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Groups acting on rooted trees

Dimension, subgroups, randomness and dynamics

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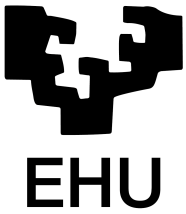

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Groups acting on rooted trees: dimension, subgroups, randomness and dynamics

Abstract

This thesis is devoted to the study of different aspects of groups acting on rooted trees and their applications.

First, we focus on the Hausdorff dimension of groups acting on rooted trees. We develop new tools that lead to the solution of several well-known open problems in the subject due to Abért-Virág, Bartholdi, Grigorchuk, Klopsch and Shalev among others.

Secondly, we study many aspects of the subgroup structure of these groups: first-order theory, rigidity of actions, profinite topology, virtual retracts, etc. The most remarkable application of the obtained results is the disproval of a well-known conjecture of Boston.

Then, we study several aspects of random elements and random subgroups of branch groups. We generalize several results of Abért and Virág.

Lastly, we develop a new ergodic theory for self-similar groups. This has found many unexpected applications, the most remarkable ones being to several problems in arithmetic dynamics. In fact, we use this new theory to obtain a long-awaited classification of the fixed-point proportion of geometric iterated Galois groups of polynomials.

Keywords

Hausdorff dimension, Hausdorff spectra, self-similar groups, branch groups, random subgroups, dynamical systems, ergodicity, fixed-point proportion, iterated monodromy groups, iterated Galois groups, polynomials

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I **Restricted Hausdorff spectra of q-adic automorphisms**

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Advances in Mathematics **472** (2025), 110294.

II **On finitely generated Hausdorff spectra in groups acting on rooted trees**

J. Fariña-Asategui, O. Garaialde Ocaña, J. Uria-Albizuri
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III **An algorithm to compute the Hausdorff dimension of regular branch groups**

J. Fariña-Asategui
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IV **The Hausdorff dimension of the generalized Brunner-Sidki-Vieira groups**

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V **On a question of Abért and Virág**

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VI **Arboreal representations of linear groups**

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VII **Branch actions and the structure lattice**

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Algebra and Discrete Mathematics **38** (**2**) (2024), 215–232.

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X **Virtual retracts in groups acting on rooted trees**

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XII **Random subgroups of branch groups**

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XIII **Cyclicity, hypercyclicity and randomness in self-similar groups**

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XIV **Markov processes associated to fractal groups of finite type**

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XV **On the fixed-point proportion of self-similar groups**

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xvi **Fixed-point proportion of geometric iterated Galois groups**

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xvii **Arboreal Galois representations of rational functions: fixed-point proportion and the extension problem**
J. Fariña-Asategui
arXiv preprint: 2601.19414 (submitted).

Populärvetenskaplig sammanfattning

Gruppteorin uppstod i studiet av polynomekvationer. Galois noterade att man kan använda symmetrierna i lösningarna av polynomekvationer för att härleda när dessa lösningar kan uttryckas. Senare, utvecklades gruppteorin som det abstrakta studiet av symmetrier och numera är den allestädes närvarande i det vetenskapliga studiet av naturen: från kristallers symmetri till drivkrafterna i universum. Dessutom, spelar grupper en nyckelroll i modern kryptografi, så det är avgörande att förstå deras egenskaper och underliggande struktur för cybersäkerhet.

Ett av de viktiga klassiska problemen inom gruppteorin är om det är möjligt att konstruera en oändlig grupp utgående från ett ändligt antal ändliga symmetrier. Zelmanov fick Fields medaljen år 1994 för lösningen av en variant av detta problem. För det allmänna problemet är svaret ja, och de flesta exemplen är grengrupper. Grengrupper uppstår som symmetrier av oändliga träd, till exempel binäraträdet som består av en nod som kallas roten som har två ättlingar och varje ättling har själv två ättlingar och så vidare. Grengrupper är fraktala objekt eftersom de innehåller mindre kopior av sig själva. Fraktaler förekommer överallt i naturen: från växter och kristaller till nervceller och blodkärlen i människokroppen. Fraktaler uppstår också i matematik i studiet av dynamiska system, där dynamiska system är en matematisk modell för hur en partikels beteende i ett rum utvecklas över tid. Dynamiska system har ett brett spektrum av tillämpningar, såsom matematisk fysik, medicin, ekonomi och biologi.

Denna avhandling studerar egenskaper och underliggande strukturen av grengrupper och självliknandegrupper, i synnerhet dessa gruppers fraktala egenskaper. Vi löser några långvariga öppna problem om Hausdorffdimensionen av grengrupper och självliknandegrupper som har varit öppna i ett kvarts sekel. Vidare upptäcker vi nya kopplingar mellan grengrupper och dynamik, och vi tillhandahåller nya perspektiv om hur grengrupper, självliknandegrupper och dynamiska sytem är relaterade. I synnerhet, löser vi flera öppna problem inom talteorin med dessa verktyg, vilket återigen visar hur gruppteori har tillämpning i oväntade områden inom matematiken.

Förlängd sammanfattning och slutsatser

Denna avhandling studierar olika aspekter av grupper som verkar på rotade träd och deras tillämpningar.

Den första delen av avhandlingen handlar om Hausdorff-dimensionen av grupper som verkar på rotade träd. Vi utvecklar nya verktyg för att beräkna Hausdorff-dimensionen av självliknande grupper. Detta har lett till lösningen av flera öppna problem om de möjliga värdena för Hausdorff-dimensionen av självliknande grupper från Grigorchuk (2005) och Bartholdi (2006). Dessutom, har vi bevisat resultat om Hausdorff-dimensionen av ändligt genererade grupper som besvarar välkända öppna problem från Klopsch (1999) och Shalev (2000). Slutligen, har vi bevisat resultat om Hausdorff-dimensionen av normala undergrupper som besvarar en fråga från Abért och Virág (2005) och om Hausdorff-dimensionen av linjära grupper och grupper som uppfyller en lag, vilket besvarar ytterligare frågor från Abért och Virág (2005).

Den andra delen av denna avhandling handlar om olika aspekter av undergruppsstrukturen hos grupper som verkar på rotade träd. Vi studerar strukturlattice hos en grengrupp, dvs. undergrupperna med ändligt många konjugat, och förbättrar ett resultat från Wilson (1971). Vi studerar vidare olika aspekter av den profinita topologin i gren- och svagt grengrupper. Vi konstruerar icke-slutna undergrupper i gren- och svagt grengrupper, och då är dessa grupper inte ERF. Dessutom visar vi att grengrupper aldrig har den LR egenskapen. Vi etablerar vidare ett samband mellan virtuella retraktioner för en itererad monodromigrupp och den orbifoldiga topologin för motsvarande komplexa polynom. Slutligen, karakteriserar vi fullständigt rigiditeten hos svagt gren verkningar, vilket har lett till att den välkända Boston-förmodanden, en av de mest grundläggande men fortfarande öppet problem om strukturen hos grengrupper, har motbevisats. De verktyg som utvecklats i denna del av avhandlingen har ytterligare tillämpningar på första ordningens teori om gren- och svagt grengrupper.

I den tredje delen av avhandlingen fokuserar vi på de probabilistiska aspekterna av grengrupper. Först återbevisar vi ett resultat av Abért (2005) för grengrupper som anger att slumpmässiga element i grengrupper är av oändlig ordning nästan säkert. Beviset är enklare än Abérts tillvägagångssätt via separerande verkningar. Sedan betraktar vi slumpmässiga undergrupper av grengrupper och studerar deras aktioner på trädets gräns. Dessa verkningar är kända för att vara nästan säkert nästan fria, men i tillämpningar är man vanligtvis intresserad av fria verkningar istället. Vi bevisar att dessa verkningar verkligen är fria och generaliserar det analoga resultatet av Abért och Virág (2005) för den itererade kransprodukten.

Den sista delen av avhandlingen handla om att utveckla en ny ergodiskteori för självliknande grupper och visa tillämpningar inom aritmetisk dynamik. Vi introducerar ett nytt originellt sätt att associera ett dynamiskt system till en fraktalgrupp, vilket låter oss studera

flera egenskaper hos fraktalgrupper via ergodiskteori. Detta har viktiga tillämpningar på undergruppsstrukturen hos fraktala grupper, men den mest anmärkningsvärda tillämpningen av denna metod är studiet av fixpunkter. Vi kan tillhandahålla ett tillräckligt (dynamiskt) villkor för att en fraktalgrupp ska ha få fixpunkter och detta har viktiga konsekvenser inom aritmetisk dynamik. Det huvudsakliga resultatet som erhållits i detta avseende är att Galoisgrupperna associerade med iterationerna av komplexa polynom fixerar väldigt få förbilder av ett transcendentalt element, såvida inte polynomet under iteration är ett Chebyshev polynom. Detta avgör ett länge öppet problem, som började med Odonis banbrytande artikel 1980. Detta resultat har viktiga tillämpningar på andelen periodiska punkter hos polynom över ändliga kroppar.

Sammanfattningsvis, fördjupar denna avhandling vår kunskap om redan väl etablerade teorier, vilket har lett till lösningen av många välkända problem inom gruppteori. Dessutom, startar den också nya originella forskningsinriktningar som visar sig vara mycket fruktbara och framgångsrika för att lösa välkända problem inom talteori.

Popular summary

Group theory originated in the study of polynomial equations. Galois noted that one may use the symmetries of the solutions of a polynomial equation to deduce when these roots may be expressed by radicals. Later on group theory evolved as the abstract study of symmetries and nowadays it is omnipresent in the scientific study of nature: from the symmetries of crystals to the driving forces in the universe. Furthermore, groups play a key role in modern cryptography, so understanding their properties and underlying structure is vital for digital safety.

One of the most important classical problems in group theory asks whether it is possible to construct an infinite group starting from a finite number of finite symmetries. A solution of a variant of this problem got Zelmanov a Fields medal in 1994. The general problem has a positive answer and the main source of examples is the class of branch groups. Branch groups consist of symmetries of infinite trees, such as the binary tree consisting of a vertex called the root who has two descendants and every descendant has itself two descendants and so on. Branch groups are fractal objects as they contain smaller copies of themselves. Fractals appear everywhere in nature: from plants and crystals to neurons and the blood vessels in the human body. Fractals also arise in mathematics in the study of dynamical systems, where dynamical systems are a mathematical model for the behaviour of a particle in a space evolving over time. Dynamical systems have a wide range of applications, such as to mathematical physics, medicine, economics and biology.

This thesis studies the properties and underlying structure of branch groups, in particular the fractal properties of these groups. We have solved several longstanding open problems on the Hausdorff dimension of branch groups and self-similar groups, which have been open for a quarter of a century. Furthermore, we have established new connections between branch groups, self-similar groups and dynamics. In particular, we solve several open problems in number theory via these tools, showing once again how group theory has application to previously unexpected areas in mathematics.

Extended summary and conclusions

This thesis is devoted to the study of different aspects of groups acting on rooted trees and their applications.

The first part of the thesis is devoted to the Hausdorff dimension of groups acting on rooted trees. The main contribution here is the development of new tools to compute the Hausdorff dimension of self-similar groups. This has led to the solution of several open problems on the possible values of the Hausdorff dimension of self-similar groups from Grigorchuk (2005) and Bartholdi (2006). Furthermore, we have obtained results on the Hausdorff dimension of finitely generated groups answering well-known open problems of Klopsch (1999) and Shalev (2000). Lastly, we have obtained results on the Hausdorff dimension of normal subgroups answering a question of Abért and Virág (2005) and on the Hausdorff dimension of linear groups and groups satisfying a law, answering further questions of Abért and Virág (2005).

The second part of this thesis is devoted to studying different aspects of the subgroup structure of groups acting on rooted trees. We study the structure lattice of a branch group, i.e. the subgroups with finitely many conjugates, and improve a result of Wilson (1971). We further study different aspects of the profinite topology in branch and weakly branch groups. We show the existence in many cases of non-closed subgroups in branch and weakly branch groups, establishing that these groups are not extended residually finite and we also show that branch groups never have the local retraction property. We further establish a connection between virtual retracts and the profinite topology of an iterated monodromy group and the orbifold topology of the corresponding complex polynomial. Lastly, we completely characterize the rigidity of weakly branch actions, which has led to the disproval of the well-known Boston Conjecture, one of the most fundamental but still open conjectures on the structure of branch groups. The tools developed in this part of the thesis have further applications to the first-order theory of branch and weakly branch groups.

In the third part of the thesis, we focus on the probabilistic aspects of branch groups. First, we reprove a result of Abért (2005) for branch groups stating that random elements in branch groups are of infinite order almost surely. The proof is much simpler than the approach of Abért via separating actions. Then, we consider random subgroups of branch groups and study their actions on the boundary of the tree. These actions are known to be almost free almost surely, but in applications one is usually interested in free actions instead. We prove that these actions are indeed free, generalizing the analogous result for iterated wreath product of Abért and Virág (2005).

The last part of the thesis is devoted to developing a new ergodic theory for self-similar groups and showing applications to arithmetic dynamics. In fact, we introduce a new original way to associate a measure-preserving dynamical system to a fractal group, which lets

us study several properties of fractal groups via ergodic theory. This has important applications to the subgroup structure of fractal groups, but the most remarkable application of this approach is to the study of fixed-points. We are able to provide a sufficient (dynamical) condition for a fractal group to have few fixed-points and this has important consequences in arithmetic dynamics. The main result obtained in this regard is that the Galois groups associated to the iterates of complex polynomials fix very few preimages of a transcendental element unless the polynomial under iteration is a Chebyshev polynomial. This settles a longstanding open problem, which started with the seminal work of Odoni in 1980. This result has important applications to the proportion of periodic points of polynomials over finite fields.

In conclusion, this thesis deepens our knowledge in already well-established theories which has led to the solution of many well-known problems in group theory and it also starts new original directions of research which are proving to be very fruitful and successful at approaching well-known problems in number theory.

Laburpena

Talde teoria ekuazio polinomikoekin hasi zen. Galois konturatu zen ekuazio polinomikoen soluzioen simetriak erabil daitezkeela soluzioak erroen bitartez noiz idatz ditzakegun zehazteko. Gaur egun, talde teoria simetrien ikerkuntza abstraktua bilakatu da, eta naturan nonahi agertzen da: kristalen simetrietatik naturako oinarrizko indarretara. Halaber, taldeak ezinbestekoak dira gaur egungo kriptografian. Beraz taldeen propietateak eta haien estruktura ezagutzea nahitaezkoa da gure babes digitelarako.

Talde teorian problema klasiko garrantzitsuenetariko bat izan da ea existitzen den talde infinitu bat simetria finitu kopuru finitu batetik abiatuta. Problema honen bertsio baten soluzioarengatik lortu zuen Zelmanovek fields domina 1994an. Orokorrean, problemak erantzun positiboa dauka eta dauzkagun adibide gehienak talde adarkatuen artean aurki ditzakegu. Talde adarkatuak zuhaitz infinituen simetriek sortutako taldeak dira, adibidez zuhaitz binarioaren simetriak, hau da erro bat daukan zuhaitza eta erroak bi ume, eta ume bakoitzak bi ume, etabar. Talde adarkatuak objetu fraktalak dira, bere buruaren kopia txikiagoak barruan dauzkate eta. Fraktalak naturan nonahi agertzen dira: landare eta kristaletatik giza gorputzean dauden neurona eta odol-hodietara. Fraktalak baita ere agertzen dira matematikan sistema dinamiko gisa; non sistema dinamiko bat partikula baten jarrera denboran zehar zehazten duen modelo matematiko bat den. Sistema dinamikoak aplikazio asko dauzkate, adibidez fisika matematikoan, medikuntzan, ekonomian edota biologian.

Tesi honetan, talde adarkatuen egitura eta propietateak aztertzen ditugu, talde hauen propietate fraktalak bereziki. Talde adarkatu eta autoantzekoen Hausdorff dimentsioari buruzko hainbat problema ebatzi ditugu, mende laurden batez irekita egon direnak. Gainera, talde adarkatuen, talde autoantzekoen eta sistema dinamikoen arteko erlazio berriak ezarri ditugu. Bereziki, zenbaki teorian agertzen diren hainbat problema ebatzi ditugu herraminta berri hauek erabilita, berriro ere talde teoriak espero ez diren matematikaren eremu ezberdinetara dituen aplikazioak erakutsiz.

Laburpen hedatua eta ondorioak

Tesi hau zuhaitz errodunetan eragiten duten taldeen alderdi desberdinak eta haien aplikazioak aztertzera bideratuta dago.

Tesiaren lehen zatian zuhaitz errodunetan eragiten duten taldeen Hausdorff dimentsioa aztertzen dugu. Gure ekarpen nagusia talde autoantzekoen Hausdorff dimentsioa kalkulatzeko tresna berriak garatzea da. Horrek talde autoantzekoen Hausdorff dimentsioaren balio posibleei buruzko hainbat problema irekien soluziora eraman gaitu, Grigorchuk (2005) eta Bartholdik (2006) proposotatutakoak. Gainera, finituki sortutako taldeen Hausdorff dimentsioa aztertu dugu, Klopsch-en (1999) eta Shalev-en (2000) problema ireki ezagunak ebaztera eraman gaituena. Azkenik, azpitalde normalen Hausdorff dimentsioa aztertu dugu, Abért eta Virág-en (2005) galdera bati erantzunez, eta talde linealen eta legeak betetzen dituzten taldeen Hausdorff dimentsioari buruzko emaitzak lortu ditugu, Abért eta Virág-en (2005) beste galdera batzuei erantzunez.

