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# ANALYSIS OF AN EXTREMAL CONTROLLER FOR HAMMERSTEIN MODELS

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## INTRODUCTION

Problems of extremal control have been frequently studied by a good number of authors, as e.g. Keviczky and Haber (1974) or Bamberger (1977). The reason for this may be the possibility to describe several interesting practical problems in this way. There are, e.g., industrial processes where the object of control is to maximize some physical variable such as an efficiency. If there exists some optimal value for the control variable giving maximal value for the output, then the problem is of this type.

One way to handle such a problem is to set up a parametric model for the process studied. Then the optimal control is determined to optimize some suitable criterion. In most cases, however, the best parameter values in the model are not known. The parameters must then be estimated. If this is done simultaneously with control we get an adaptive control problem.

In this report a special type of process model will be considered, the so called Hammerstein model. The parameters of the model will be estimated using a Least Squares estimator, and the estimates are used immediately in the control law to form an adaptive regulator. It is shown that such a scheme has to be used with great care, since the parameter estimates may well converge to some wrong values. The analyses is based on the differential equation approach developed by Ljung (1977) and is a straightforward application of his results. In a simulations section it is shown that the trajectories of the parameter estimates do really agree very well with the solutions to the corresponding differential equations. Finally some possible ways to overcome the problems are discussed.

## PROCESS MODEL AND THE KNOWN PARAMETER CASE

The model considered is of the Hammerstein type. In order to simplify notations and analyses only a special low order case will be considered. The same type of problems that appear here will of course show up also with a larger model. The system is

$$y(t) = ay(t-1) + k + bu(t-1) + cu(t-1)^2 + e(t) \tag{1}$$

The noise $\{e(t)\}$ is supposed to be a sequence of independent random variables with zero mean. The parameters a,k,b and c are supposed to be unknown and has to be estimated on-line.

This system should be controlled so that its output is kept as small as possible. To accomplish this, the criterion used is the steady state mean value of the output. To assure the existence of the mean, assume that $|a| < 1$. Admissible control laws may use all information available, i.e. u(t) may depend on y(t), u(t-1) and all previous inputs and outputs.

If the parameters of the model (1) are known, then the optimal control law is

$$u(t) = -\frac{b}{2c} \tag{2}$$

For the model (1) this controller minimizes the expected value of the output. But it also minimizes the output of the next step. With a more general system model where the output depends on the value of the input at several sampling points, the best steady state and the best one-step controllers will not coincide. For a further discussion on this point see Keviczky and Haber (1974).

The optimal control law is thus no feedback controller. This is due to the choice of model and criterion. In order to get a feedback loop the criterion could be taken to be quadratic to give a minimum variance control. This problem will not be considered in this report.

## ADAPTIVE CONTROL

When the parameters of the model (1) are unknown, the control law (2) has to be modified. One approach is to replace the parameters by their estimates to form an adaptive control law, i.e.

$$u(t) = -\frac{\hat{b}(t)}{2\hat{c}(t)} \tag{3}$$

Unfortunately, with the c-parameter unknown, its estimate may become close to zero. In such a case (3) will not work very well. It can, however, be replaced by an updating formula based on stochastic approximation

$$u(t+1) = u(t) - r(t)[\hat{b}(t) + 2\hat{c}(t)u(t)] \tag{4}$$

Several choices of r(t) could be made. A simple one is $r(t) = 1/t$. Keviczky and Haber(1974) use

$$r(t+1)^{-1} = r(t)^{-1} + 2\hat{c}(t+1) \tag{5}$$

or

$$r(t) = [2\hat{c}(t)]^{-1} \tag{6}$$

Inserting (3) into (4) gives $u(t+1) = u(t)$, and so it seems reasonable to believe that (3) and (4) will behave similarly if the estimates converge.

### Estimation

In order to use a control law such as (3) or (4) the process parameters have to be estimated. Since the process noise e in (1) is assumed to be white, it is possible to use an ordinary least squares estimator. Let $\hat{x}(t)$ be the column vector of parameter estimates obtained at time t and

$$\theta(t) = [y(t-1) \quad 1 \quad u(t-1) \quad u(t-1)^2] \tag{7}$$

Then

$$\hat{x}(t) = \hat{x}(t-1) + P(t)\theta(t)^T[y(t) - \theta(t)\hat{x}(t-1)] \tag{8}$$

where

$$P(t) = P(t-1) - P(t-1)\theta(t)^T\theta(t)P(t-1)/(1 + \theta(t)P(t-1)\theta(t)^T) \tag{9}$$

