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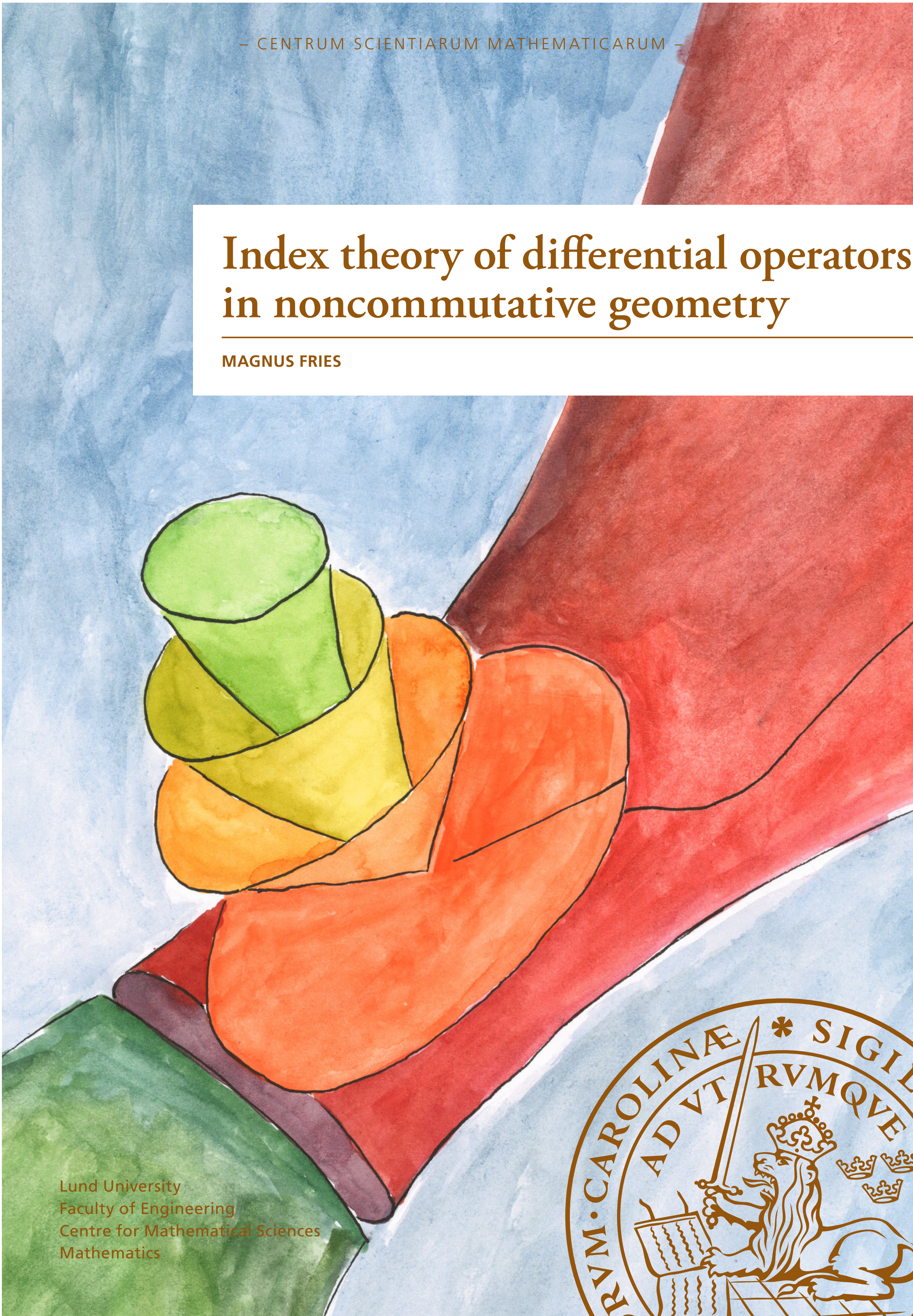
**MAGNUS FRIES**



Lund University
Faculty of Engineering
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Mathematics

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Magnus Fries

LUND UNIVERSITY

Doctoral dissertation for the degree of Doctor of Philosophy at the Faculty of Engineering at Lund University to be publicly defended on Friday the 30th of January 2026 at 13:00 in the Hörmander lecture hall at Centre for Mathematical Sciences, Lund. Faculty opponent will be Jens Kaad.

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Abstract
This thesis explores index theory for linear differential operators using tools from noncommutative geometry. We study how spectral triples can accommodate elliptic and Heisenberg-elliptic higher-order differential operators in $K$-homology, with a specific focus on manifolds with boundary. In the case of higher-order elliptic differential operators on manifolds with smooth compact boundary, we prove a generalization of the Baum-Douglas-Taylor index formula. From this, we obtain an obstruction to existence of elliptic boundary conditions. On non-compact manifolds, we revisit Gromov-Lawson's relative index theorem and show that it holds in a more general setting. In connection to this, we obtain a geometric characterization of Fredholm operators. For anisotropic geometries, we study how spectral triples can be constructed from multiple operators of different orders that together capture the geometry. We also show that any elliptic or Heisenberg-elliptic differential operator can locally reconstruct the geodesic or the Carnot-Carathéodory distance, respectively. Lastly, we present a novel approach for an eigenvalue inequality for different boundary conditions of the Laplacian.

Key words
Index theory, Boundary value problems, Hypoellipticity, Kasparov theory, Noncommutative geometry, Spectral triples, Spectral theory

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*It's not that the lows are so low and the highs are so high,*
*it's that the lows are so long and the highs are so short.*
— Ian Putnam, Trondheim 2021

# Contents

# List of publications

This thesis is based on the following papers:

I **Relative $K$-homology of higher-order differential operators**
M. Fries
Journal of Functional Analysis **288** (2025), no. 1, 110678.

II **The relative index theorem and a characterization of Fredholm operators**
M. Fries
**arXiv**:2511.06375 (2025) – Submitted.

III **Parabolic noncommutative geometry**
M. Fries, M. Goffeng and A. Masters
**arXiv**:2503.12938 (2025) – Submitted.

IV **Reproving Friedlander's inequality with the de Rham complex**
M. Fries, M. Goffeng and G. Miranda
Journal of Spectral Theory **15** (2025), no. 4, 1593–1613.

## Author contributions

I I am the sole author of Paper I. The idea for the paper came from my thesis advisor Magnus Goffeng and was written under his supervision.

II I am the sole author of Paper II. The idea for the paper was my own, but it was written under continuous discussion with my thesis advisor Magnus Goffeng.

III All three authors of Paper III were part of developing the theory and writing the article. The last two sections are due to Ada Masters.

IV Germán Miranda suggested the problem and provided bibliography for Paper IV. All three authors were part of working on the problem and writing the article.

# Popular summary in English

In the time of Euclid and many centuries thereafter, mathematics was largely based on geometrical constructions which alludes to our intuitive spatial reasoning. Since the invention of calculus by Newton and Leibniz, we have instead been able to interact with mathematical analysis more algebraically. This algebraic approach, heavily leaning on the power of abstract constructions, makes statements very precise and makes generalization of a concept a matter of relaxing assumed properties. In particular, the generalization of geometry to topology and numbers to abstract algebra have been incredibly fruitful. An intriguing question is if we algebraically can recover geometric properties. The starting seed for this question was provided in the early 19-hundreds when Gelfand showed that a topological space can be reconstructed from the algebra of continuous functions on the space. This correspondence gave way to flourishing mathematical ideas, such as Grothendieck's development of modern algebraic geometry.

At about the same time the Fredholm index was introduced as an invariant associated to an equation that tells us something about how far the equation is from being solvable. To solve an equation we want, given an output $y$ and an operator $A$, to find an input $x$ producing the output $y$ under the operation of $A$, which algebraically could be written as

$$Ax = y.$$

For such an equation, we want to know if it is solvable and if the solution is unique, that is, if any input $x$ exists given an output $y$ and if there is another input $\tilde{x}$ producing the same output. In linear algebra where $A$ is a linear operator, we can associate numbers to these questions by considering the dimension of outputs $y$ that cannot be produced by any input $x$ and the dimension of solutions $x$ giving the same output $y$. Separately these numbers are very sensitive to small perturbation of $A$, but their difference is not. This difference is called the Fredholm index of $A$,

An interesting surprise is that the index of a linear differential operator can be calculated by local geometrical quantities, as is shown by the celebrated Atiyah-Singer index theorem. A special case of this index theorem is the Gauss-Bonnet theorem, which says that the average of the curvature on a closed surface only depends on the number of holes of the surface. For instance, the surface of a ball, that is a sphere, has no holes, whereas the surface of a bagel, that is a torus, has one hole. The curvature is here a local geometric quantity whereas the number of holes is a global topological invariant. As the tools used to prove the Atiyah-Singer index theorem came from algebraic geometry, it paved the way towards a further abstract treatment of geometry.

The algebraic structures that describe classical geometry have the property that they are commutative, which means that

$$ab = ba.$$

As the algebraic descriptions of geometry could as well be applied to noncommutative algebraic structures, one can see a noncommutative algebraic structure as an analog to geometry. The subject of studying such analogs of geometry is called noncommutative geometry as coined by Connes, and was introduced partially to accommodate developments in theoretical physics by generalizing space itself.

As many examples have shown, the insights from noncommutative geometry help when studying problems in classical geometry. The main topic in this thesis is studying geometrical index problems from the viewpoint of noncommutative geometry. In particular, we study local descriptions of geometries, how we can cut out local geometries, how we can glue together geometries and how we can patch together different local geometrical behaviors.

In classical geometry, we describe a changing system locally with a differential operator, and we study how such local descriptions transfer to the language of noncommutative geometry. As a geometry on a smaller scale is still a geometry, we describe how a smaller geometry can be cut out from a larger. Mathematically, this relates to boundary value problems, meaning an equation involving a differential operator on a domain together with additional conditions that the solution needs to fulfill on the boundary of the domain. In some local descriptions of geometry, not all directions are treated equally. We also study how such descriptions can be placed in the framework of noncommutative geometry, both when the local description is described by one differential operator and when it requires many.

## Populärvetenskaplig sammanfattning på svenska

Under Euklides tid och många århundraden därefter grundade sig matematiken på geometriska konstruktioner, något som tilltalar vårt intuitiva spatiala tänkande. Sedan utvecklandet av infinitesimalkalkylens av Newton och Leibniz har vi istället kunnat behandla matematisk analys mer algebraiskt. Detta algebraiska tillvägagångssätt, vilket vilar på styrkan hos abstrakta konstruktioner, möjliggör precision i påståenden och gör att generalisering blir en fråga av att släppa efter på antagna egenskaper. Exempelvis har generaliseringen av geometri till topologi och tal till abstrakt algebra varit speciellt givande. En intressant fråga är om vi algebraiskt kan återskapa geometriska egenskaper. Startskottet till den frågan gavs i början av 1900-talet då Gelfand visade att ett topologiskt rum kan återskapas av algebran av kontinuerliga funktioner på rummet. Den korrespondensen banade väg för enastående matematiska idéer, som till exempel Grothendiecks utveckling av modern algebraisk geometri.

Ungefär samtidigt introducerades Fredholm-indexet som en invariant associerad till en ekvation som säger något om hur långt från lösbar ekvationen är. För att lösa en ekvation vill vi, givet utdata $y$ och en operator $A$, hitta indata $x$ som ger utdata $y$ under operationen $A$, vilket vi algebraiskt kan uttrycka som

$$Ax = y.$$

För en sådan ekvation vill vi veta om den är lösbar och om lösningen är unik, det vill säga om det finns någon indata $x$ givet utdata $y$ och om det finns andra indata $\tilde{x}$ som ger samma utdata. I lineär algebra där $A$ är en lineär operator kan vi associera tal till de här frågorna genom att betrakta dimensionen av utdata $y$ som inte kan ges av någon indata $x$ och dimensionen av indata $x$ som ger samma utdata $y$. Separat är dessa tal väldig känsliga för små förändringar av $A$, men inte skillnaden mellan dem. Den skillnaden kallas Fredholm-indexet av $A$

Ett oväntat resultat är att indexet av en lineär differentialoperator kan räknas ut med hjälp av lokala geometriska storheter, vilket kan ses från Atiyah-Singers hyllade indexsats. Ett specialfall av den indexsatsen är Gauss-Bonnets sats som säger att medelkrökningen på en sluten yta bara beror på antalet hål i ytan. Till exempel, ytan av en boll, vilket är en sfär, har inga hål, medan ytan av en bagel, vilket är en torus, har ett hål. Krökning är här en lokal geometrisk storhet medan antalet hål är en global topologisk invariant. Eftersom verktygen för att bevisa Aityah-Singers indexsats kom från algebraisk geometri, banade det väg för ytterligare abstrakt beskrivning av geometri.

De algebraiska strukturerna som beskriver klassisk geometri har egenskapen att de är kommutativa, vilket betyder att

$$ab = ba.$$

Eftersom de algebraiska beskrivningarna för geometri också kan appliceras på icke-kommutativ algebraiska strukturer kan vi se en icke-kommutativ algebraisk struktur som en analog till geometri. Ämnet där vi studerar den typen av analogier till geometri kallas icke-kommutativ geometri så som myntat av Connes, och introducerades delvis för att tillåta utveckling inom teoretisk fysik genom att generalisera själva rummet.

Så som många exempel visar, insikterna från icke-kommutativ geometri hjälper när vi studerar frågeställningar i klassisk geometri. Huvudtemat i den här tesen är att studera geometriska indexproblem från synvinkeln av icke-kommutativ geometri. Specifikt studerar vi lokala beskrivningar av geometrier, hur lokala geometrier kan skäras ut, hur geometrier kan limmas ihop och hur vi kan fläta ihop olika lokala geometriska beteenden.

I klassisk geometri beskriver vi ett föränderligt system med en differentialoperator, och vi undersöker hur den typen av lokala beskrivningar överförs till icke-kommutativ geometri. Eftersom en geometri på en mindre skala fortfarande är geometri, beskriver vi hur en mindre geometri kan skäras ut från en större. Matematiskt relaterar detta till randvärdesproblem, med vilket vi menar en ekvation med en differentialoperator på ett område tillsammans med ett ytterligare villkor lösningen ska uppfylla på randen av området. I vissa lokala beskrivningar av geometri behandlas inte alla riktningar på samma sätt. Vi studerar även hur sådana beskrivningar kan placeras i ramverket för icke-kommutativ geometri, både när den lokala geometrin är beskriven av en differentialoperator och när den beskrivs av flera.

## Acknowledgements

First and foremost, I would like to express my sincere gratitude to my thesis advisor, Magnus Goffeng. During these years you have introduced beautiful mathematics to me and inspired me with your vast knowledge and passion. While steadily pushing me forward academically, you have also been very supportive and understanding. I admire your incredible patience with me, and I am grateful for your teachings. It has been a true privilege.

I extend my thanks to Jens Kaad for accepting the role of opponent for my thesis as well as Francesca Arici, Matthias Lesch and Joachim Toft for accepting to act as the committee. I also wish to thank Erik Wahlén for being my co-advisor and ensuring I would remain in good hands in case anything had happened.

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I would also like to thank my other colleagues at the department. I am very grateful to the social breakes in the lunchroom, and I am in particular thankful for all my friends in the knitting gang, among the running people and in the *'briefly existing yet so perfect'* board game group. Thanks also to the other senior researchers and staff at the department for being so helpful over the years.

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I am also grateful to Jack Ekenstam, with whom I studied during my undergraduate years and who lead me to abstract mathematics, as well as my master thesis supervisor, Dennis Eriksson, whose wise teachings I still carry with me.

Finally, I wish to thank my family, loved ones and friends who have shaped me into the person I am today.

# Index theory of differential operators in noncommutative geometry

*This seems very complicated, but that's how*
*non-commutative geometry works, please respect it.*
— Grigori Rozenblum, Stockholm 2025

This preface intends to draw a path through the mathematics leading up to the research papers included in this thesis. Along the way we will give brief introductions to necessary concepts and add some context. In addition, we will present some smaller results that are not included the research papers.

# 1 Introduction

The main topic of this thesis is index theory for linear differential operators. In particular, we are interested with the Fredholm index which is an integer associated to an operator that is rather stable under small perturbations of the operator. Algebraically, the Fredholm index relates to how solvable the equation $Df = g$ is as the index of the operator $D$ is defined as

$$\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D$$

whenever these numbers are finite. Here, the kernel $\operatorname{Ker} D$ of $D$ consists of elements $f$ such that $Df = 0$ and hence $\dim \operatorname{Ker} D$ can be seen as a measure of how not unique a solution is. On the other hand, the non-zero elements of the cokernel $\operatorname{Coker} D$ of $D$ can be represented by elements $g$ that are not in the range of $D$ and hence $\dim \operatorname{Coker} D$ can be seen as a measure of how not solvable the equation.

Note that to be a solution to a differential equation is a global requirement, and hence the Fredholm index of a differential operator should be seen as a global invariant. However, a differential operator is a local operator, and it turns out that its index can at times be connected to local geometrical quantities. One of the earliest instances of this connection can be seen in Gauss-Bonnet theorem [Che90] from the early 1800-hundreds which show that for a closed surface $\Sigma$

$$2(1-g) = \frac{1}{2\pi}\int_\Sigma \kappa$$

where $\kappa$ is the curvature of $\Sigma$ and $g$ is a topological invariant counting the number of holes in the surface called the genus. Turning to complex manifolds, a similarly flavored result is the Riemann-Roch theorem [Rie57, Roc65] and its generalization to the Hirzebruch-Riemann-Roch theorem [Hir54] that states that the holomorphic Euler characteristic $\chi(M;E)$ of a holomorphic vector bundle $E$ on a closed complex manifold $M$ can be calculated from the integral

$$\chi(M;E) = \int_M \operatorname{ch}(E) \wedge \operatorname{Td}(TM)$$

where ch denotes the Chern class and Td denotes the Todd class (see also [Hir95]). Both the Gauss-Bonnet theorem and the Hirzebruch-Riemann-Roch theorem can be seen as index formulas of elliptic differential operators [AS63, AS68b]. Specifically, for the Gauss-Bonnet theorem one considers the Hodge-Dirac operator associated to the de Rham complex and for the Hirzebruch-Riemann-Roch theorem one considers the Dolbeault-Dirac operator associated to the Dolbeault complex twisted by $E$. These two index formulas among many others where generalized by the Atiyah-Singer index theorem [AS63, AS68a, AS68b] which states that for any elliptic differential operator $D$ on a closed manifold $M$

$$\operatorname{ind} D = \int_{T^*M} \operatorname{ch}(\sigma^m(D)) \wedge \operatorname{Td}(TM \otimes \mathbb{C})$$

where $\sigma^m(D)$ is the principal symbol of $D$ constructed from the leading terms of $D$ with respect to order of differentiation. With a closed manifold $M$ we mean a compact manifold without boundary. The differential operator $D$ is elliptic when its principal $\sigma^m(D)$ is invertible. The Atiyah-Singer index theorem formalizes the idea that the global invariant that is the Fredholm index can be calculated locally in the case of elliptic operators on closed manifolds [Ati70, Theorem 4.3]. The main topic of this thesis is to study index theory in other cases, such as when the manifold is non-compact, the manifold has boundary or when the differential operator is not elliptic in the classical sense.

