theta, \theta_o)$ we have

$$\| L_{\theta\theta}(\hat{\theta}^N, \theta_o) - L_{\theta\theta}(P\hat{\theta}^N, \theta_o) \| < \delta \quad \text{if} \quad \| \hat{\theta}^N - P\hat{\theta}^N \| < \epsilon_3(\delta, y)$$

But the latter condition is satisfied if $N > N_2[\epsilon_3(\delta, y), y]$.

Hence in the set, $A_1 \cap A_2$

$$\| \frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N) - L_{\theta\theta}(P\hat{\theta}, \theta_o) \| < \epsilon_1 + \delta$$

if $N > \max \{ N_1(\epsilon_1, y), N_2[\epsilon_3(\delta, y), y] \}$

Since ϵ_1 and δ are arbitrary, the lemma holds with probability $P\{A_1 \cap A_2\} = 1$

Q. E. D.

THEOREM 2

Let θ be defined by (4.3), let $\Lambda^N(y; \hat{\theta}^N)$ be the diagonal matrix of eigenvalues of $\frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N)$ and let $P^N(y; \hat{\theta}^N)$ be a matrix of corresponding (orthogonal) eigenvectors. Then

$$\lim_{N \to \infty} \| \Lambda^N(y; \hat{\theta}^N) P^{N^T}(y; \hat{\theta}^N) \hat{\theta}^N - \Lambda^N(y; \hat{\theta}^N) P^{N^T}(y; \hat{\theta}^N) \theta_o \| = 0$$

with probability one.

Comment

The theorem is a kind of consistency theorem for certain linear transformations of $\hat{\theta}^N$. Even if some or all components of $\hat{\theta}^N$ are inconsistent (which is always the case if the initial state components are included in θ), the theorem gives the linear combinations that are consistent. The fact that those combinations vary with N is due to the fact that the consistency property is expressed in terms which are observable (computable).

Corrolary

If $\frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N)$ converges to $L_{\theta\theta}$, say, then

$$\frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N)\, \hat{\theta}^N \to L_{\theta\theta}\, \theta_o$$

with probability one, and the estimate is thus strongly consistent if $L_{\theta\theta}$ is non-singular.

Comment

The difficulties of proving the theorem arise from the fact that $\frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N)$ need not converge and hence not $\Lambda^N(y; \hat{\theta}^N)$ and $P^N(y; \hat{\theta}^N)$. The quantity $\frac{1}{N} L^N_{\theta\theta}(y; \hat{\theta}^N)$ does converge in one of the following two cases:

i) $\hat{\theta}^N$ converges to some point $\theta' \in S_o$.

ii) $\hat{\theta}^N$ does not converge, but $L_{\theta\theta}(\theta', \theta_o)$ is independent of θ' in S_o.

The last case is true if the only components varying in S_o are those corresponding to the initial states of the process (2.1). This is the ordinary non-degenerate case (for conditions on the process and inputs, see lemma 4 and theorem 4). However, in the general case counter-examples have been constructed to show that ii) need not hold.

There remains the possibility that $\hat{\theta}^N$ converges. However, in the absence of a rule for choosing $\hat{\theta}^N$ in case there are several absolute maxima of $\frac{1}{N} L^N(y; \theta)$ this cannot be shown.

Proof

Assume that $\hat{\theta}^N$ is not consistent (otherwise the theorem is obviously true). Since $\hat{\theta}^N$ converges into S_o (theorem 1), there is at least one point $\theta' \neq \theta_o$ such that $\theta' \in S_o \subset S'_o$. Lemma 3 then implies that S'_o is a linear

r-dimensional set, where $r \geq 1$. Introduce the new set of coordinates Θ by an orthogonal transformation

$$\theta = P\Theta = (P_1\ P_2) \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}$$

where P_2 is of rank r and parallel to S_o'. Then P_1 is orthogonal to S_o' and any point θ can be expressed in the new coordinates as

$$\theta = P_1\Theta_1 + P_2\Theta_2 = P_1(P_1^T\theta) + P_2(P_2^T\theta) \qquad \text{(A2.12)}$$

The second partial derivative matrix of $L(P\Theta, \theta_o)$ with respect to Θ is

$$P^T L_{\theta\theta}(P\Theta, \theta_o)\, P$$

Let $\theta' = P\Theta' = P_1\Theta_1' + P_2\Theta_2' \in S_o$

Then Θ_o' is constant $= P_1^T\theta_o$ and $P_1P_1^T\theta_o + P_2\Theta_2 \in S_o$ for every Θ_2

Hence from theorem I

$$\| P_1^T\hat{\theta}^N - P_1^T\theta_o \| = \| P_1^T\hat{\theta}^N - P_1^T P\hat{\theta}^N \| \to 0 \qquad \text{(A2.13)}$$

with probability one.

