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Abstract

When the characteristics of a system, its environment and the specifications are given in terms of external models it is very natural to use algebraic system theory to design control systems. Different problems of this type are discussed in the paper. It is shown that many design problems can be solved in a uniform, simple and direct way using algebraic system theory. Relations to other methods of solving the design problem are also discussed. The importance of Blomberg's contributions to the pole-zero cancellation problem is also discussed.

1 Introduction

My first contacts with algebraic system theory were in basic courses on Heaviside operator calculus, Laplace transform theory and linear control theory. I believe that my reactions were fairly typical for a genetration who learned automatic control in a similar way. Initially it was very pleasing to see how simple it was to reduce analysis and design of control systems to pure algebra. It was also interesting to see how simple the algebraic manipulations were and how easy it was to get insight into properties of control systems using the algebraic tools. As more familiarity about the algebraic methods were obtained there was an increasing dissatisfaction about certain matters that remained obscure. The problem of cancellation of poles and zeros was the heart of the difficulties. It seemed so strange that pure algebra had to be augmented either by analysis or by rules which at that time seemed rather arbitrary in order to get a full picture of linear control systems. Intellectually it was also very dissatisfying to resort to nonalgebraic methods to obtain a sensible theory. Gradually it became clear that Laplace transform algebra was not the right framework for linear systems problems.

State space theory, particularly the notions of reachability and observability and KALMAN's decomposition theorem [1] and [2], give a good insight into the nature of the pole-zero cancellation problem. It is, however, also possible to understand the problem in other ways. I remember my first scientific contacts with Hans Blomberg in the early sixties very well. Blomberg had an unusually clear picture of the difficulties of pole-zero cancellation and also a very good feel for how the Laplace-transform algebra should be modified to obtain the desired results. Intuitively the idea is to describe linear systems

by polynomials but not to allow division. An appropriate mathematical structure is to interprete the signal space algebraically as a module over the ring of polynomial operators. The germ of the idea is expressed in [3]. It has been worked out in fine detail in many of Blomberg's later works, [4, 5, 6]. A modification is to allow division of polynomials whose factors correspond to well damped modes [7].

The purpose of this paper is to show how several design problems for control systems can be solved using algebraic system theory. The approach given here is probably the most natural one if the system to be controlled is characterized as a rational transfer function. My own first work along these lines was the development of the minimum variance control law [8]. In that case a model of a system and its environment were given in transfer function form. A model of this form was actually obtained from system identification. To determine a feedback a realization of the transfer function was first introduced and the solution was then obtained by applying standard state space theory. It was interesting to see that the problem could be solved in a much simpler way by direct application of algebraic methods to the input output model. In this paper it is shown how several design problems can be formulated and solved using pure algebraical polynomial manipulation. The discussion includes design of observers, servos and regulators. It is also shown how the algebraic methods relate to other design techniques.

2 Preliminaries

The notations used will now be discussed together with the basic assumptions. Lower case letters u, x, y, z denote signals i.e. real time functions. Time can be either the real numbers (the continuous time problem) or the integers (the discrete time problem). Upper case letters A, B, C etc. denote polynomials.

Dynamical systems are represented as

$$Ay = Bu \tag{1}$$

where the independent variable of the polynomials is a differential operator p for continuous time problems and a forward shift operator q for discrete time problems. For discrete time systems it is natural to require that

$$\deg B \leqslant \deg A \tag{2}$$

which means that the discrete time system is causal. For continuous time systems the inequality (2) means that the input output relation does not contain any pure derivatives. Since there are problems where it is meaningful to have systems which are differentiators the condition (2) can not always be imposed on continuous time systems.

Let Z be a region of the complex plane which corresponds to sufficiently well damped modes. For continuous time systems Z can be chosen as a sector in the left half plane. For discrete time systems Z can be chosen as a region well inside the unit disc. Introduce a ring $\mathcal{R}$ of rational functions of the form B/A where the denominator polynomial has all its zeros in Z. The signal space is taken as a module $\mathcal{M}$ over $\mathcal{R}$.

This means for example that if $y \in \mathfrak{M}$ and

$$Ay = 0 \qquad (3)$$

then $y = 0$ if the polynomial A has all its zeros in Z.

