Quotients of Reproducing Kernels

Applications in Complex Analysis and Operator Theory

Weiström Dahlin, Frej

2026

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

– CENTRUM SCIENTIARUM MATHEMATICARUM –

Quotients of Reproducing Kernels

Applications in Complex Analysis and Operator Theory

FREJ DAHLIN

Lund University
Faculty of Science
Centre for Mathematical Sciences
Mathematics

Quotients of Reproducing Kernels

Quotients of Reproducing Kernels

Applications in Complex Analysis and Operator Theory

by Frej Dahlin

LUND UNIVERSITY

Thesis for the degree of Doctor of Philosophy
Thesis advisors: Professor Alexandru Aleman, Professor Sandra Pott
Faculty opponent: Professor Michael T. Jury

To be presented, with the permission of the Faculty of Science of Lund University, for public criticism in MH:Hörmandersalen at the Centre for Mathematical Sciences on Friday, the 20th of March 2026 at 13:00.

Organization **LUND UNIVERSITY** Centre for Mathematical Sciences Box 118 SE–221 00 LUND Sweden	Document name **DOCTORAL DISSERTATION**
	Date of disputation 2026-03-20
Author(s) Frej Dahlin	Sponsoring organization Supported in part by the Swedish Research Council GRANT NO. 2022-05000

Title and subtitle

Quotients of Reproducing Kernels: Applications in Complex Analysis and Operator Theory

Abstract

This thesis studies reproducing kernels that are realized as pointwise quotients of two other kernels, including far-reaching generalizations of de Branges–Rovnyak spaces. In the first article, we study reproducing kernels arising from the well-known multiplier criterion. They are intimately connected to certain operator inequalities, such as the famous inequality of Shimorin in sub-Bergman spaces, which extend Sarason's sub-Hardy spaces. We develop a model reminiscent of the Sz.-Nagy–Foiaş model. As an application we resolve a conjecture regarding the density of polynomials in certain classes of weighted sub-Bergman spaces. In the second article we generalize the classical Julia–Carathéodory theorem via reproducing kernels. We develop a new boundary notion and approach regions to it, entirely in terms of reproducing kernels. We also introduce *composition factors* as a kernel-theoretic alternative to analytic selfmaps. In the third article we identify co-isometric weighted composition operators as composition factors. Moreover, we extend results of Mas, Martín, and Vukotić from the unit disk to the polydisk. Specifically, under mild regularity assumptions on a reproducing kernel k on the polydisk, we prove a dichotomy for rank 1 composition factors. The set is either all analytic automorphisms of the polydisk, in which case k is a positive power of the Szegő kernel, or exactly the rotations composed with a permutation.

DOKUMENTDATABLAD enl SIS 61 41 21

Keywords

Reproducing kernels, de Branges–Rovnyak spaces, Complex analysis, Operator theory

2020 Mathematics Subject Classification (MSC2020)

46E22, 47B32, 30H45 (PRIMARY) 32A40, 47B33 (SECONDARY)

Supplementary bibliographical information	Language
LUNFMA-1048-2026	English
ISSN and key title Doctoral Theses in Mathematical Sciences 2026:2 1404-0034	**ISBN** 978-91-8104-864-3 (print) 978-91-8104-865-0 (pdf)

Recipient's notes	Number of pages	Price
	112	
	Security classification	

Signature ____________________ Date 2026-02-02

Quotients of Reproducing Kernels

Applications in Complex Analysis and Operator Theory

by Frej Dahlin

LUND
UNIVERSITY

A doctoral thesis at a university in Sweden takes either the form of a single, cohesive research study (monograph) or a summary of research papers (compilation thesis), which the doctoral student has written alone or together with one or several other author(s).

In the latter case the thesis consists of two parts. An introductory text puts the research work into context and summarizes the main points of the papers. Then, the research publications themselves are reproduced. The research papers may either have been already published or are manuscripts at various stages (in press, submitted, or in draft).

Cover illustration: *Towards Convergence* by Una

Faculty of Science, Centre for Mathematical Sciences

Doctoral Theses in Mathematical Sciences 2026:2
ISSN: 1404-0034
ISBN: 978-91-8104-864-3 (print)
ISBN: 978-91-8104-865-0 (pdf)
LUNFMA-1048-2026

Printed in Sweden by Media-Tryck, Lund University, Lund 2026

Media-Tryck is a Nordic Swan Ecolabel certified provider of printed material. Read more about our environmental work at www.mediatryck.lu.se

Printed matter 3041 0903

MADE IN SWEDEN

Dedicated to Una

Popular summary in English

Among the most well-known structures in modern mathematics are the *Hilbert spaces*. These generalizations of Euclidean vector spaces provide a geometric framework for function spaces through an inner product. Particularly noteworthy classes of function spaces are those with a *reproducing kernel*.

These Hilbert spaces, together with their functions and inner product, are encoded by a two-variable kernel $k(x, y)$. The broad research question is the following: given algebraic properties of a reproducing kernel, what can be said about the associated Hilbert space and its functions? In this thesis, reproducing kernels that arise as pointwise quotients of two other reproducing kernels are studied.

An important example of Hilbert spaces dates back to the 1960s and is known as the *de Branges–Rovnyak spaces*. These spaces arise naturally in the study of the classical *Hardy space* on the unit disk in the complex plane, which corresponds to the Szegő kernel

$$s(z, w) = \frac{1}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D} = \{u \in \mathbb{C} : |u| < 1\}.$$

Analytic selfmaps of the unit disk are intimately connected to the Szegő kernel. Indeed, if ϕ is a selfmap of the unit disk, then the quotient

$$s^{\phi}(z, w) = \frac{s(z, w)}{s(\phi(z), \phi(w))}, \qquad z, w \in \mathbb{D},$$

is also a reproducing kernel and corresponds to a de Branges–Rovnyak space.

Studying similar quotients yields far-reaching generalizations of de Branges–Rovnyak spaces. In the first article, an investigation is carried out of such kernels and their associated spaces, where the denominator of s^{ϕ} is replaced by an arbitrary kernel.

A typical problem in complex analysis is to understand how $\phi(z)$ behaves as the variable z approaches a point on the unit circle. A classical theorem of Julia and Carathéodory gives an equivalent condition describing the behaviour of ϕ and its derivative.

In the second article, this line of investigation is taken to its logical conclusion. Here the Szegő kernel in the quotient above is allowed to be replaced in both the numerator and the denominator, by possibly distinct kernels. Three new concepts are introduced, aimed at understanding how the functions in the space behave as the variable approaches the "boundary", which is now an entirely abstract construction.

The third article studies when the analogue of s^{ϕ}, in which s is replaced by an arbitrary kernel k on the polydisk, corresponds to a one-dimensional Hilbert space. If k is not a positive power of the Szegő kernel, then ϕ must be a rotation composed with a permutation.

Populärvetenskaplig sammanfattning på svenska

Bland de mest välkända strukturerna i modern matematik finns *Hilbertrum*. Dessa generaliseringar av euklidiska vektorrum tillgodoser ett geometriskt synsätt på funktionsrum i form av en skalärprodukt. Särskilt anmärkningsvärda funktionsrum är de som har en *reproducerande kärna*.

Dessa Hilbertrum, tillsammans med sina funktioner och skalärprodukt, kan beskrivas entydigt av en kärna $k(x, y)$ av två variabler. Den breda forskningsfrågan är då följande: givet att en reproducerande kärna har vissa algebraiska egenskaper, vad kan man säga om dess Hilbertrum och funktionerna däruti? I denna avhandling studeras reproducerande kärnor som i sin tur kan skrivas som en kvot av två andra reproducerande kärnor.