It has thus been shown that if P_1 is orthogonal to S_o then the projection will converge.

It remains to show that these projections are determined by the sequence of matrices $\{\frac{1}{N} L_{\theta\theta}^N(y; \theta^N)\}$.

We have from the definition of S_o

$$L(P_1P_1^T\theta_o + P_2\Theta_2, \theta_o) = L(\theta_o, \theta_o)$$

and hence not dependent on Θ_2. It follows that the second partial derivatives with respect to the r components of Θ_2 all vanish in S_o and hence

$$P^T L_{\theta\theta}(P\Theta', \theta_o) P = \begin{bmatrix} P_1^T L_{\theta\theta} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence P_2 is a set of eigenvectors corresponding to the r eigenvalues = 0, and

$$P_2^T L_{\theta\theta}(\theta', \theta_o) = 0 \quad \text{for all } \theta' \in S_o \tag{A2.14}$$

Now consider the matrix $\frac{1}{N} L^N_{\theta\theta}(y ; \hat{\theta}^N)$. We have for every N

$$\frac{1}{N} L^N_{\theta\theta} P^N = P^N \Lambda^N \tag{A2.15}$$

where Λ^N is the diagonal matrix of eigenvalues and P^N a matrix of orthogonal eigenvectors. The arguments y and $\hat{\theta}^N$ have been dropped for convenience.

We have from (A2.14), (A2.15) and lemma 5

$$\| P_2^T P^N \Lambda^N \| = \| P_2^T P^N \Lambda^N - P_2^T L_{\theta\theta}(P\hat{\theta}^N, \theta_o) P^N \|$$

$$= \| P_2^T \frac{1}{N} L^N_{\theta\theta} P^N - P_2^T L_{\theta\theta}(P\hat{\theta}^N, \theta_o) P^N \| \to 0$$

Finally we have from (A2.12) and (A2.13)

$$\| \Lambda^N P^{N^T} \hat{\theta}^N - \Lambda^N P^{N^T} \| = \| \Lambda^N P^{N^T} P_1 P_1^T \hat{\theta}^N + \Lambda^N P^N P_2 P_2^T \hat{\theta}^N$$

$$- \Lambda^N P^{N^T} P_1 P_1^T \theta_o - \Lambda^N P^{N^T} P_2 P_2^T \theta_o \|$$

$$\to \| \Lambda^N P^N P_1 (P_1^T \hat{\theta}^N - P_1^T \theta_o) \| \to 0$$

Q.E.D.

Introduce the following concepts.

DEFINITION

A system is said to be completely identifiable if the estimates of the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta_o, \beta_1, \ldots, \beta_n, \varkappa$ and λ (i.e. all parameters except those corresponding to the initial conditions) are consistent.

DEFINITION

An input signal u is said to be persistently exciting of order m if the limits

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)$$

$$r_u(t) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} u(t)\, u(t+\tau), \qquad \tau = 0, 1, 2, \ldots,$$

exist, and if the matrix

$$\begin{bmatrix} r_u(0) & r_u(1) & \ldots & r_u(m) \\ r_u(1) & r_u(0) & & r_u(m-1) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ r_u(m) & r_u(m-1) & & r_u(0) \end{bmatrix}$$

is positive definite. We then have

THEOREM 3

The system (2.1) is completely identifiable if the input u is persistently exciting of order 2n and if the system (2.1) is completely controllable either from u or from e.

Proof

The theorem is merely a re-statement of lemma 4. The definition implies that the conditions of lemma 4 are satisfied. But then the estimates of all parameters except the initial values are consistent.

Q. E. D.

THEOREM 4

Let θ denote the (3n+3) - dimensional vector of parameters where those corresponding to the initial states have been deleted.

Assume that S_o contains only the point θ_o (e.g. under the conditions of lemma 4), so that $\hat{\theta}^N$ is consistent.

Then the stochastic variable $L_{\theta\theta}(\theta_o, \theta_o)\sqrt{N}\,(\hat{\theta}^N - \theta_o)$ is asymptotically normal $(0, -L_{\theta\theta})$.