Equation (3) represents a differential equation in the continuous time case and a difference equation in the discrete time case. In both cases the solution is a sum of decaying exponentials. The rate of decay is determined by the choice of the region Z. The chosen signal space is a convenient way to formalize the statement that solutions of differential equations that decay sufficiently fast are regarded as zero or equivalently that it is possible to cancel factors which correspond to sufficiently well damped modes. The module introduced here is a slight generalization of Blomberg's modules. A formal treatment is given in [7].

3 Observers

Consider a system with one input signal u and two output signals y and z. The input signal u is known and the output signal y is measured. It is desired to determine the output signal z. Let the signals be related through

$$Az = Bu \qquad (4)$$

$$Cy = Dz \qquad (5)$$

where A, B, C and D are polynomials in the differential operator p or the shift operator q. The pairs (A, B) and (C, D) are assumed to be coprime. It is assumed that the polynomials A and D do not have any common factors. If A and D have common factors the signal z is not observable from y and the stated problem can not be solved. Strictly speaking it is sufficient to assume that A and D are coprime in $\mathfrak{R}$. This means that z is detectable from y.

Since all signals are scalars we have

$$ACy = BDu. \qquad (6)$$

It is assumed that

$$\deg B \leqslant \deg A \qquad (7)$$

and

$$\deg BD \leqslant \deg AC. \qquad (8)$$

To determine the signal z from the signals u and y the following estimate is formed

$$A\hat{z} = Bu + (1/G)H (Cy - D\hat{z})$$

where G and H are polynomials. This equation is the result obtained if $\hat{z}$ is determined from the model (4) with feedback from the output signal. The equation can be written as

$$(GA + HD)\,\hat{z} = GBu + HCy. \tag{9}$$

Notice that the same result is obtained if the signal z is predicted from (5) with a feedback from (4) i.e.

$$D\hat{z} = Cy + (1/H)G(Bu - A\hat{z}).$$

Equation (9) is called an observer for $\hat{z}$. It is a dynamical system with u and y as inputs. The output z will be close to z as is seen in the following.

Analysis

Having obtained an observer its properties will now be analysed. Equations (4), (5) and (9) give

$$(GA + HD)\,\hat{z} = GAz + HDz = (GA + HD)\,z.$$

Hence

$$(GA + HD)\,(z - \hat{z}) = 0.$$

This shows that if the polynomial

$$F = GA + HD \tag{10}$$

is stable then the observer (9) will give an estimate $\hat{z}$ that converges to z as time goes to infinity. Moreover if $z = \hat{z}$ over a time interval then $\hat{z}$ will equal z for all times and all input signals.

Design of observers

If A and D are coprime there always exists a solution of equation (10). This solution is unique if it is required that

$$\deg H < \deg A. \tag{11}$$

It is reasonable to require that

$$\deg GB \leqslant \deg (GA + HD) \tag{12}$$

and

$$\deg HC \leqslant \deg (GA + HD). \tag{13}$$

For discrete time systems this means that the observer (9) is causal and for continuous time systems it means that the observer does not use derivatives. Equation (12) always holds because of (7).

It follows from (11) that (13) is always satisfied if

$$\deg F = \deg AC - 1. \tag{14}$$

This means that the degree of the observer polynomial F is chosen as the degree of the order of the system (6) minus one. For specific A and C it may actually be possible to choose an observer polynomial of lower degree.

The design of the observer (9) is straightforward. The observer polynomial is first chosen subject to the constraint (14). Equation (10) is then solved for G and H subject to (11).

If only detectability is assumed A and D have a common factor A_1 in Z. The problem can still be solved in this case provided that the observer polynomial is chosen so that it is divisible by A_1.

4 Pole and zero placement

The problem of designing a servo with a given closed loop transfer function will now be described and solved.