Prior to the Atiyah-Singer index theorem, there had been great developments describing topological spaces by algebraic structures, specifically Gelfand showed that there is a correspondence between locally compact Hausdorff topological spaces and commutative $C^*$-algebras [Gel41, GRS64]. This correspondence is called Gelfand duality, and it tells us that a topological space can be analyzed from the functions upon it, a viewpoint used by Grothendieck to develop modern algebraic geometry (see for instance the series starting with [Gro60]). At the core of the proof of the Atiyah-Singer index theorem was the use of $K$-theory of topological spaces [Ati67]. With Gelfand-duality, $K$-theory could be generalized to a homology theory for $C^*$-algebras. This viewpoint can be seen in Atiyah's work to develop a dual cohomology theory for topological $K$-theory in [Ati70] using abstract elliptic operators. From these abstract elliptic operators, Kasparov developed such a dual theory for $C^*$-algebras called $K$-homology [Kas75]. Furthermore, with the developments of theory for extensions of $C^*$-algebras in [BDF77] in mind, Kasparov later constructed a framework encompassing both $K$-theroy and $K$-homology which is called Kasparov's bivariant $KK$-theory for $C^*$-algebras [Kas80].

The potential of abstract elliptic operators was later extended by Connes [Con94]. He showed that the Dirac operator on a closed spin$^c$ manifold, a canonical first-order elliptic differential operator, can reconstruct some geometrical properties algebraically, such as dimension, metric, differentiation, and integration on the manifold [Con89]. As such, Connes proclaimed that an abstract elliptic operator connected to a $C^*$-algebra, also called a spectral triple, describes a noncommutative geometry and thereby refines the analogy from Gelfand duality. One aim of Connes with noncommutative geometry was to generalize the concept of a space in theoretical physics to accommodate problems arising when trying to place gravity and quantum theory on equal footing, see for instance [Con96]. Non-commutative geometry have also been used to solve problems related to the quantum Hall effect [BvESB94].

As the theory of noncommutative geometry has developed, new insights regarding index theory have been revealed. Especially the role of spectral triples and $K$-homology as a natural setting for index theory [Ati70, Kas75, BvE14, BvE18, BvE16]. More recently, there have been developments in noncommutative geometry to accommodate index theory for more than elliptic differential operators on closed manifolds [Roe93, BvE14, Moh22, Ewe23]. This is the main topic of this thesis. Some examples of non-elliptic operators which still possess good properties come from Hörmander's sum of squares theorem [Hör67]. Namely, if $X_0, X_1, \dots, X_p$ are real vector fields that together with repeated Lie brackets of the form $[X_j, [X_k, \dots, X_l]]$ span the tangent space at each point, then the differential operator

$$X_0 + \sum_{k=1}^{p} X_k^2 \tag{1}$$

is hypoelliptic. A differential operator $D$ is hypoelliptic if any distribution $f$ such that $Df$ is smooth is itself necessarily smooth. The Laplacian $\Delta = -\sum \frac{\partial^2}{\partial x_k^2}$, which is an elliptic differential operator, is of the form (1) where the vector fields themselves, without Lie brackets, span the tangent space. Another example of (1) that is not elliptic is the sub-Laplacian $\Delta_H := -X^2 - Y^2$ on the three-dimensional Heisenberg group where $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. If we allow for a complex vector field $X_0$ in (1) then we can consider $\Delta_H + i\gamma\frac{\partial}{\partial z}$ which contain non-trivial index theory [BvE14]. Such operators do not fit well in the classical framework of pseudodifferential calculus [Shu01, Hör07] used to deal with elliptic operators, and one can instead turn to the so-called Heisenberg-pseudodifferential calculus and Heisenberg-ellipticity [Mel82, BG88, Pon08, vEY19, DH22].

This thesis will study how spectral triples can accommodate higher-order differential operators, both elliptic and Heisenberg-elliptic, with a specific focus on manifolds with boundary. That is, we will use the tools from noncommutative geometry, but we are mostly concerned with problems in a geometric setting. Thus, most of the concrete $C^*$-algebras we will deal with are continuous functions on a manifold. With the presence of boundary we will also consider boundary conditions, and we will additionally study a related problem in spectral theory.

## 1.1 Spectral triples

*Think about the circle.*
— Magnus Goffeng

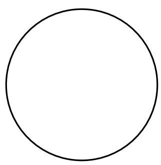

**Figure 1:** The circle.

As three of the papers presented in this thesis concern noncommutative geometry and spectral triples, we will here present the basic definition of a spectral triple. Together with examples, this will serve as a guide when we later in the text introduce the necessary concepts.

**Definition 1.1.** A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of a self-adjoint operator $D$ on a Hilbert space $\mathcal{H}$ and a unital $*$-subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ such that

1. $\operatorname{Dom} D \hookrightarrow \mathcal{H}$ is compact;
2. For all $a \in \mathcal{A}$, $a \operatorname{Dom} D \subseteq \operatorname{Dom} D$ and $[D, a]$ extends to a bounded operator on $\mathcal{H}$.

We say that $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple for the $C^*$-algebra $A$ which is the closure of $\mathcal{A}$.

The simplest example of a spectral triple comes from a derivative on the circle $S^1$. Specifically, we consider the differential operator $-i\frac{\partial}{\partial\theta}$ seen as an unbounded operator on $L^2(S^1)$ with the Sobolev space $H^1(S^1)$ as domain and obtain the spectral triple

$$\left(C^\infty(S^1), L^2(S^1), -i\frac{\partial}{\partial\theta}\right)$$

for the $C^*$-algebra $C(S^1)$. Here we let a function $a \in C^\infty(S^1)$ act by pointwise multiplication on $L^2(S^1)$, and note that

$$\left[-i\frac{\partial}{\partial\theta}, a\right] f = -i\frac{\partial}{\partial\theta}(af) - a\left(-i\frac{\partial}{\partial\theta}f\right) = -i\frac{\partial a}{\partial\theta}f$$

for any $f \in C^\infty(S^1)$ by the Leibniz rule. In particular, $\left[-i\frac{\partial}{\partial\theta}, a\right]$ extends to a bounded operator on $L^2(S^1)$ for any $a \in C^\infty(S^1)$. This also ensures that $a$ preserves the domain of $-i\frac{\partial}{\partial\theta}$. That $H^1(S^1) \hookrightarrow L^2(S^1)$ is a compact inclusion can be seen using Fourier series and is called the Rellich theorem [Rel30]. The Rellich theorem implies that $-i\frac{\partial}{\partial\theta}$ has discrete spectrum and is therefore Fredholm. However, since $\operatorname{Ker}\left(-\frac{\partial}{\partial\theta}\right)$ consists of constant functions and $\operatorname{Ran}\left(-\frac{\partial}{\partial\theta}\right)$ consists of all functions with zero mean, $\operatorname{ind}\left(-i\frac{\partial}{\partial\theta}\right) = 0$.

More generally, for any closed manifold $M$ and a self-adjoint elliptic first-order differential operator $D$ on $L^2(M; E)$ for some vector bundle $E$,

$$(C^\infty(M), L^2(M; E), D) \tag{2}$$

defines a spectral triple for the $C^*$-algebra $C(M)$. In the sense of noncommutative geometry, the way that the spectral triple (2) captures geometry is that the growth of the spectrum of $D$ captures the dimension of $M$ similarly to Weyl's law [Wey12], and differential calculus on $M$ can be constructed from commutators of the form $[D, a]$ as can be seen in the case of the circle. From derivatives, we can also obtain distances by forming a metric on $M$ (See more in section 4). One could also capture integration from the spectral triple (2) [Con89, Proposition 12] [GBVF01, Chapter 7], however we will not concern ourselves with that in this text.

A spectral triple for a $C^*$-algebra $A$ defines a $K$-homology class in either the even $K$-homology group $K^0(A)$ or the odd $K$-homology group $K^1(A)$. As Definition 1.1 stands now, we obtain an odd class. If $D$ is odd in the sense that

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

we obtain an even class, and then $D$ can contain a non-zero index in the form of ind $D_+$.

Note that a spectral triple can also be seen as an unbounded Kasparov $(A, \mathbb{C})$-module as originally done in [BJ83]. Most constructions and results regarding spectral triples in this text also apply to unbounded Kasparov $(A, B)$-modules, but in interest of a lighter exposition we refrain from dealing with Hilbert $C^*$-modules. A Hilbert $C^*$-module is a Banach space with a (right) $B$-module structure and a $B$-valued inner product for a $C^*$-algebra $B$ and should be seen a generalization of a Hilbert space. Specifically, the spectral triple constructions from Paper I can be generalized to Hilbert $C^*$-modules, Paper II is written using Hilbert $C^*$-modules, and many of the constructions in Paper III can be done for Hilbert $C^*$-modules which can be seen in the coauthor Ada Masters' PhD thesis [Mas25, Chapter 4]. The results in Paper IV have little connection to Hilbert $C^*$-modules as we are there concerned with operators with discrete spectrum.

## 1.2 Brief summaries of the papers

Here we will present a brief overview of the papers included in this thesis.

In Paper I, we construct a generalization of spectral triples suitable for higher-order differential operators on manifolds with boundary and provide a localization procedure to cut out a domain of a manifold that works well with relative $K$-homology. We also calculate a boundary map in $K$-homology in the case of a classically elliptic differential operator on a compact manifold with boundary and thereby proving a generalization of the Baum-Douglas-Taylor index formula [BDT89].

In Paper II, we revisit Gromov-Lawson's relative index theorem [GL83] and generalize the proof in [Bun95] to hold in a very general setting. In essence, the relative index theorem states that the index difference of two differential operators that partially agree can be calculated by cutting out pieces and gluing them together again. We also prove a general geometric characterization of the Fredholm property as invertibility at infinity and relates this to a model for unbounded $KK$-theory originally from [Wah07].

In Paper III, we construct a generalization of spectral triples that is suitable for anisotropic geometries called *strictly tangled spectral triples*, where one example comes from the Rumin complex on a contact manifold.

In Paper IV, we notice that the Dirichlet and Neumann boundary conditions of the Laplacian can be built from maximal/minimal extensions of the exterior differential, relating these boundary conditions to the de Rham complex. Using the de Rham complex in the framework of Hilbert complexes from [BL92], we reexamine a recent result [Roh25] concerning Friedlander's inequality [Fri91], an inequality between the eigenvalues of the Dirichlet and Neumann Laplacian.

# 2 Linear analysis and operator algebras

## 2.1 Hilbert spaces

Vector spaces with a sense of orthogonality that are topologically complete are called Hilbert spaces, and we will denote an abstract Hilbert space by $\mathcal{H}$. All our Hilbert spaces will be over the field of complex numbers. The sense of orthogonality in a Hilbert space comes from the inner product

$$\mathcal{H} \times \mathcal{H} \to \mathbb{C}, \quad (v, w) \mapsto \langle v, w \rangle$$

that is linear in $w$ and conjugate linear in $v$ such that $\|v\|^2 = \langle v, v \rangle$ defines a norm. We let $\mathcal{L}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$ which we endowed with the operator norm. For $T \in \mathcal{L}(\mathcal{H})$ we let $T^* \in \mathcal{L}(\mathcal{H})$ denote the adjoint of $T$. The set of compact operators is the closure of finite rank operators and will be denoted $\mathcal{K}(\mathcal{H})$.

For a locally compact Hausdorff space $X$ with some specified appropriate measure $\mu$, we will let $L^2(X)$ denote the Hilbert space of square integrable (complex valued) functions on $X$, where we leave the measure $\mu$ implicit. The inner product on $L^2(X)$ is defined as

$$\langle f, g \rangle = \int_X \overline{f} g \, \mathrm{d}\mu.$$

## 2.2 $C^*$-algebras

A $C^*$-algebra $A$ is a Banach algebra over $\mathbb{C}$ with a conjugate linear involution $*$ that fulfills $\|a^* a\| = \|a\|^2$ for all $a \in A$. We see that $\mathcal{L}(\mathcal{H})$ is a $C^*$-algebra with the operator norm and the adjoint as involution. In fact, by the Gelfand-Naimark-Segal theorem [GN43, Seg47, GN48] any $C^*$-algebra is isomorphic to a closed $*$-subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. For a locally compact Hausdorff space $X$, the continuous (complex valued) functions vanishing at infinity $C_0(X)$ form another natural $C^*$-algebra where the norm is the supremum norm and involution is pointwise conjugation. In particular, $C_0(X)$ can be seen as operators in $\mathcal{L}(L^2(X))$ by pointwise multiplication. One may note that $C_0(X)$ is commutative in the sense that $ab = ba$ for all $a, b \in C_0(X)$, and in fact, Gelfand duality shows that up to isomorphism all commutative $C^*$-algebras are on this form [Gel41]. More precisely, for a commutative $C^*$-algebra $A$, if we endow the so-called state space $\hat{A}$ of $A$ consisting of non-zero algebra homomorphisms $A \to \mathbb{C}$ with the weak-$*$-topology, then $A$ is isomorphic to $C_0(\hat{A})$.

In finite dimensional linear algebra the concept of eigenvalues for matrices is incredibly useful. In infinite dimensions, this is instead captured by the spectrum of an operator. For a unital

$C^*$-algebra $A$ and an element $a \in A$ we let

$$\operatorname{Spec} a := \{ \lambda \in \mathbb{C} : a - \lambda \text{ is not invertible in } A \},$$

although $\operatorname{Spec} a$ does not depend on $A$ in the sense that the spectrum of $a$ is the same in $A$ as in a $C^*$-algebra $B$ if $A$ is a $C^*$-subalgebra of $B$ [HR00, Proposition 1.3.9]. In particular, if $a \in \mathcal{L}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, then any eigenvalue of $a$ is in $\operatorname{Spec} a$. The spectrum is always closed since if $a \in A$ is invertible and $\|b - a\| < \|a^{-1}\|^{-1}$ for $b \in A$ then $b^{-1} = a^{-1}(1 + (b-a)a^{-1})^{-1}$. We see that this is well-defined using the Neumann series

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k \quad \text{for } -1 < x < 1.$$

The Neumann series, or rather its finite partial sums, is a way to bootstrap up regularity when dealing with pseudodifferential operators. This is used in abstract for the localization procedure in Paper I.

If $a \in A$ is self-adjoint, or more generally if it is normal in the sense that $aa^* = a^*a$, then the unital $C^*$-algebra $C^*(a)$ generated by $a$ is commutative. Using Gelfand duality, we can show that $C^*(a)$ is isomorphic to $C(\operatorname{Spec} a)$, and therefore for each $f \in C(\operatorname{Spec} a)$ we can form a well-defined element $f(a) \in A$ with $\operatorname{Spec} f(a) = f(\operatorname{Spec} a)$ [HR00, Definition 1.1.13]. We call this continuous functional calculus. Similar tools but for unbounded operators are used extensively in the papers included in this thesis, see [Wei80, Section 7.3] for unbounded operators on Hilbert spaces and [Lan95, Theorem 10.9] for Hilbert $C^*$-modules.

## 2.3 Linear operators

For a (linear) operator $T$ on a Hilbert space, an important note is that $T$ has closed range if and only if zero is an isolated point of the spectrum of $T$. If additionally $\operatorname{Ker} T$ is finite dimensional, then we call $T$ left-Fredholm. Since we can identify $\operatorname{Coker} T$ with $\operatorname{Ker} T^*$ if $T$ has closed range, the Fredholm index

$$\operatorname{ind} T = \dim \ker T - \dim \operatorname{Coker} T$$

is well-defined if both $T$ and $T^*$ are left-Fredholm, in which case we say that $T$ is Fredholm.

A more operator algebraic way to deal with the Fredholm property is using compact operators, and we can see why from the spectral theorem.

**Theorem 2.1** (Spectral theorem). *For $K \in \mathcal{K}(\mathcal{H})$ the spectrum of $K$ is countable and has only one possible accumulation point which is zero. Moreover, all non-zero points in the spectrum are eigenvalues of finite algebraic multiplicity.*

Note that the spectral theorem is usually stated only for self-adjoint compact operators [Wei80, Theorem 7.1]. From the spectral theorem we can conclude the Fredholm alternative [Fre03], that $1 + K$ is Fredholm for any $K \in \mathcal{K}(\mathcal{H})$ and $\operatorname{ind}(1 + K) = 0$ since zero is an isolated eigenvalue of finite multiplicity and $V\colon \ker(1 + K) \to \ker(1 + K^*)$ is an isomorphism where $V$ is the phase from the polar decomposition $1 + K = V|1 + K|$. As a consequence we obtain Atkinson's theorem [Atk51] which states that $T \in \mathcal{L}(\mathcal{H})$ is Fredholm if and only if it is invertible up to compacts and that the Fredholm index is independent of compact perturbations. We also note that the index is multiplicative and using a Neumann sum we can show that it is locally constant.

To include differential operators in Hilbert space theory we use unbounded operators. An unbounded operator is a partially defined linear operator $D\colon \operatorname{Dom} D \subseteq \mathcal{H} \to \mathcal{H}'$ where $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces. A property that makes an unbounded operator $D$ well-behaved is if it is closed, meaning that the graph of $D$

$$\{ (x, Dx) \ : \ x \in \operatorname{Dom} D \} \subseteq \mathcal{H} \times \mathcal{H}'$$

is closed. In particular, this means that $\operatorname{Dom} D$ is a Hilbert space if we endow it with the graph norm $\|x\|_D^2 := \|x\|^2 + \|Dx\|^2$ for $x \in \operatorname{Dom} D$. This is put into context by the closed graph theorem. As we will use the closed graph theorem not only for Hilbert space we will state it here for Fréchet spaces. A Fréchet space is a complete topological vector spaces whose topology is induced from a countable family of seminorms. For example, $C_c^\infty(M)$ for a manifold $M$ can be endowed with a Fréchet topology.

