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Random Walks and Forest Fire Models: Recurrence, Transience, and Phase Transitions

Random Walks and Forest Fire Models: Recurrence, Transience, and Phase Transitions

by Satyaki Bhattacharya



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Thesis for the degree of Doctor of Philosophy
Thesis advisors: Professor Stanislav Volkov, Professor Tatyana Turova
Faculty opponent: Professor Gideon Amir

To be presented, with the permission of the Faculty of Engineering of Lund University, for public criticism at MH:Hörmandersalen at the Centre for Mathematical Sciences on wednesday, the 9th of September, at 13:00.

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Abstract Random Walk is one of the holy grails of Probability Theory. This literature is born from a simple, nagging question: what happens to the drunkard's walk if his steps are no longer equal? The classical theory of random walks, a cornerstone of probability, offers elegant answers for the simple, symmetric case. But life and many stochastic models are rarely so uniform. We step into this gap by studying the <i>Rademacher walk</i>, defined by the sum $S_n = \sum_{i=1}^n a_i X_i$, where the step sizes (a_i) are a fixed, deterministic sequence and the X_i are independent Rademacher random variables. Our central concern is the ancient dichotomy of recurrence and transience: does the path return to a neighbourhood of the origin infinitely often, or does it wander off forever? We discover that the answer is a delicate balance between the growth of the step sequence and the geometry of the path. In one dimension, we find a lower bound of the threshold: if the steps grow like $n^{\alpha+o(1)}$ for $\alpha > 1/2$, the walk is transient, and this bound is exact. But we also construct sequences that grow arbitrarily fast yet still produce a weakly recurrent walk—a finding that challenges intuition. Moreover, we produce arbitrarily slowly growing sequences which result in transience. We also extend the work on 2— dimensions and prove similar results. A parallel study of a forest fire model with delays shows how relaxing standard assumptions can produce entirely new phenomena, like an “infinite fire.” A parameter of fire spreading time plays a serious role in phase transition. In addition we find how quickly fires spread.		
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Date 2026-04-21

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A doctoral thesis at a university in Sweden takes either the form of a single, cohesive research study (monograph) or a summary of research papers (compilation thesis), which the licentiate student has written alone or together with one or several other author(s).

In the latter case the thesis consists of two parts. An introductory text puts the research work into context and summarizes the main points of the papers. Then, the research publications themselves are reproduced, together with a description of the individual contributions of the authors. The research papers may either have been already published or are manuscripts at various stages (in press, submitted, or in draft).

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Popular Summary

Imagine a drunkard taking steps randomly. Sometimes he walks a step forward. Sometimes he walks a step backward. In probability theory, it is possible to study his behaviour, when his step sizes are all equal. A simple symmetric random walk in one dimension is defined this way.

Now imagine the drunk man taking steps forward, backward, towards right or towards left. A simple symmetric random walk on two dimensions can be defined through this, and the behaviour of such a walk is known.

My thesis investigates the long-term behavior of systems where randomness interacts with a fixed, changing rhythm. Most of my work focuses on a random walker whose stride length changes according to a predetermined schedule. I identify the precise mathematical thresholds where the walker shifts from being "recurrent" (doomed to return home infinitely often) to "transient" (wandering off forever). Surprisingly, I show that even if the walker's strides grow to be giant leaps, a few well-placed small stumbles can still pull them back to the origin. Extending this theme of spreading phenomena, I also analyze a Forest Fire model where flames spread with built-in delays. Here, I characterize when the fire burns out locally versus when it erupts into an "infinite fire" that engulfs the entire landscape—a parallel form of transience for a growing, random process.

These models capture a fundamental tension between randomness and rigid structure that appears across nature: from physical processes with time-varying volatility to the delayed spread of wildfires. My work provides a unified mathematical toolkit for predicting whether such systems eventually "return to base" or escape to infinity.

Publications

1. Recurrence and Transience of Rademacher Series.

ALEA, Lat. Am. J. Probab. Math. Stat. 20, 33–51 (2023)
<https://doi.org/10.30757/ALEA.v20-03>

2. Transience of Continuous-Time Conservative Random Walks.

Journal of Applied Probability, Volume 62, Issue 1, March 2025, pp. 153 - 171 DOI:
<https://doi.org/10.1017/jpr.2024.46>

3. Forest Fire Model on $\mathbb{Z}_+$ with Delays.

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4. Two Dimensional Rademacher Walk.

Journal of Theoretical Probability (2026) 39:40 <https://doi.org/10.1007/s10959-026-01497-2>

Submitted Works

1. Recurrence, Transience and Anti-Concentration of Rademacher Random Walks.

<https://arxiv.org/abs/2510.24568>

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Chapter 1

Introduction

The classic problem of probability theory is to study long term behaviours of processes. Whether a process returns to a particular state infinitely often is a fundamental question. In case of Markov processes the infinitely often return is called recurrence and the opposite is called transience. Another important long term phenomenon is about the existence of infinite clusters or connected entities. Common knowledge tells us that simple symmetric random walks on 1 and 2 dimensions are recurrent and on 3 dimensions and above are transient. In our work we investigate how different other types of Random Walks behave in terms of recurrence or transience. We work with symmetric walks but the step sizes change, thus rendering the process to be time inhomogeneous. We also work on a model where step sizes are random functions of time (random flight model). In addition we vary burning times of trees and spreading time of fires to investigate the threshold criterion of an infinite fire.

We find that the dimension of a random walk, the step sizes a_n to be the determining factor, as expected. The problem with fast growing steps is that they tend to overshoot. Take the example of $a_n = 3^n$. This walk cannot return to 0 ever since the $\sum_{i=1}^n a_i < a_{n+1}$. But surprisingly the lower or upper bound on rate of growth of the step sizes is not sufficient. The results indicate that even if the step sizes a_n are bigger than any arbitrarily fast growing function $f(n)$, if sufficiently many consecutive steps are of similar size, it can lead to recurrence.

In case of the forest fire model, the spreading time plays crucial role, which was expected. But the burning time of a tree does not play a role which is as significant.

On the first paper we introduce the idea of an a -walk. We investigate some cases with a_n growing as fast as $\ln n$ or n^α . But mostly we remain in the domain of integer step sizes. This paper acts as the foundation for most of the rest.

The work done on the first paper is extended significantly on the fourth manuscript. We break the restriction of a_n being integers and go on proving some rather counter-intuitive results, where we show that simply the growth rate of a_n is not a sufficient criterion to conclude recurrence or transience.

The third work has been an extension of both of these on 2 dimensions.

We investigate a related but somewhat different case in the second work. We work on 2 dimensions but the step sizes are not deterministic functions of time. They are rather created by a particle moving at constant speed and changing directions only at times which are drawn from an inhomogeneous Poisson process.

The fifth paper is the only work which covers the forest fire model, where we extend some previously existing work by varying burning and spreading times. This is definitely not a real world forest fire, since, first of all, it is on one dimension. But it is always the study of special cases which eventually give birth to more general results.

But one can see how the the analysis done on Paper I, II and III depend and draw motivation very much from Paper I.

Throughout our work, we very commonly use the Borel-Cantelli lemma to conclude almost sure convergence and use the Hoeffding inequality to show a significant similarity between numbers of positive and negative steps. We have explained these in details in the following chapters.

Chapter 2

Some Probabilistic Tools

2.1 Recurrence and Transience in Markov Processes

A state s in a Markov Process is called recurrent, if the probability of returning to it infinitely often is 1. It is called transient, otherwise.

For time homogeneous Markov processes, there are specific results which help determine these properties.

Define p_{ii}^n as the probability that the process started from state i returns to i at the n^{th} step.

It can be shown that the state is recurrent if $\sum_{n=0}^{\infty} p_{ii}^n = \infty$, and transient, otherwise. But showing recurrence becomes trickier.

However, not such a guarantee exists for time inhomogeneous Markov Process.

But, whenever we work with a time inhomogeneous Markov Process, we can at least apply the first Borel-Cantelli lemma to conclude, $\sum_{n=0}^{\infty} p_{ii}^n < \infty$ guarantees transience.

2.2 σ -Algebra and Filtration

σ -Algebra

A σ -algebra (or σ -algebra) on a set Ω is a collection $\mathcal{F}$ of subsets of Ω satisfying:

- $\Omega \in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$ (closed under complements)
- If $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ (closed under countable unions)

The pair $(\Omega, \mathcal{F})$ is called a **measurable space**.

A σ -algebra generated by a random variable X , means the events which are described by observing the possible values of X .

For a sequence of random variables $\{X_1, X_2, \dots, X_n, \dots\}$ defined on $(\Omega, \mathcal{F})$, the σ -algebra generated by (X_n) is:

$$\sigma(X_n : n \in \mathbb{N}) = \sigma(\{X_n^{-1}(B) : B \in \mathcal{B}(\mathbb{R}), n \in \mathbb{N}\})$$

where $\mathcal{B}(\mathbb{R})$ is the Borel σ -algebra on $\mathbb{R}$. This is the smallest σ -algebra on Ω making all X_n measurable. For a finite sequence $X_1, \dots, X_n$, we denote $\sigma(X_1, \dots, X_n)$.

Filtration

A filtration is an increasing family of σ -algebras indexed by time (discrete or continuous):

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}$$

where each $\mathcal{F}_n$ represents information available up to time n . The quadruple $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \geq 0}, \mathbb{P})$ is called a **filtered probability space**.

If a sequence of random variables is denoted by $\{X_1, X_2, \dots, X_n, \dots\}$, the σ -algebra generated by the intersections can be arranged as a filtration in the following way:

$$\sigma(X_1) \subset \sigma(X_1, X_2) \subset \dots \subset \sigma(X_1, X_2, \dots, X_n) \subset \dots$$

2.3 Stirling's Approximation

$$\ln n! = n \ln n - n + O(\ln n)$$

This implies

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

We will use it frequently to approximate the value of $\binom{n}{m}$, for different values of n and m . The most frequently used approximation would be of $\binom{2n}{n} 2^{-2n}$.

$$\binom{2n}{n} \times 2^{-2n} \sim \left(\frac{\sqrt{4\pi n} \left(\frac{2n}{e}\right)^{2n}}{2\pi n \left(\frac{n}{e}\right)^{2n}} \right) \times 2^{-2n} = \frac{1}{\sqrt{\pi n}}.$$

2.4 Almost Sure Convergence

A sequence of random variables X_n converges to a random variable X almost surely (or almost everywhere), if

$$\mathbb{P}(\lim_{n \rightarrow \infty} X_n = X) = 1.$$

If we call the sample space Ω , and outcomes ω , the random variables are functions on this space. So, almost sure convergence says that for almost all values of $\omega \in \Omega$, $X_n(\omega)$ tends to $X(\omega)$. By the phrase "for almost all $\omega \in \Omega$ ", it is meant that the probability of having an ω , such that the convergence does not occur on it is 0.

2.4.1 Kolmogorov's 0 – 1 Law

This states that the probability of a *tail event* is either 0 or 1, when the events in concern are independent. Now what is a tail event? Pick a sequence of random variables $X_1, X_2, \dots, X_n, \dots$. Define the sequence of σ -algebras in the following way :

$$\begin{aligned} \mathcal{F}_1 &= \sigma(X_1, X_2, X_3, \dots) \\ \mathcal{F}_2 &= \sigma(X_2, X_3, \dots) \\ &\dots \\ \mathcal{F}_n &= \sigma(X_n, X_{n+1}, \dots) \\ &\dots \end{aligned}$$

The **tail σ -algebra** τ is defined as $\lim_{n \rightarrow \infty} \mathcal{F}_n$. A tail event is an event which belongs to τ .

So, it means an event which does not depend on $\{X_1, X_2, \dots, X_k\}$, for any finite $k \in \mathbb{N}$, but depends on the intersection $\bigcap_{n=1}^{\infty} X_n$.

An example is whether the sum of a sample mean of random variables converges to a particular value is not determined by any finite number of events, but dependent on the infinite intersection.

To formally state the law, If $E \subset \lim_{n \rightarrow \infty} \sigma(X_n, X_{n+1}, \dots)$, where the X_i 's are independent of one another,

$$\mathbb{P}(E) = 0 \text{ or } 1.$$

2.4.2 Borel Cantelli Lemma

The first part of this lemma is the most powerful tool to conclude an almost sure convergence.

2.4.2.1 Lemma 1

For a sequence of events $\{A_1, A_2, \dots, A_n, \dots\}$,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \implies \mathbb{P}(A_n \text{ occurs infinitely often}) = 0.$$

2.4.2.2 Lemma 2

For a sequence of **independent** events $\{A_1, A_2, \dots, A_n, \dots\}$,

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \implies \mathbb{P}(A_n \text{ occurs infinitely often}) = 1.$$

2.4.3 Conditional Borel-Cantelli Lemma

Let $\mathcal{F}_n$ be a filtration and $A_n \in \mathcal{F}_n$. Define $p_n = \mathbb{P}(A_n | \mathcal{F}_{n-1})$. The lemma states:

$$1. \sum_{i=1}^{\infty} p_n < \infty \implies \mathbb{P}(A_n \text{ occurs infinitely often}) = 0$$

and

$$2. \sum_{i=1}^{\infty} p_n = \infty \implies \mathbb{P}(A_n \text{ occurs infinitely often}) = 1$$

2.4.4 Kochen-Stone Lemma

For the sequence of events $\{A_1, A_2, \dots, A_n, \dots\}$, assume

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$$

and

$$\limsup_{k \rightarrow \infty} \frac{\left(\sum_{n=1}^k \mathbb{P}(A_n)\right)^2}{\sum_{1 \leq m, n \leq k} \mathbb{P}(A_m \cap A_n)} > 0.$$

This would imply $\mathbb{P}(A_n \text{ occurs infinitely often}) > 0$.

These two lemmas are necessary when we need to work with dependent events.

2.5 Concentration Inequalities

Concentration inequalities establish an upper bound on the probability of a variable being too far from the expected value.

2.5.1 Markov Inequality

For a non negative random variable X , and a number $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

This can be extended using non decreasing functions f .

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(f(X))}{f(a)}.$$

2.5.2 Chebyshev's Inequality

This comes as a corollary of Markov inequality. And this is the first actual anti concentration inequality, in the sense that it talks about the distance of the variable from the expected value. To prove it, the variable that is used is $(X - \mathbb{E}(X))^2$ and then it is a direct application of Markov inequality.

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{\text{Var}(X)}{a^2}.$$

2.5.3 Cantelli's Inequality

For $a > 0$,

$$\mathbb{P}(X - \mathbb{E}(X) \geq a) \leq \frac{\text{Var}(X)}{\text{Var}(X) + a^2}.$$

2.5.4 Chernoff Bounds

By Markov inequality, for $t > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(e^{tX})}{e^{ta}}$$

and for $t < 0$,

$$\mathbb{P}(X \leq a) \leq \frac{\mathbb{E}(e^{tX})}{e^{ta}}.$$

This, despite being a direct application of Markov inequality, generally provides a sharper bound.

2.5.5 Hoeffding's Inequality

This is the most used inequality throughout the thesis.

For independent random variables $X_1, X_2, \dots, X_n$, such that $a_i \leq X_i \leq b_i, \forall i \in \mathbb{N}$, almost surely. Define $S_n = \sum_{i=0}^n X_i$. Then,

$$\mathbb{P}(|S_n - \mathbb{E}(S_n)| \geq t) \leq 2 \exp \left\{ - \frac{2t^2}{\sum_{i=0}^n (b_i - a_i)^2} \right\}.$$

There are many more concentration inequalities, but not all of them are mentioned here.

2.6 Anti-Concentration Inequalities

As the name suggests, these are the inequalities which talk about the minimum spread of random variables. Formally speaking, these inequalities are used to determine an upper bound of the probability of a random variable being inside a certain interval or in higher dimensions, hypercube. A quantity which is very often used is called the

concentration function. In one dimension, it is denoted by $Q_s(X)$, where X is the random variable in discussion, s is the length of the interval and

$$Q_s(X) = \sup_{a \in \mathbb{R}} \mathbb{P}(X \in [a, a + s]).$$

Obviously, the definition of Q can be adjusted accordingly in higher dimensions.

Following are a few anti-concentration inequalities. One of the very important ones we skip here is the Halász inequality, which we discuss in one of the papers.

2.6.1 Erdős-Littlewood-Offord Inequality

Let $X_1, X_2, \dots, X_n$ be random variables, such that $\forall i \in \{1, 2, \dots, n\}$

$$X_i = \begin{cases} a_i, & \text{w.p. } 1/2 \\ -a_i, & \text{w.p. } 1/2. \end{cases}$$

Assume that $a_i > c, \forall i \in \{1, 2, \dots, n\}$, for some positive constant c .

Then,

$$Q_{2c}(S_n) \leq \binom{n}{\lfloor n/2 \rfloor} 2^{-n} \sim \sqrt{\frac{2}{\pi n}}$$

, as $n \rightarrow \infty$. One must observe that if all $a_i \equiv 1$, this inequality implies that the simple symmetric random walk in one dimension takes any particular value after the n^{th} step with a probability which is bounded above by a $\sqrt{\frac{2}{\pi n}}$. In fact we know the probability that it will take the value 0 is approximately equal to $\sqrt{\frac{1}{\pi n}}$ and that is the most likely outcome.

This inequality is a very powerful tool and we would discuss how Erdős derived a clever proof of it in our fourth paper.

2.6.2 Halász Inequality

Consider n vectors $a_1, \dots, a_n \in \mathbb{R}^d$ such that for any unit vector e , we have $|\langle e, a_i \rangle| \geq 1$ for at least δn values of i , and also $\|a_i - a_j\| \geq 1$ whenever $i \neq j$. Then

$$\mathbb{P}\left(\sum_{i=1}^n \epsilon_i a_i \in B(x, 1)\right) \leq c(\delta, d) n^{-1-d/2},$$

where $c(\delta, d)$ is a constant that does not depend on n .

This inequality is used for separated step sizes. For example, the application of this on one dimension proves that a symmetric random walk with step size $a_n = n$ satisfies

$$Q_1(S_n) \leq n^{-3/2}.$$

2.6.3 Kolmogorov-Rogozin Inequality

There exists a positive constant C , such that, if $X_1, X_2, \dots, X_n$ are independent random variables, and real numbers $0 < \lambda_1, \lambda_2, \dots, \lambda_n < 2r$,

$$Q_r\left(\sum_{i=1}^n X_i\right) \leq C.r. \left(\sum_{i=1}^n \lambda_i^2 (1 - Q_{\lambda_i}(X_i))\right)^{-1/2}.$$

2.6.3.1 Kesten's Improvement of the Kolmogorov-Rogozin Inequality

Under the same setting, Harry Kesten proved the existence of a $C > 0$, such that,

$$Q_r\left(\sum_{i=1}^n X_i\right) \leq \frac{C.r. \sum_{i=1}^n \lambda_i^2 (1 - Q_{\lambda_i}(X_i)) \cdot Q_{\lambda_i}(X_i)}{\left(\sum_{i=1}^n \lambda_i^2 (1 - Q_{\lambda_i}(X_i))\right)^{3/2}}.$$

Chapter 3

Random Walk

A drunk man will find his way home, but a drunk bird may get lost forever.
~ Shizuo Kakutani.

Random walk or Drunkard's walk is a stochastic process that describes a path. A general definition is the following.

$$S_n = \sum_{i=1}^n X_i$$

where each X_i comes from a certain distribution and are independent. X_i is called the i^{th} step of the random walk S_n .

3.1 Simple Symmetric Random Walk

The word simple refers to a fixed absolute value of the step sizes. The most standard example is where $|X_i| = 1, \forall i \in \mathbb{N}$. Here,

$$X_i = \begin{cases} 1, & w.p. p \\ -1, & w.p. 1 - p \end{cases}$$

This can be adapted to higher dimensions. For dimension d , the i^{th} step is

$$X_i = \begin{cases} (1, 0, 0, 0, \dots, 0), & w.p. p_1 \\ (-1, 0, 0, 0, \dots, 0), & w.p. p_2 \\ (0, 1, 0, 0, \dots, 0), & w.p. p_3 \\ (0, -1, 0, 0, \dots, 0), & w.p. p_3 \\ \dots & \\ \dots & \\ (0, 0, 0, 0, \dots, -1), & w.p. p_{2d} \end{cases}$$

where $\sum_{i=1}^{2d} p_i = 1$.

The word symmetric refers to the probability of taking a positive and a negative step being equal and same for every direction.

3.2 Asymmetric Simple Random Walks

We would keep our analysis limited to one dimensional walks, because the results we derive from them, suffice.

Take the example where

$$X_i = \begin{cases} 1, & w.p., p \\ -1, & w.p., 1 - p \end{cases}$$

, where $p \neq (1/2)$. Note, this is a time homogeneous Markov process. Hence, to determine recurrence and transience we can apply the known technique. Also, since the process is irreducible, recurrence or transience of one state means recurrence or transience of all states, thus rendering the whole process recurrent or transient, accordingly.

$$\sum_{n=0}^{\infty} p_{00}^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} p^n (1-p)^n \approx \sum_{n=0}^{\infty} \frac{4^n}{\sqrt{\pi n}} p^n (1-p)^n = \sum_{n=0}^{\infty} \frac{c^n}{\sqrt{\pi n}} < \infty$$

Because, by A.M.-G.M. inequality, the value of $p(1-p)$ peaks at $p = (1/2)$, and for any $p \neq 1/2$, $p(1-p) < 1/4$. So, the walk is transient.

It is evident that any simple random walk in higher dimensions with asymmetric

distribution must be transient, because the embedded one dimensional walk itself is transient. Hence, we don't analyze them anymore.

3.3 Recurrence and Transience of Simple Symmetric Random walks

We will keep using **Stirling's Approximation** for estimating factorials.

3.3.0.1 Dimension 1

Since it is a time homogeneous Markov Process, we will have to check if the $\sum_{n=0}^{\infty} p_{00}^n$ is convergent or divergent.

$$\sum_{n=0}^{\infty} p_{00}^n = \sum_{n=0}^{\infty} \binom{2n}{n} (1/2)^n (1/2)^n \approx \sum_{n=0}^{\infty} \frac{4^n}{\sqrt{\pi n}} (1/2)^n (1/2)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

Hence, simple symmetric random walk in one dimension is recurrent.

3.3.1 Dimension 2

It is possible to use a condition on which steps are horizontal and which steps are vertical. Say, a total of $2m$ steps are horizontal and $2(n - m)$ steps are vertical. And we fix which steps are vertical and which are horizontal. Then the horizontal and vertical events can be treated independently and probabilities can be multiplied.

Let us call this conditional probability $p_c(2n)$, where c denotes a certain distribution of vertical and horizontal steps.

Under this condition, $p_c(2n) = \binom{2m}{m} \binom{2n-2m}{n-m} (1/2)^{2n} \approx 1/(\pi \sqrt{m(n-m)})$.

$\sqrt{m(n-m)}$ is maximized at $m = \lfloor n/2 \rfloor$. Thus p_c^{2n} is minimized when $m = \lfloor n/2 \rfloor$.

Thus, without any condition on the number and positions of vertical and horizontal states, $p_{00}^{2n} \geq 1/\pi \sqrt{n^2/4} = 2/\pi n$.

This quantity is not summable over n , thus the walk is recurrent.

3.3.2 Dimension 3 and Higher

For a three dimensional walk, a proof can be given by noting that roughly one-third of the steps will be along each axis. Then we find the conditional probability to be of order $1/n^{3/2}$. Because on each axis, the probability of return after around $2n/3$ steps is of order $1/\sqrt{n}$ and they can be multiplied. The quantity is summable over n , and hence the random walk is transient.

The claim that there would be roughly $1/3^{rd}$ of the steps along each axis is derived from a concentration inequality which we would mention later.

For higher dimension, a similar argument can be made to show transience. Or, it is sufficient to conclude that the three dimensional embedded walk in a higher dimensional walk does not return to origin infinitely often. Hence, the original walk does not return to the origin infinitely often, either.

But let us do a rigorous proof for the three dimensional case.

$$\begin{aligned} p_{000}^{2n} &= 1/6^{2n} \sum_{k,l,m:k+l+m=n} \frac{2n!}{(2k)!(2l)!(2m)!} \times \binom{2k}{k} \times \binom{2l}{l} \times \binom{2m}{m} \\ &= 1/6^{2n} \sum_{k,l,m} \frac{2n!}{k!^2 l!^2 m!^2} \sim 1/n^{3/2} \end{aligned}$$

Now this is summable over n . So, the walk is transient.

A **Rademacher** random variable is a variable which takes each of the values 1 and -1 with probability $((1/2))$. In our discussions, we will call the symmetric random walk with step sizes $a_1, a_2, \dots, a_n, ..$ a **Rademacher Walk**. We will call the sum $S_n = \sum_{i=1}^{\infty} X_i$ the **Rademacher Sum**, where X_i is a scaled Rademacher random variable with scaling factor $a_i, \forall i \in \mathbb{N}$.

3.4 Green's Function and Estimation of Probabilities

If n is a non-negative integer the Green's function $G_n(x, y)$ is defined as the expected number of visits to y starting from x in n steps.

$$G_n(x, y) = \mathbb{E}^X \left(\sum_{j=0}^n 1_{S_j=y} \right) = \sum_{j=0}^n p_j(x, y) = \sum_{j=0}^n p_j(y - x)$$

$G_n(x)$ is defined as the expected number of visits to x , starting from the origin, in n steps.

There are plenty of results based on Green's function, but we will only mention the ones necessary in our work. The only time we use this is during the analysis of a 2 dimensional random walk, hence we will mention only those results. $a(x)$ is defined as follows :

$$a(x) = \lim_{n \rightarrow \infty} |G_n(0) - G_n(x)| = \lim_{n \rightarrow \infty} \sum_{j=0}^n p_j(0) - p_j(x)$$

A theorem from [12] shows that for $d \leq 2$, the limit is finite.

The most important theorem from [12] that we will need is the following

Theorem. For $d = 2$, $\exists$ a constant k , such that, if $\alpha < 2$,

$$\lim_{|x| \rightarrow \infty} |x|^\alpha \left[a(x) - \frac{2}{\pi} \ln |x| - k \right] = 0$$

This can be improved to show

$$a(x) = \frac{2}{\pi} \ln |x| + k + O(|x|^{-2})$$

It is also easy to verify that $G_n(0) \sim \ln n$, because

$$G_n(0, 0) = \mathbb{E}^X \left(\sum_{j=0}^n 1_{S_j=0} \right) = \sum_{j=0}^n p_j(0, 0) = \sum_{j=0}^n p_j(0) \approx \sum_{j=0}^n \frac{2}{\pi \lfloor n/2 \rfloor} \sim \frac{1}{\pi} \ln n.$$

Then, the above theorems imply that the expected number of visits to a particular point at distance x from the origin is equal to $\frac{1}{\pi} \ln n - \frac{2}{\pi} \ln x + O(1)$.

3.5 Recurrence and Transience of Random Walks with Different Step sizes

3.5.1 Case 1

Consider the 1- d symmetric random walk with step sizes $\{1, 1, 4, 4, 4, 4, \dots\}$. Now we are in the zone of time inhomogeneous random walks, hence the rule of checking recurrence or transience from time homogeneous Markov Processes do not work.

Define $E := \{\text{The walk returns to 0 infinitely often almost surely}\}$.

The question is whether E is a tail event. If the walk ends up at 0 after the first two steps, the next part is just a 4 times scaled version of simple symmetric random walk, and the probability of return to 0 infinitely often is 1. But if it ends up on 2 or -2 after the first two steps, then the walk can never return to 0 again.

So, the event E depends on the first two steps. It is not a tail event.

This example clearly demonstrates that the event of return to a particular point infinitely often may not be a tail event. But there is a modification that will help us resolve the issue.

3.5.2 Case 2

Take the previous setup, but this time define $F := \{\text{The walk returns to some interval } [-C, C] \text{ infinitely often}\}$. Note, by definition the size of the interval is not determined by any finite number of steps, thus rendering the event as a tail event. Previous calculations show that the walk will return to $[-2, 2]$ infinitely often, almost surely. We will call this type of infinitely often return **weak recurrence**. And whichever walk is not weakly recurrent, will be called transient.

The way to show transience of a random walk is to use the first Borel-Cantelli lemma, because

$$\sum_{i=0}^{\infty} \mathbb{P}(S_i \in [-C, C]) < \infty \implies \mathbb{P}(\text{the walk returns to } [-C, C] \text{ infinitely often}) = 0.$$

But proving weak recurrence is much trickier. We cannot simply apply the second Borel-Cantelli lemma, since the variables $\{S_1, S_2, \dots, S_n, \dots\}$ are not independent.

