A left inverse approach to sharp Sobolev inequalities

Hindov, Raul

2025

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Hindov, R. (2025). *A left inverse approach to sharp Sobolev inequalities*. [Licentiate Thesis, Mathematics (Faculty of Sciences)]. Lund University (Media-Tryck).

Total number of authors:
1

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

A left inverse approach to sharp Sobolev inequalities

A left inverse approach to sharp Sobolev inequalities

by Raul Hindov

LUND UNIVERSITY

Thesis for the degree of Licentiate of Philosophy
Thesis advisor: Docent Jan-Fredrik Olsen
Co-advisors: Professor Sandra Pott, Doctor Eskil Rydhe
Faculty opponent: Docent Alan Sola

To be presented, with the permission of the Faculty of Science of Lund University, for public criticism in MH:Riesz auditorium at the Centre for Mathematical Sciences on Monday, the 13th of October 2025 at 10:00.

DOKUMENTDATABLAD enl SIS 61 41 21

Organization **LUND UNIVERSITY** Centre for Mathematical Sciences Box 118 SE–221 00 LUND Sweden	Document name **LICENTIATE DISSERTATION** Date of disputation 2025-10-13 Sponsoring organization
Author(s) Raul Hindov	

Title and subtitle

A left inverse approach to sharp Sobolev inequalities

Abstract

The thesis is devoted to sharp embeddings from Soblolev spaces into Lebesque spaces in one dimension. The paper included in the thesis concerns the embedding from the Sobolev space $W_0^{k,2}(-1,1)$ into $L^1(-1,1)$, for all integer orders of derivatives $k \geqslant 1$. The main result is the sharp constant and the explicit extremal functions given by the Landau kernels in this case. The result improves on the two-sided estimates obtained by Kalyabin in 2014. We survey the literature of the sharp Sobolev inequalities and discuss discrete analogues and possible extensions.

Key words

Sobolev inequalities

Classification system and/or index terms (if any)

Supplementary bibliographical information	Language English
ISSN and key title 1404-028X Licentiate theses in mathematical sciences	ISBN 978-91-8104-685-4 (print) 978-91-8104-686-1 (pdf)

Recipient's notes	Number of pages 55	Price
	Security classification	

Signature ______________________ Date 2025-9-1

A left inverse approach to sharp Sobolev inequalities

by Raul Hindov

Lund University

Cover illustration front: Norman Hindov

Faculty of Science, Centre for Mathematical Sciences
Licentiate Thesis in Mathematical Sciences 2025:2
ISSN: 1404-028X

ISBN: 978-91-8104-685-4 (print)
ISBN: 978-91-8104-686-1 (pdf)
LUNFMA-2017-2025

Printed in Sweden by Media-Tryck, Lund University, Lund 2025

NORDIC SWAN ECOLABEL
Printed matter
3041 0903

Media-Tryck is a Nordic Swan Ecolabel certified provider of printed material. Read more about our environmental work at www.mediatryck.lu.se

MADE IN SWEDEN

Contents

Publication

This thesis is based on the following publication:

1 **A sharp higher order Sobolev embedding**

R. Hindov, S. Nitzan, J.-F. Olsen, E. Rydhe
Mathematika, 2025, 71:e70012

The paper is reproduced with permission of its publisher.

Populärvetenskaplig sammanfattning på svenska

Matematiken är full av olikheter, regler som berättar hur en storhet förhåller sig till en annan. Vissa är enkla, som att ett tal är större än ett annat. Andra är djupgående och avslöjar grundläggande samband inom rymd, rörelse och till och med vågor och värme.

Bland de finns Sobolev-olikheter, som kopplar samman två till synes olika saker: hur slät en funktion är (hur väl den beter sig när man zoomar in), och hur stor den kan vara i sin helhet. Med andra ord besvarar de frågor som: Om en funktion är mycket slät, kan den ändå vara mycket stor? Det är inte bara en abstrakt fundering, det är en fråga som har format hela forskningsfält, från fysik till datavetenskap.

Den här frågan har fascinerat matematiker i över hundra år. På 1850-talet studerade den ryske matematikern Pafnutij Tjebysjev hur storleken på ett polynom kunde kontrolleras via dess koefficienter. Senare undersökte Markov-bröderna och Sergej Bernstein hur derivator, förändringshastigheter, kunde begränsa en funktions storlek. Dessa idéer lade grunden för Sobolevs genombrott år 1938, som visade att kontroll över en funktions derivator också ger kontroll över funktionen själv.

Men inte alla olikheter är lika. Vissa är skarpa, vilket betyder att de är de bästa möjliga, de kan inte förbättras. Att hitta dessa skarpa olikheter är som att upptäcka exakt var en klippkant går, man vet precis var gränsen ligger.

I min forskning fokuserar jag på en särskild skarp Sobolev-olikhet i en dimension. Den handlar om funktioner som försvinner vid intervallets gränser och vars släthet mäts via högre ordningens derivator. Olikheten ser ut så här,

$$\|f\|_{L^1(-1,1)} \leqslant c_k \left\|f^{(k)}\right\|_{L^2(-1,1)} .$$

Här mäter vänstersidan funktionens totala storlek, och högersidan mäter slätheten via dess k:te derivata. Konstanten c_k är den skarpa tröskeln, det finns inga funktioner som bryter denna gräns.

Fram till nyligen var denna konstant endast känd för det enklaste fallet. Min forskning fyller denna lucka: jag beräknar det exakta värdet på c_k för alla k, och identifierar de funktioner som uppnår likhet. Överraskande nog visar det sig att dessa extremala funktioner är Landau-kärnor, $L_k(x) = (1 - x^2)^k$, välkända från harmonisk analys.

Dessa resultat är inte bara matematiska troféer. Konstanterna i sådana olikheter kodar djup information om rummet och operatorerna som är inblandade. De dyker upp i fysik, signalbehandling, numerisk analys och till och med i algoritmdesign. Att känna till den skarpa konstanten kan hjälpa till att optimera system, lösa ekvationer mer exakt och förstå gränserna för approximation.

Även om min avhandling fokuserar på kontinuerliga funktioner, undersöker jag också diskreta analoger, versioner av problemet där funktioner lever på gitter eller nätverk. Dessa diskreta fall är mindre utforskade men erbjuder spännande möjligheter för framtida forskning.

Notation

We give some of the notation used throughout the thesis.

$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{T}$	Unit circle
$\mathbb{R}^n$	Euclidean n-space

$C(\Omega)$	Continuous functions on $\Omega \subset \mathbb{R}^n$
$\operatorname{supp} f$	Support: $\overline{\{x \in \Omega : f(x) \neq 0\}}$
$C_0(\Omega)$	Continuous functions with compact support in Ω
$C^k(\Omega)$	k-times continuously differentiable functions
$C_0^k(\Omega)$	$C^k(\Omega) \cap C_0(\Omega)$
$C^\infty(\Omega)$	Smooth functions $\bigcap_{k=1}^\infty C^k(\Omega)$
$C_0^\infty(\Omega)$	Compactly supported smooth functions (test functions)

$\|f\|_{L^p} = \left(\int |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$	for $1 \leqslant p < \infty$
$\|f\|_{L^\infty} = \operatorname{sup\,ess} |f|$	for $p = \infty$

$L^p(a,b)$	Lebesgue space of functions with $\|f\|_{L^p(a,b)} < \infty$
$W^{k,p}(a,b)$	Sobolev space of functions with weak derivatives $f^{(\alpha)} \in L^p(a,b)$ for $1 \leqslant \alpha \leqslant k$
$W_0^{k,p}(-1,1)$	Sobolev space with boundary conditions $f^{(j)}(\pm 1) = 0$ for $j = 0, 1, \dots, k-1$

$\|f\|_{W^{k,p}} = \left(\sum_{\alpha=0}^k \|f^{(\alpha)}\|_{L^p}^p\right)^{1/p}$	for $1 \leqslant p < \infty$
$\|f\|_{W^{k,\infty}} = \sum_{\alpha=0}^k \|f^{(\alpha)}\|_{L^\infty}$	for $p = \infty$
$\|f\|_{W_0^{k,p}} = \|f^{(k)}\|_{L^p}$	for $1 \leqslant p \leqslant \infty$

$P(z) = \sum_{j=0}^d a_j z^j \in \mathbb{C}[z]$	polynomial of degree d
$\|P\|_{\ell^p} = \left(\sum_{j=0}^d |a_j|^p\right)^{1/p}$	for $1 \leqslant p < \infty$
$\|P\|_{\ell^\infty} = \max_j |a_j|$	
$\|P\|_{L^p} = \left(\int_0^1 |P(e^{2\pi i\theta})|^p \,\mathrm{d}\theta\right)^{1/p}$	for $1 \leqslant p < \infty$
$\|P\|_{L^\infty} = \max_{|z|=1} |P(z)|$	

$\hat{f}(\xi) = \int f(x)e^{-2\pi i\xi x}\,\mathrm{d}x$	Fourier transform on $\mathbb{R}$
$\hat{a}(\xi) = \sum_{j\in\mathbb{Z}} a_j e^{-i\xi j}$	discrete-time Fourier transform
$\hat{a}(k) = \frac{1}{\sqrt{d+1}} \sum_{j=0}^{d} a_j e^{2\pi ikj/(d+1)}$	finite Fourier transform
$\Delta^k a_j = \sum_{m=0}^{k} \binom{k}{m}(-1)^m a_{j-m}$	discrete derivative
$\|a\|_{\ell^p} = (\sum_j \|a_j\|^p)^{1/p}$	
$\|a\|_{\ell^\infty} = \max_j \|a_j\|$	

$\Gamma(x)$	Gamma function
$B(x,y)$	Euler beta function, $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
$n!!$	Double factorial, $n!! = 2^{\frac{n}{2}}\left(\frac{n}{2}\right)!$, if n is even, and $n!! = \frac{n!}{(n-1)!!}$, if n is odd

A left inverse approach to sharp Sobolev inequalities

1 Motivation and background

I have a stern and serious secret. I study polynomial inequalities and inequalities that relate functions with their derivatives. My colleagues study weighty and deep subjects, random matrix theory, invariant subspaces, weakly branch actions, harmonic morphisms, global bifurcation theory, joint diagonalization, reproductive boundaries, complexities in the complex plane. But inequalities?

The demigods Hardy, Littlewood and Pólya wrote a comprehensive book [19] on the subject, titled ”Inequalities”, already in 1934. What can a mere mortal add?

And yet, inequalities are primal impulses within all of analysis. Inequalities can be qualitative or quantitative, and among the latter, the most difficult to obtain, are the sharp ones. They serve as the organising principles behind approximation theory, partial differential equations, and harmonic analysis. For example, they quantify the trade-off between size, smoothness, and oscillation of functions.

Among the wide variety of inequalities in analysis, Sobolev-type inequalities occupy a major position. They connect information about the derivatives of a function to information about the function itself.

Would smoothness of a function tell how large the function can be? Mathematicians have been asking versions of this question for more than a century. In 1854, Chebyshev [13] considered polynomial inequalities where the size of the polynomial was measured in the uniform norm, and the control came from its leading coefficient. Later, in 1890s and in the early 20th century, inequalities involving derivatives, such as the one from Markov brothers [29, 30] on the unit interval $[-1, 1]$ and another from Bernstein [5, 6] on the unit circle, became key tools in approximation theory. In 1938, Sobolev [40] formulated his now-famous embeddings – results that show how bounds on derivatives, in a given L^p-norm, together with boundary conditions imply bounds on the function itself in another L^q-norm.

Since then, these inequalities have been studied in many variations. For example, we can consider inequalities when the underlying domain is a continuum or when it is a discrete set. Or we can control the function via its derivatives, its Fourier transform, its moments, or its polynomial coefficients. Or we can use different norms, measuring size in L^p, l^p, or with mixed norms, or norms with weights, each leading to distinct constants and extremal problems.

The constants that appear in such inequalities are far from arbitrary, they encode information about the underlying space and the operators involved. In some cases, they can be interpreted as minimal eigenvalues of Toeplitz matrices [8], minimal eigenvalues of some higher-order ordinary differential operators [8, 34], norms of the Green kernels of these operators [8, 46, 33], or condition numbers in approximation schemes [8, 22].

Thus, a result about these inequalities are also useful in other areas of mathematics and physics. For example a precise determination of the sharp constant not only settles an extremal problem in analysis, but may also resolve a question in numerical linear algebra, spectral theory, or mathematical physics.

The present work is concerned with a specific Sobolev-type embedding in one dimension, from the Sobolev space $W_0^{k,2}(-1,1)$ into $L^1(-1,1)$, for integer $k \geqslant 1$. While classical results tell us that such an embedding exists, the sharp constant in the inequality

$$\|f\|_{L^1(-1,1)} \leqslant c_k \left\|f^{(k)}\right\|_{L^2(-1,1)}$$

was previously unknown for $k > 1$. Earlier work by Kalyabin [21] provided matching upper and lower bounds up to a gap for the embedding constant, but the exact value remained elusive.

In the paper that is included in the thesis, we close this gap by computing c_k exactly and identifying all extremal functions. Interestingly, these extremals turn out to be the Landau kernels, $L_k(x) = (1-x^2)^k$, familiar from the classical harmonic analysis.

While this thesis will primarily address the continuous case, we will also briefly discuss related discrete analogues. The discrete case can be considered as a parallel problem where similar techniques may be adapted and the landscape of known results is sparser.

In what follows, we survey the literature most relevant to our case, present our main result, placing it in context with previous work, and briefly remark on discrete analogues and possible extensions.