**Theorem 2.2** (Closed graph theorem). *For a linear map $T\colon X_1 \to X_2$ between Fréchet spaces $X_1$ and $X_2$, we have that $T$ is continuous if and only if the graph $\{ (x, Tx) \ : \ x \in X_1 \} \subseteq X_1 \times X_2$ is closed.*

The closed graph theorem creates an especially rigid structure when it comes to inclusions of Fréchet spaces.

**Corollary 2.3.** *Assume that $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$ are continuous inclusions of Fréchet spaces, then any continuous linear operator $T\colon X_1 \to X_2$ such that $TY_1 \subseteq Y_2$ is also continuous as a map $T\colon Y_1 \to Y_2$.*

*Proof.* The statement follows from the closed graph theorem since the preimage of the graph of $T\colon X_1 \to X_2$ under the inclusion $Y_1 \times Y_2 \subseteq X_1 \times X_2$ is the graph of $T\colon Y_1 \to Y_2$. □

In particular, if $X \subseteq Y \subseteq Z$ are inclusions of Fréchet spaces such that $X \subseteq Z$ and $Y \subseteq Z$ are continuous, then so is $X \subseteq Y$. We will make use of Corollary 2.3 implicitly at times both in this text and in the papers included in this thesis. For Hilbert $C^*$-modules, the analog of Corollary 2.3 is slightly more technical, see Lemma 3.1 in Paper II.

For an unbounded operator $D$ on $\mathcal{H}$ that is closed and densely defined, we will often consider the operator $(1 + D^*D)^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{H})$ which also defines a unitary map

$$(1 + D^*D)^{-\frac{1}{2}} : \mathcal{H} \to \operatorname{Dom} D.$$

In particular, $D(1 + D^*D)^{-\frac{1}{2}}$ is a bounded operator which we call the bounded transform of $D$. Also, $(D(1 + D^*D)^{-\frac{1}{2}})^* = D^*(1 + DD^*)^{-\frac{1}{2}}$ which can be seen by considering the bounded transform of the self-adjoint operator $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$.

## 2.4 Differential operators

The linear operators we are mainly concerned with in this text are differential operators. A (linear) differential operator of order (at most) $m$ on a manifold $M$ is an operator of the form

$$\sum_{|\alpha| \le m} c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

in local coordinates. Here we use the multi-index notation $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ and

$$\frac{\partial^\alpha}{\partial x^\alpha} := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

for the local coordinates $x = (x_1, x_2, \dots, x_n)$ where $n = \dim M$. The coefficients $c_\alpha$ can be taken to be smooth scalar-valued functions, but we will also allow them to take values in matrices to accommodate for differential operators acting on sections of a vector bundle.

A (complex) vector bundles over a topological space $X$ is a topological space $E$ together with a surjection $E \to X$ such that the fiber of any small enough open set $U \subseteq X$ is homeomorphic to $U \times \mathbb{C}^r$ for some $r \in \mathbb{N}$, and going between any such two trivialization consists of a linear invertible map on $\mathbb{C}^r$. Hence, a vector bundle $E \to M$ on a manifold $M$ locally takes the form $\Omega \times \mathbb{C}^r$ where $\Omega \subseteq \mathbb{R}^n$ for $n = \dim M$. We use this local form of vector bundles when defining differential operators locally. A map $f : M \to E$ is called a section if $f$ composed with the surjection $E \to M$ is the identity map on $M$, and we will denote the smooth sections of $E$ as $C^\infty(M; E)$. When needed, we will also equip our complex vector bundles with a fixed Hermitian metric.

A (real) vector bundle that is natural to consider on a manifold $M$ is the tangent bundle $TM$ consisting of local directions at each point. At each point $x \in M$, we can construct $TM$ fiberwise by derivations $X_x \in T_xM$ that are functions $X_x : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule $X_x(ab) = X_x(a)b(x) + a(x)X_x(b)$ for $a, b \in C^\infty(M)$. We call a section $X \in C^\infty(M; TM)$ a vector-field on $M$. Note that a vector field is a differential operator of order 1 on $M$. Associated to the tangent bundle $TM$ is its dual-bundle, the cotangent bundle $T^*M$, can be constructed fiberwise by covectors $\mathrm{d}x_x \in T_x^*M$ that are linear maps $\mathrm{d}x_x : T_xM \to \mathbb{R}$.

With the tangent bundle and cotangent bundle we can form a canonical coordinate independent first-order differential operator, namely the exterior differential $\mathrm{d}\colon C^\infty(M) \to C^\infty(M; T^*M)$ by

$$\mathrm{d}a(X) = X(a)$$

for any $X \in C^\infty(M; TM)$ and $a \in C^\infty(M)$. In local coordinates $x = (x_1, x_2, \dots, x_n)$, we have that

$$\mathrm{d}a = \sum \frac{\partial a}{\partial x_k} \mathrm{d}x_k,$$

and we should see the exterior differential as a metric independent gradient. A chosen metric on the tangent bundle $TM$ is called a Riemannian metric. A Riemannian metric also induces a metric on the cotangent bundle $T^*M$ and ensures that $T^*M \cong TM$ as vector bundles.

We let $\mathrm{DO}^m(M; E, F)$ denote the set of differential operators of order at most $m$ on a manifold $M$ between sections of the Hermitian vector bundles $E$ and $F$. If the vector bundles are trivial we will also use the notation $\mathrm{DO}^m(M)$. A differential operator $D \in \mathrm{DO}^m(M; E, F)$ can also be seen as a continuous map

$$D\colon C_c^\infty(M; E) \to C_c^\infty(M; F)$$

on a manifold $M$ that is local in the sense that

$$\operatorname{supp} Df \subseteq \operatorname{supp} f$$

for all $f \in C_c^\infty(M; E)$. In fact, any such operator is locally a differential operator [Shu01, Problem 2.1]. Using partial integration, any differential operator $D$ has a formal adjoint

$$D^\dagger\colon C_c^\infty(M; F) \to C_c^\infty(M; E)$$

which is a differential operator such that $\langle Df, g\rangle = \langle f, D^\dagger g\rangle$ for the $L^2$-inner product. Note that we are here implicitly using the Hermitian metric of the vector bundles and a volume density of the manifold. For instance, if $X \in C^\infty(M; TM)$ is a vector field then

$$X^\dagger = -X + c$$

for $c \in C^\infty(M)$ which is minus the divergence of $X$ with respect to the Riemannian metric.

Using the formal adjoint $D^\dagger$ we can extend $D$ to a continuous operator

$$D\colon \mathscr{D}'(M; E) \to \mathscr{D}'(M; F)$$

on distributions. Since $L^2$-sections are distributions and

$$C_c^\infty(M; E) \subseteq L^2(M; E) \subseteq \mathscr{D}'(M; E)$$

are continuous inclusions, we obtain two natural closed realizations of $D$ as an unbounded operator on $L^2$-sections,

$$\mathrm{Dom}\, D_{\min} = \overline{D|_{C_c^\infty(M;E)}}$$

and

$$\mathrm{Dom}\, D_{\max} = \{ f \in L^2(M;E) \,:\, Df \in L^2(M;F) \}$$

for which $(D^\dagger)_{\min} = (D_{\max})^*$.

For a differential operator $D \in \mathrm{DO}^m(M;E,F)$ and an open set $\Omega \subseteq M$, we will denote $D_\Omega \in \mathrm{DO}^m(\Omega;E,F)$ for the restricted map

$$D_\Omega : C_c^\infty(\Omega;E) \to C_c^\infty(\Omega;F).$$

Note in particular that if $D_{\min} = D_{\max}$ and $\partial\Omega \subseteq M$ is empty, then $D_{\Omega,\min} = D_{\Omega,\max}$. With this perspective, we will think of $M$ as having a boundary with respect to $D$ if $D_{\min} \neq D_{\max}$. In this case, it is possible to impose a boundary condition on $D$ which we will return to later when talking about boundary conditions.

## 2.5 $K$-theory

Topological $K$-theory [Ati67] is a cohomology theory of topological spaces constructed from vector bundles. Specifically, the $K$-theory group $K^0(X)$ of a compact space $X$ can be constructed by formal differences of isomorphism classes of vector bundles over $X$ with the group operation as direct sum.

Similarly to Gelfand duality which tells us that a compact topological space $X$ corresponds to the unital commutative $C^*$-algebra $C(X)$, the Serre-Swan theorem [Swa62] states that there is a correspondence between vector bundles over $X$ and finitely generated projective $C(X)$-modules. Hence, we obtain the extended definition of the $K$-theory group $K_0(A)$ of a unital $C^*$-algebra $A$ by considering formal differences of isomorphism classes of finitely generated projective $A$-modules (see for instance [Bla98]). We also obtain another description of $K_0(A)$ from projections $p \in M_n(A)$, as a finitely generated projective $A$-module can also be defined up to isomorphism by a projection $p \in M_n(A)$ for some $n$. For technical reasons, $K$-theory for non-unital algebras (and non-compact spaces) is defined as $K_0(A) := \ker(K_0(A^+) \to K_0(\mathbb{C}))$ where $A^+$ is the unitalization of $A$.

There is also the odd $K$-theory group $K_1(A)$ that can either be defined as $K_0(A \otimes C_0(\mathbb{R}))$ or as $U(A^+ \otimes \mathcal{K}(\mathcal{H}))/U(A^+ \otimes \mathcal{K}(\mathcal{H}))_0$ where $U(A^+ \otimes \mathcal{K}(\mathcal{H}))$ are unitaries and $U(A^+ \otimes \mathcal{K}(\mathcal{H}))_0$ is the path-connected component of the identity.

## 2.6 $KK$-theory and $K$-homology

Kasparov's bivariant $KK$-theory [Kas80] is a functor associating to two $C^*$-algebras $A, B$ an abelian group $KK^*(A,B)$ that is contravariant in the first argument and covariant in the second (see also [JT91]). We will recall a simplified version of Kasparov original definition, and only define $K^*(A) := KK^*(A, \mathbb{C})$ which we will call $K$-homology. We say that a Hilbert space is $\mathbb{Z}_2$-graded if it consists of two parts $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and we say that an operator $T$ on $\mathcal{H}$ is even if $T\mathcal{H}_\pm \subseteq \mathcal{H}_\pm$ and odd if $T\mathcal{H}_\pm \subseteq \mathcal{H}_\mp$.

**Definition 2.4.** [Kas80] An even bounded Kasparov $(A, \mathbb{C})$-module $(\pi, \mathcal{H}, F)$ for a $C^*$-algebra $A$ consists of a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$, a $*$-representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$ and an odd operator $F \in \mathcal{L}(\mathcal{H})$ such that

$$\pi(a)(F - F^*), \pi(a)(F^2 - 1), [F, \pi(a)] \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$. An odd bounded Kasparov $(A, \mathbb{C})$-module is defined similarly except that $\mathcal{H}$ is trivially graded and $F$ does not need to be odd. The group $K^0(A)$ is defined as even bounded Kasparov $(A, \mathbb{C})$-modules modulo unitary equivalence and homotopy, and the group operation is the direct sum. The group $K^1(A)$ is defined similarly for odd Kasparov modules.

The group $KK^*(A, B)$ can be defined similarly, except that we replace the Hilbert space with a Hilbert $C^*$-module over a $C^*$-algebra $B$. In particular, $KK^*(\mathbb{C}, B)$ is isomorphic to the $K$-theory group $K_*(B)$. We will not introduce Hilbert $C^*$-modules as it does not play a particularly big role in this work.

The operator $F$ in a bounded Kasparov module should be seen as an abstract zeroth-order elliptic pseudodifferential operator as in Atiyah's first attempt to construct a dual theory to $K$-theory [Ati70]. In particular, if $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, then $D(1 + D^2)^{-\frac{1}{2}}$ defines a bounded Kasparov $(A, \mathbb{C})$-module [BJ83]. Unbounded Kasparov $(A, B)$-modules, of which spectral triples are a special case, were first introduced in [BJ83] where they showed that the unbounded model for $KK$-theory is very suitable for computing the exterior product

$$KK^p(A, B) \times KK^q(C, D) \to KK^{p+q}(A \otimes C, B \otimes D).$$

A phenomenal feature of $KK$-theory is the interior product, also called the Kasparov product [Kas80]. The Kasparov product map

$$KK^p(A, B) \times KK^q(B, C) \to KK^{p+q}(A, C), \quad (x, y) \mapsto x \otimes_B y$$

which is a group homomorphism in each argument and associative in the sense that

$$x \otimes_B (y \otimes_C z) = (x \otimes_B y) \otimes_C z.$$

Note that the construction of the Kasparov product in the general case is nonconstructive, meaning that it can be very difficult to calculate. For methods to calculate the Kasparov product, see for instance [Kuc97, Mes14, KL13, MR16]. Certain "bad Kasparov products" are addressed in Paper III.

# 3 Ellipticity and spectral triples

## 3.1 Local spectral triples

In this section we will extend the definition of spectral triples to accommodate elliptic higher-order differential operators on possibly non-compact manifolds. Later we will show that elliptic differential operators indeed define such spectral triples, and so do Heisenberg-elliptic differential operators. Since ellipticity is a local property, we call these local spectral triples. This will be clearer still when we later talk about the boundary map in $K$-homology and localization of a spectral triple.

**Definition 3.1** (Pre-spectral triple)**.** An even pre-spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of an odd closed densely defined operator $D$ on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}$ and a dense $*$-subalgebra $\mathcal{A} \subseteq \mathcal{L}(\mathcal{H})$ such that for all $a \in \mathcal{A}$

1. $D$ is symmetric;
2. $a \operatorname{Dom} D \subseteq \operatorname{Dom} D$;
3. $[D, a](1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$;
4. $[D, a](1 + D^*D)^{-\frac{1}{2}+\frac{1}{2m_a}}$ extends to a bounded operator on $\mathcal{H}$ for some $m_a > 0$.

If there is an $m > 0$ such that condition 4 that holds independently of $a \in \mathcal{A}$, then we say that $(\mathcal{A}, \mathcal{H}, D)$ is of order $m$. An odd pre-spectral triple is defined similarly except that $\mathcal{H}$ is trivially graded and $D$ does not need to be odd.

**Definition 3.2** (Local spectral triple)**.** We say that a pre-spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a local spectral triple if it additionally satisfies that for all $a \in \mathcal{A}$

1. $a \operatorname{Dom} D^* \subseteq \operatorname{Dom} D$;
2. $a(1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$.

We split the definition in pre-spectral triple and local spectral triple since we will reuse the concept of a pre-spectral triple later in the text. If $D$ is self-adjoint, then a local spectral triple recovers the definition of higher-order spectral triple in Paper 1. In this text we will drop the naming "higher-order" as compared to Paper 1.

Similar definitions to a local spectral triple can be found in literature [Hil10, FGMR19, GM15, Wah07], for instance as a model for $KK$-theory a local spectral triple should perhaps be named a half-closed $\epsilon$-unbounded Kasparov module following [GM15, Appendix A] and [Hil10]. A closed unbounded Kasparov module would correspond to requiring the operator $D$ to be self-adjoint, and is how one would recover the usual definition of a spectral triple if one also drops the condition that $[D, a](1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ (and usually require $m = 1$).

**Remark 3.3.** Note that the condition $[D, a](1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ does not follow from the other conditions if $D$ is self-adjoint, so it is a restriction compared to the usual notion of a spectral triple. This can be seen by considering the Dirac operator $-i\frac{\partial}{\partial x}$ on $\mathbb{R}$ and the function $a(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$. Note that $a$ is smooth and $a \in C_0(\mathbb{R})$, but $[D, a] = a'$ is only in $C_b(\mathbb{R})$ and not in $C_0(\mathbb{R})$. In particular, $a$ satisfies all conditions in a local spectral triple except that $[D, a](1 + D^*D)^{-\frac{1}{2}}$ is not compact.

**Remark 3.4.** Note that for a first-order spectral triple, that is $m = 1$, $[D, a]$ extends to a bounded operator, and hence $a \operatorname{Dom} D \subseteq \operatorname{Dom} D$ implies that

$$a^* \operatorname{Dom} D^* \subseteq \operatorname{Dom} D^*$$

as noted in [BDT89, Proposition 1.3] and [Hil10, Lemma 2.1]. This makes some proofs easier in the first-order case, see for example Remark 2.23 in Paper I, but it is not generally true in the higher-order case as seen in Remark 3.10 in Paper I.

The particular conditions in a local spectral triple are to ensure that it defines a $K$-homology class.

**Proposition 3.5.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a pre-spectral triple and let $A = \overline{\mathcal{A}}$. Let $F = D(1+D^*D)^{-\frac{1}{2}}$, then*

$$[F, a] \in \mathcal{K}(\mathcal{H})$$

*for any $a \in A$. If in addition $(\mathcal{A}, \mathcal{H}, D)$ is a local spectral triple, then*

$$a(F^2 - 1) \in \mathcal{K}(\mathcal{H})$$

*for any $a \in A$, and for any extension $D_e$ of $D$ such that $D \subseteq D_e \subseteq D^*$ we have that*

$$a(F_e - F^*) \in \mathcal{K}(\mathcal{H})$$

*for any $a \in A$ where $F_e = D_e(1 + D_e^*D_e)^{-\frac{1}{2}}$. In particular, $F_e$ defines a bounded Kasparov $(A, \mathbb{C})$-module and the class $[F_e] \in K^*(A)$ is independent of choice of $D_e$.*

Proof of Proposition 3.5 can be found as Lemma 3.9, Lemma 3.8 and Lemma 3.7 in Paper I.

Note in particular that using the bounded transform $F = D(1 + D^*D)^{-\frac{1}{2}}$ to obtain a $K$-homology class forgets the spectral information of $D$. This is to be excepted, as on spaces $K$-theory (and therefore $K$-homology) is a topological theory and does not see geometrical properties. We will see more of how spectral properties capture geometry later in section 4.

For any differential operator $D$ that is locally elliptic, or more generally Heisenberg-elliptic, we shall see in subsection 3.2 and subsection 3.3 that any closed realization of $D$ defines a local spectral triple for $C_0(M)$. In particular, if $D$ is a Heisenberg-elliptic differential operator, then we obtain a class

$$\left[\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\right] \in K^0(C_0(M))$$

and if $D = D^\dagger$ then we obtain a class

$$[D] \in K^1(C_0(M))$$

which is independent of closed realization, where we are using the dense $*$-subalgebra $C_c^\infty(M) \subseteq C_0(M)$.