Ett exempel på Hilbertrum som korresponderar med sådana kärnor härstammar från 1960-talet och kallas *de Branges–Rovnyak-rum*. De uppkommer naturligt vid studiet av det klassiska Hardyrummet på enhetskivan i det komplexa talplanet, som korresponderar med Szegő-kärnan

$$s(z, w) = \frac{1}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D} = \{u \in \mathbb{C} : |u| < 1\}.$$

Analytiska självavbildningar av enhetskivan har en intim anknytning till Szegő-kärnan. Faktum är att om ϕ är en självavbildning av enhetskivan så är kvoten

$$s^{\phi}(z, w) = \frac{s(z, w)}{s(\phi(z), \phi(w))}, \qquad z, w \in \mathbb{D},$$

också en reproducerande kärna, och den korresponderar med ett de Branges–Rovnyak-rum.

Genom att studera liknande kvoter av reproducerande kärnor erhålls olika generaliseringar av de Branges–Rovnyak-rum. I den första artikeln studeras kärnor och deras korresponderande rum, där nämnaren av s^{ϕ} har bytts ut mot en godtycklig kärna.

Ett typiskt problem i komplex analys är att förstå hur $\phi(z)$ beter sig när variabeln z närmar sig en punkt på enhetscirkeln. En känd sats av Julia och Carathéodory ger ett ekvivalent villkor, som i detalj beskriver beteendet hos ϕ och dess derivata.

I den andra artikeln tas detta till dess logiska slut. Där tillåts Szegő-kärnan i kvoten ovan att bytas ut i både täljare och nämnare. Tre nya koncept introduceras, vilka alla syftar till att förstå hur funktionerna i rummet beter sig när variabeln närmar sig "randen", som numera är en helt abstrakt konstruktion.

Den tredje artikeln studerar när analogen till s^{ϕ}, där s ersätts av en godtycklig kärna k i polyskivan, korresponderar med ett endimensionellt Hilbertrum. Om k inte är en positiv potens av Szegő-kärnan måste ϕ vara en rotation komponerad med en permutation.

Acknowledgements

I am truly grateful to my supervisor, Alexandru Aleman. In the spring of 2017, you taught the *Integration Theory* course. It struck me how well you could convey deep ideas succinctly. I feel fortunate that I could develop as a mathematician under your guidance ever since then. You are always warm and welcoming whether discussing math, or life in general. And your patience in dealing with my flood of abstract hunches was tremendous. You taught me to organize, rather than blindly generalize.

A big thank-you to the faculty. In particular, to Anna-Maria Persson, who has helped me a lot during my entire education, and mentored me with my teaching; Sandra Pott, my co-supervisor, who inspired me greatly when she taught the *Functional Analysis* course, and supported me with the practical aspects of my PhD position; Kerstin Rogdahl, who has elegantly dealt with my administrative mishaps; and to all of my teachers and role models: Anitha, Erik W., Eskil, Evgeniy, Jan-Fredrik, Joachim, Marcus, Philipp, Sigmundur, and Yacin.

My colleagues have had a significant impact on me. Adem Limani, a close senior whom I could confide in. You were always ahead of me, so I find it poetic that our roles are reversed now when it comes to fatherhood. Joakim and Raul, who started their PhD alongside me. It has been a joy to witness your friendship grow from nothing. Alex, my academic younger brother and intellectual spur. You have brightened many of my days, and your integrity is infectious. Many more contributed to my academic growth, and more importantly, my personal growth. To name a few: Athanasios, Bartosz, Emil, Erik T., Felix, Germán, Giang, Hampus, Jaime, Jonathan, Jorge, Magnus, Mats, Mårten, Stefano, Teodor, Tien, Tom, Olof, and Valentina. Thank you, all of you, including anyone I may have inadvertently omitted.

I had the opportunity to meet some wonderful, genuine, and like-minded people during my conference visits. Meeting Chris Felder in Tenerife was particularly pertinent. Our shared enthusiasm for reproducing kernels reinvigorated me greatly and led me to write my favorite paper. And recurring encounters with Ryan O'Loughlin gave me a much-needed break from long technical talks. Seldom do I meet a person who is as easy to talk to.

I thank my students for allowing me to play *the sage on the stage*, briefly enabling my much-belated acting career. Thanks to my neighbors, who have gracefully overlooked the declining state of my garden during the writing of this thesis. Thanks also to my dog Luka for preserving my sanity by giving me an excuse to go for a walk.

It has been the utmost pleasure to share this ten-year journey with my friend Emil Engström. Since we met in 2015, we have had our fair share of student life, meaningful dialogues, and gym sessions. From naïve teenagers to educated men. I wonder what the future holds for our company.

Ever since moving to Lund, I have been blessed with a multitude of new friends. I am particularly grateful to my closest friends the last couple of years: Antonio, August, Elin, Ellinor, Joel, Karin, Selim, and Simon. It is remarkable that I have not lived alone at any point in these ten years; thanks in part to my former flatmates: Theodor, Robert, and Jesper. I will never forget our shared antics at *Epikureiska Akuten*. During my first year, I joined a strong friend and study group, consisting of Ante, Emil, Henrik, Owen, and me. I reminisce about the countless hours of studying and arguing over everything under (and beyond) the sun.

During my high school years in Stockholm, my mathematical interest was set ablaze by my teacher Johan Osterman. I pursued mathematics at university because of you. Furthermore, I am comforted by the fact that I still have a solid set of friends from these times: Alex, Antonio, Arvid, Axel, Carl, Julia, Leon, Niki, Nils, Olle, Peter, Simon, Sofia, and Tobias. Further back still, I am incredibly lucky to have my oldest friends David, Ming, and William. For the majority of my life, you have stuck with me through all my ups and downs. Thank you.

Above all, a heartfelt thank-you goes out to my entire family. You have all helped form the person that I am today. However, I feel the need to extend thanks to some of you individually. To my late grandmother Birgit, for teaching me to inquire enthusiastically and listen with intent. To my grandfather Jan, who sparked creativity within me from a very early age. To my late uncle, who imprinted his gentle nature upon me. To my mother, who nurtured me and continuously believed in me, regardless of my performance in school. To my future mother-in-law, who is not only an excellent grandmother and babysitter to my daughter, but also a mentor throughout my PhD. To my sister, who showed me that it is possible to do your best work while being a fantastic parent to a young child.

To my daughter, who has imbued me with a deeper sense of purpose.
Finally, to my partner, who has given me more than I can put into words.

Una, jag älskar dig.

List of publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Generalized de Branges–Rovnyak spaces**

A. Aleman, **F. Dahlin**
Journal of Functional Analysis, Volume 288, Issue 11, 1 June 2025

II **Boundary values via reproducing kernels: the Julia–Carathéodory theorem**

F. Dahlin
Canadian Journal of Mathematics, to appear

III **Rank 1 composition factors in the polydisk**

F. Dahlin
Working paper

All papers are reproduced with permission of their respective publishers.

Contents

Quotients of Reproducing Kernels:
Applications in Complex Analysis and Operator Theory

To see a world in a grain of sand
And a heaven in a wild flower,
Hold infinity in the palm of your hand
And eternity in an hour.
— William Blake

1 Introduction

Towards the tail end of my bachelor's degree I was taught the *soft* approach to analysis. After being shown how to prove Beurling's theorem through reproducing kernels, I fell in love, and the present thesis contains three works in this direction.

In the course of writing these articles, I came to realize that complex analysis has become largely a pretext for me to study reproducing kernels. Potentially uncountably many functions, united under a geometric structure, are all compressed into a single function of two variables. Simplicity is their allure.