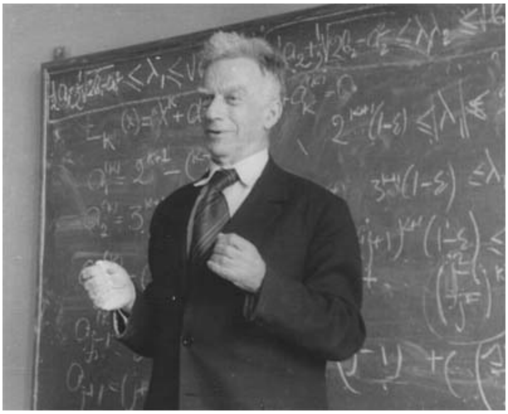

Figure 1: Sergei Sobolev (1908-1989).

2 Literature survey

Our problem belongs to the family of Sobolev inequalities for which the task can be formulated as follows.

Problem 1. *Given f in $W_0^{k,p}(-1,1)$ determine the smallest possible positive constant $c_{k,q,p}$ in the inequality*

$$\|f\|_{L^q(-1,1)} \leqslant c_{k,q,p} \|f\|_{W_0^{k,p}(-1,1)} \tag{1}$$

and indentify the extremal function.

For the qualitative character of the inequality and nested properties of the Lebesque and Sobolev spaces see 5.1, below.

Knowing the sharp constant in the inequality (1) would tell the largest size the function with compact support in $[-1,1]$ can have in L^q-norm when we know the the size of its k-th derivative in L^p-norm. In another words, the sharp constant gives the sharp upper bound for the exchange rate when we travel from a particular Sobolev space to a particular Lebesque space trading regularity to integrability.

In the case $p=2$, we have an additional interpretation. The inequality (1) can be stated as

$$\|f\|_{L^q(-1,1)} \leqslant c_{k,q,p}(2\pi)^k \left\| |\xi|^k \hat{f}(\xi) \right\|_{L^2(\mathbb{R})}, \tag{2}$$

since by the Plancherel formula

$$\int_{-1}^{1} |f^{(k)}(x)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\widehat{f^{(k)}}(\xi)|^2 \,\mathrm{d}\xi$$
$$= (2\pi)^{2k} \int_{\mathbb{R}} |\xi|^{2k} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi.$$

This means that, when we fix a L^q-norm for functions with a compact support in $[-1, 1]$, then the L^2-norm of the Fourier transform of their k-th derivative cannot be smaller than a sharp positive constant.

The family of inequalities (1) has attracted sustained attention for more than hundred years since the work of Steklov [42]. But in the case of higher order derivatives, the search for the sharp constants and the extremal functions has been particularly active and successful research area during the last 15 years.

There is a group of researchers in St. Petersburg who have systematically studied and surveyed the sharp constants and the extremal functions in Sobolev embeddings [25, 32].

Alone in the case $k = 1$, there are many contributions [4, 3, 1, 44, 2, 43, 10, 16, 17, 42, 27, 19], sometimes independent of earlier work, that have rediscovered special cases or provided alternative proofs. While the common theme in the proofs in the case $k = 1$ was Euler-Lagrange variational method, there is a notable exception by Cordero-Erausquin et al. [14] who used the mass-transportation approach.

The first known result dates back to Steklov [42] in 1896 and concerns the first order derivative, $k = 1$, and exponents $q = p = 2$. He found that $c_{1,2,2} = \frac{2}{\pi}$ and that the extremal function is $f(x) = \cos\left(\frac{\pi}{2}x\right)$. This was among the earliest inequalities with sharp constant that appeared in mathematical physics (for more to whom the result has been ascribed, see 5.4, below). Steklov applied the inequality to justify the Fourier method for the heat and wave equations.

For the first order derivative, $k = 1$, and all $q, p \in [1, \infty]$, the problem was solved by Schmidt [38] in 1940 stating that

$$c_{1,q,p} = \frac{2^{1+\frac{1}{q}-\frac{1}{p}} q(1 + p'/q)^{1/p}}{2(1 + q/p')^{1/q} B(1/q, 1/p')},$$

where B stands for the Euler beta function.

In this case, the extremal function can be expressed as a generalised trigonometric function of Lindqvist-Peetre (for the definition and properties on these functions, see 5.5, below). Interestingly, the result has been rediscovered many times [1, 44] and even as recently as in 2002 by Bennewitz and Saitō [3, 4] (for comments on their extremal function, see 5.6).

Table 1: Known sharp constants $c_{k,q,p}$ discussed in Section 2. The empty set sign indicates no known result and the algorithm of the case $q = 2$, $p = 2$ is given in Theorem 2.

q	$k=1$	$k=2$	$k=3$	$k=4$	$k \in \mathbb{Z}_+$	p
1	1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1
1	$\sqrt{\frac{2}{3}}$	$\frac{1}{3}\sqrt{\frac{2}{5}}$	$\frac{1}{15}\sqrt{\frac{2}{7}}$	$\frac{1}{105}\sqrt{\frac{2}{9}}$	$\frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}}$	2
1	1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	∞
2	$\sqrt{\frac{1}{2}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	1
2	$\frac{2}{\pi}$	$0.1795461\ldots$	$\frac{1}{\pi^3}$	$0.004322\ldots$	algorithm	2
2	$\sqrt{\frac{2}{3}}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	∞
∞	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{16}$	$\emptyset$	$\emptyset$	1
∞	$\sqrt{\frac{1}{2}}$	$\frac{1}{2\sqrt{6}}$	$\frac{1}{8\sqrt{10}}$	$\frac{1}{48\sqrt{14}}$	$\frac{1}{2^k(k-1)!\sqrt{k-\frac{1}{2}}}$	2
∞	1	$\frac{1}{4}$	$\frac{2-\sqrt{2}}{12}$	$\frac{8-3\sqrt{5}}{192}$	$\frac{k+1}{\pi k! 2^{k-1}} \int_{-1}^{+1} \frac{(1-x^2)^k}{1+(-1)^k x^{2(k+1)}} \, \mathrm{d}x$	∞

During the last 15 years we have seen a rapid development in determining the sharp embedding constants for higher order derivatives ($k > 1$). In 2010, Kalyabin [20] found the first full solution and this was on the case $q = \infty$, $p = 2$. This was followed by Petrova [34] in 2017, solving fully the case $q = 2$, $p = 2$, and Kazimirov, Sheipak [22] in 2024 solving fully the case $q = \infty$, $p = \infty$. The paper that is included in the thesis gives a full solution on the case $q = 1$, $p = 2$.

Thus, we have full solutions on four crossroads out of nine on the $\left(\frac{1}{q}, \frac{1}{p}\right)$-plane. An illustration of known sharp higher order constants is indicated on Figure 2 and in Table 1.

Kalyabin's proof of the first full solution uses a similar left inverse of the differential operator $f \mapsto f^{(k)}$ for functions $f \in W_0^{k,2}(-1,1)$ that we are using in our paper. We give an expository proof of his result.

Theorem 1 (Theorem 1 in Kalyabin [20]). *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the sharp inequality*

$$\|f\|_{L^\infty(-1,1)} \leqslant \frac{1}{2^k(k-1)!\sqrt{k-\frac{1}{2}}} \|f\|_{W_0^{k,2}(-1,1)} . \tag{3}$$

The extremal functions are given by $f(x) = \sum_{j=k}^{\infty} \left(j + \frac{1}{2}\right) \left(P_j^{(-k)}(x)\right)^2$.

Proof. We consider the left inverses of the differential operator $f \mapsto f^{(k)}$ for functions

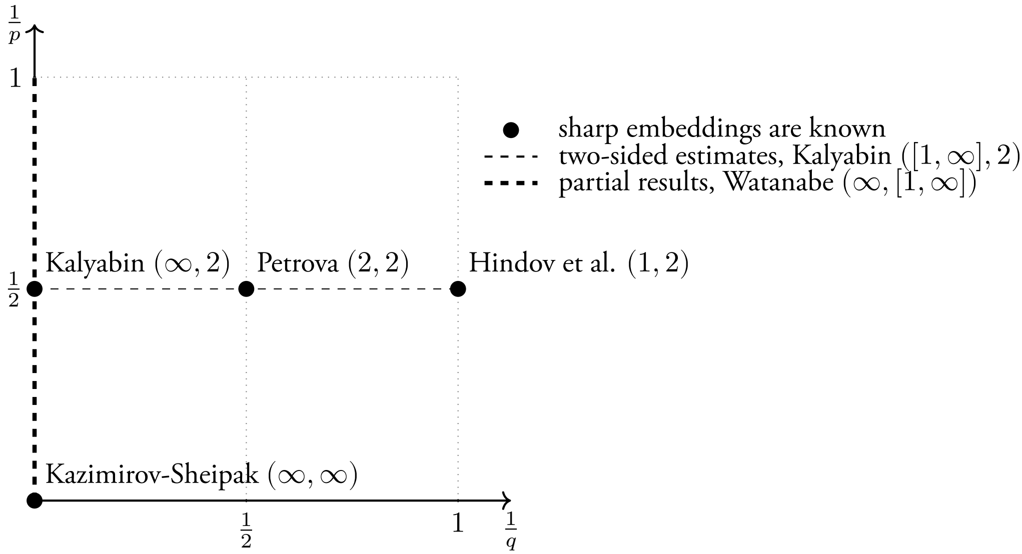

Figure 2: Sharp Sobolev embeddings for all orders of derivatives are known at the black points on the $\left(\frac{1}{q},\frac{1}{p}\right)$-plane (for the sharp embedding constants, see Table 1). Two-sided estimates are known on the dashed line. Partial results are known on the thick dashed line. In the brackets, the values of q and p are given as (q,p). $(1,2)$ is the result of the paper that is included in the thesis.

$f \in W_0^{k,2}(-1,1)$ in the following form

$$f(x) = \frac{1}{(k-1)!}\int_{-1}^{x}(x-y)^{k-1}f^{(k)}(y)\,\mathrm{d}y.$$

Given $f \in W_0^{k,2}(-1,1)$, denote $g(x) := f^{(k)}(x)$. Then $g \in L^2(-1,1)$ and repeatedly integrating by parts gives

$$\begin{aligned} g^{(-k)}(x) &:= \frac{1}{(k-1)!}\int_{-1}^{x}(x-y)^{k-1}g(y)\,\mathrm{d}y \\ &= \sum_{j=1}^{k-1}\left[\frac{(x-y)^{k-j}}{(k-j)!}f^{(k-j)}(y)\right]_{-1}^{x} + \int_{-1}^{x} f'(y)\,\mathrm{d}y = f(x) \end{aligned}$$

for all $x \in (-1,1)$, since all the terms in the sum vanish.

The boundary condition $f^{(j)}(\pm1) = 0$ for all $0 \leqslant j < k$ and integration by parts gives us a sequence of equivalences

$$
\begin{aligned}
\int_{-1}^{+1} f^{(k)}(y)\,\mathrm{d}y = 0 \iff f^{(k-1)}(\pm 1) = 0 \\
\int_{-1}^{+1} y f^{(k)}(y)\,\mathrm{d}y = 0 \iff f^{(k-1)}(\pm 1) = 0 \text{ and } f^{(k-2)}(\pm 1) = 0 \\
\dots \\
\int_{-1}^{+1} y^{k-1} f^{(k)}(y)\,\mathrm{d}y = 0 \iff f^{(j)}(\pm 1) = 0, \quad j = 0, 1, ..., k-1.
\end{aligned}
$$

Hence, we have the family of orthogonality conditions

$$\int_{-1}^{+1} y^j g(y)\,\mathrm{d}y = 0, \quad j \in \{0, 1, k-1\}.$$

Therefore, in the series expansion of g in the orthogonal Legendre polynomials,

$$g(y) = \sum_{j=0}^{\infty} a_j(g) P_j(y),$$

the coefficients $a_j(g)$ must necessarily vanish for all $j < k$.

Recall that the classical Legendre polynomials defined as

$$P_j(x) = \frac{1}{2^j j!} \frac{\mathrm{d}^j}{\mathrm{d}x^j} (1 - x^2)^j$$

with $\left\|P_j(x)\right\|^2_{L^2(-1,1)} = \frac{1}{j+\frac{1}{2}}$, form an orthogonal family on $L^2(-1,1)$. Hence, for any fixed $x \in (-1,1)$, we have by the Cauchy-Schwarz inequality

$$
\begin{aligned}
|f(x)|^2 &= |g^{(-k)}(x)|^2 \\
&= \left| \sum_{j=k}^{\infty} a_j(g) P_j^{(-k)}(x) \right|^2 \\
&= \left| \sum_{j=k}^{\infty} a_j(g) P_j^{(-k)}(x) \left(\frac{j+\frac{1}{2}}{j+\frac{1}{2}} \right)^{\frac{1}{2}} \right|^2 \\
&\leqslant \sum_{j=k}^{\infty} \frac{a_j^2(g)}{j+\frac{1}{2}} \sum_{j=k}^{\infty} \left(P_j^{(-k)}(x) \right)^2 \left(j + \frac{1}{2} \right) \\
&= \|g\|^2_{L^2(-1,1)} A_k(x),
\end{aligned}
$$

where we denote

$$
\begin{aligned}
A_k(x) &:= \sum_{j=k}^{\infty} \left(P_j^{(-k)}(x)\right)^2 \left(j+\frac{1}{2}\right) \\
&= \sum_{j=k}^{\infty} \left(\frac{1}{2^j j!}\frac{\mathrm{d}^{j-k}}{\mathrm{d}x^{j-k}}(1-x^2)^j\right)^2 \left(j+\frac{1}{2}\right).
\end{aligned}
$$

If we choose the coefficients $a_j(g) := \left(j+\frac{1}{2}\right) P_j^{(-k)}(x)$, then it follows from the above computation that

$$
|f(x)|^2 = \|g\|^2_{L^2(-1,1)} A_k(x)
$$

and we can express the extremal function as

$$
f(x) = \sum_{j=k}^{\infty} \left(j+\frac{1}{2}\right) \left(P_j^{(-k)}(x)\right)^2.
$$

It can be shown that $A_k'(x) = 0$ at $x = 0$ and $A_k''(0) < 0$. Therefore, $A_k(x)$ attains its maximum at $x = 0$. However, since

$$
\begin{aligned}
P_j^{(-k)}(x) &= \frac{1}{2^j j!}\frac{\mathrm{d}^{j-k}}{\mathrm{d}x^{j-k}}(1-x^2)^j \\
&\overset{x=0}{=} \begin{cases} (-1)^{\frac{j+k}{2}} \dfrac{(j-k-1)!!}{(j+k)!!} & \text{for even } j-k \\ 0 & \text{for odd } j-k, \end{cases}
\end{aligned}
$$

we have with the change of index $j = 2m+k$

$$
\begin{aligned}
\max_{x\in[-1,1]} A_k(x) &= A_k(0) \\
&= \sum_{m=0}^{\infty} \left(\frac{(2m)!}{2^{2(m+k)} m!(m+k)!}\right)^2 \left(2m+k+\frac{1}{2}\right) \\
&= \frac{1}{2^{2k-1}((k-1)!)^2(2k-1)}.
\end{aligned}
$$

□

Kalyabin also proved in [20] that the function $A_k(x)$ can be expressed explicitly as

$$
A_k(x) = \frac{(1-x^2)^{k-\frac{1}{2}}}{2^k (k-1)! \sqrt{k-\frac{1}{2}}} \tag{4}
$$

and then again the sharp constant follows by taking

$$\max_{x\in[-1,1]} A_k(x) = A_k(0).$$

The result was shortly after independently rediscovered by Watanabe et al. [46]. They made a connection to a boundary value problem and then calculated the sharp constant using the Green function of the differential operator. Later, in 2020, Sheipak and Garmanova [39] found the explicit form for the extremal function.

