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LIMITING DISTRIBUTION OF THE MAXIMUM DRAWDOWN FOR BROWNIAN MOTION WITH POSITIVE DRIFT

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Abstract

The maximum drawdown of a stochastic process is the largest peak-to-trough decline observed over a given horizon $[0, T]$. Using arguments from extreme value theory, we derive the limiting distribution of the maximum drawdown for a Brownian motion with positive drift as $T \to \infty$. We show that, after suitable centering and scaling, the maximum drawdown converges in distribution to the Gumbel law.

Keywords: maximum drawdown; extreme value theory; asymptotic distribution

2020 Mathematics Subject Classification: Primary 60J65; 60G70

Secondary 60G50

1. Introduction

The purpose of this paper is to study the limiting distribution of the maximum drawdown for a Brownian motion with positive drift. We let $W(t)$, $0 \le t \le T$, denote a Brownian motion and set $X(t) = \mu t + \sigma W(t)$, where both $\mu, \sigma > 0$. We also introduce the variables $M(t) = \sup_{s \le t} X(s)$ and $D(t) = M(t) - X(t)$, and refer to them as the running maximum and the drawdown, respectively. The maximum drawdown is then

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defined by

$$\bar{D}(T) = \sup_{0 \le t \le T} D(t). \tag{1}$$

Hence, this random variable quantifies the largest observed drop from a peak value to a subsequent trough up to time T. In other words, it measures the worst-case loss from a high point, before a new high is achieved, of an asset or portfolio. In the financial literature, maximum drawdown has been used both as a basis for various risk measures, see e.g. [5, 12], and as a basis for various performance measures, see e.g. [1].

The maximum drawdown has been thoroughly studied in [9], based on the results in [3]. In [9], the authors express the complementary cumulative distribution function, denoted $G_{\bar{D}(T)}(h) = \mathbb{P}(\bar{D}(T) > h)$, as an infinite series

$$G_{\bar{D}(T)}(h) = 2\sigma^4 \sum_{n=1}^{\infty} \frac{\theta_n \sin\theta_n}{\sigma^4\theta_n^2 + \mu^2 h^2 - \mu h \sigma^2} e^{-\frac{\mu h}{\sigma^2}} \left(1 - e^{-\frac{\sigma^2\theta_n^2 T}{2h^2}} e^{-\frac{\mu^2 T}{2\sigma^2}}\right) + L(h), \tag{2}$$

where $\{\theta_n\}_{n\ge 1}$ are the positive solutions of the equation $\mu h \tan\theta_n = \sigma^2\theta_n$, and

$$L(h) = \begin{cases} 0 & ; \quad \mu h < \sigma^2, \\ \frac{3}{e}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2}}\right) & ; \quad \mu h = \sigma^2, \\ \frac{2\sigma^4 \eta \sinh\eta}{\sigma^4\eta^2 - \mu^2 h^2 + \mu h \sigma^2} e^{-\frac{\mu h}{\sigma^2}} \left(1 - e^{\frac{\sigma^2\eta^2 T}{2h^2}} e^{-\frac{\mu^2 T}{2\sigma^2}}\right) & ; \quad \mu h > \sigma^2, \end{cases} \tag{3}$$

with η being the unique positive solution to the equation $\mu h \tanh\eta = \sigma^2\eta$. The authors further claim that their expression is consistent with an asymptotic Gumbel distribution, but provide no further evidence for the statement. However, the authors derive the asymptotic behavior (when T is large) for the expected maximum drawdown

$$\mathbb{E}[\bar{D}(T)] \sim \frac{\sigma^2}{2\mu}\left(\ln\frac{\mu^2 T}{2\sigma^2} + C\right), \tag{4}$$

in terms of a numerically calculated constant $C = 4 \cdot 0.49088$.

This brings us to our contribution. We prove that, when properly centered and scaled, the cumulative distribution function converges to a standard Gumbel distribution as $T \to \infty$. Hence, we derive deterministic functions a_T and b_T (Theorem 1) such that

$$\lim_{T\to\infty} \mathbb{P}(\bar{D}(T) \le a_T + b_T x) = \exp\left(-\exp(-x)\right). \tag{5}$$

In doing so, we also identify the numerical constant

$$C = 2\ln 2 + \gamma, \tag{6}$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant.