Tesi honen bigarren zatiak zuhaitz errodunetan eragiten duten taldeen azpitaldeen egituraren alderdi desberdinak aztertzen ditu. Talde adarkatu baten egitura-retikulua aztertzen dugu, hau da, konjugatu kopuru finitu bat dituzten azpitaldeak, eta Wilsonen (1971) emaitza bat hobetzen dugu. Halaber, talde adarkatuetan eta talde adarkatu ahuletan topologia profinitoa aztertzen dugu. Talde adarkatuetan eta talde adarkatu ahuletan azpitalde ez-itxiak daudela erakusten dugu, talde hauek ERF ez direla ezarriz. Baita ere talde adarkatuak inoiz ez dutela LR propietatea erakusten dugu. Gainera, monodromia talde iteratu baten topologia profinitoaren eta dagokion polinomio konplexuaren orbifold topologia erlazionatzen ditugu. Azkenik, ekintza adarkatu ahaulen zurruntasuna guztiz karakterizatzen dugu, eta horrek Bostonen Konjetura ezaguna ezeztatzera eraman gaitu, non Bostonen Konjetura talde adarkatuen egiturari buruzko problema ireki oinarrizkoenetarikoa den. Tesiaren zati honetan garatutako tresnek aplikazio gehiago dituzte talde adarkatu eta talde adarkatu ahulen lehen ordenako teoriara.

Tesiaren hirugarren zatian, talde adarkatuen alderdi probabilistikoetan zentratuko gara. Lehenik eta behin, Abért-en (2005) emaitza bat gaitzesten dugu talde adarkatuentzat, ausazko elementuek ordena infinitua dutela ia ziur esaten duena. Gure froga Abért-en ekintza bereizgarrietan oinarritutako froga baino simpleagoa da. Ondoren, talde adarkatuen ausazko azpitaldeen ekintzak zuhaitzaren mugan aztertzen ditugu. Ekintza hauek ia ziur ia askeak direla ezaguna da, baina aplikazioetan normalean ekintza askeak dira behar direnak. Ekintza hauek benetan askeak direla frogatzen dugu, Abért eta Virág-en (2005) produktu koro iteratuetarako emaitza analogoa orokortuz.

Tesiaren azken zatian talde autoantzekoetarako teoria ergodiko berri bat garatu eta aplikazioak dinamika aritmetikora erakusten ditugu. Izan ere, talde fraktal bati sistema dinamiko bat atzikitzeko modu original bat aurkezten dugu, eta horrek talde fraktalen hainbat

propietate aztertzea ahalbidetzen digu teoria ergodikoaren bitartez. Honek aplikazio garrantzitsuak ditu talde fraktalen azpitaldeen egituran, baina ikuspegi honen aplikaziorik nabarmenena puntu finkoen azterketa da. Talde fraktal batek puntu finko gutxi izateko baldintza (dinamiko) nahiko bat ematen dugu, eta horrek ondorio garrantzitsuak ditu dinamika aritmetikoan. Lortutako emaitza nagusiak dio polinomio konplexuen iterazioei atzikitutako Galoisen taldeek elementu transzendental baten aurreirudi gutxi finkatzen dituztela, iteratzen dugun polinomioa Chebyshev-en polinomio bat ez baldin bada. Horrek, Odoniren 1980ko lan garrantzitsuarekin hasi zen problema ireki garrantzitsu bat ebazten du. Emaitza honek aplikazio garrantzitsuak ditu gorputz finituen gaineko polinomioen puntu periodikoen proportzioara.

Tesi honek dagoeneko ondo ezarritako teorietan daukagun ezagutza sakontzen du, talde teorian problema ireki ezagun askoren soluziora eraman gaituena. Halaber, tesi honetan ikerketa-norabide originalak proposatzen ditugu, oso arrakastatsuak izan direnak dagoeneko zenbaki teorian.

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Introduction and short discussion

Our goal is to give a gentle introduction to the broad topic of groups acting on spherically homogeneous rooted trees. First, we motivate the definition of branch groups with several well-known problems in group theory. Secondly, we introduce the notion of self-similar groups in relation to several problems in complex and arithmetic dynamics. Then, we define the Hausdorff dimension of profinite groups and review the main problems in the area. Lastly, we discuss briefly the main results obtained in this thesis.

Burnside groups, growth and amenability

Let us introduce the reader to the class of branch groups. We review first some classical problems in group theory, which motivated Grigorchuk to construct the first branch group in 1980 [118].

The Burnside problem

Let G be a finitely generated group. If G is abelian (i.e. $gh = hg$ for every $g, h \in G$), then one of the first results one proves in a first group theory course is that G decomposes as a direct sum

$$G \cong \mathbb{Z}^n \oplus C_{q_1} \oplus \cdots \oplus C_{q_r},$$

where $n \geq 0$ and $q_1, \ldots, q_r$ are powers of distinct primes.

If G is torsion (i.e. every element in G has finite order), then necessarily $n = 0$ and G is a finite group of order $q_1 \cdots q_r$. Burnside asked in 1902 whether the same holds in a general finitely generated group:

Question 1 (The Burnside Problem, 1902). *Let G be a finitely generated torsion group. Is G necessarily finite?*

A (potential) counterexample to the Burnside problem, i.e. an infinite finitely generated torsion group, is called a *Burnside group*.

The Burnside problem was open for over 60 years, until Golod and Šafarevič provided the first example of a Burnside group in 1964 [112]. A few years later, in 1968, Adian and Novikov provided the first example of a Burnside group with finite exponent (i.e. $g^n = 1$ for every $g \in G$), answering the bounded version of Burnside problem [7]. However, it was not until 1990 that Zelmanov gave a positive solution to the so-called *restricted Burnside problem*, i.e. that for any number of generators d and any given exponent n, there exist only finitely many d-generated finite groups of exponent n [236, 238]. Zelmanov was awarded the Fields medal in 1994 for this contribution.

The Golod-Šafarevič and the Adian-Novikov examples of Burnside groups were quite complicated in nature. We shall see later that the class of branch groups provides very easy and explicit examples of Burnside groups.

The Milnor problem

Let G be a finitely generated group and let us fix a finite generating set S. We shall assume in the following that S is symmetric ($S = S^{-1}$). We define the Cayley graph $\Gamma(G, S)$ as the graph whose set of vertices and edges are given by:

(i) the set of vertices is G;

(ii) there is an edge (v, w) if and only if there is $s \in S$ such that $w = vs$.

As any graph, the Cayley graph $\Gamma(G, S)$ admits a natural metric space structure given by the usual distance $d : \Gamma(G, S) \times \Gamma(G, S) \to [0, \infty)$ in graphs:

$$d(v, w) := \min\{\#\text{ edges in a path from } v \text{ to } w\},$$

where, here and elsewhere, we write $\#$ for "number" or the cardinality of a set.

We define (with respect to S) the *growth function* $\gamma_S : \mathbb{N} \to [0, \infty)$, via

$$n \mapsto \#B_S(1, n),$$

where $B_S(1, n)$ denotes the closed ball of radius n in $\Gamma(G, S)$ centered at the identity 1.

The growth function defined above, depends on the generating set S. In order to make it independent of the generating set S we shall introduce an equivalence relation $\sim$: two

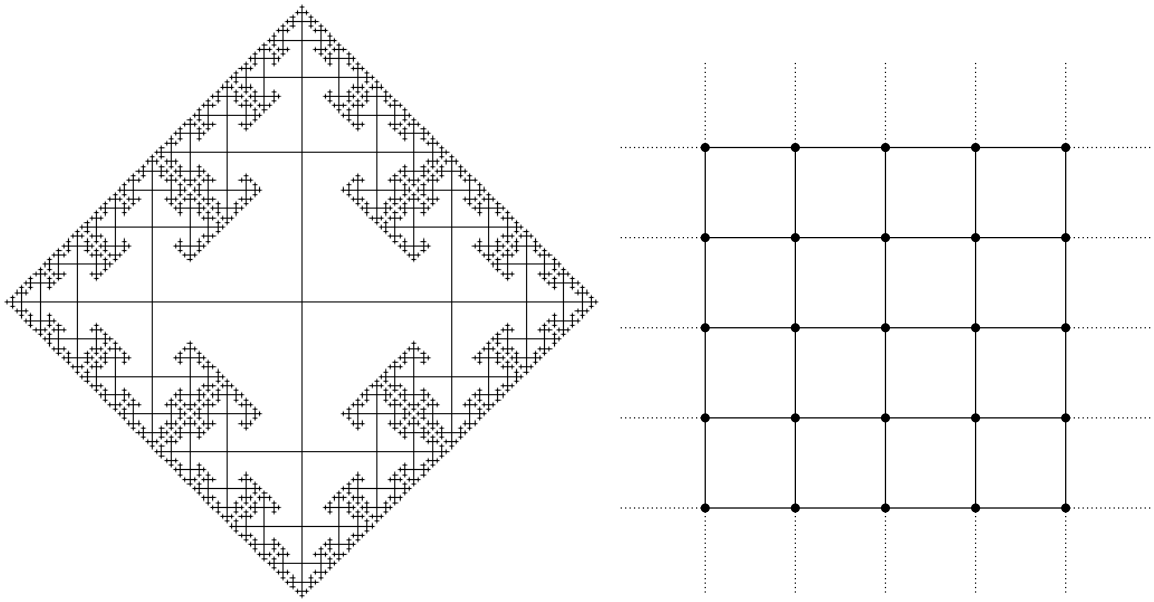

Figure 1: The Cayley graph of the free group of rank 2 (left) and of the free abelian group of rank 2 (right)

functions $f, g : [0, \infty) \to [0, \infty)$ are equivalent $f \sim g$, if there exists some real constant $C > 0$ such that for all positive integers $n \geq 1$ we have

$$f(n/C) \leq g(n) \leq f(Cn).$$

It is not difficult to see that $\gamma_S \sim \gamma_{\widetilde{S}}$ for any choice of finite generating sets $S, \widetilde{S}$.

Example 2 (Free abelian groups)**.** *Let G be a free abelian group of rank r and consider a generating set $S := \{s_1^{\pm 1}, \ldots, s_r^{\pm 1}\}$. Then, every element $g \in G$ can be written uniquely as*

$$g = s_1^{\epsilon_1} \cdots s_r^{\epsilon_r}$$

with $\epsilon_1, \ldots, \epsilon_r \in \mathbb{Z}$. Thus, it is easy to see that

$$\gamma_S(n) \sim n^r,$$

i.e. G has polynomial growth.

Example 3 (Free groups)**.** *Similarly, let G be a free group of rank $r \geq 2$ and consider a generating set $S := \{s_1^{\pm 1}, \ldots, s_r^{\pm 1}\}$. Then, every reduced word yields a distinct element in G. Hence, we get*

$$\gamma_S(n) = \#\text{ reduced words of length } n = 2r(2r-1)^{n-1} \sim 2^n,$$

i.e. G has exponential growth.

The examples above show that both polynomial and exponential growth occur among finitely generated groups. The natural question that arises is what other possible growth functions can occur. In particular, Milnor asked in 1968 whether there exist groups of *intermediate growth* [177], i.e. growing faster than any polynomial but slower than any exponential:

Question 4 (Milnor, 1968). *Does there exist a group of intermediate growth?*

Milnor also suggested in [177] that groups of polynomial growth are precisely those containing a nilpotent subgroup of finite index. A remarkable result of Gromov in [130] proved this conjecture of Milnor:

Theorem 5 (Gromov, 1981). *Let G be a finitely generated group. Then G has polynomial growth if and only if G is virtually nilpotent (i.e. G contains a nilpotent subgroup of finite index).*

Briefly afterwards, in 1984, Grigorchuk answered the Milnor problem by showing that the group he introduced in [118] is of intermediate growth [119].

The Day problem

Let G be a *discrete* group, i.e. a group equipped with the discrete topology. The *Borel algebra* of G is simply the σ-algebra generated by the open subsets of G, i.e. by all its subsets if G is discrete.

A finitely additive measure μ on the Borel algebra of G is said to be *left-invariant* if

$$\mu(gS) = \mu(S)$$

for any Borel subset $S \subseteq G$ and $g \in G$.

A discrete group G is *amenable* if G admits a left-invariant finitely additive probability measure μ.

Clearly, finite groups are amenable, as the usual counting measure is left-invariant. Abelian groups are easily shown to be amenable by constructing a left-invariant probability measure on them via the Markov-Kakutani fixed-point theorem. One can also prove that the class of amenable groups is closed under taking subgroups, quotients, extensions and directed unions. The class of groups containing both finite and abelian groups and closed under the aforementioned four operations, is called the class of *elementary amenable groups*. Day asked in 1957 whether every amenable group is elementary amenable [56]:

Question 6 (Day 1957). *Is every amenable group elementary amenable?*

The Day problem is related to the Milnor problem by the following celebrated result of Kesten [153]:

Theorem 7 (Kesten 1959). *Every group of subexponential growth is amenable. In particular, groups of intermediate growth are amenable.*

In fact, by a result of Chou [55], elementary amenable groups have either polynomial or exponential growth, so a group of intermediate growth must be amenable but not elementary amenable.

Residually finite groups

We say that a group G is *residually finite* if for every non-trivial element $g \in G$, there exists a finite-index normal subgroup $N \leq G$ such that $g \notin N$. The group G is said to be *countably-based residually finite* if G admits a filtration of finite-index normal subgroups

$$G =: H_0 \geq H_1 \geq \cdots \geq H_n \geq \cdots \geq \bigcap_{n \geq 0} H_n = 1.$$

Residually finite groups behave very well. For instance, a residually finite group G embeds into its profinite completion. Countably-based residual-finiteness is tightly connected to groups acting on rooted trees, as we shall see in Lemma 8 below.

Recall that a tree T is simply a graph with no cycles and it is called *rooted* if there is a marked vertex called the *root*, which we shall denote by $\varnothing$. A rooted tree admits a natural metric space structure given by the usual distance $d : T \times T \to [0, \infty)$ in trees:

$$d(v, w) := \#\{\text{edges in the unique path from } v \text{ to } w\}.$$

For every vertex $v \in T \setminus \{\varnothing\}$, there is a unique vertex adjacent to v and lying in the unique path from v to $\varnothing$. We call this vertex the *predecessor* of v. The remaining vertices adjacent to v are called the *descendants* of v. A vertex $w \in T$ is said to be *below* v, if the unique path from w to the root passes through v. Two distinct vertices in T are said to be *comparable* if one is below the other. Two distinct vertices are *incomparable* if they are not comparable.

A rooted tree T is said to be *spherically homogeneous*, if every vertex in T at the same distance from the root has the same number of descendants. Vertices at distance $n \geq 1$ from the root form the nth level of T denoted $\mathcal{L}_n$. In the following, we shall write T for a spherically homogeneous rooted tree unless stated otherwise. We are interested in groups of automorphisms of T.

The full automorphism group of T, which we denote $\mathrm{Aut}\, T$, consists of those graph automorphisms of T fixing the root. Note that graph automorphisms preserve adjacency, so

they are isometries with respect to the usual distance in trees. Thus, they must act by permuting vertices at the same level of T as they fix the root of T. For a vertex $v \in T$, the stabilizer of v in $\mathrm{Aut}\ T$ is denoted $\mathrm{st}(v)$. Similarly, we write $\mathrm{St}(n)$ for the nth *level-stabilizer*, i.e. the subgroup consisting of automorphisms fixing pointwise the nth level of T. The full automorphism group $\mathrm{Aut}\ T$ is countably-based residually finite. Indeed, it is a profinite group with respect to the metric topology induced by the filtration of level-stabilizers

$$\mathrm{Aut}\ T \geq \mathrm{St}(1) \geq \cdots \geq \mathrm{St}(n) \geq \cdots \geq \bigcap_{n \geq 1} \mathrm{St}(n) = 1.$$

Let us fix a subgroup $G \leq \mathrm{Aut}\ T$. Then, the vertex and the level stabilizers in G are simply given by $\mathrm{st}_G(v) := G \cap \mathrm{st}(v)$ and $\mathrm{St}_G(n) := G \cap \mathrm{St}(n)$ for a vertex $v \in T$ and a level $n \geq 1$ respectively.

There is a strong relation between countably-based residually finite groups and automorphisms of spherically homogeneous rooted trees:

Lemma 8 (see [125]). *Let G be a group. The following are equivalent:*

(i) *G is countably-based residually finite;*

(ii) *G admits a faithful action on a spherically homogeneous rooted tree.*

Proof. For (i) implies (ii), define a faithful action (via right-multiplication) of G on the tree of right-cosets $\{H_n g \mid g \in G \text{ and } n \geq 0\}$. For the converse, just consider the level-stabilizer filtration in G. □

Grigorchuk constructed a residually finite Burnside group in 1980 [118]. His group was the first group of intermediate growth (solving the Milnor problem) [119] and thus, amenable but not elementary amenable (solving the Day problem). The original description of Grigorchuk of this group in [118] was as a group acting by measure-preserving transformations on the interval $[0, 1]$. However, we shall give here the more usual description of his group as a group acting on the binary rooted tree. We need to introduce the notion of a section, which plays a central role in the theory of groups acting on rooted trees.