The next full solution for finding the sharp constant concerned the case $q = p = 2$. The sharp constant was characterised first by Petrova [34] in 2017 and then again by Carneiro [12] in 2024. Both methods use the smallest positive solution of a certain explicit determinant equation.

Petrova uses a similar left inverse of the differential operator $f \mapsto f^{(k)}$ for functions $f \in W_0^{k,2}(-1,1)$ that Kalyabin was using in [20], namely

$$f(x) = \frac{1}{(k-1)!}\int_0^x (x-y)^{k-1} f^{(k)}(y)\,\mathrm{d}y.$$

Then she connects the problem of finding the sharp constant with finding the minimal positive eigenvalue of the boundary value problem

$$\begin{aligned}(-1)^k y^{(2k)}(x) &= \lambda_k y(x),\\ y^{(j)}(0) = y^{(j)}(1) &= 0, \quad j = 0, \ldots, k-1.\end{aligned}$$

Theorem 2 (Petrova [34]). *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the inequality*

$$\|f\|_{L^2(-1,1)} \leqslant c_{k,2,2}\,\|f\|_{W_0^{k,2}(-1,1)} \tag{5}$$

with the sharp constant $c_{k,2,2} = (\lambda_k)^{-k}$, where λ_k is the least positive root of the function

$$\det D_k(\lambda) = 0,$$

where $D_k(\lambda)$ is the $k\times k$-matrix with entries

$$D_{k_{jm}}(\lambda) = (\lambda z^m)^{\frac{2j-1}{2}} J_{\frac{2j-1}{2}}(\lambda z^m), \quad j, m = 0, \ldots, k-1.$$

Here $z = e^{i\frac{\pi}{k}}$ and J_ν is the Bessel function of the first kind.

As an example, for $k = 1$, λ_1 is the first positive root of $\cos(z) = 0$, while for $k = 2$, λ_2 is the first positive root of $\tan(z) + \tanh(z) = 0$, and for $k = 3$, λ_3 is the first positive

Table 2: The first few values of the sharp embedding constant $c_{k,2,2}$.

	$k=1$	$k=2$	$k=3$	$k=4$
$c_{k,2,2}$	$\frac{2}{\pi}$	0.1795461...	$\frac{1}{\pi^3}$	0.004322...

root of $\cos(z)\left(\tan(z) - \tan\left(ze^{i\frac{pi}{3}}\right) + \tan\left(ze^{i\frac{2pi}{3}}\right)\right) = 0$, and similar conditions can be obtained for all positive integers k from the determinant equation. As an example, the first few values of the sharp embedding constant $c_{k,2,2}$ are given in Table 2.

In this $p = q = 2$ context, we would like to mention the result by Boulton and Lang [9] from 2023.

Theorem 3 (Theorem 6.1 in Boulton and Lang [9]). *For $f \in W_D^{2,p}(-1,1) = \{f \in W^{2,p}(-1,1) : f(\pm 1) = 0\}$ with $1 < p < \infty$, we have the inequality*

$$\|f\|_{L^{p'}(-1,1)} \leqslant \tilde{c}_{2,p',p} \|f\|_{W_D^{2,p}(-1,1)} \tag{6}$$

with the sharp constant

$$\tilde{c}_{2,p',p} = 2^{2+\frac{1}{p'}-\frac{1}{p}} \left(\frac{H\left(\frac{1}{2}+\frac{1}{p'}\right)}{2H\left(\frac{1}{2}\right) H\left(\frac{1}{p'}\right)} \right)^2,$$

where $H(x) = \frac{\Gamma(x+1)}{x^x}$, and p, p' are the Hölder conjugate exponents with $\frac{1}{p} + \frac{1}{p'} = 1$.

The extremal functions are constant multiples of $f(x) = \sin_{2,p'}(x)$, where $\sin_{2,p'}(x)$ is a generalised trigonometric function of Lindqvist-Peetre (for the definition and properties on these functions, see 5.5, below).

Boulton and Lang use the boundary condition that only $f(\pm 1) = 0$ and not that both $f(\pm 1) = 0$ and $f'(\pm 1) = 0$ as in Problem 1. Interestingly, it makes a noticeable difference in the case $q = p = 2$. Namely, we can easily calculate the embedding constant in this setting as

$$\tilde{c}_{2,2,2} = \frac{4}{\pi^2}.$$

In contrast, with the boundary conditions of Problem 1, we had to numerically approximate to find the least positive solution to $\tan(z) + \tanh(z) = 0$ in order to calculate the embedding constant $c_{2,2,2}$ (see the numerically approximated value in Table 2)

The next full solution for finding the sharp constant concerned the case $q = p = \infty$. Kazimirov and Sheipak [22] found the sharp constant in 2024 using L^1-approximation theory.

Table 3: The first few values of the sharp embedding constant $c_{k,\infty,\infty}$.

	$k=1$	$k=2$	$k=3$	$k=4$
$c_{k,\infty,\infty}$	1	$\frac{1}{4}$	$\frac{2-\sqrt{2}}{12}$	$\frac{8-3\sqrt{5}}{192}$

Theorem 4 (Theorem 3 in Kazimirov and Sheipak [22]). *For all integers $k \geqslant 1$ and $f \in W_0^{k,\infty}(-1,1)$, we have the inequality*

$$\|f\|_{L^\infty(-1,1)} \leqslant c_{k,\infty,\infty} \|f\|_{W_0^{k,\infty}(-1,1)}$$

with the sharp constant

$$c_{k,\infty,\infty} = \frac{k+1}{\pi k! 2^{k-1}} \int_{-1}^{+1} \frac{(1-x^2)^k}{1+(-1)^k x^{2(k+1)}} \,\mathrm{d}x.$$

For the proof, Kazimirov and Sheipak make a connection to the problem of finding the best approximation in L^1 of certain splines by polynomials. As an example, the first few values of the sharp embedding constant $c_{k,\infty,\infty}$ are given in Table 3.

Kazimirov and Sheipak also calculated the integral in the constant in terms hypergeometric functions and found the asymptotic behaviour of the constant as $k \to \infty$ to be $\frac{1}{2}\sqrt{\frac{\pi}{k}}$.

In the case $q = \infty$ and for the full range of $p \in [1, \infty]$, we have higher order results only up to $k = 3$ (except for $p = 2$ and $p = \infty$ where we have full solutions). Oshime [33] proved the cases $k = 1$ and $k = 2$, and Watanabe et al. [45] the case $k = 3$.

Oshime and Watanabe et al., both use the construction of left inverses to the differential operator. Again, they use the same left inverse approach as Kalyabin and Petrova are using.

For these orders of derivative ($k = 1$, $k = 2$, $k = 3$) the results of Oshime and Watanabe et al. are in agreement with Kalyabin [20] when $p = 2$, and with Kazimirov, Sheipak [22] when $p = \infty$.

Theorem 5 (Theorem 10 in Oshime [33], Theorem 1.1 in Watanabe et al. [45]). *For $f \in W_0^{2,p}(-1,1)$ with $1 \leqslant p \leqslant \infty$, we have the sharp inequalities*

$$\begin{aligned}
\|f\|_{L^\infty(-1,1)} &\leqslant 2^{1-\frac{1}{p}} \frac{1}{2} \|f\|_{W_0^{1,p}(-1,1)}, \\
\|f\|_{L^\infty(-1,1)} &\leqslant 2^{2-\frac{1}{p}} \frac{1}{8} (p'+1)^{-\frac{1}{p'}} \|f\|_{W_0^{2,p}(-1,1)}, \\
\|f\|_{L^\infty(-1,1)} &\leqslant 2^{3-\frac{1}{p}} \frac{1}{16} \min_{\alpha\in(0,1)} \left(\int_0^1 x^{p'} |x-\alpha|^{p'} \,\mathrm{d}x \right)^{\frac{1}{p'}} \|f\|_{W_0^{3,p}(-1,1)}.
\end{aligned}$$

Oshime gave also the explicit extremal function in the case $k = 1$ as

$$f(x) = 1 - |x|, \quad (-1 \leqslant x \leqslant 1)$$

up to the constant multiplication.

3 Our contribution

The paper included in this thesis considers a Sobolev-type embedding in one dimension, from the Sobolev space $W_0^{k,2}(-1,1)$ into $L^1(-1,1)$, for all integer orders of derivatives $k \geqslant 1$. We have the following theorem from the paper in this thesis.

Theorem 6 (H., Nitzan, Olsen, Rydhe). *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the sharp inequality*

$$\|f\|_{L^1(-1,1)} \leqslant \frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \|f\|_{W_0^{k,2}(-1,1)} . \tag{7}$$

The extremal functions are given by the Landau kernels, $L_k(x) = (1-x^2)^k$.

The theorem tells us the largest size the function with compact support in $[-1,1]$ can have in L^1-norm when we know the the size of its k-th derivative in L^2-norm. We also have an interpretation, that when we fix a L^1-norm for functions that belong to the Sobolev space $W_0^{k,2}(-1,1)$, then the L^2-norm of the Fourier transform of their k-th derivative cannot be smaller than a sharp positive constant,

$$\|f\|_{L^1(-1,1)} \leqslant \frac{(2\pi)^k}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \left\| |\xi|^k \hat{f}(\xi) \right\|_{L^2(\mathbb{R})} . \tag{8}$$

For comparison, when we also control the size of the function by the size of the moment of the function, then we have the sharp inequality on the real line conjectured by Steinerberger [41], for $\alpha \geqslant 2$

$$\|f\|_{L^1(-1,1)}^{1+\frac{1}{\alpha}} \leqslant 2\pi(\alpha+1)^{\frac{1}{\alpha}} \||x|^\alpha f(x)\|_{L^1(-1,1)}^{\frac{1}{\alpha}} \left\| |\xi| \hat{f}(\xi) \right\|_{L^\infty(\mathbb{R})} , \tag{9}$$

with the characteristic function $f(x) = \chi_{[-1,1]}(x)$ as the extremal function.

In Theorem 7, we have equality with the Landau kernel L_k as the extremal function, so that we have the L^1-norm of the Landau kernel on the left hand side, and the L^2-norm of the Legendre polynomial P_k on the right hand side. Recall that these functions are related by the Rodrigues formula,

$$P_k(x) = \frac{1}{2^k k!} \frac{\mathrm{d}^k}{\mathrm{d}x^k} \left[(x^2-1)^k\right].$$

We would like to mention that the Landau kernel is also the extremal function in a Carlson-type inequality (for the general form and other examples, see 5.7, below) by Laeng and Morpurgo [26],

$$\|f\|_{L^1(-1,1)}^5 \leqslant \frac{125}{9} \left\|x^2 f(x)\right\|_{L^1(-1,1)} \|f'(x)\|_{L^2(-1,1)}^4 .$$

In 2014, Kalyabin [21] found an upper and lower bound for the embedding constant in the inequality (7)

$$\frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \leqslant c_k \leqslant \frac{\sqrt{\pi}}{2^k \Gamma(k+\frac{1}{2})\sqrt{k-\frac{1}{2}}}.$$

Kalyabin used the Landau kernel to calculate the lower bound for the embedding constant. For the upper bound he used the function $A_k(x)$ (for the formulation see (4)) that gave the sharp inequality in the $q=\infty$, $p=2$ case. But these bounds leave a gap for the constant in the $q=1, p=2$ case.

Theorem 6 closes this gap for $q=1, p=2$. For the proof of the theorem, we construct a class of explicit left inverses of the differential operator $f \mapsto f^{(k)}$ of the form

$$f(x) = \int_{-1}^{1} B_k(x,y) f^{(k)}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

By the Hölder inequality, we obtain from this the following integral representation that

$$\|f\|_{L^1(-1,1)} \leqslant \|f\|_{W_0^{k,p'}(-1,1)} \left\| \int_{-1}^{1} |B_k(x,y)|\,\mathrm{d}x \right\|_{L^p(-1,1;\mathrm{d}y)} .$$

Then the idea is to construct the function $B_k(x,y)$ in a such way that when we minimise the $L^p(-1,1;\mathrm{d}y)$-norm we are free to choose the zeros $y_1, y_2, ..., y_k$ of the polynomial $\int_{-1}^{1} B_k(x,y)\,\mathrm{d}x$ in y, while $\left|\int_{-1}^{1} B_k(x,y)\,\mathrm{d}x\right| = \int_{-1}^{1} |B_k(x,y)|\,\mathrm{d}x$. For this construction of $B_k(x,y)$ we use the Lagrange interpolation polynomial.