But we can use the conditional version of the lemma, which does not require independence. Our works will show that calculating or bounding the conditional probabilities are not very simple. In some cases we will use the Kochen-Stone lemma to prove weak recurrence.

3.6 Conservative Random Walk

This is a model where the step sizes are not fixed, but movement along a direction continues for long time.

3.6.1 Time Homogeneous Model

- **Discrete Space Model**

After the n^{th} step the walk can redirect with probability p . In that case, for a d -dimensional walk, it chooses one of the $2d$ directions uniformly at random as its next step. With probability $(1 - p)$ it continues moving towards the same direction. Previous results show that this walk is recurrent on 1 and 2 dimensions and transient for higher dimensions.

- **Continuous Space Model**

At each step, the walk redirects with probability p , but this time, it moves along any direction, uniformly. For example, in case of a two dimensional walk, it chooses a direction uniformly at random from $[0, 2\pi]$. Similar results hold for this.

Note, both of these can be easily adapted to a Poisson process, where redirection times come from a Poisson process with intensity $\lambda > 0$, with similar results being preserved.

3.7 Time Inhomogeneous Model

- **Discrete Space Model**

After the n^{th} step the walk can redirect with probability p_n . In that case, for a d -dimensional walk, it chooses one of the $2d$ directions along the axes uniformly at random as its next step. With probability $(1 - p_n)$ it continues moving towards the same direction.

Previous results show that for a two dimensional walk $p_n < n^{-(1/2+\epsilon)}$ for all $n > N$, for some $N \in \mathbb{N}$, the walk is **strongly transient***.

- **Continuous Space Model**

At each step, the walk redirects with probability p_n , but this time, it moves along any direction from the set all possible directions, uniformly. For example, in case of a two dimensional walk, it chooses a direction uniformly at random from $[0, 2\pi]$.

These models too, can be adapted to a Poisson process. Here, the Poisson rate evolves with time.

***Strong transience** is defined by the scenario when the walk not only does not stop at the origin infinitely often, but also does not even pass through the origin infinitely often, almost surely.

Weak transience is the case which guarantees that the walk stops/redirects at the origin only finitely often almost surely. But it still can pass through the origin infinitely often.

Weak transience is the transience of the embedded walk.

Chapter 4

Forest Fire Model

In our case, we are not in fact dealing with any real forest fire, and our discussion is restricted to a very ideal and 1–dimensional case. The general model is described as follows.

On the $\mathbb{Z}^+ \cup \{0\}$ lattice, imagine there exists a fire at position 0. Any position is either vacant or occupied. At time 0, all positions are vacant. A single tree appears on a vacant node, following some exponential time, whose rate might depend on the position. When a tree catches fire, the fire spreads to the next position, only if there is a tree, and it might take some time to spread, which, too, might depend on the position.

We will study the speed of the progress of the fire, whether or not there is an infinite fire, and related questions.

4.1 Previously Known Results

4.1.1 Spread Time = 0, Rate = 1

This is the type of fire model which Stanislav Volkov discussed in [19] .

Note, in this case, after the m^{th} fire burns up to tree n , the whole process renews. Hence to find the distribution of interval between consecutive burn times, it is sufficient to find the distribution of the first burn time of a tree.

The result is the following.

Define τ_n to be the first burn time of the n^{th} tree. Then $\frac{\tau_n}{\ln n}$ converges weakly to a random variable whose distribution is $1 - \rho(x)$, where ρ is the Dickman function [4], as $n \rightarrow \infty$.

A brief idea to show the scaling factor to be of order $\ln n$:

Note that, for the tree n to catch fire, there must be a time when all the places $\{1, 2, \dots, n\}$ are occupied.

$$\mathbb{P}(\tau_n < (1 - \epsilon) \ln n) \leq \left(1 - e^{-(1-\epsilon) \ln n}\right)^n < e^{-n^\epsilon}.$$

Now, this is summable over n . By Borel Cantelli lemma, we conclude that almost surely for all sufficiently large n , the first burn time of tree n is larger than $(1 - \epsilon) \ln n$.

Similarly, we can check the probability of the first burn time of the m_k^{th} tree being larger than $(1 + \epsilon) \ln m_k$. Note, when tree m_k catches fire for the first time, position $m_k + 1$ must be vacant.

$$\mathbb{P}(\tau_{m_k} > (1 + \epsilon) \ln m_k) \leq \exp(-(1 + \epsilon) \ln m_k) = \frac{1}{m_k^{1+\epsilon}}$$

This quantity is summable over k . Hence by Borel Cantelli lemma, with probability 1, for all large enough k , m_k catches fire for the first time before time $(1 + \epsilon) \ln m_k$.

What does this show?

It does not give an exact upper bound on τ_n . It gives an almost sure lower bound on τ_n , $\forall n \in \mathbb{N}$. But it definitely shows the consecutive maxima m_k are burnt for the first time at times arbitrarily close to $\ln m_k$.

4.1.2 Bounded Rates Dependent on Location and Continuous Space

The authors in the paper [8] allow the rates at which vacant site x becomes occupied to depend on x , though they all must lie in a certain interval, and also consider a long-range mode (the terminology is consistent with use in the percolation theory), where the fires can spread further than just to the immediate neighbour of the “burning” site.

They work on a model where $\mathbb{Z}_+$ is substituted by $\mathbb{R}_+$, and obtain a similar result.

The main result concerns the expectation of the time of the first burn of the n^{th} tree, or the tree at distance x .

- In this model, the rate at position x is defined as λ_x and $\lambda_x \in [c_1, c_2]$, where c_1 and c_2 are two positive constants. Also, the fire from position x can spread to trees at the next r positions, for some $r \in \mathbb{N}$. This model is called a **non-homogeneous long range process**. .

Define τ_x as the first time when position x catches fire. Then, for any $\kappa > (\ln 2)^{-1}$ and large enough x ,

$$\mathbb{E}(\tau_x) \leq (\ln x)^\kappa.$$

- This is a continuous space model, where points appear on $\mathbb{R}_+$, according to a Poisson process. Two points x and y are called connected is $|x - y| \leq 1$. The point 0 is always occupied. Here, clusters form with connected occupied zones, and when 0 falls into the cluster, it burns instantly. The result is similar to the one in the previous case. For any $\kappa > (\ln 2)^{-1}$, and large enough x ,

$$\mathbb{E}(\tau_x) \leq (\ln x)^\kappa.$$

There are many more articles on the topic of forest fires, and we will mention them as reference. In our work, we keep our focus on models like the above ones and we will use the first result in our work.

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Chapter 5

Outline of the Papers

5.1 Paper I

For 1- d symmetric random walks, we call the n^{th} step size a_n .

- For $a_n = \lfloor \ln n \rfloor$, we prove recurrence.
- For $a_n = \ln n$, we prove that the walk will return to $(-\epsilon, \epsilon)$, for any $\epsilon > 0$, infinitely often, almost surely.
- For $a_n = \lfloor n^\alpha \rfloor$, for any $\alpha > 0$, the walk is transient.

To prove the recurrence, we improve the conditional Borel-Cantelli lemma to include the cases, where some conditions are satisfied not always, but with probability 1.

5.2 Paper II

This work is divided in two parts, a discrete and a continuous direction model. We work with $1 - d$ and $2 - d$ conservative random walk with redirection times coming from a time inhomogeneous Poisson process. We prove the following:

- For both models, the $2 - d$ walk is strongly transient if $\lambda_t < t^{-\alpha}$, for $1 > \alpha > 0$ or if $\lambda_t < (\ln t)^{-\beta}$, for $\beta > 2$.

- The embedded walk in $1-d$ is transient for $\lambda_t < t^{-\alpha}$, where $1 > \alpha > 1/3$.

5.3 Paper III

This paper is based on random walk with varying step sizes on a $2-d$ lattice. the following are the results we found.

- If the step sizes are non-decreasing integers, converging to infinity, the walk is transient.
- We found a condition on the growth rate of the step sizes a_n , for which, even non-integer step sizes result in transience.
- A way to create a recurrent sequence from a given sequence has been shown, under some conditions. An explicit example is provided.

5.4 Paper IV

This is an extension of the analysis done in the first paper. But it covers a much bigger domain. The main results are the following :

- For the step sizes a_n , if $a_n > n^{\alpha+o(1)}$, for any $\alpha > 1/2$, the walk is transient. A general model is worked with, where for all large enough n , $cn^{\alpha+\delta} < a_n < Cn^{\alpha+\delta}$, for $\alpha, \delta, c, C > 0$. An anti concentration bound on $Q_1(S_n)$ is provided with.
- If $f : \mathbb{N} \rightarrow \mathbb{R}$ is any sequence of functions, there exist step sizes $a_n > f(n)$ for all $n \in \mathbb{N}$, and $a_n \in \mathbb{N}$ such that the associated Rademacher Walk is recurrent.
- If $f : \mathbb{N} \rightarrow \mathbb{R}$ is any sequence of functions, there exist strictly increasing step sizes $a_n > f(n)$ for all $n \in \mathbb{N}$, and $a_n \in \mathbb{R}$ such that the associated Rademacher Walk is topologically recurrent. Topological recurrence refers to the property where the walk visits any interval infinitely often, almost surely. In other words, the walk is dense in $\mathbb{R}$.

- For integer sized and non-decreasing step sizes, a sufficient condition for transience is given.
- It is shown that for any $f : \mathbb{N} \rightarrow \mathbb{R}$, there exists $a_n \leq f(n)$, such that the associated Rademacher walk is irreducible and transient.
This, combined with the two previous results showed that growth rate of step sizes alone does not guarantee recurrence or transience.
- If $\sum_{i=0}^n a_n^2 = \infty$, the walk changes sign infinitely often, almost surely. If in addition, $a_n \rightarrow 0$, the walk is topologically recurrent.
- If $a_n \rightarrow \infty$ and $|a_n - a_{n-1}| \rightarrow 0$, the walk is either topologically recurrent, or transient.

5.5 Paper V

This paper deals with different forest fire models on $1 - d$ lattice. The parameters we vary are the spreading time and the burning time. We keep denoting spreading time from tree i to tree $i + 1$ by Δ_i and burning times by θ . We do not work with any case where burning times are non-identical. Some of the main results we show are the following.

We define the notion of **consecutive maxima**. Assume, the index of the tree farthest from 0, which is burnt by the first fire is m_1 . We define m_k recursively as the index of the tree farthest from the origin which is burnt by the first fire which goes beyond m_{k-1} .

- If Δ_i 's are identical and $\mathbb{P}(\Delta > 0) > 0$, there exists an infinite fire, almost surely.
- If $\theta \equiv 0$ and $0 < \mathbb{E}(\Delta) < \infty$ and $Var(\Delta) > 0$, there is only one infinite fire, almost surely.
- For $\theta \equiv 0$, we establish a condition on rate of decay of Δ_n 's, such that below a threshold, there is no infinite fire, and above it, there is an infinite fire, almost surely.

- For $\Delta \equiv 0$, almost surely there is no infinite fire.
- For $\Delta \equiv 0$, we find an almost sure lower bound for the first burning time of the n^{th} tree. We also prove an upper bound of first burn times of consecutive maxima.
- For $\Delta \equiv 0$ and constant $\theta > 0$, we prove that the consecutive maxima grow super-exponentially fast. Next, based on a conjecture, we put an upper bound on the growth rate.

Author Contributions

Paper 1

The author brought the idea to work with blocks. He provided the preliminary understanding regarding how jumps of the walks should occur. And he showed for very fast growing block sizes, what is done on the preceding block, can be undone on the next block. He also helped in coming up with the generalized version of the Conditional Borel Cantelli Lemma. He helped to come up with the proof which extended the recurrence criterion from $a_n = \lfloor \ln n \rfloor$ to $a_n = \ln n$. After the proof of the a particular transience result, he helped in finding a way to employ that in slower growth rate, to prove transience.

Paper 2

The author handled the discrete direction case and came up with the particular conditions to prove the transience. He also managed to extend the result from $t^{-\alpha}$ to $(\ln t)^{-\beta}$ in both cases.

Paper 3

The author proved the result for transience of non-integer sequences where $a_n \geq (\ln n)^{1/2+\epsilon}$. He also found the explicit example of the sequence a_n with $\limsup a_n = \infty$ which results in recurrence.

Paper 4

The author came up with the idea that step sizes $a_{n+1} - a_n \rightarrow 0$, with $\sum_{n=1}^{\infty} a_n^2 = \infty$ should imply density or transience. He was also involved in finding the argument that if $a_n \sim n^{1/2+\epsilon}$, it should imply transience. He could establish preliminary links between how to use the Erdős-Littlewood-Offord anti-concentration inequality combined with a super doubling sequence argument. In addition, he found out that step sizes $a_n = \lfloor (\ln n)^{1+\epsilon} \rfloor$ results in transience, which was later generalized. He came up with example of arbitrarily slowly growing sequence being transient.

Paper 5

The author proved the existence of infinite fire, when the spreading time has positive probability of being positive. He came up with a preliminary proof of uniqueness of infinite fire in case of 0 spreading time. He came up with the results for non identically distributed spreading times resulting in existence or non-existence of infinite fire. He found the bound on the first burn time of the n^{th} tree in case of instant spread and positive burning times. He came up with the conjecture and the related result on maximum burnt point. He showed the super-exponential growth rate of the maximum burnt points for instant spread, non-zero burning time model.

Paper I



Recurrence and transience of Rademacher series

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March 18, 2026

Abstract

We introduce the notion of $\mathbf{a}$ -walk $S(n) = a_1X_1 + \dots + a_nX_n$, based on a sequence of positive numbers $\mathbf{a} = (a_1, a_2, \dots)$ and a Rademacher sequence $X_1, X_2, \dots$. We study recurrence/transience (properly defined) of such walks for various sequences of $\mathbf{a}$. In particular, we establish the classification in the cases where $a_k = \lfloor k^\beta \rfloor$, $\beta > 0$, as well as in the case $a_k = \lceil \log_\gamma k \rceil$ or $a_k = \log_\gamma k$ for $\gamma > 1$.

Keywords: recurrence, transience, Rademacher distribution, non-homogeneous Markov chains.

AMS subject classification: 60G50, 60J10.

1 Introduction

We will say that a random variable X has a Rademacher distribution and write $X \sim \text{Rademacher}$, if $\mathbb{P}(X = +1) = \mathbb{P}(X = -1) = \frac{1}{2}$.

Let $X_i \sim \text{Rademacher}$, $i = 1, 2, \dots$, be i.i.d., and $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$ be the sigma-algebra generated by the first n members of this sequence. Let $\mathbf{a} = (a_1, a_2, \dots)$ be a non-random sequence of positive numbers. Define the $\mathbf{a}$ -walk as

$$S(n) = a_1X_1 + a_2X_2 + \dots + a_nX_n = \sum_{k=1}^n a_kX_k$$

with the convention $S(0) = 0$.

Definition 1. Let $C \geq 0$. We call the $\mathbf{a}$ -walk S defined above C -recurrent, if the event $\{|S(n)| \leq C\}$ occurs for infinitely many n s. (In case when $C = 0$, this is equivalent to the usual recurrence, i.e., $S(n) = 0$ for infinitely many n s, so we will call the walk just recurrent.)

We call the $\mathbf{a}$ -walk transient, if it is not C -recurrent for any $C \geq 0$.

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Our aim is to determine the probability that the $\mathbf{a}$ -walk for given $\mathbf{a}$ and C is recurrent; in principle, this probability may be different¹ from 0 and 1. A simplest example of an $\mathbf{a}$ -walk is when all $a_i \equiv a \in \mathbb{R}_+$. Such a random walk is obviously a.s. recurrent since it is equivalent to the one-dimensional simple random walk.

The question of recurrence is naturally related to the Littlewood-Offord problem which asks to maximize concentration probabilities such as $\mathbb{P}(S(n) = 0)$ subject to various hypotheses on $\mathbf{a}$. In particular, in [7] the authors develop an inverse Littlewood-Offord theory, using which they show that a concentration probability is large, only when the elements of $\mathbf{a}$ are contained in a generalized arithmetic progression; see also [4].

The study of $\mathbf{a}$ -walk is also somewhat relevant to the conjecture by Bogusław Tomaszewski (1986), which says that $\mathbb{P}\left(|S(n)| \leq \sqrt{a_1^2 + \dots + a_n^2}\right) \geq \frac{1}{2}$ for all sequences $\mathbf{a}$ and all n . The conjecture was recently proved by Nathan Keller and Ohad Klein in [2].

Let us first start with some general statements. First, we show that the choice of $C > 0$ is sometimes unimportant for the definition of C -recurrence.

Theorem 1. *Suppose that $a_n \rightarrow \infty$ and at the same time $|a_n - a_{n-1}| \rightarrow 0$ as $n \rightarrow \infty$. Then if $\mathbf{a}$ -walk is C -recurrent with a positive probability for some $C > 0$ then it is $\tilde{C}$ -recurrent with a positive probability for all $\tilde{C} > 0$.*

Proof. Since the notion of C -recurrence is monotone in C , i.e. if $\mathbf{a}$ -walk is C_1 -recurrent for $C_1 > 0$ then it is C_2 -recurrent for all $C_2 \geq C_1$, it suffices to prove that C -recurrence implies $\frac{2C}{3}$ -recurrence.

Indeed, suppose the $\mathbf{a}$ -walk is C -recurrent; formally, if we define the events

$$\begin{aligned} E &= \{S(n) \in [-C, C] \text{ for infinitely many } n\}, \\ \tilde{E} &= \{S(n) \in [-2C/3, 2C/3] \text{ for infinitely many } n\} \end{aligned}$$

then $\mathbb{P}(E) > 0$. We want to show that $\mathbb{P}(\tilde{E}) > 0$ as well.

Let n_1 be so large that $|a_i - a_{i-1}| < C/6$ for all $i \geq n_1$. Define the sequence n_k , $k \geq 2$, by setting

$$n_k = \min\{i \geq n_{k-1} + 1 : a_i \geq a_{n_{k-1}} + C/6\}$$

(which is well-defined since $a_i \rightarrow \infty$), then trivially

$$\frac{C}{6} \leq a_{n_{k+1}} - a_{n_k} \leq \frac{C}{3} \quad \text{for each } k = 1, 2, \dots \quad (1)$$

¹For example, if $\mathbf{a} = (1, 1, 3, 3, 3, 3, \dots)$ then the $\mathbf{a}$ -walk is recurrent with probability $1/2$.

Fix a positive integer K and for $y = (y_1, y_2, \dots, y_K) \in \Omega_K := \{-1, +1\}^K$ define

$$\bar{X}_K = \{X_1, X_2, \dots, X_K\};$$

$$s_y = a_1 y_1 + a_2 y_2 + \dots a_K y_K.$$

Let $y \in \Omega_K$ be such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$. Observe that

$$\{\bar{X}_K = y\} \cap E = \{\bar{X}_K = y\} \cap B_K(s_y)$$

where

$$B_K^+(u) = \{\exists : m_1 < m_2 < \dots \text{ such that } u + \sum_{i=K+1}^{m_j} a_i X_i \in [0, C]\};$$

$$B_K^-(u) = \{\exists : m'_1 < m'_2 < \dots \text{ such that } u + \sum_{i=K+1}^{m'_j} a_i X_i \in [-C, 0]\};$$

$$B_K(u) = B_K(u)^+ \cup B_K(u)^-.$$

Since $\{\bar{X}_K = y\}$ and $B_K(u)$ are independent, we have

$$\mathbb{P}(\{\bar{X}_K = y\} \cap B_K(s_y)) = \mathbb{P}(\bar{X}_K = y) \mathbb{P}(B_K(s_y)).$$

Consequently, $\mathbb{P}(B_K(s_y)) > 0$, and as a result, $\mathbb{P}(B_K^+(s_y)) > 0$ or $\mathbb{P}(B_K^-(s_y)) > 0$ (or both).

Let $\Omega_K^* \subseteq \Omega_K$ contain those y s for which there is an index k such that $n_{k+2} \leq K$ and $y_{n_k} = -1$, $y_{n_{k+1}} = +1$, $y_{n_{k+2}} = -1$; let k be the smallest such index. For $y \in \Omega_K^*$ define the mappings $\sigma^+, \sigma^- : \Omega_K^* \rightarrow \Omega_K$ by

$$\sigma^+(y) = \begin{cases} -y_i, & \text{if } i = n_k \text{ or } i = n_{k+1}; \\ y_i, & \text{otherwise;} \end{cases}$$

$$\sigma^-(y) = \begin{cases} -y_i, & \text{if } i = n_{k+1} \text{ or } i = n_{k+2}; \\ y_i, & \text{otherwise.} \end{cases}$$

Then for $y \in \Omega_K^*$

$$s_{\sigma^+(y)} = s_y + 2a_{n_k} - 2a_{n_{k+1}} \in [s_y - 2C/3, s_y - C/3],$$

$$s_{\sigma^-(y)} = s_y - 2a_{n_k} + 2a_{n_{k+1}} \in [s_y + C/3, s_y + 2C/3]$$

As a result, it is not hard to see that

$$\{\bar{X}_K = \sigma^+(y)\} \cap B_K^+(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } m \right\} = \tilde{E},$$

$$\{\bar{X}_K = \sigma^-(y)\} \cap B_K^-(s_y) \subseteq \left\{ \sum_{i=1}^m a_i X_i \in [-2C/3, 2C/3] \text{ for infinitely many } m \right\} = \tilde{E}.$$

Since at least one of $B_K^+(s_y)$ and $B_K^-(s_y)$ has a positive probability, $\mathbb{P}(\bar{X}_K = \sigma^\pm(y)) = 2^{-K}$ and the events on the LHS are independent, we conclude that $\mathbb{P}(\tilde{E}) > 0$.

Now it only remains to show that there exists $y \in \Omega_K^*$ such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$. Let $\kappa := \kappa(K) = \max\{k \in \mathbb{Z}_+ : n_k \leq K\}$; obviously, $\kappa(K) \rightarrow \infty$ as $K \rightarrow \infty$. If we chose y from Ω_K uniformly, we can trivially bound the probability that $y \notin \Omega_K^*$ by²

$$\left(1 - \frac{1}{8}\right)^{\lfloor \kappa/3 \rfloor} \rightarrow 0 \quad \text{as } \kappa \rightarrow \infty$$

by grouping together triples $(X_{n_1}, X_{n_2}, X_{n_3})$, $(X_{n_4}, X_{n_5}, X_{n_6})$, etc.; in each such a triple

$$\mathbb{P}((X_{n_k}, X_{n_{k+1}}, X_{n_{k+2}}) = (-1, +1, -1)) = 1/8.$$

Hence

$$\mathbb{P}(E) = \sum_{y \in \Omega_K} \mathbb{P}(\{\bar{X}_K = y\} \cap E) = \sum_{y \in \Omega_K^*} \mathbb{P}(\{\bar{X}_K = y\} \cap E) + \mathbb{P}(\{\bar{X}_K \in \Omega_K \setminus \Omega_K^*\} \cap E). \quad (2)$$

Since $\mathbb{P}(\{\bar{X}_K \in \Omega_K \setminus \Omega_K^*\} \cap E) \leq \mathbb{P}(\bar{X}_K \in \Omega_K \setminus \Omega_K^*)$, by making K sufficiently large, we can ensure that the second term on the RHS of (2) is less than $\mathbb{P}(E)$, implying that there exist some $y \in \Omega_K^*$ such that $\mathbb{P}(\{\bar{X}_K = y\} \cap E) > 0$, as required. $\square$

Next result shows that if the sequence $\mathbf{a}$ is non-decreasing, then the walk will always “jump” over 0 infinitely many times, even if the walk is not C -recurrent.

Theorem 2. *Suppose that a_i is a non-decreasing positive sequence. Then the event $\{S(n) > 0\}$ holds for infinitely many n s a.s. The same is true for the event $\{S(n) < 0\}$.*

The theorem immediately follows from symmetry and a more general

Proposition 1. *Suppose that a_i is a non-decreasing sequence, an integer m is such that $a_{m+1} > 0$, and $S(m) = A > 0$. Define*

$$\tau = \inf\{k \geq 0 : S(m+k) \leq 0\}.$$

²exact: see A005251, $\mathbb{P}(\bar{X}_K \notin \Omega_K^*) \approx \lambda^\kappa$, $\lambda = \frac{\sqrt[3]{100+12\sqrt{69}}}{6} + \frac{2}{3\sqrt[3]{100+12\sqrt{69}}} + \frac{2}{3} = 0.877\dots$, $\kappa = \kappa(K)$

Let $Y_j \sim \text{Rademacher}$ be i.i.d., and

$$\tilde{\tau} = \inf\{k \geq 0 : Y_1 + Y_2 + \cdots + Y_k \leq -r\}$$

where $r = \lceil A/a_{m+1} \rceil$; note that $\tilde{\tau} < \infty$ a.s. and that, in fact, $Y_1 + \cdots + Y_{\tilde{\tau}} = -r$. Then τ is stochastically smaller than $\tilde{\tau}$, that is,

$$\mathbb{P}(\tau > m) \leq \mathbb{P}(\tilde{\tau} > m), \quad m = 0, 1, 2, \dots$$

Proof. We will use coupling. Indeed, we can write

$$S(m+j) = A + a_{m+1}Y_1 + a_{m+2}Y_2 + \cdots + a_{m+k}Y_j, \quad j = 1, 2, \dots$$

Suppose that $\tilde{\tau} = k$, that is

$$\begin{aligned} Y_1 &> -r, \quad Y_1 + Y_2 > -r, \quad \dots, \quad Y_1 + Y_2 + \cdots + Y_{k-1} > -r; \\ Y_1 + Y_2 + \cdots + Y_{k-1} + Y_k &= -r. \end{aligned}$$

Then, recalling that a_i is a non-decreasing sequence,

$$\begin{aligned} S(m+k) &= A + a_{m+1}Y_1 + \cdots + a_{m+k-1}Y_{k-1} + a_{m+k}Y_k \\ &\leq A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-1}Y_{k-1} + a_{m+k-1}Y_k \quad (\text{since } Y_k = -1) \\ &= A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-1}[Y_{k-1} + Y_k] \\ &\leq A + a_{m+1}Y_1 + \cdots + a_{m+k-2}Y_{k-2} + a_{m+k-2}[Y_{k-1} + Y_k] \\ &= A + a_{m+1}Y_1 + \cdots + a_{m+k-2}[Y_{k-2} + Y_{k-1} + Y_k] \\ &\leq \cdots \leq A + a_{m+1}[Y_1 + \dots + Y_k] = A - ra_{m+1} \leq 0, \end{aligned}$$

since $Y_k, Y_{k-1} + Y_k, Y_{k-2} + Y_{k-1} + Y_k, \dots, Y_1 + \cdots + Y_k$ are all negative. Therefore, $\tau \leq \tilde{\tau}$. $\square$

Throughout the paper we will use a version of the Azuma-Hoeffding inequality; compare with the results of [3].

Lemma 1. Suppose that $b_1, b_2, \dots, b_m$ is a sequence of non-negative numbers and $S = b_1Y_1 + b_2Y_2 + \cdots + b_mY_m$, where $Y_j \sim \text{Rademacher}$ are i.i.d. Then

$$\mathbb{P}(|S| \geq A) \leq 2 \exp\left(-\frac{A^2}{2(b_1^2 + \cdots + b_m^2)}\right) \quad \text{for all } A > 0. \quad (3)$$

We also state the following fairly standard result.

Lemma 2. Let $T_i = Y_1 + \cdots + Y_i$ be a simple random walk. Suppose that L_k and y_k , $k = 1, 2, \dots$, are two sequences such that $L_k \rightarrow \infty$, $y_k \rightarrow \infty$ and $y_k/\sqrt{L_k} \rightarrow r > 0$. Then

$$\mathbb{P}\left(\max_{1 \leq i \leq L_k} T_i \geq y_k\right) \rightarrow 2\mathbb{P}(\eta \geq r) = 2 - 2\Phi(r)$$

where $\eta \sim \mathcal{N}(0, 1)$ and $\Phi(\cdot)$ is its CDF.

Proof. Let $\tilde{y}_k = \lceil y_k \rceil \in \mathbb{Z}_+$. By the reflection principle,

$$\begin{aligned} \mathbb{P}\left(\max_{1 \leq i \leq L_k} T_i \geq y_k\right) &= \mathbb{P}\left(\max_{1 \leq i \leq L_k} T_i \geq \tilde{y}_k\right) = 2\mathbb{P}(T_{L_k} \geq \tilde{y}_k) - \mathbb{P}(T_{L_k} = \tilde{y}_k) \\ &= 2\mathbb{P}\left(\frac{T_{L_k}}{\sqrt{L_k}} \geq \frac{\tilde{y}_k}{\sqrt{L_k}}\right) + O\left(\frac{1}{\sqrt{L_k}}\right) \rightarrow 2\mathbb{P}(\eta \geq r) \end{aligned}$$

by the Central Limit Theorem, using also the fact that $\tilde{y}_k/y_k \rightarrow 1$. $\square$

2 Integer-valued a-walks

Suppose that the sequence $\mathbf{a}$ contains only integers.