A priori, a reproducing kernel encodes all the information about the functions and operators of a particular Hilbert space; the problem lies in unpacking this knowledge, which can be difficult in general. If one fixes certain algebraic properties of a kernel, what can then be said about the corresponding Hilbert space?

In this thesis I study a subclass of reproducing kernels that are realized as a pointwise quotient of two other reproducing kernels. There are two main reasons for this constraint. The kernels studied herein should be seen as corresponding to far-reaching generalizations of de Branges–Rovnyak spaces. These classical spaces serve as guiding examples throughout my work. Moreover, there is the fundamental theorem of Schur, which says that the pointwise product of two reproducing kernels is also a reproducing kernel. I adopt a converse perspective, and hence Schur's product theorem becomes an effective tool.

To explain my disposition towards reproducing kernels, let us first visit the most iconic destination in complex analysis: the *Hardy space*.

2 The Hardy space H^2 and boundary values

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk in the complex plane, and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote its boundary, the unit circle.

The Hardy space H^2 consists of analytic functions f on $\mathbb{D}$ that satisfy

(1)
$$\sup_{0\le r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < \infty.$$

This is a classical subject, and it would be impossible to cover all of its facets here. Instead we will focus our attention on a particular circle of ideas.

Due to their analyticity, these functions are extremely well behaved when restricted to an open subset of $\mathbb{D}$. Hence a natural question arises: *for a function $f \in H^2$, what is the behaviour of $f(z)$ as z approaches the boundary of $\mathbb{D}$?* The above condition says that the quadratic means stay bounded as the radius tends to 1, suggesting that H^2 functions behave reasonably well near the boundary. However, they may even have an infinite number of singularities on $\mathbb{T}$.

Nonetheless, it is possible to say quite a lot. For a point $\zeta \in \mathbb{T}$, it is important exactly how the variable $z \in \mathbb{D}$ tends to ζ. Without additional constraints, general boundary behaviour is difficult to grasp. However, we do know that $|f(z)|$ must grow at a rate strictly slower than $(1-|z|)^{-\frac{1}{2}}$ as $|z|$ tends to 1.

The most natural constraint is along the ray towards ζ originating from 0. We say that $f(z)$ has a *radial* limit at ζ, if $f(r\zeta)$ converges as $r \to 1^-$. Surprisingly, Abel showed already in 1826 that a weaker notion of boundary limit is more appropriate. Specifically, if f has a radial limit at $\zeta \in \mathbb{T}$, then f has a *nontangential* limit at ζ, in the sense that $f(z)$ is convergent as $z \to \zeta$ when constrained to a fixed *Stolz sector*

(2)
$$\Gamma(\alpha, \zeta) = \{z \in \mathbb{D} : |\zeta - z| \le \alpha(1-|z|)\}, \qquad \alpha > 0.$$

The first theorem of boundary limits in our setting is due to Fatou in 1906. Here we state a refined and tailored version of the original, but more general variants exist [23].

Theorem (Fatou). *Every function $f \in H^2$ has nontangential limits almost everywhere on $\mathbb{T}$. The corresponding sequence of radial dilates $f_r(z) = f(rz)$ converges in $L^2(\mathbb{T})$ as $r \to 1^-$.*

This leads to an identification of H^2 as a closed subspace of $L^2(\mathbb{T})$. And at this point, I would be neglectful if I did not recall the famous *inner–outer factorization* proved by Smirnov in 1928.

Theorem (Smirnov). *Every nonzero function $f \in H^2$ can be uniquely factorized, up to a unimodular constant, as*

$$f(z) = I(z)F(z), \qquad z \in \mathbb{D}, \tag{3}$$

where I is an inner *function and F is an* outer *function.*

By definition, I is an inner function if and only if $|I|$ tends to 1 almost everywhere on $\mathbb{T}$. Moreover, it can also be factorized into two parts

$$I(z) = \exp\left(-\int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w)\right) \prod_{j=0}^{\infty} \frac{|a_j|}{a_j} \frac{a_j - z}{1 - \overline{a_j} z}, \qquad \sum_{j=0}^{\infty} 1 - |a_j| < \infty, \tag{4}$$

where μ is a positive singular measure. The first factor is called the *singular* part, and the second is a (possibly infinite) Blaschke product with zeros (a_j), which are capable of encoding surprisingly detailed geometric information through the placement of their zeros.

Figure 1: The facial silhouette of the author approximated by a finite Blaschke product. Visualized using binary domain coloring; the zeros were optimized by gradient descent.

Since outer functions do not play a role in this thesis, I omit a detailed definition; for completeness, they are all of the form

$$F(z) = \lambda \exp\left(\int_{\mathbb{T}} \frac{w+z}{w-z} G(w)\, dm(w)\right), \qquad \lambda \in \mathbb{T}, \tag{5}$$

where dm is the normalized Lebesgue measure on $\mathbb{T}$ and $G \in L^1(\mathbb{T})$.

Before we move on, I want to emphasize that the Hardy space constitutes a rich class of functions. Any choice of the singular measure μ, the zeros (a_j) satisfying the above condition, and the integrable function G will yield a unique function in H^2 through (3).

Another angle of inquiry concerns analytic selfmaps of $\mathbb{D}$. These endomorphisms are intimately related to H^2 in the following way. Every $\phi : \mathbb{D} \to \mathbb{D}$ induces a contractive *multiplication operator* by

$$M_\phi f(z) = \phi(z) f(z), \qquad z \in \mathbb{D}, \tag{6}$$

and the norm is given by the formula $\|M_\phi\|_{H^2} = \sup_{z\in\mathbb{D}} |\phi(z)|$. Except for the unimodular constant functions $z \mapsto \lambda \in \mathbb{T}$, the analytic selfmaps of $\mathbb{D}$ are the only such functions. Interestingly, in an appropriate sense deferred to the next section, H^2 is the smallest Hilbert space with the above property. Moreover, Littlewood proved in 1925, see for example [31], that every selfmap ϕ also induces a *composition operator* on H^2 defined by

$$C_\phi f(z) = f(\phi(z)), \qquad z \in \mathbb{D}. \tag{7}$$

Remarkably, the symbols for contractive multiplication operators (except for the unimodular constants) and bounded composition operators overlap in the Hardy space.

With the above consideration in mind, the second classical theorem of boundary limits that I wish to highlight is the classical Julia–Carathéodory theorem from 1929.

Theorem (Julia–Carathéodory). *Given a point $\zeta \in \mathbb{T}$ and an analytic selfmap ϕ of the unit disk, the following are equivalent:*

(i)

$$\liminf_{z \to \zeta} \frac{1 - |\phi(z)|}{1 - |z|} = c < \infty.$$

(ii) ϕ and its derivative ϕ' have nontangential limits equal to $\lambda \in \mathbb{T}$ and $c\lambda\overline{\zeta}$ at ζ, respectively.

Furthermore, if any of the above hold, then $c > 0$ and ϕ has horocyclic *limit λ at ζ, meaning the approach to ζ is constrained to a fixed* horocycle

$$E(M, \zeta) = \{z \in \mathbb{D} : |\zeta - z|^2 \le M(1 - |z|^2)\}, \; M > 0. \tag{8}$$

The strength of this theorem lies in the fact that (i) is easy to check and yields much more precise information compared to Fatou's theorem. Eventually (ii) became a definition, and it is often summarized by saying that ϕ has an *angular derivative* at ζ. An application of this notion yields a criterion for the compactness of the composition operator C_ϕ by MacCluer, Shapiro, and Taylor [25, 32].