Another way to write (9) is in terms of the inverse

$$P(t)^{-1} = P(t-1)^{-1} + \theta(t)^T\theta(t) \tag{10}$$

It is also possible to replace P(t) in (8) by just $C \cdot I \cdot t^{-1}$ to get a stochastic approximation type algorithm. This variant is easier to analyze, and will be discussed in detail in the sequel. The equation is

$$\hat{x}(t) = \hat{x}(t-1) + \frac{C}{t}\cdot\theta(t)^T[y(t) - \theta(t)\hat{x}(t-1)] \tag{11}$$

# ANALYSES

## Preliminaries

The behaviour of the algorithms outlined in the previous sections can be analysed using the technique derived by Ljung (1977). In order to apply his results a number of technical conditions must be fulfilled. Some of these conditions are quite difficult to check mathematically, but are relatively easy to accept intuitively. In this report no strict proofs of non-consistency will be given. Instead, the differential equations of Ljung will be used to show the expected paths of certain parameter estimators in connection with adaptive extremum control. The results are confirmed by simulations of the original algorithms in the next section.

There are four different combinations of input generation and parameter estimation that can be chosen from the previous section. The input may come from (3) (or, equivalently, (4) and (6)), or from (4) and (5). The latter will be called the stochastic approximation input. Parameter estimation may be of LS-type, i.e. (8) and (9), or of SA-type, eq. (11). All these different variants may be described in terms of the general recursive algorithm of Ljung (1977), which is in the time-invariant case with $\gamma(t)=1/t$

$$\hat{x}(t) = \hat{x}(t-1) + \frac{1}{t}\cdot Q(\hat{x}(t-1),y(t)) \tag{12}$$

where the measurements y are generated as one component of the vector $\varphi$ in

$$\varphi(t) = A(\hat{x}(t-1))\cdot\varphi(t-1) + B(\hat{x}(t-1))e(t) \tag{13}$$

The sequence of noise vectors $e(\cdot)$ are supposed to be independent.

According to Ljung (1977) (12) will asymptotically behave like the solution to

$$\dot{\underline{x}} = f(\underline{x}) = EQ(\underline{x},y) \tag{14}$$

In calculating the expectation of (14) $\underline{x}$ is a fixed vector and the measurement y is generated from (13) with this fixed $\underline{x}$-value.

To make (14) approximate (12), some conditions on the noise such as independence and bounded moments of some order are required. Furthermore, the parameter estimates must stay within a finite region ,and, if (3) is used, $\hat{c}$ must stay away from zero.

## The stochastic approximation case

Some of the algorithms of the previous section will now be analysed using (14). First consider SA-estimation and the input (3) which is the simplest case. Then

$$f(\underline{x}) = E\,\theta(t)^T[(a-\underline{a})y(t-1) + (k-\underline{k}) + (b-\underline{b})u(t-1) + (c-\underline{c})u(t-1)^2 + e(t)] \tag{15}$$

where

$$\underline{x} = [\,\underline{a}\quad \underline{k}\quad \underline{b}\quad \underline{c}\,]^T$$

The expectation shall be calculated for every fixed value of the vector $\underline{x}$. This gives the steady state values

$$E\,y = \frac{1}{1-a}[\,k + bu + cu^2\,] \tag{16}$$

$$E\,y^2 = \frac{\sigma^2}{1-a^2} + [E\,y]^2 \tag{17}$$

where $\sigma^2$ is the noise variance and $u = -\underline{b}/2\underline{c}$, which is deterministic since $\underline{x}$ is fixed. Let

$$\varepsilon = (a-\underline{a})(E\,y) + (k-\underline{k}) + (b-\underline{b})u + (c-\underline{c})u^2 \tag{18}$$

Then (14) is

$$\dot{\underline{a}} = \frac{\sigma^2}{1-a^2}(a-\underline{a}) + (E\,y)\cdot\varepsilon \tag{19}$$

$$\dot{\underline{k}} = \varepsilon \tag{20}$$

$$\dot{\underline{b}} = \varepsilon\cdot u \tag{21}$$

$$\dot{\underline{c}} = \varepsilon\cdot u^2 \tag{22}$$

The stationary points of (19)-(22) are defined by the conditions $\underline{a}=a$ and $\varepsilon=0$. Provided the parameter estimates converge, the dynamics will thus be correctly estimated. The other parameters, however, may converge to any point on the surface $\varepsilon=0$ for which (19)-(22) are stable. Convergence to the correct values may happen, since they satisfy $\varepsilon=0$. There are, however, infinitely many other possible convergence points.

Some of the stationary points of (19)-(22) are unstable and are thus no possible convergence points. This can be studied through a linerization around the stationary points, i.e. by looking at the derivative matrix of the right members of (19)-(22). To avoid lengthy calculations, such analyses will be done only for the special cases studied by simulation in the next section.