Proposition 2. *Let $z \in \mathbb{Z}$. Suppose that the sequence*

$$\int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt, \quad n = 1, 2, \dots$$

is summable. Then the events $\{S(n) = z\}$ occur for finitely many n s a.s.

Proof. The result follows from the standard Fourier analysis. Indeed,

$$\mathbb{E}e^{itS(n)} = \sum_{k \in \mathbb{Z}} e^{itk} \mathbb{P}(S(n) = k)$$

where the sum above goes, in fact, effectively over a finite number of ks (as $|S(n)| \leq a_1 + \dots + a_n$). At the same time,

$$\int_{-\pi}^\pi e^{it(k-z)} dt = \begin{cases} 2\pi, & \text{if } k = z; \\ 0, & \text{if } k \in \mathbb{Z} \setminus \{z\}. \end{cases}$$

By changing the order of summation and integration, we obtain

$$\frac{1}{2\pi} \int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)} dt = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{-\pi}^\pi e^{it(k-z)} \mathbb{P}(S(n) = k) dt = \mathbb{P}(S(n) = z).$$

On the other hand,

$$\frac{1}{2\pi} \int_{-\pi}^\pi \mathbb{E}e^{it(S(n)-z)} dt = \frac{1}{2\pi} \int_{-\pi}^\pi e^{-itz} \prod_{k=1}^n \mathbb{E}e^{ita_k X_k} dt = \frac{1}{\pi} \int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt$$

by the symmetry of $\cos$ and the fact that the imaginary part must equal zero. Now the result follows from the Borel-Cantelli lemma, since $\sum_n \mathbb{P}(S(n) = z) < \infty$. $\square$

Corollary 1. *Suppose that the sequence*

$$\int_0^\pi \left| \prod_{k=1}^n \cos(ta_k) \right| dt, \quad n = 1, 2, \dots$$

is summable. Then the $\mathbf{a}$ -walk is transient a.s.

Proof. From Proposition 2 we know that for each $z \in \mathbb{Z}$

$$\pi \mathbb{P}(S(n) = z) = \int_0^\pi \cos(tz) \prod_{k=1}^n \cos(ta_k) dt \leq \int_0^\pi \left| \cos(tz) \prod_{k=1}^n \cos(ta_k) \right| dt \leq \int_0^\pi \left| \prod_{k=1}^n \cos(ta_k) \right| dt.$$

Hence the event $\{S(n) = z\}$ occurs finitely often a.s. for each z . Since for each $C > 0$ there are only finitely many integers in $[-C, C]$ we conclude that the walk is not C -recurrent a.s. for every C . $\square$

An interesting and quite natural example is when $\mathbf{a} = (1, 2, 3, \dots)$, i.e., $a_i = i$. It was previously published in the IMS Bulletin [8], in the Student Puzzle Corner no. 37.

Theorem 3. *The $\mathbf{a}$ -walk with $\mathbf{a} = (1, 2, 3, \dots)$ is a.s. transient.*

This statement follows from a much stronger Theorem 4, but for the sake of completeness, we present its self-contained proof.

Proof of Theorem 3. Let $A_n = \{S(n) = 0\} = \{X_1 + 2X_2 + \dots + nX_n = 0\}$. Then $\mathbb{P}(A_n) = Q_n/2^n$, where

Q_n = number of ways to put $\pm$ in the sequence $*1*2*3*\dots*n$ such that the sum equals 0.

For example, $Q_1 = Q_2 = 0$, $Q_3 = Q_4 = 2$, $Q_5 = Q_6 = 0$, $Q_7 = 8$, $Q_8 = 14$, etc. It was essentially shown in [6] that

$$Q_n \sim \sqrt{\frac{6}{\pi}} \frac{2^n}{n^{3/2}} \quad \text{when } n \bmod 4 \in \{0, 3\}$$

(and zero otherwise) as $n \rightarrow \infty$, meaning that the ratio of the RHS and the LHS converges to one. Consequently, $\sum_n \mathbb{P}(A_n) \sim \sum_n \frac{\text{const}}{n^{3/2}} < \infty$ and the events A_n occur a.s. finitely often by the Borel-Cantelli lemma. Hence the walk is a.s. not recurrent.

Moreover, since for any $m \in \mathbb{Z}$

$$\mathbb{P}(S(n+2|m|) = S(n) - m \mid \mathcal{F}_n) \geq \frac{1}{2^{2m}}$$

(by making the signs of $X_n, X_{n+1}, X_{n+2}, \dots, X_{n+2|m|}$ alternate), we conclude that if the event $\{S(n) = m\}$ occurs infinitely often, then A_n shall also occur infinitely often a.s., leading to contradiction. As a result, $\mathbb{P}(\{S(n) = m\} \text{ i.o.}) = 0$ for all integer m s, and thus the walk is a.s. not C -recurrent for any non-negative C . $\square$

Remark 1. *Though the $(1, 2, 3, \dots)$ -walk is transient, it still jump over zero infinitely many times, as it was shown by Theorem 2.*

In fact, Theorem 3 can be generalized greatly, using the result of [5], or even a weaker result of [1].

Theorem 4. *Let $\mathbf{a}$ be such that all a_i 's are distinct integers. Then $\mathbf{a}$ -walk is a.s. transient.*

Proof. The main result of [5] implies that for any $\epsilon > 0$

$$\text{card}(\{(x_1, x_2, \dots, x_n) : \text{all } x_i = \pm 1, a_1x_1 + \dots + a_nx_n = m\}) \leq \frac{(1+\epsilon)2^{n+3}}{n^{3/2}\sqrt{\pi}}$$

for all $n \geq n_0(\epsilon)$ and all m . Setting $\epsilon = 1$, and fixing $m \in \mathbb{Z}$, we obtain that

$$\sum_{n=n_0(1)}^{\infty} \mathbb{P}(S(n) = m) \leq \sum_{n=n_0(1)}^{\infty} \frac{2 \cdot 2^{n+3}}{n^{3/2}\sqrt{\pi}} \times \frac{1}{2^n} = \frac{16}{\sqrt{\pi}} \sum_{n=n_0(1)}^{\infty} \frac{1}{n^{3/2}} < \infty.$$

Therefore, by the Borel-Cantelli lemma, only finitely many events $\{S(n) = m\}$ occur a.s. Since $S(n)$ takes only integer values, this implies that $\{|S(n)| \leq C\}$ happens finitely often a.s. for any $C > 0$. $\square$

Remark 2.

(a) *It is not difficult to see that in the condition of Theorem 4 it suffices that all a_k 's are distinct only starting from some $k_0 \geq 1$.*

(b) *If $a_k = \lfloor k^\beta \rfloor$ with $\beta \geq 1$, then we immediately have a.s. transience by Theorem 4.*

3 A non-trivial recurrent example

We assume here that $\mathbf{a} = (B_1, B_2, B_3, \dots)$ where each B_k is a consecutive block of k 's of length precisely $L_k \geq 1$. Denote also by $i_k = 1 + L_1 + L_2 + \dots + L_{k-1}$ the index of the first element of the k -th block. For example, if $L_k = 2^k$, then $i_k = 2^k - 1$ and

$$\mathbf{a} = (1, 1, \underbrace{2, 2, 2, 2}_{L_2 \text{ times}}, \underbrace{3, 3, 3, 3, 3, 3, 3, 3}_{L_3 \text{ times}}, \underbrace{4, \dots, 4}_{L_4 \text{ times}}, \dots),$$

one can also notice that $a_i = \lfloor \log_2(i+1) \rfloor = \lceil \log_2(i+2) \rceil - 1$.

Theorem 5. *Suppose that for some $\varepsilon > 0$, $r > 0$, and k_0 we have*

$$\begin{aligned} \frac{L_k}{L_1 + L_2 + \dots + L_{k'}} &\geq (2 + \varepsilon) \ln k; \\ \frac{L_k}{L_{k'+1} + L_{k'+2} + \dots + L_{k-1}} &\geq 2r; \\ L_k &\geq k^4, \end{aligned} \tag{4}$$

whenever $k - k' \geq \frac{k}{\ln k} - 2$ and $k, k' \geq k_0$. Then $\mathbf{a}$ -walk described above is a.s. recurrent.

Remark 3. One can easily check that the condition of the theorem are satisfied if $a_k = \lfloor (\log_\gamma k)^\beta \rfloor$, where $\gamma > 1$ and $\beta \in (0, 1]$.

Proof of Theorem 5. We will proceed in FIVE steps.

Step 1: Preliminaries

First, we need the following lemma, which is probably known.

Lemma 3. Let $m \in \mathbb{Z}_+$ and T_m be a simple symmetric random walk on $\mathbb{Z}^1$, that is, $T_m = Y_1 + \dots + Y_m$, where $Y_i \sim \text{Rademacher}$ are i.i.d. There exists a universal constant $c_1 > 0$ such that for all integers z such that $|z| \leq 2\sqrt{m}$, assuming that m is sufficiently large and $m + z$ is even,

$$\mathbb{P}(T_m = z) \geq \frac{c_1}{\sqrt{m}}.$$

Proof. W.l.o.g. assume $z \geq 0$. We have

$$\mathbb{P}(T_m = z) = \mathbb{P}\left(\frac{T_m + m}{2} = \frac{z + m}{2}\right) = \mathbb{P}(\tilde{T} = w)$$

where $\tilde{T} \sim \text{Bin}(m, 1/2)$ and $w = \frac{z+m}{2} \in \mathbb{Z}_+$. Note that $\tilde{m} \leq w \leq \tilde{m} + \sqrt{m}$ where $\tilde{m} = m/2$. So

$$\begin{aligned} \mathbb{P}(\tilde{T} = w) &= \binom{m}{w} 2^{-m} = \binom{2\tilde{m}}{\tilde{m}} \frac{1}{2^{2\tilde{m}}} \frac{\tilde{m}! \tilde{m}!}{w!(m-w)!} = \frac{1+o(1)}{\sqrt{\pi\tilde{m}}} \frac{(2\tilde{m}-w+1)(2\tilde{m}-w+2)\dots\tilde{m}}{(\tilde{m}+1)(\tilde{m}+2)\dots w} \\ &= \frac{1+o(1)}{\sqrt{\pi\tilde{m}}} \left(1 - \frac{w-\tilde{m}}{\tilde{m}+1}\right) \left(1 - \frac{w-\tilde{m}}{\tilde{m}+2}\right) \dots \left(1 - \frac{w-\tilde{m}}{w}\right) \\ &\geq \frac{1+o(1)}{\sqrt{\pi\tilde{m}}} \left(1 - \frac{\sqrt{m}}{\tilde{m}+1}\right)^{w-\tilde{m}} \geq \frac{1+o(1)}{\sqrt{\pi\tilde{m}}} \left(1 - \frac{\sqrt{2}+o(1)}{\sqrt{\tilde{m}}}\right)^{\sqrt{2\tilde{m}}} = \frac{e^{-2}+o(1)}{\sqrt{\pi m/2}} \geq \frac{0.1}{\sqrt{m}} \end{aligned}$$

for large enough m . □

Corollary 2. Let T_m , $m = 0, 1, 2, \dots$, be as simple symmetric random walk as in Lemma 3. Assume that m and k are positive integers such that $k^2 \leq m$. Let $u \in \mathbb{Z}$, and either k is odd, or both k and $m - u$ are even. Then for large ks

$$\mathbb{P}(T_m - u \bmod k = 0) \geq \frac{c_1}{2k}$$

where c_1 is the constant from Lemma 3.

Proof of Corollary 2. First, assume that m , and hence a , are even. Since $(T_m - u) \bmod k = 0 \iff T_m = \tilde{u} \bmod k$, where $\tilde{u} = (u \bmod k) \in \{0, 1, 2, \dots, k-1\}$, it suffices to show the statement for $\tilde{u}$.

Let $M = \lfloor 2\sqrt{m} \rfloor \in (2\sqrt{m} - 1, 2\sqrt{m}]$ and define

$$\begin{aligned}\mathbb{I} &= [-M, -M+1, \dots, -1, 0, 1, \dots, M] = \mathbb{I}_0 \cup \mathbb{I}_1; \\ \mathbb{I}_0 &= \{z \in \mathbb{I} : z \text{ is even}\}; \quad \mathbb{I}_1 = \{z \in \mathbb{I} : z \text{ is odd}\}.\end{aligned}$$

There are at least M elements in each $\mathbb{I}_0$ and $\mathbb{I}_1$.

If k is odd, then each of these two sets contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$. If m is even (odd, resp.) for all z either in $\mathbb{I}_0$ (in $\mathbb{I}_1$, resp.) by Lemma 3 for large ks (and hence large m) we have $\mathbb{P}(T_m = z) \geq c_1/\sqrt{m}$. Consequently,

$$\begin{aligned}\mathbb{P}(T_m = \tilde{u} \bmod k) &\geq \sum_{z \in \mathbb{I}, z = \tilde{u} \bmod k} \mathbb{P}(T_m = z) \geq \left\lfloor \frac{M}{k} \right\rfloor \times \frac{c_1}{\sqrt{m}} \geq \left(\frac{M}{k} - 1 \right) \times \frac{c_1}{\sqrt{m}} \\ &\geq \left(\frac{2\sqrt{m} - 1}{k} - 1 \right) \times \frac{c_1}{\sqrt{m}} \geq \left(1 - \frac{1}{k} \right) \times \frac{c_1}{\sqrt{m}} = \frac{c_1}{k} - O(k^{-2})\end{aligned}$$

since $m \geq k^2$.

If k is even, then if m is even (and thus a is also even) then $\mathbb{I}_0$ contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$ and at the same time Lemma 3 is applicable for $z \in \mathbb{I}_0$. On the other hand, if m (and so u) is odd then $\mathbb{I}_1$ contains at least $\lfloor \frac{M}{k} \rfloor$ elements z such that $z = \tilde{u} \bmod k$ and Lemma 3 is applicable for $z \in \mathbb{I}_1$. The rest of the proof is the same as for the case when k is odd. $\square$

Step 2: Splitting $S(n)$

Recall that that i_k denotes the first index of block k and note that the sum of all the steps within block k can be represented as

$$S(i_{k+1} - 1) - S(i_k - 1) = k \cdot T_k, \quad T_k = X_1^{(k)} + \dots + X_{L_k}^{(k)}$$

where $X_j^{(k)}$'s are i.i.d. *Rademacher* random variables.

For $m = 2, \dots$, let

$$k_m = \begin{cases} \lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is odd;} \\ \lfloor m \ln m \rfloor + 1 & \text{if } \lfloor m \ln m \rfloor \text{ is even.} \end{cases} \quad (5)$$

Thus k_m is *always* odd; $k_m, m = 2, 3, \dots$ equal 1, 3, 5, 7, 9, 13, 15, 19, 23, etc. Define also

$$A_m = \{S(j) = 0 \text{ for some } i_{k_m} \leq j < i_{k_m+1}\},$$

the event that $S(j)$ hits zero for the step sizes inside of block B_{k_m} , and the sequence of sigma-algebras

$$\mathcal{G}_m = \mathcal{F}_{i_{k_m+1}-1} = \sigma \left(\bigcup_{\ell=1}^{k_m} \left(\bigcup_{i=1}^{L_\ell} X_i^{(\ell)} \right) \right).$$

Intuitively, $\mathcal{G}_m$ contains all the information about the walk during its steps corresponding to the first k_m blocks.

To simplify notations, let us now write $k = k_m$ and $k' = k_{m-1}$, and observe that

$$k - k' = k_m - k_{m-1} \geq m \ln m - (m-1) \ln(m-1) - 2 = \ln m - 1 + O\left(\frac{1}{m}\right) \geq \ln m - 2 \quad (6)$$

for large m .

Let us split $S(j)$ where $j \in [i_k, i_{k+1})$, as follows:

$$\begin{aligned} S(j) &= S(i_{k'}) + \sum_{n=k'+1}^{k-1} \left(X_1^{(n)} + \cdots + X_{L_n}^{(n)} \right) + k \cdot \left(X_1^{(k)} + \cdots + X_{j-i_k}^{(k)} \right) \\ &= S(i_{k'}) + \left[\sum_{n=k'+1}^{k-2} \left(X_1^{(n)} + \cdots + X_{L_n}^{(n)} \right) + (k-1) \sum_{\ell=1}^{i_k-2k^2-1} X_\ell^{(k-1)} \right] \\ &\quad + (k-1) \cdot \Sigma_3 + k \cdot \left(X_1^{(k)} + \cdots + X_{j-i_k}^{(k)} \right) \\ &= \Sigma_1 + \Sigma_2 + (k-1) \cdot \Sigma_3 + k \cdot \Sigma_4 \end{aligned}$$

where $\Sigma_1 = S(i_{k'})$ and

$$\begin{aligned} \Sigma_2 &= \sum_{n=k'+1}^{k-2} n T_n + (k-1) T'_{k-1}, \quad T'_{k-1} = \sum_{\ell=i_{k-1}}^{i_k-1-2k^2} X_\ell^{(k-1)}; \\ \Sigma_3 &= X_{i_k-2k^2}^{(k-1)} + X_{i_k-2k^2+1}^{(k-1)} + \cdots + X_{i_k-2}^{(k-1)} + X_{i_k-1}^{(k-1)}; \\ \Sigma_4 &= X_1^{(k)} + X_2^{(k)} + \cdots + X_{j-i_k}^{(k)}. \end{aligned}$$

Note that Σ_i , $i = 1, 2, 3, 4$ are independent, and Σ_3 has precisely $2k^2$ terms.

Step 3: Estimating Σ_1

Recall that $k = k_m$, $k' = k_{m-1}$ and let

$$E_{m-1} = \left\{ |\Sigma_1| < k \sqrt{L_k} \right\} \in \mathcal{G}_{m-1}.$$

By Lemma 1 and (4), assuming k is large,

$$\begin{aligned} \mathbb{P}(E_{m-1}^c) &\leq \mathbb{P}(|S(k')| \geq k' \sqrt{L_k}) \leq 2 \exp \left(- \frac{k'^2 \cdot L_k}{2(L_1 + 2^2 \cdot L_2 + 3^2 \cdot L_3 + \cdots + k'^2 \cdot L_{k'})} \right) \\ &\quad (7) \\ &\leq 2 \exp \left(- \frac{L_k}{2(L_1 + L_2 + L_3 + \cdots + L_{k'})} \right) \leq 2 \exp \left(-(1 + \varepsilon/2) \ln k \right) = \frac{2}{k^{1+\varepsilon/2}} =: \varepsilon_m. \end{aligned}$$

Step 4: Estimating Σ_2

Again, by Lemma 1 and (4), assuming that k is sufficiently large,

$$\begin{aligned}\mathbb{P}\left(|\Sigma_2| \geq k\sqrt{\frac{L_k}{r}}\right) &\leq 2 \exp\left(-\frac{k^2 r^{-1} L_k}{2[(k'+1)^2 L_{k'+1} + \dots + (k-2)^2 L_{k-2} + (k-1)^2 (L_{k-1} - 2k^2)]}\right) \\ &\leq 2 \exp\left(-\frac{r^{-1} L_k}{2[L_{k'+1} + \dots + L_{k-1} - 2k^2]}\right) \leq 2 \exp(-1) = 0.7357588824 \dots\end{aligned}$$

Consequently,

$$\mathbb{P}\left(|\Sigma_2| < k\sqrt{L_k/r}\right) \geq 0.2 \quad \text{for large } k. \quad (8)$$

Step 5: Finishing the proof

We have a trivial lower bound

$$\begin{aligned}\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) &\geq \mathbb{P}\left(A_m \mid |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1}, \mathcal{G}_{m-1}\right) \times \mathbb{P}\left(|\Sigma_2| < k\sqrt{\frac{L_k}{r}} \mid E_{m-1}, \mathcal{G}_{m-1}\right) \\ &=: (*) \times 0.2 \quad \text{for large } k\end{aligned} \quad (9)$$

by (8), since the second multiplier equals $\mathbb{P}\left(|\Sigma_2| < k\sqrt{L_k/r}\right)$ by independence.

Let

$$\text{Div}_k = \{\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3 = 0 \bmod k\} = \{\Sigma_1 + \Sigma_2 - \Sigma_3 = 0 \bmod k\}.$$

Since only on the event Div_k , it is possible that $S(j) = 0$ for some j (since the step sizes are $\pm k$ in the block B_k), we conclude that for large k

$$\begin{aligned}(*) &= \mathbb{P}\left(A_m \mid \text{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1}, \mathcal{G}_{m-1}\right) \times \mathbb{P}\left(\text{Div}_k \mid |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1}, \mathcal{G}_{m-1}\right) \\ &\geq \mathbb{P}\left(A_m \mid \text{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1}, \mathcal{G}_{m-1}\right) \times \frac{c_1}{2k}\end{aligned} \quad (10)$$

due to the fact that by Corollary 2, $\mathbb{P}(\text{Div}_k \mid \mathcal{F}_{i_k-2k^2-1}) \geq c_1/(2k)$. On the other hand,

$$\mathbb{P}\left(A_m \mid \text{Div}_k, |\Sigma_2| < k\sqrt{\frac{L_k}{r}}, E_{m-1}, \mathcal{G}_{m-1}\right) \geq \mathbb{P}(\bar{A}_k) \geq \beta \quad (11)$$

where

$$\begin{aligned}\bar{A}_k &= \{\text{a simple random walk, starting no further than } (r^{-1/2} + 3)\sqrt{L_k} \\ &\quad \text{from the origin, will hit zero within } L_k \text{ steps}\},\end{aligned}$$

$$\text{and } \beta = 1 - \Phi(r^{-1/2} + 3) > 0,$$

because under our assumptions, using the last part of (4),

$$|\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3| \leq k\sqrt{L_k} + k\sqrt{L_k/r} + 2(k-1)k^2 < (1+r^{-1/2}+2)k\sqrt{L_k}$$

for large k , $S(j) = [\Sigma_1 + \Sigma_2 + (k-1)\Sigma_3] + k \cdot \Sigma_4$, and by Lemma 2

$$\liminf_{k \rightarrow \infty} \mathbb{P}(\bar{A}_k) \geq 2\mathbb{P}(\eta > r^{-1/2} + 3) = 2\beta$$

so $\mathbb{P}(\bar{A}_k) \geq \beta$ for sufficiently large k .

Finally, from (9), (10), and (11) we get that

$$\sum_m \mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq \sum_m \frac{0.2 c_1 \beta}{2m \log m} = +\infty$$

and $\mathbb{P}(E_m^c)$ is summable by (7), so we can apply Lemma 5 to conclude that events A_m occur infinitely often and thus our **a**-walk is recurrent. $\square$

4 Continuous example

The example of an **a**-walk described in Theorem 5 roughly corresponds to the case $a_k = \lceil \log_\gamma k \rceil$, $k = 1, 2, \dots$. But what if a_k 's take non-integer values, but, for example, equal

$$a_k = \log_\gamma k \equiv c \ln k, \quad k = 1, 2, \dots,$$

where $\gamma = e^{1/c} > 1$? In this Section we will study this example. It is unreasonable to assume that such **a**-walk is recurrent, because of the irrationality of the step sizes, however, we might want to investigate if this walk is C -recurrent for *some* $C > 0$. Our main result is the following

Theorem 6. *Let $c > 0$ and $a_k = c \ln k$. Then the **a**-walk is a.s. C -recurrent for every $C > 0$.*

To prove this theorem, it is sufficient to show that whatever the value $c > 0$ is, $\{|S(n)| \leq 3\}$ happens i.o. almost surely. Indeed, take any $C > 0$. Then the statement that **a**'-walk with $a'_k = \frac{3c}{C} \ln k$, $k = 1, 2, \dots$, is 3-recurrent is equivalent to the statement that **a**-walk with $a_k = c \ln k$ is C -recurrent.

The proof will proceed similarly to that of Theorem 5. Let us define k_m slightly differently from (5); namely, let

$$k_m = \begin{cases} \lfloor m \ln m \rfloor & \text{if } \lfloor m \ln m \rfloor \text{ is even;} \\ \lfloor m \ln m \rfloor - 1 & \text{if } \lfloor m \ln m \rfloor \text{ is odd.} \end{cases}$$

Thus now k_m are always *even*. As before, set $k = k_m$, and $k' = k_{m-1}$, and define

$$i_k = \lceil \gamma^k \rceil = \max\{i \geq 1 : a_i < k\} + 1 = \min\{i \geq 1 : a_i \geq k\} \in [\gamma^k, \gamma^k + 1),$$

i.e., the first index when a_i starts exceeding k . For $i \in J_k := [i_k, i_{k+1})$ write

$$\begin{aligned} S(i) &= S(i_{k'} - 1) + [S(i_k - 1) - S(i_{k'} - 1) - \Sigma_3] + \Sigma_3 + [S(i) - S(i_k - 1)] \\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4(i) \end{aligned} \quad (12)$$

where

$$\Sigma_3 = [S(i_{k-1} + k^2 - 1) - S(i_{k-1} - 1)] + [S(i_k) - S(i_k - k^2)].$$

Note that Σ_i , $i = 1, 2, 3, 4$ are independent, and Σ_3 has $2 \cdot k^2$ terms, and contains the first k^2 and the last k^2 steps of the walk, when the step sizes lie in $[k, k + 1)$.

Let

$$E_{m-1} = \{|\Sigma_1| < k\sqrt{i_k}\} = \{S(i_{k_{m-1}}) < k_m \sqrt{i_{k_m}}\} \quad (13)$$

By Lemma 1, since $a_i < k' < k$ for $i < i_{k'}$,

$$\begin{aligned} \mathbb{P}(E_{m-1}^c) &= \mathbb{P}(|\Sigma_1| \geq k\sqrt{i_k}) \leq 2 \exp\left(-\frac{i_k k^2}{2 \sum_{j=1}^{i_{k'}} a_j^2}\right) \leq 2 \exp\left(-\frac{i_k}{2i_{k'}}\right) \leq 2 \exp\left(-\frac{\gamma^k - 1}{2\gamma^{k'}}\right) \\ &= 2 \exp\left(-\frac{\gamma^{k-k'}(1 + o(1))}{2}\right) = 2 \exp\left(-\frac{\gamma^{\ln m - 2}}{2 + o(1)}\right) = 2 \exp\left(-\frac{m^{\ln \gamma}}{2\gamma^2 + o(1)}\right) =: \varepsilon_{m-1} \end{aligned} \quad (14)$$

using (6) for k sufficiently large³. Observe that ε_m is summable.

Similarly, by Lemma 1

$$\mathbb{P}(|\Sigma_2| \geq 2k\sqrt{i_k}) \leq 2 \exp\left(-\frac{4k^2 i_k}{2k^2(i_k - i_{k'} - 2k^2)}\right) < 2e^{-2} = 0.27 \dots$$

Hence,

$$\mathbb{P}(F_m) \geq 0.72 \quad \text{where } F_m = \{|\Sigma_2| < 2k\sqrt{i_k}\}. \quad (15)$$

Lemma 4. *Let $n = k^2$ where k is an even positive integer, and assume also that k is sufficiently large. Suppose that $X_i, Y_i, i = 1, 2, \dots, n$, are i.i.d. Rademacher. Let*

$$T = (k - 1)(X_1 + X_2 + \dots + X_n) + k(Y_1 + Y_2 + \dots + Y_n). \quad (16)$$

³Note that (6) was stated for k_m defined slightly differently, however, it holds here as well.

Then

$$\mathbb{P}(T = j) \geq \frac{c_1^2}{4n} \quad \text{for each } j = 0, \pm 2, \pm 4, \dots, \pm n$$

where c_1 is the constant from Lemma 3.