## 3.2 Ellipticity and regularity

Elliptic differential operators are operators that can control the norm of any differential operator of lower or equal order. The theory is best understood using pseudodifferential operators. As the theory for pseudodifferential operator is extensive, we shall only give a brief description and state what we need from it. For a thorough introduction, see for example [Shu01] or [Hör07].

Given a differential operator $D \in \mathrm{DO}^m(M; E, F)$, we can express it locally as

$$D = \sum_{|\alpha| \le m} c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}.$$

Using the Fourier transform, we see that this local expression can be written as

$$Df(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\xi(x-y)} \sigma(D)(x,\xi) f(y) \,\mathrm{d}y\mathrm{d}\xi \tag{3}$$

where

$$\sigma(D)(x,\xi) = \sum_{|\alpha| \le m} c_\alpha(x)(i\xi)^\alpha$$

is called the full symbol of $D$. A pseudodifferential operator $T \in \Psi\mathrm{DO}^m(M; E, F)$ for $m \in \mathbb{R}$ is a continuous operator $T : C_c^\infty(M; E) \to C^\infty(M; F)$ which locally is on the form (3) with a full symbol $\sigma(T)(x,\xi) \in \mathrm{Hom}(E_x, F_y)$ such that all matrix entries $\sigma(T)_{ij}(x,\xi)$ satisfies that

$$\left| \frac{\partial^{\alpha+\beta}}{\partial \xi^\alpha \partial x^\beta} \sigma_{ij}(T)(x,\xi) \right| \lesssim (1 + |\xi|)^{m-|\alpha|}$$

for all multi-indices $\alpha, \beta \in \mathbb{N}^n$. For more detail we refer to [Shu01] or [Hör07]. Note that any differential operator is a pseudodifferential operator although pseudodifferential operators are not necessarily local operators in contrast to differential operators.

We say that $T \in \Psi\mathrm{DO}^m(M; E, F)$ is properly supported if $T : C_c^\infty(M; E) \to C_c^\infty(M; F)$, in which case we can define a distributional adjoint $T^\dagger : \mathscr{D}'(M; F) \to \mathscr{D}'(M; E)$ as $\langle T^\dagger f, g \rangle := \langle f, Tg \rangle$ for $f \in \mathscr{D}'(M; F)$ and $g \in C_c^\infty(M; E)$. We have that

- $\mathrm{DO}^m(M;E,F) \subseteq \Psi\mathrm{DO}^m(M;E,F)$;
- If $T \in \Psi\mathrm{DO}^m(M;E,F)$ is properly supported then $T^\dagger \in \Psi\mathrm{DO}^m(M;F,E)$;
- If $T \in \Psi\mathrm{DO}^{m_1}(M;E,F)$ is properly supported then $ST \in \Psi\mathrm{DO}^{m_1+m_2}(M;E,G)$ for any $S \in \Psi\mathrm{DO}^{m_2}(M;F,G)$;
- Let $T \in \Psi\mathrm{DO}^m(M;E,F)$ and $a \in C_c^\infty(M)$. If $m \leq 0$ then $aT$ extends to a bounded operator $aT\colon L^2(M;E) \to L^2(M;F)$ and if $m < 0$ this operator is compact.

The full symbol of a differential operator as in (3) is only a local description. However, coordinate changes will only change the terms of lower order differentiation in a differential operator, and we can define a global symbol by only taking into account the highest order terms. For a differential operator $D \in \mathrm{DO}^m(M;E,F)$ that locally takes the form $D = \sum_{|\alpha|\leq m} c_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$ we define the principal symbol as the map

$$\sigma^m(D)\colon T^*M \to \mathrm{Hom}(E,F)$$

using the local expression

$$\sigma^m(D)(x,\xi) = \sum_{|\alpha|=m} c_\alpha(x)(i\xi)^\alpha$$

for $(x,\xi) \in T^*M$ [Shu01, Section 4]. We say that $D$ is elliptic of order $m$ if $\sigma^m(D)(x,\xi)$ is invertible for all $(x,\xi) \in T^*M$ such that $\xi \neq 0$.

For instance, $\mathrm{d}^\dagger\mathrm{d} \in \mathrm{DO}^2(M)$ is elliptic where $\mathrm{d}$ is the exterior differential on scalar functions. Locally in orthonormal coordinates $\mathrm{d}^\dagger\mathrm{d}$ takes the form of a Laplacian on $\mathbb{R}^n$

$$\mathrm{d}^\dagger\mathrm{d} = -\sum_k \frac{\partial^2}{\partial x_k^2} + \text{lower order terms}$$

and therefore has principal symbol $\sigma^2(\mathrm{d}^\dagger\mathrm{d})(x,\xi) = |\xi|^2$ for $\xi \in T_xM$.

Using that the symbol of an elliptic differential operator is invertible, one can construct a so called parametrix which is an inverse up to arbitrarily low order. That is, if $D \in \mathrm{DO}^m(M;E,F)$ is elliptic, then for any $s > 0$ there is a properly supported $Q \in \Psi\mathrm{DO}^{-m}(M;F,E)$ such that

$$1 - QD \in \Psi\mathrm{DO}^{-s}(M;E,E) \quad \text{and} \quad 1 - DQ \in \Psi\mathrm{DO}^{-s}(M;F,F)$$

[Shu01, Theorem 5.1].

To express results regarding regularity, for integers $s \geq 0$ we define the compactly supported Sobolev spaces

$$H^s_{\mathrm{comp}}(M;E) := \{\, f \in \mathscr{D}'(M;E) \;:\; Pf \in L^2(M;F) \quad \text{for any} \quad P \in \mathrm{DO}^s(M;E,F)\,\}$$

and the local Sobolev spaces

$$H^s_{\mathrm{loc}}(M;E) := \{ f \in \mathscr{D}'(M;E) \ : \ af \in H^s_{\mathrm{comp}}(M;E) \quad \text{for any} \quad a \in C^\infty_c(M) \}.$$

These definitions are not the same as in [Shu01], but they are equivalent. The maps that define these Sobolev spaces also induce a Fréchet spaces topology and $C^\infty_c(M;E) \subseteq H^s_{\mathrm{comp}}(M;E)$ is dense for any $s \geq 0$. Hence, $H^m_{\mathrm{comp}}(M;E) \subseteq \operatorname{Dom} D_{\min}$ for any $D \in \mathrm{DO}^m(M;E,F)$. Also, the inclusion $H^s_{\mathrm{comp}}(M;E) \subseteq L^2(M;E)$ is compact for $s > 0$ and for any properly supported $T \in \Psi\mathrm{DO}^m(M;E,F)$ we have that

$$TH^s_{\mathrm{comp}}(M;E) \subseteq H^{s-m}_{\mathrm{comp}}(M;F)$$

and

$$TH^s_{\mathrm{loc}}(M;E) \subseteq H^{s-m}_{\mathrm{loc}}(M;F)$$

if $s \geq 0$ and $s - m \geq 0$.

**Proposition 3.6** (Interior regularity). *If $D \in \mathrm{DO}^m(M;E,F)$ is elliptic of order $m$ then*

$$\operatorname{Dom} D_{\max} \subseteq H^m_{\mathrm{loc}}(M;E)$$

*and in particular*

$$a \operatorname{Dom} D_{\max} \subseteq \operatorname{Dom} D_{\min}$$

*for any $a \in C^\infty_c(M)$.*

*Proof.* Let $Q \in \Psi\mathrm{DO}^{-m}(M;F,E)$ be such that $1 - QD \in \Psi\mathrm{DO}^{-m}(M;E,E)$. If $f \in \operatorname{Dom} D_{\max}$, then $f \in L^2(M;E)$ and $Df \in L^2(M;F)$ and hence

$$f = (1 - QD)f + QDf \in H^m_{\mathrm{loc}}(M;E).$$

The second statement now follows from that $aH^m_{\mathrm{loc}}(M;E) \subseteq H^m_{\mathrm{comp}}(M;E)$ by definition and that $H^m_{\mathrm{comp}}(M;E) \subseteq \operatorname{Dom} D_{\min}$. □

Note in particular that if $M$ a closed manifold then $D_{\min} = D_{\max}$ for an elliptic differential operator $D$ since then $H^m_{\mathrm{comp}}(M;E) = H^m_{\mathrm{loc}}(M;E)$.

**Proposition 3.7.** *If $D \in \mathrm{DO}^m(M;E,F)$ is elliptic of order $m$ then*

$$\operatorname{Ran}(1 + (D_{\min})^* D_{\min})^{-\frac{1}{2}t} \subseteq H^{tm}_{\mathrm{loc}}(M;E)$$

*for any $0 \leq t \leq 1$.*

*Proof.* By definition, it is enough to show that

$$\operatorname{Ran} a(1+(D_{\min})^* D_{\min})^{-\frac{1}{2}t} \subseteq H^{tm}_{\mathrm{comp}}(M;E)$$

for any and $a \in C^\infty_c(M)$.

Fix an $a \in C^\infty_c(M)$ and let $\Omega \subseteq M$ be an open pre-compact domain containing the support of $a$. Then

$$aL^2(M;E) \subseteq L^2(\Omega;E)$$

and by density of smooth sections

$$a \operatorname{Dom} D_{\min} \subseteq \operatorname{Dom} D_{\Omega,\min}$$

where $D_\Omega$ is the restriction of $D$ seen as an unbounded operator on $L^2(\Omega;E)$. Using interpolation of Hilbert spaces we then have that

$$a[L^2(M;E), \operatorname{Dom} D_{\min}]_t \subseteq [L^2(\Omega,;E), \operatorname{Dom} D_{\Omega,\min}]_t$$

for $0 \le t \le 1$ where $\operatorname{Ran}(1+(D_{\min})^* D_{\min})^{-\frac{1}{2}t} = [L^2(M;E), \operatorname{Dom} D_{\min}]_t$ by [See71, Theorem 3]. Since $D$ is elliptic, $\operatorname{Dom} D_{\Omega,\min}$ agrees with $H^m_0(\Omega;E)$ which is the Sobolev space vanishing at the boundary $\partial\Omega$. By embedding $\Omega$ in a closed manifold we can refer to classical results to obtain that $[L^2(\Omega,;E), H^m_0(\Omega;E)]_t \subseteq H^{tm}_0(\Omega;E)$, see for instance [Tay11, Section 4.2]. Lastly, we note that $H^{tm}_0(\Omega;E) \subseteq H^{tm}_{\mathrm{comp}}(M;E)$ which concludes the proof. □

**Proposition 3.8.** *Let $D \in \mathrm{DO}^m(M;E,E)$ be formally self-adjoint, that is $D^\dagger = D$, and elliptic of order $m$, then $(C^\infty_c(M), L^2(M;E), D_{\min})$ defines a local spectral triple of order $m$.*

*Proof.* Fix an $a \in C^\infty_c(M)$ and let $\chi \in C^\infty_c(M)$ be such that $a = a\chi$. Note that $[D,a] \in \mathrm{DO}^{m-1}(M;E,E)$ by the Leibniz rule and $[D,a] = \chi[D,a]$ since $D$ is a local operator.

By Proposition 3.6

$$\operatorname{Ran}(1+(D_{\min})^* D_{\min})^{-\frac{1}{2}} = \operatorname{Dom} D_{\min} \subseteq H^m_{\mathrm{loc}}(M;E).$$

Hence, by using that $aH^m_{\mathrm{loc}}(M;E) \subseteq H^m_{\mathrm{comp}}(M;E)$ and that $H^m_{\mathrm{comp}}(M;E) \subseteq L^2(M;E)$ is compact, we obtain that

$$a(1+(D_{\min})^* D_{\min})^{-\frac{1}{2}} \in \mathcal{K}(L^2(M;E)).$$

Similarly, by using that $[D,a] = \chi[D,a]$ and therefore $[D,a]H^m_{\mathrm{loc}}(M;E) \subseteq H^{m-1}_{\mathrm{comp}}(M;E)$, we obtain that

$$[D_{\min}, a](1+(D_{\min})^* D_{\min})^{-\frac{1}{2}} \in \mathcal{K}(L^2(M;E)).$$

By Proposition 3.7

$$\operatorname{Ran}(1 + (D_{\min})^* D_{\min})^{-\frac{1}{2}+\frac{1}{2m}} \subseteq H^{m-1}_{\mathrm{loc}}(M, E)$$

and using that $[D, a] = \chi[D, a]$ we see that $[D_{\min}, a](1 + (D_{\min})^* D_{\min})^{-\frac{1}{2}+\frac{1}{2m}}$ is bounded. Lastly, the domain inclusion

$$a \operatorname{Dom} D_{\max} \subseteq \operatorname{Dom} D_{\min}$$

is proven in Proposition 3.6. □

For an elliptic differential operator $D$, using a parametrix as in proof of Proposition 3.6 one can also show that if $Df \in H^s_{\mathrm{loc}}(M; F)$ then $f \in H^{s+m}_{\mathrm{loc}}(M; E)$. Since $\bigcap H^s_{\mathrm{loc}}(M; E) = C^\infty(M; E)$ this shows that elliptic operator are hypoelliptic in the sense that if $Df \in C^\infty(M; F)$ then $f \in C^\infty(M; E)$.

## 3.3 Hörmander's sum of squares and Heisenberg-elliptic operators

Not all hypoelliptic differential operators are elliptic. For example, we can consider those operators stemming from Hörmander's sum of squares theorem [Hör67] that essentially classifies all hypoelliptic scalar second-order real-valued operators.

**Theorem 3.9** (Hörmander's sum of squares). *For an open set $\Omega \subseteq \mathbb{R}^n$, let $X_0, X_1, \dots, X_m \in C^\infty(\Omega; T\Omega)$ and $c \in C^\infty(\Omega)$, then the differential operator*

$$c + X_0 + \sum_{k=1}^{m} X_k^2$$

*is hypoelliptic if the vector fields of the form*

$$X_{k_1}, [X_{k_1}, X_{k_2}], [X_{k_1}, [X_{k_2}, X_{k_3}]], \dots, [X_{k_1}, [X_{k_2}, \dots, X_{k_r}]], \dots$$

*span $T\Omega$ at each point.*

We call the condition on the vector fields in Theorem 3.9 Hörmander's condition. An example of such vector fields on $\mathbb{R}^3$ could be $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ for which $[X, Y] = \frac{\partial}{\partial z}$. These are the left invariant vector fields on the Heisenberg group $\mathbb{H}^3$.

To obtain a notion of ellipticity for differential operators as in Theorem 3.9, we turn to the Heisenberg-pseudodifferential calculus on filtered manifolds [Mel82, BG88, Pon08, vEY19, DH22]. A filtered manifold $M$ is a manifold together with a sequence of vector sub-bundles

$$H = H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots \subseteq TM$$

such that $H_p = TM$ if $p \geq r$ for some $r$ and $[C^\infty(M; H_p), C^\infty(M; H_q)] \subseteq C^\infty(M; H_{p+q})$ [Mor93, vEY19]. Additionally, we will say that $M$ is a bracket-generated filtered manifold if $H$ is bracket generating in the sense that

$$C^\infty(M; H_{p+1}) = [C^\infty(M; H_p), C^\infty(M; H_1)] + C^\infty(M; H_1).$$

In literature, a filtered manifold is sometimes instead called a Carnot manifold [CP19], although in [Has14] which we will reference, the term Carnot manifold is used to mean a bracket-generated filtered manifold. A bracket-generated filtered manifold can be called an equiregular Carnot-Carathéodory manifold [Gro96, p. 98].

We can construct a globally defined differential operator

$$\Delta_H := \mathrm{d}_H^\dagger \mathrm{d}_H \in \mathrm{DO}^2(M)$$

called the sub-Laplacian on a filtered manifold $M$, where

$$\mathrm{d}_H \colon C^\infty(M) \to C^\infty(M; H^*)$$

is the horizontal exterior differential defined as $\mathrm{d}_H a(X) = X(a)$ for $X \in C^\infty(M; H)$. If $M$ is bracket-generated then $\Delta_H$ is hypoelliptic by Theorem 3.9, but it is not elliptic in the classical sense.

To define Heisenberg ellipticity for differential operators we will follow constructions found in [DH22]. For a filtered manifold $M$ we let $\mathrm{DO}_H^m(M; E, F)$ denote differential operators that locally are a sum of terms of the form

$$cX_1 X_2 \dots X_l \tag{4}$$

for $c \in C^\infty(M; \mathrm{Hom}(E, F))$ and $X_k \in C^\infty(M; H_{p_k})$ are vector fields such that $\sum_{k=1}^l p_k \leq m$. Note that $\mathrm{DO}_H^m(M; E, F) \subseteq \mathrm{DO}^m(M; E, F)$. Towards defining the Heisenberg principal symbol of a differential operator, consider the vector bundle

$$\mathfrak{t}_H M := \bigoplus H_p / H_{p-1}.$$

From the filtration of $TM$ we have that $\mathfrak{t}_H M$ inherits the structure of a graded nilpotent Lie algebra bundle since for each point $x \in M$

$$[\cdot, \cdot]_x \colon C^\infty(M; TM) \times C^\infty(M; TM) \to H_{p,x} / H_{p-1,x}$$

is $C^\infty(M)$-linear and $[X_1, X_2]_x \in H_{p-1,x}$ for $X_k \in C^\infty(M; H_{p_k})$ such that $p_1 + p_2 < p$ [vEY17, Section 3]. Let $T_H M$ denote for the bundle of simply connected nilpotent Lie groups integrating to $\mathfrak{t}_H M$. We call $T_H M$ the oscillating Lie groupoid. Let $\mathcal{U}(\mathfrak{t}_{H,x} M)$ denote the universal enveloping algebra of $\mathfrak{t}_{H,x} M$, that is, the algebra generated by elements in the Lie algebra with its commutator rules imposed. Note that $\mathcal{U}(\mathfrak{t}_{H,x} M)$ inherits the grading from $\mathfrak{t}_{H,x} M$, and we let $\mathcal{U}_m(\mathfrak{t}_H M)$ denote the degree $m$ part. For a differential operator $D \in \mathrm{DO}^m_H(M; E, F)$ we define the Heisenberg principal symbol as the map

$$\sigma^m_H(D) \colon M \to \mathrm{Hom}(E; F) \otimes \mathcal{U}_m(\mathfrak{t}_H M)$$

obtained linearly from local expressions as $\sigma^m_H(cX_1X_2 \dots X_l) = c \otimes [X_1][X_2] \dots [X_l]$ where $X_k \in C^\infty(M; H_{p_k})$ such that $\sum p_k = m$. For instance, locally $\sigma^2_H(\Delta_H) = -\sum [X_k]^2$ for some orthogonal vector fields with values in $H$.

