Performance Limits in Adaptive Control

Sternby, Jan

1978

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Sternby, J. (1978). *Performance Limits in Adaptive Control*. (Technical Reports TFRT-7145). Department of Automatic Control, Lund Institute of Technology (LTH).

Total number of authors:
1

CODEN: LUTFD2/(TFRT-7145)/1-009/(1978)

PERFORMANCE LIMITS IN ADAPTIVE CONTROL

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June 1978

Dokumentutgivare
Lund Institute of Technology
Handläggare Dept of Automatic Control
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Författare
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Dokumentnamn
REPORT
Utgivningsdatum
June 1978

Dokumentbeteckning
LUTFD2/(TFRT-7145)/1-009/(1978)
Ärendebeteckning
STU 76-3804

10T4

Dokumenttitel och undertitel
18T0

Performance Limits in Adaptive Control

Referat (sammandrag)
26T0
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This work was financially supported by the Swedish Board for Technical Development under contract No. 76-3804.

Referat skrivet av
Author

Förslag till ytterligare nyckelord
44T0

Klassifikationssystem och -klass(er)
50T0

Indextermer (ange källa)
52T0

Omfång
9 pages

Övriga bibliografiska uppgifter
56T2

Språk
English

Sekretessuppgifter
60T0

ISSN
60T4

ISBN
60T6

Dokumentet kan erhållas från
62T0
Department of Automatic Control
Lund Institute of Technology
P O Box 725, S-220 07 LUND 7, Sweden

Mottagarens uppgifter
62T4

Pris
66T0

DOKUMENTDATABLAD enligt SIS 62 10 12

SIS-DB 1

Blankett LU 11:25 1976-07

1. INTRODUCTION

The optimal adaptive control law is still in most cases impossible to calculate, particularly if the solution has the dual features first described by Feldbaum. Therefore, a large number of suboptimal control laws have been suggested. Their performance are usually tested by comparison with other suboptimal strategies or with the known parameter case in a simulation study. The case of known parameters is then used to give a lower limit for the achievable loss. This limit, however, is frequently much lower than the optimal loss, and is consequently of limited value.

In this correspondence,minimum variance control of difference equations systems with time-varying stochastic parameters are studied and a tighter lower bound for the optimal performance is derived. This is achieved by assuming that information about the actual parameter values is always delayed one step. The effect of this decrease in available information is to increase the loss, thus giving a better lower limit for the optimal loss. On the other hand, with the previous parameter values known, the probability law for the current parameters will be independent of the control signal. This fact simplifies considerably the optimal control problem. Apart from giving lower limits to the loss, the resulting control law and optimal system also reveal the following interesting facts.

With time-varying stochastic system parameters, there exist bounds on the uncertainties, that must not be exceeded for the optimal system to be mean square stable over an infinite time horizon. For white noise parameters this uncertainty threshold principle (UTP) was earlier discussed by Ku and Athans(1977). The present correspondence shows that in such cases with more general parameters there exists no control law that can stabilize the system in the sense of mean square.

Also, the optimal system may show an unsatisfactory behaviour, since the distribution of the output may have large tails and lack higher moments. This may show up as sudden periods of large outputs, and causes jumps in the accumulated loss. A simple continuous time problem exhibiting this behaviour was given in Åström(1965). In such cases, it is, of course, very difficult to use simulation to test suboptimal control laws.

2. PROBLEM STATEMENT

Consider a system with the dynamics described by the linear difference equation

$$y(t) = \theta(t)^T x(t) + e(t) \tag{2.1}$$

where

$$\theta(t)^T = [-y(t-1) \;\ldots\ldots\; -y(t-n) \quad u(t-1) \;\ldots\ldots\; u(t-m)]$$

$$x(t)^T = [\, a_1(t) \;\ldots\ldots\ldots\; a_n(t) \quad b_1(t) \;\ldots\ldots\; b_m(t)]$$

and $e(\cdot)$ is a sequence of independent random variables with zero mean and variance σ^2. The parameter vector $x(t)$ is a stochastic process evolving as

$$x(t+1) = f(x(t)) + v(t+1) \tag{2.2}$$

with $v(\cdot)$ a sequence of independent random variables with zero mean and covariance matrix R. Also, $e(t)$ and $v(s)$ are independent for every t and s. The function $f(\cdot)$ is completely arbitrary, but is assumed to be known.