Let $v \in T$ be a vertex and let $g \in \mathrm{Aut}\ T$. Let us denote also by T_v the subtree rooted at v, i.e. the induced subtree consisting of v and all the vertices in T below v. The *section of g at v* is the unique automorphism $g|_v \in \mathrm{Aut}\ T_v$ such that for every $w \in T_v$ we have

$$(vw)^g = v^g w^{g|_v},$$

where we identify the subtrees rooted at the same level as T is spherically homogeneous. Here, we use the notation vw to refer to the vertex $w \in T_v$ as a vertex in T. Note that we

consider right actions on the tree and we use exponential notation for these actions, here and elsewhere. We also define the *label of g at v* as the unique permutation $g|_v^1$ describing the action of g on the immediate descendants of v (equivalently, on the first level of T_v).

We have an isomorphism $\psi : \mathrm{Aut}\ T \to (\prod_{v\in\mathcal{L}_1} \mathrm{Aut}\ T_v) \rtimes \mathrm{Sym}(\mathcal{L}_1)$ given by

$$g \mapsto (g|_v)_{v\in\mathcal{L}_1} \cdot g|_\varnothing^1.$$

The isomorphism ψ lets us define automorphisms of T recursively.

The first Grigorchuk group

Let T_2 be the binary rooted tree, i.e. the rooted tree whose vertices all have exactly 2 immediate descendants. Let $\sigma := (1\,2) \in \mathrm{Sym}(2)$ be the non-trivial transposition acting on the set consisting of two atoms. We define the automorphisms $a, b, c, d \in \mathrm{Aut}\ T_2$ recursively via ψ as

$$\psi(a) = (1,1)\sigma, \quad \psi(b) = (a,c), \quad \psi(c) = (a,d) \quad \text{and} \quad \psi(d) = (1,b).$$

The *first Grigorchuk group* $\mathcal{G}$ is the group generated by a, b, c and d. The group $\mathcal{G}$ is an instance of an automata group, as its generators may be described by a finite-state automaton. We summarize some of the main properties of the group $\mathcal{G}$ in the following proposition:

Proposition 9 (Properties of the first Grigorchuk group). *The first Grigorchuk group $\mathcal{G}$ satisfies:*

(i) *$\mathcal{G}$ is a Burnside group;*

(ii) *$\mathcal{G}$ is of intermediate growth;*

(iii) *$\mathcal{G}$ is amenable but not elementary amenable;*

(iv) *$\mathcal{G}$ is just-infinite, i.e. it is infinite but every proper quotient of $\mathcal{G}$ is finite.*

Branch groups

The class of branch groups was introduced by Grigorchuk during a conference in Bath in 1997, and first appeared in print in [117].

Just-infinite groups were classified by Wilson in 1971 in terms of their structure lattice [230], where the *structure lattice* is a lattice constructed from subnormal subgroups under an equivalence relation in terms of centralizers. In 2000, Grigorchuk described Wilson's classification of just-infinite groups in terms of branch groups and hereditarily just-infinite groups

in [117]. A detailed description of the structure lattice of a (non-necessarily just-infinite) branch group has been obtained recently by myself and Grigorchuk in [83].

Let T be a spherically homogeneous rooted tree and let us fix a subgroup $G \leq \mathrm{Aut}\, T$. For a vertex $v \in T$, we define the *rigid vertex stabilizer* $\mathrm{rist}_G(v) \leq G$ as

$$\mathrm{rist}_G(v) := \{g \in G \mid w^g = w \text{ for every } w \in T \setminus T_v\}.$$

In other words, the automorphisms in $\mathrm{rist}_G(v)$ only move vertices which are below v. Let $n \geq 1$, then we define the *rigid level stabilizer* $\mathrm{Rist}_G(n) \leq G$ via

$$\mathrm{Rist}_G(n) := \prod_{v \in \mathcal{L}_n} \mathrm{rist}_G(v).$$

It can be deduced that the above product is an internal direct product from

$$\mathrm{rist}_G(v) \cap \mathrm{rist}_G(w) = [\mathrm{rist}_G(v), \mathrm{rist}_G(w)] = 1,$$

whenever v and w are incomparable (in particular when they are distinct vertices at the same level of T). Furthermore, the rigid level stabilizers are normal subgroups as

$$\mathrm{rist}_G(v)^g = \mathrm{rist}_G(v^g)$$

for every $v \in T$ and $g \in G$.

As automorphisms of T must permute vertices at the same level of T, the most "transitive" the action of G on T can be is transitive on each level of T. We say that G is *level-transitive*, precisely when the action on each level of T is transitive.

Now, we are in a position to define the class of branch groups. A group G is called *branch* if G is level-transitive and for every $n \geq 1$, the rigid level stabilizer $\mathrm{Rist}_G(n)$ has finite index in G.

Example 10 (The first Grigorchuk group). *The first Grigorchuk group $\mathcal{G}$ was the first example of a branch group. As we shall see later on, it even satisfies a stronger property called regular branchness.*

The class of branch groups was further generalized to *weakly branch* groups by Grigorchuk in [115], by relaxing the condition of the rigid level stabilizers $\mathrm{Rist}_G(n)$ being all of finite index in G to just being infinite. As we shall see, the class of weakly branch groups includes further examples of groups with exotic properties, like the Basilica group, introduced by Grigorchuk and Żuk in [128], which was proved by Bartholdi and Virág in [26] to be the first example of an amenable but not elementary amenable group of exponential growth.

Complex and arithmetic dynamics

We follow the presentation in [23]. Let us consider a branched covering f of $\mathbb{S}^2$ and its set of critical points C_f. We say that f is a *Thurston map* (or *post-critically finite*) if the post-critical set P_f is finite, where

$$P_f := \bigcup_{n\geq 1} f^n(C_f).$$

Two Thurston maps f and g are *equivalent* if there exist orientation-preserving homeomorphisms $\phi_0, \phi_1 : (\mathbb{S}^2, P_f) \to (\mathbb{S}^2, P_g)$ such that ϕ_0 and ϕ_1 are isotopic relative to P_f and $\phi_0 f = g\phi_1$.

A major goal in complex dynamics is to classify Thurston maps up to equivalence. A celebrated result of Douady and Hubbard (1993) proved the so-called Thurston's criterion, which characterizes when a (hyperbolic) Thurston map is equivalent to a rational function, and asserts that if this is the case then the rational function is unique; see [65]. However, Thurston's criterion is not constructive and it does not yield an algorithm to determine to which rational function a Thurston map is equivalent to. Neither does it say anything about *obstructed Thurston maps*, i.e. Thurston maps which are not equivalent to a rational function.

Hubbard's twisted rabbit problem

As discussed above, Thurston's criterion is not effective at providing the rational function equivalent to the initial Thurston map. Pilgrim illustrated this issue with the twisted rabbit problem in [198], posed by Hubbard via private communication with Pilgrim.

Let us consider the rabbit f_R, corabbit f_C and airplane f_A polynomials, given by

$$f_R := z^2 + c_R, \quad f_C := z^2 + c_C \quad \text{and} \quad f_A := z^2 + c_A,$$

where

$$c_R \approx 0.1226 + 0.7449i, \quad c_C \approx 0.1226 - 0.7449i \quad \text{and} \quad c_A \approx 1.7549.$$

The polynomials f_R, f_C and f_A are the three non-equivalent quadratic complex polynomials with ramification graph (i.e. the graph corresponding to the orbits of the critical points)

$$0 \longrightarrow c \longrightarrow c^2 + c \longrightarrow 0$$

for $c \in \{c_R, c_C, c_A\}$, where 0 is the unique critical point. In other words, they are the unique periodic polynomials with exactly three post-critical points.

Now, consider a Dehn twist D around the two non-critical post-critical points of f_R. Then $f_R \cdot D^m$ is a Thurston map with the same ramification graph as f_R for every $m \in \mathbb{Z}$. As the ramification graph is a cycle, there are no possible obstructions and by Thurston's criterion $f_R \cdot D^m$ is equivalent to f_R, f_C or f_A, but which one?

Question 11 (Hubbard's twisted rabbit problem). *Deduce in terms of m which polynomial among f_R, f_C and f_A is the Thurston map $f_R \cdot D^m$ equivalent to.*

Hubbard's twisted rabbit problem was answered by Bartholdi and Nekrashevych in their brilliant work in [23, Theorem 4.7]:

Theorem 12 (Bartholdi-Nekrashevych, 2006). *If the 4-adic expansion of the number m has digits 1 or 2, then the branched covering $f_R \cdot D^m$ is equivalent to the airplane f_A. Otherwise, it is equivalent to the rabbit f_R for non-negative m and to the corabbit f_C for negative m.*

Furthermore, the approach of Bartholdi and Nekrashevych also allowed them to treat the obstructed cases; see [23, Theorem 6.1]. Their main insight was to use the so-called iterated monodromy groups associated to Thurston maps.

Iterated monodromy groups

Let us introduce briefly the iterated monodromy group of a Thurston map f; see [180] for a detailed exposition on this topic.

Let $t \in \mathbb{S}^2 \setminus P_f$, and let γ be a loop starting and ending at t. Let $\widetilde{t}_0 \in f^{-n}(t)$. Then, there exists a unique lift $\widetilde{\gamma}$ of γ starting at $\widetilde{t}_0$. This lift $\widetilde{\gamma}$ must be a path from $\widetilde{t}_0$ to another point $\widetilde{t}_1 \in f^{-n}(t)$, as γ was a loop starting and ending at t. For each natural number $n \geq 1$, we obtain a well-defined action of the fundamental group $\pi_1(\mathbb{S}^2 \setminus P_f, t)$ on $f^{-n}(t)$ via

$$(\widetilde{t}_0)^\gamma := \widetilde{t}_1.$$

Indeed, if γ' is another loop starting and ending at t and homotopically equivalent to γ, then its unique lift starting at $\widetilde{t}_0$ ends also at $\widetilde{t}_1$, so this action is well defined.

Furthermore, as for each natural number $n \geq 1$ the action of $\pi_1(\mathbb{S}^2 \setminus P_f, t)$ on the finite tree of preimages $f^{-n}(t)$ preserves both the root t and adjacency and these actions are compatible, we obtain a well-defined action of $\pi_1(\mathbb{S}^2 \setminus P_f, t)$ by automorphisms on the infinite tree of preimages

$$T := \bigsqcup_{n \geq 0} f^{-n}(t).$$

This action is called the *iterated monodromy action* of $\pi_1(\mathbb{S}^2 \setminus P_f, t)$.

Since $t \notin P_f$, the set $f^{-n}(t)$ has cardinality d^n for each $n \geq 1$, where d denotes the degree of f. Hence, we may identify the tree of preimages $T = \bigsqcup_{n\geq 0} f^{-n}(t)$ with the d-adic tree T_d. Consequently, we may think of the iterated monodromy action of $\pi_1(\mathbb{S}^2 \setminus P_f, t)$ as an action by automorphisms on T_d. The iterated monodromy action is not faithful in general. Quotienting $\pi_1(\mathbb{S}^2 \setminus P_f, t)$ with the kernel of the iterated monodromy action yields a subgroup of $\mathrm{Aut}\ T_d$. This subgroup of $\mathrm{Aut}\ T_d$ is called the *iterated monodromy group* of f, which we denote $\mathrm{IMG}(f)$.

Note that the group $\mathrm{IMG}(f)$ is only well defined up to conjugation in $\mathrm{Aut}\ T_d$, as it depends on the choice of the base point t (only up to conjugation as $\mathbb{S}^2 \setminus P_f$ is path-connected) and the identification of the tree of preimages with T_d.

The iterated monodromy action is an instance of a self-similar action on T_d. It was indeed iterated monodromy actions which motivated Nekrashevych to introduce the class of self-similar groups in the form which is presented today.

Self-similar groups

Let T_d denote the d-adic tree. Then, the subtrees rooted at any vertex of T_d are isomorphic to T_d as graphs. Thus, for any $g \in \mathrm{Aut}\ T_d$ and any $v \in T_d$, we may consider the section $g|_v$ again as an element of $\mathrm{Aut}\ T_d$ via this identification.

Let us consider now a subgroup $G \leq \mathrm{Aut}\ T_d$. We say that G is *self-similar* if $g|_v \in G$ for every $g \in G$ and $v \in T_d$. In other words, G is invariant under taking sections. Let us further define for each $v \in T_d$, the homomorphism $\varphi_v : \mathrm{st}(v) \to \mathrm{Aut}\ T_d$ given by $g \mapsto g|_v$. Then, we define the following notions of fractality:

(i) G is *fractal* if G is level-transitive, self-similar and $\varphi_v(\mathrm{st}_G(v)) = G$ for every $v \in T_d$;

(ii) G is *super strongly fractal* if G is level-transitive, self-similar and $\varphi_v(\mathrm{St}_G(|v|)) = G$ for every $v \in T_d$, where $|v|$ denotes the level at which v lies.

There exist further notions of fractality in self-similar groups, but the ones above are the most natural ones from a dynamical perspective. In fact, for a Thurston polynomial f (i.e. a Thurston map such that $f^{-1}(\infty) = \{\infty\}$) the corresponding iterated monodromy group is always fractal [180]. Furthermore, super strongly fractal groups play an important role in arithmetic dynamics, as observed recently by myself and Radi in [87].

Let us show a motivating example of a self-similar group: the Basilica group B. The Basilica group B was introduced as an automata group acting on the binary rooted tree by Grigorchuk and Żuk in [128]. The Basilica group coincides with the iterated monodromy group of the complex polynomial $z^2 - 1$. We give its realization in $\mathrm{Aut}\ T_2$ as a group

given by a kneading automaton; see [22] for further details on kneading automata. Let $a, b \in \operatorname{Aut} T_2$ be the automorphisms given by

$$\psi(a) = (1, b)\sigma \quad \text{and} \quad \psi(b) = (1, a),$$

where $\sigma := (1\,2) \in \operatorname{Sym}(2)$. Then B is the group generated by a and b.

Note that B is self-similar, as the sections of the generators a and b at any vertex are either $1, a$ or b, i.e. elements in B. Then, the sections at any vertex of any word in a and b, and thus of any element of B, must be words in a and b and hence elements of B. Now, note that

$$\psi(a^2) = (b, b).$$

Hence, by considering the appropriate conjugate of a power of a or b, we obtain

$$a, b \in \varphi_v(\operatorname{St}_G(|v|)),$$

which implies by the self-similarity of B that

$$B = \langle a, b \rangle \leq \varphi_v(\operatorname{St}_G(|v|)) \leq B.$$

Also note that ab and ba are both level-transitive elements (i.e. elements acting transitively on every level of the tree) as both ab and ba act transitively on level 1 and

$$\psi((ab)^2) = (ab, ba) \quad \text{and} \quad \psi((ba)^2) = (ab, ab).$$

Therefore, the Basilica group G is self-similar, level-transitive and $\varphi_v(\operatorname{St}_G(|v|)) = B$ for every $v \in T_2$, i.e. B is super strongly fractal.

The Basilica group is named after the Julia set of its defining polynomial $z^2 - 1$, which resembles the basilica of San Marco mirrored in the water; see Figure 2 below.

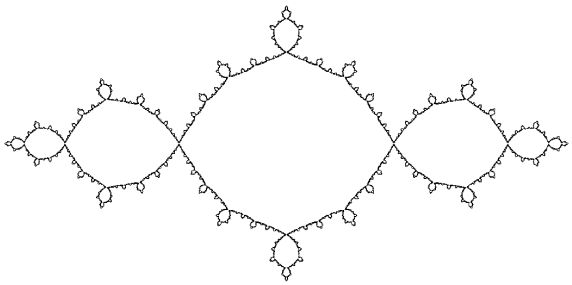

Figure 2: The Basilica Julia set, i.e. the fractal Julia set of the complex polynomial $z^2 - 1$.

Density problems in arithmetic dynamics

Let $a_0 \in \mathbb{Q}$ and consider a polynomial $f \in \mathbb{Q}[z]$. We define the sequence $\mathcal{I}_{f,a_0}$ of iterates of the polynomial f on a_0, i.e.

$$\mathcal{I}_{f,a_0} := \{f^n(a_0)\}_{n\geq 0}.$$

Many interesting families of numbers arise in this manner:

(i) Fermat numbers correspond to $\mathcal{I}_{z^2-2z+2,3}$;

(ii) Mersenne numbers correspond to $\mathcal{I}_{2z+1,1}$.

One of the main long-standing open problems in arithmetic dynamics is to prove which of the terms in $\mathcal{I}_{f,a_0}$ are prime. A related easier problem is to determine the density of the primes dividing a non-zero term in $\mathcal{I}_{f,a_0}$. If $P_f(a_0)$ denotes the set of primes dividing a non-zero term in $\mathcal{I}_{f,a_0}$, we would like to compute its natural density

$$\mathcal{N}(P_f(a_0)) := \lim_{k\to\infty} \frac{\#\{p \in P_f(a_0) : p \leq k\}}{\#\{p \text{ prime} : p \leq k\}}.$$

More generally, we may consider a rational function $f \in K(z)$ for a number field K, any point $a_0 \in \mathcal{O}_K$ in its ring of integers, and we would like to compute the Dirichlet density of $P_f(a_0)$ in K, i.e.

$$\mathcal{D}(P_f(a_0)) := \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in P_f(a_0)} |\mathcal{O}_K/\mathfrak{p}|^{-s}}{\sum_{\mathfrak{p} \text{ prime in } \mathcal{O}_K} |\mathcal{O}_K/\mathfrak{p}|^{-s}},$$

where $\mathcal{O}_K/\mathfrak{p}$ is the residue field corresponding to the prime $\mathfrak{p}$.