Thus, the problem reduces to the classical problem in approximation theory to determine polynomials of degree k with leading coefficient one that have the minimal L^p-norm on the interval $[-1,1]$, $1 \leqslant p \leqslant \infty$. The solution to this problem is well known in the cases $p=1,2,\infty$ (see, e.g., [36, Chapter II]).

- For $p = 2$, it is the Legendre polynomial and we get the result in Theorem 6.
- For $p = \infty$, it is the Chebyshev polynomial of first kind and we get Corollary 1 below.
- For $p = 1$, it is the Chebyshev polynomial of second kind and we get Corollary 2 below.

Less is known about the cases $1 < p < 2$ and $2 < p < \infty$, apart from the fact established by Kroó and Peherstorfer [24] that the minimal polynomials in these cases have zeros that interlace with those of the Chebyshev polynomials of the first and second kind. However, the known results of approximation theory for $p = \infty$ and $p = 1$ give us the two corollaries below.

Corollary 1. *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the inequality*

$$\|f\|_{L^1(-1,1)} \leqslant \frac{1}{k!2^{k-1}} \|f\|_{W_0^{k,1}(-1,1)} . \tag{10}$$

Proof. The approach we developed for the case $p = 2$ is easily adapted for $p = 1$. We just need to find the minimal monic polynomial in $L^\infty(-1,1)$ of degree k, as described above. We have

$$\min_{\substack{p \text{ monic} \\ \deg p = k}} \|p\|_{L^\infty(-1,1)} = \left\| \frac{1}{2^{k-1}} T_k \right\|_{L^\infty(-1,1)} = \frac{1}{2^{k-1}},$$

where T_k is the Chebyshev polynomial of first kind of degree k,

$$T_k(x) = \frac{1}{2}\left[(x + \sqrt{x^2-1})^k + (x - \sqrt{x^2-1})^k\right].$$

$\square$

In 2019, Guessab and Milovanović [18, Theorem 4] studied the $q = p = 1$ inequality. They developed similar theory with a left inverse to the differential operator, but they did not complete the calculations to find the sharp embedding constant.

Corollary 2. *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the inequality*

$$\|f\|_{L^1(-1,1)} \leqslant \frac{1}{k!2^{k-1}} \|f\|_{W_0^{k,\infty}(-1,1)} . \tag{11}$$

Proof. Similarly to the corollary above, we can also use our left inverse approach in the case $p = \infty$. Now we need to find the minimal monic polynomial in $L^1(-1,1)$ of degree k, as described above. We have

$$\min_{\substack{p \text{ monic} \\ \deg p = k}} \|p\|_{L^1(-1,1)} = \left\| \frac{1}{2^k} U_k \right\|_{L^1(-1,1)} = \frac{1}{2^{k-1}},$$

where U_k is the Chebyshev polynomial of second kind of degree k,

$$U_k(x) = \frac{(x+\sqrt{x^2-1})^{k+1} - (x-\sqrt{x^2-1})^{k+1}}{2\sqrt{x^2-1}}.$$

□

As already mentioned in the introduction, the progress and results from the continuous case might be useful to making progress in the discrete case. For example, in the paper included in this thesis, we show that the sharp embedding constant in (7) is equal to the minimal eigenvalue of the following boundary value problem,

$$\begin{cases} (-1)^k u^{(2k)}(x) = \mu \operatorname{sgn}(u(x)), & x \in [-1,1], \\ \displaystyle\int_{-1}^{1} |u(x)|\,\mathrm{d}x = 1, & \\ \qquad u^{(j)}(\pm 1) = 0, & j = 0, \ldots, k-1. \end{cases}$$

If in this setting we could show that u, that corresponds to the minimal eigenvalue, does not change its sign inside $(-1,1)$, then we know that in the discrete sampled form of the inequality, the extremal sequence does not change sign either. We think this would help to prove the discrete analogue. Our progress in the discrete case, although not yet applying the connection put forward here, is presented next in Chapter 4.

4 Related problem in the discrete setting

Recall that the Sobolev inequality in Problem 1 for the case $p = 2$ can be formulated through the Fourier transform as

$$\|f(x)\|_{L^q(-1,1)} \leqslant c_{k,q,p}(2\pi)^k \left\|\widehat{f^{(k)}}(\xi)\right\|_{L^2(\mathbb{R})}. \tag{12}$$

Given this, we are also inspired to study the related problem in the discrete case. The discrete analogue is presented in Problem 2, in the case $p = 2$. We can also consider the Problem 2 for all $1 \leqslant p \leqslant \infty$ for its own right.

Problem 2. *Given a sequence $a : \{0,1,2,\ldots,d\} \mapsto \mathbb{C}$ determine the smallest possible positive constant $C_{d,k,q,p}$ in the inequality*

$$\left\|a_j\right\|_{l^q} \leqslant C_{d,k,q,p} \left\|\widehat{\Delta^k a}(\xi)\right\|_{L^p(\mathbb{T})}$$

and identify the extremal sequence.

Since we have that

$$\widehat{\Delta^k a}(\xi) = (e^{i\xi} - 1)^k \hat{a}(\xi)$$

we can express the inequality also as

$$\left\|a_j\right\|_{l^q} \leqslant C_{d,k,q,p} \left\|(e^{i\xi} - 1)^k \hat{a}(\xi)\right\|_{L^p(\mathbb{T})}$$

and therefore as a polynomial inequality as well

$$\|P(z)\|_{l^q} \leqslant C_{d,k,q,p} \left\|(z-1)^k P(z)\right\|_{L^p(\mathbb{T})}.$$

The l^q-norm of a polynomial $P(z) = \sum_{j=0}^{d} a_j z^j \in \mathbb{C}[z]$ is defined as

$$\|P(z)\|_{l^q} = \left(\sum_{j=0}^{d} |a_j|^q\right)^{\frac{1}{q}}, \text{ for } 1 \leq q < \infty \text{ with } \|P(z)\|_{l^\infty} = \max_j\{|a_j|\}.$$

4.1 An example of implication

The continuous Problem 1 and the discrete Problem 2 are related in the case $p = 2$, in the sense that knowing the sharp constant in the discrete case allows us to calculate the sharp constant in the continuous case. We present the cases $k = 1$ and $k = 2$ from our ongoing work for $q = \infty$, $p = 2$. The result for $k = 1$ follows from a simple calculation presented below, but for $k = 2$ we only provide a conjecture. We also demonstrate how the lemma and conjecture give asymptotically the continuous results.

Lemma 1. *Let $a : \{0, 1, 2, ..., d\} \mapsto \mathbb{C}$ be a sequence with d even. Then we have the sharp inequality*

$$\left\|a_j\right\|_{l^\infty} \leqslant \frac{\sqrt{d+2}}{2} \left\|\widehat{\Delta a}(\xi)\right\|_{L^2(\mathbb{T})}. \tag{13}$$

The extremal sequence is a constant multiple of the triangle sequence

$$a = \left\{1, 2, ..., \frac{d}{2} - 1, \frac{d}{2}, \frac{d}{2} - 1, ..., 2, 1\right\}.$$

Proof. We have for every $j \in \{0, 1, 2, ..., d\}$,

$$|a_j| \leqslant |a_j - a_{j-1}| + |a_{j-1} - a_{j-2}| + ... + |a_1 - a_0| + |a_0|$$

and also

$$|a_j| \leqslant |a_j - a_{j+1}| + |a_{j+1} - a_{j+2}| + ... + |a_{d-1} - a_d| + |a_d|.$$

Adding these expressions, we get for every $j \in \{0, 1, 2, ..., d\}$,

$$|a_j| \leqslant \frac{1}{2}\left(|a_0| + |a_d| + \sum_{k=1}^{d} |a_k - a_{k-1}|\right).$$

Hence by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |a_j| &\leqslant \frac{\sqrt{d+2}}{2}\left(\sum_{j=0}^{d+1} |\Delta a_j|^2\right)^{\frac{1}{2}} \\ &= \frac{\sqrt{d+2}}{2}\left\|\widehat{\Delta a}(\xi)\right\|_{L^2(\mathbb{T})}. \end{aligned}$$

$\square$

For $k = 2$ we have the following conjecture.

Conjecture 1. *Let $a : \{0, 1, 2, ..., d\} \mapsto \mathbb{C}$ be a sequence with d even. Then we have the sharp inequality*

$$\left\|a_j\right\|_{l^\infty} \leqslant \frac{1}{8\sqrt{3}}\sqrt{(d+2)(d+3)(d+4) + 3\frac{(d+2)(d+4)}{d+3}}\left\|\widehat{\Delta^2 a}(\xi)\right\|_{L^2(\mathbb{T})}.$$

These two discrete versions of the $q = \infty, p = 2$ inequalities of the order 1 and 2 imply the corresponding continuous cases on the real line. Indeed, suppose that $f(x)$ has support on $[-1, 1]$. Then we can apply the inequality (13) with $d = 2N$ to get

$$\max_{j\in\{0,1,2,...,2N\}}\left|f\left(\frac{j}{N}-1\right)\right| \leqslant \frac{\sqrt{2N+2}}{2}\left(\sum_{j=0}^{2N+1}\left|\Delta f\left(\frac{j}{N}-1\right)\right|^2\right)^{\frac{1}{2}}.$$

Manipulating the right-most sum to make it a Riemann-sum, we arrive at

$$\max_{j\in\{0,1,2,...,2N\}}\left|f\left(\frac{j}{N}-1\right)\right| \leqslant \frac{1}{\sqrt{N}}\frac{\sqrt{2N+2}}{2}\left(\sum_{j=0}^{2N+1}\left|\frac{\Delta f\left(\frac{j}{N}-1\right)}{1/N}\right|^2\frac{1}{N}\right)^{\frac{1}{2}}.$$

Taking the limit $N \to \infty$, we arrive at the inequality

$$\max_{x\in[-1,1]}|f(x)| \leqslant \frac{1}{\sqrt{2}}\left(\int_{-1}^{+1}|f'(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Similarly with the case $k = 2$, we observe that

$$\sum_{j=0}^{2N+2} \left| \frac{\Delta^2 f\left(\frac{j}{N} - 1\right)}{1/N^2} \right|^2 \frac{1}{N} \overset{N\to\infty}{\longrightarrow} \int_{-1}^{+1} |f''(x)|^2 \, \mathrm{d}x,$$

whence

$$\max_{x\in[-1,1]} |f(x)| \leqslant \frac{1}{2\sqrt{6}} \left(\int_{-1}^{+1} |f''(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

These two last inequalities match exactly with the inequality (3) in Theorem 1, and with the inequalities given in Theorem 5 with $k = 1$ and $k = 2$.

4.2 Subplot with three equivalent problems

In 2021 Kravitz and Steinerberger [23] posed the problem of finding averaging function, $u : \{-n, ..., n\} \mapsto \mathbb{R}$ normalised to $\sum_{j=-n}^{n} u_j = 1$, such that when convolving it with functions $f \in l^2(\mathbb{Z})$ we minimise

$$\sup_{0\neq f\in l^2(\mathbb{Z})} \frac{\left\|\Delta^k (f * u)\right\|_{l^2(\mathbb{Z})}}{\|f\|_{l^2(\mathbb{Z})}}.$$

Kravitz and Steinerberger showed that the minimisation problem is equivalent with determining the smallest positive constant in the Problem 2 with $p = \infty$, in this form,

$$\sum_{j=-n}^{n} u_j \leqslant C_{n,k} \left\|\widehat{\Delta^k u}(\xi)\right\|_{L^\infty(\mathbb{T})}.$$

This follows by using the Plancherel formula

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |(\Delta^k (u * f))(k)|^2 &= \frac{1}{2\pi} \int_{\mathbb{T}} |e^{i\xi} - 1|^{2k} |\hat{u}(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \\
&\leqslant \left\|(e^{i\xi} - 1)^{2k} \hat{u}(\xi)^2\right\|_{L^\infty(\mathbb{T})} \cdot \frac{1}{2\pi} \int_{\mathbb{T}} |\hat{f}(\xi)|^2 d\xi \\
&= \left\|(e^{i\xi} - 1)^{2k} \hat{u}(\xi)^2\right\|_{L^\infty(\mathbb{T})} \cdot \sum_{k\in\mathbb{Z}} |f(k)|^2.
\end{aligned}$$

After taking a square root we get

$$\left\|\Delta^k (u * f)\right\|_{l^2(\mathbb{Z})} \leqslant \left\|(e^{i\xi} - 1)^k \hat{u}(\xi)\right\|_{L^\infty(\mathbb{T})} \cdot \|f\|_{l^2(\mathbb{Z})}.$$

By choosing f so that $\hat{f}(\xi)$ has L^2 mass concentrated at the ξ in which the function $|e^{i\xi} - 1|^{2k}|\hat{u}(\xi)|^2$ achieves its maximum, we can make the above inequality arbitrary close to an equality. Hence, we have two equivalent minimisation problems

$$\min_u \sup_{f\in l^2(\mathbb{Z})} \frac{\left\|\Delta^k(f*u)\right\|_{l^2(\mathbb{Z})}}{\|f\|_{l^2(\mathbb{Z})}} = \min_u \left\|(e^{i\xi}-1)^k\hat{u}(\xi)\right\|_{L^\infty(\mathbb{T})}.$$

Kravitz and Steinerberger also showed that there is a third way to formulate the problem by reducing the setting to the unit interval with substitution $x = \cos\xi$,

$$\left\|(e^{i\xi}-1)^k\hat{u}(\xi)\right\|_{L^\infty(\mathbb{T})} = \max_{x\in[-1,1]} 2^{\frac{k}{2}}|1-x|^{\frac{k}{2}}|p(x)|,$$

where $p(x)$ is a polynomial of degree n that satisfies $p(1) = 1$.