The object of control is to minimize the criterion

$$V^N = E \sum_{s=1}^{N} [\, y(s) - y_r \,]^2 \tag{2.3}$$

where N is a given number, and y_r is a constant reference value. To find a sequence of control laws minimizing (2.3), it is also necessary to specify the information state available to admissible control laws. A natural rule in adaptive control is that $u(t)$ may use all measurements up to time t. In view of (2.1)-(2.2) this can be summarized as $u(t)$ measurable w.r.t. $I_1(t)$ with

$$I_1(t) = [\, \theta(t) \,,\, y(t) \,,\, \text{conditional distribution for } x(t) \,]$$

With $I_1(t)$ given,the conditional distribution of $x(t+1)$ which is needed for the optimal control, can in principle be calculated using only (2.2). The resulting optimization problem is however in general very difficult, being of the dual nature first discussed by Feldbaum(1960).

By increasing the information state beyond the possible, lower limits for adaptive control of (2.1) can be obtained. Introduce

$$I_2(t) = [\, I_1(t) \,,\, x(t) \,]$$

The problem is thus defined as to minimize (2.3) for the system (2.1)-(2.2) by choosing a sequence of control laws such that $u(t)$ is measurable w.r.t. $I_2(t)$ for every t. This obviously gives a lower limit to the ordinary optimal adaptive control problem, since $u(t) \in I_1(t) \Rightarrow u(t) \in I_2(t)$.

3. OPTIMAL CONTROL

By introducing the information state I_2 the estimation part of the optimal control problem is much simplified. The estimate of x(t+1), i.e. the conditional mean, is just f(x(t)), and its covariance is R. The estimate can thus not be improved by choosing the input, and the problem is not dual in Feldbaum's sense. That is why dynamic programming can be used in a straightforward way. Introduce

$$V_t^N(I_2(t)) = \min E\left[\sum_{k=t+1}^{N} [\, y(k) - y_r]^2 \,\Big|\, I_2(t)\right] \qquad (3.1)$$

The minimization is w.r.t. the present and future admissible control laws. The superscript N for V_t will be omitted in the sequel. Some additional notations are needed to describe the result from dynamic programming. Introduce

$$\bar{\theta}(t)^T = [\; -y(t-1) \ldots\ldots -y(t-n) \quad 0 \quad u(t-2) \ldots\ldots u(t-m)\;]$$

$$\bar{\ell}^T = [\quad 0 \ldots\ldots\ldots\ldots 0 \quad 1 \quad 0 \ldots\ldots\ldots\ldots 0 \quad]$$

$$\ell^T = [\quad 1 \quad 0 \ldots\ldots\ldots 0 \quad 0 \quad 0 \ldots\ldots\ldots\ldots 0 \quad]$$

and a shift matrix S so that

$$\theta(t)^T = \bar{\theta}(t)^T + \bar{\ell}^T u(t-1)$$

$$\bar{\theta}(t)^T = \theta(t-1)^T S - \ell^T y(t-1)$$

Using the last two equations to take out y for the expectation and u for the minimization dynamic programming shows that $V_t(I_2(t))$ takes the form

$$V_t(I_2(t)) = \bar{\theta}(t+1)^T H_t(x(t))\bar{\theta}(t+1) + \bar{\theta}(t+1)^T g_t(x(t)) + r_t(x(t)) \qquad (3.2)$$

Furthermore

$$H_t(x(t)) = A_t - (A_t\bar{\ell})(\bar{\ell}^T A_t)/\bar{\ell}^T A_t \bar{\ell} \qquad (3.3)$$

$$g_t(x(t)) = b_t - (A_t\bar{\ell})(\bar{\ell}^T b_t)/\bar{\ell}^T A_t \bar{\ell} \qquad (3.4)$$

$$r_t(x(t)) = E[\, r_{t+1}(x(t+1)) \mid I_2(t)] + \sigma^2 \cdot E[1 + \ell^T H_{t+1}(x(t+1))\ell \mid I_2(t)] + y_r^2 - (\bar{\ell}^T b_t)^2 / 4\bar{\ell}^T A_t \bar{\ell} \qquad (3.5)$$