Formulation

Consider a process characterized by the rational operator

$$G_p = \frac{B}{A} \tag{15}$$

where A and B are polynomials. It is assumed that A and B are coprime and that

$$\deg B < \deg A. \tag{16}$$

It is desired to find a controller such that the closed loop is stable and that the transfer function from the command input u_c to the output is given by

$$G_M = \frac{Q}{P} \tag{17}$$

where P and Q are coprime and

$$\deg P - \deg Q \geqslant \deg A - \deg B. \tag{18}$$

Design procedure

A general linear regulator can be described by

$$Ru = Tu_c - Sy. \tag{19}$$

The regulator can be thought of as a combination of a feedback having the transfer function $-S/R$ with a feedforward with the transfer function T/R. The closed loop system is then characterized by the operator

$$G = \frac{TB}{AR + BS} \cdot$$

Since G should equal the desired closed loop response G_M given by (17) we get

$$\frac{TB}{AR + BS} = \frac{Q}{P} \cdot \tag{20}$$

The design problem is thus equivalent to the algebraic problem of finding polynomials R, S and T such that (20) holds. It follows from (20) that factors of B which are not also factors of Q must be factors of R. Factors of B which are not desired closed loop zeros are thus cancelled by corresponding factors of R. Factor B as

$$B = B^+B^-, \tag{21}$$

where all the zeros of B^+ are in Z and all zeros of B^- outside Z. This means that all zeros of B^+ correspond to well damped modes and all zeros of B^- correspond to unstable or poorly damped modes.

A necessary condition for solvability of the servo problem is thus that the specifications are such that B divides Q in $\mathcal{R}$ i.e.

$$Q = Q_1 B^-. \tag{22}$$

Since $\deg P$ is normally less than $\deg(AR + BS)$ it is clear that there are factors in (20) which cancel. In state space theory it can be shown that the regulator (19) corresponds to a combination of an observer and a state feedback. See [9]. It is natural to assume that the observer is designed in such a way that changes in command signals do not generate errors in the observer. This means that the factor which cancels in the right hand side of (20) can be interpreted as the observer polynomial F.

The design procedure can be formulated as follows.

Data: Given the desired response specified by the polynomials P and Q, subject to $\deg A = \deg P$ and the conditions (18), (22), and the desired observer polynomial F.

Step 1: Solve the equation

$$AR_1 + B^-S = PF \tag{23}$$

with respect to R_1 and S.

Step 2: The regulator which gives the desired closed loop response is then given by (19) with

$$R = R_1 B^+ \tag{24}$$

and

$$T = FQ_1. \tag{25}$$

The equation (23) can always be solved because it was assumed that A and B were relatively prime. This implies of course that A and B^- are also relatively prime. Equation (23) has infinitely many solutions. The unique solution determined by

$$\deg S < \deg A \tag{26}$$

is chosen here. Since $\deg P = \deg A$ it then follows that $\deg R_1 = \deg F$. If the observer polynomial is chosen in such a way that

$$\deg F = \deg A - \deg B^+ - 1 \tag{27}$$

then

$$\deg R = \deg A - 1 \geqslant \deg S$$
$$\deg T = \deg A - 1 = \deg R. \tag{28}$$

This means in the continuous time case that the regulator does not include any pure derivatives and in the discrete time case that the regulator is causal. Notice that in special cases the regulator may still have the property (28) even if (27) does not hold. Also notice that the choice (27) corresponds to a Luenberger observer in the state space interpretation. Further discussions including examples are found in [9].

Analysis

A direct calculation gives

$$\frac{TB}{AR + BS} = \frac{FQ_1 B^+ B^-}{B^+ (AR_1 + B^- S)} = \frac{FQ_1 B^-}{PF} = \frac{Q}{P}$$

which shows that the regulator gives the desired closed loop response. Notice that in this calculation we have divided with the factors B^+ and F. This is permitted since it was assumed that all their zeros are well damped.

A direct calculation shows that the closed loop system has the characteristic polynomial $B^+ FP$. The polynomial B^+ has all its zeros in Z by definition. Since the observer polynomial F and the polynomial P were chosen to have all their zeros in Z it follows that the closed loop system has all its poles in Z.