Theorem (MacCluer–Shapiro–Taylor). *Let ϕ be an analytic selfmap. If C_ϕ is compact, then ϕ has no angular derivative at any point of $\mathbb{T}$. In addition, the converse holds if ϕ is univalent (meaning injective).*

After giving you this glimpse into the Hardy space, let us now reconnect to reproducing kernels. Here comes the staggering truth: H^2 is *equivalently reproduced* by the Szegő kernel

$$s(z, w) = \frac{1}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D}. \tag{9}$$

In principle, all the complexity of H^2 discussed above is encapsulated in this simple formula. The challenge, however, lies in decoding it.

Mathematicians have been able to prove, and generalize, many classical theorems through this point of view. Some representative examples include: Fatou's theorem by Hartz and Chalmoukis [9]. A version of the inner-outer factorization by Aleman, Hartz, McCarthy, and Richter [3] continuing the work of Jury and Martin [22]. As well as the Julia–Carathéodory theorem by Sarason [29], Jury [17], and myself in Paper II.

Describing my contribution requires the introduction of the de Branges–Rovnyak spaces, but first, let us make the connection between the Hardy space and the Szegő kernel precise.

3 Definitions and fundamental theory of reproducing kernels

The theory of reproducing kernels is rich and beautiful. For a complete overview and history, see the famous article from 1950 of Aronszajn [5] or the more contemporary introduction by Paulsen and Raghupathi [28]. Here I will only give the necessary background for the reader to understand the broader context of my research.

There are two main ways of defining *reproducing kernels*. Let X be a nonempty set and fix $k : X \times X \to \mathbb{C}$. We will often write $k_y = x \mapsto k(x, y), x, y \in X$.

Definition 1 (Via Hilbert spaces)**.** k is a *reproducing kernel* if and only if there exists a Hilbert space $\mathcal{H}(k)$ of functions on X such that for all $y \in X$, $k_y \in \mathcal{H}(k)$ and

$$f(y) = \langle f, k_y \rangle, \qquad \forall f \in \mathcal{H}(k). \tag{10}$$

In this case k is called the reproducing kernel of $\mathcal{H}(k)$.

Remark 1. Most authors introduce reproducing kernels using a variation of the above. Suppose that $\mathcal{H}$ is a Hilbert space of functions on X that satisfies the following property. For every $y \in X$ there exists a constant C_y such that

$$|f(y)| \le C_y \|f\|, \qquad \forall f \in \mathcal{H}.$$

This means that the *evaluation functional* $E_y : f \mapsto f(y), f \in \mathcal{H}$ is continuous. Hence, by the Riesz representation theorem there exists a function g_y that satisfies (10). In this case the reproducing kernel of $\mathcal{H}$ is the function $x, y \mapsto g_y(x), x, y \in X$.

It is in this sense that the Szegő kernel corresponds to the Hardy space. But what about the converse: Starting with the reproducing kernel, can we recreate the Hilbert space? The following proposition hints that this is indeed possible.

Proposition 1. *Let k be a reproducing kernel, then the linear span of $\{k_x : x \in X\}$ is dense in $\mathcal{H}(k)$.*

Proof. Let $f \in \mathcal{H}(k)$ with $f \perp k_x$ for all $x \in X$. Then $f \equiv 0$ by (10), and the result follows. □

However, to fully answer the question above, we first require a different definition.

Definition 2 (Via matrices). k is a *reproducing kernel* if and only if k is Hermitian, meaning $k(x, y) = \overline{k(y, x)}$, $\forall x, y \in X$, and for every finite subset $\{x_1, \dots, x_n\} \subset X$, the matrix

$$G_k = [k(x_i, x_j)]_{i,j=1}^n = \begin{pmatrix} k(x_1, x_1) & \dots & k(x_1, x_n) \\ \vdots & \ddots & \vdots \\ k(x_n, x_1) & \dots & k(x_n, x_n) \end{pmatrix} \tag{11}$$

is positive semi-definite, i.e. $a^* G_k a \geq 0$ for every $a \in \mathbb{C}^n$. In this case we write $k \geq 0$, or

$$k(x, y) \geq 0, \qquad x, y \in X. \tag{12}$$

It is not difficult to prove that if k satisfies Definition 1, then it also satisfies Definition 2. The following fundamental theorem by Moore proves the converse, which unites the two definitions and therefore asserts an intimate relationship between reproducing kernels and its corresponding Hilbert space.

Theorem (Moore). *Let k satisfy Definition 2, then there exists a unique Hilbert space $\mathcal{H}(k)$ of functions on X, such that k satisfies Definition 1 within $\mathcal{H}(k)$.*

From now on throughout this section, we fix k to be a reproducing kernel on X.

Thanks to Moore's theorem it is possible, in theory, to answer any question about a Hilbert space through its reproducing kernel, even in a completely abstract setting. As a brief aside, let me illustrate with a proposition that generalizes the $o((1 - |z|)^{-\frac{1}{2}})$ estimate for H^2 that I mentioned in the previous section. This is extended in Paper II.

Proposition 2. *Let (x_n) be a sequence in X such that $0 \neq \|k_{x_n}\|^2 = k(x_n, x_n) \to \infty$ as $n \to \infty$, and suppose that for all $y \in X$ there exists a constant M_y such that $|k(x_n, y)| < M_y$. Then the sequence of normalized kernel functions $\hat{k}_{x_n} = \frac{k_{x_n}}{\|k_{x_n}\|}$ converges weakly to 0 in $\mathcal{H}(k)$. Equivalently,*

$$\lim_{n \to \infty} \frac{f(x_n)}{k(x_n, x_n)^{\frac{1}{2}}} = 0, \qquad \forall f \in \mathcal{H}(k). \tag{13}$$

Proof. Indeed, $\|\hat{k}_{x_n}\| = 1$, meaning they are norm-bounded, therefore it is sufficient to prove that $(\hat{k}_{x_n})$ tends pointwise to 0 by Proposition 1. But $|\hat{k}(y, x_n)| = \frac{|k(y,x_n)|}{\|k_{x_n}\|}$, and the result then follows by assumption. □

A more concrete answer to our original question is provided by the following criterion, which says exactly what functions belong to $\mathcal{H}(k)$ only in terms of k.

Proposition (Inclusion criterion). *Let k be a reproducing kernel on a nonempty set X. A function $f : X \to \mathbb{C}$ belongs to $\mathcal{H}(k)$ if and only if there is a constant $A > 0$ such that*

$$Ak(x, y) - f(x)\overline{f(y)} \geq 0, \qquad x, y \in X. \tag{14}$$

Moreover, the least constant admissible is $A = \|f\|^2$.

Often it is reasonable to impose that k is *normalized* at a point $x_0 \in X$, which means that $x \mapsto k(x, x_0) \equiv 1$. For these kernels, there is an immediate consequence of the inclusion criterion that is useful in practice.

Proposition 3. *Let k be a normalized reproducing kernel on a nonempty set X, then*

$$k(x, y) \geq 1, \qquad x, y \in X. \tag{15}$$

My favorite theorem of reproducing kernels, which I apply frequently, is the following.

Theorem (Schur's product theorem). *In addition to k, let t also be a reproducing kernel on X, then their pointwise product is also a reproducing kernel,*

$$k(x, y)t(x, y) \geq 0, \qquad x, y \in X. \tag{16}$$

3.1 Symbols inducing operators

Throughout this subsection, fix k and t to be reproducing kernels on some nonempty sets X and Y, respectively.

Reproducing kernels are frequently used to study operators on, or between, their corresponding spaces. The primary examples are the *multiplication* and *composition* operators mentioned previously in the context of H^2, for brevity, we shall consider them as special cases of *weighted composition* operators.