where

$$A_t = E\Big[\, x(t+1)x(t+1)^T + (S - x(t+1)\ell^T)\cdot H_{t+1}(x(t+1))\cdot(S - x(t+1)\ell^T)^T \,\Big|\, I_2(t)\Big] \qquad (3.6)$$

$$b_t = E\Big[\, S g_{t+1}(x(t+1)) - x(t+1)[\ell^T g_{t+1}(x(t+1)) + 2y_r] \,\Big|\, I_2(t)\Big] \qquad (3.7)$$

and the optimal control law is

$$u(t) = -\frac{[\bar{\ell}^T A_t \bar{\theta}(t+1) + \bar{\ell}^T b_t/2]}{\bar{\ell}^T A_t \bar{\ell}} \qquad (3.8)$$

The "loss to go" is thus quadratic in the control variable despite the nonlinear parameter equation, as long as the old parameter values are measured exactly. This makes the minimization in the dynamic programming possible. Note that with information I_1 only, A_t and b_t would also depend on the control.

The parameter uncertainties are reduced to an unrealistic minimum with the preceding parameter values known. Even then a large part of the loss increase compared to the known parameter case is still present. To show this we will consider some special cases.

4. SPECIAL CASES

In order to simplify the expression (3.2) for V_t further it is necessary to have H_t, g_t and r_t independent of x, since the expectations will otherwise be too difficult. Some cases of this type are given below.

Known b-parameters

Suppose now that R is a constant diagonal matrix with zero elements for all the b-parameters variances. This means that the b-parameters are known and that the a-parameters change independently. Then

$$V_t(I_2(t)) = \bar{\theta}(t+1)^T H_t \bar{\theta}(t+1) + r_t \tag{4.1}$$

with

$$H_t = \text{diag}\left[H_t^{11} \; \ldots\ldots \; H_t^{nn} \; 0 \; \ldots\ldots \; 0\right] \tag{4.2}$$

$$r_t = r_{t+1} + \sigma^2(1 + H_{t+1}^{11}) + y_r^2 \cdot \frac{H_{t+1}^{11}}{1 + H_{t+1}^{11}} \tag{4.3}$$

The evolution of H_t is described by

$$h_t = \Phi \cdot h_{t+1} + \Gamma \tag{4.4}$$

where

$$h_t^T = [H_t^{11} \; \ldots\ldots \; H_t^{nn}] \qquad \Gamma^T = [R_{11} \; \ldots\ldots \; R_{nn}]$$

$$\Phi = \left[\begin{array}{c|c} \Gamma & I \\ \hline & 0 \end{array}\right]$$

Now consider the average loss per step as $t \to -\infty$. It will exist if and only if h_t of (4.4) converges, i.e. if and only if Φ is stable. This is the case if and only if $\text{tr}(R) < 1$. The following theorem can be shown.

Theorem 1: Consider the system (2.1)-(2.2) with the criterion (2.3) and suppose that R is diagonal with zero b-parameter variances. Then

$$\lim_{N\to\infty} \frac{1}{N} V^N = y_r^2 \, tr(R) + \sigma^2/(1 - tr(R)) \qquad (4.5)$$

if and only if $tr(R) < 1$. The limit does not exist if $tr(R) \geq 1$.

This is a result of the same type as the Uncertainty Threshold Principle of Athans, Ku and Gershwin(1976) in that it gives a limit to the stabilizability of a stochastic system (in a mean-square sense). The most interesting aspect of the above result, however, is that it gives a lower limit for the optimal steady-state loss with any control law using information state I_1.

Corrollary: Consider the system of theorem 1. If $tr(R) \geq 1$ then no regulator at all, based only on the ordinary information state I_1, can stabilize the system (in a mean-square sense).

The corrollary is a clear indication to the limits of adaptive control of time-varying stochastic systems. The fixed limit value one for the covariances may seem surprising, but has to do with the unit circle as a stability region for discrete-time systems. Note that many parameters varying just a little may be more dangerous than one parameter with larger variations. Note also the cost of a non-zero reference value in (4.5) for unknown parameters.