Our approach is based on extreme value theory. However, while classical theory (that is, the weak convergence of the maximum of independent, identically distributed random variables) has been well understood for some time [4, 6], much less is known about the maximum of stochastic processes. Even weak convergence of the maximum of stationary stochastic processes poses significant technical difficulties; see [8] for extensive details on all aspects of extreme value theory. We overcome those difficulties by constructing an approximating sequence to which the classical theory can be applied. Having derived candidate deterministic functions (a_T, b_T), we then prove that the approximating sequence and the maximum drawdown are asymptotically equal in law. We also study the convergence rates of the approximating sequence and the maximum drawdown towards the asymptotics.

Throughout the paper, we use the notations Φ and ϕ for the cumulative and probability distribution functions, respectively, of a standard Gaussian random variable. We also introduce the risk-related constant

$$R = \frac{\sigma^2}{2\mu}, \tag{7}$$

as it frequently appears. In fact, as noted in Appendix A, R is equal to the expected long-term drawdown $\mathbb{E}[D(\infty)]$.

2. Approximating sequence

Let $D_1(\infty), D_2(\infty), \ldots, D_n(\infty)$ be independent copies of the random variable $D(\infty)$. Heuristically, we think of the sequence as observations at time points $T_1, T_2, \ldots, T_n$. Then, as shown in Appendix A, for a given point in time $D_i(\infty)$ has first-order stochastic dominance over $D(T_i)$, $1 \leq i \leq n$, and can therefore be seen as a worst-case outcome. We set

$$D^{max}(n\bar{\tau}) = \max_{1\leq i\leq n} D_i(\infty), \tag{8}$$

and choose the constant

$$\bar{\tau} = \frac{\sigma^2}{2\mu^2}, \tag{9}$$

to match the long-term average drawdown time, see Appendix B. This enables us to identify the time points $T_n = n\bar{\tau}$, such that $D^{max}(n\bar{\tau})$ can be regarded as a worst-case approximation of $\bar{D}(n\bar{\tau})$, when n is large.

We now apply classical extreme value theory to the random variable $D^{max}(n\bar{\tau})$. It follows that

$$\mathbb{P}(D^{max}(n\bar{\tau}) \leq x) = [\mathbb{P}(D(\infty) \leq x)]^n = \left(1 - e^{-x/R}\right)^n, \tag{10}$$

see Appendix A. Note that the largest order statistic of independent exponentially distributed random variables, like $D(\infty)$, is not exponentially distributed. Nevertheless, the expected value can be derived by integrating the complementary cumulative distribution function

$$\mathbb{E}[D^{max}(n\bar{\tau})] = \int_0^\infty \mathbb{P}\left(D^{max}(n\bar{\tau}) > x\right) dx = \int_0^\infty \left(1 - \left(1 - e^{-x/R}\right)^n\right) dx. \tag{11}$$

This integral can be evaluated by expanding the binomial $(1 - e^{-x/R})^n$. Alternatively, we can use the result that $D^{max}(n\bar{\tau})$ is identical in distribution to the weighted sum of n independent and exponentially distributed random variables, as shown below.

Lemma 1. *For any integer $n \geq 1$, we have*

$$\mathbb{E}\left[D^{max}(n\bar{\tau})\right] = RH_n,$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the n'th harmonic number.

Proof. It follows from [11] that $D^{max}(n\bar{\tau})/R$ has the same law as $\sum_{j=1}^n Z_j/j$, where $\{Z_j\}_{j\leq n}$ is a sequence of independent, identically distributed standard exponential random variables. Consequently, since $\mathbb{E}[Z_j] = 1$, for all j, the proof concludes. □

The harmonic number H_n further admits the expansion

$$H_n = \ln n + \gamma + \frac{1}{2n} - \varepsilon_n, \tag{12}$$

where $\gamma \approx 0.577216$ is the Euler–Mascheroni constant and $0 \leq \varepsilon_n \leq 1/(8n^2)$; see [2].

Proposition 1. *Let $\bar{\tau} = \frac{\sigma^2}{2\mu^2}$ and set $T_n = n\bar{\tau}$, then*

$$\lim_{T_n\to\infty} \mathbb{P}\left(D^{max}(T_n) \leq R\ln\left(T_n/\bar{\tau}\right) + Rx\right) = \exp(-\exp(-x)),$$

with

$$\lim_{T_n\to\infty} \left(\mathbb{E}\left[D^{max}(T_n)\right] - R\ln\left(T_n/\bar{\tau}\right)\right) = R\gamma.$$

Proof. We set $F_n(x) = \mathbb{P}\left(D^{max}(T_n) \le R\ln(T_n/\bar{\tau}) + Rx\right)$ and use (10) to obtain

$$F_n(x) = \left(1 - \exp\left(-\frac{R\ln n + Rx}{R}\right)\right)^n = \left(1 - \frac{1}{n}\exp(-x)\right)^n.$$