A map $\phi : X \to Y$ and function $\delta : X \to \mathbb{C}$ induce a weighted composition operator from $\mathcal{H}(t)$ to $\mathcal{H}(k)$ if and only if $x \mapsto \delta(x)f(\phi(x))$ belongs to $\mathcal{H}(k)$ for every function $f \in \mathcal{H}(t)$. By the closed graph theorem, a simple argument proves that the operator $W_{\delta,\phi} : \mathcal{H}(t) \to \mathcal{H}(k)$ defined by $W_{\delta,\phi}f(x) = \delta(x)f(\phi(x))$ is bounded.

In terms of reproducing kernels, there is the following criterion that is analogous to the inclusion criterion.

Proposition 4. *A map $\phi : X \to Y$ and function $\delta : X \to \mathbb{C}$ induce a weighted composition operator from $\mathcal{H}(t)$ to $\mathcal{H}(k)$ if and only if there is a constant $A > 0$ such that the function*

$$Ak(x,y) - \delta(x)\overline{\delta(y)}t(\phi(x),\phi(y)) \geq 0, \qquad x,y \in X. \tag{17}$$

Moreover, the least constant admissible above is $A = \|W_{\delta,\phi}\|^2$.

When $\delta \equiv 1$, then we say that ϕ induces a composition operator C_ϕ. If $X = Y$ and ϕ is the identity map, then we say that δ is a multiplier, i.e. it induces a multiplication operator M_δ. Furthermore, in this case (17) is often referred to as the 'multiplier criterion'

$$Ak(x,y) - \delta(x)\overline{\delta(y)}t(x,y) \geq 0, \qquad x,y \in X. \tag{18}$$

In my research I encountered a third important special case. The case when the operator $W_{\delta,\phi}$ is co-isometric, i.e. $W_{\delta,\phi}W^*_{\delta,\phi} = I$. To my knowledge, the first study of co-isometric weighted composition operators in $\mathbb{D}$ was due to Mas, Martín, and Vukotić [26]. One should also mention the work of Hartz and Tornes [16] that extends their results to the unit ball in several variables. Given that $W_{\delta,\phi}$ is co-isometric, a computation shows that

$$k(x,y) = \delta(x)\overline{\delta(y)}t(\phi(x),\phi(y)), \qquad x,y \in X. \tag{19}$$

Note that $x,y \mapsto \delta(x)\overline{\delta(y)}$ and $x,y \mapsto t(\phi(x),\phi(y))$ are themselves reproducing kernels. In other words, k is factored by $t\circ\phi$, and the quotient kernel $\delta\overline{\delta}$ corresponds to a one-dimensional Hilbert space. Therefore, I call ϕ a *rank* 1 *composition factor* of k with t, and I continue the work of the above authors using this definition in Paper III. In the general case, we can consider the equation (19) where $\delta\overline{\delta}$ is replaced by any reproducing kernel. In particular, when $k = t$ the theory is quite elegant.

Proposition 5. *The set of all composition factors of k forms a semigroup under composition.*

Proof. For simplicity, suppose that k is zero-free, the general case is proved in a similar way. Let ϕ and ψ be composition factors of k. By definition, this means that the quotients satisfy

$$k^\theta(x,y) = \frac{k(x,y)}{k(\theta(x),\theta(y))} \geq 0, \qquad \theta = \phi,\psi, \quad x,y \in X. \tag{20}$$

Note that $x, y \mapsto k^{\psi}(\phi(x), \phi(y))$ is also a reproducing kernel, and a computation reveals that

$$k^{\psi\circ\phi}(x,y) = \frac{k(x,y)}{k(\psi(\phi(x)), \psi(\phi(y)))} = k^{\phi}(x,y)k^{\psi}(\phi(x), \phi(y)), \qquad x, y \in X. \tag{21}$$

It then follows by Schur's product theorem that $k^{\psi\circ\phi} \geq 0$. □

It is often a stronger requirement for a selfmap to be a composition factor than inducing a composition operator. However, sometimes the two notions overlap.

3.2 Back to H^2

Now I want to uphold a promise made in the prior section.

Proposition 6 (Minimality of H^2). *Let k be a reproducing kernel on $\mathbb{D}$ that is normalized at* 0. *Suppose that every analytic selfmap of $\mathbb{D}$ is a contractive multiplier of $\mathcal{H}(k)$, then $H^2 \subset \mathcal{H}(k)$.*

Proof. In particular, the identity function is a contractive multiplier of $\mathcal{H}(k)$. Therefore by the multiplier criterion, $z, w \mapsto k(z,w)(1-z\overline{w})$, $z, w \in \mathbb{D}$, is a normalized reproducing kernel. Hence by Proposition 3

$$k(z,w)(1 - z\overline{w}) \geq 1, \qquad z, w \in \mathbb{D}. \tag{22}$$

Using Schur's product theorem, multiplying by the Szegő kernel yields

$$k(z,w) \geq s(z,w), \qquad z, w \in \mathbb{D}, \tag{23}$$

which finishes the proof by using the inclusion criterion once more. □

While we are discussing H^2 again, I want to show a proof of the essential part of Littlewood's theorem.

Proposition 7 (Littlewood's subordination principle). *Let ϕ be an analytic selfmap of $\mathbb{D}$, with $\phi(0) = 0$. Then for every function $f \in H^2$,*

$$C_{\phi}f \in H^2 \qquad \textit{and} \qquad \|C_{\phi}f\| \leq \|f\|. \tag{24}$$

Proof. Note first that

$$s(\phi(z), \phi(w)) = \frac{1}{1 - \phi(z)\overline{\phi(w)}} = \sum_{j=0}^{\infty} (\phi(z)\overline{\phi(w)})^j \tag{25}$$

is a reproducing kernel normalized at 0. Moreover, the multiplier critierion yields that

$$\frac{1 - \phi(z)\overline{\phi(w)}}{1 - z\overline{w}} \geq 1, \qquad z, w \in \mathbb{D}, \tag{26}$$

since the above reproducing kernel is normalized at 0. By multiplying with $s \circ \phi$, applying Schur's product theorem, and reordering, we obtain

$$s(z, w) - s(\phi(z), \phi(w)) \geq 0, \qquad z, w \in \mathbb{D}, \tag{27}$$

and the result follows by the composition criterion, i.e. Proposition 4. □

Remark 2. It is notable that the starting point for Littlewood's original proof also uses the fact that ϕ induces a contractive multiplication operator.

Remark 3. The above line of reasoning can be extended to directly prove that C_ϕ maps H^2 into itself, even if $\phi(0) \neq 0$. In fact, this is performed within a completely abstract setting in Paper II.

The key observation is that for every analytic selfmap ϕ of $\mathbb{D}$, due to the multiplier criterion, we can define a class of reproducing kernels by

$$s^\phi(z, w) = \frac{1 - \phi(z)\overline{\phi(w)}}{1 - z\overline{w}} = \frac{s(z, w)}{s(\phi(z), \phi(w))} \qquad z, w \in \mathbb{D}. \tag{28}$$

Hence we obtain three classifications of all analytic selfmaps of $\mathbb{D}$. As the contractive multipliers of H^2 (disregarding the unimodular constants), as the symbols inducing composition operators on H^2, as well as the composition factors of the Szegő kernel!

The reproducing kernels in (28) are by no means as well behaved as the Szegő kernel. Nonetheless, the study of their corresponding Hilbert spaces has attracted a lot of attention over the last 50 years. Let us now look into the primary vessel for my work: the shadows cast by H^2.