Even though there are infinitely many stationary points of (19)-(22), the trajectories might be such that the parameters would tend to the correct values from most starting points. This is unfortunately not the case. With u(t) given by (3) the $\underline{b}$- and $\underline{c}$-variables of (21)-(22) will stay on a certain closed curve determined by the initial conditions so that

$$\underline{b}(\tau)^2 + 2\underline{c}(\tau)^2 = \underline{b}(0)^2 + 2\underline{c}(0)^2 \qquad (23)$$

This is seen by taking the derivative and insert (21), (22) and $u = -\underline{b}/2\underline{c}$. The correct values will thus be reached only for very special initial values, although random effects will prevent (23) from being exact for the estimates.

LS-estimation

If LS-estimation is used instead of stochastic approximation, the differential equations for analyses have to be modified. Some additional equations are needed to handle the elements of the P-matrix. The P-equation (9) or (10) must then be rewritten in a form to fit in (12). This is done by introducing

$$R(t) = [t \cdot P(t)]^{-1}$$

as shown in Ljung(1977). The differential equations (14) then are

$$\dot{\underline{x}} = \underline{R}^{-1} \cdot f(\underline{x}) \qquad (24)$$

$$\dot{\underline{R}} = G(\underline{x}) - \underline{R} \qquad (25)$$

where $\underline{x}$ and $f(\underline{x})$ are as before and

$$G(\underline{x}) = E\theta(t)^T\theta(t) = \begin{pmatrix} Ey^2 & Ey & u \cdot Ey & u^2 \cdot Ey \\ Ey & 1 & u & u^2 \\ u \cdot Ey & u & u^2 & u^3 \\ u^2 \cdot Ey & u^2 & u^3 & u^4 \end{pmatrix} \qquad (26)$$

A necessary condition for stationarity is still $f(\underline{x})=0$, but this might not be sufficient. The stationary value of $\underline{R}$ from (25) is $\underline{R} = G(\underline{x})$, and since G is not invertible it is necessary to study $\underline{R}^{-1}f$ in (24), and not just f. These problems are avoided if a term $\varepsilon \cdot I$ is added to the right member of (25). This corresponds to adding $\varepsilon \cdot I$ to the right member of (10), i.e. in the updating of $P^{-1}$. That may also be motivated from a numerical point of view to keep $P^{-1}$ well-conditioned. It can be shown, that with this modification the stationary points and their stability properties are the same for LS-estimation as for stochastic approximation.

Stochastic approximation input

If the input is generated according to (4)-(5) instead of (3), two more states $\underline{u}$ and $\underline{r}$ are needed in the differential equation system for analyses. The equations corresponding to (4) and (5) are

$$\dot{\underline{u}} = -\underline{r}^{-1}(\underline{b} + 2\underline{c}\,\underline{u}) \tag{27}$$

$$\dot{\underline{r}} = 2\underline{c} - \underline{r} \tag{28}$$

The other differential equations are the same as before, except that $u = -\underline{b}/2\underline{c}$ is replaced by $\underline{u}$ from (27). Provided $\underline{c} \neq 0$, the only stationary point of (27)-(28) is

$$\underline{r} = 2\underline{c} \quad ; \quad \underline{u} = -\underline{b}/2\underline{c}$$

It can be shown that inclusion of (27)-(28) does not change the stationary points of the differential equation system or their stability conditions. But the trajectories will of course be different. The $\underline{b}$- and $\underline{c}$-variables will e.g. no longer stay on the closed curve given by (23).

## SIMULATIONS

The analysis of the previous section will now be illustrated by a few simulated examples. It is shown that convergence to the correct parameter values is an exceptional event that cannot be expected. To facilitate graphical presentation, the first two examples deal with the simplest case of SA-estimation according to (11) and the input given by (3).

Example 1:

Consider the system (1) with

$$a = 0 \quad ; \quad k = b = 0.4 \quad ; \quad c = 0.2$$

The standard deviation of the noise is $\sigma = 0.03$. Let the a- and c-parameters be known. The differential equations (20)-(21) for k and b are then

$$\dot{\underline{k}} = \varepsilon \tag{29}$$

$$\dot{\underline{b}} = -\varepsilon \cdot \underline{b}/2\underline{c} \tag{30}$$

with

$$\varepsilon = k-\underline{k} - (b-\underline{b}) \cdot \underline{b}/2\underline{c} \tag{31}$$

The stationary points $\varepsilon = 0$ thus form a parabola in the $\underline{b}$-$\underline{k}$-plane. The trajectories of (29)-(30) are easily found by dividing (29) by (30). They satisfy

$$\underline{b}(\tau) = \underline{b}(0) \cdot \exp\{[\underline{k}(0) - \underline{k}(\tau)]/2c\} \tag{32}$$

Figs. 1 and 2 show parameter phase-planes for the algorithm and its associated differential equations respectively. Note the parabola of stationary points (dashed in both figs.).