If in addition $L_{\theta\theta}(\theta_o, \theta_o)$ is non-singular, then $\hat{\theta}^N$ is asymptotically normal $(\theta_o, \frac{1}{N} L_{\theta\theta}^{-1})$.

Proof

This theorem is very similar to one of the standard theorems for the maximum likelihood estimator. See e.g. Wilks [83, p. 360]. The difference is that in our case the samples are dependent. Again the proof is obtained by an extension of the standard results. As $\hat{\theta}^N \to \theta_o$ with probability one we can always to given $\epsilon > 0$ and $\delta > 0$ find N_o and a set A_N in N-dimensional Euclidian space defined by

$$A_N = \{ y \mid \| \hat{\theta}^N(y) - \theta_o \| < \delta \}$$

such that

$$P \{ (y) \in A_N \text{ for every } N > N_o \} > 1 - \epsilon \qquad \text{(A4. 1)}$$

In A_N we have

$$L_\theta^N(y;\theta_o) = L_\theta^N(y;\hat{\theta}^N) + L_{\theta\theta}^N(y;\theta^*)\,(\theta_o - \hat{\theta}^N) \qquad \text{(A4.2)}$$

where

$$\| \theta_o - \theta^* \| < \| \theta_o - \hat{\theta}^N \|$$

But

$$L_\theta^N(y;\hat{\theta}^N) = 0 \qquad \text{(A4.3)}$$

The equations (A4.1), (A4.2) and (A4.3) now give

$$P\{ \frac{1}{N} L_\theta^N(y;\theta_o) = \frac{1}{N} L_{\theta\theta}^N(y;\theta^*) \cdot \frac{1}{N} (\theta_o - \hat{\theta}^N) \text{ for every } N > N_o \} > 1 - \epsilon$$

But this implies that the random variables

$$\frac{1}{N} L_\theta^N(y;\theta_o) \qquad \text{(A4.4)}$$

and

$$\frac{1}{N} L_{\theta\theta}^N(y;\theta^*)\ \frac{1}{N}\ (\theta_o - \hat{\theta}^N) \qquad \text{(A4.5)}$$

converge together in probability. But

$$\frac{1}{N} L_{\theta\theta}^N(y;\theta^*) \to L_{\theta\theta}(\theta_o, \theta_o)$$

with probability one. Hence it only remains to show that the vector (A4.4) is asymptotically normal. We have

$$\frac{\partial L^N}{\partial \lambda} = \frac{N}{\lambda} - \frac{1}{\lambda^3} \sum_{t=1}^{N} [\epsilon^1(t)]^2$$

But for $\theta = \theta_o$ the $\epsilon^1(t)$ are independent, equally distributed, random variables and the asymptotic normality of the last component of (A4.5) follows directly from the central limit theorem.

The asymptotic normality of the remaining components is equivalent to the asymptotic normality of $\frac{1}{N} V_\theta$ at the point $\theta = \theta_o$. From (4.10) we see that the various components can be written on the common form

$$\epsilon_o C_o^{T^{-1}} I_r C_o^{-\nu} \xi^T$$

where

$$\xi = -y, u, i, i_1, \text{ or } \epsilon_o \equiv \lambda e C_o^T$$

put

$$\xi = \bar{\xi} + e P_\xi^T$$

then

$$\begin{cases} \bar{y} = (u B_o^T + i K_o^T + i_1 Y_{oo}^T) A_o^{T^{-1}}, & P_y^T = \lambda C_o^T A_o^{T^{-1}} \\ \bar{u} = u, \bar{i} = i, \bar{i}_1 = i_1, & P_u = P_i = P_{i_1} = 0 \\ \bar{\epsilon} = 0, & P_\xi^T = \lambda C_o^T \end{cases}$$

now

$$\frac{1}{N} \epsilon_o C_o^{T^{-1}} I_r C_o^{-\nu} \xi = \frac{1}{N} \lambda e I_r C_o^{-\nu} \bar{\xi}^T + \frac{1}{N} \lambda e I_r C_o^{-\nu} P_\xi e^T$$

$$= \frac{1}{N} \lambda e I_r C_o^{-\nu} \bar{\xi}^T + \frac{1}{N} e Q e^T$$

where

$$Q = \frac{\lambda}{2} [I_r C_o^{-\nu} P_\xi^T C_o^{T^{-\nu}} I_r^T]$$

The first term is always normal and converges if the second moment

$$\frac{\lambda^2}{N} E_o e I_r C_o^{-\nu} \bar{\xi}^T \bar{\xi} C_o^{T^{-\nu}} I_r^T e^T = \frac{\lambda^2}{N} \| \bar{\xi} C_o^{T^{-\nu}} I_r^T \|^2$$

converges (Euclidian norm).