4 de Branges–Rovnyak spaces

In 1966, de Branges and Rovnyak introduced their eponymous Hilbert spaces [7, 8]. From the operator-theoretic view, these spaces are classified by the property that the backward shift operator defined by

$$Lf(z) = \frac{f(z) - f(0)}{z}, \qquad z \in \mathbb{D}.$$

is contractive, see [4, 2]. This provides a functional model for certain contractive operators that is distinct from the well-known Sz.-Nagy–Foiaş model, see for example [27].

In this thesis, I take the kernel-centric view instead. Each de Branges–Rovnyak space corresponds to a reproducing kernel s^ϕ given by (28) through an analytic selfmap ϕ of $\mathbb{D}$. What is the relation of the symbol ϕ, the reproducing kernel s^ϕ, and the Hilbert space $\mathcal{H}(s^\phi)$? A pioneer in this direction is the late Donald Sarason. He invented many new techniques and discovered much of the fundamental structure of the $\mathcal{H}(s^\phi)$ spaces, or *sub-Hardy spaces* as he called them. His work culminated in a concise yet dense ~100-page book [30] that has greatly inspired me. The interested reader may also consider the recent books by Fricain and Mashreghi [11, 12].

Let us begin our short exposition with a simple proposition.

Proposition 8. *The space $\mathcal{H}(s^\phi)$ is contractively contained in H^2. Meaning $\mathcal{H}(s^\phi) \subset H^2$ with the inclusion map being contractive.*

Proof. The difference between the Szegő kernel and the corresponding de Branges–Rovnyak kernel is easily seen to be

$$s(z,w) - s^\phi(z,w) = \frac{\phi(z)\overline{\phi(w)}}{1 - z\overline{w}}, \qquad z, w \in \mathbb{D}. \tag{29}$$

It is immediate that $z, w \mapsto \phi(z)\overline{\phi(w)}$ is a reproducing kernel. Therefore, the above is also a reproducing kernel due to Schur's product theorem. The result then readily follows by the inclusion criterion. □

The above answers what functions may possibly reside in $\mathcal{H}(s^\phi)$. Apart from the kernel functions $s_w^\phi, w \in \mathbb{D}$, it is difficult in general to provide explicit examples of functions that belong to $\mathcal{H}(s^\phi)$. For instance, it is not always true that the polynomials reside in $\mathcal{H}(s^\phi)$, even though they constitute an orthonormal basis of H^2. Sarason gave a fantastic structural response to this inquiry. First recall that H^∞ denotes the bounded analytic functions on $\mathbb{D}$.

Theorem (Sarason's dichotomy). *For an analytic selfmap ϕ of $\mathbb{D}$, the following are equivalent:*

(i) *ϕ is not an extreme point in the unit ball of H^∞,*

(ii) $\phi \in \mathcal{H}(s^\phi)$,

(iii) *$\mathcal{H}(s^\phi)$ is invariant under the shift operator defined by $M_z f(z) = zf(z)$,*

(iv) *the polynomials are dense in $\mathcal{H}(s^\phi)$.*

I want to discuss one more contribution from Sarason, namely his proof, and refinement, of the Julia–Carathéodory theorem.

Theorem (Sarason). *Given a point $\zeta \in \mathbb{T}$ and an analytic selfmap ϕ of $\mathbb{D}$, then the following are equivalent:*

(i)

$$\liminf_{z\to\zeta} \frac{1 - |\phi(z)|}{1 - |z|} = c < \infty,$$

(ii) *There is a $\lambda \in \mathbb{T}$ such that the function defined by*

$$q_\zeta(z) = \frac{1 - \phi(z)\overline{\lambda}}{1 - z\overline{\zeta}}$$

belongs to $\mathcal{H}(s^\phi)$.

(iii) *All functions in $\mathcal{H}(s^\phi)$ have nontangential limits at ζ.*

Furthermore, if any of the above hold, then ϕ has horocyclic limit λ at ζ and $\|q_\zeta\|^2 = c$.

Once again, Sarason links a function theoretic condition on ϕ to a statement about all functions in $\mathcal{H}(s^\phi)$; this is a theorem in its own right, not just a novel proof.

I defer a more detailed discussion to Paper II. However, a brief note on how the proof starts is in order. The brilliant observation that Sarason made is twofold. Firstly, (i) is equivalent to

$$\liminf_{z\to\zeta} \frac{1 - |\phi(z)|^2}{1 - |z|^2} = c < \infty. \tag{30}$$

Secondly, the quotient in (30) is exactly equal to $s^\phi(z,z) = \|s_z^\phi\|^2$. Consequently, supposing that (i) holds, then there is a sequence $z_n \to \zeta$ such that the sequence of kernel functions $s_{z_n}^\phi$ is norm-bounded. By invoking weak compactness to obtain a convergent subsequence of the kernel functions, the stage is set for the rest of the proof.

With these two results of Sarason as a backdrop, let us turn to some generalizations of de Branges–Rovnyak spaces. I am aware of two different generalizations that precede our work in Paper I.

4.1 Sub-Bergman spaces

The *sub-Bergman spaces* were introduced by Zhu [35] in 1996. Just as in the classical de Branges–Rovnyak space, each space stems from an analytic selfmap ϕ of $\mathbb{D}$, written as $A^2(\phi)$. They correspond to reproducing kernels of the form

$$\frac{1-\phi(z)\overline{\phi(w)}}{(1-z\overline{w})^2}, \qquad z,w \in \mathbb{D}. \tag{31}$$

Just as in the case of de Branges–Rovnyak spaces, one may observe sub-Bergman spaces from operator-theoretic viewpoint. The link is provided by the famous inequality of Shimorin [33],

$$\|M_z f + g\|^2 \leq 2(\|f\|^2 + \|M_z g\|^2), \qquad \forall f,g \in A^2(\phi), \tag{32}$$

which we extend in Paper I.

Sub-Bergman spaces garnered some attention and led to subsequent papers [34] and further extensions. For instance, one may also consider the kernels

$$\frac{1-\phi(z)\overline{\phi(w)}}{(1-z\overline{w})^\alpha}, \qquad \alpha > 1, \quad z,w \in \mathbb{D}, \tag{33}$$

which are weighted variants $A_\alpha^2(\phi)$, studied by Abkar and Jafarzadeh [1] as well as Shuaibing and Zhu [24].

Sarason's dichotomy disappears in this setting. For example as proved by Chu [10], when $\alpha = 2$, then the polynomials are always dense in $A^2(\phi)$, no matter what ϕ is. An explanation of this phenomenon can be obtained from the reproducing kernel. The kernel in (31) can be factorized as $s^\phi \cdot s$, hence, by virtue of being factored by the Szegő kernel, the corresponding space inherits much of the regularity of H^2.

From this viewpoint, it is perhaps not surprising that the same result holds for $\alpha \geq 2$. Chu's proof relied on function-theoretic methods, which did not extend to the range $1 < \alpha < 2$. This case was for a short time an open problem, see [13], that we resolved in Paper I.

Much less studied is the case when $\alpha < 1$, and it seems quite difficult. Not every analytic selfmap of $\mathbb{D}$ is a contractive multiplier of the so-called standard weighted *Dirichlet spaces*. I invite the reader to compare with the case of several variables below. However, using Sarason's methods, it is possible to extract a refined version of the Julia–Carathéodory theorem in this case. Specifically, the Julia condition is modified to

$$\liminf_{z \to \zeta} \frac{1 - |\phi(z)|^2}{(1 - |z|^2)^\alpha} = c < \infty, \tag{34}$$

which is weaker than the classical condition. For an exact statement and details see Paper II. It would be very interesting to try to extend this result to every analytic selfmap of $\mathbb{D}$.