On a filtered manifold $M$, we say that $D \in \mathrm{DO}^m_H(M; E, F)$ satisfies the *Rockland condition* if for every point $x \in M$ and every non-trivial unitary irreducible representation $\pi \colon T_{H,x}M \to U(\mathcal{H})$ the map

$$\mathrm{d}\pi(\sigma^m_H(D)(x)) \colon E_x \otimes \mathcal{H}^\infty \to F_x \otimes \mathcal{H}^\infty$$

is injective, where $\mathcal{H}^\infty \subseteq \mathcal{H}$ denotes the smooth vector with respect to $\pi$. Melin showed in his seminal unpublished preprint [Mel82] that if $D$ satisfies the Rockland condition then $D$ is hypoelliptic (see also [CGGP92, DH22]). We say that $D$ is Heisenberg-elliptic if $\mathrm{d}\pi(\sigma^m_H(D)(x))$ is bijective at every point and every representation, thereby assuring that $D$ and $D^\dagger$ are hypoelliptic.

Note that if $M$ is trivially filtered, then Heisenberg-ellipticity recovers classical ellipticity. In this case, $T_H M \cong TM$ as Lie group bundles and $\mathfrak{t}_H M \cong TM$ as Lie algebra bundles where each fiber of $TM$ has the trivial Lie bracket. Therefore, for each point $x \in M$ the irreducible unitary representations of $T_x M \cong \mathbb{R}^n$ can be parametrized by $\xi \in T^*_x M \cong (\mathbb{R}^n)^*$ so that

$$\pi_\xi(x) = e^{i\xi(x)} \quad \text{and} \quad \mathrm{d}\pi_\xi(X) = i\xi(X)$$

acting on $\mathbb{C}$ for $X \in T_x M \cong \mathbb{R}^n$. So if $D \in \mathrm{DO}^m(M; E, F)$ locally takes the form $D = \sum_{|\alpha| \leq m} c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$ then for $\xi \in T^*_x M$ we have that

$$\mathrm{d}\pi_\xi(\sigma^m_H(D)(x)) = \sum_{|\alpha|=m} c_\alpha(x) \mathrm{d}\pi_\xi \left( \frac{\partial^\alpha}{\partial x^\alpha} \right) = \sum_{|\alpha|=m} c_\alpha(x)(i\xi)^\alpha = \sigma^m(D)(x, \xi) \tag{5}$$

as a map $E_x \to F_x$.

For another example, we can consider the situation when $T_H M$ is isomorphic to the Heisenberg group $\mathbb{H}^3$ at each point. By Darboux theorem, this is true if and only if $M$ is a three-dimensional contact manifold (see later in subsection 7.3). The Heisenberg group $\mathbb{H}^3$ is a simply connected nilpotent Lie group and its associated Lie algebra $\mathfrak{h}_3$ is spanned by three elements $X, Y, Z \in \mathfrak{h}_3$ such that $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. By the Stone-von Neumann theorem [Fol89, Chapter 1], there are two types of irreducible unitary representations of $\mathbb{H}^3$. The abelian representations $\pi_\xi$ of $\mathbb{H}^3$ are parametrized by $\xi \in \mathbb{R}^2$ and acts on $\mathbb{C}$ for which

$$\mathrm{d}\pi_\xi(X) = i\xi_1, \quad \mathrm{d}\pi_\xi(Y) = i\xi_2 \quad \text{and} \quad \mathrm{d}\pi_\xi(Z) = 0.$$

The Schrödinger representations $\pi_\hbar$ of $\mathbb{H}^3$ are parametrized by $\hbar \in \mathbb{R} \setminus 0$ and acts on $L^2(\mathbb{R})$ for which

$$\mathrm{d}\pi_\hbar(X) = ix, \quad \mathrm{d}\pi_\hbar(Y) = -\hbar\frac{\partial}{\partial x} \quad \text{and} \quad \mathrm{d}\pi_\hbar(Z) = i\hbar$$

with the smooth vectors $\mathcal{H}^\infty$ being the space of Schwartz functions $\mathcal{S}(\mathbb{R})$. See Figure 2 for a schematic picture of the set of irreducible unitary representations of the Heisenberg group.

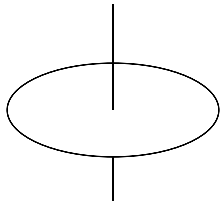

**Figure 2:** The irreducible unitary representations on the Heisenberg group $\mathbb{H}^3$. The plane represents the abelian representations and the line represents the Schrödinger representations.

For instance, we can now ensure that the sub-Laplacian $\Delta_H$ on a contact manifold of three dimensions is Heisenberg-elliptic. For the abelian representations we obtain

$$\mathrm{d}\pi_\xi(\sigma_H^2(\Delta_H)(x)) = |\xi|^2$$

acting on $\mathbb{C}$, and see that $\mathrm{d}\pi_\xi(\sigma_H^2(\Delta_H)(x))$ is invertible if $\xi \neq 0$, and for the Schrödinger representations we obtain harmonic oscillators

$$\mathrm{d}\pi_\hbar(\sigma_H^2(\Delta_H)(x)) = -\hbar^2\frac{\partial^2}{\partial x^2} + x^2$$

acting on $\mathcal{S}(\mathbb{R})$, and see that $\mathrm{d}\pi_\hbar(\sigma_H^2(\Delta_H)(x))$ is invertible if $\hbar \neq 0$.

Note that the parameter $\xi$ for the abelian representations $\pi_\xi$ for the Heisenberg group $\mathbb{H}^3$ can be seen as coming from $(x, \xi) \in H^* \subseteq \mathfrak{t}_H M^* = \mathfrak{h}_3^*$ using the theory from [CG90]. In particular, the abelian representations capture the classical principal symbol restricted to $H^* \subseteq T^*M$ for a differential operator $D \in \mathrm{DO}_H^m(M; E, F)$ since for $\xi \in H_x^* \subseteq T_x^*M$

$$\mathrm{d}\pi_\xi(\sigma_H^m(D)(x)) = \sigma^m(D)(x, \xi) \tag{6}$$

which can be seen in the same way as in (5). Moreover, (6) holds for any filtration since any simply connected graded Lie group will have a strand of irreducible representations corresponding to its lowest degree in the grading as can be seen from [CG90, Theorem 2.2.1]. Hence, any differential operator $D \in \mathrm{DO}_H^m(M; E, F)$ that is Heisenberg-elliptic of order $m$ must satisfy that $\sigma^m(D)(x, \xi) \in \mathrm{Hom}(E_x, F_x)$ is invertible for all $(x, \xi) \in H^*$ such that $\xi \neq 0$, where we are using the classical principal symbol.

For a general simply connected graded nilpotent Lie group, one can draw a similar schematic picture of the irreducible unitary representations as in Figure 2, see for instance Figure 3. The theory behind this can be found in [CG90] which is based on Kirillov's orbit method [Kir04], and states that equivalence classes of irreducible unitary representations of a simply connected nilpotent Lie group $G$ are naturally parametrized by the orbits of $\mathfrak{g}^*$ acted on by $G$ via the co-adjoint action.

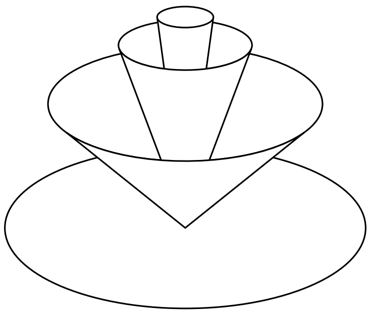

**Figure 3:** A schematic picture of the irreducible representations a simply connected graded Lie group. The flat disc at the bottom here represents the abelian representations and cones each higher order in the grading. In the illustration, the filtration has length four. Note that angles between the cones in the picture are purely for visualization purposes and are not suggested to have mathematical meaning.

The Heisenberg-pseudodifferential calculus $\Psi\mathrm{DO}_H^m(M;E,F)$ as defined in [vEY19] (see also [DH22]) satisfies that $\mathrm{DO}_H^m(M;E,F) \subseteq \Psi\mathrm{DO}_H^m(M;E,F)$ and similar properties to the classical pseudodifferential calculus $\Psi\mathrm{DO}^m(M;E,F)$ that we presented in previous section. In particular, with the Sobolev spaces defined in [DH22] we can prove analogous statements to Proposition 3.6, Proposition 3.7 and Proposition 3.8 for any Heisenberg-elliptic differential operator with one caveat. To follow the proof of Proposition 3.7 for a Heisenberg-elliptic differential operator $D$ we need to be able to embed any arbitrarily small pre-compact open set $\Omega \subseteq M$ in a closed filtered manifold $\widetilde{M}$ with matching filtration to $M$ such that we can extend $D_\Omega$ to a Heisenberg-elliptic differential operator on $\widetilde{M}$. Then we can refer to the results for complex powers in [DH20] to obtain the interpolation results we require. We expect that such $\widetilde{M}$ can always be found, although we will not attempt to show this in this text. Taking this caveat into consideration, the analog of Proposition 3.8 for Heisenberg-elliptic differential operators that we obtain is the following.

**Proposition 3.10.** *On a filtered manifold $M$, let $D \in \mathrm{DO}_H^m(M;E,E)$ be formally self-adjoint, that is $D^\dagger = D$, and Heisenberg-elliptic of order $m$. If we assume that either*

1. *$M$ is a domain in a closed manifold over which $D$ extends Heisenberg-elliptically, or*
2. *$M$ is locally filter isomorphic to a nilpotent Lie group,*

*then* $(C_c^\infty(M), L^2(M;E), D_{\min})$ *defines a local spectral triple of order m.*

# 4 Reconstructing geometry

In [Con94], Connes proposes that a spectral triple is the correct analog of a Riemannian manifold. This is motivated by considering the Dirac operator on a closed Riemannian spin$^c$ manifold, from which Connes is able to reconstruct several geometric phenomena [Con94, Chapter 6]. Here we will study two geometrical properties that are still captured by higher-order spectral triples build from elliptic or Heisenberg-elliptic differential operators, namely dimension and metric.

The results in this section are not present in any of the papers included in this thesis, and they are included to give more context and further motivate the studied objects. To the author's knowledge, the results in this section are novel.

## 4.1 Weyl's law and dimension

The dimension is meant to be seen from spectral asymptotics of the operator $D$ in a spectral triple and the inspiration comes from Weyl's law. To understand why, we will recall Weyl's in its simplest form.

Consider the Dirichlet Laplacian $\Delta_D$ on the unit hypercube in $\mathbb{R}^n$. By a separation of variables, we can see that the eigenfunctions of $\Delta_D$ consists of $\prod \sin\left(\frac{k_i x_i}{\pi}\right)$ for each $\{ k_i \} \in \{ 1, 2, \dots \}^n$ with corresponding eigenvalues $\prod \pi^2 k_i^2$. Note that $\prod \pi^2 k_i^2 \leq \lambda$ if and only if the grid point $\{ k_i \} \in \mathbb{R}^n$ lies inside the ball of radius $\frac{1}{\pi}\sqrt{\lambda}$, and that the ball of radius $\frac{1}{\pi}\sqrt{\lambda}$ must contain fewer grid points with positive coordinates then its volume divided by $2^n$. Hence, if we introduce the counting function

$$N(\Delta_D, \lambda) := \#\{ k : \lambda_k(\Delta_D) \leq \lambda \}$$

where $\lambda_k(\Delta_D)$ is the $k$:th eigenvalue of $\Delta_D$ counting multiplicities, we see that

$$N(\Delta_D, \lambda) \leq \frac{\omega_n}{(2\pi)^n} \lambda^{\frac{n}{2}}$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. In the other hand, the volume of the ball of radius $\frac{1}{\pi}\sqrt{\lambda} - 1$ is guaranteed to be less than the number grid points $\{ k_i \} \in \mathbb{R}^n$ inside the ball of radius $\frac{1}{\pi}\sqrt{\lambda}$. Hence,

$$\lim_{\lambda \to \infty} \frac{N(\Delta_D, \lambda)}{\lambda^{\frac{n}{2}}} = \frac{\omega_n}{(2\pi)^n} \tag{7}$$

which is Weyl's law in the case of the Dirichlet Laplacian $\Delta_D$ on the unite hypercube in $\mathbb{R}^n$.

Weyl's law states that (7) holds for the Dirichlet Laplacian on any bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ with the asymptotic constant in (7) multiplied by the volume of $\Omega$ [Hör07, Corollary

17.5.8]. In particular, since $N(\Delta_D, \lambda_k(\Delta_D)) \geq k$ we obtain from Weyl's law that $\lambda_k(\Delta_D) \gtrsim k^{\frac{2}{n}}$ and hence

$$\sum^{\infty} |\lambda_k(\Delta_D)|^{-p} < \infty$$

for any $p > \frac{n}{2}$. We can phrase this as $\Delta_D^{-1} \in \mathcal{L}^p(L^2(\Omega))$ where

$$\mathcal{L}^p(\mathcal{H}) := \{T \in \mathcal{K}(\mathcal{H}) \ : \ \sum_k^{\infty} |\lambda_k(T)|^p \leq \infty\}$$

are called the Schatten $p$-ideal. We have that $\mathcal{L}^p(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$ that is not closed in the operator norm.

The same type of eigenvalue asymptotics as Weyl's law also holds for any elliptic differential operator $D$ order $m > 0$ on a closed manifold $M$ [Shu01, Theorem 15.2] in the sense that

$$\lim_{\lambda\to\infty} \frac{N(D^\dagger D, \lambda)}{\lambda^{\frac{\dim M}{2m}}} \tag{8}$$

is a non-zero finite value. In particular, $(1 + D^*D)^{-\frac{1}{2}} \in \mathcal{L}^p(L^2(M; E))$ for any $p > \frac{\dim M}{m}$. An interesting note is that the order of a pseudodifferential operator can in some cases be directly determined by this type of eigenvalue asymptotics, see [BT10].

A local spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be $p$-summable if $(1+D^*D)^{-\frac{1}{2}} \in \mathcal{L}^p(\mathcal{H})$ [Con94, p. 559]. As an analog to (8) we will say that the spectral dimension of $(\mathcal{A}, \mathcal{H}, D)$ is the infimum of all $mp$ such that $(\mathcal{A}, \mathcal{H}, D)$ is of order $m$ and $p$-summable. In particular, if $M$ is a closed manifold then any spectral triple for $C(M)$ defined from an elliptic differential operator has spectral dimension $\dim M$. See Theorem 2.2 in Paper III for the relation to summability of bounded Kasparov $(A, \mathbb{C})$-modules.

On a closed filtered manifold $M$ with filtration $H = H_1 \subseteq H_2 \subseteq \ldots H_r = TM$, we can introduce the homogenous dimension of $M$ as $\dim_H M = \sum_k k \operatorname{rank}(H_k)$. If $D \in \mathrm{DO}_H^m(M; E, F)$ is Heisenberg-elliptic of order $m$ then by [DH20, Corollary 3]

$$\lim_{\lambda\to\infty} \frac{N(D^\dagger D, \lambda)}{\lambda^{\frac{\dim_H M}{2m}}}$$

is a finite non-zero value. Hence, any spectral triple for $C(M)$ defined from a Heisenberg-elliptic operator has spectral dimension $\dim_H M$. In Paper III, we present similar summability results for the strictly tangled spectral triples in Corollary 5.14.

## 4.2 Kantorovich metric and distance

The geodesic distance between two points $x, y \in M$ on a Riemannian manifold $M$ is defined as the length of the shortest (smooth) path between $x$ and $y$. That is, we have a metric

$$d_{\mathrm{geo}}(x, y) := \inf\left\{ \int_0^1 |\dot{\gamma}(t)|\, \mathrm{d}t \ : \ \gamma \in C^\infty([0,1], M), \gamma(0) = x, \gamma(1) = y \right\}$$

where $\dot{\gamma}(t) \in T_{\gamma(t)}M$. Now, for any function $a \in C^\infty(M)$ and a path $\gamma$ form $x$ to $y$, we have that

$$\begin{aligned} a(y) - a(x) &= a(\gamma(1)) - a(\gamma(2)) \\ &= \int_0^1 \frac{\partial}{\partial t} a(\gamma(t))\,\mathrm{d}t \\ &= \int_0^1 \mathrm{d}a(\dot{\gamma}(t))\,\mathrm{d}t \\ &\leq \int_0^1 |\mathrm{d}a(\gamma(t))|\,|\dot{\gamma}(t)|\,\mathrm{d}t \\ &\leq \|\mathrm{d}a\|_\infty \int_0^1 |\dot{\gamma}(t)|\,\mathrm{d}t \end{aligned}$$

and hence $|a(y) - a(x)| \leq \|\mathrm{d}a\|_\infty\, d_{\mathrm{geo}}(x, y)$ for any $x, y \in M$. With more work one can for each $\epsilon > 0$ find an $a \in C^\infty(M)$ such that $|a(y) - a(x)| + \epsilon > \|\mathrm{d}a\|_\infty\, d_{\mathrm{geo}}(x, y)$. Therefore, the geodesic distance can also be expressed as

$$d_{\mathrm{geo}}(x, y) = \sup_{a \in C^\infty(M)} \{\, |a(x) - a(y)| \,:\, \|\mathrm{d}a\|_\infty \leq 1 \,\}$$

which is also called the Kantorovich metric. On probability measures, a dual formulation is called the Monge-Kantorovich metric or the Wasserstein distance.

For a spin$^c$ Dirac operator $\not{D}$ and a function $a \in C^\infty(M)$, $[\not{D}, a] = c(\mathrm{d}a)$ where $c\colon TM \to \mathrm{End}(\not{S})$ is the Clifford action. In particular, $|c(\mathrm{d}a)_x| = |\mathrm{d}a_x|$ at each point $x \in M$ and we obtain that $\|[\not{D}, a]\| = \|\mathrm{d}a\|_\infty$. Hence, we recover the geodesic distance from the spectral triple $(C^\infty(M), L^2(M; \not{S}), \not{D})$ as

$$d_{\mathrm{geo}}(x, y) = \sup_{a \in C^\infty(M)} \{\, |a(x) - a(y)| \,:\, \|[\not{D}, a]\| \leq 1 \,\}.$$

With this in mind, Connes extend the notion of metric to noncommutative geometry by showing that a first-order spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for $A$ induces an extended metric on the state space $\hat{A}$ using the so-called called the Connes metric

$$d(\mu, \nu) := \sup_{a \in \mathcal{A}} \{\, |\mu(a) - \nu(a)| \,:\, \|[D, a]\| \leq 1 \,\}$$

for $\mu, \nu \in \hat{A}$ [Con89]. Rieffel later examined when such a metric induces the natural topology $\hat{A}$, the weak $*$-topology, and introduced the related notion of compact quantum metric spaces [Rie98, Rie99]. For resent results on compact quantum metric spaces see for instance [Kaa24].