Hence, we have three equivalent minimisation problems

$$\min_u \sup_{f\in\ell^2(\mathbb{Z})} \frac{\left\|\Delta^k(u*f)\right\|_{l^2(\mathbb{Z})}}{\|f\|_{l^2(\mathbb{Z})}} = \min_u \left\|\widehat{\Delta^k u}(\xi)\right\|_{L^\infty(\mathbb{T})} = \min_p \max_{x\in[-1,1]} 2^{\frac{k}{2}}|1-x|^{\frac{k}{2}}|p(x)|.$$

For $k = 1$, Kravitz and Steinerberger [23] solved the problem in the third formulation and found that,

$$C_{n,1} = \frac{2n+1}{2}$$

with the constant sequence $u_j = \frac{1}{2n+1}$ as the extremal sequence.

For $k = 2$, they also solved the problem in the third formulation, but with the additional requirement that $\hat{u}(\xi) \geqslant 0$, and found that,

$$C_{n,2} = \frac{(n+1)^2}{4}$$

with the triangle sequence as the extremal sequence $u_j = \frac{n+1-|j|}{(n+1)^2}$.

In 2023, Richardson [35] removed the restriction that u has non-negative Fourier transform and proved the following theorem.

Theorem 7 (Richardson [35]). *Let $u : \{-n, ..., n\} \mapsto \mathbb{R}$ be a sequence with d even. Then we have the sharp inequality*

$$\left|\sum_{j=-n}^{n} u_j\right| \leqslant \frac{n+1}{4}\cot\frac{\pi}{4(n+1)}\left\|\widehat{\Delta^2 u}(\xi)\right\|_{L^\infty(\mathbb{T})}.$$

Richardson also gave the extremal sequence by defining u to be symmetric $u_j = u_{-j}$, then for $j \geqslant 0$ setting

$$u_j = \frac{1}{\pi}\int_{-1}^{1} S_n(x) T_j(x) \frac{dx}{\sqrt{1-x^2}}$$

where $T_j(x)$ is the j-th Chebyshev polynomial, and

$$S_{n-1}(x) = \frac{1}{x-1} \cdot \frac{2\sin\left(\frac{\pi}{2n}\right)}{n\left(1+\cos\left(\frac{\pi}{2n}\right)\right)} \cdot T_n\left(\frac{1+\cos\left(\frac{\pi}{2n}\right)}{2}(x+1)-1\right).$$

Richardson comments that the extremal sequence resembles sampling from a parabola, but the extremal sequence does not quite lie on any parabola. However, he shows also that choosing the discrete Epanechnikov kernel, $E_n : \{-n, ..., n\} \mapsto \mathbb{R}$, defined by

$$E_n(j) = n^2 - j^2 + 1$$

is asymptotically close to the true extremiser.

We suggest here that perhaps it is somewhat easier to calculate the extremal sequence as follows. Take the coefficients of the following polynomial

$$p_n(z) = \prod_{k=1}^{n} \left(z^2 - 2zR(k,n) + 1\right),$$

where

$$R(k,n) = \frac{2\cos\left(\frac{\pi(2k+1)}{2n+2}\right) - \cos\left(\frac{\pi}{2n+2}\right) + 1}{\cos\left(\frac{\pi}{2n+2}\right) + 1}.$$

Then we can calculate for the sharp constant as follows. First, we have

$$\begin{aligned}
\left|\sum_{j=-n}^{n} u_j\right| &= |p_n(1)| \\
&= \prod_{k=1}^{n} 2\left(1 - R(k,n)\right) \\
&= \left(\frac{2}{\cos\left(\frac{\pi}{2(n+1)}\right)+1}\right)^n \frac{n+1}{\sin\left(\frac{\pi}{2(n+1)}\right)}
\end{aligned}$$

and also

$$\left\|(z-1)^2 p_n(z)\right\|_{L^\infty} = 2\left(\cos\frac{\pi}{4(n+1)}\right)^{-2(n+1)},$$

whence

$$\frac{\left|\sum_{j=-n}^{n} u_j\right|}{\|(z-1)^2 p_n(z)\|_{L^\infty}} = \frac{n+1}{4}\cot\frac{\pi}{4(n+1)}.$$

In the next few steps, we prove an extended and improved version of Kravitz and Steinerberger, in the case of $k=1$. First we prove the discrete problem for $q=2, p=\infty$, then extend it for all $q\in[1,2]$, and so, in particular, we get the $q=1, p=\infty$ sharp inequality.

Theorem 8. *For a sequence $a:\{0,1,2,...,d\}\mapsto\mathbb{C}$ we have the sharp inequality*

$$\left\|a_j\right\|_{l^2} \leqslant \frac{\sqrt{d+1}}{2}\left\|\widehat{\Delta a}(\xi)\right\|_{L^\infty(\mathbb{T})}, \tag{14}$$

with equality if and only if a is a constant sequence.

Proof. Let the finite Fourier transform of the sequence be defined in this form

$$\hat{a}(k) = \frac{1}{\sqrt{d+1}}\sum_{j=0}^{d} a_j e^{2\pi i\frac{k+1/2}{d+1}j}, \quad k=0,1,...,d.$$

By the Plancherel formula, we have

$$\begin{aligned}
\sum_{j=0}^{d}|a_j|^2 &= \sum_{k=0}^{d}|\hat{a}(k)|^2\\
&= \frac{1}{d+1}\sum_{k=0}^{d}\left|\sum_{j=0}^{d} a_j e^{2\pi i\frac{k+1/2}{d+1}j}\right|^2\\
&= \frac{1}{d+1}\sum_{k=0}^{d}\left|\sum_{j=0}^{d} a_j e^{2\pi i\frac{k+1/2}{d+1}j}\right|^2\frac{\left|1-e^{2\pi i\frac{k+1/2}{d+1}}\right|^2}{\left|1-e^{2\pi i\frac{k+1/2}{d+1}}\right|^2}\\
&\leqslant \frac{1}{d+1}\max_{\xi\in\mathbb{T}}|(e^{i\xi}-1)\hat{a}(\xi)|^2\sum_{k=0}^{d}\frac{1}{\left|1-e^{2\pi i\frac{k+1/2}{d+1}}\right|^2}\\
&= \frac{1}{d+1}\max_{\xi\in\mathbb{T}}|(e^{i\xi}-1)\hat{a}(\xi)|^2\frac{(d+1)^2}{4}\\
&= \frac{d+1}{4}\max_{\xi\in\mathbb{T}}|(e^{i\xi}-1)\hat{a}(\xi)|^2.
\end{aligned}$$

□

Corollary 3. *Let $a : \mathbb{Z} \mapsto \mathbb{C}$ be a finitely supported sequence of length $d+1$. Then we have the sharp inequality for all $q \in [1,2]$*

$$\left\|a_j\right\|_{l^q} \leqslant \frac{(d+1)^{\frac{1}{q}}}{2} \left\|\widehat{\Delta a}(\xi)\right\|_{L^\infty(\mathbb{T})}, \tag{15}$$

with equality if and only if a is a constant sequence.

Proof. By the Hölder's inequality

$$\sum_{j=0}^{d} 1 \cdot |a_j|^q \leqslant \left(\sum_{j=0}^{d} 1^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \left(\sum_{j=0}^{d} |a_j|^{q\frac{2}{q}}\right)^{\frac{q}{2}}$$

and therefore

$$\begin{aligned}\left(\sum_{j=0}^{d} |a_j|^q\right)^{\frac{1}{q}} &\leqslant (d+1)^{\frac{1}{q}-\frac{1}{2}} \left(\sum_{j=0}^{d} |a_j|^2\right)^{\frac{1}{2}} \\ &= \frac{(d+1)^{\frac{1}{q}}}{2} \max_{\xi\in\mathbb{T}} |(e^{i\xi}-1)\hat{a}(\xi)|.\end{aligned}$$

$\square$

In particular, for the case $q = 1$, we have the sharp inequality

$$\left\|a_j\right\|_{l^1} \leqslant \frac{d+1}{2} \left\|\widehat{\Delta a}(\xi)\right\|_{L^\infty(\mathbb{T})}. \tag{16}$$

5 Endnotes

5.1

In a Lebesque space on $\mathbb{R}$ all that matters in calculating the norm of a function is the width of the domain and the height or amplitude of the function. For example, a component of a simple step function, a constant function $f(x) = A$, we have its L^p-norm, $\|f\|_{L^p(a,b)} = |A||b-a|^{\frac{1}{p}}$ that combines the width and height of the function. Note that in the limit $p \to \infty$ the width loses its significance.

On one hand, we have a nice property that all L^p spaces on a compact domain are nested in each other

$$L^\infty \subset ... \subset L^2 \subset ... \subset L^1,$$

but on the other hand we have to live with the fact that taking a derivative of function in a L^p-space might take the function out of all Lebesque spaces.

In a Sobolev space, the norm of a function captures not only the width and height of the function, but also the regularity of the function. Regularity of a function tells us how many times we can differentiate the function before it ceases to be a function.

In a contrast to the Lebesque spaces, taking a derivative of a function in a Sobolev space takes the function to another Sobolev space. For example, if $u \in W^{k,p}(a,b)$, then $u' \in W^{k-1,p}(a,b)$.

Similarly to the Lebesque spaces, the Sobolev spaces are nested in each other, so that $W^{k,p}$ lies automatically in every other Sobolev space $W^{m,r}$ with $m < k$ and $r > p$. This means we can give up regularity to gain integrability by moving from one Sobolev space to another. In particular, we can embed a Sobolev space into a Lebesque space as shown by the next Lemma.

Lemma 2. *For a function $f \in W^{k,p}(a,b)$, there exists a positive constant C such that*

$$\|f\|_{L^\infty(a,b)} \leqslant C \|f\|_{W^{k,p}(a,b)}.$$

Proof. For all $k \geqslant 1$ and $1 \leqslant p \leqslant \infty$ it suffices to show that

$$\|f\|_{L^\infty(a,b)} \leqslant C \|f\|_{W^{1,1}(a,b)}.$$

By the fundamental theorem of calculus, we have

$$|f(x) - f(a)| = \left| \int_a^x f'(t)\,\mathrm{d}t \right| \leqslant \|f'\|_{L^1(a,b)} = \|f\|_{W^{1,1}(a,b)}$$

for all x. Then, by the triangle inequality, we have

$$|f(x)| \leqslant |f(a)| + \|f\|_{W^{1,1}(a,b)}.$$

By using $x > b$ and that f has compact support, we also have that

$$|f(a)| \leqslant \|f\|_{W^{1,1}(a,b)}.$$

□

This means that for functions with compact support on the real line there exists an embedding constant from every Sobolev space to every Lebesque space. In the one-dimensional case, on the real line, there are no critical exponents in contrast to the case of the Euclidean space $\mathbb{R}^n$ with $n \geqslant 2$ (for the critical exponent, see 5.2, below, and for the unbounded functions in Sobolev spaces, see 5.3, below).

5.2

When we want to embed a Sobolev space $W^{k,p}$ into a Lebesque space L^q in the Euclidean space $\mathbb{R}^n$ with $n \geqslant 2$, then there is a limit for the exponent of the Lebesque space above which we can not go, in the case $k < \frac{n}{p}$. We have an embedding constant in the inequality

$$\|f\|_{L^q} \leqslant C \|f\|_{W^{k,p}}$$

only for

$$q \leqslant \frac{np}{n-kp}.$$

Usually the critical exponent is denoted as $q = p^* := \frac{np}{n-kp}$.

5.3

If we think about functions on a compact domain, then intuitively, it seems reasonable that if we have a control over the derivative then the function itself cannot build up a large size. But it turns out that the answer is more subtle than that, and it depends on how we measure the smoothness and the size of functions, and also how many dimensions the domain has.

There are unbounded functions in Sobolev spaces when the domain is two or more dimensional. For example, let $u : \{x \in \mathbb{R}^n : \|x\| \leqslant 1\} \mapsto \mathbb{R}$ be given by

$$u(x) = \begin{cases} \log\left(\log\left(1 + \dfrac{1}{|x|}\right)\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $u \in W^{1,n}(\{x \in \mathbb{R}^n : \|x\| \leqslant 1\})$ when $n \geqslant 2$, but $u \notin L^\infty(\{x \in \mathbb{R}^n : \|x\| \leqslant 1\})$. Thus, in general the functions in $W^{1,p}$, $1 \leqslant p \leqslant n$, $n \geqslant 2$ are not continuous. In contrast, every $W^{1,p}$ function with $p > n$ coincides with a continuous function almost everywhere.

5.4

Mitrinović et al. [31] suggest that the first proof of the inequality (the case $q = p = 2$ and $k = 1$) may be attributed to Scheeffer [37]. Kuznetsov and Nazarov [25] have also examined the historical development of the inequality, observing that while Scheeffer derived an identity from which the inequality follows, he appears not to have emphasized its significance or stated it explicitly. Hardy et al. [19], on the other hand, attribute the inequality to Wirtinger [47].

5.5

The generalised trigonometric function $\sin_{p,q}$ of Lindqvist-Peetre [28] is defined as the inverse of the strictly increasing function $F_{p,q} : [0,1] \mapsto [0, \pi_{p,q}/2]$ given by

$$F_{p,q}(x) = \int_0^x \frac{\mathrm{d}t}{(1-t^q)^{\frac{1}{p}}}, \quad x \in [0,1].$$

We have the connection to the usual sine function when $p = q = 2$ since $F_{2,2}(x) = \arcsin(x)$. We extend $\sin_{p,q}$ to $[0, \pi_{p,q}]$ by defining

$$\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q} - x), \text{ for } x \in [\pi_{p,q}/2, \pi_{p,q}]$$

and further extension to $[-\pi_{p,q}, \pi_{p,q}]$ is made by oddness, and finally to the whole of $\mathbb{R}$ by $2\pi_{p,q}$-periodicity.