One b-parameter unknown

With the b-parameters unknown H_t, g_t and r_t of (3.3)-(3.5) depend on x(t) in general, and the recursion cannot be handled analytically. But a special case that can be treated is the case of only one a-parameter and one b-parameter, both white noise. The resulting system then falls within the framework of Ku and Athans(1977), where a stability limit for this case is given.

With one b-parameter but several a-parameters, H_t does not depend on x(t), but the recursion becomes nonlinear, and is difficult to solve. A partial result is the following. If all parameters are mutually independent and white with mean values $E(a_1)=m_1$, $E(b)=m_b$ and the rest zero, then the optimal system is mean square stable if and only if

$$tr(R) - R_b + \frac{m_1^2 R_b}{m_b^2 + R_b} < 1 \qquad (4.6)$$

The left member is exactly the sum of the expressions in the conditions of known b/unknown a and known a/unknown b. This result cannot be obtained from Ku and Athans(1977), since their system is essentially a set of dynamically independent first order equations with common parameter variation(B assumed nonsingular).

Discrete state Markovian b-parameter

The white noise parameter situation is of course not very realistic. It would be interesting to have some results when the b-parameter sequence is correlated, as it would hopefully be in practise. To this end, consider the simplified situation with b taking one of only two states, namely +d or -d. The probability for a change of state is p. With the same parameter assumptions otherwise as in the previous example, the stability condition is identical to (4.6). Then m_b and R_b should be taken as conditional means, i.e. $m_b^2 = (1-2p)^2d^2$ and $R_b = d^2-m_b^2$.

General situation

To handle other examples than the ones treated above it is usually necessary to run the equations (3.3)-(3.7) on a computer. This is a much less time-consuming task than to calculate the optimal dual control, mainly because the minimization is already taken care of, and the number of parameters to discretize is much smaller. In the ordinary dual control problem parameters, their covariances and old inputs and outputs must be discretized. For the lower limits of this correspondence it is just the parameters, since the covariances are constant and old inputs and outputs enter the loss in a known fashion. On the other hand there are more functions to calculate (the elements of H_t, g_t and r_t), but this is cheaper. The obtained lower limits can then be used instead of the known parameter case to evaluate the performance of suboptimal dual control laws.

5. EXTENSIONS AND IMPLICATIONS

Sofar, only the second moment of the output has been considered. The distribution of the output from the optimally controlled system is, however, also interesting to study. It may well happen, that the second moment is quite small, but the tails of the distribution is so large that the overall performance of the system becomes unacceptable. An example of this is given in figures 1 and 2, showing simulations of a system with one known b-parameter (b=1) and one unknown a-parameter. The noises are Gaussian with covariances $\sigma^2 = 0.01$ and R= 0.96, which is near the stability limit. The reference value is zero, and the optimal loss then 0.25 per step.

Fig. 1 shows the accumulated loss in 10 runs. The accumulated loss is about the same in 7 of the runs. The 3 remaining ones show large jumps in the loss due to the large tails of the distribution of the optimal input. Note that the expected mean value of the loss is greater than the mean value from these 10 runs. Fig. 2 shows the output for the run with the largest jump. The output is fairly close to zero most of the time, but assumes large values at to short periods around t= 80 and t=180.