But $\bar{\xi}$ and hence $\bar{\xi} C_o^{T^{-\nu}} I_r^T$ are solutions of linear difference equations driven by u, and hence are Cesàro summable (together with its cross-products) under the assumption A, and provided the difference equations are stable. It is immediately clear that the stability of those equations is determined by the parameters $\{\alpha_{io}\}$ and $\{\gamma_{io}\}$. But these parameters generate stable difference equations ($\| A_o^{-1} \|$, $\| C_o^{-1} \|$ bounded), since $\theta_o \in R$.

The second term

$$\frac{1}{N} e Q e^T = \frac{1}{N} e M \Lambda M^T e^T = \frac{1}{N} v \Lambda v^T = \frac{1}{N} \sum_{i=1}^{N} (v_i^2 - 1) \lambda_i + \frac{1}{N} \sum_{i=1}^{N} \lambda_i$$

where λ_i are the eigenvalues of Q, and v_i are independent and normal (0, 1).

But

$$\frac{1}{N} \sum_{i=1}^{N} \lambda_i = \frac{1}{N} T_r(\Lambda) = \frac{1}{N} Tr(Q) = \frac{\lambda}{N} Tr(I_r C_o^{-\nu} P_\xi) \equiv 0$$

since C_o^{-1} and $P_\xi \in T$, and $r > 0$ (or $P_\xi = 0$).

Then Liapounoff's criterion for the central limit theorem [18, p 215] can be applied, so that

$$\frac{1}{N} \sum_{i=1}^{N} (v_i^2 - 1) \lambda_i$$

is asymptotically normal, provided that

$$\lim_{N \to \infty} \frac{\varrho}{\sigma} = 0$$

where

$$\rho = \sqrt[3]{\sum_{i=1}^{N} |\lambda_i^3| E_o |v_i^2 - 1|^3} = \text{const} \sqrt[3]{\sum_{i=1}^{N} |\lambda_i|^3}$$

is the absolute third moment, and

$$\sigma = \sqrt{\sum_{i=1}^{N} \lambda_i^2 E_o (v_i^2 - 1)^2} = \text{const} \sqrt{\sum_{i=1}^{N} \lambda_i^2}$$

is the second moment.

But

$$\frac{\rho}{\sigma} \leq \text{const} \frac{\sqrt[3]{N} \cdot \|Q\|}{\sqrt{Tr(Q^2)}}$$

(maximum norm) and we get after some calculation

$$Tr(Q^2) = \text{const} \; Tr[P_\xi^T C_o^{T^{-\nu}} I_r C_o^{-\nu} P_\xi]$$

Consider separatelv the three cases, which cover all components of V_θ

1. $\xi = u, i, i_1$: $\Rightarrow$ $Q \equiv 0$ (no quadratic term)

2. $\xi = -\nu, \nu = 1$: $P_\xi = A_o^{-1} C_o$

We have

$$Tr(Q^2) = \text{const} \; Tr\{A_o^{T^{-1}} I_r^T I_r A_o^{-1}\} \sim \text{const} \cdot N$$

if the parameters $\{a_{io}\}$ generate a stable difference equation, and

$$\|Q\| \leq \text{const} \; \|A_o^{-1}\| \leq \text{const}$$

under the same condition. Then

$$\lim_{N\to\infty} \frac{\rho}{\sigma} \leq \text{const} \lim_{N\to\infty} \frac{N^{1/3}}{N^{1/2}} = 0$$

3. $\underline{\xi = \epsilon,\ \nu=2}$: $\qquad P_\xi = \lambda C_o^T$

$$\mathrm{Tr}(Q^2) = \text{const}\ \mathrm{Tr}\left[C_o^{T^{-1}} I_r^T I_r C_o^{-1} \right] \sim \text{const} \cdot N$$

and $\| Q \| \leq \text{const} \| C_o^{-1} \| \leq \text{const}$

if the parameters $\{\gamma_{io}\}$ generate a stable difference equation.

It has thus been shown that the random vector $\frac{1}{N} L_\theta^N(y;\theta_o)$ is asymptotically normal for $N\to\infty$ if $\theta_o \in R'$.

The random vector (A4.4) has asymptotically zero mean and hence it only remains to calculate the asymptotic covariance of the random vectors (A4.4) and (A4.5).