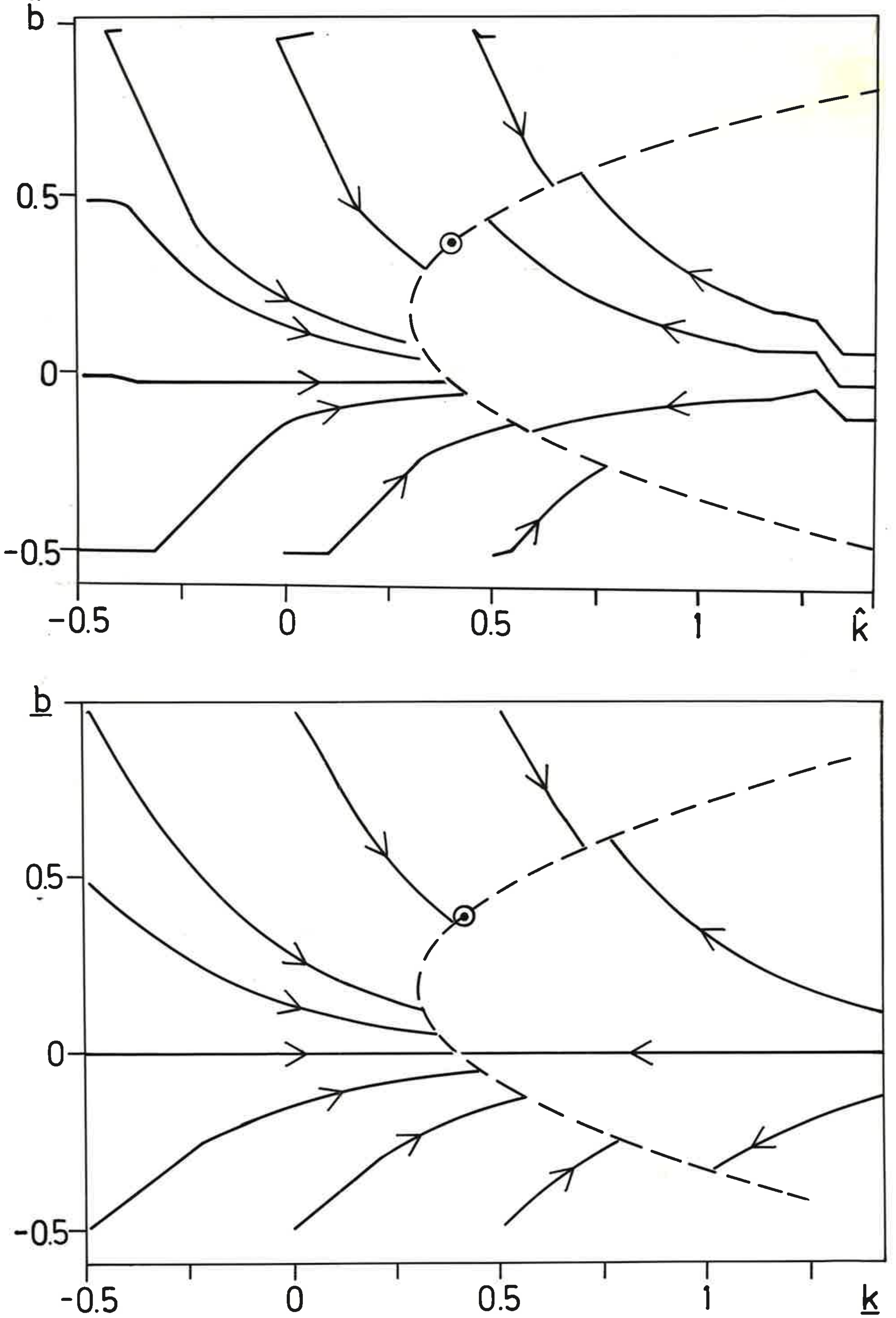
For small c-values, part of the stationary points are unstable. This will happen if

$$32c^2 < b^2 \tag{33}$$

The unstable points then satisfy

$$b - \sqrt{b^2 - 32c^2} < \underline{b} < b + \sqrt{b^2 - 32c^2} \tag{34}$$

A phaseplane for (29)-(30) in this case is shown in fig. 3.