The density $\mathcal{D}(P_f(a_0))$ is related to self-similar actions of Galois groups, by the pioneering work of Odoni in [184].

Let K be any field. Let us consider the infinite Galois group $G_\infty(K, f, \alpha)$ associated to the preimages of $\alpha \in K$ by all the iterates of the polynomial f, i.e. the Galois group of the splitting field $K_\infty(f, \alpha) \subseteq K^{\text{sep}}$ of $\{f^n - \alpha\}_{n\geq 1}$ over K, for some fixed separable closure K^{sep} of K. The group $G_\infty(K, f, \alpha)$ has a natural faithful action on the tree of preimages $\bigsqcup_{n\geq 0} f^{-n}(\alpha)$. Similarly, we may consider a transcendental element t over K and define $G_\infty(K, f, t)$ as the Galois group of the splitting field $K_\infty(f, t) \subseteq K(t)^{\text{sep}}$ of $\{f^n - t\}_{n\geq 1}$ over $K(t)$. If α is not strictly post-critical, we may see $G_\infty(K, f, \alpha)$ as the specialization of $G_\infty(K, f, t)$ to $t = \alpha \in K$, i.e. $G_\infty(K, f, \alpha)$ may be embedded into $G_\infty(K, f, t)$ as

the decomposition subgroup of a prime $\mathfrak{P}$ in $K_\infty(f,t)$ over the maximal ideal $(t-\alpha)$ in $K[t]$.

Note that the tree of preimages $\bigsqcup_{n\geq 0} f^{-n}(t)$ is isomorphic to T_d, where d is the degree of f. Therefore, we may view $G_\infty(K,f,t)$ as a closed (profinite) subgroup of $\mathrm{Aut}\ T_d$. We usually refer to the group $G_\infty(K,f,t)$ as the *arithmetic iterated Galois group* or the *arithmetic iterated monodromy group*.

Let us consider the normal subgroup $G_\infty(K^{\mathrm{sep}},f,t)$ of $G_\infty(K,f,t)$ corresponding to those automorphisms restricting to the identity on $K_\infty := K_\infty(f,t)\cap K^{\mathrm{sep}}$. We call $G_\infty(K^{\mathrm{sep}},f,t)$ the *geometric iterated Galois group* or the *geometric iterated monodromy group*. Then, Galois theory yields the short exact sequence

$$1 \longrightarrow G_\infty(K^{\mathrm{sep}},f,t) \longrightarrow G_\infty(K,f,t) \longrightarrow \mathrm{Gal}(K_\infty/K) \longrightarrow 1.$$

For a group $G \leq \mathrm{Aut}\ T$ and $n \geq 1$, we write

$$G_n := \frac{G}{\mathrm{St}_G(n)}$$

and call G_n the *nth congruence quotient* of G. The quotient G_n yields the action of G on the nth level of T.

If $f \in K(x)$ is a rational function over a number field K, then we may consider $K^{\mathrm{sep}} = \mathbb{C}$. In this case, it is well known that the congruence quotients of $G_\infty(K^{\mathrm{sep}},f,t)$ coincide (up to a relabelling of the tree T_d) with the congruence quotients of $\mathrm{IMG}(f)$ if f is post-critically finite [180, Proposition 6.4.2]. Therefore, the groups

$$G_\infty(K^{\mathrm{sep}},f,t) \quad \text{and} \quad \overline{\mathrm{IMG}(f)}$$

coincide up to conjugation in $\mathrm{Aut}\ T_d$, so we may rewrite the short exact sequence above as

$$1 \longrightarrow \overline{\mathrm{IMG}(f)} \longrightarrow G_\infty(K,f,t) \longrightarrow \mathrm{Gal}(K_\infty/K) \longrightarrow 1.$$

Thus, in order to understand the arithmetic monodromy group $G_\infty(K,f,t)$, it is enough to understand $\overline{\mathrm{IMG}(f)}$ and the outer action of $\mathrm{Gal}(K_\infty/K)$ on it. As a first step, we shall focus on understanding $\overline{\mathrm{IMG}(f)}$.

Using the Čebotarev density theorem, one can give an upper bound of $\mathcal{D}(P_f(a_0))$ via the so-called fixed-point proportion of $G_\infty(K,f,0)$; see [144, Theorem 2.3]. Other problems in arithmetic dynamics related to the proportion of periodic points of f over finite fields, are directly related to the fixed-point proportion of $\mathrm{IMG}(f)$; see [44, 86, 150].

Fixed-point proportion

Let $G \leq \operatorname{Aut} T$ be a group. We define the *fixed-point proportion* of G, denoted $\operatorname{FPP}(G)$, via the limit

$$\operatorname{FPP}(G) := \lim_{n\to\infty} \frac{\#\{g \in G_n \mid g \text{ fixes a vertex at level } n\}}{|G_n|}.$$

First note that the limit in the definition exists. Indeed, as fixing a vertex at level $n+1$ implies fixing a vertex at level n, the fraction in the limit is non-decreasing on n so the limit exists by monotone convergence.

Now, note that the fixed-point proportion of G and its closure $\overline{G}$ in $\operatorname{Aut} T$ coincide, as the definition only depends on the congruence quotients of G. Since $\overline{G}$ is a closed subgroup of the profinite group $\operatorname{Aut} T$, it is itself profinite. As a profinite group, $\overline{G}$ admits a unique Haar probability measure, which we shall denote by μ_G. This Haar probability measure satisfies

$$\mu_G(\operatorname{St}_G(n)) = |G : \operatorname{St}_G(n)|^{-1}$$

for every $n \geq 1$. We shall give an alternative definition of the fixed-point proportion in terms of the Haar measure μ_G.

Let ∂T be the space of *ends* in T, i.e. the Cantor set consisting of infinite rooted paths in T. The following observation will be useful for us:

Lemma 13. *Let $g \in \operatorname{Aut} T$. The following are both equivalent:*

(i) *the automorphism g fixes infinitely many vertices in T;*

(ii) *the automorphism g fixes an end $\gamma \in \partial T$.*

Let us write S for the set of elements in $\overline{G}$ fixing an end in ∂T, and S_n for the set of elements in $\overline{G}$ fixing a vertex at level n. By Lemma 13, an automorphism $g \in G$ fixes an end in ∂T if and only if it fixes a vertex at every level; hence

$$S = \bigcap_{n\geq 1} S_n.$$

This yields an alternative definition of the fixed-point proportion in terms of the Haar measure:

$$\begin{aligned}\operatorname{FPP}(G) &= \lim_{n\to\infty} \frac{\#\{g \in G_n \mid g \text{ fixes a vertex at level } n\}}{|G_n|} \\ &= \lim_{n\to\infty} \mu_G(S_n) = \mu_G\Big(\bigcap_{n\geq 1} S_n\Big) = \mu_G(S).\end{aligned}$$

An example: the quadratic Chebyshev polynomial

Let us consider the quadratic Chebyshev polynomial $f(z) := 2z^2 - 1$. The Chebyshev polynomial f is post-critically finite, with ramification graph

$$0 \longrightarrow -1 \longrightarrow 1 \circlearrowleft$$

where 0 is the unique critical point of f. We follow the approach of Jones in [144] to compute the fixed-point proportion of $G := \mathrm{IMG}(f)$.

The iterated monodromy group G is generated by the automorphisms $a, b \in \mathrm{Aut}\ T_2$ defined via

$$\psi(a) = (1,1)\sigma \quad \text{and} \quad \psi(b) = (a,b),$$

where $\sigma := (1\,2) \in \mathrm{Sym}(2)$.

Let D_∞ be the infinite dihedral group, i.e. the group given by the presentation

$$D_\infty := \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

It is easy to see that $G \cong D_\infty$, as $a^2 = b^2 = 1$ and ab has infinite order. Furthermore

$$G = a\langle ab \rangle \sqcup \langle ab \rangle.$$

The coset $a\langle ab \rangle$ corresponds to the reflections, while the coset $\langle ab \rangle$ corresponds to the rotations. Note that the only rotation fixing an end in ∂T_2 is the identity, so the Haar measure of the subset of rotations fixing an end in ∂T_2 is zero.

Now, the reflection a moves every vertex in T_2 and thus it does not fix any end in ∂T_2. On the other hand, the reflection b fixes precisely one end in ∂T_2. Now note that

$$a\langle ab \rangle = a\langle (ab)^2 \rangle \sqcup b\langle (ab)^2 \rangle.$$

The elements in $a\langle (ab)^2 \rangle$ are all conjugate to a, while the elements in $b\langle (ab)^2 \rangle$ are all conjugate to b. Indeed, we have

$$a(ab)^{2k} = a^{(ba)^{k-1}b} \quad \text{and} \quad b(ab)^{2k} = b^{(ab)^k}$$

for $k \geq 1$ and, as $(ab)^{-1} = ba$, we have similarly

$$a(ba)^{2k} = a^{(ba)^k} \quad \text{and} \quad b(ba)^{2k} = b^{(ab)^{k-1}a}$$

for $k \geq 1$. Thus, none of the elements in $a\langle (ab)^2 \rangle$ fix an end in ∂T_2, while every element in $b\langle (ab)^2 \rangle$ fixes exactly one end in ∂T_2. Hence

$$\mathrm{FPP}(G) = \mu_G(b\langle (ab)^2 \rangle) = \mu_G(\langle (ab)^2 \rangle) = |G : \langle (ab)^2 \rangle|^{-1} = \frac{1}{4}.$$

The iterated monodromy group of a Chebyshev polynomial of any degree has positive fixed-point proportion, by the same argument as above; see [144, Proposition 1.2]. However, it is widely believed that these are the only exceptional cases with positive fixed-point proportion:

Conjecture 14. *If $f \in \mathbb{C}[z]$ is a post-critically finite complex polynomial not linearly conjugate to (up to a sign) a Chebyshev polynomial, then* $\mathrm{FPP}(\mathrm{IMG}(f)) = 0$.

More generally, Chebyshev polynomials are expected to be the only polynomials whose geometric iterated Galois groups over a number field have positive fixed-point proportion.

Hausdorff dimension

Let (M, d) be a non-empty metric space. For any non-empty subset $S \subseteq M$, we define its diameter

$$\operatorname{diam} S := \sup\{d(x, y) \mid x, y \in S\},$$

and we set $\operatorname{diam} \varnothing := 0$.

Let $s \in [0, \infty)$. For any $\delta > 0$, we define

$$\mathcal{H}^s_\delta(S) := \inf_{\mathcal{U}}\{\sum_{U\in\mathcal{U}} (\operatorname{diam} U)^s \mid S \subseteq \bigcup_{u\in\mathcal{U}} U \text{ and } \operatorname{diam} U < \delta\},$$

where the infimum is taken over all such countable covers $\mathcal{U}$ of S. We define the *s-Hausdorff outer measure* $\mathcal{H}^s(\cdot)$ as

$$\mathcal{H}^s(S) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(S),$$

which is well defined by monotone convergence. The outer measure $\mathcal{H}^s(\cdot)$ yields a well-defined measure on the Borel algebra of (M, d).

We define the *Hausdorff dimension* $\operatorname{hdim}_M(S)$ of S in M via

$$\operatorname{hdim}_M(S) := \inf_{s\geq 0}\{s \mid \mathcal{H}^s(S) = 0\} = \sup_{s\geq 0}\{s \mid \mathcal{H}^s(S) = \infty\}.$$

Hausdorff dimension is a standard tool in fractal geometry [73]. Here, we shall restrict ourselves to its use in the study of profinite groups.

Hausdorff dimension in profinite groups

The study of the Hausdorff dimension in profinite groups was initiated by Abercrombie in 1994 [1]. Let G be a countably based profinite group, and

$$G =: N_0 \geq N_1 \geq \cdots \geq N_k \geq \cdots \geq \bigcap_{k\geq 0} N_k = 1$$

a filtration $\mathcal{N}$ of open normal subgroups of G. We may endow G with an ultrametric $d_{\mathcal{N}} : G \times G \to [0, \infty)$ given by

$$d_{\mathcal{N}}(g, h) := \inf_{k\geq 0}\{|G : N_k|^{-1} \mid gh^{-1} \in N_k\}.$$

Therefore, we obtain a metric space $(G, d_{\mathcal{N}})$ and we may define a Hausdorff dimension $\mathrm{hdim}_G^{\mathcal{N}}(\cdot)$ with respect to $d_{\mathcal{N}}$ for the Borel subsets of G.

Some properties of $\mathrm{hdim}_G^{\mathcal{N}}(\cdot)$ may be deduced directly from the definition. For example, for any filtration $\mathcal{N}$, we have

$$\mathrm{hdim}_G^{\mathcal{N}}(G) = 1.$$

Indeed, for any $k \geq 1$, the group G may be covered minimally by $|G : N_k|$ sets of diameter $|G : N_k|^{-1}$ corresponding to each right (or left) coset of N_k, as every set of diameter at most $|G : N_k|^{-1}$ is contained in a coset of N_k. Then

$$\mathcal{H}^s(G) = \lim_{k\to\infty} \frac{|G : N_k|}{|G : N_k|^s} = \lim_{k\to\infty} |G : N_k|^{1-s}.$$

Thus $\mathrm{hdim}_G^{\mathcal{N}}(G) = 1$, as for any $s < 1$ we have

$$\mathcal{H}^s(G) = \lim_{k\to\infty} |G : N_k|^{1-s} = \infty,$$

but

$$\mathcal{H}^1(G) = \lim_{k\to\infty} |G : N_k|^{1-1} = 1.$$

In general, working directly with the definition of the Hausdorff dimension is not easy. We shall introduce another fractal dimension, which, even if it is not as well behaved as the Hausdorff dimension, it is easier to work with.

For any $\delta > 0$, let us define $N_\delta(S)$ to be the minimal number of sets of diameter at most δ needed to cover a subset $S \subseteq G$. We define the *lower-box dimension* $\underline{\dim}_B^{\mathcal{N}}(\cdot)$ in G via

$$\underline{\dim}_B^{\mathcal{N}}(S) := \liminf_{\delta\to 0} \frac{\log |N_\delta(S)|}{-\log \delta}$$

for any $S \subseteq G$.

Arguing as above, for each $\delta = |G : N_k|^{-1}$, we have $N_\delta(S) = |SN_k : N_k|$, and we obtain

$$\underline{\dim}_B^{\mathcal{N}}(S) = \liminf_{k\to\infty} \frac{\log |SN_k : N_k|}{\log |G : N_k|}$$

as the diameters can only take values in $\{|G : N_k|^{-1}\}_{k\geq 0}$ by construction.

It is a standard result in fractal geometry [73] that

$$\mathrm{hdim}_G^{\mathcal{N}}(S) \leq \underline{\dim}_B^{\mathcal{N}}(S)$$

for any Borel subset $S \subseteq G$. For a closed subgroup $H \leq G$, the inequality

$$\mathrm{hdim}_G^{\mathcal{N}}(H) \geq \underline{\dim}_B^{\mathcal{N}}(H)$$

follows from the work of Abercrombie in [1] and it was first stated explicitly by Barnea and Shalev in [13]:

Proposition 15 (Abercrombie 1994, Barnea and Shalev 1997). *Let G be a countably based profinite group endowed with a filtration $\mathcal{N}$. Then $\mathrm{hdim}_G^{\mathcal{N}}(H) = \underline{\dim}_B^{\mathcal{N}}(H)$ for any closed subgroup $H \leq G$.*

A recent short proof of a more general version of Proposition 15 for equilibrated and fundamental sets due to Alexandru and Vâjâitu can be found in [9]. From now, we shall focus on the Hausdorff dimension of closed subgroups of a profinite group.

Before discussing the Hausdorff dimension in groups acting on rooted trees, we define the notion of strong Hausdorff dimension. A closed subgroup $H \leq G$ has *strong Hausdorff dimension* in G (with respect to the filtration $\mathcal{N}$) if the limit

$$\mathrm{hdim}_G^{\mathcal{N}}(H) = \lim_{k\to\infty} \frac{\log |HN_k : N_k|}{\log |G : N_k|}$$

exists. For example, closed subgroups of G with Hausdorff dimension 1 have strong Hausdorff dimension in G as the limit superior of the sequence

$$\{\log |HN_k : N_k| / \log |G : N_k|\}_{k\geq 1}$$

is at most 1.

Hausdorff dimension in groups acting on rooted trees

Let T be a spherically homogeneous rooted tree. The automorphism group $\mathrm{Aut}\ T$ is a countably based profinite group endowed with a natural filtration $\mathcal{S}$ of open normal subgroups:

$$\mathrm{Aut}\ T \geq \mathrm{St}(1) \geq \cdots \geq \mathrm{St}(k) \geq \cdots \geq \bigcap_{k\geq 1} \mathrm{St}(k) = 1.$$

Thus, we consider the Hausdorff dimension of a closed subgroup $G \leq \mathrm{Aut}\ T$ with respect to the level-stabilizer filtration $\mathcal{S}$, unless stated otherwise. We shall write $\mathrm{hdim}_T(\cdot)$ for $\mathrm{hdim}^{\mathcal{S}}_{\mathrm{Aut}\ T}(\cdot)$ to simplify notation.

If one considers an arbitrary closed subgroup $G \leq \mathrm{Aut}\ T$, there is little one can say about $\mathrm{hdim}_T(G)$ in general. Therefore, it is natural to consider the Hausdorff dimension of closed subgroups satisfying additional properties. Let us introduce a class of groups which is well-behaved in terms of the Hausdorff dimension.