4.2 Sub-Drury-Arveson spaces

For a fixed dimension $d > 0$ one may consider many different analogues to the Szegő kernel in several variables. Even so, from the kernel-centric view, the Drury-Arveson kernel is the clear choice, see for example the recent survey by Hartz [14]. It is defined on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ by

$$S(z, w) = \frac{1}{1 - \langle z, w \rangle}, \qquad z, w \in \mathbb{B}_d. \tag{35}$$

Consider then, analytic selfmaps Φ of $\mathbb{B}_d$ such that

$$S^\Phi(z, w) = \frac{1 - \langle \Phi(z), \Phi(w) \rangle}{1 - \langle z, w \rangle} = \frac{S(z, w)}{S(\Phi(z), \Phi(w))}, \qquad z, w \in \mathbb{B}_d \tag{36}$$

defines a reproducing kernel. Importantly, only the contractive (vector-valued) multipliers of the Drury-Arveson space are permissible above; these are a strict subset of all analytic selfmaps of $\mathbb{B}_d$, and are often referred to as the *Schur-Agler class*.

I cannot definitively answer the question of who first studied the $\mathcal{H}(S^\Phi)$ spaces, as the condition (36) was surely known for quite some time. But some early references are by Jury [19] in 2007, and by the authors Ball, Bolotnikov, and Fang [6] in 2008. Given that this is a very natural generalization, I expect that multiple authors arrived at it independently.

Regardless, I first encountered them through the later work of Jury. He has been successful in carrying out large parts of Sarason's program in this more intricate setting, see for example [19, 18], and together with Martin [20, 21]. In particular, the last paper, together with a result of Hartz [15], gives an analogue to Sarason's dichotomy. Jury also has an unpublished generalization of the Julia–Carathéodory theorem [17], which caused me a lot of headaches.

I independently arrived at the same generalization around 2022, and subsequently began drafting a paper to be published. At the time, I was unaware of the terminology "Schur-Agler class", which explains why I did not encounter the relevant article earlier. Discovering this during a later and more careful search was, while initially disconcerting, ultimately a fortunate development as it led me to my best work in Paper II.

5 Summary of the research papers

This section provides a summary of the three research articles that constitute this thesis. Each article concerns different, but sometimes overlapping, generalizations of de Branges–Rovnyak spaces. However, their respective reproducing kernels are all realized as quotients of two other reproducing kernels. In fact, all major results are derived from the theory of reproducing kernels.

Paper I

The first paper is concerned with a generalization of de Branges–Rovnyak spaces via the multiplier criterion. Let k be a reproducing kernel on a nonempty set X, and let b be a contractive multiplier of $\mathscr{H}(k)$. We study reproducing kernels of the form

$$k^b(x,y) = k(x,y)(1 - b(x)\overline{b(y)}) = \frac{k(x,y)}{s(b(x),b(y))}, \qquad x,y \in X, \tag{37}$$

as well as their spaces. This frames the reproducing kernels of sub-Bergman spaces and sub-Drury-Arveson spaces as quotients. In each case, the denominator is given by the Szegő kernel or the Drury-Arveson kernel, composed with a function. Some of these kernels are classified via operator inequalities, such as a famous inequality of Shimorin that we extend. In this setting, we develop a model for $\mathscr{H}(k^b)$ reminiscent of the Sz.-Nagy–Foiaş model for contractions. It yields sufficient conditions for the containment and density of the linear span of $\{k_y : y \in X\}$. In particular, it resolves a conjecture in [13] regarding the density of polynomials in spaces with reproducing kernel $\frac{(1-b(z)b(w)^*)^m}{(1-z\overline{w})^\beta}$, $1 \le m \le \beta, m \in \mathbb{N}$ on $\mathbb{D}$.

Paper II

The second paper is motivated by Sarason's more abstract version and proof of the Julia–Carathéodory theorem. Let k be a reproducing kernel on a nonempty set X. We define and study a notion of boundary for X using the reproducing kernel k. Approach regions to the *reproductive boundary* are also given in this setting. Furthermore, composition factors are introduced as a kernel-theoretic alternative to analytic selfmaps of $\mathbb{D}$. This leads to a generalization of the Julia–Carathéodory theorem stated entirely in terms of reproducing kernels on arbitrary sets. As a consequence, we obtain an abstract form of Julia's lemma, which also yields sufficient conditions for the iterates of a composition factor to converge in an appropriate sense. As an application, Jury's own generalization is extended. This refines the classical theorem for contractive multipliers of standard weighted Dirichlet spaces.

Paper III

The third paper reframes co-isometric weighted composition operators as rank 1 composition factors. We extend the work of Mas, Martín, and Vukotić to the polydisk. Specifically, for a reproducing kernel k on the polydisk that is normalized, permutation- and rotation-invariant, zero-free, and diverges on the diagonal when approaching the topological boundary, we recover their dichotomy. The set of rank 1 composition factors of k is either just the rotations composed with permutations. Otherwise, it is the set of all automorphisms of the polydisk. In this case k must be a positive power of the Szegő kernel. Future work includes removing the assumptions that k is zero-free and that it blows up on the diagonal.

This thesis is my attempt at a way of organizing mathematics.
At least it succeeded in structuring my own mind.
That is enough.

References

[1] Ali Abkar and B. Jafarzadeh. “Weighted sub-Bergman Hilbert spaces in the unit disk”. In: *Czechoslovak Math. J.* 60(135).2 (2010), pp. 435–443.

[2] Alexandru Aleman and Frej Dahlin. “Generalized de Branges–Rovnyak spaces”. In: *J. Funct. Anal.* 288.11 (2025), Paper No. 110860, 28.

[3] Alexandru Aleman, Michael Hartz, John E. McCarthy and Stefan Richter. “Free outer functions in complete Pick spaces”. In: *Trans. Amer. Math. Soc.* 376.3 (2023), pp. 1929–1978.

[4] Alexandru Aleman and Bartosz Malman. “Hilbert spaces of analytic functions with a contractive backward shift”. In: *J. Funct. Anal.* 277.1 (2019), pp. 157–199.

[5] Nachman Aronszajn. “Theory of reproducing kernels”. In: *Trans. Amer. Math. Soc.* 68 (1950), pp. 337–404.

[6] Joseph A. Ball, Vladimir Bolotnikov and Quanlei Fang. “Schur-class multipliers on the Arveson space: de Branges–Rovnyak reproducing kernel spaces and commutative transfer-function realizations”. In: *J. Math. Anal. Appl.* 341.1 (2008), pp. 519–539.

[7] Louis de Branges and James Rovnyak. “Canonical models in quantum scattering theory”. In: *Perturbation Theory and its Applications in Quantum Mechanics (Proc. Adv. Sem. Math. Res. Center, U.S. Army, Theoret. Chem. Inst., Univ. of Wisconsin, Madison, Wis., 1965).* Wiley, New York-London-Sydney, 1966, pp. 295–392.

[8] Louis de Branges and James Rovnyak. *Square summable power series*. Holt, Rinehart and Winston, New York-Toronto-London, 1966, pp. viii+104.

[9] Nikolaos Chalmoukis and Michael Hartz. *Potential theory and boundary behavior in the Drury–Arveson space*. 2024.

[10] Cheng Chu. “Density of polynomials in sub-Bergman Hilbert spaces”. In: *J. Math. Anal. Appl.* 467.1 (2018), pp. 699–703.

[11] Emmanuel Fricain and Javad Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 1*. Vol. 20. New Mathematical Monographs. Cambridge University Press, Cambridge, 2016, pp. xix+681.