Figures 1 & 2 - Phaseplanes for example 1 for the estimation algorithm (above) and its associated differential equations (below). The stationary points are indicated by the dashed line, and the true value is the encircled dot.

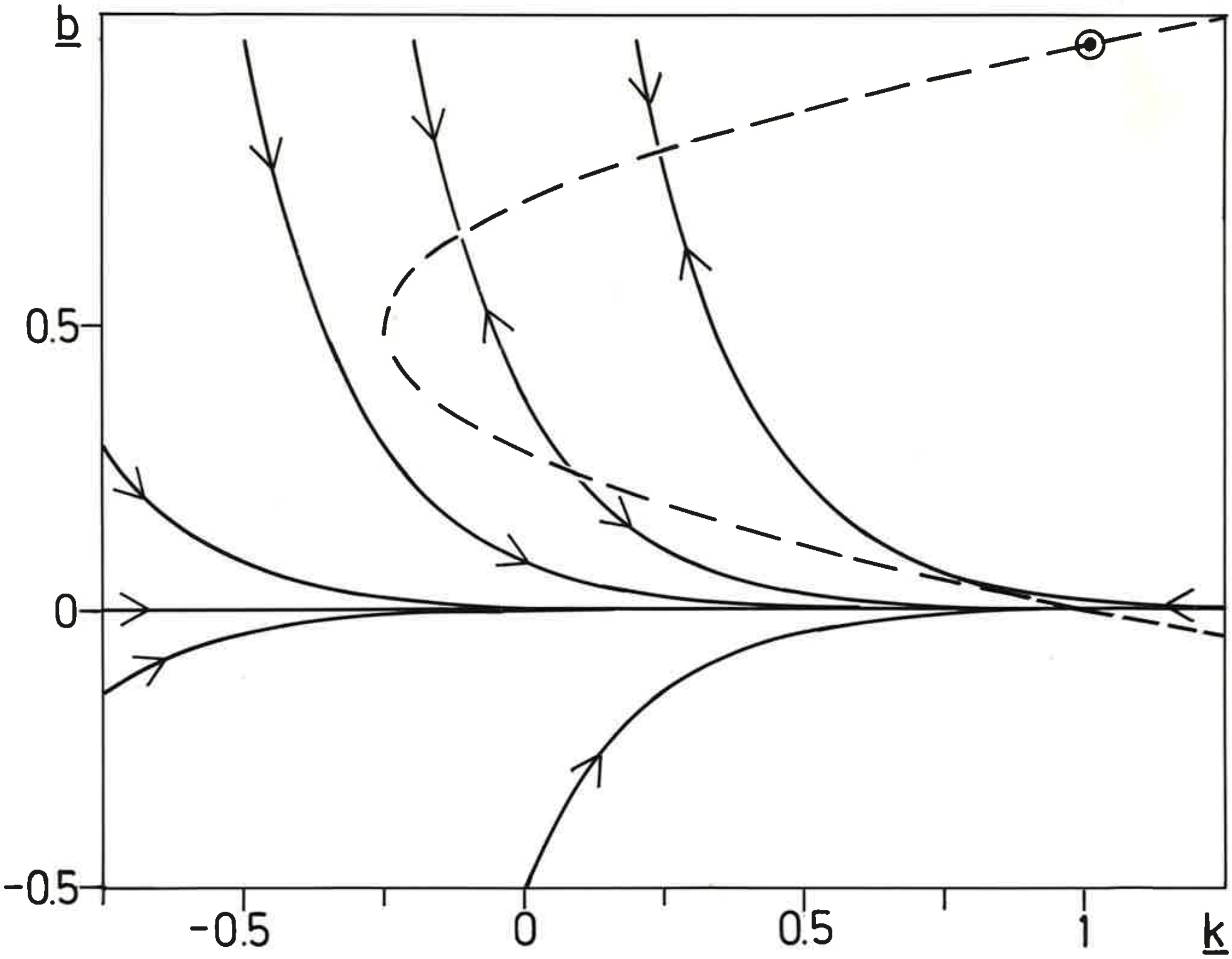



Figure 3 - Phaseplane corresponding to that of fig. 2, but for a smaller value of c.