We have the identity [83, p 348]

$$E_o \frac{1}{N} L_\theta^N(y;\theta_o) \left[L_\theta^N(y;\theta_o) \right]^T = - E_o \frac{1}{N} L_{\theta\theta}^N(y;\theta_o)$$

But the right member converges to

$$- L_{\theta\theta}(\theta_o, \theta_o)$$

The random vectors (A4.4) and (A4.5) are thus asymptotically normal.

$$(0,\ - L_{\theta\theta}(\theta_o, \theta_o))$$

Making a linear transformation, we find that the vector

$$L_{\theta\theta}(\theta_o, \theta_o)\sqrt{N}\ (\theta_o - \hat{\theta}^N)$$

is asymptotically normal $(0,\ - L_{\theta\theta}(\theta_o, \theta_o))$

Q. E. D.

APPENDIX B - SYMBOLS AND NOTATIONS

Some special mathematical symbols are used. They have the following meaning

$A \Rightarrow B$	A implies B
$A \Leftrightarrow B$	A is equivalent to B
$\omega \in R$	ω is an element in R
$R' \subset R$	R' is contained in R
$A \cup B$	the set of elements in one of A or B or both
$A \cap B$	the set of elements in both A and B
$\{\omega \mid (\text{conditions on } \omega)\}$	the set of elements ω such that the conditions are satisfied
$\omega\ \exists$	ω exists
$A = \{a_{ij}\}$	m x n matrix of elements a_{ij}, i = row index,
$= \begin{cases} a_{ij}; & i = 1, \ldots, m \\ & j = 1, \ldots, n \end{cases}$	j = column index
$\{a_i\} = \{a_i, i = 1, \ldots, m\}$	m x 1 matrix = column vector (if the index is j, the matrix is a row vector)
$\{u(t)\} = \{u(t), t = 1, \ldots, N\}$	sequence of u(t) (of length N)
$\{A\}_{ij}$	element (i, j) of the matrix A
A^T	transpose of A
$\|A\|$	norm of A
$Tr(A)$	trace of $A = \sum_i a_{ii}$
I	unit matrix
δ_i	$\begin{cases} 1 & i = 0 \\ 0 & i \neq 0 \end{cases}$
$\sim$	denotes "asymptotically equal to", i.e. $a_n \sim b_n \Leftrightarrow \lim_{n \to \infty} a_n / b_n = 1$

R	stability region = the θ - region inside which the likelihood function converges (see lemma 1)
R'	closed region $\subset R$
S_o	$\{\theta \mid L(\theta, \theta_o) = L(\theta_o, \theta_o)\}$ = equivalence class of parameter values
$P\theta$	projection of θ on the set $S_o \cap R'$, the nearest point in $S_o \cap R'$ from θ

$N(m, \sigma)$	normal distribution, mean = m, variance $= \sigma^2$
$\bar{x} = Ex$	expectation = mean of x
$D^2 x$	$Ex^2 - (Ex)^2$ = variance of x (if x is a scalar)
cov (x, y)	$Exy^T - ExEy^T$ = covariance matrix of x and y y (if x and y are column vectors)
E_o	expectation with respect to the distribution defined by the "true" parameter values θ_o
Subscript $_o$	attached to a parameter or a function of parameters, this means that the parameters have their "true" value i.e. the values that define the distribution function of the observed sample

The attempt to define a set of notations with a unique meaning has not been completely successful. A few notations have a dual meaning, and several are not listed. Those occur, however, only locally and are defined, where they are introduced. Functions are sometimes written with explicit arguments, particularly when the dependence of the arguments is stressed. The following set of notations consitutes a compromise between the demands of common practice and non-ambiguity.

u(t)	process input at time t
y(t)	observed output at time t
e(t)	independent random variables N(0, 1) generating disturbances of the process
v(t)	independent random variables N(0, 1) generating measurement errors
$u = \{u(j), j = 1, \ldots, N\}$	row vector of available inputs
$y = \{y(j), j = 1, \ldots, N\}$	row vector of available observed outputs
$e = \{e(j), j = 1, \ldots, N\}$	row vector of unknown random variables

n	order of model and process
N	length of input/output record
$\alpha = \{\alpha_i, i = 0, \ldots, n\}$	
$\beta = \{\beta_i, i = 0, \ldots, n\}$	process parameters in the representation
$\gamma = \{\gamma_i, i = 0, \ldots, n\}$	(2.1) ($\alpha_o = \gamma_o = 1$)
λ	
$\varkappa$	
a_i	
b_i	process parameters in the representation
c_i	(2.11)
k	
d_i	process parameters in the representation (2.15)
$z(t) = \{z_i(t)\}$	state vectors in state space representation
z	translation operator: $zy(t) = y(t + 1)$
$R(z)$	
$Q(z)$	pulse transfer functions
$P_i(z)$	
$Q_i(z)$	
$\varphi(\omega) = \Psi(\exp i\omega)$	power spectrum of disturbances
$y_o(t)$	initial conditions of representation (2.1) defined by (4.2)
Φ	
Γ	parameter matrices defined by (3.2)
Δ	