Proof. It follows from Corollary 2 that

$$\mathbb{P}(X_1 + \dots + X_n = \ell) \geq \frac{c_1}{2k}, \quad \mathbb{P}(Y_1 + \dots + Y_n = \ell) \geq \frac{c_1}{2k} \quad \text{for all even } \ell \text{ such that } |\ell| \leq 2k. \quad (17)$$

Let $j = 2\tilde{j} \in \{0, 2, 4, \dots, n-2, n\}$. Consider the sequence of $k-1$ numbers

$$\tilde{j}, \tilde{j} - k, \tilde{j} - 2k, \tilde{j} - 3k, \dots, \tilde{j} - (k-2)k;$$

they all give different remainders when divided by $k-1$. Hence there must be an $m \in \{0, 1, \dots, k-2\}$ such that $\tilde{j} - mk = b(k-1)$ and b is an integer; moreover, $b \in \left[-\frac{k(k-2)}{k-1}, \frac{n}{2(k-1)}\right]$ since $0 \leq \tilde{j} \leq n/2$. For such m and b we have $j = 2\tilde{j} = (2m)k + (2b)(k-1)$, and, since both $|2m|$ and $|2b| \leq 2k$,

$$\mathbb{P}(T = j) \geq \mathbb{P}(X_1 + \dots + X_n = 2b) \cdot \mathbb{P}(Y_1 + \dots + Y_n = 2m) \geq \left(\frac{c_1}{2k}\right)^2 = \frac{c_1^2}{4n}$$

by (17). The result for negative j follows from the symmetry. $\square$

Corollary 3. Let $\varepsilon = \frac{2ck^4}{\gamma^{k-1}}$. Then for large even k

$$\mathbb{P}(\Sigma_3 \in [j - \varepsilon, j + \varepsilon]) \geq \frac{c_1^2}{4k^2} \quad \text{for each } j = 0, \pm 2, \pm 4, \dots, \pm k^2.$$

Proof. Σ_3 has the same distribution as

$$\sum_{\ell=1}^{k^2} c \ln(i_{k-1} - 1 + \ell) X_\ell + \sum_{\ell=1}^{k^2} c \ln(i_k - \ell) Y_\ell$$

for some i.i.d. $X_\ell, Y_\ell \sim \text{Rademacher}$. At the same time, for $\ell \geq 1$,

$$\begin{aligned} |c \ln(i_{k-1} - 1 + \ell) - (k-1)| &= |c \ln(\lceil \gamma^{k-1} \rceil + \ell - 1) - (k-1)| \\ &\leq |c \ln(\gamma^{k-1} + \ell) - (k-1)| = c \ln \left(1 + \frac{\ell}{\gamma^{k-1}}\right) \leq \frac{c\ell}{\gamma^{k-1}} \end{aligned}$$

Similarly,

$$k - c \ln(i_k - \ell) = k - c \ln(\lceil \gamma^k \rceil - \ell) = k - c \ln(\gamma^k - \ell') = -c \ln \left(1 - \frac{\ell'}{\gamma^k}\right) \in \left[0, \frac{c\ell}{\gamma^{k-1}}\right]$$

for some $\ell' \in [\ell - 1, \ell]$, assuming $\ell = o(\gamma^k)$. As a result, for T defined by (16),

$$|\Sigma_3 - T| \leq \sum_{\ell=1}^{k^2} \frac{2c\ell}{\gamma^{k-1}} = \frac{ck^2(k^2 + 1)}{\gamma^{k-1}} \leq \frac{2ck^4}{\gamma^{k-1}}.$$

Now the result follows from Lemma 4. $\square$

Proof of Theorem 6. Recall that $J_k = [i_k, i_{k+1})$ and define

$$A_m = \{S(i) \text{ for some } i \in J_{k_m}\}.$$

Then

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq 0.72 \times \mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) \quad (18)$$

(please see the definition of F_m in (15)). Recall formula (12) and write

$$\tilde{S}(i) = S(i) - \Sigma_3 = \Sigma_1 + \Sigma_2 + \Sigma_4(i).$$

From now on assume that $|\Sigma_1| < k\sqrt{i_k}$ and $|\Sigma_2| < 2k\sqrt{i_k}$, that is, E_{m-1} and F_m occur. Also assume w.l.o.g. that $\Sigma_1 + \Sigma_2 \geq 0$. Let

$$L_k = i_{k+1} - i_k - k^2 = (\gamma - 1)\gamma^k + o(\gamma^k).$$

Consider a simple random walk with steps $Y_i \sim \text{Rademacher}$ during its first L_k steps. The probability that its minimum will be equal to or below the level $-3\sqrt{i_k} = -\frac{3+o(1)}{\sqrt{\gamma-1}}\sqrt{L_k}$ converges by Lemma 2 to

$$2\mathbb{P}\left(\eta > \frac{3}{\sqrt{\gamma-1}}\right) = 2 - 2\Phi\left(\frac{3}{\sqrt{\gamma-1}}\right) =: 2c_2 \in (0, 1)$$

as $k \rightarrow \infty$ (recall that $\eta \sim \mathcal{N}(0, 1)$). As a result, by Proposition 1, the probability that for some $j_0 \in \{i_k, i_k + 1, i_k + 2, \dots, i_k + L_k - 1\}$ we have the downcrossing, that is,

$$\tilde{S}(j_0 - 1) \geq 0 > \tilde{S}(j_0)$$

is bounded below by c_2 for k sufficiently large. Formally, let

$$\begin{aligned} j_0 &= \inf\{j > i_k : \tilde{S}(j) < 0\}; \\ \mathcal{C}_0 &= \{i_k \leq j_0 \leq i_k + L_k - 1\}, \end{aligned}$$

so we have showed that on $E_{m-1} \cap F_m \cap \{\Sigma_1 + \Sigma_2 > 0\}$ we have $\mathbb{P}(\mathcal{C}_0) > c_2$ for large k .

Now assume that event $\mathcal{C}_0$ occurred and define additionally

$$\begin{aligned} b_0 &= \tilde{S}(j_0) \in (-k - 1, 0]; \\ \mathcal{C} &= \left\{ \max_{0 \leq h \leq k^2} \sum_{g=1}^h X_{j_0+g} \geq k \right\}. \end{aligned}$$

Again, from Lemma 2,

$$\mathbb{P}(\mathcal{C}) = 2\mathbb{P}(X_{j_0+1} + X_{j_0+1} + \dots + X_{j_0+k^2} \geq k) \rightarrow 2(1 - \Phi(1)) = 0.3173 \dots \quad \text{as } k \rightarrow \infty.$$

From now on assume that k is so large that $\mathbb{P}(\mathcal{C}) > 0.2$. On the event $\mathcal{C}$ there exists an increasing sequence $j_1, j_2, \dots, j_k$ such that

$$j_0 < j_1 < j_2 < \dots < j_k \leq j_0 + k^2 < i_{k+1}$$

such that $X_{j_0+1} + X_{j_0+1} + \dots + X_{j_\ell} = \ell$ for each $\ell = 1, 2, \dots, k$, since the random walk must pass through each integer in $\{1, 2, \dots, k\}$ in order to reach level k .

For $\ell = 1, 2, \dots, k$, define

$$\begin{aligned} b_\ell &:= \tilde{S}(j_\ell) = b_0 + \sum_{h=j_0+1}^{j_\ell} a_h X_h = b_0 + a_{j_0} \left[\sum_{h=j_0+1}^{j_\ell} X_h \right] + \sum_{h=j_0+1}^{j_\ell} (a_h - a_{j_0}) X_h \\ &= b_0 + a_{j_0} \ell + O\left(\frac{k^4}{\gamma^k}\right) \end{aligned}$$

since for $h \in [j_0, j_0 + k^2] \subseteq [i_k, i_{k+1})$ we have

$$|a_h - a_{j_0}| = c \left| \ln \frac{a_h}{a_{j_0}} \right| \leq c \left| \ln \frac{i_k + k^2}{j_k} \right| = O\left(\frac{k^2}{\gamma^k}\right).$$

As a result,

$$-(k+1) < b_0 < b_1 < b_2 < \dots < b_{k-1} < (k-1)(k+1) < k^2$$

and moreover the distance between consecutive b_g 's is at least two (provided k is large). For $\ell = 0, 1, \dots, k-1$ define

$$\tilde{b}_\ell = \sup \left\{ x \in 2\mathbb{Z} : x + b_\ell \in \left[\frac{1}{2}, 3 - \frac{1}{2} \right) \right\} \equiv -2 \left\lfloor \frac{b_\ell}{2} - \frac{1}{4} \right\rfloor.$$

Then $\tilde{b}_\ell$'s are all distinct even integers satisfying $|\tilde{b}_\ell| \leq k^2$.

As a result,

$$\begin{aligned} \mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) &\geq c_2 \times 0.2 \times \mathbb{P}(S(i) \in [0, 3] \text{ for some } i \in J_k \mid \mathcal{C}, \mathcal{C}_0) \\ &\geq c_2 \times 0.2 \times \mathbb{P}(\tilde{S}(i_\ell) + \Sigma_3 \in [0, 3] \text{ for some } \ell = 0, 1, \dots, k-1 \mid \mathcal{C}, \mathcal{C}_0) \\ &= c_2 \times 0.2 \times \mathbb{P}(b_\ell + \Sigma_3 \in [0, 3] \text{ for some } \ell = 0, 1, \dots, k-1 \mid \mathcal{C}, \mathcal{C}_0) \\ &\geq c_2 \times 0.2 \times \mathbb{P}(\tilde{\Sigma}_3 \in [\tilde{b}_\ell - \varepsilon, \tilde{b}_\ell + \varepsilon] \text{ for some } \ell = 0, 1, \dots, k-1 \mid \mathcal{C}, \mathcal{C}_0) \\ &= c_2 \times 0.2 \times \sum_{\ell=0}^{k-1} \mathbb{P}(\Sigma_3 \in [\tilde{b}_\ell - \varepsilon, \tilde{b}_\ell + \varepsilon] \mid \mathcal{C}, \mathcal{C}_0) \geq c_2 \times 0.2 \times k \times \frac{c_1^2}{4k^2} = \frac{c_1^2 c_2}{20k} \end{aligned}$$

assuming that ε in Corollary 3 is sufficiently small. Finally,

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq 0.72 \times \mathbb{P}(A_m \mid F_m, E_{m-1}, \mathcal{G}_{m-1}) \geq \frac{0.72 c_1^2 c_2}{20k_m} \geq \frac{0.036 c_1^2 c_2}{m \ln m}$$

the sum of which diverges. Hence, recalling (14), we can again apply Lemma 5. $\square$

5 Sublinear growth of step sizes

Throughout this Section we assume

$$a_k = \lfloor k^\beta \rfloor, \quad 0 < \beta < 1.$$

Proposition 3. *Let $S(n) = a_1X_1 + \dots + a_nX_n$ where $a_k = \lfloor k^\beta \rfloor$, $0 < \beta < 1$. Then*

$$\mathbb{P}(|S(n)| = z) \leq \frac{\nu}{n^{1/2+\beta}} \quad \text{for all large } n,$$

for some $\nu > 0$.

Proof. Let $F_n(t) = \prod_{k=1}^n |\cos(ta_k)|$. For all $z \in \mathbb{Z}$ we have

$$\mathbb{P}(S(n) = z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itz} \mathbb{E} e^{itS(n)} dt \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\mathbb{E} e^{itS(n)}| dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} F_n(t) dt \leq \frac{\nu}{n^{1/2+\beta}}$$

by Lemma 6 for some $\nu > 0$, for all large n . $\square$

Theorem 7. *Suppose that $a_k = \lfloor k^\beta \rfloor$, $0 < \beta < 1$. Then the $\mathbf{a}$ -walk is a.s. transient.*

Proof. In the case $\beta > 1/2$ the a.s. transience follows immediately from Borel-Cantelli lemma and Proposition 3, as $\sum_n \mathbb{P}(|S(n)| \leq C) < \infty$ for each $C \geq 1$.

Assume from now on that $0 < \beta \leq 1/2$. Define k_m, Δ_m, m_n as in Case 3 of the proof of Lemma 6. Fix a positive integer M and consider now only those n for which $m_n = M$; assume without losing much rigor that $0.6M$ is an integer. Let $I_M = \{k_M, k_M + 1, \dots, k_{M+1} - 1\}$. Note that the elements of I_M are precisely those n s for which $a_n = M$, and that I_M has of order $M^{1/\beta-1}$ elements. Next, fix some $z \in \mathbb{Z}$ and define

$$E_M = E_M(z) = \{S(n) = z \text{ for some } n \in I_M\}.$$

For each z we will show that $\sum_M \mathbb{P}(E_M) < \infty$, and so by the Borel-Cantelli lemma a.s. only finitely many events E_M occur. Since $S(n)$ takes only integer values, this will imply that the walk is not C -recurrent for any $C \geq 0$.

So, fix z from now on, write $S(n) = S(k_M) + R(n)$ where

$$R(n) = \sum_{i=k_M}^n a_i X_i = M \sum_{i=k_M}^n X_i,$$

Observe also that $S(k_M)$ and $R(n)$ are independent. In order $S(n) = z$ for some $n \in I_M$, we need that $S(k_M) = z \bmod M$. Let $Q = M^{\frac{1}{2\beta}+1-\varepsilon}$ for an $\varepsilon \in (0, 1/2)$.

Assuming M is so large that $Q \geq 2|z|$,

$$\begin{aligned}\mathbb{P}(|R(n)| \geq Q - |z|) &\leq 2 \exp\left(-\frac{(Q/2)^2}{2M^2 \cdot (k_{M+1} - k_M)}\right) = 2 \exp\left(-\frac{\beta Q^2}{(8 + o(1))M^2 \cdot M^{\frac{1}{\beta}-1}}\right) \\ &= 2 \exp\left(-\frac{\beta M^{1-2\varepsilon}}{8 + o(1)}\right) \quad \text{for all } n \in I_M\end{aligned}$$

by Lemma 1; hence

$$\mathbb{P}(\max_{n \in I_M} |R(n)| \geq Q - |z|) \leq |I_M| \times 2 \exp\left(-\frac{\beta M^{1-2\varepsilon}}{8 + o(1)}\right) =: \alpha_M = O\left(M^{\frac{1}{\beta}-1} e^{-\frac{\beta M^{1-2\varepsilon}}{8+o(1)}}\right),$$

which is summable in M . So,

$$\begin{aligned}\mathbb{P}(E_M) &\leq \mathbb{P}\left(E_M, \max_{n \in I_M} |R(n)| < Q - |z|\right) + \mathbb{P}\left(\max_{n \in I_M} |R(n)| \geq Q - |z|\right) \\ &= \mathbb{P}\left(E_M, S(k_M) = z \bmod M, \max_{n \in I_M} |R(n)| < Q - |z|\right) + \alpha_M = (*) + \alpha_M\end{aligned}$$

where the term α_M is summable since $1 - 2\varepsilon > 0$. Since E_M implies $-S(k_M) = R(n) - z$ for some $n \in I_M$,

$$\begin{aligned}(*) &\leq \mathbb{P}(|S(k_M)| < Q, S(k_M) = z \bmod M) = \sum_{j: |j| < Q, j \equiv z \bmod M} \mathbb{P}(|S(k_M)| = j) \\ &\leq \frac{\nu}{k_M^{1/2+\beta}} \times |\{j: |j| < Q, j \equiv z \bmod M\}| \leq \frac{\nu + o(1)}{M^{1+\frac{1}{2\beta}}} \times \left[\frac{2Q+1}{M} + 1\right] = \frac{(\nu + o(1))}{\pi M^{1+\varepsilon}}\end{aligned}$$

by Proposition 3. The RHS is summable in M , which concludes the proof. $\square$

Remark 4. By setting $\varepsilon = 1/2 - \delta/2$, where $\delta > 0$ is very close to zero, we can ensure that

$$\begin{aligned}\mathbb{P}(|S(n)| < M^{1/2-\delta} \text{ for some } n \in I_M) &\leq \sum_{z: |z| < M^{1/2-\delta}} \mathbb{P}(E_M(z)) \leq \left[\frac{\text{const}}{M^{1+\varepsilon}} + \alpha_M\right] \times 2M^{1/2-\delta} \\ &= \frac{2 \text{const}}{M^{1+\delta/2}} + 2M^{1/2-\delta} \alpha_M\end{aligned}$$

is summable. Hence, a.s. eventually $|S(n)|$ will be larger than $n^{\beta/2-\delta}$ for any $\delta > 0$.

Appendix 1: Modified conditional Borel-Cantelli lemma

Lemma 5. Suppose that we have an increasing sequence of sigma-algebras $\mathcal{G}_m$ and a sequence of $\mathcal{G}_m$ -measurable events A_m and E_m such that

$$\mathbb{P}(A_m \mid E_{m-1}, \mathcal{G}_{m-1}) \geq \alpha_m, \quad \mathbb{P}(E_m^c) \leq \varepsilon_m \quad \text{a.s.}$$

where the non-negative α_n and ε_n satisfy

$$\sum_m \alpha_m = \infty, \quad \sum_m \varepsilon_m < \infty. \quad (19)$$

Then $\mathbb{P}(A_m \text{ i.o.}) = 1$.

Proof. Let $m > \ell \geq 1$ and $B_{\ell,m} = \bigcap_{i=\ell}^m A_i^c$. We need to show that for any $\ell \geq 1$

$$\mathbb{P}(B_{\ell,\infty}) = \mathbb{P}(A_\ell^c \cap A_{\ell+1}^c \cap A_{\ell+2}^c \cap \dots) = 0. \quad (20)$$

We have for $m \geq \ell + 1$

$$\begin{aligned} \mathbb{P}(B_{\ell,m}) &= \mathbb{P}(A_m^c \cap B_{\ell,m-1}) \leq \mathbb{P}(A_m^c \cap B_{\ell,m-1} \cap E_{m-1}) + \mathbb{P}(E_{m-1}^c) \\ &= \mathbb{P}(A_m^c \mid B_{\ell,m-1} \cap E_{m-1}) \mathbb{P}(B_{\ell,m-1} \cap E_{m-1}) + \mathbb{P}(E_{m-1}^c) \\ &\leq \mathbb{P}(A_m^c \mid B_{\ell,m-1} \cap E_{m-1}) \mathbb{P}(B_{\ell,m-1}) + \varepsilon_{m-1} \leq (1 - \alpha_m) \mathbb{P}(B_{\ell,m-1}) + \varepsilon_{m-1}. \end{aligned} \quad (21)$$

By induction over $m, m-1, m-2, \dots, \ell+1$ in (21), we get that

$$\begin{aligned} \mathbb{P}(B_{\ell,m}) &\leq \varepsilon_{m-1} + (1 - \alpha_m) [(1 - \alpha_{m-1}) \mathbb{P}(B_{\ell,m-2} \mid \mathcal{G}_m) + \varepsilon_{m-2}] \leq \dots \\ &\leq \varepsilon_{m-1} + (1 - \alpha_m) \varepsilon_{m-2} + (1 - \alpha_m)(1 - \alpha_{m-1}) \varepsilon_{m-3} + \dots \\ &\quad + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{\ell+2}) \varepsilon_\ell + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{\ell+1}). \end{aligned}$$

Hence, for any integer $M \in (\ell, m)$

$$\begin{aligned} \mathbb{P}(B_{\ell,m}) &\leq \varepsilon_{m-1} + \varepsilon_{m-2} + \dots + \varepsilon_M \\ &\quad + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{M+1}) \varepsilon_{M-1} + \dots + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{\ell+2}) \varepsilon_\ell \\ &\quad + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{\ell+1}) \\ &\leq [\varepsilon_{m-1} + \varepsilon_{m-2} + \dots + \varepsilon_M] \\ &\quad + (1 - \alpha_m)(1 - \alpha_{m-1}) \dots (1 - \alpha_{M+1}) [\varepsilon_{M-1} + \varepsilon_{M-2} + \dots + \varepsilon_\ell + 1] \end{aligned}$$

Fix any $\delta > 0$. By (19) we can find an M be so large that $\sum_{i=M}^\infty \varepsilon_i < \delta/2$. Then, again by (19), there exists an $m_0 > M$ such that $\prod_{i=M+1}^{m_0} (1 - \alpha_i) < \frac{\delta}{2(1 + \sum_{i=\ell}^{M-1} \varepsilon_i)}$.

Hence, for all $m \geq m_0$ we have $\mathbb{P}(B_{\ell,m}) \leq \delta/2 + \delta/2 = \delta$. Since $\delta > 0$ is arbitrary, and $B_{\ell,m}$ is a decreasing sequence of events in m , we conclude that $\mathbb{P}(B_{\ell,\infty}) = 0$, as required. $\square$

Appendix 2: Generalization of Blair Sullivan's results

Let $a_k = \lfloor k^\beta \rfloor$, where $0 < \beta < 1$.

Lemma 6. Let $F_n(t) = \prod_{k=1}^n |\cos(ta_k)|$. Then

$$\int_{-\pi}^{\pi} F_n(t) dt = \frac{\sqrt{8\pi(1+2\beta)} + o(1)}{n^{\beta+1/2}} \quad \text{as } n \rightarrow \infty.$$

Remark 5. Note that for $\beta = 1$ we would have obtained the same result as in [6].

Proof. We will proceed in the spirit of [6]. Note that by symmetry

$$\int_{-\pi}^{\pi} F_n(t) dt = 2 \int_0^{\pi} F_n(t) dt = 2 \int_0^{\pi/2} F_n(t) dt + 2 \int_0^{\pi/2} F_n(\pi-t) dt = 4 \int_0^{\pi/2} F_n(t) dt$$

since $|\cos((\pi-t)a_k)| = |\cos(\pi a_k - ta_k)| = |\cos(ta_k)|$ as a_k is an integer. Let $\varepsilon > 0$ be very small and define

$$I_0 = \left[0, \frac{1}{n^{\beta+1/2-\varepsilon}}\right], \quad I_1 = \left[\frac{1}{n^{\beta+1/2-\varepsilon}}, \frac{1}{n^{\beta}}\right], \quad I_2 = \left[\frac{1}{n^{\beta}}, \frac{c_1}{n^{\beta}}\right], \quad I_3 = \left[\frac{c_1}{n^{\beta}}, \frac{\pi}{2}\right].$$

for some $c_1 > 1$ to be determined later. Then

$$\int_0^{\pi/2} F_n(t) dt = \int_{I_0} F_n(t) dt + \int_{I_1} F_n(t) dt + \int_{I_2} F_n(t) dt + \int_{I_3} F_n(t) dt.$$

We will show that the contribution of all the integrals except the first one is negligible, and estimate the value of the first one.

First, observe that when $0 \leq ta_k \leq \pi/2$ for all $k \leq n$, by the elementary inequality $|\cos u| \leq e^{-u^2/2}$ valid for $|u| \leq \pi/2$, we have

$$F_n(t) \leq \prod_{k=1}^n \exp\left(-\frac{t^2 a_k^2}{2}\right) = \exp\left(-\frac{t^2}{2} \sum_{k=1}^n a_k^2\right) = \exp\left(-\frac{t^2 n^{2\beta+1}(1+o(1))}{2(1+2\beta)}\right) \quad (22)$$

since $a_k^2 = k^{2\beta}(1+o(1))$.

Case 0: $t \in I_0$

Here $ta_k \leq n^{-1/2+\varepsilon} \ll 1$, hence for n large enough

$$\frac{(ta_k)^2}{2} \leq -\ln \cos(ta_k) = \frac{(ta_k)^2}{2} + \frac{(ta_k)^4}{12} + O((ta_k)^6) \leq (1+o(1)) \frac{(ta_k)^2}{2}$$

yielding $F_n(t) = \exp\left(-\frac{t^2 n^{2\beta+1} \rho_{n,t}}{2(1+2\beta)}\right)$ where $\rho_{n,t} = 1+o(1)$ for large n (compare with (22)). Since

$$\begin{aligned} \int_0^{n^{-\beta-1/2+\varepsilon}} \exp\left(-\frac{t^2 n^{2\beta+1} \rho}{2(1+2\beta)}\right) dt &= \frac{1}{n^{1/2+\beta}} \int_0^{n^{\varepsilon}} \exp\left(-\frac{s^2 \rho}{2(1+2\beta)}\right) ds \\ &= \frac{1}{n^{1/2+\beta}} \left[\sqrt{\frac{\pi(1+2\beta)}{2\rho}} + o(1) \right] \end{aligned}$$

by substituting $t = s/n^{1/2+\beta}$, we conclude that

$$\int_{I_0} F_n(t) dt = \frac{1}{n^{1/2+\beta}} \left[\sqrt{\pi(1/2 + \beta)} + o(1) \right].$$

Case 1: $t \in I_1$

Since $ta_k \leq 1 < \pi/2$, by (22) for some $C_2 > 0$ we have $F_n(t) \leq \exp\left(-\frac{n^{2\varepsilon}(1+o(1))}{2(1+2\beta)}\right) \leq e^{-C_2 n^{2\varepsilon}}$, so $\int_{I_1} F_n(t) dt \leq e^{-C_2 n^{2\varepsilon}}$, which decays faster than polynomially.

Case 2: $t \in I_2$

As in Case 2 in [6], we will use monotonicity of $F_n(t)$ in n . Let $r = c_1^{-1/\beta} \in (0, 1)$ then $\lfloor rn \rfloor^\beta \leq (rn)^\beta = \frac{n^\beta}{c_1}$, consequently by (22), since $t \leq \frac{c_1}{n^\beta}$,

$$F_{\lfloor rn \rfloor}(t) \leq \exp\left(-\frac{t^2 (rn)^{2\beta+1}(1+o(1))}{2(1+2\beta)}\right)$$

and since $F_n(t) \leq F_{\lfloor rn \rfloor}(t)$, we get a similar bound as in Case 1.

Case 3: $t \in I_3$

Let

$$k_m = \inf\{k \in \mathbb{Z}_+ : k^\beta \geq m\} = \lceil m^{1/\beta} \rceil, \quad m = 1, 2, \dots;$$

$$\Delta_m = k_{m+1} - k_m = \beta^{-1} m^\gamma + O(m^{1/\beta-2}) + \rho_0, \quad \gamma := \frac{1-\beta}{\beta},$$

where $|\rho_0| \leq 1$. Then

$$a_k = m \quad \text{if and only if} \quad k \in \{k_m, k_m + 1, \dots, k_m + \Delta_m - 1 (\equiv k_{m+1} - 1)\}.$$

For $n \in \mathbb{Z}_+$ let

$$m_n = \max\{m : k_m \leq n\} = n^\beta (1 + o(1)), \quad n \in [k_{m_n}, k_{m_n} + \Delta_{m_n} - 1].$$

By the inequality between the mean geometric and the mean arithmetic,

$$F_n(t) = \sqrt{\prod_{k=1}^n \cos^2(ta_k)} = \left(\sqrt[n]{\prod_{k=1}^n \cos^2(ta_k)} \right)^{n/2} \leq \left(\frac{\sum_{k=1}^n \cos^2(ta_k)}{n} \right)^{n/2}$$

$$= \left(\frac{1}{2} + \frac{U_n(t)}{2n} \right)^{n/2} \quad \text{where } U_n(t) = \sum_{k=1}^n \cos(2ta_k).$$

We will show that if t is not too small, then for some $0 \leq c < 1$ we have $U_n(t) \leq cn$ and hence $F_n(t) \leq \left(\frac{1+c}{2}\right)^{n/2}$. In order to do that, first note that

$$U_n(t) \leq \sum_{k=1}^{k_{m_n}} \cos(2ta_k) + (n - k_{m_n}) = \sum_{m=1}^{m_n} \Delta_m \cos(2tm) + (n - k_{m_n}).$$

Let $r \in (0, 1)$ and assume w.l.o.g. that rm_n is an integer. For $m \in [rm_n + 1, m_n]$ we have

$$A \leq \Delta_m \leq \bar{A} \text{ where } A = \beta^{-1}(rm_n)^\gamma + O(1), \quad \bar{A} = \beta^{-1}m_n^\gamma + O(1).$$

Consequently,

$$\begin{aligned} \sum_{m=rm_n+1}^{m_n} \Delta_m \cos(2tm) &\leq \sum_{m=rm_n+1}^{m_n} [\bar{A} \cdot \mathbf{1}_{\cos(2tm) \geq 0} + A \cdot \mathbf{1}_{\cos(2tm) < 0}] \cos(2tm) \\ &= \sum_{m=rm_n+1}^{m_n} [\bar{A} - A] \cos(2tm) \mathbf{1}_{\cos(2tm) \geq 0} + \sum_{m=rm_n+1}^{m_n} A \cos(2tm) \\ &\leq (1-r)m_n (\bar{A} - A) + A \sum_{m=rm_n+1}^{m_n} \cos(2tm) = (1-r)m_n (\bar{A} - A) \\ &\quad + A \left(\cos^2(rtm_n + t) - \cos^2(tm_n + t) + \frac{\cos t}{2 \sin t} [\sin(2t(m_n + 1)) - \sin(2t(rm_n + 1))] \right) \\ &\leq \frac{m_n^{1/\beta}}{\beta} (1-r) [1 - r^\gamma + O(m_n^{-\gamma})] + m_n^\gamma \left(\frac{r^\gamma}{\beta} + O(m_n^{-\gamma}) \right) \left(1 + \frac{1}{|\sin t|} \right) \end{aligned}$$

Hence, since $m_n^{1/\beta} = n + o(n)$,

$$\begin{aligned} U_n(t) &\leq \sum_{m=1}^{rm_n} \Delta_m + \sum_{m=rm_n+1}^{m_n} \Delta_m \cos(2tm) + (n - k_{m_n}) = k_{rm_n+1} + \sum_{m=rm_n+1}^{m_n} \Delta_m \cos(2tm) + O(\Delta_{m_n}) \\ &\leq r^{1/\beta} n + \frac{m_n^{1/\beta}}{\beta} (1-r) [1 - r^\gamma + O(m_n^{-\gamma})] + m_n^\gamma \left(\frac{r^\gamma}{\beta} + O(m_n^{-\gamma}) \right) \left(1 + \frac{1}{|\sin t|} \right) + O(m_n^{\gamma/\beta}) \\ &\leq n \left(r^{1/\beta} + \frac{(1-r)(1-r^\gamma)}{\beta} \right) + \frac{4n^{1-\beta} \beta^{-1} r^\gamma}{|\sin t|} + o(n) \end{aligned}$$

Consider now the the function

$$h(r, \beta) := r^{1/\beta} + \frac{(1-r)(1-r^\gamma)}{\beta} = r^{1/\beta} + \frac{(1-r)(1-r^{1/\beta-1})}{\beta}$$

and note that

$$h(1-\beta, \beta) = (1-\beta)^{1/\beta} + 1 - (1-\beta)^{1/\beta-1} = 1 - \beta(1-\beta)^{1/\beta-1} \leq 1 - e^{-1} \beta < 1 - \beta/3$$

since $\sup_{\beta \in (0,1)} (1-\beta)^{1/\beta-1} = e^{-1}$ by elementary calculus. So if we set $r = 1 - \beta \in (0, 1)$, by noting $t \leq 2 \sin t$ for $t \in [0, \pi/2]$, we conclude that $U_n(t) \leq \left(1 - \frac{\beta}{4}\right) n$ provided that $t \geq \frac{c_1}{n^\beta}$ for some $c_1 > 0$. Consequently, $\int_{I_3} F_n(t) dt \leq \left(1 - \frac{\beta}{8}\right)^{n/2}$ for large n , which converges to zero exponentially fast. $\square$

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Paper II



Transience of continuous-time conservative random walks

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Abstract

We consider two continuous-time generalizations of conservative random walks introduced in [10], an orthogonal and a spherically symmetrical one; the latter model is also known as *random flights*. For both models, we show the transience of the walks when $d \geq 2$ and the rate of changing of direction follows power law $t^{-\alpha}$, $0 < \alpha \leq 1$, or the law $(\ln t)^{-\beta}$ where $\beta > 2$.