Example 2:

In this second example, the b- and c-parameters are assumed unknown. The true parameter values and the noise variance are the same as in example 1. The associated differential equations (21)-(22) become

$$\dot{\underline{b}} = -\varepsilon\cdot\underline{b}/2\underline{c} \qquad (35)$$

$$\dot{\underline{c}} = \varepsilon\cdot(\underline{b}/2\underline{c})^2 \qquad (36)$$

where

$$\varepsilon = -(b-\underline{b})\cdot\underline{b}/2\underline{c} + (c-\underline{c})\cdot(\underline{b}/2\underline{c})^2 \qquad (37)$$

The stationary points $\varepsilon = 0$ are

$$\underline{b} = 0 \quad \text{and} \quad \underline{c} = c\cdot\underline{b}/(2b-\underline{b}) \quad (\text{for } \underline{b} \neq 2b)$$

Some of the stationary points are unstable if $\underline{c}$ has the wrong sign. This is the case if

$$-c \quad \underline{c} \quad 0 \tag{38}$$

or if

$$\underline{b}^2 \cdot c < -2\underline{c}^2 \cdot (\underline{c} + c) \qquad (\text{for } \underline{c} < -c) \tag{39}$$

Fig. 4 shows some phase-plane trajectories for the estimation algorithm and for its associated differential equations. The stationary points are indicated by dashed lines. Note that the differential equations are not defined for $\underline{c} = 0$, since the right members are infinite. This fact shows up in the estimation algorithm as well, with large steps in the estimates when $\hat{c}$ is small.

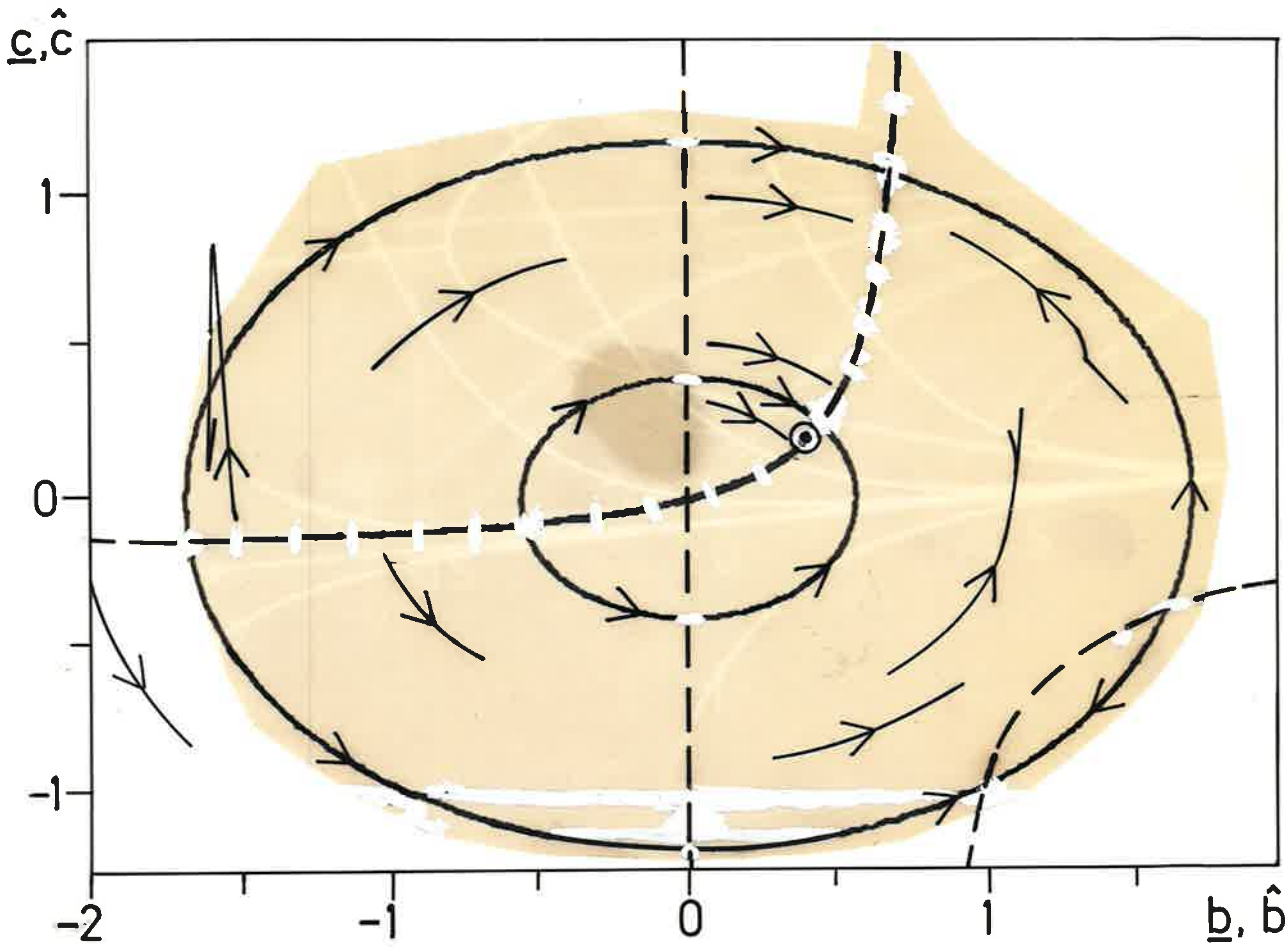



Figure 4 - Phaseplane for example 2. The dashed lines are the stationary points, and the encircled dot is the true values.