By sending T_n to infinity, via n, the first part of the proof is concluded, while the second part is imminent from Lemma 1 and (12). □

Remark 1. The preceding arguments were based on knowledge of the cumulative distribution function (10) at discrete points in time $T_n = n\bar{\tau}$, where n is a positive integer. We now postulate that for any time T there exists a real number $n = T/\bar{\tau}$ such that (10) holds. It follows that Lemma 1 extends to $\mathbb{E}\left[D^{max}(T)\right] = R(\psi(T/\bar{\tau}+1)+\gamma)$, where ψ is the digamma function defined as the logarithmic derivative of the gamma function $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$. Since asymptotically $\psi(z) \sim \ln z - \frac{1}{2z}$, we therefore conclude that Proposition 1 is valid for any time $T > 0$, with a corresponding positive real number $n = T/\bar{\tau}$. In particular, $\mathbb{E}\left[D^{max}(T)\right] \sim R\left(\ln(T/\bar{\tau}) + \gamma\right)$, for T large enough, which we interestingly compare with (4).

3. Main result

In this section, we show that the maximum drawdown $\bar{D}(\infty)$ has the same law as $D^{max}(\infty)$. Hence, from (2) it follows that we want to study the term

$$1 - \lim_{T\to\infty} G_{\bar{D}(T)}\left(R\ln(T/\bar{\tau}) + Rx\right), \quad \bar{\tau} = \frac{\sigma^2}{2\mu^2}.$$

Since the argument tends to infinity, as $T \to \infty$, we start by considering $G_{\bar{D}(T)}(h)$ for large values of h. In this case, the sequence $\{\theta_n\}_{n\ge1}$, associated with the positive solutions to the equation $\tan\theta_n/\theta_n = 0$, is given by $\theta_n = n\pi$. Consequently, when h is large, $G_{\bar{D}(T)}(h) \sim L(h)$ and it is clear which branch of the function L, in (3), to use.

Lemma 2. *Let $h > \sigma^2/\mu$ and define η as the unique positive solution to the equation $\mu h \tanh\eta = \sigma^2\eta$. Then*

$$\lim_{h\to\infty}\left(\eta - \frac{\mu}{\sigma^2}h\right) = 0, \quad \lim_{h\to\infty}\frac{\eta}{h} = \frac{\mu}{\sigma^2}.$$

Proof. We note that when h approaches infinity, so does η. Next, we find that

$$\eta - \frac{\mu}{\sigma^2}h = \eta\left(1 - \frac{1}{\tanh\eta}\right) = -\frac{2\eta}{e^{2\eta}-1},$$

from which the proof concludes. □

Using the results above, it follows that

$$2\sinh\eta e^{-\frac{\mu h}{\sigma^2}} = \left(e^{\eta} - e^{-\eta}\right) e^{-\frac{\mu h}{\sigma^2}} \to 1, \quad \text{as } h \to \infty, \tag{13}$$

$$\frac{\sigma^4\eta}{\sigma^4\eta^2 - \mu^2 h^2 + \mu h\sigma^2} \to 1, \quad \text{as } h \to \infty, \tag{14}$$

which leaves us with

$$G_{\bar{D}(T)}(h) \sim 1 - e^{\frac{\sigma^2\eta^2 T}{2h^2}} e^{-\frac{\mu^2 T}{2\sigma^2}}, \tag{15}$$

for h sufficiently large. The following is the main result of the paper:

Theorem 1. *Let $\bar{\tau} = \frac{\sigma^2}{2\mu^2}$, then*

$$\lim_{T\to\infty} \mathbb{P}\left(\bar{D}(T) \le R\ln\left(T/\bar{\tau}\right) + Rx\right) = \exp(-\exp(-x)),$$

with

$$\lim_{T\to\infty} \left(\mathbb{E}\left[\bar{D}(T)\right] - R\ln\left(T/\bar{\tau}\right)\right) = R\gamma.$$

Proof. Let $h_T(x) = R\ln(T/\bar{\tau}) + Rx$ and define $\eta_T(x)$ as the solution to the equation

$$\frac{\tanh\eta_T(x)}{\eta_T(x)} = \frac{2R}{h_T(x)}, \quad x \in \mathbb{R}.$$