AMS subject classification: 60G50, 60J05, 60J75

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1 Introduction

Conservative random walk (discrete time) was introduced in [10] as time- nonhomogeneous Markov chain X_n , $n = 1, 2, \dots$, defined as a process on $\mathbb{Z}^d$ such that for given (non-random) sequence $p_1, p_2, \dots$, where all $p_i \in (0, 1)$, the walk at time n with probability p_n randomly picks one of the $2d$ directions parallel to the axis, and otherwise it continues moving in the direction it was going before. This walk can be viewed as generalization of Gillies random walk [7]. A special interesting case studied in [10] was when $p_n \rightarrow 0$, and in particular when $p_n \sim n^{-\alpha}$ where $\alpha \in (0, 1]$; note that the case $\alpha > 1$ is trivial as the walk would make only finally many turns. The question of recurrence vs. transience of this walk was one of the main questions of that paper.

Similar processes were known in the literature under different names. Some of the earliest papers which mention the continuous process with memory of this type were probably [8, 12]. The term *persistent random* was used in [3, 4]; in these papers some very general criteria of recurrence vs. transience were investigated. A planar motion with just three directions was studied in [6]. A book on *Markov random flights* was recently published by Kolesnik [14]. *Planar random motions with drifts* with four directions/speeds, switching at Poisson times, were studied in [16]. Applications of telegraph processes to option pricing can be found in [18]. Characteristic functions of correlated random walks were studied in [5].

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The main difference between the conservative random walk and most of the models studied in the literature (except, perhaps, [19], which has a more applied focus) is that the underlying process of switching the direction is *time-inhomogeneous*, thus creating various new phenomena. The recurrence/transience of discrete space conservative random walk on $\mathbb{Z}^1$ was thoroughly studied in [9] (please also see the references therein), and we believe that the continuous time version in one dimension will have very similar features. The establishment of recurrence in one dimension is more or less equivalent to finding that $\limsup$ of the process is $+\infty$ and $\liminf$ of the process is $-\infty$, while in higher dimensions the situation is much more intricate. Hence, we concentrate on the case when the dimension of the space is at least 2, except Theorem 1 which deals with the embedded process.

Below we introduce formally the two versions of a *continuous-time* conservative random walk on $\mathbb{R}^d$, $d \geq 1$.

Model A (orthogonal model). Let $\lambda(t)$ be a non-negative function such that

$$\begin{aligned} \Lambda(T) &= \int_0^T \lambda(t) dt < \infty \quad \text{for all } T \geq 0; \\ \lim_{T \rightarrow \infty} \Lambda(T) &= \infty. \end{aligned} \tag{1}$$

Let $\tau_1 < \tau_2 < \dots$ be the consecutive points of a non-homogeneous Poisson point process (PPP for short) on $[0, \infty)$ with rate $\lambda(t)$, and $\tau_0 = 0$. Then conditions (1) will guarantee that there will be finitely many τ_i 's in every finite interval and that $\tau_n \rightarrow \infty$. Let $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2$ be an i.i.d. sequence of vectors, each of which has a uniform distribution on the set of $2d$ unit vectors $\{\pm \mathbf{e}_1, \pm \mathbf{e}_2, \dots, \pm \mathbf{e}_d\}$ in $\mathbb{R}^d$.

The (orthogonal, continuous-time) conservative walk generated by the rate function $\lambda(\cdot)$ is a process $Z(t)$, $t \geq 0$, in $\mathbb{R}^d$, $d \geq 1$, such that $Z(0) = 0$ and at each time τ_k , $k \geq 0$, the walk starts moving in the direction $\mathbf{f}_k$, and keeps moving in this direction until time τ_{k+1} , when it updates its direction. Formally, define

$$N(t) = \sup\{k \geq 0 : \tau_k \leq t\},$$

as the number of points of the PPP by time t , then

$$Z(t) = \sum_{k=0}^{N(t)-1} (\tau_{k+1} - \tau_k) \mathbf{f}_k + (t - \tau_{N(t)}) \mathbf{f}_{N(t)}.$$

We can also define the embedded process $W_n = Z(\tau_n)$ so that $W(0) = 0$ and for $n \geq 1$

$$W_n = \sum_{k=0}^{n-1} (\tau_{k+1} - \tau_k) \mathbf{f}_k.$$

The process $Z(t)$ can be viewed as a continuous equivalent of the conservative random walk introduced in [10].

Model B (von Mises–Fisher model). This model is defined similarly to the

previous one, except that now random vectors $\mathbf{f}_k$, $k = 1, 2, \dots$, have a uniform distribution on a d -dimensional unit sphere $\mathcal{S}^{d-1}$, often called *von Mises–Fisher distribution*, instead of just on $2d$ unit vectors of $\mathbb{R}^d$.

Please note that this model is similar to the “random flights” model studied e.g. in [17], however, their results are only for a time-homogeneous Poisson process, unlike our case.

The results that we obtain in the current paper are somewhat different for the two models, however, since they share a lot of common features, certain statements will hold for both of them. The main goal is to establish transience vs. recurrence of the walks, defined as follows.

Definition 1. Let $\rho \geq 0$. We say that the walk $Z(t)$ is ρ -recurrent, if there is an infinite sequence of times $t_1 < t_2 < \dots$ converging to infinity, such that $Z(t_i) \in [-\rho, \rho]^d$ for all $i = 1, 2, \dots$.

We say that the walk $Z(t)$ is transient, if it is not ρ -recurrent for any $\rho > 0$, i.e. equivalently $\lim_{t \rightarrow \infty} \|Z(t)\| = \infty$.

Recurrence and transience of the embedded process W_n are defined analogously, with the exception that instead of $t_1, t_2, t_3, \dots$ in the above definition, we have a strictly increasing sequence of positive integers n_i , $i = 1, 2, \dots$.

Remark 1. Note that a priori it is unclear if transience and recurrence are zero-one events, neither can we easily rule out a possibility of “intermediate” situations (e.g. ρ -recurrence only for some ρ).

Our main result, which shows transience for two type of rates, are presented in Theorems 1, 2, 3 and 4.

2 Preliminaries

Throughout the paper, we will use the following notations. We write $X \sim \text{Poi}(\mu)$ when X has a Poisson distribution with parameter $\mu > 0$. For any set A , $|A|$ denotes the cardinality of the set A , and for $x \in \mathbb{R}^d$, $\|x\|$ denotes the usual Euclidean norm of x .

First, we state Kesten’s generalization of the Kolmogorov-Rogozin inequality. Let $S_n = \xi_1 + \dots + \xi_n$ where ξ_i are independent, and for any random variable Y define $Q(Y; a) = \sup_x \mathbb{P}(Y \in [x, x + a])$.

Lemma 1 ([13]). *There exists $C > 0$ such that for any real numbers $0 < a_1, \dots, a_n \leq 2L$ one has*

$$Q(S_n; L) \leq \frac{CL \sum_{i=1}^n a_i^2 (1 - Q(\xi_i; a_i)) Q(\xi_i; a_i)}{[\sum_{i=1}^n a_i^2 (1 - Q(\xi_i; a_i))]^{3/2}}$$

Secondly, if $D_1, \dots, D_m$ is a sequence of independent events each with probability p , and $\varepsilon > 0$, and $N_D(m) = \text{card}(\{i \in \{1, \dots, m\} : D_i \text{ occurs}\}) = \sum_{i=1}^m \mathbf{1}_{D_i}$, then

$$\mathbb{P}(|N_D(m) - pm| \geq \varepsilon m) \leq 2 \exp(-2\varepsilon^2 m); \quad (2)$$

by Hoeffding’s inequality (see e.g. [11]).

Suppose we have an inhomogeneous PPP with rate

$$\lambda(t) = \frac{1}{t^\alpha}, \quad t > 0, \quad (3)$$

where $0 < \alpha < 1$ is constant; thus $\Lambda(T) = \int_0^T \lambda(t) dt = \frac{T^{1-\alpha}}{1-\alpha}$ and conditions (1) are fulfilled. Let $0 < \tau_1 < \tau_2 < \dots$ denote the points of the PPP in the increasing order.

The following statement is probably known, but for the sake of completeness, we provide its short proof.

Claim 1. *Let Z be a Poisson random variable with rate $\mu > 0$. Then*

$$\begin{aligned} \mathbb{P}\left(Z \geq \frac{3\mu}{2}\right) &\leq e^{-\frac{3\ln(3/2)-1}{2}\mu} = e^{-0.108\dots\mu}, \\ \mathbb{P}\left(Z \leq \frac{\mu}{2}\right) &\leq e^{-\frac{1-\ln 2}{2}\mu} = e^{-0.153\dots\mu}. \end{aligned}$$

Proof. By Markov inequality, since $\mathbb{E}e^{uZ} = e^{\mu(e^u-1)}$, we have for $u > 0$

$$\mathbb{P}\left(Z \geq \frac{3\mu}{2}\right) \leq \mathbb{P}\left(e^{uZ} \geq e^{\frac{3\mu u}{2}}\right) \leq e^{-\frac{3\mu u}{2}} \mathbb{E}e^{uZ} = e^{-\mu\left[\frac{3u}{2} - e^u + 1\right]}.$$

Setting $u = \ln(3/2)$ yields the first inequality in the claim.

For the second inequality, we use

$$\mathbb{P}\left(Z \leq \frac{\mu}{2}\right) \leq \mathbb{P}\left(e^{-uZ} \geq e^{-\frac{\mu u}{2}}\right) \leq e^{\frac{\mu u}{2}} \mathbb{E}e^{-uZ} = e^{-\mu\left[-\frac{u}{2} - e^{-u} + 1\right]}.$$

Now let $u = \ln 2$. □

Lemma 2. *Suppose that the rate of the PPP is given by (3). For some $c_1 > c_0 > 0$, depending on α only, we have*

$$\mathbb{P}\left(\tau_k \leq c_0 k^{\frac{1}{1-\alpha}}\right) \leq e^{-k/15}, \quad \mathbb{P}\left(\tau_k \geq c_1 k^{\frac{1}{1-\alpha}}\right) \leq e^{-k/15}.$$

Proof. Recall that $N(s)$ denote the number of points of the PPP by time s . Then $N(s) \sim \text{Poi}(\Lambda(s))$ and

$$\mathbb{P}(\tau_k \leq s) = \mathbb{P}(N(s) \geq k).$$

Let $T_n = \Lambda^{(-1)}(n) = {}^{1-\alpha}\sqrt{(1-\alpha)n}$. Noting that $N(T_n) \sim \text{Poi}(n)$ for all n , we have

$$\mathbb{P}(\tau_k \leq T_{2k/3}) = \mathbb{P}(N(T_{2k/3}) \geq k) = \mathbb{P}\left(Z \geq \frac{3}{2}\mu\right) \leq e^{-0.07213\dots k}.$$

by Claim 1, with $Z \sim \text{Poi}(\mu)$ where $\mu = 2k/3$. Similarly,

$$\mathbb{P}(\tau_k \geq T_{2k}) = \mathbb{P}(N(T_{2k}) \leq k) \leq \mathbb{P}\left(Z \leq \frac{1}{2}\mu\right) \leq e^{-0.30685\dots k}.$$

by Claim 1, with $Z \sim \text{Poi}(\mu)$ where $\mu = 2k$. Note that $0.07213 > \frac{1}{15}$, $0.30685 > \frac{1}{15}$.

Now the statement follows with $c_0 = {}^{1-\alpha}\sqrt{2(1-\alpha)/3}$ and $c_1 = {}^{1-\alpha}\sqrt{2(1-\alpha)}$. □

3 Analysis of Model A

Theorem 1. *Let $d = 1$, $\alpha \in (1/3, 1)$, and the rate is given by (3). Then the embedded walk W_n is transient a.s.*

Proof. Assume w.l.o.g. that n is even. We will show that for any $\rho > 0$ the walk W_n visits $[-\rho, \rho]$ finitely often a.s. With probabilities close to one, both events

$$\begin{aligned}\mathcal{E}_1 &= \left\{ \tau_{n/2} \geq c_0 (n/2)^{\frac{1}{1-\alpha}} \right\}, \\ \mathcal{E}_2 &= \left\{ \tau_n \leq c_1 n^{\frac{1}{1-\alpha}} \right\}\end{aligned}\tag{4}$$

occur; indeed,

$$\mathbb{P}(\mathcal{E}_1^c) \leq e^{-n/30}, \quad \mathbb{P}(\mathcal{E}_2^c) \leq e^{-n/15}\tag{5}$$

by Lemma 2.

Since rate $\lambda(t) = t^{-\alpha}$ of the Poisson process is monotonously decreasing, random variables $\tau_i - \tau_{i-1}$, $n/2 \leq i \leq n$, under the condition stated in the event $\mathcal{E}_1$, are stochastically larger than i.i.d. exponential random variables ζ_i with rates equal to $[c_0(n/2)^{\frac{1}{1-\alpha}}]^\alpha = \tilde{c}_0 n^{-\frac{\alpha}{1-\alpha}}$ for some $\tilde{c}_0 > 0$. For ζ , there exists $\beta = \beta(c_0, \alpha) > 0$ such that

$$\mathbb{P}\left(\zeta_i > \beta n^{\frac{\alpha}{1-\alpha}}\right) = \exp\left(-\tilde{c}_0 n^{-\frac{\alpha}{1-\alpha}} \cdot \beta n^{\frac{\alpha}{1-\alpha}}\right) = e^{-\tilde{c}_0 \beta} = \frac{2}{3}.$$

Let

$$I_n = \left\{ i \in [n/2, n] : i \text{ is even, } \tau_i - \tau_{i-2} > \beta n^{\frac{\alpha}{1-\alpha}} \right\},$$

note that $\text{card}(I_n) \leq n/4$. Then, since $\tau_i - \tau_{i-2} > \tau_i - \tau_{i-1}$, by stochastic monotonicity and Hoeffding's inequality (2) with $m = \frac{n}{4}$, $p = \frac{2}{3}$, and $\varepsilon = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

$$\mathbb{P}\left(\text{card}(I_n) < \frac{1}{2} \cdot \frac{n}{4} \mid \mathcal{E}_1\right) \leq \mathbb{P}\left(\text{card}\left(\left\{i \in [n/2, n] : i \text{ is even, } \zeta_i > \beta n^{\frac{\alpha}{1-\alpha}}\right\}\right) < \frac{n}{8}\right) \leq 2e^{-n/72}.\tag{6}$$

Let

$$J_n = \{i \in I_n : \mathbf{f}_{i-1} = -\mathbf{f}_{i-2}\}.$$

Since $\mathbf{f}_i$ are i.i.d. and independent of $\{\tau_1, \tau_2, \dots\}$, and $\mathbb{P}(\mathbf{f}_{i-1} = -\mathbf{f}_i) = 1/2$, on the event $\{\text{card}(I_n) \geq n/8\}$ we have by (2) with $m = \text{card}(I_n)$, $p = 1/2$, $\varepsilon = 1/34$

$$\begin{aligned}\mathbb{P}\left(\text{card}(J_n) < \frac{n}{17} \mid \text{card}(I_n) \geq \frac{n}{8}\right) &\leq \mathbb{P}\left(\left|\text{card}(J_n) - \frac{\text{card}(I_n)}{2}\right| > \frac{\text{card}(I_n)}{34} \mid \text{card}(I_n) \geq \frac{n}{8}\right) \\ &\leq 2 \exp\left\{-2 \cdot \frac{1}{34^2} \cdot \frac{n}{8}\right\} = 2e^{-c_* n}\end{aligned}\tag{7}$$

where $c_* = \frac{1}{4624}$.

Our proof will rely on conditioning over the *even stopping times*, that is, on the event

$$\mathcal{D} = \mathcal{D}_0 \cap \mathcal{D}_1 \cap \mathcal{E}_2,$$

where

$$\begin{aligned}\mathcal{D}_0 &= \{\tau_{n/2} = t_{n/2}, \tau_{n/2+2} = t_{n/2+2}, \dots, \tau_n = t_n\}, \\ \mathcal{D}_1 &= \{\text{card}(J_n) \geq n/17\} \cap \mathcal{E}_1\end{aligned}\tag{8}$$

for some strictly increasing sequence $0 < t_{n/2} < t_{n/2+2} < \dots < t_n$. Note that

$$\begin{aligned}\mathbb{P}(\mathcal{D}_1^c) &\leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}\left(\text{card}(J_n) < \frac{n}{17} \mid \mathcal{E}_1\right) \\ &\leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}\left(\text{card}(J_n) < \frac{n}{17} \mid \text{card}(I_n) \geq \frac{n}{8}, \mathcal{E}_1\right) + \mathbb{P}\left(\text{card}(I_n) < \frac{n}{8} \mid \mathcal{E}_1\right) \\ &\leq e^{-n/30} + 2e^{-c_* n} + 2e^{-n/72} \leq 5e^{-c_* n}\end{aligned}\tag{9}$$

by (5), (6), and (7).

We denote the i^{th} step of the embedded walk by $X_i = W_i - W_{i-1}$, $i = 1, 2, \dots$, and

$$\xi_i := X_{i-1} + X_i = (\tau_{i-1} - \tau_{i-2})\mathbf{f}_{i-2} + (\tau_i - \tau_{i-1})\mathbf{f}_{i-1}.$$

Conditioned on $\mathcal{D}$, random variables $\xi_2, \xi_4, \xi_6, \dots$ are then independent.

Lemma 3. *For $i \in J_n$ we have*

$$\sup_{x \in \mathbb{R}} \mathbb{P}(\xi_i \in [x - \rho, x + \rho] \mid \mathcal{D}) \leq \frac{c\rho}{n^{\frac{\alpha}{1-\alpha}}}$$

for some $c = c(\alpha) > 0$.

Proof of Lemma 3. Given $\tau_{i-2} = t_{i-2}$ and $\tau_i = t_i$, τ_{i-1} has the distribution of the only point of the PPP on $[t_{i-2}, t_i]$ with rate $\lambda(t)$ conditioned on the fact that there is exactly one point in this interval. Hence the conditional density of τ_{i-1} is given by

$$f_{\tau_{i-1}|\mathcal{D}}(x) = \begin{cases} \frac{\lambda(t)}{\int_{t_{i-2}}^{t_i} \lambda(u) du}, & \text{if } x \in [t_{i-2}, t_i]; \\ 0, & \text{otherwise.} \end{cases}$$

Assume w.l.o.g. that $X_{i-1} > 0 > X_i$ (recall that $i \in J_n$). Then

$$\xi_i = [\tau_{i-1} - \tau_{i-2}] - [\tau_i - \tau_{i-1}] = 2\tau_{i-1} - (t_{i-2} + t_i)$$

so that the maximum of the conditional density of ξ_i equals one-half of the maximum of the conditional density of τ_{i-1} . At the same time, since λ is a decreasing function,

$$\begin{aligned}\sup_{x \in \mathbb{R}} f_{\tau_{i-1}|\mathcal{D}}(x) &= \frac{\lambda(t_{i-2})}{\int_{t_{i-2}}^{t_i} \lambda(u) du} \leq \frac{\lambda(t_{i-2})}{(t_i - t_{i-2})\lambda(t_i)} \leq \frac{\lambda(t_{i-2})}{\beta n^{\frac{\alpha}{1-\alpha}} \lambda(t_i)} = \frac{1}{\beta n^{\frac{\alpha}{1-\alpha}}} \cdot \left(\frac{t_i}{t_{i-2}}\right)^\alpha \\ &\leq \frac{1}{\beta n^{\frac{\alpha}{1-\alpha}}} \cdot \left(\frac{\tau_n}{\tau_{n/2}}\right)^\alpha \leq \frac{1}{\beta n^{\frac{\alpha}{1-\alpha}}} \cdot \left(\frac{c_1}{c_0 2^{-\frac{1}{1-\alpha}}}\right)^\alpha\end{aligned}$$

since the events $\mathcal{E}_1$ and $\mathcal{E}_2$ occur. This implies the stated result with $c = \beta^{-1} (c_0^{-1} c_1^{1-\alpha} \sqrt[1-\alpha]{2})^\alpha$. $\square$

Now we divide W_n into two portions:

$$\begin{aligned} A &= \sum_{i \in J_n} \xi_i; \\ B &= W_n - A = \sum_{i \in \{2, 4, \dots, n\} \setminus J_n} \xi_i; \end{aligned}$$

Lemma 4. *We have*

$$\sup_{x \in \mathbb{R}} \mathbb{P}(A \in [x - \rho, x + \rho] \mid \mathcal{D}) \leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}}$$

for some $C = C(\alpha, \rho) > 0$.

Proof of Lemma 4. The result follows immediately from Lemma 1 with $a_i \equiv 2\rho = L$, using Lemma 3, and the fact that $\text{card}(J_n) \geq n/17$. $\square$

So,

$$\begin{aligned} \mathbb{P}(|W_n| \leq \rho \mid \mathcal{D}) &= \mathbb{P}(A + B \in [-\rho, \rho] \mid \mathcal{D}) = \int \mathbb{P}(A + b \in [-\rho, \rho] \mid \mathcal{D}) f_{B|\mathcal{D}}(b) db \\ &\leq \int \sup_x \mathbb{P}(A \in [x - \rho, x + \rho] \mid \mathcal{D}) f_{B|\mathcal{D}}(b) db \leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}} \end{aligned}$$

where $f_{B|\mathcal{D}}(\cdot)$ is the density of B conditional on $\mathcal{D}$.

Finally, using (5) and (9) we get

$$\mathbb{P}(W_n \in [-\rho, \rho]) \leq \mathbb{P}(W_n \in [-\rho, \rho] \mid \mathcal{D}) + \mathbb{P}(\mathcal{D}_1^c) + \mathbb{P}(\mathcal{E}_2^c) \leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}} + 5e^{-c_* n} + e^{-n/15}$$

which is summable over n , so we can apply the Borel-Cantelli lemma to show that $\{|W_n| \leq \rho\}$ occurs finitely often a.s. $\square$

Theorem 2. *Let $d \geq 2$, $\alpha \in (0, 1)$, and the rate of the PPP is given by (3). Then $Z(t)$ is transient a.s.*

Remark 2. *The above result holds also for $\alpha = 1$, and the proof is more or less identical to that of Theorem 5.2 in [10], once we establish that a.s. $\tau_n > e^{cn}$ for some $c > 0$ and all large n ; the latter follows from the arguments similar to that of Lemma 2.*

Proof. We will provide the proof only for the case $d = 2$ and $\rho = 1$; it can be easily generalized for all $d \geq 3$ and $\rho > 0$. Denote the coordinates of the embedded walk by X_n and Y_n , thus $W_n = (X_n, Y_n) \in \mathbb{R}^2$. Fix some small $\varepsilon > 0$ and consider the event

$$\mathcal{R}_n = \{Z(t) \in [-1, 1]^2 \text{ for some } t \in (\tau_n, \tau_{n+1}]\}$$

We will show that events $\mathcal{R}_n$ occur finitely often a.s., thus ensuring the transience of $Z(t)$.

For the walk $Z(t)$ to hit $[-\rho, \rho]^2$ between times τ_n and τ_{n+1} , we have to have either

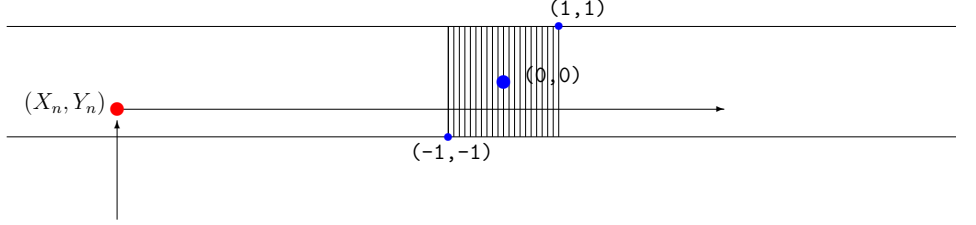


Figure 1: Model A: recurrence of conservative random walk on $\mathbb{R}^2$

- $|X_n| \leq \rho$, $\mathbf{f}_n = -\text{sign}(Y_n)\mathbf{e}_2$, and $\tau_{n+1} - \tau_n \geq |Y_n| - 1$, or
- $|Y_n| \leq \rho$, $\mathbf{f}_n = -\text{sign}(X_n)\mathbf{e}_1$, and $\tau_{n+1} - \tau_n \geq |X_n| - 1$

(see Figure 1.)

Using the fact that $\mathbf{f}_n$ are independent of anything, we get

$$\mathbb{P}(\mathcal{R}_n) \leq \frac{1}{4} \mathbb{P}(|X_n| \leq \rho, \tau_{n+1} - \tau_n \geq |Y_n| - 1) + \frac{1}{4} \mathbb{P}(|Y_n| \leq \rho, \tau_{n+1} - \tau_n \geq |X_n| - 1).$$

We will show that $\mathbb{P}(|X_n| \leq \rho, |Y_n| \leq \tau_{n+1} - \tau_n + 1)$ is summable in n (and the same holds for the other summand by symmetry), hence by the Borel-Cantelli lemma that $\mathcal{R}_n$ occur finitely often a.s.