Thick lines - differential equations
Thin lines - estimation algorithm

In these two examples the parameter estimates can converge not only to the true parameter values, but to any point along a line through these values. As discussed earlier, the same thing will happen with LS-estimation or with a stochastic approximation input. This will result in a non-optimal performance of the overall system, depending on the steady state value of $u = -\hat{b}/2\hat{c}$.

Example 3:

A final example is given to show the behaviour of the LS-estimates of all four parameters of (1) at the same time. The stochastic approximation input (4)-(6) is used, and the system parameters are

$$a = 0.8 \quad ; \quad k = b = 0.4 \quad ; \quad c = 0.2$$

With the standard deviation of the noise $\sigma$=0.03, this is the system considered by Keviczky and Haber(1974).

Figs. 5 and 6 show one simulation of this system with the starting values

$$\hat{a}(0) = 0 \qquad \hat{k}(0) = 5 \qquad \hat{b}(0) = -0.8 \qquad \hat{c}(0) = 0.5$$

$$P_a(0) = 1 \qquad P_k(0) = 10 \qquad P_b(0) = 1 \qquad P_c(0) = 0.1$$

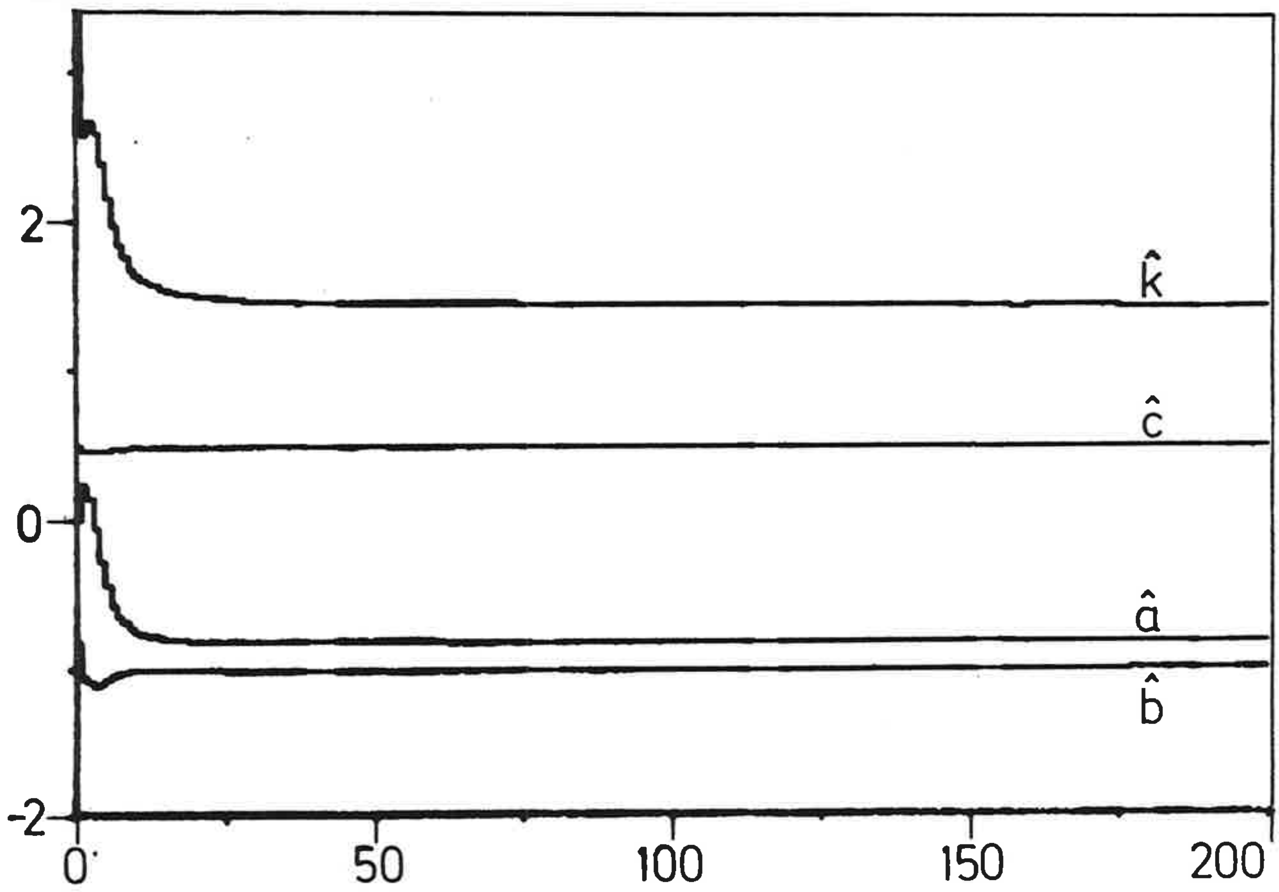



Figure 5 - Parameter estimates in one simulation of example 3.

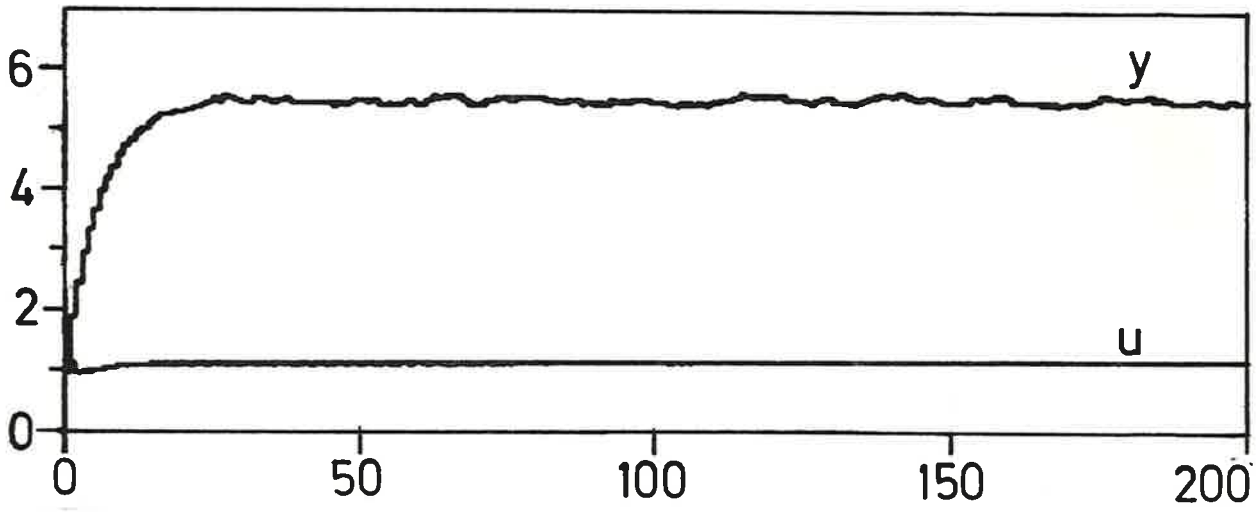



Figure 6 - Input and output in the simulation of fig. 5

The variance of the initial c-parameter estimate had to be taken small to prevent $\hat{c}$ from passing through zero.

The estimates shown in fig. 5 converge quickly, but, as expected, only the a-parameter estimate is correct. This results in a non-optimal input of $u \simeq 1$ shown in fig. 6. The optimal value is $u = -1$. This optimal input would reduce the average output value to unity.