Clearly, both $h_T(x)$ and $\eta_T(x)$ tend to infinity as T goes to infinity. According to (15), the first part of the proof follows once we show that

$$\lim_{T\to\infty} \left(\frac{\mu^2}{\sigma^4} - \frac{\eta_T^2(x)}{h_T^2(x)}\right) \frac{\sigma^2 T}{2} = \exp(-x), \quad x \in \mathbb{R}.$$

In order to tackle this problem, we start with the observation that

$$\lim_{T\to\infty} (1 - \tanh\eta_T(x)) \ln T = \lim_{T\to\infty} 2R\left(\eta_T(x) - \frac{\mu}{\sigma^2} h_T(x)\right) \frac{\ln T}{h_T(x)} = 0,$$

The result follows from Lemma 2 and the fact that $\ln T/h_T(x) \to 1/R$, as $T \to \infty$. Next, note that

$$\frac{\mu^2}{\sigma^4} - \frac{\eta_T^2(x)}{h_T^2(x)} = \frac{\mu^2}{\sigma^4}\left(1 - \tanh^2\eta_T(x)\right) = \frac{\mu^2}{\sigma^4} \frac{4}{e^{2\eta_T(x)} + e^{-2\eta_T(x)} + 2},$$

where

$$e^{2\eta_T(x)} = e^{\tanh\eta_T(x) h_T(x)/R} = \left(\frac{T}{\bar{\tau}} e^x\right)^{\tanh\eta_T(x)}.$$

Therefore, for T large, we find that

$$\begin{aligned}\left(\frac{\mu^2}{\sigma^4}-\frac{\eta_T^2(x)}{h_T^2(x)}\right)\frac{\sigma^2 T}{2} &= \frac{1}{\frac{\bar\tau}{T}\left(\frac{T}{\bar\tau}e^x\right)^{\tanh\eta_T(x)}+\frac{\bar\tau}{T}e^{-2\eta_T(x)}+\frac{2\bar\tau}{T}}, \\ &\sim \left(\frac{T}{\bar\tau}\right)^{1-\tanh\eta_T(x)} e^{-x\tanh\eta_T(x)}, \\ &\to e^{-x},\end{aligned}$$

since $(1-\tanh\eta_T(x))\ln T \to 0$, as $T\to\infty$. Because the random variables $\bar D(\infty)$ and $D^{max}(\infty)$ are shown to be equal in law, the second part follows directly from Proposition 1. □

Summing up, we have proved the weak convergence of

$$\frac{\bar D(T)-R\ln(T/\bar\tau)}{R}, \quad \bar\tau=\frac{\sigma^2}{2\mu^2}, \quad R=\frac{\sigma^2}{2\mu},$$

to a standard Gumbel random variable as $T\to\infty$. It is now straightforward to identify the constant C in (4), which appears in [9], with $2\ln 2+\gamma$.

4. Rate of convergence

From a practical point of view, it is of interest to know the convergence rate of the normalized stochastic processes

$$\bar Z(n\bar\tau)=\frac{\bar D(n\bar\tau)-R\ln n}{R}, \quad Z^{max}(n\bar\tau)=\frac{D^{max}(n\bar\tau)-R\ln n}{R}.$$

Since both terms converge weakly to a standard Gumbel random variable, as $n\to\infty$, we know that

$$\lim_{n\to\infty}\mathbb{E}[\bar Z(n\bar\tau)]=\lim_{n\to\infty}\mathbb{E}[Z^{max}(n\bar\tau)]=\gamma, \tag{16}$$

$$\lim_{n\to\infty}\mathbb{V}[\bar Z(n\bar\tau)]=\lim_{n\to\infty}\mathbb{V}[Z^{max}(n\bar\tau)]=\tfrac{\pi^2}{6}. \tag{17}$$

But which process converges faster, and what does infinity mean in real life; is it tens, hundreds, thousands, or maybe millions of years? We are also interested in knowing the convergence rate for the quantiles

$$\lim_{n\to\infty}F^{-1}_{\bar Z(n\bar\tau)}(p)=\lim_{n\to\infty}F^{-1}_{Z^{max}(n\bar\tau)}(p)=-\ln(-\ln(p)), \quad p\in[0,1], \tag{18}$$

since this quantity is important for financial risk management via the concept of value-at risk; see [5, 12] for additional details on risk measures based on the maximum drawdown. Although the convergence rate associated with higher moments can also be derived, we have decided to focus only on the variables above.