The constant $\pi_{p,q}$ is given by

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p}\,\mathrm{d}t$$

with natural extensions

$$\pi_{p,q} = \begin{cases} \dfrac{2p}{p-1}, \text{ if } 1 \leqslant p \leqslant \infty, & q = 1, \\ 2, \text{ if } 1 \leqslant p \leqslant \infty, & q = \infty, \\ \infty, \text{ if } p = 1, & 1 \leqslant q < \infty, \\ 2, \text{ if } p = \infty, & 1 \leqslant q \leqslant \infty. \end{cases}$$

The function $\cos_{p,q}$ is defined to be the derivative of $\sin_{p,q}$, and it follows that for all $x \in \mathbb{R}$,

$$\left|\sin_{p,q}\right|^q + \left|\cos_{p,q}\right|^p = 1.$$

5.6

Bennewitz and Saitō study in their papers [3] and [4] the Sobolev embedding for the first derivative

$$\|f\|_{L^q(-1,1)} \leqslant c_{1,q,p} \|f\|_{W_0^{1,p}(-1,1)} \tag{17}$$

and rediscover correctly the sharp embedding constant $c_{1,q,p}$.

While calculating the sharpness of the constant they use the correct extremal function $\sin_{p,q}\left(\frac{\pi_{p,q}x}{2}\right)$ in both papers, but unfortunately they use $\cos_{p,q}\left(\frac{\pi_{p,q}x}{2}\right)$ for the explicit calculation of the extremal functions. Therefore the extremals given in Theorem 4.2 in [3] and in the claim on p. 246 in [4] are not correct. For example, in the case $q = 1$ and $1 < p \leqslant \infty$, the extremal function is

$$1 - (1-x)^{\frac{p}{p-1}}, \text{ not } (1-x)^{\frac{1}{p-1}}, \text{ etc.}$$

Also, the calculations of the norms $\|F\|_{q,I_1}$ and $\|F'\|_{p,I_1}$ on p. 257 in [4] contain typos.

$$\|F\|_{q,I_1} = \left(\frac{p'}{p'+q}\right)^{\frac{1}{q}}, \text{ not } \frac{p'}{p'+q}$$

$$\|F'\|_{p,I_1} = \left(\frac{q\pi_{p,q}}{2(p'+q)}\right)^{\frac{1}{p}} \text{ not } \frac{q\pi_{p,q}}{2p'+q}.$$

5.7

In 1934, Fritz Carlson [11] found a sharp inequality, that

$$\left(\int_0^\infty f(x)\,\mathrm{d}x\right)^4 \leqslant \pi^2 \int_0^\infty f^2(x)\,\mathrm{d}x \int_0^\infty x^2 f^2(x)\,\mathrm{d}x$$

holds for any measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$. Now, in general, by a Carlson-type inequality, we mean an inequality of the form

$$\|f\|_X \leqslant K \prod_{i=1}^m \|f\|_{A_i}^{\theta_i},$$

where X and A_i are normed vector spaces, and θ_i are such that

$$\sum_{i=1}^m \theta_i = 1.$$

The constant K is independent of f. Typically, we have $m = 2$, and the spaces involved are Lebesgue spaces, weighted Lebesgue spaces or Sobolev spaces, etc.

In 1938, Beurling [7] found a similar sharp inequality to the original Carlson's inequality,

$$\|f\|_{L^1(\mathbb{R})} \leqslant \sqrt{2\pi}\left(\|f\|_{L^2(\mathbb{R})} \|xf\|_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.$$

We mention some other interesting examples. In 1999, Laeng and Morpurgo [26] proved the sharp inequality for functions with compact support,

$$\|f\|_{L^2(\mathbb{R})} \leqslant \frac{2\pi}{\sqrt{\Lambda_0}} \|f\|_{L^1(\mathbb{R})}^{-\frac{1}{2}} \|f'\|_{L^2(\mathbb{R})} \left\|x^2 f\right\|_{L^1(\mathbb{R})}^{\frac{1}{2}},$$

where $\Lambda_0 = 0.428368....$ In 1984, Cowling and Price [15] proved the qualitative inequality for any $\alpha > 0$ and $\beta > 1/2$,

$$\|f\|_{L^2(\mathbb{R})}^{\alpha+\beta} \leqslant K \|x^\alpha f\|_{L^1(\mathbb{R})}^{\beta-\frac{1}{2}} \left\|\xi^\beta \hat{f}\right\|_{L^\infty(\mathbb{R})}^{\alpha+\frac{1}{2}},$$

and in 2020, Steinerberger [41] proved the qualitative inequality for any $\alpha > 0$ and $\beta > 1/2$,

$$\|f\|_{L^1(\mathbb{R})}^{\alpha+\beta} \leqslant K \|x^\alpha f\|_{L^1(\mathbb{R})}^{\beta} \left\|\xi^\beta \hat{f}\right\|_{L^\infty(\mathbb{R})}^{\alpha}.$$

5.8

Classical inequalities can be used to build new ones, but we might loose sharpness, as we show in the following example.

We have the classical inequality of Bernstein [5, 6] that for a polynomial $P(z) = \sum_{j=0}^{d} a_j z^j \in \mathbb{C}[z]$ of degree d, we have

$$\|P'(z)\|_{L^\infty} \leqslant d\,\|P(z)\|_{L^\infty}.$$

If we apply the inequality on the polynomial $(z-1)P(z)$, then we get

$$|P(1)| \leqslant \|P(z) + (z-1)P'(z)\|_{L^\infty} \leqslant (d+1)\,\|(z-1)P(z)\|_{L^\infty}.$$

But Kravitz and Steinerberger [23] showed that the sharp constant is $\frac{d+1}{2}$, instead. That is

$$\left|\sum_{j=0}^{d} a_j\right| = |P(1)| \leqslant \frac{d+1}{2}\,\|(z-1)P(z)\|_{L^\infty},$$

with the extremal polynomial $P(z) = \sum_{j=0}^{d} z^j$.

6 References

[1] Thierry Aubin. Problèmes isoperimetriques et espaces de Sobolev. *Journal of Differential Geometry*, 11:573–598, 1976.

[2] Paul R. Beesack. Hardy's inequality and its extensions. *Pacific Journal of Mathematics*, 11(1):39–61, 1961.

[3] Christer Bennewitz and Yoshimi Saitō. An embedding norm and the Lindqvist trigonometric functions. *Electron. J. Differ. Equ.*, 2002:6, 2002. Id/No 86.

[4] Christer Bennewitz and Yoshimi Saitō. Approximation numbers of Sobolev embedding operators on an interval. *Journal of the London Mathematical Society*, 70(1):244–260, 2004.

[5] Serge Bernstein. Sur les polynômes orthogonaux relatifs à un segment fini. *Journal de Mathématiques Pures et Appliquées*, 9:127–178, 1930.

[6] Serge Bernstein. Sur les polynômes orthogonaux relatifs à un segment fini (seconde partie). *Journal de Mathématiques Pures et Appliquées*, 10:219–286, 1931.

[7] Arne Beurling. Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle. In *Compte Rendu du Neuvième Congrès des Mathématiciens Scandinaves*, pages 345–366, Helsingfors, 1939. Mercator. Congrès tenu à Helsinki en 1938.

[8] Albrecht Böttcher and Harold Widom. From Toeplitz eigenvalues through Green's kernels to higher-order Wirtinger-Sobolev inequalities. In *The extended field of operator theory. Containing lectures of the 15th international workshop on operator theory and its applications, IWOTA 2004, Newcastle, UK, July 12–16, 2004*, pages 73–87. Basel: Birkhäuser, 2006.

[9] Lyonell Boulton and Jan Lang. The eigenvalues and eigenfunctions of the non-linear equation associated to second order Sobolev embeddings. *Nonlinear Analysis: Theory, Methods & Applications*, 236:113362, 2023.

[10] David W. Boyd. Best constants in a class of integral inequalities. *Pacific Journal of Mathematics*, 30(2):367–383, 1969.

[11] Fritz Carlson. Une inégalité. *Arkiv för Matematik, Astronomi och Fysik*, 25B(1):1–5, 1934.

[12] Emanuel Carneiro, Cristian González-Riquelme, Lucas Oliveira, Andrea Olivo, Sheldy Ombrosi, Antonio Pedro Ramos, and Mateus Sousa. Sharp embeddings between weighted Paley-Wiener spaces. *Nonlinear Analysis*, 233:113589, 2024.

[13] Pafnuty L. Chebyshev. Théorie des mécanismes connus sous le nom de parallélogrammes. *Mémoires de l'Académie Impériale des Sciences de St.-Pétersbourg*, 7:539–568, 1854.

[14] Dario Cordero-Erausquin, Bruno Nazaret, and Cédric Villani. A mass–transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities. *Advances in Mathematics*, 182(2):307–332, 2004.

[15] Michael Cowling and John F. Price. Bandwidth versus time concentration: The Heisenberg–Pauli–Weyl inequality. *SIAM Journal on Mathematical Analysis*, 15(1):151–165, 1984.

[16] Pavel Drábek and Raúl Manásevich. On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian. *Differential and Integral Equations*, 12(6):773–788, 1999.

[17] David E. Edmunds and Jan Lang. Behaviour of the approximation numbers of a Sobolev embedding in the one-dimensional case. *Journal of Functional Analysis*, 206(1):149–166, 2004.

[18] Allal Guessab and Gradimir V. Milovanović. Best constants for Poincaré–type inequalities in $W_1^n(0,1)$. In *Differential and Integral Inequalities*, volume 151 of *Springer Optimization and Its Applications*, pages 391–402. Springer, 2019.

[19] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1934.

[20] Gennadiy A. Kalyabin. Sharp estimates for derivatives of functions in the Sobolev classes $\mathring{W}_2^r(-1,1)$. *Proc. Steklov Inst. Math.*, 269:137–142, 2010.

[21] Gennadiy A. Kalyabin. On two-sided and asymptotic estimates for the norms of embedding operators of $\mathring{W}_2^n(-1,1)$ into $L_q(\mathrm{d}\mu)$. *Proc. Steklov Inst. Math.*, 284:161–167, 2014.

[22] Danil D. Kazimirov and Igor A. Sheipak. Exact estimates of functions in Sobolev spaces with uniform norm. *Dokl. Math.*, 109(2):107–111, 2024.

[23] Noah Kravitz and Stefan Steinerberger. The smoothest average: Dirichlet, Fejér and Chebyshev. *Bulletin of the London Mathematical Society*, 53(6):1801–1815, 2021.

[24] András Kroó and Franz Peherstorfer. On the zeros of polynomials of minimal L^p-norm. *Proceedings of the American Mathematical Society*, 101(4):651–657, 1987.

[25] Nikolay G. Kuznetsov and Alexander I. Nazarov. Sharp constants in the Poincaré, Steklov and related inequalities (a survey). *Mathematika*, 61(2):328–344, 2015.

[26] Enrico Laeng and Carlo Morpurgo. An uncertainty inequality involving L^1 norms. *Proceedings of the American Mathematical Society*, 127(12):3565–3572, 1999.

[27] Vitalii I. Levin. Notes on inequalities. II: On a class of integral inequalities. *Rec. Math. Moscou, n. Ser.*, 4:309–324 (in Russian), 1938.

[28] Peter Lindqvist and Jaak Peetre. p-arclength of the q-circle. *Math. Stud.*, 72(1-4):139–145, 2003.

[29] Andrei A. Markov. On a question by D. I. Mendeleev. *Zapiski Imperatorskoi Akademii Nauk St. Petersburg*, 62:1–24, 1890. Original in Russian; English translation available by Carl de Boor and Olga Holtz.

[30] Vladimir A. Markov. On functions of least deviation from zero in a given interval. *Zapiski Imperatorskoi Akademii Nauk St. Petersburg*, 67:1–76, 1892. Original in Russian; foundational work on extremal properties of polynomials.

[31] Dragoslav S. Mitrinović, Josip E. Pečarić, and Alan M. Fink. *Inequalities involving functions and their integrals and derivatives*, volume 53 of *Math. Appl., East Eur. Ser.* Dordrecht etc.: Kluwer Academic Publishers, 1991.

[32] Alexander I. Nazarov and Alexandra P. Shcheglova. Steklov-type 1d inequalities (a survey). *arXiv preprint arXiv:2101.10752*, May 2025.

[33] Yorimasa Oshime. On the best constant for L^p Sobolev inequalities. *Scientiae Mathematicae Japonicae*, 68(3):333–344, 2008.

[34] Yulia P. Petrova. Spectral asymptotics for problems with integral constraints. *Math. Notes*, 102(3):369–377, 2017.

[35] Sean Richardson. A sharp Fourier inequality and the Epanechnikov kernel. *arXiv preprint arXiv:2310.09713*, 2023.

[36] Theodore J. Rivlin. *The Chebyshev Polynomials*, volume 40 of *Pure and Applied Mathematics*. Wiley-Interscience, New York, 1974.

[37] Ludwig Scheeffer. Über die Bedeutung der Begriffe Maximum und Minimum in der Variationsrechnung. *Mathematische Annalen*, 26:197–208, 1886.

[38] Erhard Schmidt. Über die Ungleichung, welche die Integrale über eine Potenz einer Funktion und über eine andere Potenz ihrer Ableitung verbindet. *Mathematische Annalen*, 117(1):301–326, Dec 1940.

[39] Igor A. Sheipak and Tatyana A. Garmanova. An explicit form for extremal functions in the embedding constant problem for Sobolev spaces. *Trans. Moscow Math. Soc.*, 80:189–210, 2019.