Only 200 time steps are shown in fig. 5, and a slow convergence of the estimates to the true values would not be detected. However, long simulations of 5000 steps were also performed, and the estimates did not move in that time. As a further test, the initial estimates were in one run picked to give $\varepsilon=0$, but away from the true values. With unity initial covariance matrix, the estimates then stayed at the initial values, just as predicted by the differential equations.

## DISCUSSION

This report shows that direct application of the certainty equivalence principle to adaptive extremum control of a Hammerstein type model may cause identification problems. The estimates will converge to a certain hypersurface in the parameter space, but in most cases to other than the true parameter values.

The analysis has been done using the differential equation approach developed by Ljung(1977). The simulation examples show how remarkably well the differential equations describe the behaviour of the estimation algorithm, and thus the overall system. The differential equations are therefore a very valuable tool for analysis.

A possible way to avoid the identification problems would be to add an extra disturbance signal to the input. A few simulations have been performed to test this. The indication was, that it is a possible method, but may, at least in some cases, lead to slow convergence.

The best method to avoid all problems is probably to change the system model or the criterion. Already the optimal control law for known parameters (2) indicates possible problems, since the dynamics of the system does not come in at all. In other words, there is no feedback in the system. This can be introduced by a change of criterion. One possibility is to minimize the second moment of the output around its maximal expected value.

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