Proposition 2. *Let $\bar{\tau} = \frac{\sigma^2}{2\mu^2}$, then*

$$\begin{aligned}
\mathbb{E}[Z^{max}(n\bar{\tau})] &= \gamma + \psi(n+1) - \ln n, \\
\mathbb{V}[Z^{max}(n\bar{\tau})] &= \tfrac{\pi^2}{6} - \psi^{(1)}(n+1), \\
F^{-1}_{Z^{max}(n\bar{\tau})}(p) &= -\ln\left(n\left(1 - p^{1/n}\right)\right), \quad p \in [0,1],
\end{aligned}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is the digamma function and $\psi^{(1)}(z) = \frac{d}{dz}\psi(z)$ the polygamma function of order one.

Proof. For integer values of n, we make use of [11] as in the proof of Lemma 1. That is, since $D^{max}(n\bar{\tau})/R$ has the same law as $\sum_{j=1}^n Z_j/j$, where $\{Z_j\}_{j\leq n}$ is a sequence of independent and identically distributed standard exponential random variables, we obtain

$$\frac{1}{R^2}\mathbb{E}\left[D^{max}(n\bar{\tau})^2\right] = \sum_{j=1}^n\sum_{k=1}^n \frac{1}{jk}\mathbb{E}\left[Z_j Z_k\right] = \sum_{k=1}^n \frac{1}{k^2}\mathbb{E}\left[Z_k^2\right] + \sum_{\substack{j,k=1\\ j\neq k}}^n \frac{1}{jk}\mathbb{E}\left[Z_j\right]\mathbb{E}\left[Z_k\right].$$

Because $\mathbb{E}\left[Z_k\right] = 1$ and $\mathbb{E}\left[Z_k^2\right] = 2$, for all k, and $\mathbb{E}\left[D^{max}(n\bar{\tau})/R\right] = H_n$, we get

$$\frac{1}{R^2}\mathbb{V}\left[D^{max}(n\bar{\tau})\right] = 2\sum_{k=1}^n \frac{1}{k^2} + \sum_{\substack{j,k=1\\ j\neq k}}^n \frac{1}{jk} - \sum_{j,k=1}^n \frac{1}{jk} = \sum_{k=1}^n \frac{1}{k^2}.$$

Similar to Remark 1, we then extend the results to real values of n. □

In order to compare Z^{max} with the normalized maximum drawdown $\bar{Z}$, we have chosen, for efficiency reasons, to use a Monte Carlo simulation rather than to work with the infinite series expansion in (2). The parameters (μ, σ) used in the simulation are uniquely set so that $R = \bar{\tau} = 1$; that is, $\mu = 1$ and $\sigma = \sqrt{2}$. For precision, we use 1 million paths with $10,000$ points per year, over an interval spanning 100 years. Other combinations of (μ, σ) are subsequently obtained through appropriate scaling.

Since we are initially interested in the convergence rates of the mean and the variance of the normalized processes $\bar{Z}$ and Z^{max}, we are looking for the exponents (α_1, α_2) in

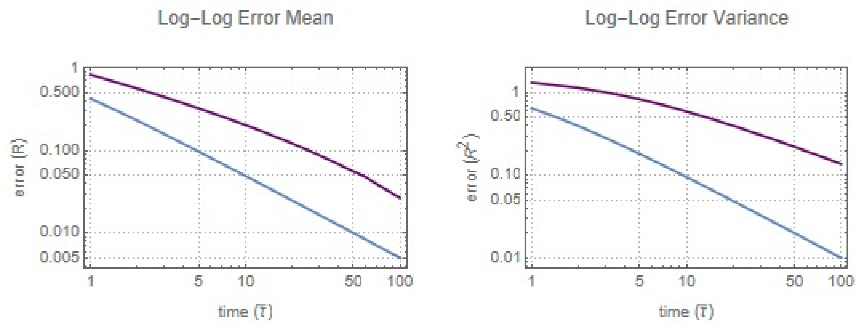

FIGURE 1: This figure shows the magnitude of the differences between the mean and the variance against their Gumbel limits. The result based on $\bar{Z}$ are plotted in magenta, while those based on Z^{max} are plotted in blue. Note that time is measured in units of $\bar{\tau}$ and error in units of R (R^2) for the mean (variance).