Let T_d be the d-adic tree and $K \leq \mathrm{Aut}\ T_d$ be a subgroup. We say that K is *branching*, if

$$\psi(K) \geq K \times \cdots \times K,$$

i.e. if K contains copies of itself. A group $G \leq \mathrm{Aut}\ T_d$ is *regular branch* if G is self-similar, level-transitive and it contains a branching subgroup K of finite index in G. More generally G is said to be *weakly regular branch* if it is self-similar, level-transitive and it contains a non-trivial branching subgroup K. Note that (weakly) regular branch groups are (weakly) branch.

Let p denote a prime. We call the Sylow pro-p subgroup W_p of $\mathrm{Aut}\ T_p$ the *group of p-adic automorphisms*. Bartholdi proved in 2006 the following result on the Hausdorff dimension of regular branch groups [15]:

Proposition 16 (Bartholdi, 2006). *Let $G \leq \mathrm{Aut}\ T_d$ be a regular branch group. Then*

$$\mathrm{hdim}_T(G) > 0.$$

Furthermore, if $G \leq W_p$ then

$$0 < \mathrm{hdim}_{W_p}(G) \in \mathbb{Z}[1/p].$$

Grigorchuk in 2005 and Šunić in 2007 asked whether more generally self-similar subgroups of W_p have rational Hausdorff dimension [120, 221]:

Question 17 (Grigorchuk 2005, Šunić 2007). *Let $G \leq W_p$ be a self-similar group. Then, is it true that $\mathrm{hdim}_{W_p}(G)$ is rational?*

The Fontaine-Mazur conjecture, a conjecture of Abért and Virág and Boston's conjecture

A main motivation to study the Hausdorff dimension of subgroups of W_p comes from arithmetic geometry.

Let K be a number field, K^{sep} a separable closure of K and S a finite set of primes not above p. Let us consider $L \subseteq K^{\text{sep}}$ the maximal pro-p extension unramified outside the primes above S and let $G_{K,S}$ denote its Galois group. A central conjecture in arithmetic geometry claims that the p-adic Galois representations of $G_{K,S}$ must have finite image [94]:

Conjecture 18 (The Fontaine-Mazur conjecture, 1993). *Let $G_{K,S}$ be as above. Then, any p-adic Galois representation*

$$\rho : G_{K,S} \to \mathrm{GL}_n(\mathbb{Z}_p)$$

has finite image.

Boston proposed in [37] a further generalization of the Fontaine-Mazur conjecture for linear representations over a pro-p domain.

How is the Fontaine-Mazur conjecture related to the Hausdorff dimension of subgroups of W_p?

In their seminal paper [4], Abért and Virág showed that solvable subgroups of W_p are zero-dimensional and suggested as a possible generalization that groups satisfying a law are zero-dimensional. Furthermore, they further conjectured that pro-p groups which are linear over a pro-p domain are also zero-dimensional in W_p:

Conjecture 19 (Abért and Virág, 2005). *Let $G \leq W_p$ be a pro-p group, which is linear over a pro-p domain R. Then* $\mathrm{hdim}_{W_p}(G) = 0$.

The critical cases of the Fontaine-Mazur conjecture are precisely those such that $G_{K,S}$ is just-infinite; see Boston's discussion in [35]. Thus, if Conjecture 19 holds, then it is enough to show that $G_{K,S}$ admits a positive-dimensional embedding into W_p to prove the Fontaine-Mazur conjecture (for just-infinite pro-p groups).

Furthermore, motivated by the classification of just-infinite pro-p groups, Boston conjectured that the just-infinite pro-p quotients of $G_{K,S}$ are branch. As all known branch subgroups of W_p have positive Hausdorff dimension in W_p and the just-infinite pro-p quotients of $G_{K,S}$ are expected to have positive-Hausdorff dimension in W_p, Boston posed in 2000 the following purely geometric group-theoretic conjecture [35]:

Conjecture 20 (Boston's conjecture, 2000). *A just-infinite pro-p group G is branch if and only if it admits a positive-dimensional embedding into W_p.*

It was shown by Ershov and Jaikin-Zapirain in [71] that pro-p groups with positive weighted deficiency admit non-branch just-infinite quotients. Thus, as many of the Galois groups $G_{K,S}$ have positive weighted deficiency, they are not branch and the initial conjecture of Boston is false in many instances. However, the above purely group-theoretic Boston's conjecture has remained open.

Hausdorff spectra and the conjectures of Klopsch and Shalev

One of the main results of Barnea and Shalev in [13] was the computation of the Hausdorff dimension (with respect to the p-power filtration) of the closed subgroups of a p-adic analytic pro-p group. Given a countably based profinite group G and a filtration $\mathcal{N}$, we define its *Hausdorff spectrum* $\mathrm{hspec}^{\mathcal{N}}(G)$ as

$$\mathrm{hspec}^{\mathcal{N}}(G) := \{\mathrm{hdim}_G^{\mathcal{N}}(H) \mid H \leq G \text{ closed}\}.$$

We further define the *finitely generated Hausdorff spectrum* of G $\mathrm{hspec}_{\mathrm{fg}}^{\mathcal{N}}(G)$ as

$$\mathrm{hspec}_{\mathrm{fg}}^{\mathcal{N}}(G) := \{\mathrm{hdim}_G^{\mathcal{N}}(H) \mid H \leq G \text{ closed and (topologically) finitely generated}\}.$$

Barnea and Shalev proved in [13] that $\mathrm{hspec}^{\mathcal{P}}(G)$ is finite, where $\mathcal{P}$ is the p-power filtration, if G is a p-adic analytic pro-p group. Furthermore, they conjectured that this is indeed a characterization of p-adic analytic groups among pro-p groups, a conjecture which is still only known to hold for solvable groups; see [160].

Motivated by this conjecture, the study of the Hausdorff spectrum of other classes of groups started soon after. One of the first results in this direction was obtained by Klopsch in his PhD thesis [158]. He proved that branch groups have complete Hausdorff spectrum (with respect to the stabilizer filtration), i.e.

$$\mathrm{hspec}(G) = [0, 1]$$

for $G \leq \mathrm{Aut}\, T$ a branch group. However, only countably many of the closed subgroups constructed by Klopsch in [158] were (topologically) finitely generated, which motivated him to pose the following question:

Question 21 (Klopsch, 1999). *Does there exist a topologically finitely generated branch profinite group whose finitely generated Hausdorff spectrum is uncountable?*

Around the same time, Shalev also considered in [211] the finitely generated Hausdorff spectrum of general (topologically) finitely generated pro-p groups. Like Klopsch, he also asked about the finitely generated Hausdorff spectrum:

Question 22 (Shalev, 2000). *Are there topologically finitely generated pro-p groups with uncountable finitely generated Hausdorff spectrum? Can the finitely generated Hausdorff spectrum of such a group contain irrational numbers? Intervals?*

The question of Shalev regarding the irrationality of the Hausdorff dimension was answered by the remarkable work of Abért and Virág in [4]. Indeed, they proved that three independent Haar-random elements in W_p generate (topologically) a closed subgroup of Hausdorff dimension 1 in W_p, and used a generalization of this result to show that for every $\alpha \in [0,1]$, the group W_p contains 3-generated subgroups whose closure has Hausdorff dimension α in W_p. As a result, they obtained (topologically) finitely generated pro-p groups whose finitely generated Hausdorff spectrum contains irrational numbers. However, as W_p itself is not (topologically) finitely generated, their results did not answer the remaining questions of Shalev.

Motivated by the remarkable result of Abért and Virág on the Hausdorff dimension of the closure of 3-generated Haar-random subgroups of W_p, Bartholdi further posed the following question in [15]:

Question 23 (Bartholdi, 2006). *Does there exists a finitely generated self-similar group whose closure in* $\mathrm{Aut}\ T_d$ *has Hausdorff dimension* 1*?*

We note that the random subgroups of Abért and Virág are very far from being self-similar.

Main results of the thesis

Now, we discuss briefly the main results in this thesis. The goal of this discussion is to give the reader an overview of the main results obtained in this thesis. We have skipped all the proofs here and we invite the reader to read the subsequent chapters for detailed proofs and in-depth discussions of the results reviewed in this section.

Hausdorff dimension (Chapters I-VI)

The first part of the thesis is devoted to the study of the Hausdorff dimension in groups acting on rooted trees.

Self-similar groups

The first goal of **Chapter I** is to develop tools to compute the Hausdorff dimension of self-similar groups. We show that the Hausdorff dimension of a self-similar group is strong and

that it may be computed explicitly from an ordinary generating function as follows.

Let us fix $G \leq \mathrm{Aut}\, T_d$ a self-similar group acting on the d-adic tree T_d. Let us consider the sequence $\{r_n(G)\}_{n\geq 1}$ given by

$$r_n(G) := d \cdot \log |G : \mathrm{St}_G(n-1)| - \log |G : \mathrm{St}_G(n)| + \log |G : \mathrm{St}_G(1)|$$

and its forward-gradient

$$\{s_n(G)\}_{n\geq 1} := \nabla\{r_n(G)\}_{n\geq 1} = \{r_{n+1}(G) - r_n(G)\}_{n\geq 1}.$$

We consider the ordinary generating function

$$S_G(x) := \sum_{n\geq 1} s_n(G) x^n.$$

For any $H \leq \mathrm{Sym}(d)$, we define the *iterated wreath product* W_H as

$$W_H := \{g \in \mathrm{Aut}\, T_d \mid g|_v^1 \in H\} \leq \mathrm{Aut}\, T_d.$$

The first main result in **Chapter I** reads as follows:

Theorem 24 (Chapter I). *Let $H \leq \mathrm{Sym}(d)$ and $G \leq W_H$ a self-similar group. Then, the closure of G in W_H has strong Hausdorff dimension given by*

$$\mathrm{hdim}_{W_H}(\overline{G}) = \frac{1}{\log |H|} \left(\log |G : \mathrm{St}_G(1)| - S_G(1/d)\right).$$

Theorem 24 has direct applications. For example, we may answer Question 23 as follows:

Corollary 25 (Chapter I). *Let $G \leq \mathrm{Aut}\, T_d$ be a self-similar group such that*

$$\mathrm{hdim}_{T_d}(\overline{G}) = 1.$$

Then G is not finitely generated.

Furthermore, we show that a self-similar closed subgroup $G \leq \mathrm{Aut}\, T_d$ is regular branch if and only if the sequence $\{r_n(G)\}_{n\geq 1}$ is eventually constant. Thus, the Hausdorff dimension of a regular branch group is computable from finitely many levels. It was proved by Bartholdi and Noce in [24] that there exists an algorithm to compute the Hausdorff dimension of a regular branch group. Such an algorithm is immediate if we know a branch structure over a level stabilizer. However, this is not always the case in practice.

In **Chapter III**, we deduce, in terms of the index of an arbitrary branching subgroup, an upper bound on the number of levels one has to consider to compute the Hausdorff dimension of a regular branch group. This yields an explicit algorithm to compute the Hausdorff dimension of a regular branch group from any finite-index branching subgroup. Let us write $\Omega(G : K)$ for the number of prime factors of a subgroup index $|G : K|$ with multiplicity.

Theorem 26 (Chapter III). *Let $G \leq W_H \leq \operatorname{Aut} T_d$ be regular branch over K. Then*

$$\operatorname{hdim}_{W_H}(\overline{G}) = \frac{1}{\log|H|}\left(\log|G_1| - \sum_{n=1}^{\Omega(G:K)} \frac{s_n(G)}{d^n}\right).$$

Theorem 24 is very useful to compute the Hausdorff dimension of explicit examples of self-similar groups. Together with Garciarena, we apply Theorem 24 to a particular family of self-similar groups in **Chapter IV** to obtain the first examples of self-similar subgroups of $W_{\langle\sigma\rangle}$ with transcendental Hausdorff dimension, where $\sigma := (1\,2\cdots d) \in \operatorname{Sym}(d)$. Let us consider the automorphisms $a, b \in W_{\langle\sigma\rangle}$ defined recursively as

$$a = (1, \ldots, 1, a)\sigma \quad \text{and} \quad b = (1, \ldots, 1, b^{-1})\sigma.$$

The group $H := \langle a, b\rangle$ is called the *generalized Brunner-Sidki-Vieira group*. For $d = 2$, this group was introduced in [47], and later generalized to $d \geq 3$ in [212]. The main result of **Chapter IV** is the following:

Theorem 27 (Chapter IV). *Let $H \leq \operatorname{Aut} T_d$ be the generalized Brunner-Sidki-Vieira group. Then*

$$\operatorname{hdim}_{W_{\langle\sigma\rangle}}(\overline{H}) = \frac{d - \tau(d-1)\log_d 2}{d+1},$$

where

$$\tau := \begin{cases} 0, & \text{if } d \text{ is odd;} \\ 1, & \text{if } d \text{ is even.} \end{cases}$$

Lastly, Boston's conjecture suggests a direct connection between branch groups and positive Hausdorff dimension for just-infinite groups. More generally, we might wonder whether there is a connection between weakly branch groups and positive Hausdorff dimension. In **Chapter V**, we establish the following:

Theorem 28 (Chapter V). *Let $G \leq \operatorname{Aut} T_d$ be a level-transitive self-similar closed subgroup such that*

$$\operatorname{hdim}_{T_d}(G) > 0.$$

Then G is weakly branch.

We also show in **Chapter V** that Theorem 28 does not hold if one does not assume self-similarity; see Theorem 34 below.

Hausdorff Spectra

We write W_q for the iterated wreath product corresponding to a subgroup of $\mathrm{Sym}(q)$ generated by a q-cycle for a prime power q. We call W_q the *group of q-adic automorphisms*.

One of the main goals of **Chapter I** is to answer Question 17, i.e. whether self-similar groups always have rational Hausdorff dimension in W_p. We prove a much stronger result, where we compute the Hausdorff spectrum of W_q restricted to the main subclasses of self-similar groups:

Theorem 29 (Chapter I). *The group W_q has the following four restricted Hausdorff spectra:*

(i) *self-similar, super strongly fractal and level-transitive Hausdorff spectrum:* $[0,1]$;

(ii) *weakly regular branch Hausdorff spectrum:* $(0,1]$;

(iii) *regular branch Hausdorff spectrum:* $\mathbb{Z}[1/p] \cap (0,1]$;

(iv) *self-similar branch Hausdorff spectrum:* $[0,1]$.

The self-similar branch Hausdorff spectrum suggests that Boston's conjecture might not be true. In fact, the last part of **Chapter I** is devoted to constructing just-infinite branch pro-p groups of zero Hausdorff dimension. We summarize the main result here:

Theorem 30 (Chapter I). *For every prime $p \geq 2$, there exists a just-infinite branch pro-p group $G \leq W_p$ such that*

$$\mathrm{hdim}_{W_p}(G) = 0.$$

Note that Theorem 30 does not disprove Boston's conjecture. We need to understand the distinct actions of the groups in Theorem 30 on the p-adic tree T_p. As we shall see, this is the main content of **Chapter VIII**, which is highly non-trivial.

In **Chapter II**, together with Garaialde Ocaña and Uria-Albizuri, we answer Questions 21 and 22. For a profinite group G, we define the *bounded Hausdorff spectrum* $\mathrm{hdim}_{\mathrm{b}}(G)$ as

$$\mathrm{hdim}_{\mathrm{b}}(G) := \{\mathrm{hdim}_G(H) \mid H \leq G \text{ and } d(H) < b\},$$

where $d(H)$ denotes the minimum number of topological generators of H.

The first main result in **Chapter II** is the following:

Theorem 31 (Chapter II). *Let $G \leq \mathrm{Aut}\, T_d$ be a finitely generated regular branch group. Then*

$$\mathrm{hdim}_{\mathrm{b}}(\overline{G}) = \mathrm{hdim}_{\mathrm{fg}}(\overline{G}) = \mathrm{hdim}(\overline{G}) = [0,1].$$

Theorem 31 further applies to weakly regular branch groups with a finitely generated normal branching subgroup, like the iterated monodromy groups of periodic quadratic complex polynomials; see **Chapter X**. Theorem 31 answers Questions 21 and 22 simultaneously.

Theorem 31 for $G = W_p$ was proved by Abért and Virág in their seminal paper [4]. Their probabilistic proof fails for more general iterated wreath product. A second main result in **Chapter II** is a generalization of the result of Abért and Virág to arbitrary iterated wreath products:

Theorem 32 (Chapter II). *Let $W_H \leq \mathrm{Aut}\, T_d$ be an iterated wreath product for a transitive group $H \leq \mathrm{Sym}(d)$. Then*

$$\mathrm{hdim}_{\mathrm{b}}(W_H) = \mathrm{hdim}_{\mathrm{fg}}(W_H) = \mathrm{hdim}(W_H) = [0,1].$$

We remark that the proof of Theorem 32 is constructive and explicit.

In **Chapter II**, we also consider Hausdorff spectra of branch groups for other filtrations. We introduce the notion of *nice filtrations*, as those filtrations that satisfy a key property satisfied by the level-stabilizer filtration; see **Chapter II**. In particular, this includes the rigid vertex stabilizer filtration. Most of the results we obtain in **Chapter II** for the level-stabilizer filtration hold more generally for arbitrary nice filtrations.