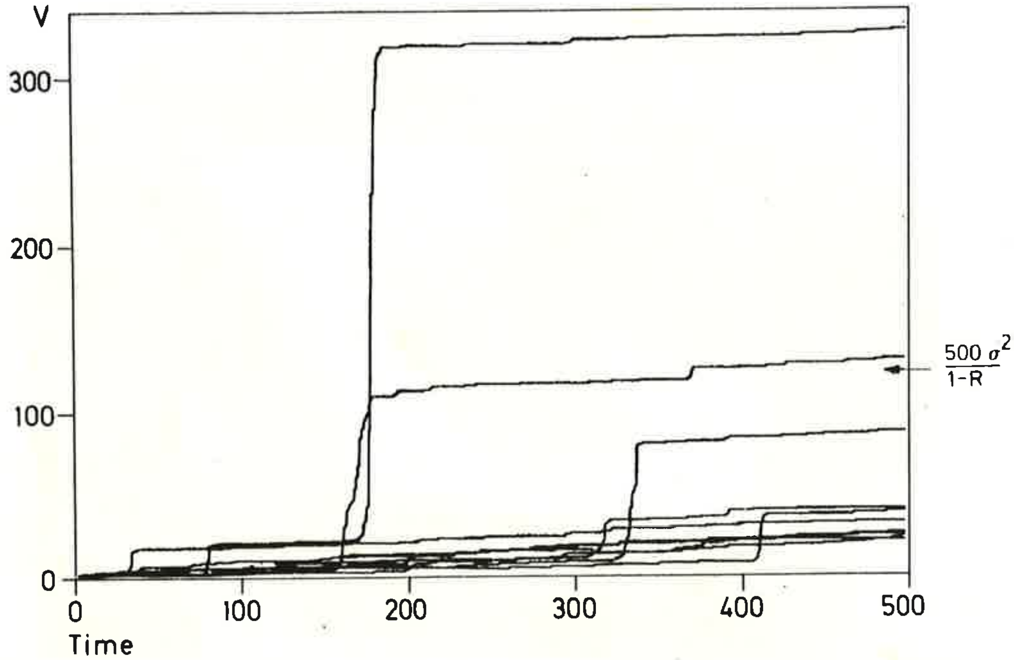

Figure 1 - The accumulated loss in 10 runs. Note the expected value for 500 steps.

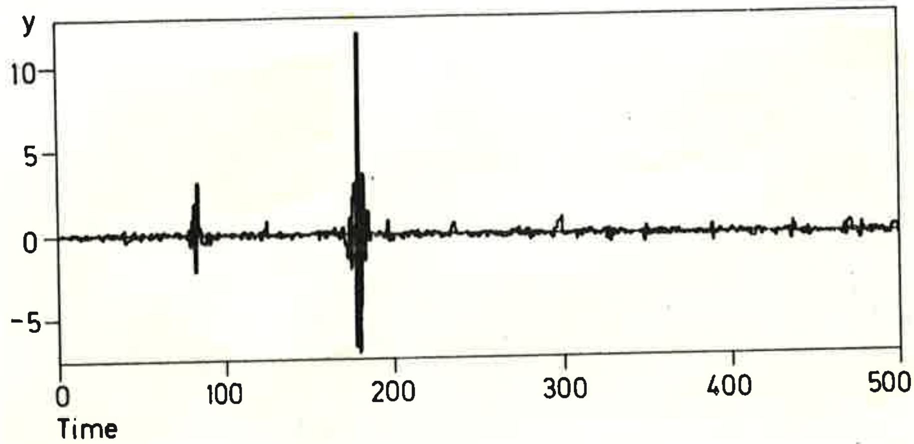

Figure 2 - The output in one of the runs of fig. 1

A system behaviour as that of fig. 1-2 is of course not desirable from a practical point of view. But it also gives another type of problem. Suboptimal dual control laws are often tested by simulation. If jumps are present, but less frequent than in the above simulation, the regulator may seem better than it is. The mean value of the loss may be small, and the estimated covariance of the loss close to zero, and yet if longer and more simulations were made a completely different result would appear. It is thus necessary to be extremely careful when evaluating the behaviour of stochastic systems by simulation. The results of a single run is not even an indication to the final result, and even 100 runs could give large errors.

For the system of fig. 1-2 further information can be gained from calculating higher moments of the output. It turns out that for such a simple system, the same control law minimizes any moment of the output. Denoting the n^{th} steady state moment of y by M_y^n the condition for existence is

$$M_y^{2n} < \infty \quad \Leftrightarrow \quad M_v^{2n} < 1 \qquad (5.1)$$

The fourth moment does thus not exist in the example above. Even if it did, the output would probably have sudden large periods, but less frequently. From (5.1) follows that with v(•) Gaussian there is always only a finite number of finite moments of the output. This means that there is a large probability for y to be large some time. Furthermore it is not possible to decrease the tails of the distribution by using e.g. $\exp\{\mu \cdot y(t)^2\}$ with $\mu>0$ in the loss function instead of $y(t)^2$.