the expressions

$$|\mathbb{E}\left[\bar{Z}(n\bar{\tau})\right] - \gamma| \leq C_1 n^{\alpha_1}, \quad |\mathbb{V}\left[\bar{Z}(n\bar{\tau})\right] - \tfrac{\pi^2}{6}| \leq C_2 n^{\alpha_2}, \tag{19}$$

and similarly for Z^{max}. In Fig. 1, we plot the results in a log-log diagram, where we highlight that $(R, \bar{\tau})$ are the fundamental parameters, rather than (μ, σ). As anticipated from proposition 2, we find that $\alpha_1 = \alpha_2 = -1$ for Z^{max}. However, for $\bar{Z}$ the convergence is slower, with α_1 approaching -1 and $\alpha_2 \approx -0.7$. We notice that $\bar{Z}$ has lower variance than Z^{max}, which is further confirmed by plotting the quantiles, see Fig. 2. However, this time we consider

$$F^{-1}_{\bar{D}(n\bar{\tau})}(p) = RF^{-1}_{\bar{Z}(n\bar{\tau})}(p) + R\ln n, \quad F^{-1}_{D^{max}(n\bar{\tau})}(p) = RF^{-1}_{Z^{max}(n\bar{\tau})}(p) + R\ln n, \tag{20}$$

since these are the variables that ultimately matter. From Fig. 2, we see that the left-side tails agree less well than the right-side tails, especially for short time horizons.

4.1. Application to financial risk management

We conclude with a short discussion on the implications for risk management. First, as a reasonable approximation, we let the process X represent the logarithmic return of an asset with price process S, so that

$$\ln\left(S(t)/S(0)\right) = X(t). \tag{21}$$

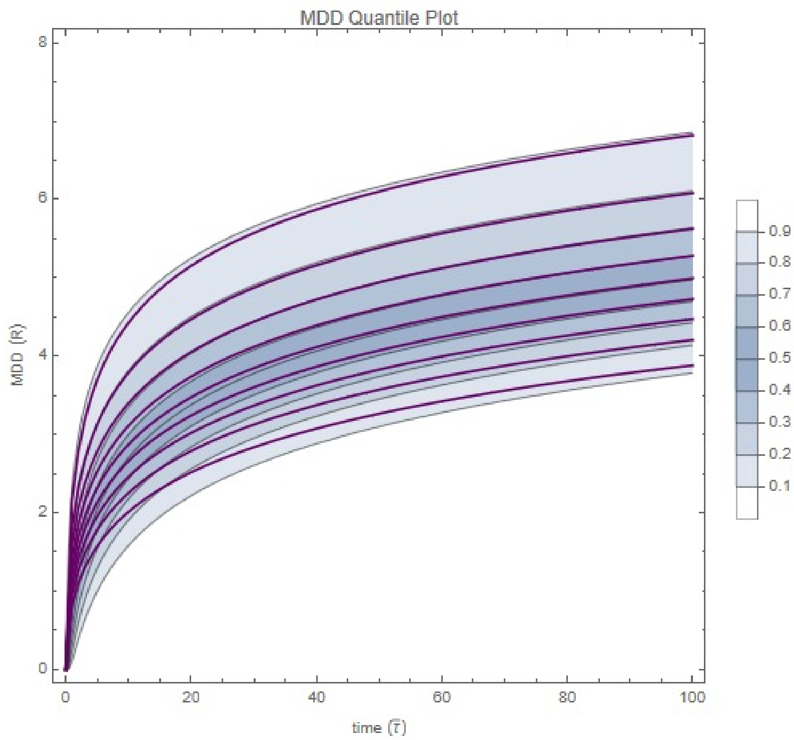

FIGURE 2: This figure shows the inverse cumulative distribution function of $\bar{D}$ (magenta) and D^{max} (blue) for various confidence levels p as a function of time. Note that time is measured in units of $\bar{\tau}$ and the output in units of R.

It then follows that the maximum relative drawdown (for S) equals

$$\bar{\Psi}(T) \triangleq \sup_{t\leq T} \frac{\sup_{s\leq t} S(s) - S(t)}{\sup_{s\leq t} S(s)} = 1 - \exp\left(-\bar{D}(T)\right). \tag{22}$$

Similar to the concept of Value-at-Risk, we now look at the risk variable

$$Q_p^{\bar{\Psi}}(T) \triangleq \inf\{y \in [0,1] : \mathbb{P}\left(\bar{\Psi}(T) > y\right) \leq 1-p\}, \quad p \in [0,1], \tag{23}$$

for some confidence level p; typically around 0.90 or higher. It follows that

$$Q_p^{\bar{\Psi}}(T) = 1 - \exp\left(-F^{-1}_{\bar{D}(T)}(p)\right), \quad \lim_{T\to\infty} Q_p^{\bar{\Psi}}(T) = 1, \tag{24}$$

where we have used (18) and (20) when calculating the limit.