However, for general topologically finitely generated pro-p groups, there are several natural filtrations such as the lower-p central series $\mathcal{L}$ and the Jennings-Zassenhaus series $\mathcal{D}$ which are not nice filtrations. In **Chapter II**, we further compute the bounded and finitely generated Hausdorff spectra of the first Grigorchuk group with respect to these filtrations:

Theorem 33 (Chapter II). *Let $\mathcal{G} \leq \mathrm{Aut}\, T_2$ be the first Grigorchuk group. Then*

$$\mathrm{hdim}_{\mathrm{b}}^{\mathcal{N}}(\overline{\mathcal{G}}) = \mathrm{hdim}_{\mathrm{fg}}^{\mathcal{N}}(\overline{\mathcal{G}}) = \mathrm{hdim}^{\mathcal{N}}(\overline{\mathcal{G}}) = [0,1],$$

for $\mathcal{N} \in \{\mathcal{L}, \mathcal{D}\}$.

The main obstruction to generalizing Theorem 33 to general branch groups is that the filtrations $\mathcal{L}$ and $\mathcal{D}$ are not well understood for arbitrary branch groups.

Normal subgroups and zero-dimensional subgroups

In 2005, Abért and Virág showed that if a level-transitive subgroup $G \leq W_p$ has Hausdorff dimension 1 in W_p, then the normal subgroups of G all have Hausdorff dimension 1 in W_p. They further asked whether this holds replacing $\mathrm{hdim}_{W_p}(G) = 1$ with $\mathrm{hdim}_{W_p}(G) > 0$. In **Chapter V**, we answer this question in the negative:

Theorem 34 (Chapter V). *For any $d \geq 2$, any transitive $H \leq \mathrm{Sym}(d)$ and any $\alpha \in [0, 1/d]$, there exists a closed subgroup $G \leq W_H$ such that*

(i) *G is level-transitive;*

(ii) *G has strong Hausdorff dimension α in W_H;*

(iii) *there is a non-trivial normal subgroup $1 \neq N \trianglelefteq G$ with $\mathrm{hdim}_G(N) = 0$;*

(iv) *G is not weakly branch.*

However, we answer the question of Abért and Virág in the positive if G is assumed to be self-similar:

Theorem 35 (Chapter V). *Let $G \leq W_H$ be a self-similar closed subgroup with*

$$\mathrm{hdim}_{W_H}(G) > 0.$$

Then $\mathrm{hdim}_G(N) = 1$ for any non-trivial $1 \neq N \trianglelefteq G$ normal subgroup.

The proof of Theorem 35 is based on a result on the perfectness of the Hausdorff dimension:

Theorem 36 (Chapter V). *Let $G \leq W_H$ be a closed subgroup. Then*

$$\mathrm{hdim}_{W_H}(G') = \mathrm{hdim}_{W_H}(G).$$

Note that Theorem 36 was proved for subgroups of W_p by Abért and Virág in [4]. In particular, Theorem 36 yields that solvable subgroups of W_H are zero-dimensional. We generalize this result to groups satisfying a law in **Chapter VI**, but with very different methods.

Let T be a bounded tree, i.e. a spherically homogeneous rooted tree where every vertex has at most M immediate descendants for some fixed $M \geq 1$. The groups constructed in Theorem 34 are not weakly branch, but almost all (i.e. all but finitely many) the projections are weakly branch. In **Chapter VI**, we prove that this holds for all positive-dimensional groups:

Theorem 37 (Chapter VI). *Let $G \leq \mathrm{Aut}\, T$ be a closed subgroup with $\mathrm{hdim}_T(G) > 0$. Then, for almost all $v \in T$, the projection $\varphi_v(\mathrm{st}_G(v))$ is weakly branch in $\mathrm{Aut}\, T_v$.*

Theorem 37 imposes strong restrictions on positive-dimensional subgroups of $\mathrm{Aut}\, T$. We use this restrictions in **Chapter VI** to generalize the zero-dimensionality of solvable groups to groups satisfying a law, as suggested by Abért and Virág in [4]:

Theorem 38 (Chapter VI). *Let $G \leq \mathrm{Aut}\, T$ be a closed subgroup such that either*

(i) *G satisfies a group law; or*

(ii) *G has infinite center.*

Then $\mathrm{hdim}_T(G) = 0$.

Finally, we answer Conjecture 19 in the positive. We prove the following stronger result:

Theorem 39 (Chapter VI). *Let $G \leq \mathrm{GL}_n(R)$, where R is an integral domain. Then for any embedding $\rho : G \hookrightarrow \mathrm{Aut}\, T$, the image $\rho(G)$ has strong zero Hausdorff dimension in* $\mathrm{Aut}\, T$.

The proof of Theorem 39 uses Theorem 37 and non-commuting representations of graphs.

Subgroup structure (Chapters VII-X)

The second part of this thesis is devoted to understanding the subgroup structure of groups acting on rooted trees.

The structure lattice of a branch group

If $G \leq \mathrm{Aut}\, T$ is a branch group, let $L(G)$ be the collection of subnormal subgroups of G with finitely many conjugates. We write $H \leq_{\mathrm{va}} K$ if H contains the commutator subgroup of a finite-index subgroup of K. We put a Boolean algebra structure on $L(G)$ via $L \wedge K := L \cap K$ and $L \vee K := \langle L, K \rangle$ and we consider the quotient $L(G)/\sim$ where

$$L \sim K \iff L \cap K \leq_{\mathrm{va}} L, K.$$

The quotient $\mathcal{L}(G) := L(G)/\sim$ is called the *structure lattice* of G.

In 1971, Wilson showed that the structure lattice of a just-infinite branch group is isomorphic to the Boolean algebra of clopens of Cantor's ternary set. In **Chapter VII**, together with Grigorchuk, we give a complete description of the structure lattice of a (not necessarily just-infinite) branch group. Indeed, we obtain a G-equivariant and explicit canonical isomorphism of Boolean algebras characterising the structure lattice of a branch group G, generalizing the result of Wilson:

Theorem 40 (Chapter VII). *Let $\rho : G \to \mathrm{Aut}\, T$ be a branch action of a group G on a spherically homogeneous rooted tree T and* $\mathrm{Bool}(\partial T)$ *the Boolean algebra of clopen subsets of*

the boundary of T. Then there exists a canonical G-equivariant isomorphism $\Phi : \mathcal{L}(G) \to \mathrm{Bool}(\partial T)$ with inverse $\Phi^{-1} : \mathrm{Bool}(\partial T) \to \mathcal{L}(G)$ given by

$$\Phi : \mathcal{L}(G) \to \mathrm{Bool}(\partial T) \qquad \text{and} \qquad \Phi^{-1} : \mathrm{Bool}(\partial T) \to \mathcal{L}(G)$$
$$[H] \mapsto \mathrm{Supp}(H), \qquad\qquad C \mapsto \Big[\prod_{v \in C} \mathrm{rist}_{\rho(G)}(v) \Big],$$

where $\mathrm{Supp}(H)$ is the support of H, i.e. the elements in ∂T not fixed by $\rho(H)$.

Theorem 40 shows that for a branch group G every G-action on ∂T induced by a branch action $\rho : G \to \mathrm{Aut}\ T$ is equivalent to the action of G by conjugation on its structure lattice $\mathcal{L}(G)$. In particular the action of G on the Stone space of its structure lattice is equivalent to the action of G on ∂T. Theorem 40 also gives an explicit description of the structure lattice analogous to the one of the structure graph by Hardy in his PhD thesis [138].

First-order theory of weakly branch groups

The first part of **Chapter VIII** is devoted to generalizing the structure graph of a branch group to weakly branch groups. Recall that the *structure graph* of a branch group G is the restriction of the structure lattice to basal subgroups, where a subgroup $B \leq G$ is *basal* if it has finitely many conjugates and

$$B^G = B^{g_1} \times B^{g_2} \times \cdots \times B^{g_r}$$

for some $g_1, g_2, \ldots, g_r \in G$. For basal subgroups, the equivalence relation $\sim$ may be described in terms of centralizers. In **Chapter VIII**, we propose an equivalence relation in terms of normalizers, which makes sense also for weakly branch groups.

Let $G \leq \mathrm{Aut}\ T$ be a weakly branch group and let $B(G)$ be the collection of basal subgroups of G. We define the equivalence relation $\sim$ in $B(G)$ given by

$$B \sim \widetilde{B} \iff B \cap \widetilde{B} \neq 1 \text{ and } N_G(B) = N_G(\widetilde{B}).$$

We define the *structure graph* of G to be $\mathcal{B}_G := B(G)/\sim$.

A *first-order sentence* in G is a formula built up from the quantifiers $\neg, \wedge, \vee, \to, \exists, \forall$ and equations $f(x_1, \ldots, x_k) = 1$, with variables taking values in G. A *first-order definable* (subset) subgroup of G is a (subset) subgroup $H \leq G$ given by

$$H = \{g \in G \mid \varphi(g)\},$$

where $\varphi(g)$ is a first-order sentence in G. A collection of subgroups $\{H_i\}_{i\in I}$ is said to be *first-order definable in* G if the set I is first-order definable in G and for every $i \in I$ the subgroup H_i is first-order definable in G.

In **Chapter VIII**, following the approach of Wilson in [234] for branch groups, we use the structure graph of a weakly branch group to obtain the following result on the first-order theory of weakly branch groups:

Theorem 41 (Chapter VIII). *Let $G \leq \mathrm{Aut}\ T$ be a weakly branch group. Then, the following are all first-order definable in G:*

(i) *the congruence topology in G;*

(ii) *the structure graph $\mathcal{B}_G$.*

Furthermore, if the action of G on T as a weakly branch group is unique up to conjugation in $\mathrm{Aut}\ T$*, then*

(iii) *each level-stabilizer $\mathrm{St}_G(n)$ is also first-order definable in G.*

We note that a group-theoretic definition of the congruence topology in a branch group and a weakly branch group was given by Garrido in [106] and in her doctoral thesis [105]. The first-order definability in (i) and (iii) of Theorem 41 are new even for branch groups. In fact, (iii) extends a result of Nekrashevych and Lavreniuk stating that $\mathrm{St}_G(n)$ is characteristic if G is a level-transitive iterated wreath product [181, Theorem 8.2].

Rigidity of weakly branch actions and Boston's conjecture

The main motivation in **Chapter VIII** was to disprove Boston's conjecture. With this goal in mind, the second part of **Chapter VIII** is devoted to studying rigidity of weakly branch actions.

Let G be a group. We say that an embedding $\rho : G \to \mathrm{Aut}\ T$ is a *weakly branch action* of G on T, if $\rho(G)$ is weakly branch. We say that G is *T-rigid*, if it admits a weakly branch action $\rho : G \to \mathrm{Aut}\ T$ and for any other weakly branch action $\tau : G \to \mathrm{Aut}\ T$, the image $\tau(G)$ is conjugate to $\rho(G)$ in $\mathrm{Aut}\ T$.

The structure graph $\mathcal{B}_G$ encodes all the possible weakly branch actions of G on spherically homogeneous rooted trees. The next main result in **Chapter VIII** yields a complete characterization of T-rigidity in terms of the structure graph $\mathcal{B}_G$. This extends previous partial results of Grigorchuk and Wilson (for branch groups) in [127], of Garrido and Hardy (for

branch groups) in their respective doctoral theses [105, 138], and of Wilson (also for branch groups) in [233]. It also unifies the approach of Wilson to study branch actions via the structure graph and the alternative approach of Nekrashevych and Lavreniuk in [181] to study automorphisms of weakly branch groups, as T-rigidity of a weakly branch group $G \leq \mathrm{Aut}\, T$ implies that its automorphism group coincides with its normalizer in $\mathrm{Aut}\, T$. The connection between the ideas in [181] and the approach of Wilson was explicitly asked by Garrido in her doctoral thesis [105].

Let us define the collection $\mathcal{L}_G$ of subgroups of G:

$$\mathcal{L}_G = \{\mathrm{N}_G(B) \mid B \in B(G)\}.$$

We characterize T-rigidity in terms of the new notion of T-filtrations.

Definition 42 (T-filtration). A collection of subgroups $\{H_k\}_{k\geq 0}$ of G is defined to be a *T-filtration* if $H_0 = G$ and, for every $k \geq 0$:

(i) $H_k \geq H_{k+1}$;

(ii) $H_k \in \mathcal{L}_G$;

(iii) $|G : H_k| = N_k$, where N_k is the number of vertices at the kth level of T.

We write $\mathcal{F}_T$ for the collection of all subgroups of G which belong to some T-filtration in G.

Let G be a group admitting a weakly branch action $\rho : G \to \mathrm{Aut}\, T$. For each $v \in T$, we write $\mathrm{st}_\rho(v)$ for the subgroup of G corresponding, via ρ, to the stabilizer of v in $\rho(G)$. We also write

$$\mathcal{S}_\rho := \{\mathrm{st}_\rho(v)\}_{v\in T}.$$

The characterization of T-rigidity obtained in **Chapter VIII** is the following:

Theorem 43 (Chapter VIII). *Let G be a group admitting a (weakly) branch action $\tau : G \to \mathrm{Aut}\, T$. The following are all equivalent:*

(i) *the group G is T-rigid;*

(ii) *for every (weakly) branch action $\rho : G \to \mathrm{Aut}\, T$, we have $\mathcal{F}_T = \mathcal{S}_\rho$;*

(iii) *there exists a (weakly) branch action $\rho : G \to \mathrm{Aut}\, T$, such that $\mathcal{F}_T = \mathcal{S}_\rho$.*

Theorem 43 is the first result on the rigidity of a (weakly) branch group where the structure graph is not required to be isomorphic to the tree T. In fact, Theorem 43 shows that T-rigidity is equivalent to the structure graph containing a unique rooted copy of T.

We obtain some direct applications of Theorem 43. For example, we show that T-rigidity is a property of the closure:

Theorem 44 (Chapter VIII). *Let G be a group admitting a weakly branch action $\tau : G \to \mathrm{Aut}\, T$. Then G is T-rigid if and only if its completion $\overline{G}$ with respect to the congruence topology is T-rigid.*

The drawback of Theorem 43 is that it is not very explicit and easy to apply to particular examples. However, in **Chapter VIII** we further specialize to weakly branch actions of fractal groups of p-adic automorphisms and give a characterization for T_p-rigidity which is very easy to check in this context.

Grigorchuk and Wilson introduced the following sufficient conditions $(*)$ and $(**)$ for rigidity in [127]:

- $(*)$ for each vertex $v \in T$ the stabilizer $\mathrm{st}_G(v)$ acts as a (transitive) cyclic group of prime order on the immediate descendants of v;
- $(**)$ whenever u and u' are incomparable vertices and v is a descendant of (not equal to) u, there is an element $g \in G$ fixing u' and moving v.

Note that if a fractal group $G \leq \mathrm{Aut}\, T_d$ satisfies condition $(*)$, then $d = p$ is prime and $G \leq W_p$. We greatly generalize conditions $(*)$ and $(**)$ and deduce a very simple characterization of T_p-rigidity for fractal groups of p-adic automorphisms:

Theorem 45 (Chapter VIII). *Let $G \leq W_p$ be a fractal weakly branch group. Then G is T_p-rigid if and only if*

$$\frac{G}{\mathrm{St}_G(2)} \ncong C_p \times C_p.$$

Theorem 45 yields easily T_p-rigidity for the most studied weakly branch groups in the literature.

An interesting consequence of Theorem 45 is that having large Hausdorff dimension in W_p is an obstruction for fractal weakly branch subgroups of W_p to not be T_p-rigid:

Corollary 46 (Chapter VIII). *Let $G \leq W_p$ be a fractal weakly branch group such that its closure satisfies*

$$\mathrm{hdim}_{W_p}(\overline{G}) > \frac{1}{p}.$$

Then G is T_p-rigid.

To the best of our knowledge, Corollary 46 is the first result in the literature relating the Hausdorff dimension of a weakly branch group to the rigidity of its weakly branch actions.

The last part of **Chapter VIII** is devoted to disproving Boston's conjecture. Corollary 46 suggests that being zero-dimensional may be an obstruction for T_p-rigidity. Let $G \leq W_p$ be the zero-dimensional just-infinite branch pro-p group in Theorem 30. The group G is not T_p-rigid, so in order to disprove Boston's conjecture, we need to find a more subtle approach.

There is a natural tree where G acts on: a tree T obtained from the p-adic tree T_p by deletion of levels. In fact, the proof of the zero-dimensionality of G in W_p does not use the action of G on the p-adic tree, but its action on this "growing tree" T. To disprove Boston's conjecture, we only need to show T-rigidity of G. The last part of **Chapter VIII** is devoted to proving the T-rigidity of G, which finally yields the disproval of Boston's conjecture initiated in **Chapter I**:

Theorem 47. *For each prime $p \geq 2$, there is a just-infinite branch pro-p group $G \leq W_p$ which is a counterexample to Boston's conjecture.*

In order to prove T-rigidity of G, one needs the full power of the characterization in Theorem 43. The previous results on rigidity of Hardy, Garrido, Grigorchuk and Wilson, do not apply to branch actions on the growing tree T.

ERF and non-ERF subgroups of weakly branch groups

A finitely generated group G is said to be *ERF (extended residually finite)* if every subgroup of G is closed in the profinite topology.

The goal of **Chapter IX** is to construct non-ERF subgroups and non-closed ERF subgroups in weakly branch groups. This is joint work with Leemann and Nagnibeda. The first main result in **Chapter IX** generalizes the construction of de Cornulier in [57] of a non-ERF subgroup of the first Grigorchuk group:

Theorem 48 (Chapter IX). *Let T be a spherically homogeneous rooted tree such that there are only finitely many primes dividing the number of immediate descendants of each vertex. Let $G \leq \mathrm{Aut}\, T$ be a weakly branch group. Then G contains a continuum of pairwise distinct non-closed subgroups that are not ERF. In particular G is not ERF.*

In fact, the construction in Theorem 48 works in the more general context of groups with micro-supported actions.