Lemma 5. *Assume that $\varepsilon \in (0, 1)$. Then there exists $c_1^* = c_1^*(\varepsilon)$ such that for all sufficiently large n*

$$\mathbb{P}(\tau_{n+1} - \tau_n + 1 \geq n^{\frac{\alpha}{1-\alpha} + \varepsilon}) \leq 3e^{-c_1^* n^\varepsilon}$$

Proof of Lemma 5. For $s, t \geq 0$ we have

$$\mathbb{P}(\tau_{n+1} - \tau_n \geq s \mid \tau_n = t) = \exp\{-[\Lambda(t+s) - \Lambda(t)]\} = \exp\left\{-\frac{(t+s)^{1-\alpha} - t^{1-\alpha}}{1-\alpha}\right\},$$

so if $t < c_1 n^{\frac{1}{1-\alpha}}$, where c_1 is the constant from Lemma 2, and $s = n^{\frac{\alpha}{1-\alpha} + \varepsilon} - 1 = o\left(n^{\frac{1}{1-\alpha}}\right)$, we have

$$\begin{aligned} \mathbb{P}(\tau_{n+1} - \tau_n \geq s \mid \tau_n = t) &\leq \exp\left\{-\frac{(c_1 n^{\frac{1}{1-\alpha}} + s)^{1-\alpha} - c_1^{1-\alpha} n}{1-\alpha}\right\} \\ &\leq \exp\left\{-\frac{nc_1^{1-\alpha}}{1-\alpha} \left[\left(1 + \frac{n^{\frac{\alpha}{1-\alpha} + \varepsilon} - 1}{c_1 n^{\frac{1}{1-\alpha}}}\right)^{1-\alpha} - 1\right]\right\} = \exp\left\{-\frac{n^\varepsilon(1 + o(1))}{c_1^\alpha}\right\} \end{aligned}$$

Consequently, using Lemma 2,

$$\begin{aligned} \mathbb{P}(\tau_{n+1} - \tau_n \geq n^{\frac{\alpha}{1-\alpha} + \varepsilon} - 1) &\leq \mathbb{P}(\tau_{n+1} - \tau_n \geq n^{\frac{\alpha}{1-\alpha} + \varepsilon} - 1 \mid \tau_n < c_1 n^{\frac{1}{1-\alpha}}) + \mathbb{P}(\tau_n \geq c_1 n^{\frac{1}{1-\alpha}}) \\ &\leq e^{-c_1^* n^\varepsilon} + e^{-n/15} \leq 2e^{-c_1^* n^\varepsilon} \end{aligned} \tag{10}$$

for some $c_1^* \in (0, 1/15)$. $\square$

Now we will modify the proof of Theorem 1 slightly to adapt to our needs. Let the events $\mathcal{E}_1$ and $\mathcal{E}_2$ be the same as in the proof of this theorem. We will also use the set I_n , but instead of J_n we will introduce the sets

$$\begin{aligned} J_n^1 &= \{i \in I_n : \mathbf{f}_{i-1} = -\mathbf{f}_{i-2}, \mathbf{f}_{i-1} \in \{\mathbf{e}_1, -\mathbf{e}_1\}\}; \\ J_n^2 &= \{i \in I_n : \mathbf{f}_{i-1} = -\mathbf{f}_{i-2}, \mathbf{f}_{i-1} \in \{\mathbf{e}_2, -\mathbf{e}_2\}\}. \end{aligned}$$

Similarly to the proof of Theorem 1, inequality (9), we immediately obtain that

$$\begin{aligned} \mathbb{P}(\text{card}(J_n^1) \leq n/34) &\leq \mathbb{P}(\text{card}(J_n^1) \leq n/34 \mid \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_1^c) \leq 6e^{-c'_*n}, \\ \mathbb{P}(\text{card}(J_n^2) \leq n/34) &\leq \mathbb{P}(\text{card}(J_n^2) \leq n/34 \mid \mathcal{E}_1) + \mathbb{P}(\mathcal{E}_1^c) \leq 6e^{-c'_*n} \end{aligned} \quad (11)$$

for some $c'_* > 0$.

Fix a deterministic sequence of unit vectors $\mathbf{g}_1, \dots, \mathbf{g}_n$ such that each $\mathbf{g}_i \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. We now also define the event

$$\begin{aligned} \mathcal{D} &= \{\mathbf{f}_1 = \mathbf{g}_1, \dots, \mathbf{f}_n = \mathbf{g}_n\} \cap \{\tau_k = t_k \text{ for all } k \leq n : k \text{ is even or } \mathbf{f}_k \perp \mathbf{f}_{k+1}\} \\ &\cap \left\{ \text{card}(J_n^1) \geq \frac{n}{34} \right\} \cap \left\{ \text{card}(J_n^2) \geq \frac{n}{34} \right\} \cap \mathcal{E}_1 \cap \mathcal{E}_2. \end{aligned}$$

Therefore, repeating the previous arguments for each of the horizontal and vertical components of W_n we immediately obtain

$$\begin{aligned} \mathbb{P}(|X_n| \leq 1 \mid \mathcal{D}) &\leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}}; \\ \sup_{x \in \mathbb{R}} \mathbb{P}(|Y_n - x| \leq 1 \mid \mathcal{D}) &\leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}}. \end{aligned}$$

The second inequality implies that

$$\mathbb{P}(|Y_n| \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon} \mid \mathcal{D}) \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon} \cdot \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}} = \frac{C}{n^{\frac{1}{2} - \varepsilon}}.$$

Now, by Lemma 5

$$\begin{aligned} \mathbb{P}(|X_n| \leq 1, |Y_n| \leq \tau_{n+1} - \tau_n + 1) &\leq \mathbb{P}\left(|X_n| \leq 1, |Y_n| \leq \tau_{n+1} - \tau_n + 1 \mid \tau_{n+1} - \tau_n + 1 \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon}\right) \\ &\quad + \mathbb{P}\left(\tau_{n+1} - \tau_n + 1 \geq n^{\frac{\alpha}{1-\alpha} + \varepsilon}\right) \\ &\leq \mathbb{P}\left(|X_n| \leq 1, |Y_n| \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon}\right) + 3e^{-c_1^* n^\varepsilon}. \end{aligned}$$

Observing that X_n and Y_n are actually independent given $\mathcal{D}$, we conclude that

$$\begin{aligned} \mathbb{P}\left(|X_n| \leq 1, |Y_n| \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon}\right) &\leq \mathbb{P}\left(|X_n| \leq 1, |Y_n| \leq n^{\frac{\alpha}{1-\alpha} + \varepsilon} \mid \mathcal{D}\right) \\ &\quad + \mathbb{P}\left(\text{card}(J_n^1) < \frac{n}{34} \text{ or } \text{card}(J_n^2) < \frac{n}{34}\right) + \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c) \\ &\leq \frac{C}{n^{\frac{\alpha}{1-\alpha} + \frac{1}{2}}} \cdot \frac{C}{n^{\frac{1}{2} - \varepsilon}} + 12e^{-c'_*n} + e^{-n/15} + e^{-n/15} = \frac{C^2 + o(1)}{n^{1 + [\frac{\alpha}{1-\alpha} - \varepsilon]}} \end{aligned}$$

by (5) and (11). Assuming $\varepsilon \in (0, \frac{\alpha}{1-\alpha})$, the RHS is summable, and thus $\mathbb{P}(|X_n| \leq \rho, |Y_n| - 1 \leq \tau_n)$ is also summable in n . $\square$

Theorem 3. Let $d \geq 2$, and the rate be given by

$$\lambda(t) = \begin{cases} \frac{1}{(\ln t)^\beta}, & t \geq e; \\ 0, & \text{otherwise.} \end{cases} \quad (12)$$

Then $Z(t)$ is transient as long as $\beta > 2$.

Proof. The proof is analogous to the proof of Theorem 2; we will only indicate how that prove should be modified to this case. As before, we assume that $\rho = 1$, $d = 2$, without loss of generality. First, we prove the following

Lemma 6. Let

$$\Lambda(T) = \int_0^T \lambda(s) ds = \int_e^T \frac{ds}{(\ln s)^\beta}.$$

Then

$$\lim_{T \rightarrow \infty} \frac{\Lambda(T)}{T(\ln T)^{-\beta}} = 1$$

Proof of Lemma 6. Fix an $\epsilon \in (0, 1)$, then

$$\Lambda(T) = \int_e^T \frac{ds}{(\ln s)^\beta} = \int_e^{T^{1-\epsilon}} \frac{ds}{(\ln s)^\beta} + \int_{T^{1-\epsilon}}^T \frac{ds}{(\ln s)^\beta} < T^{1-\epsilon} + \frac{T}{(\ln T)^\beta(1-\epsilon)^\beta}.$$

At the same time, trivially,

$$\Lambda(T) > \frac{T - e}{(\ln T)^\beta}.$$

Now the limit of the ratio of the upper and the lower bounds of $\Lambda(T)$ can be made arbitrarily close to 1, by choosing a small enough ϵ . Hence the statement of the lemma follows. $\square$

The rest of the proof goes along the same lines as that of Theorem 2

- (1) First note that for some $c > 0$ the event $\{\tau_n \leq cn(\ln n)^\beta\}$ occurs finitely often almost surely. Indeed,

$$\tilde{\Lambda} = \Lambda(cn(\ln n)^\beta) = \frac{cn(\ln n)^\beta}{[\ln(cn(\ln n)^\beta)]^\beta}(1+o(1)) = \frac{(\ln n)^\beta}{(1+o(1))(\ln n)^\beta}cn(1+o(1)) = (1+o(1))cn$$

by Lemma 6, and thus

$$\begin{aligned} \mathbb{P}(\tau_n \leq cn(\ln n)^\beta) &= \mathbb{P}(N(cn(\ln n)^\beta) \geq n) = e^{-\tilde{\Lambda}} \left(\frac{\tilde{\Lambda}^n}{n!} + \frac{\tilde{\Lambda}^{n+1}}{(n+1)!} + \frac{\tilde{\Lambda}^{n+2}}{(n+2)!} + \dots \right) \\ &= e^{-\tilde{\Lambda}} \frac{\tilde{\Lambda}^n}{n!} \left(1 + \frac{\tilde{\Lambda}}{(n+1)} + \frac{\tilde{\Lambda}^2}{(n+1)(n+2)} + \dots \right) \\ &\leq e^{-\tilde{\Lambda}} \frac{\tilde{\Lambda}^n}{n!} \left(1 + \frac{\tilde{\Lambda}}{1!} + \frac{\tilde{\Lambda}^2}{2!} + \dots \right) = \frac{\tilde{\Lambda}^n}{n!} = \frac{[(1+o(1))cn]^n}{n!} = \mathcal{O}\left(\frac{[(1+o(1))cn]^n}{n^n e^{-cn}}\right) \end{aligned}$$

This quantity is summable as long as $ce < 1$, and the statement follows from the Borel-Cantelli lemma.

- (2) For any positive ϵ , the event $\{\tau_n \geq c^* n (\ln n)^\beta\}$, where $c^* = 1 + \epsilon$, occurs finitely often, almost surely. Indeed,

$$\bar{\Lambda} := \Lambda(c^* n (\ln n)^\beta) = \frac{c^* n (\ln n)^\beta}{[\ln(c^* n (\ln n)^\beta)]^\beta} (1 + o(1)) = c^* n (1 + o(1)) \geq (1 + \epsilon/2)n$$

for large enough n by Lemma 6. Hence, since $\bar{\Lambda} > n$,

$$\begin{aligned} \mathbb{P}(\tau_n \geq c^* n (\ln n)^\beta) &= \mathbb{P}(N(c^* n (\ln n)^\beta) \leq n) = e^{-\bar{\Lambda}} \left(1 + \bar{\Lambda} + \frac{\bar{\Lambda}^2}{2!} + \cdots + \frac{\bar{\Lambda}^n}{n!}\right) \\ &\leq e^{-\bar{\Lambda}} (n+1) \frac{\bar{\Lambda}^n}{n!} = e^{-\bar{\Lambda}} (n+1) \frac{\bar{\Lambda}^n}{n^n e^{-n} \sqrt{2\pi n}} (1 + o(1)) \\ &= (1 + o(1)) \sqrt{\frac{n}{2\pi}} \exp \left\{ -n \left[\frac{\bar{\Lambda}}{n} - 1 - \ln \frac{\bar{\Lambda}}{n} \right] \right\} \end{aligned}$$

which is summable in n , as $c^* = 1 + \epsilon$ implies that the expression in square brackets is strictly positive (this follows from an easy fact that $c - 1 - \ln c > 0$ for $c > 1$). Hence, $\{\tau_n \geq c^* n (\ln n)^\beta\}$ happens finitely often, almost surely, as stated.

- (3) The event $\{\tau_{n+1} - \tau_n \geq 2(\ln n)^{1+\beta}\}$ occurs finitely often, almost surely. This holds because, for all sufficiently large n , we have $\tau_{n+1} \leq c^*(n+1)(\ln(n+1))^\beta$, hence for $t \leq \tau_{n+1}$

$$\lambda(t) \geq \frac{1}{[\ln(c(n+1)(\ln(n+1))^\beta)]^\beta} = \frac{1 + o(1)}{(\ln n)^\beta}$$

and thus $(\tau_{n+1} - \tau_n)$ is stochastically smaller than an exponential random variable $\mathcal{E}$ with parameter $\frac{1 + o(1)}{(\ln n)^\beta}$. So for all sufficiently large n ,

$$\mathbb{P}(\tau_{n+1} - \tau_n \geq 2(\ln n)^{1+\beta}) \leq \mathbb{P}(\mathcal{E} \geq 2(\ln n)^{1+\beta}) = \frac{1}{n^{2-o(1)}} \quad (13)$$

which is summable in n , and we can apply the Borel-Cantelli lemma.

Using previous arguments for horizontal and vertical components, we obtain,

$$\begin{aligned} \mathbb{P}(|X_n| \leq 1) &\leq \frac{C}{\sqrt{n}(\ln n)^\beta} \\ \mathbb{P}(|Y_n| \leq (\ln n)^{1+\beta}) &\leq (\ln n)^{1+\beta} \times \frac{C}{\sqrt{n}(\ln n)^\beta} = \frac{C \ln n}{\sqrt{n}} \end{aligned}$$

Again, similar to Theorem 2, the event $\{|X_n| \leq \rho, |Y_n| \leq 2(\ln n)^{1+\beta}\}$ has to happen infinitely often almost surely, for $Z(t)$ to be recurrent.

Thus it will follow, that

$$\begin{aligned} \mathbb{P}(Z(t) \text{ visits } [-1, 1]^2 \text{ for } t \in [\tau_n, \tau_{n+1}]) &\leq \mathbb{P}(|X_n| \leq 1, |Y_n| \leq 2(\ln n)^{1+\beta}) + f(n) \\ &\leq \frac{C^2}{n(\ln n)^{\beta-1}} + f(n) + g(n), \end{aligned}$$

where f and g are two summable functions over n , similar to Theorem 2. The RHS is summable over n as long as $\beta > 2$. Hence $Z(t)$ visits $[-1, 1]^2$ finitely often, almost surely. □

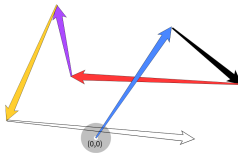


Figure 2: Model B: $\hat{W}_n$, (the projection of) conservative random walk on $\mathbb{R}^2$

4 Analysis of Model B

Recall that in Model B, vectors $\mathbf{f}_n$ are uniformly distributed over the unit sphere in $\mathbb{R}^d$.

Throughout this section, we will again suppose that $\rho = 1$ and we will show that the process is not 1-recurrent. The proof for the general ρ is analogous and is omitted.

Let $W_n = (X_n, Y_n, *, *, \dots, *)$ be the embedded version of the process $Z(t) = (X(t), Y(t), *, *, \dots, *)$; here X_n and Y_n ($X(t)$ and $Y(t)$ respectively) stand for the process' first two coordinates. We denote the projection of $Z(t)$ (W_n respectively) on the two-dimensional plane by $\hat{W}(t) = (X(t), Y(t))$ ($\hat{W}_n = (X_n, Y_n)$ respectively.) See Figure 2.

The following statement is quite intuitive.

Lemma 7. *Suppose $d \geq 3$, and let $\mathbf{f} = (f_1, f_2, \dots, f_d)$ be a random vector uniformly distributed on the unit sphere $\mathcal{S}^{d-1}$ in $\mathbb{R}^d$. Then for some $\gamma > 0$ we have*

$$\mathbb{P}\left(\sqrt{f_1^2 + f_2^2} \geq \gamma\right) \geq \frac{2}{3}.$$

Remark 3. *The statement is trivially true for the case $d = 2$ as well.*

Proof. We will use the following well-known representation (see e.g. [2], Section 2.5) of $\mathbf{f}$:

$$\mathbf{f} = \left(\frac{\eta_1}{\|\eta\|}, \dots, \frac{\eta_d}{\|\eta\|} \right)$$

where η_i , $i = 1, 2, \dots$, are i.i.d. standard normal and $\|\eta\| = \sqrt{\eta_1^2 + \dots + \eta_d^2}$. For some large enough $A > 0$ we have

$$\mathbb{P}\left(\max_{i=3, \dots, d} |\eta_i| < A\right) = (\Phi(A) - \Phi(-A))^{d-2} \geq \sqrt{\frac{2}{3}},$$

where $\Phi(\cdot)$ is the distribution function of the standard normal random variable. Also, for some small enough $a \in (0, A)$

$$\mathbb{P}\left(\max_{i=1, 2} |\eta_i| > a\right) \geq \sqrt{\frac{2}{3}}.$$

On the intersection of these two independent events we have

$$f_1^2 + f_2^2 = \frac{\eta_1^2 + \eta_2^2}{(\eta_1^2 + \eta_2^2) + (\eta_3^2 + \dots + \eta_d^2)} \geq \frac{a^2}{a^2 + (d-2)A^2} =: \gamma^2.$$

□

Remark 4. In fact, one can rigorously compute

$$\begin{aligned}\mathbb{P}(f_1^2 + f_2^2 \geq \gamma^2) &= \mathbb{P}\left(\frac{\eta_1^2 + \eta_2^2}{\eta_1^2 + \eta_2^2 + \dots + \eta_d^2} \geq \gamma^2\right) = \mathbb{P}\left(\eta_1^2 + \eta_2^2 \geq \frac{\gamma^2}{1 - \gamma^2}(\eta_3^2 + \dots + \eta_d^2)\right) \\ &= \mathbb{P}\left(\chi^2(2) \geq \frac{\gamma^2}{1 - \gamma^2} \chi^2(d - 2)\right) = \iint_{x \geq \frac{\gamma^2}{1 - \gamma^2} y \geq 0} \frac{e^{-x/2}}{2} \times \frac{y^{d/2-2} e^{-y/2}}{2^{d/2-1} \Gamma(d/2 - 1)} dx dy \\ &= \int_0^\infty e^{-\frac{y}{2} \cdot \frac{\gamma^2}{2(1-\gamma^2)}} \times \frac{y^{d/2-2} e^{-y/2}}{2^{d/2-1} \Gamma(d/2 - 1)} dy = (1 - \gamma^2)^{d/2-1},\end{aligned}$$

however, we do not really need this exact expression.

Lemma 8. Suppose $d \geq 2$, and let $R_k = \{(x, y) \in \mathbb{R}^2 : k^2 \leq x^2 + y^2 \leq (k+1)^2\}$ be the ring of radius k and width 1 centred at the origin. For some constant $C > 0$, possibly depending on d and α ,

$$\begin{aligned}\mathbb{P}\left(\hat{W}_n^{(1)} \in R_k\right) &\leq \frac{Ck}{n^{1+\frac{2\alpha}{1-\alpha}}}; \\ \mathbb{P}\left(\hat{W}_n^{(2)} \in R_k\right) &\leq \frac{Ck}{n(\ln n)^{2\beta}}\end{aligned}$$

for all large n , where $\hat{W}_t^{(1)}$ is the walk with rate (3) and $\hat{W}_t^{(2)}$ is the walk with rate with rate (12) respectively.

Proof. Assume that the event $\mathcal{E}_1$ defined by (4) has occurred. We can write

$$\hat{W}_n = (X_n, Y_n) = \sum_{k=1}^n (\tau_k - \tau_{k-1}) \tilde{\mathbf{f}}_k \quad \text{where} \quad \tilde{\mathbf{f}}_k = \ell_k [\mathbf{e}_1 \cos(\phi_k) + \mathbf{e}_2 \sin(\phi_k)]$$

and ϕ_k , $k = 1, 2, \dots$, are uniformly distributed on $[-\pi, \pi]$, and ℓ_k is a length of projection $\tilde{\mathbf{f}}_k$ of $\mathbf{f}_k$ on the two-dimensional plane. Note that elements of the set $\{\ell_1, \eta_2, \eta_3, \dots, \phi_1, \phi_2, \phi_3, \dots\}$ are all independent. Also, define

$$\mathcal{D}_2 = \{\text{for at least half of the integers } i \in [n/2, n] \text{ we have } \ell_i \geq \gamma\}$$

where γ is the constant from Lemma 7. Then $\mathbb{P}(\mathcal{D}_2^c) \leq 2e^{-n/36}$ by (2) and Lemma 7.

Let $\tilde{W}_n$ be the distribution of $\hat{W}_n \in \mathbb{R}^2$ conditioned on $\mathcal{E}_1 \cap \mathcal{D}_2$, and

$$\varphi_{\tilde{W}_n}(t) = \mathbb{E} e^{it \cdot \tilde{W}_n} = \mathbb{E} \exp \left\{ i \sum_{k=1}^n t \cdot \tilde{\mathbf{f}}_k (\tau_k - \tau_{k-1}) \right\}$$

be its characteristic function (here $t \cdot \tilde{\mathbf{f}}_k = \ell_k (t_1 \cos(\phi_k) + t_2 \sin(\phi_k))$). We will use the Lévy inversion formula, which allows us to compute the density of $\tilde{W}_n$, provided $|\varphi_{\tilde{W}_n}(t)|$ is integrable:

$$f_{\tilde{W}_n}(x, y) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{-i(t_1 x + t_2 y)} \varphi_{\tilde{W}_n}(t) dt_1 dt_2 \leq \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} |\varphi_{\tilde{W}_n}(t)| dt_1 dt_2. \quad (14)$$

Let $\Delta_k = \tau_k - \tau_{k-1}$. Since $\phi_1, \dots, \phi_n$ are i.i.d. Uniform $[-\pi, \pi]$ and independent of anything, we have

$$\begin{aligned} \varphi_{\tilde{W}_n}(t) &= \mathbb{E} \left[\prod_{k=1}^n \mathbb{E} \left(e^{i\Delta_k \ell_k (t_1 \cos \phi_k + t_2 \sin \phi_k)} \mid \tau_1, \dots, \tau_n; \ell_1, \dots, \ell_n \right) \right] = \mathbb{E} \left[\prod_{k=1}^n J_0(\|t\| \Delta_k \ell_k) \right] \\ &\Rightarrow \\ |\varphi_{\tilde{W}_n}(t)| &\leq \mathbb{E} \left[\prod_{k=1}^n |J_0(\|t\| \Delta_k \ell_k)| \right] \end{aligned}$$

where $\|t\| = \sqrt{t_1^2 + t_2^2}$, and $J_0(x) = \sum_{m=0}^{\infty} \frac{(-x^2/4)^m}{m!^2}$ is Bessel J_0 function. Indeed, for any $x \in \mathbb{R}$, setting $\tilde{x} = x\sqrt{t_1^2 + t_2^2}$ and $\beta = \arctan\left(\frac{t_2}{t_1}\right)$, we get

$$\begin{aligned} \mathbb{E} \left[e^{ix(t_1 \cos \phi_k + t_2 \sin \phi_k)} \right] &= \frac{1}{2\pi} \int_0^{2\pi} e^{ix(t_1 \cos \phi + t_2 \sin \phi)} d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x}(\cos \beta \cos \phi + \sin \beta \sin \phi)} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x} \cos(\phi + \beta)} d\phi = \frac{1}{2\pi} \int_0^{2\pi} e^{i\tilde{x} \cos(\phi)} d\phi = \frac{1}{2\pi} \int_0^{2\pi} \cos(\tilde{x} \cos(\phi)) d\phi = \end{aligned}$$

due to periodicity of $\cos(\cdot)$ and the fact that the function $\sin(\cdot)$ is odd, and formula (9.1.18) from [1].

Let ξ_k be the random variable with the distribution of Δ_k given τ_{k-1} . Then, recalling that we are on the event $\mathcal{E}_1$, we get

$$\begin{aligned} \xi_k &\succ \tilde{\xi}^{(1)} \text{ for } \hat{W}_n^{(1)}; \\ \xi_k &\succ \tilde{\xi}^{(2)} \text{ for } \hat{W}_n^{(2)} \end{aligned}$$

where $\tilde{\xi}^{(1)}$ ($\tilde{\xi}^{(2)}$ respectively) is an exponential random variable with rate $1/a_1$ ($1/a_2$ respectively) with

$$\begin{aligned} a_1 &= \tilde{c}_1 n^{\frac{\alpha}{1-\alpha}}, \\ a_2 &= \tilde{c}_2 (\ln n)^\beta \end{aligned}$$

for some constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$, and “ $\zeta_a \succ \zeta_b$ ” denotes that random variable ζ_a is stochastically larger than random variable ζ_b . Consequently, setting $a = a_1$, $\xi = \tilde{\xi}^{(1)}$ or $a = a_2$, $\tilde{\xi} = \tilde{\xi}^{(2)}$ depending on which of the two models we are talking about, and $\mathcal{F}_k = \sigma(\tau_1, \dots, \tau_k)$, we get

$$\begin{aligned} \mathbb{E} [|J_0(\|t\| \Delta_k \ell_k)| \mid \mathcal{F}_{k-1}, \ell_k = \ell] &= \mathbb{E} [|J_0(\|t\| \xi_k \ell)|] \leq \mathbb{E} G(\|t\| \xi_k \ell) \leq \mathbb{E} G(\|t\| \tilde{\xi} \ell) \\ &= \int_0^\infty a^{-1} e^{-y/a} G(\|t\| y \ell) dy = \int_0^\infty e^{-u} G(su) du = \int_0^\infty \frac{e^{-u}}{\sqrt[4]{1 + s^2 u^2}} du =: h(s, \ell) \end{aligned} \tag{15}$$

where

$$s = a\ell\|t\|,$$

since $|J_0(x)| \leq G(x)$ by (18) and the fact that $G(\cdot)$ is a decreasing function and $\xi_k \succ \tilde{\xi}$.

We will now estimate the function $h(t, \ell)$. Since $|J_0(x)| \leq 1$, we trivially get $0 \leq h(s, \ell) \leq 1$. Additionally, for all $s > 0$

$$h(s, \ell) \leq \int_0^\infty \frac{e^{-u}}{\sqrt{su}} du = \sqrt{\frac{\pi}{s}} = \sqrt{\frac{\pi}{a\ell\|t\|}}.$$

Let

$$n/2 \leq j_1 < j_2 < \dots < j_m \leq n$$

be the indices $i \in [n/2, n]$ for which $\ell_i \geq \gamma$. Since $|J_0(x)| \leq 1$, we have from (15)

$$\begin{aligned} \mathbb{E} \left[\prod_{k=1}^n |J_0(\|t\| \Delta_k \ell_k)| \right] &\leq \mathbb{E} \left[\prod_{i=1}^m |J_0(\|t\| \Delta_{j_i} \ell_{j_i})| \right] = \mathbb{E} \left[\mathbb{E} (|J_0(\|t\| \Delta_{j_m} \ell_{j_m})| \mid \mathcal{F}_{j_m-1}, \ell_{j_m}) \prod_{i=1}^{m-1} |J_0(\|t\| \right. \\ &\leq \sqrt{\frac{\pi}{a\gamma\|t\|}} \cdot \mathbb{E} \left[\prod_{i=1}^{m-1} |J_0(\|t\| \Delta_{j_i} \ell_{j_i})| \right] \end{aligned}$$

By iterating this argument for $i = j_{m-1}, j_{m-2}, \dots, j_1$, we get

$$|\varphi_{\tilde{W}_n}(t)| \leq \mathbb{E} \left[\prod_{k=1}^n |J_0(\|t\| \Delta_k \ell_k)| \right] \leq \left(\sqrt{\frac{\pi}{a\gamma\|t\|}} \right)^{n/4}$$

(recall that $m \geq n/4$ on $\mathcal{D}_1$).

Consider now two cases. For $\|t\| \geq \frac{2\pi}{a\gamma}$ a part of the inversion formula gives

$$\begin{aligned} \iint_{\|t\| \geq \frac{2\pi}{a\gamma}} |\varphi_{\tilde{W}_n}(t)| dt_1 dt_2 &\leq \iint_{\|t\| \geq \frac{2\pi}{a\gamma}} \left(\frac{\pi}{\gamma\|t\|a} \right)^{n/8} dt_1 dt_2 = \left(\frac{\pi}{a\gamma} \right)^2 \int_0^{2\pi} d\theta \int_2^\infty \frac{r dr}{r^{n/8}} \\ &= \frac{2\pi^3}{a^2\gamma^2} \times \frac{r^{2-n/8}}{2-n/8} \Big|_2^\infty = o(2^{-n/8}) \end{aligned} \tag{16}$$

by changing the variables $t_1 = \frac{\pi}{a\gamma} r \cos \theta$, $t_2 = \frac{\pi}{a\gamma} r \sin \theta$.