6. CONCLUSIONS

The main contribution of this correspondence is the extension of the uncertainty threshold principle (UTP) described by Ku and Athans(1977) to more general systems with stochastic parameters. In some cases, necessary conditions can be given for a system to be mean square stabilizable. It is also shown that simulation must be used with a great care when evaluating the performance of such systems.

The method used can be applied also to other and more complicated systems, but will then, in general, require the evaluation of certain expectations on a computer to go through the recursion.

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PERFORMANCE LIMITS IN ADAPTIVE CONTROL

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Abstract

Minimum variance control of difference equations systems with time-varying stochastic parameters are studied. The optimal control law is derived under the assumption that all previous but not the current parameter values are known. This is used to give performance limits for dual control. In particular, it is shown that too much parameter uncertainty may cause the system to be unstable in the sense of mean square, whatever control law is used. Furthermore, it is shown that the optimal system may show a behaviour that makes it difficult to test suboptimal control laws using simulation studies.

This work was financially supported by the Swedish Board for Technical Development under contract No. 76-3804.

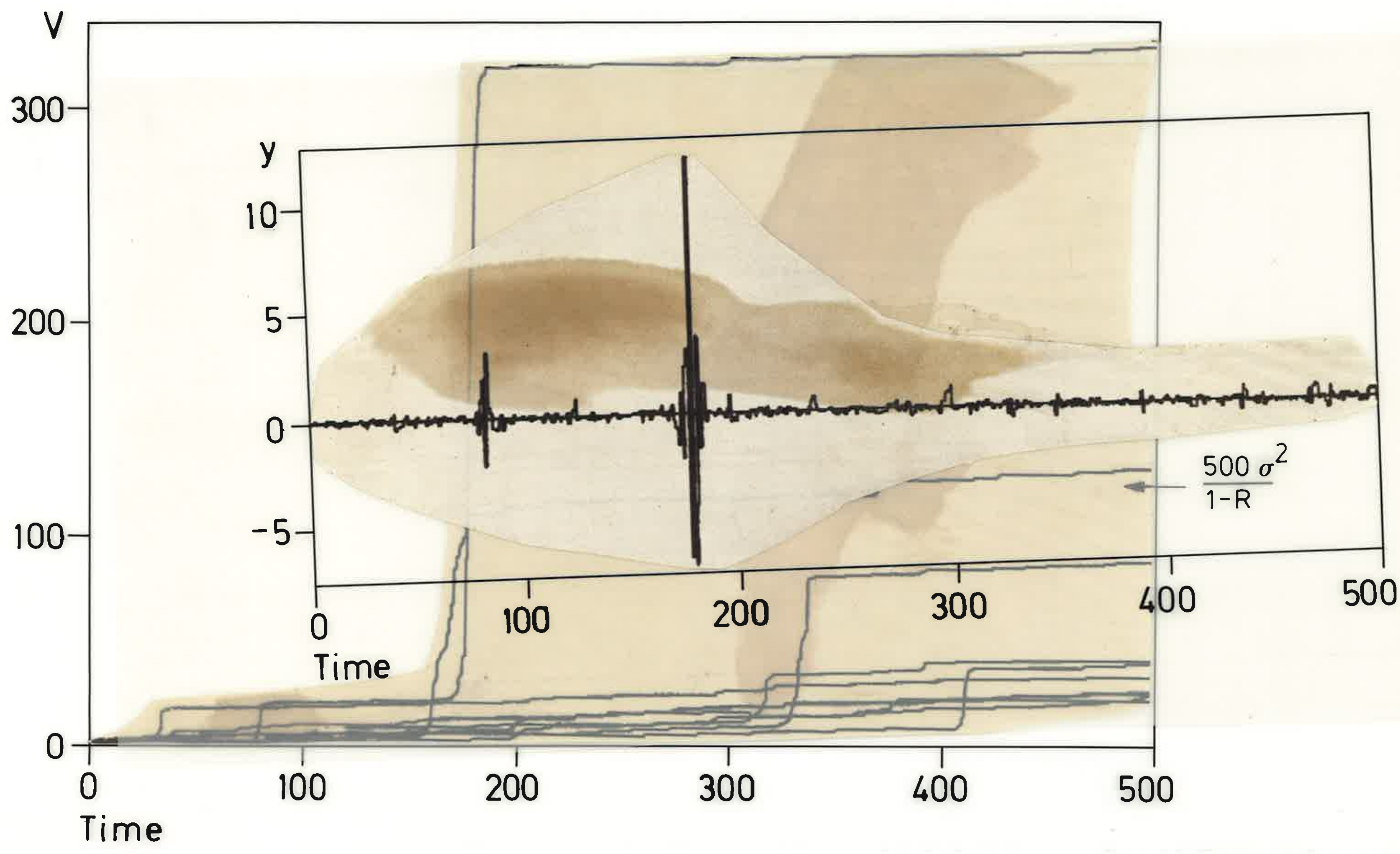
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Fig. 2