Example 1. Let $p = 0.9$ and consider an asset that is very similar to S&P500 with $\mu = 0.1$ and $\sigma = 0.2$. This gives us $R = 0.2$ and $\bar{\tau} = 2.0$. We now consider the times

$T_1 = 5\bar{\tau} = 10$ years and $T_2 = 50\bar{\tau} = 100$ years. From Fig. 2 we obtain:

$$F^{-1}_{\bar{D}(5\bar{\tau})}(0.9) \approx 3.70R, \quad F^{-1}_{\bar{D}(50\bar{\tau})}(0.9) \approx 6.10R,$$

which yields $Q^{\bar{\Psi}}_{0.9}(10) \approx 52.3\%$ and $Q^{\bar{\Psi}}_{0.9}(100) \approx 70.5\%$. By repeating the same procedure for the term

$$Q^{\Psi^{max}}_p(T) = 1 - \exp\left(-F^{-1}_{D^{max}(T)}(p)\right) = 1 - \left(1 - p^{\bar{\tau}/T}\right)^R,$$

we get $Q^{\Psi^{max}}_{0.9}(10) \approx 53.9\%$ and $Q^{\Psi^{max}}_{0.9}(100) \approx 70.8\%$. Hence, by using D^{max} instead of $\bar{D}$, we obtain a conservative and accurate estimate of the maximum relative drawdown risk.

Appendix A. Drawdown distribution

We show that the terminal drawdown $D(\infty)$ has first-order stochastic dominance over any other drawdown $D(t)$, when the underlying process $X(t) = \mu t + \sigma W(t)$ has positive parameters $\mu, \sigma > 0$. Letting $M(t) = \sup_{s \le t} X(s)$ it follows from the well-known joint distribution

$$\mathbb{P}\left(X(t) \le x, M(t) \le y\right) = \Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - e^{y/R}\Phi\left(\frac{x - 2y - \mu t}{\sigma\sqrt{t}}\right), \quad R = \frac{\sigma^2}{2\mu},$$

that $D(t) = M(t) - X(t)$ has the law

$$\mathbb{P}\left(D(t) \le x\right) = \Phi\left(\frac{x + \mu t}{\sigma\sqrt{t}}\right) - e^{-x/R}\Phi\left(\frac{-x + \mu t}{\sigma\sqrt{t}}\right).$$

We leave the proof to the reader, but note that $D(\infty)/R$ has the law of a standard exponential distribution. Hence, the constant R equals the expected long-term drawdown $\mathbb{E}[D(\infty)]$.

Proposition 3. *The drawdown $D(\infty)$ has first-order stochastic dominance over any other drawdown $D(t)$, $t < \infty$.*

Proof. Define $H(x) = \mathbb{P}\left(D(t) \le x\right) - \mathbb{P}\left(D(\infty) \le x\right)$ and evaluate

$$H(x) = e^{-x/R}\Phi\left(\frac{x - \mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{-x - \mu t}{\sigma\sqrt{t}}\right).$$

We need to show that $H(x) \ge 0$, for all $x \ge 0$, with strict inequality for some x. First, note that $H(0) = \lim_{x\to\infty} H(x) = 0$. Hence, it suffices to show that $H'(0) > 0$ and that H has a unique local maximum point.

By using the identity

$$\phi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right)=\phi\left(\frac{x+\mu t}{\sigma\sqrt{t}}\right)=e^{-x/R}\phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right),$$

we express the derivative of H as

$$H'(x)=\frac{1}{R}e^{-x/R}\left(\frac{2R}{\sigma\sqrt{t}}\phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right)-\Phi\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right)\right).$$

We analyze the derivative over the interval $[0,\mu t]$ using the auxiliary function

$$h(y)=H'(\mu t+\sigma\sqrt{t}\Phi^{-1}(y))=\frac{1}{R}e^{-\frac{1}{R}(\mu t+\sigma\sqrt{t}\Phi^{-1}(y))}\left(\frac{2R}{\sigma\sqrt{t}}\phi(\Phi^{-1}(y))-y\right),$$

where now $y\in[\Phi(-\mu\sqrt{t}/\sigma),\Phi(0)]$. Hence, the sign of h is determined by the sign of

$$\tilde{h}(y)=\frac{2R}{\sigma\sqrt{t}}\phi(\Phi^{-1}(y))-y.$$

Since

$$\tilde{h}'(y)=\frac{2R}{\sigma\sqrt{t}}\frac{\phi'(\Phi^{-1}(y))}{\phi(\Phi^{-1}(y))}-1=-\frac{2R}{\sigma\sqrt{t}}\Phi^{-1}(y)-1,$$

it is clear that $\tilde{h}'$ is a strictly decreasing function with $\tilde{h}'(\Phi(-\mu\sqrt{t}/\sigma))=0$ and $\tilde{h}'(\Phi(0))=-1$; thus negative over the chosen interval. Therefore, $\tilde{h}$ is also strictly decreasing and consequently so is h.