On the other hand, for $\|t\| \leq \frac{2\pi}{a\gamma}$, when $\ell \geq \gamma$ and thua $\sigma := a\gamma\|t\| \leq s$,

$$\begin{aligned} h(s, \ell) &= \int_0^\infty \frac{e^{-u} du}{\sqrt{1+s^2u^2}} \leq \int_0^\infty \frac{e^{-u} du}{\sqrt{1+\sigma^2u^2}} \leq \int_0^1 \frac{e^{-u} du}{\sqrt{1+\sigma^2u^2}} + \int_1^\infty \frac{e^{-u} du}{\sqrt{1+\sigma^2u^2}} \\ &\leq \int_0^1 \left(1 - \frac{\sigma^2 u^2}{50} \right) e^{-u} du + \int_1^\infty e^{-u} du \\ &= - \int_0^1 \frac{\sigma^2 u^2}{50} e^{-u} du + \int_0^\infty e^{-u} du = 1 - 0.0016... \sigma^2 \leq \exp \left\{ -\frac{\sigma^2}{700} \right\} = \exp \left\{ -\frac{\gamma^2 \|t\|^2 a^2}{700} \right\} \end{aligned}$$

since $(1+x^2)^{-1/4} \leq 1-x^2/50$ for $0 \leq x \leq 7$, and $\sigma \leq 2\pi < 7$ by assumption. By iterating the same argument as before, we get

$$\varphi_{\tilde{W}_n} \leq h(s, \ell)^{n/4} \leq \exp \left\{ -\frac{n\gamma^2 \|t\|^2 a^2}{2800} \right\}.$$

Consequently,

$$\begin{aligned} \iint_{\|t\| \leq \frac{2\pi}{a\gamma}} |\varphi_{\tilde{W}_n}(t)| dt_1 dt_2 &\leq \iint_{\|t\| \leq \frac{2\pi}{a\gamma}} \exp \left\{ -\frac{n\gamma^2 \|t\|^2 a^2}{2800} \right\} dt_1 dt_2 \\ &\leq \frac{1400}{a^2\gamma^2} \int_0^{2\pi} d\theta \int_0^\infty \exp \left\{ -\frac{nr^2}{2} \right\} r dr = \frac{2800\pi}{na^2\gamma^2} \end{aligned} \tag{17}$$

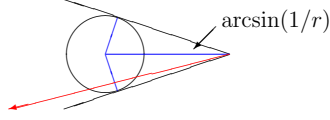


Figure 3: A ray from $\hat{W}_n$ passes through the unit circle

by changing the variables $t_1 = \frac{10\sqrt{14}}{a\gamma}r \cos \theta$, $t_2 = \frac{10\sqrt{14}}{a\gamma}r \sin \theta$.

Finally, recalling that $a_1 \sim n^{\frac{\alpha}{1-\alpha}}$, for $\hat{W}_t^{(1)}$ and $a_2 \sim (\ln n)^\beta$, for $\hat{W}_t^{(2)}$, from (14), (16) and (17) we obtain that

$$\begin{aligned} f_{\hat{W}_n^{(1)}}(x, y) &\leq O\left(\frac{1}{n^{1+\frac{2\alpha}{1-\alpha}}}\right); \\ f_{\hat{W}_n^{(2)}}(x, y) &\leq O\left(\frac{1}{n(\ln n)^{2\beta}}\right). \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}(\hat{W}_n \in R_k) &\leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{D}_2^c) + \mathbb{P}(\tilde{W}_n \in R_k) = 4e^{-n/36} + \iint_{(x,y) \in R_k} f_{\tilde{W}_n}(x, y) \, dx \, dy \\ &= 2\pi k \times \begin{cases} O\left(\frac{1}{n^{1+\frac{2\alpha}{1-\alpha}}}\right), & \text{for } W_n^{(1)}; \\ O\left(\frac{1}{n(\ln n)^{2\beta}}\right), & \text{for } W_n^{(2)} \end{cases} \end{aligned}$$

since the area of R_k is $\pi(2k+1)$. $\square$

Lemma 9. Suppose $d \geq 2$, and $\hat{W}_n = r > 1$. Then

$$\mathbb{P}\left(\|\hat{W}(t)\| \leq 1 \text{ for some } t \in [\tau_n, \tau_{n+1})\right) \leq \frac{1}{2r}.$$

Proof. First, note that $\hat{W}_n$ lies outside of the unit circle on $\mathbb{R}^2$. The projection $\tilde{\mathbf{f}}_{n+1}$ of $\mathbf{f}_{n+1}$ on the first two coordinates plane has an angle ϕ uniformly distributed over $[0, 2\pi]$. The probability in the statement of the lemma is monotone increasing in τ_{n+1} , hence it is bounded above by

$$\mathbb{P}(\text{the infinite ray from } \hat{W}_n \text{ in the direction } \tilde{\mathbf{f}}_{n+1} \text{ passes through the unit circle}) = \frac{2 \arcsin(1/r)}{2\pi}$$

(see Figure 3.) Finally, $\arcsin(x) \leq \frac{\pi x}{2}$ for $0 \leq x \leq 1$. $\square$

Theorem 4. Let $d \geq 2$ and the walk $Z(t)$ has either the rate (3) with $\alpha \in (0, 1)$ or rate (12) with $\beta > 2$. Then $Z(t)$ is transient a.s.

Proof. As mentioned before, we will only show that the walk is not ρ -recurrent for $\rho = 1$. To do that, it will suffice that a.s. there will be only finitely many n such that the event

$$A_n = \left\{ \|\hat{W}(t)\| \leq 1 \text{ for some } t \in [\tau_n, \tau_{n+1}) \right\}$$

occurs. Indeed, fix a positive integer n . At this time, $\hat{W}_n \in R_k$ for some $k \in \{0, 1, 2, \dots\}$. According to Lemma 9,

$$\mathbb{P}(A_n \mid \hat{W}_n \in R_k) \leq \begin{cases} 1, & \text{if } k = 0; \\ \frac{1}{2k}, & \text{if } k \geq 1. \end{cases}$$

Fix some very small $\varepsilon > 0$. Then

$$\begin{aligned} \mathbb{P}(A_n) &\leq \sum_{k=1}^{n^{\frac{\alpha}{1-\alpha}+\varepsilon}} \mathbb{P}\left(A_n \mid \hat{W}_n^{(1)} \in R_k\right) \mathbb{P}\left(\hat{W}_n^{(1)} \in R_k\right) + \mathbb{P}\left(A_n \mid \|\hat{W}_n^{(1)}\| \geq n^{\frac{\alpha}{1-\alpha}+\varepsilon}\right) \\ &\leq \frac{Cn^{\frac{\alpha}{1-\alpha}+\varepsilon}}{n^{1+\frac{2\alpha}{1-\alpha}}} + \mathbb{P}\left(\tau_{n+1} - \tau_n \geq n^{\frac{\alpha}{1-\alpha}+\varepsilon} - 1\right) \leq \frac{C}{n^{1+\frac{\alpha}{1-\alpha}-\varepsilon}} + 3e^{-c_1^* n^\varepsilon} \end{aligned}$$

by (10) and Lemma 8. Hence $\sum_n \mathbb{P}(A_n) < \infty$ and the result follows from the Borel-Cantelli lemma.

Similarly, in the other case

$$\begin{aligned} \mathbb{P}(A_n) &\leq \sum_{k=1}^{2(\ln n)^{1+\beta}} \mathbb{P}(A_n, \hat{W}_n^{(2)} \in R_k) + \mathbb{P}\left(A_n \mid \|\hat{W}_n^{(2)}\| \geq 2(\ln n)^{1+\beta}\right) \\ &\leq \frac{2C(\ln n)^{1+\beta}}{n(\ln n)^{2\beta}} + \mathbb{P}\left(\tau_{n+1} - \tau_n \geq 2(\ln n)^{1+\beta} - 1\right) \leq \frac{2C}{n(\ln n)^{\beta-1}} + \frac{1}{n^2} \end{aligned}$$

for all sufficiently large n by (13) and Lemma 8. Hence $\sum_n \mathbb{P}(A_n) < \infty$ for $\beta > 2$, and the result again follows from the Borel-Cantelli lemma. $\square$

Appendix: properties of Bessel function J_0

Let

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{(m!)^2}$$

be the Bessel function of the first kind. The following statement might be known, but we could not find it in the literature.

Claim 2. *For all $x \geq 0$*

$$|J_0(x)| \leq \frac{1}{\sqrt[4]{1+x^2}} =: G(x). \quad (18)$$

Proof. From Theorem 1 and formula (1) in [15] we get (with $\nu = 0$, $\mu = 3$)

$$J_0^2(x) \leq \frac{4(4x^2 - 5)}{\pi((4x^2 - 3)^{3/2} - 3)}, \quad x \geq 1.13.$$

This inequality also implies that

$$|J_0(x)| \leq \frac{1}{\sqrt[4]{1+x^2}}, \quad x \geq 1.13.$$

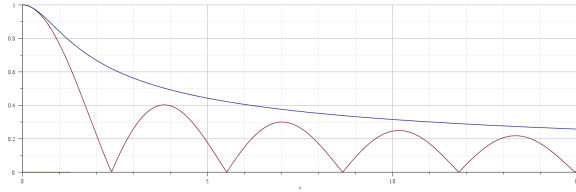


Figure 4: $|J_0(x)|$ (red) and its upper bound $G(x)$ (blue).

At the same time for $0 \leq x \leq 2$ (using [1] 9.1.14)

$$J_0^2(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{(k!)^4} \left(\frac{x}{2}\right)^{2k} \leq 1 - 2\left(\frac{x}{2}\right)^2 + \frac{3}{2}\left(\frac{x}{2}\right)^4 - \frac{5}{9}\left(\frac{x}{2}\right)^6 + \frac{35}{288}\left(\frac{x}{2}\right)^8$$

and at the same time

$$\frac{1}{\sqrt{1+x^2}} \geq 1 - 2\left(\frac{x}{2}\right)^2 + \frac{3}{2}\left(\frac{x}{2}\right)^4 - \frac{5}{9}\left(\frac{x}{2}\right)^6 + \frac{35}{288}\left(\frac{x}{2}\right)^8$$

yielding 18. □

Observe as well that $0 \leq G(x) \leq 1/\sqrt{x}$ and that G is a decreasing function for $x \geq 0$; see Figure 4

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Paper III



Two-dimensional Rademacher walk

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Abstract

We study a generalisation of the one-dimensional Rademacher random walk introduced in [2] to $\mathbb{Z}^2$ (for $d \geq 3$, the Rademacher random walk is always transient, as follows from Theorem 8.8 in [4]). This walk is defined as the sum of a sequence of independent steps, where each step goes in one of the four possible directions with equal probability, and the size of the n th step is a_n where $\{a_n\}$ is a given sequence of positive integers. We establish some general conditions under which the walk is recurrent or transient.

Keywords: recurrence, transience, Rademacher distribution, non-homogeneous Markov chains.

AMS subject classification: 60G50, 60J10.

1 Introduction

A one-dimensional Rademacher walk was introduced in [2], as a generalisation of a simple symmetric random walk on $\mathbb{Z}$ with step sizes following some pre-determined sequence of integers $a_1, a_2, \dots$ (as opposed to all step sizes being one). It was shown that this walk is recurrent for some very slowly growing sequences, and transient, for example, if $a_k = \lfloor k^\gamma \rfloor$ for all $\gamma > 0$. Some further generalisations of the Rademacher walk with non-integer step sizes were also considered, and they were studied further in [1]. This paper studies the extension of the walk to higher dimensions.

We say that ξ has a *two-dimensional Rademacher distribution*, if ξ takes one of the values $\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_1, -\mathbf{e}_2\}$ where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ are the unit vectors in $\mathbb{R}^2$ with equal probability $1/4$.

Let $\xi_n, n = 1, 2, \dots$ be a sequence of i.i.d. two-dimensional Rademacher random variables. Let a_n be a deterministic sequence of positive real numbers. The two-dimensional Rademacher walk is defined as $S_n = Z_1 + Z_2 + \dots + Z_n$ where $Z_i = a_i \xi_i$.

Definition 1. We say that the walk S_n is transient if $\|S_n\| \rightarrow \infty$ a.s. We say that the walk S_n is recurrent, if $S_n = z$ for infinitely many n a.s., for each $z \in \mathbb{Z}^2$.

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Note that the transience and recurrence are zero-one events. Also, in general, we do not have the dichotomy between recurrence vs. transience; for example, if all $a_n \equiv 2$ then the walk is neither recurrent nor transient according to our definition, as it returns to the origin (denoted by $\mathbf{0} = (0, 0)$) infinitely often, yet never visits points with odd coordinates.

We study the conditions for transience in Section 2 and for recurrence in Section 3.

2 Transience

2.1 Non-decreasing steps

Theorem 1. *Suppose that a_n is a non-decreasing sequence of integers converging to infinity. Then S_n is transient.*

First, we prove a necessary technical result.

Lemma 1. *Let $d_1, d_2, d_3 \dots$ be a sequence of distinct positive integers, and let $T_k = d_1 \varepsilon_1 + \dots + d_k \varepsilon_k$ where ε_i are i.i.d. Rademacher. Then for every integer $m \geq \max_{1 \leq i \leq k} d_i$ and every $M \in \mathbb{Z}$ we have*

$$\mathbb{P}(T_k - M \equiv 0 \pmod{m}) \leq \frac{C \log k}{k}$$

for some universal constant $C > 0$.

Proof. It is sufficient to prove the statement for $M \in \{0, 1, \dots, m-1\}$, which we will assume from now on. By the result of [7], for each $z \in \mathbb{Z}$ we have $\mathbb{P}(T_k = z) \leq \frac{C_1}{k^{3/2}}$. On the other hand, by Hoeffding's inequality [6]

$$\mathbb{P}(|T_k| \geq m\sqrt{2k \log k}) \leq 2 \exp\left(-\frac{4m^2 k \log k}{4d_1^2 + \dots + 4d_k^2}\right) \leq 2 \exp(-\log k) = \frac{2}{k}.$$

Consequently,

$$\begin{aligned} \mathbb{P}(T_k \equiv M \pmod{m}) &\leq \mathbb{P}(T_k \equiv M \pmod{m}, |T_k| < m\sqrt{2k \log k}) + \mathbb{P}(|T_k| \geq m\sqrt{2k \log k}) \\ &\leq \mathbb{P}(T_k = M + \ell m \text{ for some } \ell \in \mathbb{Z} \text{ s.t. } |M + m\ell| < m\sqrt{2k \log k}) + \frac{2}{k} \\ &\leq \frac{1 + 2m\sqrt{2k \log k}}{m} \sup_{z \in \mathbb{Z}} \mathbb{P}(T_k = z) + \frac{2}{k} \leq \frac{3m\sqrt{k \log k}}{m} \times \frac{C_1}{k^{3/2}} + \frac{2}{k} \\ &= \frac{3C_1\sqrt{\log k}}{k} + \frac{2}{k} \leq \frac{C \log k}{k} \end{aligned}$$

for $C = 3C_1 + 2 > 0$. □

Corollary 1. *Let $d_1, d_2, d_3 \dots, d_n$ be a sequence of positive integers, of which at least k are distinct, say $d_{\ell_1}, d_{\ell_2}, \dots, d_{\ell_k}$, and let $T_n = \sum_{i=1}^n d_i \varepsilon_i$ where ε_i are i.i.d. Rademacher. Then for every integer $m \geq \max_{1 \leq i \leq k} d_{\ell_i}$ we have*

$$\mathbb{P}(T_n \equiv 0 \pmod{m}) \leq \frac{C \log k}{k}$$

where C is the constant from the statement of Lemma 1.

Proof. Suppose w.l.o.g. that $d_1, d_2, \dots, d_k$ are distinct. Let $T_n = \tilde{T}_k + U$ where $\tilde{T}_k = \sum_{i=1}^k d_{\ell_i} \varepsilon_{\ell_i}$. Then, since $\tilde{T}_k$ and U are independent,

$$\begin{aligned} \mathbb{P}(T_n \equiv 0 \pmod{m}) &= \sum_M \mathbb{P}(\tilde{T}_k - M \equiv 0 \pmod{m}) \mathbb{P}(U = M) \\ &\leq \left(\sup_M \mathbb{P}(\tilde{T}_k - M \equiv 0 \pmod{m}) \right) \sum_M \mathbb{P}(U = M) \leq \frac{C \log k}{k} \end{aligned}$$

by Lemma 1. $\square$

Proof of Theorem 1. Let $b_1 = a_1 \geq 1$, $b_2 = \min\{a_n : a_n > b_1\} \geq a_1 + 1$, $b_3 = \min\{a_n : a_n > b_2\} \geq a_2 + 1$, etc. Denote by $m_i = \text{card}\{n : a_n = b_i\}$, $i = 1, 2, \dots$. Then the sequence $\{a_n\}$ can be represented as

$$\underbrace{b_1, b_1, \dots, b_1}_{m_1}, \underbrace{b_2, b_2, \dots, b_2}_{m_2}, \underbrace{b_3, b_3, \dots, b_3}_{m_3}, \dots, \underbrace{b_j, b_j, \dots, b_j}_{m_j}, b_{j+1}, \dots$$

Let $k_{j+1} = m_1 + m_2 + \dots + m_j + 1 = \inf\{n : a_n = b_{j+1}\}$ be the first index of the sequence $\{a_n\}$ equal to b_{j+1} .

Let $\kappa_i = 1$ if $\xi_i \in \{\mathbf{e}_1, -\mathbf{e}_1\}$ and $= 0$ otherwise. Let $\varepsilon_i = \pm 1$ equal to the non-zero coordinate of ξ_i . Then $\{\kappa_i\}$ are i.i.d. Bernoulli and $\{\varepsilon_i\}$ are i.i.d. Rademacher; moreover, these two sequences are independent. The values of $i \in \{1, 2, \dots, n\}$ for which $\kappa_i = 1$ ($= 0$ resp.) correspond to horizontal (vertical, resp.) steps. Formally, let

$$H_n = \{1 \leq i < n : \kappa_i = 1\}, \quad V_n = \{1 \leq i < n : \kappa_i = 0\}$$

be the indices corresponding to horizontal (resp. vertical) steps. Note that $Z_i = X_i \mathbf{e}_1 + Y_i \mathbf{e}_2$ where

$$X_i = \begin{cases} a_i \varepsilon_i, & \text{if } \kappa_i = 1; \\ 0, & \text{if } \kappa_i = 0 \end{cases} \quad Y_i = \begin{cases} 0, & \text{if } \kappa_i = 1; \\ a_i \varepsilon_i, & \text{if } \kappa_i = 0, \end{cases}$$

and denote

$$S_n = S_n^X \mathbf{e}_1 + S_n^Y \mathbf{e}_2 \quad \text{where } S_n^X = X_1 + \dots + X_n, \quad S_n^Y = Y_1 + \dots + Y_n.$$

Fix a large integer j and let

$$A_j = \left\{ \text{card}\{a_i, i \in H_{k_j}\} < \frac{j}{3} \text{ or } \text{card}\{a_i, i \in V_{k_j}\} < \frac{j}{3} \right\}.$$

For each $1 \leq i < k_j$, κ_i equals 0 or 1 with equal probability $\frac{1}{2}$. Hence, by Hoeffding's inequality, there exists a constant c_0 such that

$$\mathbb{P}(A_j) \leq 2e^{-c_0 j},$$

i.e., there are at least $j/3$ horizontal and $j/3$ vertical steps with probability tending to one. As a result, on A_j^c , we can write

$$S_{k_j-1}^X = \sum_{i=1}^{h_j} b'_i \varepsilon'_i, \quad S_{k_j-1}^Y = \sum_{i=1}^{v_j} b''_i \varepsilon''_i$$

where $h_j = \text{card}\{H_{k_j}\}$, $v_j = \text{card}\{V_{k_j}\}$, the positive integer sequences $\{b_i\}_{i=1}^{h_j}$ and $\{b_i''\}_{i=1}^{v_j}$ are non-decreasing and each has at least $j/3$ distinct elements, while $\varepsilon'_i, \varepsilon''_i$ are i.i.d. Rademacher.

Now, by Corollary 1,

$$\mathbb{P}(S_{k_j-1}^X \equiv 0 \pmod{b_j} \mid A_j^c) \leq \frac{C \log(j/3)}{j/3}, \quad \mathbb{P}(S_{k_j-1}^Y \equiv 0 \pmod{b_j} \mid A_j^c) \leq \frac{C \log(j/3)}{j/3}$$

and, moreover, conditioned on the sequence κ_i and A_j^c , the events above are independent. Since for the walk to hit $\mathbf{0}$ at time n , $k_j \leq n \leq k_{j+1} - 1$, we need that both its coordinates are divisible by b_j at time $k_j - 1$, we have

$$\mathbb{P}(S_n = \mathbf{0} \text{ for some } n \in [k_j, k_{j+1})) \leq \mathbb{P}(A_j) + \left(\frac{C \log(j/3)}{j/3} \right)^2 = O\left(\frac{\log^2 j}{j^2} \right). \quad (1)$$

Since the quantity in (1) is summable in j , by the Borel-Cantelli lemma we conclude that the event $\{S_n = \mathbf{0}\}$ occurs finitely often. By the same token, we get that $\{S_n = \mathbf{w}\}$ occurs finitely often for each $\mathbf{w} \in \mathbb{Z}^2$. Hence, the transience follows. $\square$

2.2 Transience for non-integer sequences $\{a_n\}$

In this section, we allow a_n to take any positive real values (not only integers). We will use the same Definition 1 for the transience of the walk.

Definition 2. Let $a_n \in \mathbb{R}_+$, $n = 1, 2, \dots$. We call the sequence $\{a_n\}$ good, if for each large n there is a positive integer K_n such that

$$\frac{K_n}{\log_2 a_n} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

and a subset of $\{1, 2, \dots, n\}$ with K_n distinct elements, say $\{i_1, i_2, \dots, i_{K_n}\}$, such that $i_{K_n} = n$ and $a_{i_{k+1}} \geq 2a_{i_k}$ for every $k = 1, 2, \dots, K_n - 1$.

Later in the Section, in Proposition 1, we present a sufficient condition for a sequence to be good.

Definition 3. Let $r, s \geq 1$ be some constants. We call the sequence $\{a_n\}$ (r, s) -monotone, if for all large n we have $a_n \leq sa_m$ whenever $m \geq rn$.

Remark 1. The case $r = s = 1$ corresponds in the above definition to monotonicity in the usual sense.

Theorem 2. Assume that the sequence a_n is good and (r, s) -monotone, and for some $\epsilon > 0$ we have $a_n \geq (\log n)^{1/2+\epsilon}$ for all sufficiently large n . Then the two-dimensional Rademacher walk is transient.

Proof. To show transience, it suffices to show that for each $(x, y) \in \mathbb{R}^2$ we have $S_n \in (x, x+1) \times (y, y+1)$ for at most finitely many ns a.s. W.l.o.g. we can assume that $a_n > 1$ for all n (this is, of course, true for all large n due to the assumption of the theorem, and a few initial values of the sequence $\{a_n\}$ do not affect transience).

The proof will follow an argument involving anti-concentration inequalities on two different scales, similarly to the method used in [1]. Define the event

$$A_n = \{\text{The walk makes at least } n/5 \text{ horizontal and } n/5 \text{ vertical steps} \\ \text{among the last } n/2 \text{ steps of the } n\text{-step walk}\}$$

By Hoeffding's inequality, it is easy to check that $\mathbb{P}(A_n^c) \leq 2e^{-0.01n}$.

Let T_m be a one-dimensional Rademacher random walk with step sizes $d_1, d_2, \dots, d_m \geq D$ for some $D > 0$. By the Erdős–Littlewood–Offord inequality (see [5] or Lemma 14 in [1]),

$$\mathbb{P}(T_m \in (x - D, x + D]) \leq \frac{0.8}{\sqrt{m}} \quad (2)$$

for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}_+$.

Let $n_1 := \lfloor n/2 \rfloor$ and consider the walk S_n during the times from $n_1 + 1$ to n . Define

$$D = D_n = \min_{n_1 < i \leq n} a_i. \quad (3)$$

Let

$$A'_n = \left\{ \sum_{i=n_1+1}^n Z_i \cdot \mathbf{e}_1 \in (x - D, x + D] \right\} \\ A''_n = \left\{ \sum_{i=n_1+1}^n Z_i \cdot \mathbf{e}_2 \in (x - D, x + D] \right\}$$

Conditionally on A_n and the sequence (κ_i) defined in the proof of Theorem 1, these two events are independent, and by (2) each has a (conditional) probability less than $\frac{0.8}{\sqrt{n/5}}$. Consequently, for some $c_1 > 0$ we have

$$\mathbb{P}(S' \in (x - D, x + D) \times (y - D, y + D)) \leq \frac{c_1^2}{n} + \mathbb{P}(A_n^c) = \frac{c_1^2 + o(1)}{n}. \quad (4)$$

where $S' = \sum_{i=n_1+1}^n Z_i$.

Now let us observe the walk during its first $n_0 := \lfloor \frac{n_1}{r} \rfloor \leq n_1$ steps. By the assumption of the theorem, it is possible to extract a subsequence $i_1, i_2, \dots, i_K$ from $\{1, 2, \dots, n_0\}$ of size

$$K = (1 + \gamma_{n_0}) \log_2 a_{n_0} \quad \text{where } \gamma_{n_0} \rightarrow 0$$

such that $a_{i_{k+1}} \geq 2a_{i_k}$ for $k = 1, 2, \dots, K - 1$, and $i_K = n_0$.

Let $I = (i_1, i_2, \dots, i_{K-\ell})$ where ℓ is the smallest positive integer such that $2^{\ell-1} \geq s$, and $S'' = \sum_{j \in I} Z_j$. Every step Z_j where $j \in I$, can be in one of the four directions, creating $4^{K-\ell}$ possible values. Since $a_{i_1} > 1$ and the sequence is “super-doubling”, no two such values lie in the same square with side 1; let us denote by B the set of all possible values of S'' . As a result, for any $(x', y') \in \mathbb{R}^2$

$$\bigcup_{(x,y) \in B} (x' - x, x' - x + 1) \times (y' - y, y' - y + 1) \subseteq (x' - \tilde{D}, x' + \tilde{D}) \times (y' - \tilde{D}, y' + \tilde{D})$$

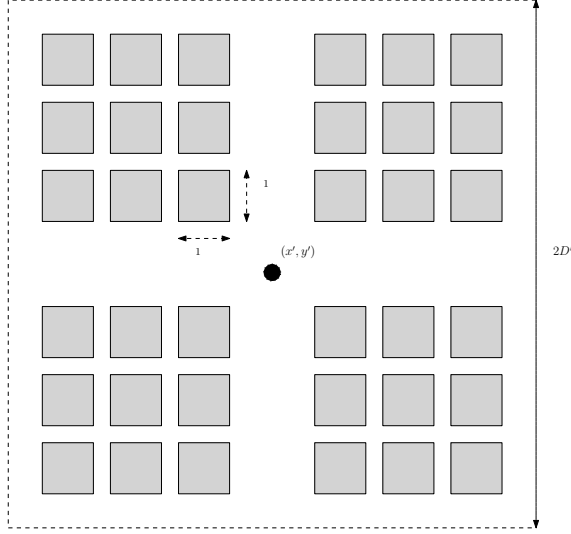


Figure 1: The disjoint union of unit squares in case $K = 2$; the left-bottom corners of each square are the elements of B .

is a collection of **disjoint** unit squares (see Figure 1); here

$$\tilde{D} = 1 + a_{i_1} + a_{i_2} + \cdots + a_{i_{K-\ell}} \leq 2a_{i_{K-\ell}} \leq 2^{1-\ell}a_{i_K} = 2^{1-\ell}a_{n_0} \leq \frac{a_{n_0}}{s} \leq D$$

where the last inequality follows from the (r, s) -monotonicity, since $n_1 \geq rn_0$ implies that $a_i \geq a_{n_0}/s$ for each $i \geq n_1$ and hence the same is true for D by (3). Moreover, note that S'' has a uniform distribution over the set B and

$$\begin{aligned} \text{card}(B) &= 4^{K-\ell} = 4^{(1+\gamma_{n_0})\log_2 a_{n_0} - \ell} = \frac{a_{n_0}^{2(1+\gamma_{n_0})}}{4^\ell} \\ &\geq \frac{(\log \lfloor n/(2r) \rfloor)^{(\frac{1}{2}+\epsilon) \cdot 2(1+\gamma_{n_0})}}{4^\ell} > (\log n)^{1+\epsilon} \end{aligned}$$

where the last inequality holds for all large n since $|\gamma_{n_0}| \rightarrow 0$ as the sequence $\{a_n\}$ is good.