[40] Sergei L. Sobolev. On a theorem of functional analysis. *American Mathematical Society Translations, Series 2*, 34:39–68, 1963. Originally published in Russian in 1938.

[41] Stefan Steinerberger. Fourier uncertainty principles, scale space theory and the smoothest average. *arXiv preprint arXiv:2005.01665*, 2020.

[42] Vladimir A. Steklov. The problem of cooling of an heterogeneous rigid rod. *Commun. Kharkov Math. Soc. Ser. 2*, (5):136–181 (in Russian), 1896.

[43] Volker Strassen. An invariance principle for the law of the iterated logarithm. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 3(3):211–226, 1964.

[44] Giorgio Talenti. Best constant in Sobolev inequality. *Annali di Matematica Pura ed Applicata*, 110(1):353–372, Dec 1976.

[45] Kohtaro Watanabe, Yoshinori Kametaka, Atsushi Nagai, Hiroyuki Yamagishi, and Kazuo Takemura. Symmetrization of functions and the best constant of 1-dim L^p Sobolev inequality. *J. Inequal. Appl.*, 2009:12, 2009. Id/No 874631.

[46] Kohtaro Watanabe, Yoshinori Kametaka, Hiroyuki Yamagishi, Atsushi Nagai, and Kazuo Takemura. The best constant of Sobolev inequality corresponding to clamped boundary value problem. *Boundary Value Problems*, 2011:Article ID 875057, 17 pages, 2011.

[47] Wilhelm Wirtinger. Zur formalen Theorie der Funktionen von mehreren reellen Veränderlichen. *Monatshefte für Mathematik und Physik*, 41:298–302, 1934.

Paper 1

Received: 15 November 2024 | Accepted: 15 January 2025

DOI: 10.1112/mtk.70012

RESEARCH ARTICLE

Mathematatika

A sharp higher order Sobolev embedding

Raul Hindov[1] | Shahaf Nitzan[2] | Jan-Fredrik Olsen[1] | Eskil Rydhe[1]

[1]Centre for Mathematical Sciences, Lund University, Lund, Sweden

[2]School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia, USA

Correspondence
Eskil Rydhe, Centre for Mathematical Sciences, Lund University, Box 118, SE-22100 Lund, Sweden.
Email: eskil.rydhe@math.lu.se

Funding information
NSF CAREER, Grant/Award Number: DMS 1847796; VR, Grant/Award Number: 2022-04307

Abstract

We obtain sharp embeddings from the Sobolev space $W_0^{k,2}(-1,1)$ into the space $L^1(-1,1)$ and determine the extremal functions. This improves on a previous estimate of the sharp constants of these embeddings due to Kalyabin.

MSC 2020
46E35 (primary)

1 | INTRODUCTION

In this paper, we give a proof of a Sobolev-type inequality with a sharp constant and an explicit extremal function. This is motivated by the more general problem of calculating sharp constants and identifying extremal functions for Sobolev embeddings $W_0^{k,p}(-1,1) \subset L^q(-1,1)$. That is, inequalities of the form

$$\left(\int_{-1}^{1} |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \leqslant c_{k,p,q}\left(\int_{-1}^{1} \left|f^{(k)}\right|^p \,\mathrm{d}x\right)^{\frac{1}{p}}$$

for functions $f : \mathbb{R} \to \mathbb{R}$ with support in $[-1,1]$ and that satisfy $f^{(k)} \in L^p(\mathbb{R})$. In particular, for integers $k \geqslant 1$, f satisfies the boundary conditions $f^{(j)}(\pm 1) = 0$ for all $0 \leqslant j < k$.

We shall consider the case $q = 1$, $p = 2$, and integers $k \geqslant 1$, for which the sharp constants and extremal functions do not seem to be known. See, for example, the surveys by Mitrinović et al. [5, Chapter II], Kuznetsov and Nazarov [3], and Nazarov and Shcheglova [6].

Our main result is as follows.

Theorem 1. *For all integers $k \geqslant 1$ and $f \in W_0^{k,2}(-1,1)$, we have the sharp inequality*

$$\int_{-1}^{1} |f(x)|\,\mathrm{d}x \leqslant \frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \left(\int_{-1}^{1} |f^{(k)}(x)|^2\,\mathrm{d}x \right)^{\frac{1}{2}}. \tag{1}$$

The extremal functions are given by the Landau kernels, $L_k(x) = (1-x^2)^k$.

This theorem improves a bound due to Kalyabin [2]. Indeed, for $p = 2$, $q \in (0,\infty)$ and $k \in \mathbb{Z}_+$, Kalyabin obtained that

$$\frac{\sqrt{k+\frac{1}{2}}}{2^k k!} (\mathcal{L}(kq))^{1/q} \leqslant c_{k,2,q} \leqslant \frac{\left(\mathcal{L}((k-\frac{1}{2})q)\right)^{1/q}}{2^k (k-1)! \sqrt{k-\frac{1}{2}}},$$

where $\mathcal{L}(s) = \int_{-1}^{1} (1-x^2)^s\,\mathrm{d}x$. A straightforward calculation reveals that the constant from Theorem 1 is identical to Kalyabin's lower estimate in the case $q = 1$, implying that it was indeed sharp.

It is well known that sharp constants of Sobolev embeddings can be connected to minimal eigenvalues of certain boundary value problems (see, e.g., [1]). In our case, the sharp constant in Theorem 1 is connected to the minimal eigenvalue of the boundary value problem

$$\begin{cases} (-1)^k u^{(2k)}(x) = \lambda \dfrac{u(x)}{|u(x)|} \displaystyle\int_{-1}^{1} |u(t)|\,\mathrm{d}t, & x \in [-1,1], \\[2ex] u^{(j)}(\pm 1) = 0, & j \in \{0,1,\dots,k-1\}. \end{cases}$$

Indeed, we have

$$\begin{aligned} \min_{u\neq 0} \frac{\left((-1)^k u^{(2k)}, u\right)}{\left(\frac{u}{|u|}\|u\|_{L^1(-1,1)}, u\right)} &= \min_{u\neq 0} \frac{\left(u^{(k)}, u^{(k)}\right)}{\|u\|^2_{L^1(-1,1)}} \\ &= \min_{u\neq 0} \frac{\left\|u^{(k)}\right\|^2_{L^2(-1,1)}}{\|u\|^2_{L^1(-1,1)}} = \left(\frac{1}{c_{k,2,1}}\right)^2, \end{aligned}$$

where $(\cdot,\cdot)$ is the inner product in $L^2(-1,1)$ and the first equality follows upon k times partially integrating and using the boundary conditions. In [1], a similar example is provided for the case $p = q = 2$.

2 | PROOF OF THEOREM 1

The main idea of the proof is to consider a class of explicit left inverses of the differential operator $f \mapsto f^{(k)}$ of the form

$$f(x) = \int_{-1}^{1} B_k(x,y) f^{(k)}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

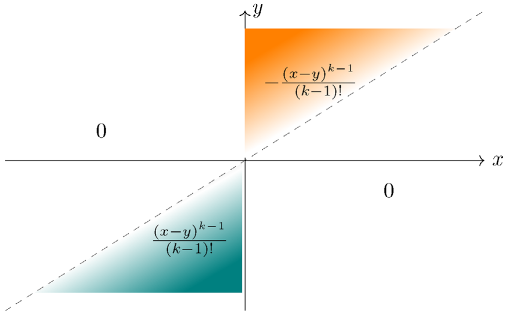

FIGURE 1 Illustration of the support and values of $b_k(x, y)$. The thin dashed line indicates $y = x$. $b_k(x, y)$ is equal to zero outside of the shaded area between the y-axis and the line $y = x$.

By the Cauchy–Schwarz inequality, we obtain from this integral representation that

$$\int_{-1}^{1} |f(x)|\,\mathrm{d}x \leqslant \left\| f^{(k)} \right\|_{L^2(-1,1)} \left\| \int_{-1}^{1} |B_k(x, y)|\,\mathrm{d}x \right\|_{L^2(-1,1;\mathrm{d}y)}.$$

The inequality of Theorem 1 is then obtained by connecting the expression involving the norm of $B_k(x, y)$ to a minimization problem having Legendre polynomials as minimizers. Finally, sharpness is obtained by noting that Landau kernels are extremal functions for the resulting inequality.

Before proceeding with a proof, we discuss some notation. For integers $k \geqslant 1$ and $p \in [1, \infty)$, we define the Sobolev space

$$W_0^{k,p}(-1, 1) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid \operatorname{supp} f \subseteq [-1, 1],\ f^{(k)} \in L^p(\mathbb{R}) \right\}.$$

For each integer $0 \leqslant j < k$, the derivative $f^{(j)}$ is absolutely continuous since $f^{(j-1)}(x) = \int_{-1}^{x} f^{(j)}(t)\,\mathrm{d}t$ and $L^p(-1, 1) \subset L^1(-1, 1)$. Hence, as mentioned in the introduction, since f has support in $[-1, 1]$, it follows that $f^{(j)}(\pm 1) = 0$ for each such j. The norm of f in $W_0^{k,p}(-1, 1)$ is defined by $\|f\|_{W_0^{k,p}(-1,1)} = \|f^{(k)}\|_{L^p(-1,1)}$. By $\mathbb{1}_X$, we denote the function that equals 1 if condition X is satisfied, and 0 otherwise. By δ_{nm}, we denote the standard Kronecker delta function.

Now we give the details of the proof.

2.1 | Construction of the explicit left inverses

For integers $k \geqslant 1$, define the functions $b_k : \mathbb{R}^2 \to \mathbb{R}$ by

$$b_k(x, y) = \frac{(x - y)^{k-1}}{(k - 1)!} \left[\mathbb{1}_{y<x<0} - \mathbb{1}_{y>x\geqslant 0} \right].$$

The support of b_k is indicated in Figure 1.

Proposition 1. *For all integers $k \geqslant 1$, the following holds.*

(i) $\int_{-\infty}^{\infty} b_k(x,y)\,\mathrm{d}x = \frac{(-y)^k}{k!}$ *for all* $y \in \mathbb{R}$.
(ii) *The following one-sided limits hold.*

$$\lim_{x\to 0^-} b_k(x,y) = \begin{cases} 0 & \text{for} \quad y > 0, \\ \frac{(-y)^{k-1}}{(k-1)!} & \text{for} \quad y \leqslant 0. \end{cases}$$

$$b_k(0,y) = \lim_{x\to 0^+} b_k(x,y) = \begin{cases} \frac{-(-y)^{k-1}}{(k-1)!} & \text{for} \quad y > 0, \\ 0 & \text{for} \quad y \leqslant 0. \end{cases}$$

(iii) *Let* $f \in W_0^{k,2}(-1,1)$ *and Q be a polynomial with* $\deg Q < k$. *Then, for all* $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} [b_k(x,y) - Q(y)] f^{(k)}(y)\,\mathrm{d}y = f(x).$$

Remark. By part (iii) of the proposition, the integral operator induced by b_k is a left-inverse to the differential operator $f \mapsto f^{(k)}$. Moreover, such a left-inverse is not unique; the polynomial Q, which may depend on x, provides a parametrization of a family of left-inverses. By part (ii), $b_k(x,y)$ is discontinuous across the y-axis whenever $y \neq 0$.

Proof. Property (i) follows by direct computation, and property (ii) follows immediately from the definition of $b_k(x,y)$.

To prove the reproducing property (iii), we use integration by parts repeatedly. Indeed, for $x < 0$, we have

$$\int_{\mathbb{R}} b_k(x,y) f^{(k)}(y)\,\mathrm{d}y = \sum_{j=1}^{k-1} \left[\frac{(x-y)^{k-j}}{(k-j)!} f^{(k-j)}(y) \right]_{-1}^{x} + \int_{-1}^{x} f'(y)\,\mathrm{d}y = f(x),$$

where all the terms in the sum vanish due to the boundary conditions on f. The case $x \geqslant 0$ is treated similarly. Finally, since Q is a polynomial of $\deg Q < k$, it follows that

$$\int_{\mathbb{R}} Q(y) f^{(k)}(y)\,\mathrm{d}y = 0. \qquad \square$$

Lemma 1. *Given any integer $k \geqslant 1$, let*

$$f_k(y) = y^{k-1}\mathbb{1}_{y>0}, \quad y \in \mathbb{R}.$$

Moreover, let Q be a polynomial of degree at most $k-1$, such that $f_k(y_n) = Q(y_n)$ for $k+1$ distinct real numbers $y_1 < \cdots < y_{k+1}$. Then, either $Q(y) \equiv y^{k-1}$ or $Q(y) \equiv 0$. In particular, this implies that either $y_1 \geqslant 0$ or $y_{k+1} \leqslant 0$.

Proof. If $k = 1$, then Q is a constant function. From this, it follows immediately that if $Q(y) - \mathbb{1}_{y>0}$ vanishes at two distinct points, then either $Q(y) \equiv 1$ or $Q(y) \equiv 0$. Moreover, these points have to be both nonpositive or both nonnegative. This argument is easily extended to $k = 2$ using the linearity of Q and the hypothesis that $Q(y) - y\mathbb{1}_{y>0}$ is to vanish at three distinct points.

For general $k > 2$, we proceed by induction. To this end, suppose that Q is a polynomial of degree at most $k-1$ and that $g(y) = Q(y) - y^{k-1}\mathbb{1}_{y>0}$ vanishes at $k+1$ distinct points. As g has a continuous derivative, the mean value theorem implies that $g'(y) = Q(y) - (k-1)y^{k-2}\mathbb{1}_{y>0}$ vanishes at at least k distinct points. It follows by the induction hypothesis, that these k points are either all nonpositive or all nonnegative. In particular, this means that $Q(y) - y^{k-1}\mathbb{1}_{y>0}$ vanishes at k points that are either all nonpositive or all nonnegative. Since Q is of degree $k-1$, the conclusion follows. □

Remark. For $k \geqslant 1$, let $f_k(y)$ be as above. Then, the lemma implies that if a polynomial Q of degree at most $k-1$ is equal to $f_k(x^* - y)$ at $k+1$ distinct points $y_1 < \cdots < y_{k+1}$, then either $y_1 \geqslant x^*$ or $y_{k+1} \leqslant x^*$.