Interpretation as model following

The results obtained will now be used to give an interpretation of the regulator (19). It follows from (23), (24) and (25) that

$$\frac{T}{R} = \frac{FQ_1}{B^+R_1} = \frac{(AR_1 + B^-S)Q_1}{PB^+R_1} = \frac{AQ_1}{B^+P} + \frac{SB^-Q_1}{B^+R_1P} = \frac{A}{B} \cdot \frac{Q}{P} + \frac{S}{R} \cdot \frac{Q}{P}.$$

The feedback law (19) can thus be written as

$$u = \frac{A}{B}y_c + \frac{S}{R}(y_c - y) \tag{29}$$

where

$$y_c = \frac{Q}{P}u_c. \tag{30}$$

The signal y_c can be interpreted as the output obtained when the command signal u_c is applied to the model Q/P. When the regulator (19) is re-written as (29) it is clear that it can be thought of as composed of two parts, one feedforward term $(A/B)y_c = (A/B)(Q/P)u_c$ and one feedback term $(S/R)(y_c - y)$. The feedforward is a combination of the ideal model and an inverse of the process model. Notice, however, that the system A/B is not realizable although the combination $AQ/(BP)$ is realizable because of (18). The feedback term is obtained by feeding the error $y_c - y$ through a dynamical system characterized by the operator S/R. It is thus clear that the regulator can be interpreted as a model following servo.

The MISO regulator

So far it has been assumed that there is only one measured output signal. In practice it often happens that there are additional signals which are measured and that the regulator can be simplified and improved by using these additional signals. This will in general require a complete multivariable theory. It will be shown, however, that the problem can also be handled by marginal extensions to the single output theory. Let it be assumed that a regulator has been determined for the case of one measured output y and that the feedback operator S/R has been determined. Furthermore assume that one additional measurement y_1 is available. Let this signal be related to y as

$$y_1 = \frac{D_1}{B_1} \tag{31}$$

where D_1 and B_1 are coprime. It is easy to show that B_1 must divide B in (15). To find a feedback from y and y_1 which is identical to the signal $(S/R)y$ the polynomial R is first factored as

$$R = R_1R_2. \tag{32}$$

It is then attempted to find two polynomials S_1 and S_2 such that

$$\frac{S}{R}y = \frac{S_1}{R_1}y + \frac{S_2}{R_2}y_1 = \left(\frac{S_1}{R_1} + \frac{S_2 D_1}{R_2 B_1}\right)y.$$

If this is possible B_1 must divide S_2. Hence

$$S_2 = B_1 S_3 \tag{33}$$

and S_1 and S_3 satisfy

$$S = S_1 R_2 + S_3 D_1 R_1. \tag{34}$$

This equation can be solved with respect to S_1 and S_3 if $D_1 R_1$ and R_2 are relatively prime. Since the factoring (32) may be done in many ways there may be several solutions. The choice among the different possibilities can be made with respect to simplicity, disturbance sensitivity, and causality.

It is possible to proceed recursively in a similar way if there are several measured signals. The regulator obtained when this procedure is used is of the form

$$u = \frac{T}{R}u_c - \frac{S_1}{R_1}y_1 - \frac{S_2}{R_2}y_2 - \frac{S_3}{R_3}y_3 + \dots \tag{35}$$

where

$$R = R_1 R_2 R_3 \dots$$

This regulator is called the MISD regulator because it has many inputs, $u_c, y_1, y_2, \dots,$ and a single output u. An implementation of the MISO regulator with a highly interactive operator communication is described in [10].

5 Linear quadratic control

In the problems discussed in Section 4 the desired closed loop polynomial P and desired observer polynomial F are considered as given. There are many ways to determine P and F. They can e.g. be regarded as tuning parameters which are adjusted to obtain a suitable performance or the polynomial P can be chosen from tables of standard forms. Another possibility is to formulate the control problem as an optimization problem. The polynomials P and F can then be determined from the optimization criterion. This problem will be briefly discussed in this section.

Preliminaries

It is first assumed that the process to be controlled is governed by

$$Ay = Bu \tag{36}$$

and that the control problem is formulated as to minimize the quadratic criterion

$$J = \int_0^\infty [y^2(t) + \rho u^2(t)]\, dt. \tag{37}$$

It is well known [11] that the solution of this optimization problem gives a closed loop system whose characteristic polynomial P is given by

$$PP_\star = \rho AA_\star + BB_\star \tag{38}$$

where $A_\star$ denotes the conjugate of the polynomial A. If

$$A(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$$

then the conjugate polynomial is defined by

$$A_\star(x) = \begin{cases} a_0 + a_1 x + \dots + a_n x^n & \text{discrete time systems} \\ \\ A(-x) & \text{continuous time systems.} \end{cases}$$

The design of a regulator which minimizes (37) is thus reduced to the following algebraic problems:

— Find a polynomial P which satisfies (38). This is called spectral factorization problem.