Returning to the original variables, we have shown that H' is strictly decreasing over $[0,\mu t]$. Next, consider

$$H'(0)=\frac{1}{R}\left(\frac{1}{z}\phi\left(-z\right)-\Phi\left(-z\right)\right),\quad z=\frac{\mu}{\sigma}\sqrt{t}>0.$$

Since $z<\infty$ it follows from the estimate

$$\Phi(-z)=1-\Phi(z)=\int_z^\infty 1\cdot\phi(u)du<\int_z^\infty\frac{u}{z}\cdot\phi(u)du=-\frac{1}{z}\int_z^\infty\phi'(u)du,$$

that $H'(0)>0$. We also see that $\lim_{x\to\infty}H'(x)=0$. Now, two cases can occur: either $H'(\mu t)\geq 0$ or $H'(\mu t)<0$. We treat each case separately, but first we set $f(x)=(x-\mu t)/(\sigma\sqrt{t})$.

If $H'(\mu t)\geq 0$ it follows that there is no point $x^*<\mu t$ such that $H'(x^*)=0$ because H' is strictly decreasing over $[0,\mu t]$. Therefore, there is a unique point $x^*\geq\mu t$ such that $H'(x^*)=0$ because the function $\phi\circ f$ is decreasing and the function $\Phi\circ f$ increases over $[\mu t,\infty)$.

If $H'(\mu t) < 0$ it follows that there is no point $x^* \geq \mu t$ such that $H'(x^*) = 0$ because the function $\phi \circ f$ is decreasing while the function $\Phi \circ f$ increases over $[\mu t, \infty)$. However, there is a unique point $x^* \in [0, \mu t]$ such that $H'(x^*) = 0$ because H' is strictly decreasing in this interval. □

Appendix B. Drawdown time

We consider the time since the last maximum of the process $X(t) = \mu t + \sigma W(t)$ over the interval $[0, T]$. In particular, we evaluate the expectation as $T \to \infty$ under the assumption that $\mu, \sigma > 0$. Following [10] we set

$$\tau_T = \sup_{0 \leq t \leq T} \{T - t : M(t) = M(T)\}, \quad M(t) = \sup_{0 \leq s \leq t} X(s),$$

and notice that

$$\mathbb{E}[\tau_T] = \int_0^T \mathbb{P}(\tau_T > t) dt.$$

It follows (through standard arguments) that

$$\lim_{T \to \infty} \mathbb{E}[\tau_T] = \int_0^\infty \left(1 - \lim_{T \to \infty} \mathbb{P}(\tau_T \leq t)\right) dt,$$

where

$$\lim_{T \to \infty} \mathbb{P}(\tau_T \leq t) = 1 + \frac{2\mu\sqrt{t}}{\sigma} \phi\left(\frac{\mu\sqrt{t}}{\sigma}\right) - 2\left(1 + \frac{\mu^2 t}{\sigma^2}\right) \Phi\left(\frac{-\mu\sqrt{t}}{\sigma}\right),$$

according to [10]. We now set $\bar{\tau} = \lim_{T \to \infty} \mathbb{E}[\tau_T]$ and refer to $\bar{\tau}$ as the long-term average drawdown time, see [7] for further information in this vein.

Proposition 4. *The long-term average drawdown time satisfies*

$$\bar{\tau} = \frac{\sigma^2}{2\mu^2}.$$

Proof. We start by a change of variables

$$\bar{\tau} = \frac{4\sigma^2}{\mu^2} \int_0^\infty \left((v + v^3)\Phi(-v) - v^2 \phi(v)\right) dv,$$

and recall the well-known expressions

$$\int_0^\infty v^{2n} \phi(v) dv = \frac{1}{2} \prod_{i=1}^n \left(2(n-i) + 1\right),$$

$$\int_0^\infty v^{2n-1} \Phi(-v) dv = \frac{1}{2n} \int_0^\infty v^{2n} \phi(v) dv,$$

which follow from iterated usage of the integration by parts formula, the identity $\phi'(v) = -v\phi(v)$, and the symmetry of the Gaussian density. Straightforward calculations conclude the proof. $\square$

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