We now fix k distinct real numbers $y_1 < y_2 < \cdots < y_k$, and consider the corresponding Lagrange interpolation basis $\{p_n\}_{n=1}^k$ given by

$$p_n(y) = \prod_{\substack{1\leqslant j\leqslant k;\\ j\neq n}} \frac{y - y_j}{y_n - y_j}.$$

These polynomials are of degree $k-1$ and satisfy $p_n(y_m) = \delta_{nm}$. Using these polynomials, we define functions $B_k : \mathbb{R}^2 \to \mathbb{R}$ by

$$B_k(x,y) = b_k(x,y) - \sum_{n=1}^{k} p_n(y) b_k(x, y_n).$$

Proposition 2. *For all integers $k \geqslant 1$, the following holds.*

(i) *For all $n \in \{1, 2, \ldots, k\}$ and $x \in \mathbb{R}$, we have $B_k(x, y_n) = 0$.*
(ii) *For all $f \in W_0^{k,2}(-1,1)$ and $x \in \mathbb{R}$, we have $\int_{-\infty}^{\infty} B_k(x,y) f^{(k)}(y)\,\mathrm{d}y = f(x)$.*
(iii) *For all $y \in \mathbb{R}$, the function $x \mapsto B_k(x,y)$ is continuous.*
(iv) *For all $y \in \mathbb{R}$,*

$$B_k(x,y) = \begin{cases} \dfrac{(x-y)^{k-1}}{(k-1)!}\mathbb{1}_{y<x}, & x \leqslant y_1 \\[2ex] -\dfrac{(x-y)^{k-1}}{(k-1)!}\mathbb{1}_{y>x}, & x \geqslant y_k \end{cases}.$$

(v) *For all $y \in \mathbb{R}$, either $B_k(x,y) \geqslant 0$ or $B_k(x,y) \leqslant 0$ for all $x \in \mathbb{R}$.*

Proof. Property (i) is immediate from $p_n(y_m) = \delta_{nm}$. Note that for fixed x, $B_k(x,y) = b_k(x,y) - Q(y)$ for a polynomial Q of degree at most $k-1$. Hence, (ii) follows from Proposition 1 (iii).

To establish (iii), we note that by Proposition 1 (ii), for any fixed $y \in \mathbb{R}$, $x \mapsto b_k(x,y)$ is continuous apart from a jump discontinuity at $x = 0$. Consequently, $x \mapsto B_k(x,y)$ can only fail to be continuous at $x = 0$. We therefore compare $\lim_{x\to 0^-} B_k(x,y)$ with $B_k(0,y) = \lim_{x\to 0^+} B_k(x,y)$.

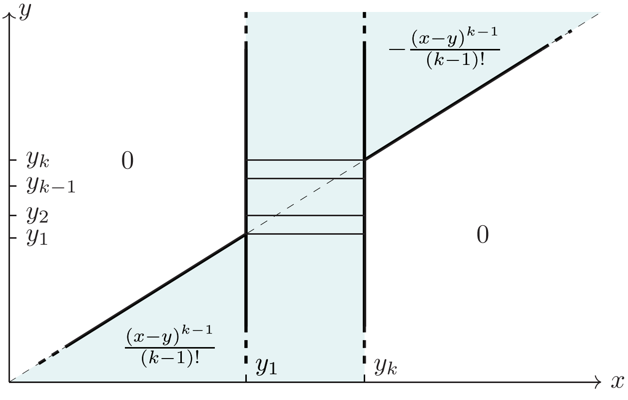

FIGURE 2 Indication of the values of B_k in various regions separated by thick lines, when $y_1 > 0$. The thin dashed line indicates $y = x$. The horizontal lines at $y \in \{y_1, \dots, y_k\}$ indicate the zero set of B_k for x in the convex hull of $\{y_1, y_k, y\}$.

Assuming first that $y > 0$, we obtain from Proposition 1(ii) that

$$\lim_{x\to 0^-} B_k(x,y) = -\sum_{n=1}^{k} p_n(y)\frac{(-y_n)^{k-1}}{(k-1)!}\mathbb{1}_{y_n<0},$$

$$\lim_{x\to 0^+} B_k(x,y) = -\frac{(-y)^{k-1}}{(k-1)!} + \sum_{n=1}^{k} p_n(y)\frac{(-y_n)^{k-1}}{(k-1)!}\mathbb{1}_{y_n>0}.$$

By similar calculations for $y \leqslant 0$, it therefore holds for all $y \in \mathbb{R}$ that

$$\lim_{x\to 0^-} B_k(x,y) - \lim_{x\to 0^+} B_k(x,y) = \frac{(-y)^{k-1}}{(k-1)!} - \sum_{n=1}^{k} p_n(y)\frac{(-y_n)^{k-1}}{(k-1)!}.$$

As a function of y, the above right-hand side vanishes at each of the points $y_1, y_2, \dots, y_k$, and since it is a polynomial of degree at most $k-1$, it must be identically equal to 0.

To establish (iv) for the case $x \leqslant y_1$, we note that the desired conclusion is immediate from the definition of B_k if $x < 0$. For the remaining case, we fix x so that $0 \leqslant x \leqslant y_1$ and consider the corresponding expression

$$B_k(x,y) = -\frac{(x-y)^{k-1}}{(k-1)!}\mathbb{1}_{y>x} + \sum_{n=1}^{k} p_n(y)\frac{(x-y_n)^{k-1}}{(k-1)!}.$$

As a function of y, the sum in the above right-hand side is a polynomial of degree at most $k-1$. Moreover, it is equal to the polynomial $\frac{(x-y)^{k-1}}{(k-1)!}$ at each y_n, and so, these polynomials must coincide. This establishes the case $x \leqslant y_1$. For the case $x \geqslant y_k$, the result follows by analogous arguments.

To establish (v), we first note that by (i) and (iii), the function $x \mapsto B_k(x,y)$ is continuous for all $y \in \mathbb{R}$ and identically equal to zero for each $y \in \{y_1, \dots, y_k\}$. Moreover, by (iv), $B_k(x,y)$ does not change sign on the sets $\{(x,y) : x \leqslant y_1\}$ and $\{(x,y) : x \geqslant y_k\}$, respectively (cf. Figure 2). Therefore, if $x \to B_k(x,y)$ changes sign for any fixed $y^* \notin \{y_1, \dots, y_k\}$, there must exist a point $x^* \in (y_1, y_k)$ so that $B_k(x^*, y^*) = 0$. In particular, this means that $y \mapsto B_k(x^*, y)$ vanishes at the $k+1$ distinct points $\{y_1, \dots, y_k, y^*\}$. To show that this leads to a contradiction, we consider the cases $x^* \geqslant 0$ and $x^* < 0$ separately. In the first case, it holds that

$$y \longmapsto B_k(x^*, y) = -\frac{(x^* - y)^{k-1}}{(k-1)!}\mathbb{1}_{y > x^*} - Q(y),$$

where Q is some polynomial of degree at most $k-1$. Hence, by Lemma 1 and the remark following it, this implies that the points $\{y_1, \dots, y_k, y^*\}$ are either all smaller than x^* or all greater than x^*. In particular, either $y_k \leqslant x^*$ or $y_1 \geqslant x^*$. This contradicts the assumption that $x^* \in (y_1, y_k)$. The case $x^* < 0$ may be treated similarly. □

2.2 | Connection to a minimization problem that leads to the Sobolev-type inequality (1)

Suppose that $f \in W_0^{k,2}(-1,1)$ and $B_k(x,y)$ as above. Then, we have

$$f(x) = \int_{-1}^{1} B_k(x,y) f^{(k)}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Applying the Cauchy–Schwarz inequality, we obtain that

$$\begin{aligned}\|f\|_{L^1(-1,1)} &\leqslant \left\|f^{(k)}\right\|_{L^2(-1,1)} \left\| \int_{-1}^{1} |B_k(x,y)|\,\mathrm{d}x \right\|_{L^2(-1,1;\mathrm{d}y)} \\ &= \left\|f^{(k)}\right\|_{L^2(-1,1)} \left\| \int_{-1}^{1} B_k(x,y)\,\mathrm{d}x \right\|_{L^2(-1,1;\mathrm{d}y)},\end{aligned}$$

where the final equality is immediate from Proposition 2(v).

It follows from Proposition 1(i) that

$$k! \int_{-1}^{1} B_k(x,y)\,\mathrm{d}x = (-y)^k - \sum_{n=1}^{k} (-y_n)^k p_n(y).$$

Since the polynomials p_n are of degree $k-1$ and satisfy $p_n(y_m) = \delta_{nm}$, for $n, m \in \{1, 2, \dots, k\}$, the right-hand side of the above expression is a monic polynomial of degree k with distinct zeros $y_1, y_2, \dots, y_k$. Since we can choose the zeros $y_1, y_2, \dots, y_k$ freely, any monic polynomial with distinct zeros can be obtained in this way. It follows that

$$\|f\|_{L^1(-1,1)} \leqslant \frac{\|f^{(k)}\|_{L^2(-1,1)}}{k!} \min_{\substack{p \text{ monic} \\ \deg p = k}} \|p\|_{L^2(-1,1)}.$$

It is well known that the unique minimizers are given by the monic Legendre polynomials

$$P_k(y) = \frac{k!}{(2k)!} \frac{\mathrm{d}^k}{\mathrm{d}y^k}\left[(y^2-1)^k\right].$$

Since we could not find a convenient reference explaining this fact, we point out that the minimizing property in the $L^2(-1,1)$ norm follows from the construction of the monic Legendre polynomials as the orthogonalization of powers $1, x, x^2, \ldots$ on the interval $-1 \leqslant x \leqslant 1$ with respect to the Lebesgue measure. From this, it follows that any monic polynomial p of order k can be written in the form $p(x) = P_k + c_{k-1}P_{k-1}(x) + \cdots + c_0$. Hence,

$$\|p\|^2_{L^2(-1,1)} = \|P_k\|^2_{L^2(-1,1)} + \sum_{n=1}^{k-1} c_n^2 \|P_n\|^2_{L^2(-1,1)}.$$

Clearly, this expression is minimized by choosing $c_0 = c_1 = \cdots = c_{k-1} = 0$. Since the $L^2(-1,1)$ norm of the monic Legendre polynomial (see, e.g., [4]) is

$$\|P_k\|^2_{L^2(-1,1)} = \left(\frac{k!}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \right)^2.$$

we conclude that,

$$\|f\|_{L^1(-1,1)} \leqslant \frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}} \left\| f^{(k)} \right\|_{L^2(-1,1)}.$$

2.3 | Sharpness of the inequality in Theorem 1

While the proof of the sharpness of the inequality in Theorem 1 is implicitly contained in [2], we provide a proof for the sake of completeness.

Lemma 2. *For all integers $k \geqslant 0$, we have equality in (1) if and only if f is equal to a constant multiple of the Landau kernel $L_k(x)$. In particular, the best constant of the equality is given by*

$$\frac{\|L_k\|_{L^1(-1,1)}}{\left\| L_k^{(k)} \right\|_{L^2(-1,1)}} = \frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}}.$$

Proof. Recall that $L_k(x) = (1-x^2)^k$. Writing $L_k(x) = (1-x)^k(1+x)^k$, we obtain by repeated integration by parts that

$$\int_{-1}^{1} |L_k(x)| \, \mathrm{d}x = \frac{(k!)^2}{(2k)!} \frac{2^{2k+1}}{2k+1},$$

and moreover, that

$$\int_{-1}^{1} \left(L_k^{(k)}(x) \right)^2 \mathrm{d}x = (k!)^2 \frac{2^{2k+1}}{2k+1}.$$

We conclude that

$$\frac{\|L_k\|_{L^1(-1,1)}}{\left\|L_k^{(k)}\right\|_{L^2(-1,1)}} = \frac{k!}{(2k)!}\sqrt{\frac{2^{2k+1}}{2k+1}} = \frac{1}{(2k-1)!!\sqrt{k+\frac{1}{2}}}. \qquad \square$$

ACKNOWLEDGMENTS

The second author is supported by NSF CAREER grant DMS 1847796. The fourth author is supported by VR grant 2022-04307.

JOURNAL INFORMATION

Mathematika is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society's research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Eskil Rydhe https://orcid.org/0000-0001-7275-028X

REFERENCES

1. A. Böttcher and H. Widom, *From Toeplitz eigenvalues through Green's kernels to higher-order Wirtinger-Sobolev inequalities*, The extended field of operator theory. Containing lectures of the 15th international workshop on operator theory and its applications, IWOTA 2004, Newcastle, Uk, July 12–16, 2004, Birkhäuser, Basel, 2006, pp. 73–87.
2. G. A. Kalyabin, *On two-sided and asymptotic estimates for the norms of embedding operators of* $\overset{\circ}{W}{}_2^n(-1,1)$ *into* $L_q(d\mu)$, Proc. Steklov Inst. Math. **284** (2014), 161–167.
3. N. G. Kuznetsov and A. I. Nazarov, *Sharp constants in the Poincaré, Steklov and related inequalities (a survey)*, Mathematika **61** (2015), no. 2, 328–344.
4. F. M. S. Lima, *Lecture notes on Legendre polynomials: their origin and main properties* (2022). https://arxiv.org/abs/2210.10942
5. D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Math. Appl., East Eur. Ser., vol. 53, Kluwer Academic Publishers, Dordrecht, 1991.
6. A. I. Nazarov and A. P. Shcheglova, *Steklov-type 1d inequalities (a survey)* (2021). https://arxiv.org/abs/2101.10752.