We will instead keep to the geometric situation and consider higher-order differential operators. For a differential operator $D$ of order higher than one, we cannot use the same reformulation of the metric as $[D, a]$ is not a bounded operator. However, repeated commutators will eventually be a bounded operator, and we have the following identity.

**Lemma 4.1.** *For a differential operator $D \in \mathrm{DO}^m(M; E, F)$ and $a \in C^\infty(M)$ we have that*

$$\frac{1}{m!}[\dots[D,a]\dots,a] = \sigma^m(D)(\mathrm{d}a) \in C^\infty(M; \mathrm{Hom}(E;F))$$

*where we are taking $m$ consecutive commutators with $a$.*

*Proof.* For linear operators $a$ and $D_i$ for $i = 1, \dots, m$ we have that

$$\left[\prod_i D_i, a\right] = \sum_j \left(\prod_{i<j} D_i\right)[D_j, a]\left(\prod_{i>j} D_i\right).$$

If such that $[[D_i, a], a] = 0$, then which by induction we can show that

$$\frac{1}{m!}\left[\dots\left[\prod_i D_i, a\right]\dots, a\right] = \prod_i [D_i, a]$$

where we are taking $m$ consecutive commutators. Note also that if we take $m+1$ commutators the expression becomes zero. Expressing a differential operator $D \in \mathrm{DO}^m(M; E, F)$ locally as $D = \sum_{|\alpha|\le m} c_\alpha(x)\frac{\partial^\alpha}{\partial x^\alpha}$ we see that

$$\frac{1}{m!}[\dots[D,a]\dots,a] = \sum_{|\alpha|=m} c_\alpha(x)\frac{\partial^\alpha a}{\partial x^\alpha}$$

which globally is equal to $\sigma^m(D)(\mathrm{d}a)$. □

For a higher-order spectral triple defined from a differential operator $D$ of order $m$, we propose a generalization of the Connes metric as

$$d_D(x,y) := \sup_{a\in C^\infty(M)}\left\{ |a(x) - a(y)| \;:\; \left\|\frac{1}{m!}[\dots[D,a]\dots,a]\right\| \le 1\right\}. \tag{9}$$

For instance, $d_{\mathrm{d}^\dagger\mathrm{d}}$ for the Laplace-type operator $\mathrm{d}^\dagger\mathrm{d} \in \mathrm{DO}^2(M)$ recovers the geodesic metric (which is implicitly shown in [Has14, Theorem 7.2.2]). We will now show that any elliptic differential operator $D \in \mathrm{DO}^m(M; E, F)$ defines a metric $d_D$ on $M$ that is locally equivalent to the geodesic metric on $M$.

**Lemma 4.2.** *Let $D \in \mathrm{DO}^m(M; E, F)$ be elliptic of order $m$, then for any compact $K \subseteq M$ there are constants $A_K, B_K > 0$ such that for any $a \in C^\infty(M)$*

$$A_K\|\mathrm{d}a\|_\infty^m \le \left\|\frac{1}{m!}[\dots[D,a]\dots,a]\right\| \le B_K\|\mathrm{d}a\|_\infty^m$$

*as operators on $L^2$-sections over $K$. In particular, for any $x, y \in K$*

$$A_K^{\frac{1}{m}} d_{\mathrm{geo}}(x,y) \le d_D(x,y) \le B_K^{\frac{1}{m}} d_{\mathrm{geo}}(x,y)$$

*where $d_D$ and $d_{\mathrm{geo}}$ are the metrics defined on $K^\circ$.*

*Proof.* For any differential operator $D \in \mathrm{DO}^m(M; E, F)$ we can find upper and lower bounds, $A_K$ and $B_K$ respectively, to $|\sigma^m(D)(v)|$ for $v \in S^*K$ since $K$ is compact, where $S^*K \subseteq T^*K$ is the co-sphere bundle. In particular, using that the principal symbol is homogenous of order $m$, for any $a \in C^\infty(M)$ we obtain that

$$A_K |\mathrm{d}a_x|^m \leq |\sigma^m(D)(x, \mathrm{d}a_x)| \leq B_K |\mathrm{d}a_x|^m \tag{10}$$

for all $x \in K$. If $D$ is elliptic, then $\sigma^m(D)(x, \xi) \neq 0$ if $\xi \neq 0$ and hence $A_K > 0$ since $K$ is compact. The statement we want to show now follows from the identity in Lemma 4.1. □

We will now see that we obtain a metric from Heisenberg-elliptic differential operators on a bracket-generated filtered manifold as well, although this is not the standard geodesic metric.

On a bracket-generated filtered manifold $M$, Chow's theorem [Cho39] says that any two points on a connected component of $M$ can be joined by a horizontal path $\gamma$ (see also for instance [ABB20, Chapter 3]). A path $\gamma$ is horizontal if $\dot{\gamma} \subseteq H$ where $H \subseteq TM$ is the first degree in the filtering of $M$. The Carnot-Carathéodory metric $d_{CC}$ is defined as the shortest horizontal path between to two points. That is,

$$d_{CC}(x, y) := \inf\left\{ \int_0^1 |\dot{\gamma}(t)|\, \mathrm{d}t \;:\; \gamma \in C^\infty([0,1], M), \gamma(0) = x, \gamma(1) = y \text{ and } \dot{\gamma}(t) \in H \right\}.$$

The corresponding formulation using functions is

$$d_{CC}(x, y) := \sup_{a \in C^\infty(M)} \{\, |a(x) - a(y)| \;:\; \|\mathrm{d}_H a\|_\infty \leq 1 \,\}$$

where the use of smooth function is proven in [Has14, Corollary 3.3.7]. Previous literature have found that $d_{\Delta_H} = d_{CC}$ on contact manifolds, see [Has14, Theorem 7.2.2] and Remark 6.5 in Paper III. We will show that the same indeed holds locally for any Heisenberg-elliptic differential operator.

**Lemma 4.3.** *Let $D \in \mathrm{DO}^m_H(M; E, F)$ be Heisenberg-elliptic of order $m$ on a filtered manifold $M$, then for any compact $K \subseteq M$ there are constants $A_K, B_K > 0$ such that for any $a \in C^\infty(M)$*

$$A_K \|\mathrm{d}_H a\|_\infty^m \leq \left\| \frac{1}{m!} [\dots[D, a]\dots, a] \right\| \leq B_K \|\mathrm{d}_H a\|_\infty^m$$

*as operators on $L^2$-sections over $K$. In particular, for any $x, y \in K$*

$$A_K^{\frac{1}{m}} d_{CC}(x, y) \leq d_D(x, y) \leq B_K^{\frac{1}{m}} d_{CC}(x, y)$$

*where $d_D$ and $d_{CC}$ are the metrics defined on $K^\circ$.*

*Proof.* As discussed after (6) on 23, the abelian representations of the oscillating Lie groupoid $T_H M$ ensures that $\sigma^m(D)(x,\xi)$ is invertible for any $(x,\xi) \in H^* \subseteq T^*M$ if $D$ is Heisenberg-elliptic. We can therefore follow the proof of Lemma 4.3 to obtain that

$$A_K |\mathrm{d}_H a_x|^m \leq |\sigma^m(D)(x, \mathrm{d}_H a_x)| \leq B_K |\mathrm{d}_H a_x|^m$$

for any $a \in C^\infty(M)$ where $A_K > 0$, analogously to (10). Lastly, we note that $\sigma^m(D)(x, d_x a) = \sigma^m(D)(x, \mathrm{d}_H a_x)$ by using Lemma 4.1 since if we look back at the definition of $\mathrm{DO}^m_H(M; E, F)$ we see that $\sigma^m(D)(x, d_x a)$ is a sum of terms on the form $c(x) X_1(a) X_2(a) \dots X_m(a)$ for some vector fields $X_1, X_2, \dots, X_m \in C^\infty(M; H)$. □

Hence, for a Heisenberg-elliptic differential operator $D$ on a bracket-generated filtered manifold $M$, $d_D$ is locally equivalent $d_{CC}$. To further motivate our definition of the higher-order Connes' metric, we will relate it back to dimension. On a metric space we can define Hausdorff dimension which for the Carnot-Carathéodory metric on a filtered manifold is $\dim_H M$ [Mit85, Theorem 2]. Hence, using that the Hausdorff dimension is locally defined we obtain the following result.

**Theorem 4.4.** *For a Heisenberg-elliptic differential operator $D$ on a bracket-generated filtered manifold $M$, the higher-order Connes' metric of $D$ is locally equivalent to the Carnot-Carathéodory metric $d_{CC}$. Moreover, both the spectral dimension of $D$ and the Hausdorff dimension of the metric constructed from $D$ is equal to the homogenous dimension of the filtration* $\dim_H M$.

# 5 Manifolds with boundary and localization

## 5.1 Boundary map in $K$-homology

$K$-homology enjoys a long exact sequence similar to Mayer-Vietoris sequence in the cohomology of spaces [HR00, Section 5.3]. Consider an ideal $J \triangleleft A$ in a $C^*$-algebra $A$ such that $A \to A/J$ has a completely positive right-inverse [HR00, Definition 3.1.1]. Then associated to the short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

there is a long exact sequence

$$\cdots \longrightarrow K^*(A) \longrightarrow K^*(J) \xrightarrow{\partial} K^{*-1}(A/J) \longrightarrow K^{*-1}(A) \longrightarrow \cdots$$

where we refer to $\partial\colon K^*(J) \to K^{*-1}(A/J)$ as the boundary map in $K$-homology. If $\Omega$ is a manifold with boundary and we let $\overline{\Omega}$ denote $\Omega$ including its boundary $\partial\Omega$, then

$$C_0(\overline{\Omega})/C_0(\Omega) = C_0(\partial\Omega)$$

and we can easily construct a completely positive map $C_0(\partial\Omega) \to C_0(\overline{\Omega})$ that is a right-inverse of the restriction map. In particular, for a differential operator $D$ that defines a $K$-homology class $[D] \in K^*(C_0(\Omega))$, the boundary map $\partial[D] \in K^{*-1}(C_0(\partial\Omega))$ is a class on the boundary. The topic of Paper I is examining the boundary map in $K$-homology for differential operators with a focus on higher-order operators.

However, the boundary map $\partial x \in K^{*-1}(A/J)$ of a class $x \in K^*(J)$ can in general not be computed from the class $x$ alone as $x$ does not contain any information about $A$. We can instead consider relative $K$-homology $K^*(J \lhd A)$. The group $K^*(J \lhd A)$ is isomorphic to $K^*(J)$ by excision [HR00, p. 216] but ensures that it is feasible to compute the boundary map [HR00, Proposition 8.5.6]. The only difference between how $K^*(J \lhd A)$ and $K^*(J)$ are defined is that for $K^*(J \lhd A)$ we additionally require a cycle $(\mathcal{H}, F)$ in Definition 2.4 to satisfy

$$[F, a] \in \mathcal{K}(\mathcal{H})$$

for all $a \in A$. Looking at how local spectral triples gave $K$-homology classes in Proposition 3.5, we naturally arrive at the definition of relative spectral triples, for which $D(1 + D^*D)^{-\frac{1}{2}}$ give a relative $K$-homology class.

**Definition 5.1** (Relative spectral triples). A relative spectral triple $(\mathcal{J} \lhd \mathcal{A}, \mathcal{H}, D)$ for $J \lhd A$ consists of a $*$-ideal $\mathcal{J} \subseteq \mathcal{A}$ such that

1. $(\mathcal{A}, \mathcal{H}, D)$ is a pre-spectral triple as in Definition 3.1;
2. $(\mathcal{J}, \mathcal{H}, D)$ is a local spectral triple as in Definition 3.2.

This definition of relative spectral triples is given as higher-order relative spectral triples in Paper I.

If $D$ defines an even relative spectral triple for $J \lhd A$, $\operatorname{Ran} D_-$ is closed and $a(1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ for all $a \in A$, then

$$\partial[D] = [P_{\ker(D_-)^*}] \in K^1(A/J)$$

where we use the description of $K^1(A/J)$ by Toeplitz extensions and $P_{\ker(D_-)^*} \in \mathcal{L}(\mathcal{H}_+)$ denotes the orthogonal projection onto $\ker(D_-)^*$. More details can be found in Paper I. In particular, for any differential operator $D$ such that $\operatorname{Ran}(D^\dagger)_{\min}$ is closed and $\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$ defines a relative spectral triple for $C_0(\Omega) \lhd C_0(\overline{\Omega})$ we have that

$$\partial\left[\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\right] = [P_{\ker D_{\max}}] \quad \text{in} \quad K^1(C_0(\partial\Omega)).$$

The inspiration for Paper I comes from the Baum-Douglas-Taylor index theorem in [BDT89], where they consider the boundary map of first-order elliptic differential operators on a compact manifold $\Omega$ with smooth boundary. They show that

$$\partial\left[\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\right] = [P_C]$$

where $P_C$ is the Calderón projection [See66], a pseudodifferential operator on $\partial\Omega$. They further showed for a spin$^c$ manifold $\Omega$ with boundary that

$$\partial[\Omega] = [\partial\Omega]$$

where $[\Omega] \in K^*(C_0(\Omega))$ is the fundamental class of $\Omega$ defined from a spin$^c$ Dirac operator. One of the main results of Paper I is generalizing Baum-Douglas-Taylor index theorem to classically elliptic differential operators of any order and obtain the following.

**Theorem 5.2.** *For a classically elliptic differential operator $D$ on a manifold $\Omega$ with smooth compact boundary $\partial\Omega$, then*

$$\partial\left[\begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}\right] = \varphi_*\left([E_+(D)] \cap [S^*\partial\Omega]\right) \quad \text{in} \quad K^1(C(\partial\Omega)),$$

*where $\varphi\colon S^*\partial\Omega \to \partial\Omega$ is the fiber map, $E_+(D) \to S^*\partial\Omega$ is the vector bundle associated to the principal symbol of the Calderón projector (see Definition 4.8 in Paper I), $\cap$ is the cap product and $[S^*\partial\Omega] \in K^1(C(S^*\partial\Omega))$ is the fundamental class of the co-spherebundle $S^*\partial\Omega$ with the spin$^c$-structure induced from $T^*\partial\Omega$.*

In comparison to Theorem 1.5 in Paper 1, Theorem 5.2 does not assume that $\Omega$ is compact, only that $\partial\Omega$ is. This slight generalization comes from that the boundary map factor as

$$\begin{array}{ccc} K^*(C_0(\Omega)) & \longrightarrow & K^*(C_0(\partial\Omega \times (0,1))) \\ & \overset{\partial}{\searrow} & \Big\downarrow{\scriptstyle \partial} \\ & & K^{*-1}(C_0(\partial\Omega)) \end{array}$$

by naturality of the boundary map, where $\partial\Omega \times (0,1) \subseteq \Omega$ is a collar neighborhood of $\partial\Omega$. That is, the boundary map of a differential operator only depends on the differential operator close to the boundary. In particular, we obtain Theorem 5.2 as an extension of Theorem 1.5 in Paper 1 by noting that for any manifold $\Omega$ with compact boundary, $\partial\Omega \times (0,1)$ can be embedded into $\partial\Omega \times S^1$ over which it is possible to extend a differential operator elliptically.

The Baum-Douglas-Taylor index formula, and therefore Theorem 5.2, is related to the Boutet de Monvel index theorem. We return to the Boutet de Monvel index theorem later in subsection 7.2.

## 5.2 Domains in manifolds and localization

To calculate the boundary map in $K$-homology we need to construct relative $K$-homology cycles. Here will present a localization procedure from Paper I to obtain relative spectral triples from domains in a manifold. We will adopt the view of a manifold $\Omega$ with boundary as an open set $\Omega \subseteq M$ in some manifold $M$. Then $\Omega$ including its boundary $\overline{\Omega} := \overline{\Omega} \setminus \Omega$ and the boundary $\partial\Omega$ can be seen as subsets of $M$. In the rest of this section we will not care about the regularity of the boundary, and $\Omega \subseteq M$ will denote any open set.

If $(C_c^\infty(M), L^2(M; E), D_{\min})$ defines a local spectral triple such that $D_{\min} = D_{\max}$, then we show in Paper I that

$$(C_c^\infty(\Omega) \lhd C_c^\infty(\overline{\Omega}), L^2(\Omega; E), D_{\Omega,\min}) \tag{11}$$

defines a relative spectral triple for $C_0(\Omega) \lhd C_0(\overline{\Omega})$ using the following abstract localization procedure, where $D_\Omega$ denotes the restriction of $D$ to $\Omega$.

**Theorem 5.3.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a local spectral triple such that $D = D^*$ and let $\mathcal{J} \lhd \mathcal{A}$ be a $*$-ideal. If $D$ is local with respect to $\mathcal{J}$ in the sense that for each $j \in \mathcal{J}$ there is a $k$ such that $j = kj$ and $Dj = kDj$, then*

$$(\mathcal{J} \lhd \mathcal{A}, \mathcal{H}_{\mathcal{J}}, D_{\mathcal{J}})$$

*defines a relative spectral triple with the additional property that $a(1 + D_{\mathcal{J}}^* D_{\mathcal{J}})^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H}_{\mathcal{J}})$ for all $a \in \mathcal{A}$, where $D_{\mathcal{J}} := \overline{D|_{\mathcal{J}\,\mathrm{Dom}\,D}}$ on the module $\mathcal{H}_{\mathcal{J}} := \overline{\mathcal{J}\mathcal{H}}$.*

See Theorem 2.12 in Paper I for proof. The condition $D = D^*$ corresponds to $M$ not having a boundary with respect to $D$, although $M$ can still be non-compact.

Note in particular that $\mathcal{A}$ acting on $\mathcal{H}_{\mathcal{J}}$ in Theorem 5.3 factors through $\mathcal{A}/(\mathcal{J}^\perp \cap \mathcal{A})$ where $\mathcal{J}^\perp = J^\perp = \{a \in A : aj = 0 \text{ for all } j \in J\}$ where $A = \overline{\mathcal{A}}$ and $J = \overline{\mathcal{J}}$. In the case of differential operators, $(C_c^\infty(M), L^2(M; E), D_{\min})$ can be localized to

$$(C_c^\infty(\Omega) \lhd C_c^\infty(M), L^2(\Omega; E), D_{\Omega,\min})$$

from which we obtain (11) by noting that $C^\infty(\overline{\Omega}) = C_c^\infty(M)/C_0^\infty(M \setminus \Omega)$.