The second main result in **Chapter IX** is a construction of ERF subgroups of weakly branch groups which are not closed in the congruence topology:

Theorem 49 (Chapter IX). *Let $G \leq \mathrm{Aut}\, T$. Suppose that there exists a prime $p \geq 2$ such that for every $v \in T$, the subgroup $\mathrm{rist}_G(v)$ has an element of order divisible by p. Then G contains a continuum of pairwise distinct ERF subgroups which are non-closed in the congruence topology.*

Note that the assumption in Theorem 48 of only finitely many primes dividing the number of immediate descendants of each vertex, together with every rigid vertex stabilizer containing torsion elements, imply the assumption of Theorem 49.

Virtual retracts in groups acting on rooted trees

A subgroup $H \leq G$ is said to be a *retract* of G if there exists a group homomorphism $f : G \to H$ which restricts to the identity in H. A subgroup $H \leq G$ is a *virtual retract* of G if there exists a finite-index subgroup $K \leq_f G$ such that H is a retract of K. The group G has the *local retraction property* (LR) if every finitely generated subgroup of G is a virtual retract of G and the (VRC) *property* if every cyclic subgroup is a virtual retract of G. The properties (LR) and (VRC) imply subgroup separability and cyclic subgroup separability (i.e. cyclic subgroups are closed in the profinite topology) respectively, since virtual retracts are known to be closed in the profinite topology.

The first goal of **Chapter X**, joint work with Jon Merladet Urigüen, is to study virtual retracts of branch and weakly branch groups. We obtain the following result:

Theorem 50 (Chapter X). *Let G be a finitely generated branch group. Then G is not* (LR).

The proof of Theorem 50 involves several steps. First, we generalize the example of Minasyan in [179, Example 3.6 and Proposition 3.7] to prove that branch groups with torsion cannot have (LR). Secondly, we characterize virtual retracts among normal subgroups of a weakly branch group. Indeed, we show that normal subgroups are virtual retracts if and only if they are of finite index. Lastly, we characterize the (VRC) property in just-infinite branch groups, where we show that (VRC) is equivalent to the group being torsion, concluding the proof of Theorem 50. The remainder of **Chapter X** is devoted to an application of our results on virtual retracts to the orbifold topology of complex polynomials.

The *Thurston orbifold* of a post-critically finite rational function $f \in \mathbb{C}(x)$ is the pair $(\mathbb{P}^1_{\mathbb{C}}, \nu_f)$, where the map $\nu_f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{N} \cup \{\infty\}$ is defined via

$$\nu_f(z) := \mathrm{lcm}\{\nu_f(w)e_f(w) : w \in f^{-n}(z) \text{ for some } n \geq 1\},$$

unless z belongs to a super-attracting periodic orbit, in which case $\nu_f(z) = \infty$. The *Euler characteristic* χ of the orbifold $(\mathbb{P}^1_{\mathbb{C}}, \nu_f)$ is defined as

$$\chi(\nu_f) := 2 - \sum_{z \in P_f} \left(1 - \frac{1}{\nu_f(z)}\right).$$

A rational function $f \in \mathbb{C}(x)$ is *euclidean* if $\chi(\nu_f) = 0$ and *hyperbolic* if $\chi(\nu_f) < 0$.

The Thurston orbifold of a post-critically finite rational function $f \in \mathbb{C}(x)$ captures to some extent the topology of the rational function f. Similarly, virtual retracts capture the topology of the group $\mathrm{IMG}(f)$ to some extent. We wonder how the topology of the orbifold is related to the virtual retracts of $\mathrm{IMG}(f)$. As hyperbolic rational maps are expected to yield iterated monodromy groups with weakly branch closures, we propose that (LR) property characterizes the complex rational functions with euclidean orbifolds in view of Theorem 50. In fact, for a rational function $f \in \mathbb{C}(x)$ with a euclidean orbifold, the group $\mathrm{IMG}(f)$ can be identified with a virtually abelian group of affine transformations in the complex plane and thus it has (LR).

In **Chapter X**, we confirm this conjecture for quadratic complex polynomials. Note that complex polynomials with a euclidean orbifold are linearly conjugate (up to a sign) to either the powering map or the Chebyshev polynomial [65]:

Theorem 51 (Chapter X). *Let $f \in \mathbb{C}[x]$ be a post-critically finite quadratic polynomial. Then, the following are equivalent:*

(i) $\mathrm{IMG}(f)$ *has* (LR)*;*

(ii) $\mathrm{IMG}(f)$ *is virtually abelian;*

(iii) f *is linearly conjugate (up to a sign) to the powering map z^2 or the Chebyshev polynomial $2x^2 - 1$.*

To prove Theorem 51, we use the (weakly) branch structure of the iterated monodromy groups of a post-critically finite hyperbolic quadratic complex polynomial obtained by Bartholdi and Nekrashevych in [22], and the results on virtual retracts of (weakly) branch groups obtained in the first half of **Chapter X**.

Combining the technical results on the finite generation of normal subgroups of iterated monodromy groups in **Chapter X**, with the results in **Chapter II** and **Chapter V**, we further obtain that for a periodic quadratic complex polynomial the closure $\overline{\mathrm{IMG}(f)}$ has complete finitely generated Hausdorff spectrum:

Corollary 52 (Chapter X). *Let $f \in \mathbb{C}[x]$ be a periodic quadratic polynomial. Then, there exists $b \geq 1$ such that*

$$\mathrm{hspec}_b(\overline{\mathrm{IMG}(f)}) = [0, 1].$$

Corollary 52 yields the first infinite family of weakly branch groups with complete bounded Hausdorff spectrum, generalizing the analogous result for the Basilica group $\mathrm{IMG}(z^2-1)$ given in **Chapter II**.

Random elements and subgroups (Chapters XI-XII)

The third part of this thesis focuses on the random elements and random subgroups of groups acting on rooted trees.

Torsion in branch pro-p groups

Chapter XI is joint work with Radi. The main contribution in **Chapter XI** is an elementary proof of random elements of branch pro-p groups being of infinite order:

Theorem 53 (Chapter XI). *Let $G \leq \mathrm{Aut}\, T_p$ be a closed branch pro-p group and μ_G its unique Haar probability measure. If $F \subseteq G$ denotes the set of torsion elements in G, then*

$$\mu_G(F) = 0.$$

The proof is elementary, in contrast with the probabilistic proof of a more general result of Abért in [2] that we mention below.

Random subgroups of branch groups

In [2], Abért showed that random subgroups of weakly branch groups are free and act almost freely on the boundary of the tree almost surely. A stronger result was obtained by Abért and Virág in [4] for iterated wreath products: a random subgroup of an iterated wreath product is a free group acting freely on the boundary of the tree almost surely. A free action is much stronger than almost free actions, and free actions are particularly relevant in applications to arithmetic dynamics. The main goal of **Chapter XII** is to establish, together with Radi, freeness of the action of random subgroups for branch groups. The main result in **Chapter XII** reads as follows:

Theorem 54 (Chapter XII). *Let $G \leq \mathrm{Aut}\, T_d$ be a super strongly fractal branch closed group. Then, the random subgroup H_k, generated by k independent Haar random elements, is free and acts freely on ∂T_d almost surely.*

In particular, Theorem 54 yields that the action of H_k on the boundary of the tree is not amenable almost surely if $k \geq 2$.

The proof of Theorem 54 is based on the ergodic theory of self-similar groups initiated in **Chapter XIII**.

Ergodic theory and arithmetic dynamics (Chapters XIII-XVII)

The last part of this thesis is devoted to developing a new ergodic theory for self-similar groups and showing applications to certain well-known problems in arithmetic dynamics.

Measure-preserving dynamical systems associated to fractal groups

The main contribution in **Chapter XIII** is to develop an ergodic theory for self-similar groups. This ergodic theory has found many different applications to randomness in self-similar groups and even to several problems in number theory.

Let (X, μ) be a probability space, A a monoid and $\mathcal{T}$ an action of A on (X, μ). We say $\mathcal{T}$ is *measure-preserving* if for any measurable subset $S \subseteq X$ and any $a \in A$ we have

$$\mu(\mathcal{T}_a^{-1}(S)) = \mu(S),$$

where $\mathcal{T}_a$ is the operator associated to the action of a. Then the tuple $(X, \mu, \mathcal{T}, A)$ is called a *measure-preserving dynamical system*.

A measurable subset $S \subseteq X$ such that $\mathcal{T}_a^{-1}(S) = S$ for every $a \in A$ is called $\mathcal{T}$*-invariant*. We say that a measure-preserving dynamical system $(X, \mu, \mathcal{T}, A)$ is *ergodic* if for every $\mathcal{T}$-invariant measurable subset $S \subseteq X$ we have $\mu(S) \in \{0, 1\}$.

Let F be a free monoid. We define a dynamical system $(X, \mu, \mathcal{T}, F)$ to be *strongly mixing* if for any pair S_1, S_2 of measurable subsets and any sequence $\{a_k\}_{k\geq 1} \subseteq F \setminus \{1\}$ the following limit exists:

$$\lim_{k\to\infty} \mu(\mathcal{T}_{a_1\cdots a_k}^{-1}(S_1) \cap S_2) = \mu(S_1) \cdot \mu(S_2).$$

Clearly, strong mixing implies ergodicity.

If $G \leq \mathrm{Aut}\ T_d$ is a closed self-similar group, it admits a unique Haar probability measure μ_G. We may identify T_d with the free monoid of rank $d \geq 2$, and define a monoid operator $\mathcal{T}$ of T_d on (G, μ_G) via sections, i.e.

$$\mathcal{T}_v(g) := g|_v \in G$$

for any $v \in T_d$ and any $g \in G$.

The first main result in **Chapter XIII** yields a sufficient condition for $\mathcal{T}$ to be measure-preserving and a sufficient condition for the resulting dynamical system to be ergodic and strongly mixing:

Theorem 55 (Chapter XIII). *Let $G \leq \mathrm{Aut}\, T$ be a self-similar profinite group:*

(i) *if G is fractal then $(G, \mu_G, \mathcal{T}, T)$ is a measure-preserving dynamical system;*

(ii) *if G is super strongly fractal then $(G, \mu_G, \mathcal{T}, T)$ is ergodic and strongly mixing.*

Cyclicity, hypercyclicity and self-similar subgroups

In **Chapter XIII**, we introduce the notion of cyclicity and hypercyclicity in self-similar groups as an analogue of cyclic and hypercyclic vectors in functional analysis.

Let $G \leq \mathrm{Aut}\, T_d$ be a self-similar group. Given a subgroup $H \leq G$, we define the *self-similar closure* of H in G, denoted by H^{SS}, as the smallest self-similar subgroup of G containing H.

We define cyclicity and hypercyclicity in self-similar closed groups:

Definition 56 (Cyclicity and hypercyclicity in self-similar closed groups). Let $G \leq \mathrm{Aut}\, T_d$ be a self-similar closed subgroup. We say $g \in G$ is *cyclic* in G if $\overline{\langle g \rangle}^{\mathrm{SS}} = G$ and *hypercyclic* if $\overline{\{g|_v\}_{v \in T_d}} = G$.

Similarly, we define the notion of cyclicity in self-similar discrete groups and study self-similar subgroups of automata groups in **Chapter XIII**.

The main result in **Chapter XIII** concerning cyclicity and hypercyclicity is the following:

Theorem 57. *Let $G \leq \mathrm{Aut}\, T_d$ be a super strongly fractal closed group. Then a Haar-random element of G is hypercyclic almost surely.*

Theorem 57 is obtained as an application of Birkhoff's ergodic theorem for free semigroup actions; see [48, 116, 182].

Markov processes

Let $(X, \mu, \mathcal{T}, A)$ be a measure-preserving dynamical system. Given a finite measurable partition α of (X, μ), we call the tuple $(X, \mu, \mathcal{T}, A, \alpha)$ an *A-process*. Two A-processes $(X, \mu, \mathcal{T}, A, \alpha)$ and $(\widetilde{X}, \nu, \widetilde{\mathcal{T}}, A, \beta)$ are *isomorphic* if there exist conull sets $X' \subseteq X$ and $\widetilde{X}' \subseteq \widetilde{X}$ and a measurable map $\phi : X' \to \widetilde{X}'$ with measurable inverse $\phi^{-1} : \widetilde{X}' \to X'$ such that:

(i) ϕ is measure-preserving;

(ii) $\phi(\mathcal{T}_a(x)) = \widetilde{\mathcal{T}}_a\phi(x)$ for all $a \in T$ and $x \in X'$;

(iii) ϕ induces a bijection between the partitions α and β.

The f-invariant was introduced by Bowen in [39] as a measure-conjugacy invariant for free group measure-preserving action. Let us recall its definition in the free semigroup case [40].

Let T_d be the free monoid on the set $\{1, \dots, d\}$ and let (X, μ) be a probability space, where T_d acts on X via a measure-preserving action $\mathcal{T}$. We write $\mathcal{P}$ for the set of all measurable finite partitions of (X, μ). For a partition $\alpha \in \mathcal{P}$ and a finite subset $Q \subseteq T_d$ we write

$$\alpha^Q := \bigvee_{q \in Q} \mathcal{T}_q^{-1}\alpha,$$

where the join of two partitions $\alpha \vee \beta$ is the partition into the sets $A \cap B$ with $A \in \alpha$ and $B \in \beta$. In the special case when $Q = B_T(\varnothing, n)$, i.e. the ball of radius n centered at the identity (the root $\varnothing$), we simply write $\alpha^n := \alpha^{B_T(\varnothing,n)}$.

For a partition $\alpha \in \mathcal{P}$, its *Shannon entropy* $H(\alpha)$ is defined as

$$H(\alpha) := -\sum_{A \in \alpha} \mu(A) \log(\mu(A)).$$

We further define the quantity $F(T_d, \alpha)$ as

$$F(T_d, \alpha) := (1 - 2d)H(\alpha) + \sum_{i=1}^{d} H(\alpha \vee \mathcal{T}_i^{-1}(\alpha))$$

and $f(T_d, \alpha)$ as

$$f(T_d, \alpha) := \inf_{n \geq 1} F(T_d, \alpha^n).$$

Finally, we define the *f-invariant* of the measure-preserving dynamical system $(X, \mu, \mathcal{T}, T_d)$ as $f(X) := f(T_d, \alpha)$ for any generating partition α (if such a partition exists).

We shall identify T_d with the d-adic tree.

Definition 58 (Past of a vertex). Given a vertex $v \in T_d$, we define its *past* $\mathrm{Past}(v)$ as the set of vertices in the unique path from v to the root $\varnothing$ (including both v and the root $\varnothing$).

If we write a vertex $v = x_1 \cdots x_n \in T_d$ with each $x_i \in X$, then

$$\mathrm{Past}(v) = \{x_1 \dots x_i \mid 1 \leq i \leq n\} \cup \{\varnothing\}.$$

Definition 59 (Markov process [40, Definition 20]). A T_d-process $(X, \mu, \mathcal{T}, T_d, \alpha)$ is a *Markov* process if for every $x \in X$, every $v \in T_d$ and every $A \in \alpha$ we have

$$\mu\Big(\mathcal{T}_{vx}^{-1}(A) \,\Big|\, \bigvee_{w \in \mathrm{Past}(v)} \mathcal{T}_w^{-1}(\alpha)\Big) = \mu(\mathcal{T}_{vx}^{-1}(A) \mid \mathcal{T}_v^{-1}(\alpha)) = \mu(\mathcal{T}_x^{-1}(A) \mid \alpha),$$

where we write $\mu(\cdot \mid \mathcal{F})$ for the conditional probability on a sub-σ algebra $\mathcal{F}$. Note that the second equality always holds as $\mathcal{T}$ is measure-preserving.

The f-invariant characterizes Markov processes:

Theorem 60 (see [40, Theorem 11.1]). *A T_d-process $(X, \mu, \mathcal{T}, T_d, \alpha)$ is Markov if and only if $f(X) = F(T_d, \alpha)$.*

In **Chapter XIV**, we study the f-invariant of the associated measure-preserving dynamical system $(G, \mu_G, \mathcal{T}, T_d)$ for a fractal branch closed group $G \leq \mathrm{Aut}\, T_d$. Remarkably, the sequence $\{r_n(G)\}_{n\geq 1}$ essentially gives the f-invariant:

Theorem 61 (Chapter XIV). *Let $G \leq \mathrm{Aut}\, T_d$ be a fractal branch closed group. Then, there exists some $D \geq 1$ such that the f-invariant of the measure-preserving dynamical system $(G, \mu_G, \mathcal{T}, T_d)$ is given by*

$$f(G) = F(T_d, \alpha_s^D) = \log|G_1| - r_D(G).$$

In particular, the T_d-process $(G, \mu_G, \mathcal{T}, T_d, \alpha_s^D)$, where α_s denotes the standard partition of (G, μ_G), is Markov.

The fact that the sequence $\{r_n(G)\}_{n\geq 1}$ plays such a big role in the f-invariant of the measure-preserving dynamical system $(G, \mu_G, \mathcal{T}, T_d)$, suggests that the Hausdorff dimension is also related to the measure-preserving dynamical system $(G, \mu_G, \mathcal{T}, T_d)$. We make this precise in **Chapter XIV**:

Theorem 62 (Chapter XIV). *Let $G, H \leq \mathrm{Aut}\, T_d$ be two fractal branch closed groups such that:*

(i) $f(G) = f(H)$;

(ii) $\mathrm{hdim}_{T_d}(G) = \mathrm{hdim}_{T_d}(H)$.