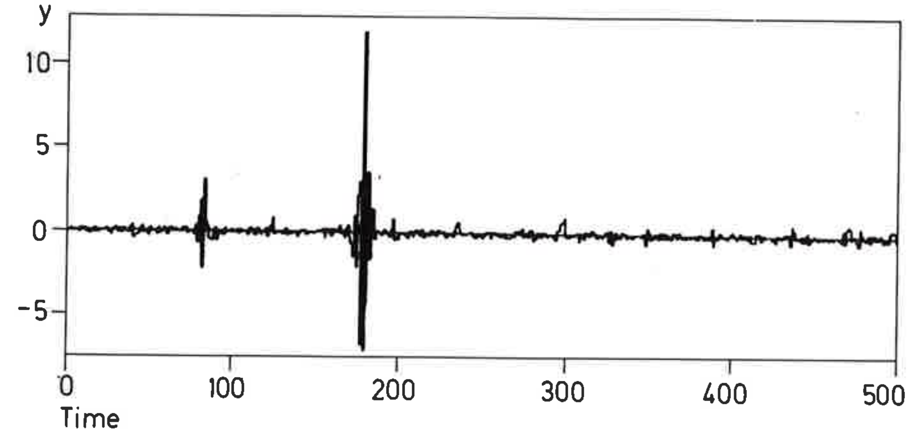
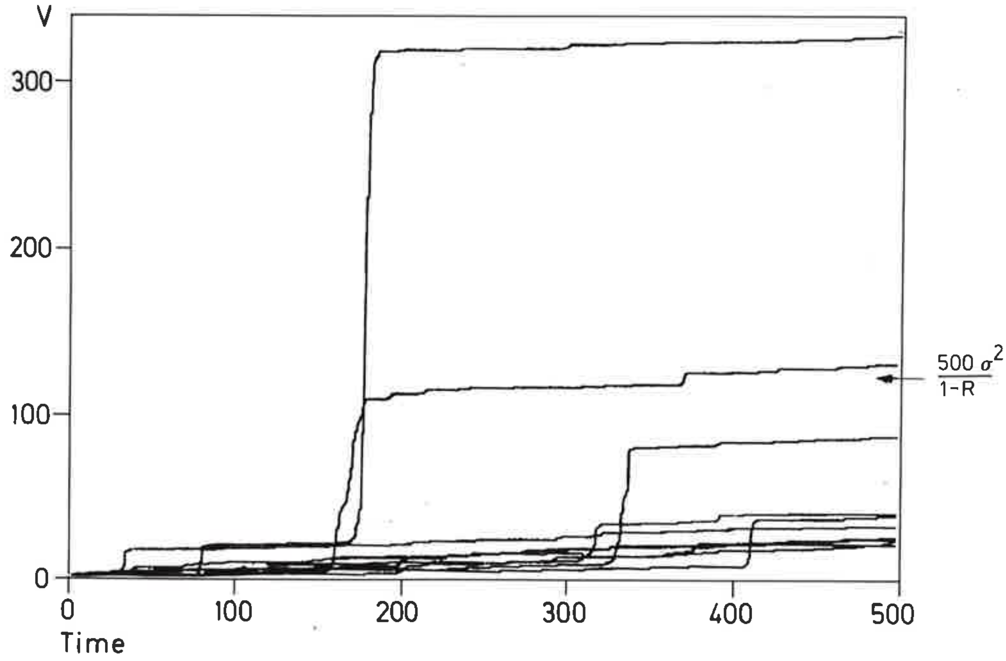


Fig. 1



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