It should be noted that since $D_{\mathcal{J}} \subseteq D$, the only non-trivial part in proving Theorem 5.3 is to show that

$$j\,\mathrm{Dom}\,D_{\mathcal{J}}^* \subseteq \mathrm{Dom}\,D_{\mathcal{J}} \tag{12}$$

holds for any $j \in \mathcal{J}$. In the case of differential operators, this corresponds to the interior regularity property

$$j\,\mathrm{Dom}\,D_{\Omega,\max} \subseteq \mathrm{Dom}\,D_{\Omega,\min}$$

for $j \in C_c^\infty(\Omega)$ presented in Proposition 3.6. In fact, in Paper I we prove (12) by constructing a parametrix in an abstract pseudodifferential calculus similarly as was needed to prove Proposition 3.6. Since the closed graph theorem implies that the map $j$ in (12) is continuous, we obtain a Gårding type inequality.

In Paper I, we also examine how $\operatorname{Dom} D_{\mathcal{J}}^*$ relates to $\operatorname{Dom} D$. Specifically, Lemma 2.24 in Paper I displays a condition that we call *smooth approximation* since in the geometric case of differential operators this ensures that $C_c^\infty(\overline{\Omega}; E)$ is a core of $D_{\Omega,\max}$.

## 5.3 Approaching the boundary from different directions

A point not addressed in Paper I is that for an open set $\Omega \subseteq M$ there are a boundary maps to $\partial\Omega$ both from $\Omega$ and from $M \setminus \overline{\Omega}$. Abstractly, for $J \lhd A$ we have that $(A/J^\perp)/J \cong A/(J+J^\perp) \cong (A/J)/J^\perp$, and if $A \to A/(J+J^\perp)$ has a completely positive right-inverse we can ask how the boundary maps $K^*(J) \to K^{*-1}(A/(J+J^\perp))$ and $K^*(J^\perp) \to K^{*-1}(A/(J+J^\perp))$ relate.

**Lemma 5.4.** *For $J \lhd A$, the diagram*

$$\begin{array}{ccccc} & & K^*(A) & & \\ & \swarrow & & \searrow & \\ K^*(J) & & & & K^*(J^\perp) \\ & \overset{\partial}{\searrow} & & \overset{-\partial}{\swarrow} & \\ & & K^{*-1}(A/(J+J^\perp)) & & \end{array}$$

*commutes.*

*Proof.* Note that $J+J^\perp \cong J \oplus J^\perp$ and $K^*(J) \oplus K^*(J^\perp) \cong K^*(J+J^\perp)$ where $x_1 \oplus x_2 \mapsto x_1 + x_2$. Hence, for $x \in K^*(A)$ such that $x = x_1$ in $K^*(J)$ and $x = x_2$ in $K^*(J^\perp)$ we have that $x = x_1 + x_2$ in $K^*(J+J^\perp)$. By naturality of the boundary map the diagram

$$\begin{array}{ccc} K^*(J) & \longleftarrow & K^*(J+J^\perp) \\ & \overset{\partial}{\searrow} & \downarrow \partial \\ & & K(A/(J+J^\perp)) \end{array}$$

commutes, and similarly for $J^\perp$. Therefore,

$$\partial x_1 + \partial x_2 = \partial x = 0$$

in $K^{*-1}(A/(J+J^\perp))$ since $K^*(A) \to K^*(J+J^\perp) \to^{\partial} K^{*-1}(A/(J+J^\perp))$ is exact. □

In regard to localizations of spectral triples, we obtain the corresponding result.

**Proposition 5.5.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a local spectral triple for $A$ and let $\mathcal{J}, \mathcal{J}_\perp \lhd \mathcal{A}$ be $*$-ideals such that $\overline{\mathcal{J}_\perp} = J^\perp$ for $J = \overline{\mathcal{J}}$. Assume that $D$ is local with respect to $\mathcal{J}$ and with respect to $\mathcal{J}_\perp$ in the sense of Theorem 5.3, then*

$$\partial[D_{\mathcal{J}}] = -\partial[D_{\mathcal{J}_\perp}]$$

*in $K^{*-1}(A/(J+J^\perp))$.*

For differential operators, Proposition 5.5 corresponds to approaching the boundary from two different directions, that is, the boundary maps

$$\partial\colon K^*(C_0(\Omega)) \to K^{*-1}(C_0(\partial\Omega))$$

and

$$\partial\colon K^*(C_0(M \setminus \overline{\Omega})) \to K^{*-1}(C_0(\partial\Omega)).$$

In the context of the Baum-Douglas-Taylor index theorem, the sign change in Lemma 5.4 is to be expected since it corresponds to taking the opposite spin$^c$ orientation of $\partial\Omega$.

# 6 Fredholm operators and boundary conditions

## 6.1 Fredholm spectral triples

Looking back at the definition of a local spectral triple $(\mathcal{A}, \mathcal{H}, D)$, note that if $\mathcal{A}$ is unital then $D$ is self-adjoint and $(1 + D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ which implies that $D$ has compact resolvent. However, compact resolvent is a rather restrictive condition when considering differential operator on non-compact manifolds, and we might wonder if we weaken this condition. This question is discussed in Paper II in the form of truly unbounded Kasparov modules [Wah07, Definition 2.4]. As we do not deal with Hilbert modules in this text and to keep in line with its theme, we will here call them Fredholm spectral triples instead.

**Definition 6.1** (Fredholm spectral triples)**.** A Fredholm spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is pre-spectral triple such that

1. $D$ is Fredholm,
2. $D$ is self-adjoint;

**Proposition 6.2.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a Fredholm spectral triple for $A$, then*

$$D(k + D^2)^{-\frac{1}{2}}$$

*defines a bounded Kasparov $(A, \mathbb{C})$-module for any positive $k \in \mathcal{L}(\mathcal{H})$ such that $k(1 + D^2)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ and $k + D^2$ is invertible. Such a k always exists and the class $[D(k + D^2)^{-\frac{1}{2}}] \in K^*(A)$ is independent of choice of k.*

For a proof of Proposition 6.2, see section 5.2 in Paper II. In particular, that such a $k$ always exists is the first part of a characterization of Fredholm operators presented in Paper II and the second part says that such a $k$ can be chosen freely in the following sense.

**Proposition 6.3.** *Let $D$ be an unbounded operator on a Hilbert space $\mathcal{H}$ and $\{\chi_n\} \subseteq \mathcal{L}(\mathcal{H})$ a self-adjoint strong approximate unit, that is, a sequence strongly converging to the identity in the sense that*

$$\|v - \chi_n v\| \to 0 \textit{ for any } v \in \mathcal{H}.$$

*If* $\chi_n(1+D^*D)^{-\frac{1}{2}} \in \mathcal{K}(\mathcal{H})$ *for all n, then D is left-Fredholm if and only if there is an n such that*

$$\chi_n^2 + D^*D$$

*is invertible.*

Proposition 6.3 is a slight reformulation of Proposition 4.4 in Paper II. For instance, we can let $k \in C_c^\infty(M)$ in Proposition 6.2 if $D$ is a differential operator. From this we also obtain a generalization of the classical result Persson's lemma in spectral theory [Per60] that describes the essential spectrum of a self-adjoint operator bounded from below, found as Corollary 4.7 in Paper II. Another possible choice of $k$ that was used in Lafforgue's work on the Baum-Connes conjecture [Laf02, Proposition 3.1.2] is $k = P_{\ker D}$, in which case $D(k+D^2)^{-\frac{1}{2}} = \operatorname{sgn}(D)$.

Note that if $(\mathcal{A}, \mathcal{H}, D)$ is a Fredholm spectral triple for $A$, then it is also a Fredholm spectral triple for the unitalization of $A$ by adding the unit to $\mathcal{A}$. We see this as the conditions including $\mathcal{A}$ is domain inclusion and commutators, and the unit trivially satisfies these conditions. However, we do not assume $D$ to have compact resolvent and a Fredholm spectral triple is therefore a weaker condition then a spectral triple for a unital $C^*$-algebra.

See Paper II for further properties of Fredholm spectral triples. Here we want to add additional comments on the relation between Fredholm spectral triples and the boundary map in $K$-homology. For $J \lhd A$ such that $A \to A/J$ has a completely positive right-inverse, the boundary map in $K$-homology satisfies that

$$K^*(A) \longrightarrow K^*(J) \xrightarrow{\partial} K^{*-1}(A/J)$$

is exact, meaning that for $x \in K^*(J)$ we have that $\partial x = 0$ if and only if there is a preimage $y \in K^*(A)$ of $x$. In particular, if $(\mathcal{J}, \mathcal{H}, D)$ is a relative spectral triple for $J$ and we can find a Fredholm self-adjoint extension $D_e$ of $D$ and a dense $*$-subalgebra $\mathcal{A} \subseteq A$ such that $(\mathcal{A}, \mathcal{H}, D_e)$ defines a pre-spectral triples, then $\partial[D] = 0$ since $D_e$ defines a Fredholm spectral triple that give us a preimage of $[D]$ in $K^*(A)$. For instance, in Example 2.15 in Paper I we regard a spin Dirac operator $\not{D}$ on a complete Riemannian spin manifold $M$ and the Higson compactification. Using the characterization of Fredholmness in Paper II and Fredholm spectral triples, we see that

$$\partial[\not{D}] \in K^*(C(\nu M))$$

is an obstruction of existence of scalar curvature strictly positive outside a compact, where $\nu M$ is the Higson corona of $M$ [Roe93, Chapter 5].

## 6.2 Boundary value problems

Consider an open set $\Omega \subseteq M$ where $M$ is a manifold without boundary and a differential operator $D$ on $M$. A boundary value problem for $D$ on $\Omega$ takes the form

$$\begin{cases} Du = f & \text{in } \Omega \\ Bu = g & \text{on } \partial\Omega \end{cases}$$

where $f$ is a section on $\Omega$, $g$ is a section on $\partial\Omega$ and $B\colon C^\infty(\overline{\Omega}; E) \to C(\partial\Omega; G)$ is a boundary operator in the sense that $B(C_c(\Omega)) = 0$. Since we are concerned with linear differential operators we can divide a boundary value problem into two problems, when $f = 0$ and when $g = 0$.

If $g = 0$, we are interested in $D$ acting on $u \in \operatorname{Ker} B$. With our theory of unbounded operators this entails considering the realization $D_B$ of $D_\Omega$ with domain

$$\operatorname{Dom} D_B = \{ u \in \operatorname{Dom} D_{\Omega,\max} \, : \, Bu = 0 \}.$$

We have that $D_{\Omega,\min} \subseteq D_B \subseteq D_{\Omega,\max}$ and if $B$ is a well-behaved boundary condition, for instance if $D$ is elliptic and $B$ is an elliptic boundary conditions for $D$, then $D_B$ is closed. In particular, if $D$ defines a local spectral triple then the class $[D] \in K^*(C_0(\Omega))$ is in this sense independent of choice of boundary condition.

If $f = 0$, we are interested in the space $\ker D_{\Omega,\max}$. We can see $\ker D_{\Omega,\max}$ as an analog to the Bergman space $A(\Omega) := \mathcal{O}(\Omega) \cap L^2(\Omega)$ consisting of holomorphic $L^2$-functions on a complex manifold with boundary since $A(\Omega) = \ker \overline{\partial}_{\Omega,\max}$ for the anti-complex differential $\overline{\partial}$ acting on scalar-valued functions. In connection to the boundary map in $K$-homology, we saw that for a differential operator $D$ that defines a local spectral triple then

$$\partial \left[ \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \right] = [P_{\ker D_{\Omega,\max}}]. \tag{13}$$

Now, if there is a boundary condition $B$ such that $[D_B] \in K^*(C_0(\Omega))$ lifts to a class in $K^*(C_0(\overline{\Omega}))$ then $\partial[D] = 0$. In particular, such a $B$ cannot exist if $\partial[D] \neq 0$. In what is next, we will show that (13) is an obstruction to existence of elliptic boundary conditions for $D$, although we will show this without constructing a class in $K^0(C_0(\overline{\Omega}))$.

## 6.3 Elliptic boundary conditions

For an elliptic differential operator $D$ on a manifold $\Omega$ with smooth boundary, we will consider a particular type of boundary conditions called elliptic boundary conditions. These are boundary conditions constructed from differential operators in the sense that $B = \gamma_0 B'$ for some differential operator $B'$ on $\Omega$ where $\gamma_0$ denote the restriction to the boundary, also called the trace map. Following the argument found in [Hör07, p. 233], if $D$ is elliptic of order

$m$ we can reduce a boundary condition constructed from differential operators to $B = b\gamma$ where $b$ is a differential operator on $\partial\Omega$ and

$$\gamma = \begin{pmatrix} \gamma_0 \\ \gamma_0 \partial_r \\ \vdots \\ \gamma_0 \partial_r^{m-1} \end{pmatrix}$$

denote the full trace map for some normal direction $r$ to the boundary.

**Definition 6.4** (Shapiro-Lopatinskii elliptic boundary condition)**.** [Hör07, Definition 20.1.1] Let $\Omega$ be a compact manifold with smooth boundary and let $D \in \mathrm{DO}^m(\Omega; E, F)$ be elliptic of order $m$. Consider a boundary condition $B = b\gamma$ where $b = (b_{jk})$ is a $t \times m$ matrix of differential operators $b_{jk} \in \mathrm{DO}^{s_j - k}(\partial\Omega; E, G_j)$ for some $s_1, \dots, s_t \geq 0$. Let $\sigma(b) := (\sigma^{s_j-k}(b_{jk}))$ denote the combined principal symbol of $b$ and let $G := \bigoplus G_j$, then $B$ is said to be a Shapiro-Lopatinskii elliptic boundary condition for $D$ if

$$\sigma(b)\colon E_+(D) \to \varphi^* G \tag{14}$$

is an isomorphism of vector bundles where $E_+(D) \to S^*\partial\Omega$ is the vector bundle associated to the principal symbol of the Calderón projector (see Definition 4.8 in Paper 1) and $\varphi\colon S^*\partial\Omega \to \partial\Omega$ is the fiber map.

In particular, $D_B$ is Fredholm if $B$ is an elliptic boundary condition for $D$ [Hör07, Theorem 20.1.2]. Moreover, we have the following result.

**Theorem 6.5** (The Atiyah-Bott index theorem)**.** *[AB64] Let $\Omega$ be a compact manifold with smooth boundary, $D \in \mathrm{DO}^m(\Omega; E, F)$ elliptic of order $m$ and $B$ an elliptic boundary condition for $D$. Then*

$$\mathrm{ind}(D \oplus B) = \int_{\partial(B^*\Omega)} \mathrm{ch}(\sigma^m(D, B)) \wedge \mathrm{Td}(T\Omega \otimes \mathbb{C})$$

*where $\sigma^m(D, B)$ is an invertible extension $\sigma^m(D)$ from $S^*\Omega$ to $\partial(B^*\Omega) = B^*\partial\Omega \cup S^*\Omega$ corresponding to the boundary condition $B$, and we see $D \oplus B$ as a map $D \oplus B\colon C^\infty(\Omega; E) \to C^\infty(\Omega; F) \oplus C^\infty(\partial\Omega; \bigoplus G_j)$.*

Note that the index of $D_B$ and $D \oplus B$ is different. That is, $\ker D_B = \ker(D \oplus B)$ but $\mathrm{Coker}\, D_B \oplus 0 \subseteq \mathrm{Coker}(D \oplus B)$ is not necessarily an equality.

The following result was mentioned in Paper 1 but not shown.

**Theorem 6.6.** *For an elliptic differential operator $D$ on a manifold $\Omega$ with smooth compact boundary,*

$$\partial \left[ \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \right]$$

*is an obstruction of existence of elliptic boundary conditions for $D$.*

*Proof.* By Theorem 5.2, $\partial[D] = \varphi_*([E_+(D)] \cup [S^*\partial\Omega])$. Now, if an elliptic boundary condition $B$ exists, then by definition there is a vector bundle $G \to \partial\Omega$ such that $E_+(D) \cong \varphi^* G$, the isomorphism in Definition 6.4 constructed from $B$ in (14). Then

$$\begin{aligned}\partial[D] &= \varphi_*([E_+(D)] \cup [S^*\partial\Omega]) \\ &= \varphi_*([\varphi^* G] \cup [S^*\partial\Omega]) \\ &= [G] \cup \varphi_*[S^*\partial\Omega].\end{aligned}$$

By the Baum-Douglas-Taylor index theorem $[S^*\partial\Omega] = \partial[B^*\partial\Omega]$, and $K^1(C(\overline{B^*}\partial\Omega)) \cong K^1(C(\partial\Omega))$ since $\overline{B}^*\partial\Omega$ contracts to $\partial\Omega$. In particular, the following diagram

$$\begin{array}{ccccc} & & & & K^1(C(\partial\Omega)) \\ & & & \overset{\varphi_*}{\nearrow} & \downarrow \cong \\ K^0(C_0(B^*\partial\Omega)) & \xrightarrow{\partial} & K^1(C(S^*\partial\Omega)) & \longrightarrow & K^1(C(\overline{B^*}\partial\Omega)) \end{array}$$

commutes, and obtain that $\varphi_*[S^*\partial\Omega] = \varphi_*\partial[B^*\partial\Omega] = 0$ by exactness of the bottom row. □

Although it relies on more heavy machinery, we can also obtain a proof of Theorem 6.6 from Lemma 4.20 in Paper I since we obtain that $\varphi_*([\varphi^* G] \cup [S^*\partial\Omega]) = [\mathbb{1}_{L^2(\partial\Omega;G)}] = 0$.

# 7 Operators from complexes

This section will introduce three differential complexes.

1. The de Rham complex which is central in Paper IV,
2. the Dolbeault complex which connects to Paper I as the Baum-Douglas-Taylor index theorem can be used to prove the Boutet de Monvel index formula,
3. and lastly the Rumin complex which served as one of the main examples that motivated Paper III.