Let α_s and β_s denote the standard partitions of (G, μ_G) and (H, μ_H) respectively. Then, there exists some $D \geq 1$ such that the Markov processes $(G, \mu_G, \mathcal{T}, T_d, \alpha_s^D)$ and $(H, \mu_H, \mathcal{T}, T_d, \beta_s^D)$ are isomorphic.

Theorem 62 yields a new application of the Hausdorff dimension of self-similar closed groups. For instance, in **Chapter XIV**, we deduce that symmetric (respectively non-symmetric) defining vectors whose circulant matrix are of the same rank yield GGS-groups with isomorphic Markov processes.

Fixed-point proportion of self-similar groups

In **Chapter XV**, together with Radi, we apply the ergodic theory developed in **Chapter XIII** to compute the fixed-point proportion of super strongly fractal groups. The main result in **Chapter XV** reads as follows:

Theorem 63 (Chapter XV). *Let* $G \leq \mathrm{Aut}\, T_d$ *be a super strongly fractal group. Then*

$$\mathrm{FPP}(G) = 0.$$

?? is greatly generalized, together with Radi, in **Chapter XVI**. We need the following key definition from **Chapter XVI**.

A fractal group $G \leq \mathrm{Aut}\, T_d$ is *mixing* if

(i) for every $n \geq 1$ the level stabilizer $\mathrm{St}_G(n)$ acts level-transitively on the subtrees rooted at level n;

(ii) for every $n, m \geq 1$, there exists $N := N(m) \geq 0$ such that for any v with $|v| \geq n + N$ we have

$$H_m = G_m,$$

where $H := \varphi_v(\mathrm{St}_G(n) \cap \mathrm{st}_G(v)) \leq G$.

We say $N := N(m)$ is the *delay constant* of G.

In particular, super strongly fractal groups correspond to mixing groups with delay constant $N(m) = 0$. In **Chapter XVI**, we prove the following key result, generalizing Theorem 63:

Theorem 64 (Chapter XVI). *Let* $G \leq \mathrm{Aut}\, T_d$ *be a mixing group. Then* $\mathrm{FPP}(G) = 0$.

The proof of Theorem 64 is based on the ergodic theory developed in **Chapter XIII** and in the martingale approach of Jones. These martingales are studied more in detail in **Chapter XVII**.

The *fixed-point process* of a group $G \leq \mathrm{Aut}\, T$, introduced by Jones in [146], is the real stochastic process $\{X_n\}_{n\geq 1}$, where the random variables $X_n : (G, \mu_G) \to \mathbb{N} \cup \{0\}$ are given by

$$X_n(g) := \#\{\text{vertices at level } n \text{ fixed by } g\}.$$

If the level stabilizers act transitively on the subtrees rooted at that level, the fixed point process is a discrete non-negative martingale and thus it becomes eventually constant almost surely [44, Theorem 5.7]. In all the known examples, we have $\lim_{n\to\infty} X_n = r$ with $r \in \{0, 1, 2\}$ almost surely. Furthermore, $r = 2$ only appears with positive probability in the geometric iterated Galois groups of Chebyshev polynomials of even degree. A natural question suggested by Tucker, via private communication, is the following:

Question 65 (Tucker). *What values $r \geq 0$ can the fixed-point process of a level-transitive group $G \leq \mathrm{Aut}\, T$ converge to with positive probability?*

In **Chapter XVII**, we completely answer Question 65:

Theorem 66 (Chapter XVII). *Let $d \geq 2$ and T_d the d-adic tree. There exists a fractal branch closed subgroup $G_H \leq \mathrm{Aut}\, T_d$ such that*

(i) $\mathrm{hdim}_T(G_H) = 1/d > 0$;

(ii) $\mathrm{FPP}(G_H) \geq (d-1)!/d^d > 0$;

(iii) *there is $S \subseteq G$ with $\mu_G(S) > 0$ such that $X_n(g) = d$ for all $n \geq 1$ and $g \in S$.*

Theorem 66 shows that the family of groups with positive fixed-point proportion is far richer than anticipated and provides further counterexamples to a conjecture of Jones in [143]; see also [36]. The groups G_H in Theorem 66 generalize a construction of Radi in [201]. These groups G_H play a key role in the extension problem related to arithmetic iterated Galois groups as we shall see.

Fixed-point proportion of geometric iterated Galois groups

In the second part of **Chapter XV**, we apply Theorem 63 to iterated monodromy groups of post-critically finite polynomials.

In [174], Makarov and Smirnov defined the notion of exceptional sets for complex rational functions. Given a rational function $f \in \mathbb{C}(x)$, a set $\Sigma \subseteq \mathbb{P}^1_{\mathbb{C}}$ is said to be *exceptional* if

$$\Sigma = f^{-1}(\Sigma) \setminus C_f.$$

A rational map f is said to be *exceptional* if it admits a non-empty exceptional set Σ_f. Riemann-Hurwitz shows that for polynomials

$$\#\Sigma_f \leq 2,$$

and it is well known that $\#\Sigma_f = 2$ if and only if f is linearly conjugate (up to a sign) to a Chebyshev polynomial. Thus, exceptional polynomials with two exceptional points yield positive fixed-point proportion [144]. In contrast, Jones showed in [144] that non-exceptional post-critically finite complex polynomials yield zero fixed-point proportion. However, his approach is flawed and no other approach is known for post-critically finite polynomials; see [84] for a detailed discussion on the failure of the approach in [144].

For complex polynomials with a single exceptional point nothing is known on the fixed-point proportion. In **Chapter XV**, we construct the first examples of exceptional polynomials f of every degree $d \geq 3$ with zero fixed-point proportion:

Theorem 67 (Chapter XV). *Let $d \geq 3$. Then, there exists a polynomial $f \in \mathbb{C}[x]$ of degree d such that $\#\Sigma_f = 1$ and*

$$\mathrm{FPP}(\mathrm{IMG}(\mathrm{f})) = 0.$$

In **Chapter XVI**, together with Radi, we greatly generalize Theorem 67. In fact, we completely solve the longstanding open problem of determining the fixed-point proportion of geometric iterated Galois groups of polynomials.

Recall that given a post-critically finite rational function $f \in K(x)$, its *Thurston orbifold* is the pair $(\mathbb{P}^1_K, \nu_f)$, where the map $\nu_f : \mathbb{P}^1_K \to \mathbb{N} \cup \{\infty\}$ is given by

$$\nu_f(z) := \mathrm{lcm}\{\nu_f(w)e_f(w) : w \in f^{-n}(z) \text{ for some } n \geq 1\},$$

unless z belongs to a super-attracting periodic orbit, in which case $\nu_f(z) = \infty$. The finite tuple $(\nu_f(z))_{z \in P_f}$ is called the *type* of the orbifold.

The main result in **Chapter XVI** reads as follows:

Theorem 68 (Chapter XVI). *Let K be a field and $f \in K[x]$ a polynomial of degree $d \geq 2$, such that either $\mathrm{char}(K) = 0$ or $\mathrm{char}(K)$ does not divide the local degree of any critical point of f. Then, for t transcendental over K, either*

(i) *f has a euclidean orbifold of type $(2, 2, \infty)$ and*

$$\mathrm{FPP}(G_\infty(K^{\mathrm{sep}}, f, t)) = \begin{cases} 1/2 & \text{if } d \text{ is odd,} \\ 1/4 & \text{if } d \text{ is even;} \end{cases}$$

(ii) *or $\mathrm{FPP}(G_\infty(K^{\mathrm{sep}}, f, t)) = 0$ otherwise.*

If $K = \mathbb{C}$, the complex polynomials with euclidean orbifold of type $(2, 2, \infty)$ are precisely those linearly conjugate (up to a sign) to a Chebyshev polynomial. This confirms the well-known conjecture that Chebyshev polynomials are the only polynomials whose geometric iterated Galois groups have positive fixed-point proportion.

The proof of Theorem 68 is quite involved and it relies on techniques from group theory, ergodic theory, martingale theory and complex dynamics. One of the key ingredients of the proof is a generalization of the exceptional sets of Makarov and Smirnov defined above. Furthermore, we prove a stronger version of Theorem 68, which applies to a large class of rational functions.

Periodic points of polynomials over finite fields

We now discuss several direct applications of Theorem 68 in **Chapter XVI**.

Let K be a number field and $f \in K(x)$ a rational function of degree $d \geq 2$. Given $\mathfrak{p}$ a prime ideal of good reduction for f, denote by $f_\mathfrak{p}$ the reduction of the coefficients of f modulo $\mathfrak{p}$ and $\mathbb{F}_\mathfrak{p}$ the residue field of $\mathfrak{p}$. As $\mathbb{F}_\mathfrak{p}$ is a finite field, every point in $\mathbb{P}^1_{\mathbb{F}_\mathfrak{p}}$ is either periodic or strictly preperiodic for f. We write $\mathrm{Per}(f_\mathfrak{p}, \mathbb{P}^1_{\mathbb{F}_\mathfrak{p}}) \subseteq \mathbb{P}^1_{\mathbb{F}_\mathfrak{p}}$ for the set of periodic points. It was shown by Juul, Kurlberg, Madhu and Tucker in [150] that

$$\liminf_{N(\mathfrak{p})\to\infty} \frac{\#\mathrm{Per}(f_\mathfrak{p}, \mathbb{P}^1_{\mathbb{F}_\mathfrak{p}})}{N(\mathfrak{p})+1} \leq \mathrm{FPP}(G_\infty(\mathbb{C}, f, t)). \tag{1}$$

They asked in [150, Question 1.4] whether one can classify all the rational functions such that the lower limit in Equation (1) is non-zero. Chebyshev polynomials are known to yield a non-zero lower limit in Equation (1) if and only if they are of prime power degree; see [150, Example 7.2]. Thus, Theorem 68 answers [150, Question 1.4] for all polynomials:

Corollary 69 (Chapter XVI). *Let K be a number field and $f \in K[x]$ a polynomial of degree $d \geq 2$. Then,*

$$\liminf_{N(\mathfrak{p})\to\infty} \frac{\#\mathrm{Per}(f_\mathfrak{p}, \mathbb{P}^1_{\mathbb{F}_\mathfrak{p}})}{N(\mathfrak{p})+1} = 0$$

if and only if f is not linearly conjugate to (up to a sign) to a Chebyshev polynomial of degree d with d a prime power.

If $K = \mathbb{F}_q$ is a finite field of characteristic p, we may compute the geometric iterated Galois group over $\mathbb{F}_q^{\mathrm{sep}}$. Recently, Bridy, Jones, Kelsey and Lodge proved in [44] that if $\mathbb{F}_q$ is a finite field of characteristic p and $f \in \mathbb{F}_q(x)$ is a rational function of degree $2 \leq d < p$, then

$$\liminf_{n\to\infty} \frac{\#\mathrm{Per}(f, \mathbb{P}^1_{\mathbb{F}_{q^n}})}{q^n+1} \leq \mathrm{FPP}(G_\infty(\mathbb{F}_q^{\mathrm{sep}}, f, t)). \tag{2}$$

For a polynomial f, the authors in [44] expect the lower limit in Equation (2) to be zero if f is not linearly conjugate (up to a sign) to a Chebyshev polynomial.

A polynomial $f \in K[x]$ is a *standard twisted Chebyshev polynomial* of degree $d \geq 2$ if it satisfies

$$f\left(x + \frac{\zeta}{x}\right) = x^d + \frac{\zeta}{x^d}$$

for some ζ satisfying $\zeta^{d-1} = 1$. A polynomial f linearly conjugate to a standard twisted Chebyshev polynomial is called a *twisted Chebyshev polynomial*.

In **Chapter XVI**, we show that polynomials over K with a euclidean orbifold of type $(2, 2, \infty)$ are twisted Chebyshev polynomials. Thus, Theorem 68 also yields the following corollary:

Corollary 70 (Chapter XVI). *Let $f \in \mathbb{F}_q[x]$ be a polynomial of degree $2 \leq d < p$, where p is the characteristic of $\mathbb{F}_q$. If f is not a twisted Chebyshev polynomial, then*

$$\liminf_{n\to\infty} \frac{\#\mathrm{Per}(f, \mathbb{P}^1_{\mathbb{F}_{q^n}})}{q^n + 1} = 0.$$

Extension problem and the fixed-point proportion of arithmetic iterated Galois groups

To conclude, we discuss the extension problem and the fixed-point proportion of arithmetic iterated Galois groups. The discussed results appeared in **Chapter XVII**.

Let us recall the short exact sequence

$$1 \longrightarrow G_\infty(K^{\mathrm{sep}}, f, t) \longrightarrow G_\infty(K, f, t) \longrightarrow \mathrm{Gal}(K_\infty/K) \longrightarrow 1.$$

The extension K_∞/K is quite mysterious in general; see [145, Section 3.1] for a detailed discussion. The following, has become a standard problem in the field:

Question 71 (Extension problem). *What conditions guarantee that the extension K_∞/K is finite?*

Adams and Hyde asked in [6, page 51] whether the geometric iterated Galois group being branch is a sufficient condition for K_∞/K to be finite. In **Chapter XVII**, we answer their question in the positive:

Theorem 72 (Chapter XVII). *Let K be a field and $f \in K(x)$ a rational function. Furthermore, set $G := G_\infty(K, f, t)$ and $H := G_\infty(K^{\mathrm{sep}}, f, t)$. Then*

(i) *H is branch if and only if G is branch;*

(ii) *if either H or G is branch, then*

$$[K_\infty : K] = |G : H| = |G_D : H_D| < \infty,$$

where D is the depth of H as a group of finite type (or equivalently G is regular branch over $\mathrm{St}_G(D-1)$). In particular

$$K_\infty = K_D(f,t) \cap K^{\mathrm{sep}},$$

where $K_D(f,t)$ is the splitting field of $\{f^n\}_{n=1}^D$ over $K(t)$.

Theorem 72 applies to a large class of groups. In particular, it applies to the post-critical preperiodic quadratic polynomials considered by Pink in [200] and also, to the best of our knowledge, to all the cases where Question 71 is known to have a positive answer, compare [6, 199, 201]. Indeed, Theorem 72 applies to unicritical preperiodic polynomials by [6, Proposition 4.14].

A recurrent question since the pioneering work of Odoni in the 1980s [184, 185], has been to understand what conditions on $f \in K(x)$ and $\alpha \in K$ guarantee that

$$|\mathrm{Aut}\ T_d : G_\infty(K,f,\alpha)| < \infty.$$

Theorem 72 yields a necessary condition for this to happen:

Corollary 73 (Chapter XVII). *Let $f \in K(x)$ a rational function and let $\alpha \in K$ not strictly post-critical. Then if*

$$|\mathrm{Aut}\ T_d : G_\infty(K,f,\alpha)| < \infty,$$

we necessarily have

$$G_\infty(K^{\mathrm{sep}},f,t) = G_\infty(K,f,t) = \mathrm{Aut}\ T_d.$$

In **Chapter XVII**, we also study the fixed-point proportion of arithmetic iterated Galois groups and their specializations, even when we do not understand neither the extension problem nor the specialization problem (the latter being the embedding of the specialization into the arithmetic iterated Galois group).

We say that a rational function $f \in K(x)$ has *bad monodromy* if there is a non-empty subset of cosets

$$\varnothing \neq M \subseteq Q := \frac{G_\infty(K,f,t)_1}{G_\infty(K^{\mathrm{sep}},f,t)_1} \cong \mathrm{Gal}(K_1/K)$$

whose elements all fix a vertex at the first level of T_d. Note that $K_1 := K_1(f,t) \cap K^{\mathrm{sep}}$, where $K_1(f,t)$ is the splitting field of f over K.

In **Chapter XVII**, we prove that the arithmetic iterated Galois groups of rational functions with bad monodromy and many of their specializations have positive fixed-point proportion. This shows that positive fixed-point proportion is more abundant than previously expected:

Theorem 74 (Chapter XVII). *Let $f \in K(x)$ be a rational function with bad monodromy. Then*

$$\mathrm{FPP}(G_\infty(K,f,t)) \geq \frac{\#M}{|Q|} > 0.$$

If $\alpha \in K$ is not strictly post-critical and such that

$$G_\infty(K,f,t)_1 = G_\infty(K,f,\alpha)_1,$$

we further get

$$\mathrm{FPP}(G_\infty(K,f,\alpha)) \geq \frac{\#M}{|Q|} > 0.$$

A direct application of Theorem 74 is to unicritical polynomials of odd degree. This yields the following corollary, greatly generalizing a recent result of Radi for the polynomial z^d+1 in [201, Theorem E]:

Corollary 75 (Chapter XVII). *Let $f \in \mathbb{Q}[x]$ be a unicritical polynomial of odd degree $d \geq 3$. Then*

$$\mathrm{FPP}(G_\infty(\mathbb{Q},f,t)) \geq \prod_{p|d}\left(\frac{p-2}{p-1}\right) > 0.$$

Furthermore, if $\alpha \in \mathbb{Q}$ is not a strictly post-critical point, then

$$\mathrm{FPP}(G_\infty(\mathbb{Q},f,\alpha)) \geq \prod_{p|d}\left(\frac{p-2}{p-1}\right) > 0.$$

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