A	
B	
C	parameter matrices defined by (4.4)
K	
Y_o	
$\varepsilon(t)$	variables defined by (3.1) occuring in sections 3 and 5
$\theta = \{\theta_i\}$	column vector of any $4n + 3$ parameters defining the model. Also used as notation for the first $4n + 2$ parameters remaining when $\theta_{4n+3} \equiv c_o \equiv \lambda$ has been excluded, particularly as the argument of the function $V(\theta)$ (see below)
θ_o	the "true" value of θ
$\hat{\theta} = \hat{\theta}^N = \hat{\theta}^N(y)$	maximum likelihood estimate of θ_o
θ^k	k^{th} approximation when computing $\hat{\theta}$
$\ell^N(y;\theta)$	likelihood function of θ = probability density function of y given θ
$L = L(y;\theta) = L^N(y;\theta)$	$\log \ell^N(y;\theta)$
$L^N_\theta(y;\theta)$	column vector of partial derivatives of L
$L^N_{\theta\theta}(y;\theta)$	matrix of second partial derivatives of L
$L(\theta, \theta_o)$	$\lim_{N\to\infty} \frac{1}{N} E_o L^N(y;\theta)$
$L_{\theta\theta}(\theta, \theta_o)$	$\lim_{N\to\infty} \frac{1}{N} E_o L^N_{\theta\theta}(y;\theta)$
$V(\theta)$	function defined by (3.5) or (4.8) = the loss function of the problem
$V_\theta(\theta)$	column vector of partial derivatives of $V(\theta)$
$V_{\theta\theta}(\theta)$	matrix of second partial derivatives of $V(\theta)$

$V^{*}_{\theta\theta}(\theta)$	approximate second derivative matrix defined by (6.2)
$I(\theta) = I^{N}(\theta) = \{ I^{N}_{ij}(\theta) \}$	Fisher's information matrix
$x(t) = \{x_i(t)\}$	state vector of algorithm I for the computation of V
$z^{i}(t)$	state vectors for the computation of V_θ and $V_{\theta\theta}$
$s_{ij}(t)$	auxiliary variables for the computation of $V_{\theta\theta}$ by (3.32)
e^{i}	$\{\delta_j - n + i - 1\}^T$ (column vectors)
i	$\{1,\ j = 1, \ldots, N\}$ = unit step input
i_1	$\{\delta_{j-1}, j = 1, \ldots, N\}$ = unit pulse input
$\epsilon = \{\epsilon(j), j = 1, \ldots, N\}$	the row vector of disturbances of the output in sections 4 and appendix A
$\epsilon^{\nu} = \epsilon C^{T^{-\nu}},\ \nu = 1, 2, 3$	
$u^{\nu} = u C^{T^{-\nu}},\ \nu = 1, 2$	
$y^{\nu} = y C^{T^{-\nu}},\ \nu = 1, 2$	row vectors of auxiliary state variables of algorithm II for the computation of V, V_θ, $V_{\theta\theta}$
$i^{\nu} = i C^{T^{-\nu}},\ \nu = 1, 2$	
$i_1^{\nu} = i_1 C^{T^{-\nu}},\ \nu = 1, 2$	
$\psi^{\nu\mu}(\xi, \eta, t)$	auxiliary functions for the computation of V, V_θ, and $V_{\theta\theta}$ defined by (4.17)
F	
G	system matrices in state space representation
H	
I_k	$\{\delta_{i-j-k},\ i, j = 1, \ldots, N\}$ = shift matrix
T	the class of left triangular and Toeplitz matrices $= \{A \mid a_{ij} = a_{i-j},\ a_i = 0 \text{ for } i < 0\}$

AUTOMATIC CONSTRUCTION OF LINEAR STOCHASTIC DYNAMIC MODELS FOR STATIONARY INDUSTRIAL PROCESSES WITH RANDOM DISTURBANCES USING OPERATING RECORDS

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