## 7.1 The de Rham complex and the Laplacian

A fundamental property of partial derivatives is that they commute. To be precise, directional derivatives at a point commute, vector fields do not. This can be captured algebraically by extending the exterior differential $\mathrm{d}\colon C^\infty(M) \to C^\infty(M; T^*M)$ to an operator

$$\mathrm{d}\colon \Lambda^k(M) \to \Lambda^{k+1}(M)$$

where $\Lambda^k(M) := C^\infty(M; \bigwedge^k T^*M)$ are called differential $k$-forms. For consistency, we also denote $\Lambda^0(M) := C^\infty(M)$. For $\omega \in \Lambda^1(M)$ and vector fields $X, Y$ we define

$$\mathrm{d}\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

and extend to $\Lambda^k(M)$ by imposing the condition that

$$\mathrm{d}(f \wedge g) = (\mathrm{d}f) \wedge g + (-1)^a f \wedge (\mathrm{d}g)$$

for $f \in \Lambda^a(M)$ and $g \in \Lambda^b(M)$ such that $a + b = k$. Note in particular that

$$\mathrm{d}^2 f(X, Y) = X(Y(f)) - Y(X(f)) - [X, Y](f) = 0,$$

which in locally coordinates expresses that partial derivatives commute. Since we used a recursive definition, $\mathrm{d}^2 = 0$ also hold for d acting on any differential forms.

We arrive at the first example of a differential complex, namely the so called de Rham complex

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{\mathrm{d}} \Lambda^1(M) \xrightarrow{\mathrm{d}} \Lambda^2(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^n(M) \longrightarrow 0$$

where $n = \dim M$. The de Rham complex is an elliptic complex, meaning that the corresponding complex of the symbols is exact, or equivalently, the so called Hodge-Dirac operator

$$\not{D} = \mathrm{d} + \mathrm{d}^\dagger$$

acting on $\Lambda^*(M) := \bigoplus \Lambda^k(M)$ is elliptic. Note that the usage of adjoint is the first choice we made in constructing the de Rham complex that depends on metric and measure. Squaring the Hodge-Dirac operator we obtain the Hodge-Laplacian

$$(\mathrm{d} + \mathrm{d}^\dagger)^2 = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$$

of which the part acting on scalar function we saw earlier as an example of en elliptic operator.

For a domain $\Omega \subseteq M$ we can introduce a boundary condition on the Hodge-Dirac operator as

$$\not{D}_a = \mathrm{d}_{\Omega,\max} + \mathrm{d}^\dagger_{\Omega,\min}$$

which we call the absolute boundary condition of $\not{D}$. In Paper IV, we consider the associated boundary condition

$$\not{D}_a^2 = \mathrm{d}^\dagger_{\Omega,\min}\mathrm{d}_{\Omega,\max} + \mathrm{d}_{\Omega,\max}\mathrm{d}^\dagger_{\Omega,\min}$$

of the Hodge-Laplacian on a domain $\Omega$ and note that on scalar functions this realization acts as the Neumann realization

$$\operatorname{Dom}\Delta_N = \{\, f \in H^2(\Omega) \;:\; \partial_r f|_{\partial\Omega} = 0 \,\}$$

of the Laplacian for the normal direction $r$ to the boundary, and on $n$-forms as the Dirichlet realization

$$\operatorname{Dom}\Delta_D = \{\, f \in H^2(\Omega) \;:\; f|_{\partial\Omega} = 0 \,\}.$$

In Paper IV, we use this identification to reformulate the recent result [Roh25] that

$$\lambda_{k+2}(\Delta_N) \leq \lambda_k(\Delta_D)$$

on any simply connected bounded domain $\Omega \subseteq \mathbb{R}^2$ with Lipschitz boundary. This inequality relates to Friedlander's inequality $\lambda_{k+1}(\Delta_N) \leq \lambda_k(\Delta_D)$ [Fri91] which we prove hold for any bounded domain $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary if $n \geq 2$. A key observation in the method of Paper IV is that if $\Omega$ is a domain of $\mathbb{R}^n$, then acting on $k$-forms we can identify

$$\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d} = \bigoplus^{\binom{n}{k}} \Delta$$

where $\Delta = \sum \frac{\partial^2}{\partial^2 x_k}$ is the Laplacian on $\mathbb{R}^n$.

For the Hodge-Dirac operator, the absolute boundary condition is an elliptic boundary condition for $\not{D}$ [BL92, Theorem 4.1], and $\not{D}_a$ has discrete spectrum on precompact domains. In particular,

$$\ker \not{D}_a = \bigoplus H^k_{\mathrm{dR}}(\Omega)$$

where $H^k_{\mathrm{dR}}(\Omega)$ is the de Rham cohomology groups. Since $\not{D}$ is first-order, the absolute boundary condition is necessarily a local boundary condition in the sense of [BDT89, (3.4)], and we see that $[\not{D}] \in K^0(C_0(\Omega))$ lifts to $[\not{D}_a] \in K^0(C(\overline{\Omega}))$. In particular, $\partial[\not{D}] = 0$ for the Hodge-Dirac operator. In the next section we will consider the Dolbeault complex where this is not true, showing that the methods of Paper IV cannot be directly applied to the Dolbeault complex.

## 7.2 The Dolbeault complex and the Boutet de Monvel index theorem

On a complex manifold $M$ of complex dimension $n$ the de Rham differential can be decomposed into two parts by using complex derivatives to produce the Dolbeault complex. In this section we will introduce the Dolbeault complex and clarify the connection between the results in Paper I and the Boutet de Monvel index theorem.

Consider the complexification

$$\mathrm{d}\colon \Lambda^0(M) \otimes \mathbb{C} \to \Lambda^1(M) \otimes \mathbb{C}$$

of the exterior differential d. Then we can choose complex coordinates in the sense that $z_i = x_i + iy_i$ and $\bar{z}_i = x_i - iy_i$ to obtain a natural split of the 1-forms as

$$\Lambda^1(M) \otimes \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$$

by considering the basis $\mathrm{d}z_i$ and $\mathrm{d}\bar{z}_i$ respectively. We have that

$$\frac{\partial}{\partial x_i}\mathrm{d}x_i + \frac{\partial}{\partial y_i}\mathrm{d}y_i = \frac{\partial}{\partial z_i}\mathrm{d}z_i + \frac{\partial}{\partial \bar{z}_i}\mathrm{d}\bar{z}_i$$

and hence we also obtain a split $\mathrm{d} = \partial + \overline{\partial}$ which is preserved by holomorphic coordinate changes. We can similarly obtain a split of the $k$-forms as

$$\Lambda^k(M) \otimes \mathbb{C} = \bigoplus_{p+q=k} \Lambda^{p,q}(M)$$

on which $\partial\colon \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)$ and $\overline{\partial}\colon \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$. In particular, since $\overline{\partial}^2 = 0$ we obtain a differential complex

$$0 \longrightarrow \Lambda^{0,0}(M) \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \xrightarrow{\overline{\partial}} \Lambda^{0,2}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Lambda^{0,n}(M) \longrightarrow 0$$

which is called the Dolbeault complex. Associated to the Dolbeault complex we have the Dolbeault-Dirac operator

$$\not{\partial} = \overline{\partial} + \overline{\partial}^{\dagger}$$

which in fact is the canonical Dirac operator associated with the spin$^c$ structure of the complex manifold $M$.

Let $\Omega$ be a precompact strictly pseudoconvex domain in a complex manifold and let $H(\partial\Omega)$ denote the Hardy space consisting of the $L^2$-functions that are restrictions of holomorphic functions in the interior $\Omega$. From the Szegö projection $P_S \in \mathcal{L}(L^2(\partial\Omega))$, the orthogonal projection onto $H(\partial\Omega)$, we can construct the Toeplitz operator $P_S a P_S$ for $a \in \mathrm{GL}_k(C(\partial\Omega))$ that is Fredholm, and the Boutet de Monvel index theorem [BdM79] states that

$$\mathrm{ind}(P_S a P_S) = \int_{\partial\Omega} \mathrm{ch}(a) \wedge \mathrm{Td}(\partial\Omega). \tag{15}$$

This index theorem should be seen as an index pairing between the $K$-theory class $[a] \in K_1(C(\partial\Omega))$ and the $K$-homology class $[P_S] \in K^1(C(\partial\Omega))$ in the sense that

$$\langle [a], [P_S] \rangle = \mathrm{ind}(P_S a P_S)$$

for the pairing $\langle \cdot, \cdot \rangle\colon K_1(C(\partial\Omega)) \times K^1(C(\partial\Omega)) \to \mathbb{Z}$. With this view, the Boutet de Monvel index theorem states that

$$[P_S] = [\partial\Omega] \tag{16}$$

in $K^1(C(\partial\Omega))$, where $[\partial\Omega]$ is the fundamental class of $K^1(C(\partial\Omega))$ defined from its spin$^c$-structure.

As done by Baum-Douglas-Taylor [BDT89] (see also [BvE21]), (16) can be shown using relative cycles and the Baum-Douglas-Taylor index formula. This has a surprising connection to Paper IV in that they use the Dolbeault-Dirac operator $\not{\partial}_a = \overline{\partial}_{\Omega,\max} + \overline{\partial}^{\dagger}_{\Omega,\min}$ with absolute boundary condition, also called the $\overline{\partial}$-Neumann boundary condition. We have that

$$\ker \not{\partial}_a = A(\Omega) \oplus \bigoplus_{q>0}^{n} H^{0,q}(\Omega)$$

where $A(\Omega) := \mathcal{O}(\Omega) \cap L^2(\Omega)$ is the Bergmann space consisting of holomorphic $L^2$-functions and $H^{0,q}(\Omega)$ are the Hodge-cohomology groups (vector spaces) which are finite dimensional if $\Omega$ is strictly pseudoconvex. In particular, $\bar{\partial}\!\!\!/_{-,a}$ has compact resolvent, and since the absolute boundary condition is a local boundary condition we can, by checking some regularity conditions, see that

$$\begin{pmatrix} 0 & \bar{\partial}\!\!\!/_{-,a} \\ \bar{\partial}\!\!\!/_{+,\min} & 0 \end{pmatrix}$$

defines an even relative spectral triple for $C_0(\Omega) \lhd C(\overline{\Omega})$. Then $\bar{\partial}\!\!\!/_a$ also defines a relative $K$-homology class with $\partial[\bar{\partial}\!\!\!/_a] = [P_{\ker \bar{\partial}\!\!\!/_{+,a}}]$. If one now connects the Szegö projection and the Bergmann projection $P_{\mathcal{O}(\Omega)} \in \mathcal{L}(L^2(\Omega))$ we arrive at

$$[P_S] = [P_{\mathcal{O}(\Omega)}] = [P_{\ker \bar{\partial}\!\!\!/_{a,+}}] = \partial[\bar{\partial}\!\!\!/_a] = \partial[\bar{\partial}\!\!\!/] = \partial[\Omega] = [\partial\Omega]$$

where we use that $\bar{\partial}\!\!\!/$ is constructed from the spin$^c$ structure of $\Omega$ and therefore $[\Omega] = [\bar{\partial}\!\!\!/]$.

The analog of the Boutet de Monvel index formula presented in Paper I states that for any elliptic differential operator $D$ on a manifold $\Omega$ with smooth compact boundary

$$\operatorname{ind}(P_{\ker D_{\max}} a P_{\ker D_{\max}}) = \int_{S^*\partial\Omega} \operatorname{ch}(a) \wedge \operatorname{ch}(E_+(D)) \wedge \operatorname{Td}(S^*\partial\Omega) \tag{17}$$

for $a \in M_k(C(\overline{\Omega}))$ such that $a|_{\partial\Omega} \in \mathrm{GL}_k(C(\partial\Omega))$. This is the index pairing between $[a|_{\partial\Omega}] \in K_1(C(\partial\Omega))$ and $[P_{\ker D_{\max}}] \in K^1(C(\partial\Omega))$ rewritten using Theorem 5.2. A note here might be that $P_{\ker D_{\max}}$ is more similar to the Bergman projection than the Szegö projection, and in the case $D = \bar{\partial}\!\!\!/_+$ one would need to argue that $\varphi^*([E_+(\bar{\partial}\!\!\!/_+)] \cap [S^*\partial\Omega]) = [\partial\Omega]$ for the Dolbeault-Dirac operator on a strictly pseudoconvex domain in order for the formulas (15) and (17) to match.

### 7.3 Contact manifolds and the Rumin complex

The boundary of a strictly pseudoconvex domain is an example of a contact manifold. Another example would be the Heisenberg group $\mathbb{H}^3$. A contact manifold is a bracket-generated filtered manifold of $2n+1$ dimensions such that the bundle of horizontal directions $H \subseteq TM$ has rank $2n$ and any non-trivial 1-form $\theta$ such that $\ker\theta = H$ satisfies that $\theta \wedge (\mathrm{d}\theta)^n$ is a volume form. Another way to view this is that $\mathrm{d}\theta$ restricted to $H$ is a positive definite symplectic form, and in fact there is tight connection between symplectic manifolds and contact manifolds.

On a contact manifold we can construct differential complex called the Rumin complex [Rum94]. The first-operator of the complex is the horizontal exterior differential $\mathrm{d}_H$ seen previously, but not all operator in the complex are first-order. The exact construction can be found in Section 6 in Paper III or [Rum94].

The Rumin complex was one of the main examples in mind that motivated Paper III. In Paper III, we build a framework for spectral triples associated to these type of situations where we have multiple operators that together control derivatives but can have different orders.

# 8 Outlook

In this section we will present potential future research topics stemming from this thesis.

## 8.1 The boundary map for non-elliptic differential operators

The theory of spectral triples in Paper I is well-suited also for Heisenberg-elliptic differential operator, and it is of interest to calculate the boundary map in $K$-homology in such cases. For instance, one could consider the $\Delta_H + i\gamma Z$ on a contact manifold.

## 8.2 The odd boundary map

In Paper I, only the even-to-odd boundary map in $K$-homology is examined. It would also be interesting to calculate $\partial[D]$ for $[D] \in K^1(C_0(\Omega))$ in terms of boundary data. On a speculation level, we suspect that this can be expressed using $\mathfrak{A}$ from Lemma 4.1 in Paper I.

## 8.3 Elliptic boundary conditions

The Atiyah-Bott index theorem [AB64] in Theorem 6.5 uses an extension $\sigma^m(D, B)$ of the principal $\sigma^m(D)$ over the ball bundle $B^*\partial\Omega$ of the co-sphere bundle on the boundary. In particular, $\sigma^m(D, B)$ defines a $K$-theory class in $K_0(C_0(T\Omega))$. We also expect $D_B$ to define a class in $K^0(C(\overline{\Omega}))$. It is of interest to see how these classes relate through Poincaré duality. Note that by Theorem 6.5 these classes are not expected to be the same as $\operatorname{ind} D_B \neq \operatorname{ind} D \oplus B$.

## 8.4 An index theorem for non-compact manifolds

In Example 2.4 in Paper II, we calculate the index of an elliptic operator on a manifold that is Euclidean at infinity. As is explained in the example, it bears resembles to the Atiyah-Bott index theorem. Using the characterization of Fredholmness in Paper II, we wonder if a Fredholm elliptic differential operator $D$ on a non-compact manifold $M$ can be restricted to a domain $\Omega$ where $D$ automatically has a "canonical" elliptic boundary condition $B$ such that $\operatorname{ind} D = \operatorname{ind} D_{\Omega,B}$.

## 8.5 Look further at reconstructing geometry for higher-order operators

In section 4 we recreated a metric on a manifold using elliptic or Heisenberg-elliptic differential operators. However, this does not fit into the framework of compact quantum metric spaces in the sense of Mark Rieffel in an obvious way. This is because

$$a \mapsto \left\| \frac{1}{m!}[\dots[D, a]\dots, a] \right\|^{\frac{1}{m}}$$

does not seem to be sub-additive and hence it does not define a semi-norm. Also, one might wonder how the condition in a local spectral triple that $[D, a](1 + D^*D)^{-\frac{1}{2}+\frac{1}{2m}}$ is bounded relates to $\frac{1}{m!}[\dots[D, a]\dots, a]$.

## 8.6 Higson compactification and constructing boundaries

The Higson compactification $C_h(M)$ of $C_0(M)$ is a unitalization of $C_0(M)$ and with Fredholm spectral triples we can obtain a generalization of this. Namely, given a Fredholm spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for $A$, what is the largest algebra $\mathcal{A}_D$ such that $\mathcal{A}$ is an ideal in $\tilde{\mathcal{A}}$ and $(\mathcal{A}_D, \mathcal{H}, D)$ is still a Fredholm spectral triple. One might see $A_D = \overline{\mathcal{A}_D}$ as the Higson unitalization of $A$ with respect to $D$. This type of construction would be interesting to study.

In particular, we recover the usual Higson compactification $C_h(M)$ from certain elliptic differential operators $D$ on $M$ since we can reconstruct the geodesic distance from $D$ by considering $\frac{1}{m!}[\dots[D, a]\dots, a]$. However, it is unclear how the proposed generalization of the Higson compactification $C_0(M)_D$ relates to $C_h(M)$ as there is no clear relation between $[D, a](1 + D^*D)^{-\frac{1}{2}+\frac{1}{2m}}$ and $\frac{1}{m!}[\dots[D, a]\dots, a]$.

## 8.7 Eigenvalue inequalities and other complexes

The approach to show Friedlander's inequality using the de Rham complex can be seen as very suitable in two dimension as shown in Paper IV. For higher dimensions we would need additional tricks to obtain better results.

One could also turn to other complexes. The Dolbeault complex in two complex dimensions also have length three, however, the resulting Laplacian does not have discrete spectrum. The Rumin complex would also be of interest as there are known results for eigenvalue inequalities for the sub-Laplacian [FL10], however this would require separate analysis of the operators in the middle of the complex as they do not take the same form as the sub-Laplacians on the ends.

## 8.8 A true generalization of the Boutet de Monvel index theorem

For the Dolbeault-Dirac operator with its absolute boundary condition $\not\partial_a$, we saw in the Boutet de Monvel index theorem that $[\not\partial_a]$ give a relative cycle in $K^0(C_0(\Omega) \lhd C(\overline{\Omega}))$ at the same time as $\partial[\not\partial] = [\partial\Omega] \neq 0$. Hence, we obtain a class $[P_{\ker \not\partial_{+,a}}] \neq 0$. We wonder if one can find another boundary condition for a differential operator where a similar index theorem can be constructed.

Similarly to how we motivated that the Dolbeault complex cannot be used to produce eigenvalue inequalities, we note that since there are known results for eigenvalue inequalities for the sub-Laplacian [FL10] we do not expect to find an analog to the Boutet de Monvel index theorem on a contact manifold with boundary using the Rumin complex.

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