— Apply the algebraic design procedure given in Section 4.

Notice that in the second step it is necessary to choose the observer polynomial and to choose an appropriate solution of the equation (23). It will next be shown how the observer polynomial can be determined if mathematical models of the disturbances are included.

Deterministic control

Assume that the process to be controlled is described by

$$Ay = Bu + Ce \tag{39}$$

where e is an impulse disturbance and C a stable polynomial. Let the criterion be to minimize (37).

Representing the signals as infinite power series and invoking Parseval's theorem the criterion (37) can be written as

$$J = \frac{1}{2\pi i} \oint (yy_\star + \rho uu_\star)\frac{dz}{z}\ .$$

Using (36) the integrand can be written as

$$I = \frac{(Bu + Ce)(Bu + Ce)_\star}{AA_\star} + \rho uu_\star = \frac{PP_\star uu_\star + (CB_\star + C_\star B)ue_\star + CC_\star ee_\star}{AA_\star}$$

where P is stable and given by (38). To complete the squares the polynomial S defined by

$$PS_\star + P_\star S = CB_\star + C_\star B \tag{40}$$

is introduced. Then

$$I = \frac{PP_\star uu_\star + PS_\star ue_\star + P_\star Su_\star e + CC_\star ee_\star}{AA_\star} =$$

$$= \frac{(Pu + Se)(Pu + Se)_\star + (CC_\star - SS_\star)ee_\star}{AA_\star}$$

The loss function (37) is thus minimized for the control signal

$$u = -\frac{S}{P} e. \tag{41}$$

This is an open loop control or a control program. To find the corresponding control law introduce this signal into (39). Hence

$$APy = (CP - BS)e.$$

The polynomial $CP - BS$ is divisible by A. Hence

$$AR = CP - BS$$

or

$$AR + BS = CP. \tag{42}$$

This implies

$$y = \frac{R}{P} e.$$

The feedback law is thus

$$u = -\frac{S}{R} y. \tag{43}$$

It is easy to find all solutions to (42). The particular solution which also satisfies (40) is given by

$$\deg S = \deg B + \deg C - \deg P. \tag{44}$$

If the polynomial C is not stable it can be factored as

$$C = C^+ C^-$$

provided that it has no zeros on the critical line. Equation (42) is then replaced by

$$AR + BS = C^+ C_\star^- P \tag{45}$$

and the optimal feedback is given by (43).

The solution of the factorization problem is not unique, if P_1 is a stable solution then $z^k P_1$ is also a solution. The integer k can then e.g. be chosen as the smallest integer which gives a causal feedback or as the feedback which does not include any derivatives.

Notice that the solution can also be interpreted as follows. Choose the observer polynomial C and the polynomial P as the solution to the factorization problem. Then apply the design procedure of the previous section with $B^+ = 1$, where the appropriate solution to equation (23) is determined by condition (44).

Stochastic control

In this case it is assumed that the process to be controlled is described by (36) where e is a white noise stochastic process and that C is stable. The purpose of control is to minimize

$$E(y^2 + \rho u^2) \tag{46}$$

in steady state. It is straightforward to show that the control law (43) minimizes (46).

For discrete time minimum variance control we have $\rho = 0$ and

$$P = z^{\deg A - \deg B^-} B_\star^- B^+.$$

See [12].

6 Conclusions

It has been shown that algebraic system theory is a simple and convenient tool for solving several control system design problems. Using the algebraic tools the design problems are simply reduced to polynomial manipulation. In particular it was demonstrated that design of observers and regulators and servos could be reduced to the solution of the diophantine equation

$$AX + BY = C \tag{47}$$

and that linear quadratic optimal control problems can be reduced to solving the factorization problem

$$PP_\star = \rho AA_\star + BB_\star. \tag{48}$$

Relations between the algebraic methods and state space theory has also been given. For further discussions see [6, 7, 9, 13–17].

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