[12] Emmanuel Fricain and Javad Mashreghi. *The theory of $\mathcal{H}(b)$ spaces. Vol. 2*. Vol. 21. New Mathematical Monographs. Cambridge University Press, Cambridge, 2016, pp. xix+619.

[13] Caixing Gu, In Sung Hwang, Woo Young Lee and Jaehui Park. "Higher-order de Branges–Rovnyak and sub-Bergman spaces". In: *Adv. Math.* 428 (2023), Paper No. 109143, 49.

[14] Michael Hartz. "An invitation to the Drury–Arveson space". In: *Lectures on analytic function spaces and their applications*. Vol. 39. Fields Inst. Monogr. Springer, Cham, [2023] ©2023, pp. 347–413.

[15] Michael Hartz. "Every complete Pick space satisfies the column-row property". In: *Acta Math.* 231.2 (2023), pp. 345–386.

[16] Michael Hartz and Maximilian Tornes. "Weighted composition operators on Hilbert function spaces on the ball". In: *Results Math.* 80.6 (2025), Paper No. 180, 16.

[17] Michael T. Jury. *An improved Julia-Caratheodory theorem for Schur-Agler mappings of the unit ball*. arXiv:0707.3423. 2007.

[18] Michael T. Jury. "Clark theory in the Drury–Arveson space". In: *J. Funct. Anal.* 266.6 (2014), pp. 3855–3893.

[19] Michael T. Jury. "Reproducing kernels, de Branges–Rovnyak spaces, and norms of weighted composition operators". In: *Proc. Amer. Math. Soc.* 135.11 (2007), pp. 3669–3675.

[20] Michael T. Jury and Robert T. W. Martin. "Aleksandrov–Clark theory for Drury–Arveson space". In: *Integral Equations Operator Theory* 90.4 (2018), Paper No. 45, 42.

[21] Michael T. Jury and Robert T. W. Martin. "Extremal multipliers of the Drury–Arveson space". In: *Proc. Amer. Math. Soc.* 146.10 (2018), pp. 4293–4306.

[22] Michael T. Jury and Robert T. W. Martin. "Factorization in weak products of complete pick spaces". In: *Bull. Lond. Math. Soc.* 51.2 (2019), pp. 223–229.

[23] Steven G. Krantz. "The boundary behavior of holomorphic functions: global and local results". In: *Asian J. Math.* 11.2 (2007), pp. 179–199.

[24] Shuaibing Luo and Kehe Zhu. "Sub-Bergman Hilbert spaces on the unit disk III". In: *Canad. J. Math.* 76.5 (2024), pp. 1520–1537.

[25] Barbara D. MacCluer and Joel H. Shapiro. "Angular derivatives and compact composition operators on the Hardy and Bergman spaces". In: *Canad. J. Math.* 38.4 (1986), pp. 878–906.

[26] María J. Martín, Alejandro Mas and Dragan Vukotić. “Co-isometric weighted composition operators on Hilbert spaces of analytic functions”. In: *Results Math.* 75.3 (2020), Paper No. 128, 21.

[27] Javad Mashreghi and Dan Timotin. “Nonextreme de Branges–Rovnyak spaces as models for contractions”. In: *Integral Equations Operator Theory* 80.1 (2014), pp. 137–152.

[28] Vern I. Paulsen and Mrinal Raghupathi. *An introduction to the theory of reproducing kernel Hilbert spaces*. Vol. 152. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2016, pp. x+182.

[29] Donald Sarason. “Angular derivatives via Hilbert space”. In: *Complex Variables Theory Appl.* 10.1 (1988), pp. 1–10.

[30] Donald Sarason. *Sub-Hardy Hilbert spaces in the unit disk*. Vol. 10. University of Arkansas Lecture Notes in the Mathematical Sciences. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1994, pp. xvi+95.

[31] Joel H. Shapiro. *Composition operators and classical function theory*. Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993, pp. xvi+223.

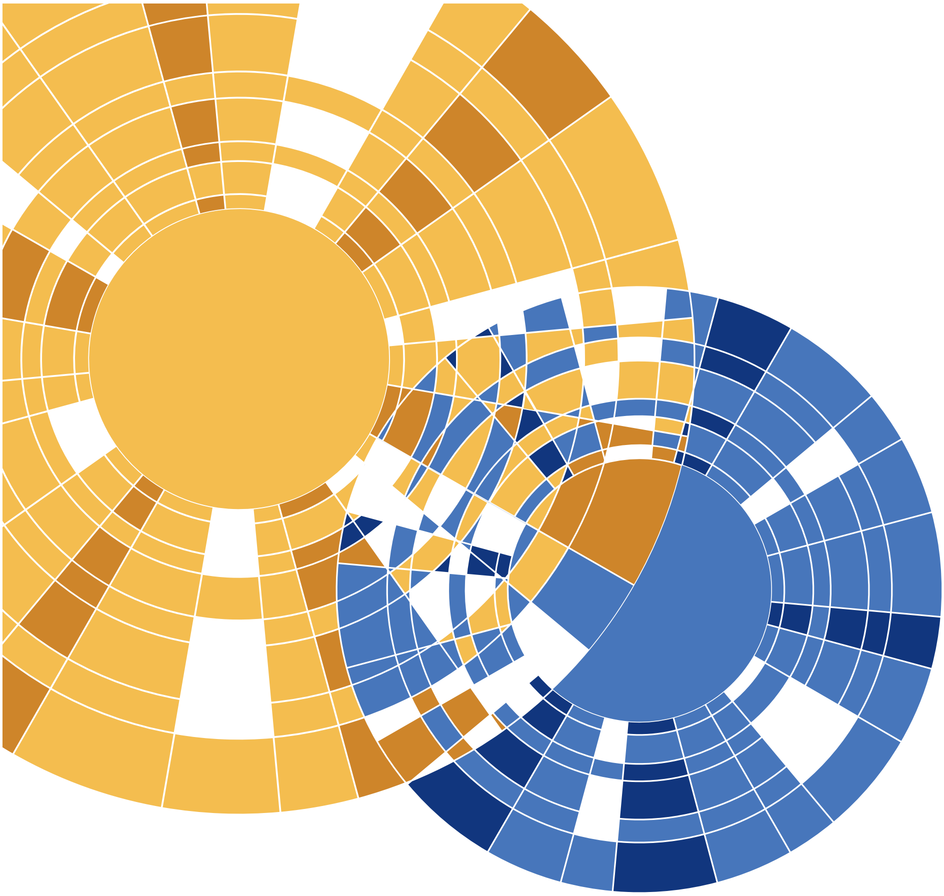
[32] Joel H. Shapiro and Peter D. Taylor. “Compact, nuclear, and Hilbert–Schmidt composition operators on H^2”. In: *Indiana Univ. Math. J.* 23 (1973/74), pp. 471–496.

[33] Sergei Shimorin. “Wold-type decompositions and wandering subspaces for operators close to isometries”. In: *J. Reine Angew. Math.* 531 (2001), pp. 147–189.

[34] Kehe Zhu. “Sub-Bergman Hilbert spaces in the unit disk. II”. In: *J. Funct. Anal.* 202.2 (2003), pp. 327–341.

[35] Kehe Zhu. “Sub-Bergman Hilbert spaces on the unit disk”. In: *Indiana Univ. Math. J.* 45.1 (1996), pp. 165–176.

Scientific publications



Printed by Media-Tryck, Lund 2026 NORDIC SWAN ECOLABEL 3041 0903

LUND UNIVERSITY

Doctoral Theses in Mathematical Sciences 2026:2
ISBN 978-91-8104-864-3
ISSN 1404-0034