Next, for every $(x', y') \in \mathbb{R}^2$

$$\begin{aligned}
& \mathbb{P}(S' + S'' \in (x', x' + 1) \times (y', y' + 1)) \\
&= \sum_{(x, y) \in B} \mathbb{P}(\{S'' = (x, y)\} \cap \{S' \in (x' - x, x' - x + 1) \times (y' - y, y' - y + 1)\}) \\
&= \sum_{(x, y) \in B} \mathbb{P}(S'' = (x, y)) \cdot \mathbb{P}(S' \in (x' - x, x' - x + 1) \times (y' - y, y' - y + 1)) \\
&\leq \frac{1}{(\log n)^{1+\epsilon}} \cdot \sum_{(x, y) \in B} \mathbb{P}(S' \in (x' - x, x' - x + 1) \times (y' - y, y' - y + 1)) \\
&= \frac{1}{(\log n)^{1+\epsilon}} \cdot \mathbb{P}\left(S' \in \bigcup_{(x, y) \in B} (x' - x, x' - x + 1) \times (y' - y, y' - y + 1)\right) \\
&\leq \frac{1}{(\log n)^{1+\epsilon}} \cdot \mathbb{P}\left(S' \in (x' - \tilde{D}, x' + \tilde{D}) \times (y' - \tilde{D}, y' + \tilde{D})\right) \leq \frac{c_1^2 + o(1)}{n(\log n)^{1+\epsilon}}
\end{aligned}$$

by (4) and the fact that $\tilde{D} \leq D$.

Finally, denoting by $S''' = \sum_{i \in \{1, \dots, \lfloor n/2 \rfloor\} \setminus I} Z_i$ the sum of the remaining steps, since S', S'', S''' are independent, for all $(x'', y'') \in \mathbb{R}^2$ we get

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^n Z_i \in (x'', x'' + 1) \times (y'', y'' + 1)\right) &= \mathbb{P}(S' + S'' + S''' \in (x', x' + 1) \times (y', y' + 1)) \\
&\leq \sup_{(x', y') \in \mathbb{R}^2} \mathbb{P}(S' + S'' \in (x', x' + 1) \times (y', y' + 1)) \\
&\leq \frac{c_1^2 + o(1)}{n(\log n)^{1+\epsilon}},
\end{aligned}$$

which is summable over n as $\epsilon > 0$. The result now follows from the Borel-Cantelli lemma. $\square$

The next statement tells us when we can apply Theorem 2.

Proposition 1. *Suppose there is some $C > 0$ such that $a_1 = 1$, $a_n \geq 1$, $|a_{n+1} - a_n| \leq C$ for all n , and $a_n \rightarrow \infty$. Then the sequence a_n is good.*

Remark 2. *The above proposition covers the case when $a_n = n^\alpha$, $\alpha \in (0, 1]$.*

Proof of Proposition 1. Fix an n and let $j_1 = n$. Now let us find j_2 such that $a_{j_2} \approx \frac{1}{2}a_{j_1}$. Since the distances between consecutive a_i s do not exceed C , we can find some j_2 such that

$$\frac{a_n}{2} - C = \frac{a_{j_1}}{2} - C < a_{j_2} \leq \frac{a_{j_1}}{2}.$$

Similarly, we can find j_3 such that $a_{j_3} \leq a_{j_2}/2$ and at the same time $a_{j_3} > \frac{a_{j_2}}{2} - C \geq \frac{a_n}{2^2} - \frac{C}{2} - C$. By induction, we get that

$$a_{j_k} \geq \frac{a_n}{2^{k-1}} - C \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{k-2}}\right) = \frac{a_n + 2C}{2^{k-1}} - 2C,$$

and we can continue the process as long as $a_{j_k} \geq 1$. Hence, the maximum possible k for which the process is feasible is

$$\left\lfloor \log_2 \left(\frac{a_n + 2C}{C + 1/2} \right) \right\rfloor = (1 + \gamma_n) \log_2 a_n$$

where $\gamma_n \rightarrow 0$ since $\log_2 a_n \rightarrow \infty$ as $n \rightarrow \infty$. $\square$

3 Recurrence for some general sequences

Let $B = \{b_1, b_2, \dots\} \subseteq \mathbb{Z}_+ \setminus \{0\}$. We say that the set B is *good* if for any $b \in B$ there are infinitely many coprimes of b in B .

Theorem 3. *Suppose that $B \subseteq \mathbb{Z}_+ \setminus \{0\}$ is good. Then there exists a sequence $a_1, a_2, \dots$ which contains all elements of the set B , and each element only finitely many times, such that the a -walk S_n is recurrent.*

Lemma 2. *Let $C > 0$ and $z \in \mathbb{Z}^2$ with $\|z\| \leq C$, and b' and b'' are positive integers which are coprime. Then there exist three positive integers c', c'', N_0 such that N_0 depends on c', c'' and C only and*

$$\mathbb{P}(T_m = z \text{ for some } m \leq N_0) \geq 1/2$$

where

$$T_m = \sum_{i=1}^m \tilde{Z}_i \text{ with } \tilde{Z}_i = b'(\xi'_{i,1} + \dots + \xi'_{i,c'}) + b''(\xi''_{i,1} + \dots + \xi''_{i,c''})$$

and $\xi'_{i,k}, \xi''_{i,k}$, are i.i.d. two-dimensional Rademacher random variables.

Proof. Let $c', c'' > 0$ be such that $c'b' - c''b'' = 1$; the existence of such c', c'' follows from Bézout's identity. This implies that $\mathbb{P}(\tilde{Z}_i = \mathbf{f}) > 0$ for each $\mathbf{f} = \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_2$ and thus the Markov chain T_m is irreducible on $\mathbb{Z}^2$.

Next, by the Chung–Fuchs theorem (see [3]) T_m is recurrent and thus there exists N_0 such that

$$\inf_{z' \in \mathbb{Z}^2: \|z'\| \leq C} \mathbb{P}(T_m \text{ visits } z' \text{ for some } m \leq N_0) \geq \frac{1}{2}$$

which concludes the proof. $\square$

Proof of Theorem 3. The construction of the sequence a_i will be done recursively. Let $n_0 = 0$ and suppose that we have already constructed the sequence up to time $i = n_k$. Let $\alpha_k = \max\{a_1, a_2, \dots, a_{n_k}\}$, then $\|S_{n_k}\| \leq \alpha_k n_k$. Find the element of B not already used, with the smallest index, call it b' , and let b'' be another element of B , not already used so far, such that b' and b'' are coprimes. By Lemma 2 there exist c', c'', N_0 corresponding to b', b'' and $C := \alpha_k n_k$. Now define

$$a_{n_k+i+1} = \begin{cases} b', & \text{if } i = \ell \pmod{c' + c''} \text{ for some } \ell \in \{0, 1, \dots, c' - 1\}, \\ b'', & \text{if } i = \ell \pmod{c' + c''} \text{ for some } \ell \in \{c', c' + 1, \dots, c' + c'' - 1\} \end{cases}$$

for $i = 0, 1, 2, \dots, (c' + c'')N_0 - 1$. Define also $n_{k+1} = n_k + (c' + c'')N_0$. It is clear that by following this procedure, we will eventually use all elements of B , and, moreover, we will use each element of B only finitely many times.

Now, by Lemma 2, since $\|S_{n_k}\| \leq C$,

$$\mathbb{P}(S_n = 0 \text{ for some } n \in [n_k + 1, n_{k+1}] \mid \mathcal{F}_{n_k}) \geq \frac{1}{2}.$$

Consequently, $S_n = 0$ for infinitely many n s a.s., by Lévy's extension of the Borel-Cantelli lemma (see e.g. [8]). $\square$

3.1 An explicit recurrent sequence

In this section, we present an explicit example of a sequence a_n such that $a_n \rightarrow \infty$ and yet the walk is recurrent. The elements of this sequence will consist of blocks numbered $1, 2, \dots$, and the k th block will be a consecutive sequence of exactly k sub-blocks, denoted by $B_{k,1}, B_{k,2}, \dots, B_{k,k}$, where

$$\begin{aligned} B_{k,i} &= \underbrace{k, k, \dots, k}_{L_{k,i} \text{ elements}}, \underbrace{k-1, k-1, \dots, k-1}_{3k^2 \text{ elements}} \quad \text{for } i = 1, 2, \dots, k-1; \\ B_{k,k} &= \underbrace{k, k, \dots, k}_{L_{k,k} \text{ elements}}, \quad \text{where } L_{k,i} = 2^{2^{k^2+i-1}}, \quad i = 1, 2, \dots, k. \end{aligned}$$

Now suppose that the sequence a_n is given by $\{a_n\} = B_1 B_2 \dots$ where $B_k = B_{k,1} B_{k,2} \dots B_{k,k}$.

Theorem 4. *A two-dimensional Rademacher walk with the above sequence is recurrent.*

We begin with a (probably) well-known result.

Lemma 3. *Consider a simple random walk $T_n = (X_n, Y_n)$ on $\mathbb{Z}^2$ with steps $\pm \mathbf{e}_1, \pm \mathbf{e}_2$ and $T_0 = (x_0, y_0)$. Let $\tau_0 = \inf\{n : T_n = \mathbf{0}\}$. Then there exists a universal constant $\alpha > 0$ such that*

$$\mathbb{P}(\tau_0 \leq r^3) \geq \alpha, \quad \text{where } r = \|(x_0, y_0)\|.$$

Proof. The fact that $\tau_0 < \infty$ a.s. follows from the recurrence of the two-dimensional walk. Let $R = r^{1.25}$ and

$$\tau = \inf\{n : T_n = \mathbf{0} \text{ or } \|T_n\| \geq R\}.$$

Define

$$f(x, y) = \begin{cases} \log(x^2 + y^2 - 1/2), & \text{if } (x, y) \neq \mathbf{0}; \\ -5, & \text{otherwise.} \end{cases}$$

An easy calculation¹ shows that $\xi_n = f(T_n)$ is a supermartingale as long as $T_n \neq \mathbf{0}$, hence by the optional stopping theorem, as $r \rightarrow \infty$,

$$\mathbb{P}(T_r \neq \mathbf{0}) \leq \frac{\ln(r)}{\ln(R)} + o(1) = \frac{4}{5} + o(1). \quad (5)$$

¹Indeed, let $\Delta_{x,y} := \frac{1}{4}(f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1)) - f(x, y)$. Then $\exp(4\Delta_{x,y}) = 1 - \frac{64(x^2-y^2)^2}{(2x^2+2y^2-1)^4} \leq 1$ when $|x| + |y| \geq 2$, and $\exp(4\Delta_{x,y}) = 126e^{-5} < 0.85$ when $|x| + |y| = 1$.

On the other hand,

$$\mathbb{P}(\tau > r^3) = o(1) \quad \text{as } r \rightarrow \infty.$$

Indeed, since during $3\lfloor R^{2.2} \rfloor \ll r^3$ steps, with probability very close to 1, the walk T_n will make at least $\lfloor R^{2.2} \rfloor$ horizontal steps. Then, by the reflection principle,

$$\mathbb{P}(\max_{i \leq n} (X_i - X_0) \geq m) = 2\mathbb{P}(X_n - X_0 > m) + \mathbb{P}(X_n - X_0 = m).$$

By choosing $m = \lceil R \rceil$ and $n = \lfloor R^{2.2} \rfloor$ we get

$$\mathbb{P}(\tau > r^3) \leq \mathbb{P}\left(\max_{i \leq r^3} \|T_i\| < R\right) \leq \mathbb{P}\left(\max_{i \leq n} (X_i - X_0) < m\right) = \mathbb{P}(|\eta| < R^{-0.1}) + o(1) = o(1)$$

where $\eta \sim \mathcal{N}(0, 1)$.

Finally,

$$\begin{aligned} \mathbb{P}(\tau_0 > r^3) &= \mathbb{P}(\tau_0 > r^3, \tau < \tau_0) + \mathbb{P}(\tau_0 > r^3, \tau \geq \tau_0) \\ &\leq \mathbb{P}(\tau < \tau_0) + \mathbb{P}(\tau > r^3) = \mathbb{P}(T_\tau \neq \mathbf{0}) + o(1) \leq \frac{4}{5} + o(1) \end{aligned}$$

by (5). □

Proof of Theorem 4. Let n_k be the index of the last element in the k th block and $n_{k,j}$ be the index of the last element in the j th sub-block of the k th block.

Let $E_{k,j} = \{\|S_{n_{k,j}}\| < k\sqrt{L_{k,j} \log L_{k,j}}\}$. Since $L_{k,j} = L_{k,j-1}^2$ for $j \geq 2$, and also $L_{k+1,1} \gg L_{k,k}^2$, by Hoeffding's inequality

$$\mathbb{P}(E_{k,j}^c) \leq \frac{2 + o(1)}{L_{k,j}^2}, \quad j = 2, 3, \dots, k. \quad (6)$$

Since $L_{k,j+1} = L_{k,j}^2 > (\sqrt{L_{k,j} \log L_{k,j}})^3$, on $E_{k,j}$ by Lemma 3, with probability at least α , the walk T_n , $n \in [n_{k,j}, n_{k,j+1} - 3k^2]$ will visit the square $[0, k-1]^2$ at least once (as this is essentially a two-dimensional simple random walk with step sizes equal to k .)

Now, Corollary 3.2 from [2] implies that with probability of order $\frac{1}{k^2}$, the walk starts this sub-block at a position whose x - and y -coordinates are both divisible by k . This follows from the fact that, with high probability, at least k^2 steps were taken in each of the two directions during the last $3k^2$ steps of the preceding block. As a result,

$$\mathbb{P}(A_{k,j+1} \mid E_{k,j}) \geq \frac{c + o(1)}{k^2} \quad \text{where } A_{k,j+1} = \{T_n = \mathbf{0} \text{ for some } n \in [n_{k,j}, n_{k,j+1}]\}$$

for some $c > 0$ independent of k and j .

We will use the extension of the conditional Borel-Cantelli lemma from [2] to conclude recurrence. Define $\mathcal{F}_{k,j}$ as the sigma-algebra generated by the walk up to the time $n_{k,j}$. We have

$$\sum_k \sum_{j=1}^{k-1} \mathbb{P}(A_{k,j+1} \mid E_{k,j}, \mathcal{F}_{k,j}) \geq \sum_k \sum_{j=1}^{k-1} \frac{c + o(1)}{k^2} = \infty.$$

At the same time,

$$\sum_k \sum_{j=1}^{k-1} \mathbb{P}(E_{k,j}^c) < \infty.$$

by (6). Hence, the events $A_{k,j+1}$ occur infinitely often a.s. by Lemma 5.1 in [2], thus implying recurrence of site $\mathbf{0}$. The fact that the walk visits all other $\mathbf{w} \in \mathbb{Z}^2$ a.s. follows from the same argument. $\square$

Remark 3. *The result of Theorem 3 can be extended for faster growing sequences, for example, with the sub-block sizes equal to $2^{2^{k^{1+\nu}+j}}$, for $\nu \in (0, 1)$.*

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

Data availability statement

Not applicable.

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Paper IV



Recurrence, transience and anti-concentration of Rademacher random walks

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Abstract

The Rademacher random walk associated with a deterministic sequence $(a_n)_{n \geq 1}$ is the walk which starts at zero and, at step i , independently steps either up or down by a_i with equal probability. We continue the study begun by Bhattacharya and Volkov in 2023 of the transience or recurrence of one-dimensional Rademacher random walks. In particular, we show that if the sequence of step sizes is bounded, the walk is weakly recurrent, meaning that it returns infinitely often to a random finite interval, while if the step sizes tend to infinity arbitrarily slowly the walk may be transient. On the other hand, we show that the step sizes may grow arbitrarily fast and still give a weakly recurrent random walk, using a construction with integer step sizes. We show the same conclusion holds if we restrict to strictly increasing step sizes, now allowing non-integer steps. However, if $a_n = n^{\alpha+o(1)}$ for some $\alpha > 1/2$, we show that the walk is transient. We also show that the bound on α is tight by giving an example where $a_n = \Theta(n^{1/2})$ and the walk is weakly recurrent.

Keywords: inhomogeneous random walk, anti-concentration.

1 Introduction

Throughout this paper, let $\epsilon_1, \epsilon_2, \dots$ be a sequence of i.i.d. Rademacher(1/2) random variables, taking the values ± 1 each with probability 1/2. We will refer to these simply as Rademacher random variables (dropping the 1/2). Let $(a_n)_{n \geq 1}$ be a deterministic sequence of non-negative real numbers, and define the associated *Rademacher random walk* $(X_n)_{n \geq 0}$ by

$$X_n = \sum_{k=1}^n \epsilon_k a_k.$$

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When all of the a_k take the value 1, the associated Rademacher random walk is exactly the simple symmetric random walk in one dimension, and the recurrence of this random walk is a fundamental result in any introductory probability course. However, by simply changing the sequence of step sizes (a_n) , the problem becomes much less elementary and the behaviour of the Rademacher random walk is still unknown in many cases, including the simple-looking case where $a_n = n^\alpha$.

We mention that recurrence and transience of two-dimensional Rademacher random walks has been studied recently by Bhattacharya and Volkov in [4]. In this paper we will focus on the one-dimensional case, continuing the study begun by Bhattacharya and Volkov in [3].

If a particular sample of a Rademacher random walk satisfies $\{X_n \leq C\}$ infinitely often (i.o.), we call it *C-recurrent*. When X is almost surely 0-recurrent we say X is *recurrent*. Note that *C*-recurrence is not necessarily a tail event. Indeed, if we take the sequence $(a_n)_{n \geq 1}$ where $a_1 = 1$, $a_2 = 1$ and $a_i = 3$ for all $i \geq 2$, then whether the walk returns to zero i.o. depends on the value after two steps. However, the event $\{\exists C : |X_n| \leq C \text{ i.o.}\}$ is a tail event, and we shall call a Rademacher random walk *weakly recurrent* if the probability of this event is 1. Otherwise, we will call the walk *transient*. Finally, we say that a random walk is *topologically recurrent* if its range is almost surely dense in $\mathbb{R}$.

Note that weak recurrence of the Rademacher random walk is unaffected by making any finite number of changes, deletions, or insertions to $(a_n)_{n \geq 1}$.

One interesting class of Rademacher random walks is those whose step sizes $(a_n)_{n \geq 1}$ satisfy $\sum_n a_n^2 < \infty$ but $\sum_n a_n = \infty$. In this case, X_n almost surely converges to a random limit by the $p = 2$ case of Doob's L^p -martingale convergence theorem. However, the distribution of the limit has unbounded support, so X is weakly recurrent but there is no $C < \infty$ such that X is almost surely *C*-recurrent.

Our main motivation is the following problem.

Problem 1. *What conditions on the growth rate of (a_n) guarantee that the associated Rademacher random walk is transient? What growth conditions guarantee that the Rademacher random walk is weakly recurrent?*

In Bhattacharya and Volkov [3] it is shown that the Rademacher random walk is *C*-recurrent for every $C > 0$, and hence weakly recurrent, in the following cases:

- $a_n = \log n$, and
- $a_n = \lfloor (\log_\gamma n)^\beta \rfloor$ for constants $\gamma > 1$ and $0 < \beta \leq 1$.

Two transient cases with integer step sizes are also given:

- the steps $(a_n)_{n \geq 1}$ are all distinct integers, and
- $a_n = \lfloor n^\beta \rfloor$ for any constant $0 < \beta < 1$.

One particularly natural case of Problem 1 is to assume that $a_n \approx n^\alpha$ for some $\alpha > 0$. With this constraint, we can ask whether there are values of α for which

we can guarantee that the walk is transient or weakly recurrent. Combining the two transience results from [3], we see that the Rademacher walk with step sizes $a_n = \lfloor n^\alpha \rfloor$ is transient for all $\alpha > 0$. Hence, there is no $\alpha > 0$ for which the condition $a_n \approx n^\alpha$ implies the walk is weakly recurrent. In contrast, our main result shows that there are values of α for which this growth rate guarantees that the walk is transient.

Theorem 1. *Let $(a_n)_{n \geq 1}$ be a sequence and suppose that $a_n = n^{\alpha+o(1)}$ for some $\alpha > 1/2$. Then the associated Rademacher random walk is transient.*

The condition on α in Theorem 1 is best possible, even under the stronger assumption that $a_n = \Theta(n^\alpha)$ for some $\alpha > 0$ as $n \rightarrow \infty$: there is a sequence of step sizes $(a_n)_{n \geq 1}$ where $a_n = \Theta(n^{1/2})$ and the associated Rademacher random walk is weakly recurrent.

Theorem 2. *There exists a sequence $(a_n)_{n \geq 1}$ of integers such that $a_n = \Theta(n^{1/2})$ as $n \rightarrow \infty$ and the associated Rademacher random walk is weakly recurrent.*

We remark that from our proof of Theorem 1 we can already relax the condition that $a_n = n^{\alpha+o(1)}$ slightly to allow $n^{\alpha-\delta} \leq a_n \leq n^{\alpha+\delta}$ for some sufficiently small $\delta = \delta(\alpha)$. However, we still need both upper and lower bounds on a_n . Perhaps surprisingly, the upper bound is necessary: the sequence (a_n) can grow arbitrarily fast and still yield a recurrent Rademacher walk.

Theorem 3. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any non-decreasing function. There is an integer sequence $(a_n)_{n \geq 1}$ such that $a_n \geq f(n)$ for all n and the associated Rademacher random walk is recurrent.*

Our proof of Theorem 3 constructs the sequence in blocks of increasing length such that within each block the terms alternate between two consecutive integers. Allowing non-integer step sizes, we can get topological recurrence from a strictly increasing sequence that grows as fast as we like:

Theorem 4. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any non-decreasing function. There is a strictly increasing real sequence $(a_n)_{n \geq 1}$ such that $a_n \geq f(n)$ for all n and the associated Rademacher random walk is topologically recurrent.*

Recall that Bhattacharya and Volkov showed that if (a_n) consists of distinct integers then the Rademacher random walk is transient. In particular, the walk associated to any strictly increasing integer sequence (a_n) is transient, and there cannot be a single sequence that demonstrates both Theorem 3 and Theorem 4.

These results leave open the possibility that there could be a powerful sufficient condition for transience of the Rademacher random walk, in terms of the asymptotic behaviour of the sequence (a_n) , when we restrict to non-decreasing integer sequences. The asymptotically fastest-growing non-decreasing integer sequences that we know to yield weakly recurrent Rademacher random walks are the examples $a_n = \lfloor c \log n \rfloor$ from [3]. On the other hand, increasing the growth rate slightly, but still using all the non-negative integers, we can obtain a transient Rademacher random walk.

Theorem 5. Let $(a_n)_{n \geq 1}$ be a non-decreasing sequence of integers, and let L_i be the number of times i appears in the sequence. Suppose that, for all large enough n ,

$$\sum_{\substack{i \leq n \\ \gcd(i, n) = 1}} L_i \geq 2n^2 \quad (1)$$

and

$$\sum_{i=1}^{n-1} i^2 L_i \geq 4n^2 \log^3(n) \cdot L_n. \quad (2)$$

Then the Rademacher random walk X associated with $(a_n)_{n \geq 1}$ is transient.

Corollary 6. For any $\alpha > 1$, the Rademacher random walk with step sizes $a_n = \lfloor \log^\alpha(n) \rfloor$ is transient

The (Lévy) concentration function Q of a real valued random variable A is defined by¹

$$Q_r(A) = \sup_{x \in \mathbb{R}} \mathbb{P}(x < A \leq x + r).$$

Any upper bound for a value of the concentration function is called an *anti-concentration* bound, while a lower bound is called a *concentration* bound. Some known anti-concentration bounds for sums of independent random variables are discussed in Section 2.

To prove Theorem 1 we show, using the theorem below, that the position of the Rademacher random walk at each time n is sufficiently anti-concentrated, and then apply the Borel–Cantelli lemma. We remark that, as alluded to earlier, the following theorem allows us to slightly relax the condition that $a_n = n^{\alpha+o(1)}$.

Theorem 7. Let $(a_n)_{n \geq 0}$ be a sequence and suppose that there are constants $c, C, \alpha > 0$ and $\delta \geq 0$ such that, for all large enough n ,

$$cn^\alpha \leq a_n \leq Cn^{\alpha+\delta}.$$

Then, for any $\gamma > 0$, we have the anti-concentration bound

$$Q_1\left(\sum_{i=1}^n \epsilon_i a_i\right) = O\left(n^{-(\frac{1}{2} + \alpha f(\alpha, \delta) - \gamma)}\right),$$

where

$$f(\alpha, \delta) = \begin{cases} \frac{\alpha^2}{(\alpha+\delta)(\alpha+2\delta+2\sqrt{\delta^2+\alpha\delta})} & \text{if } \delta \leq \frac{\sqrt{\alpha^2+1}-\alpha}{2}, \\ \frac{\alpha^2}{(\alpha+\delta)(1+2\delta)(\alpha+1/2+\delta)} & \text{if } \delta \geq \frac{\sqrt{\alpha^2+1}-\alpha}{2}. \end{cases}$$

From this it is easy to deduce Theorem 1.

¹We warn the reader that authors disagree about the strictness of the inequalities in the definition of the concentration function, so care is needed in interpreting anti-concentration inequalities in the literature. We have made the same choice as in [10], because it gives $Q_1(A)$ the meaning that we want in the case of an integer-valued random variable A .

Proof of Theorem 1 from Theorem 7. Fix $\lambda > 0$ which is small enough that $1/2 + \alpha - \lambda > 1$. For any $\delta > 0$, we have

$$n^{\alpha-\delta} \leq a_n \leq n^{\alpha+\delta}$$

for all sufficiently large n . As $f(\alpha, \delta) \rightarrow 1$ as $\delta \rightarrow 0$, there is some $\delta > 0$ such that this condition on a_n is enough to get the anti-concentration bound $Q_1(X_n) = O(n^{-(1/2+\alpha-\lambda)})$.

For any fixed C , the probability that $|X_n| \leq C$ is $O(n^{-(1/2+\alpha-\lambda)})$ and this is summable by our choice of λ . Hence, by the Borel–Cantelli lemma, the probability that $|X_n| \leq C$ infinitely often is 0, and the walk is transient. $\square$

We remark that by substituting $\delta = 0$ into Theorem 7, we get the bound $O(n^{-(\alpha+1/2-\gamma)})$, which is easily seen to be tight up to the γ term by considering the sequence $a_n = n^\alpha$. Also of interest is what happens as $\delta \rightarrow \infty$. In this case, the anti-concentration gets close to $O(n^{-1/2})$, and it is also straightforward to show that there must be points where this is the correct behaviour. More generally, we have the following easy lower bounds which complement Theorem 7.

Proposition 8. *Fix $\alpha > 0$. Then*

$$Q_1\left(\sum_{i=1}^n \epsilon_i n^\alpha\right) = \Omega(n^{-(1/2+\alpha)}).$$

Moreover, for any $\delta \geq 1/2$, there exists a sequence $(a_n)_{n \geq 1}$ of step sizes and a sequence of times $(n_i)_{i \geq 1}$ such that

$$Q_1\left(\sum_{i=1}^{n_i} \epsilon_i a_i\right) \geq n_i^{-(1/2+\frac{\alpha}{2\delta}+o(1))}.$$

Let us now turn to the second part of Problem 1: are there conditions on the growth rate of (a_n) that guarantee that the associated Rademacher random walk is weakly recurrent?

There is no unbounded non-decreasing function f for which the condition $a_n \leq f(n)$ suffices to imply weak recurrence, as the following example shows. Let $f : \mathbb{N} \rightarrow [1, \infty)$ be unbounded and non-decreasing. Take $a_n = 2^{b_n}$ where $b_n = \lfloor \log_2 f(n) \rfloor$ for all n , so that $a_n \leq f(n)$. Once $b_n \geq k$, the congruence class of X_n modulo 2^k becomes constant and it is not difficult to deduce that the random walk is transient. In light of this example, one might ask for the walk to be “irreducible”. We say an integer-valued random walk is *irreducible* if, for any $a, b \in \mathbb{Z}$ and any $n \in \mathbb{N}$ such that $\mathbb{P}(X_n = a) > 0$, there is some $m > n$ such that $\mathbb{P}(X_m = b \mid X_n = a) > 0$. Clearly, the Rademacher random walk that we just constructed is not irreducible. However, imposing irreducibility does not change the situation, as the following lemma shows.

Lemma 9. *Let $f : \mathbb{N} \rightarrow [1, \infty)$ be any function such that $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists a non-decreasing sequence $(a_n)_{n \geq 1}$ of integer step sizes for which $a_n \leq f(n)$ for all n and the associated Rademacher random walk is both irreducible and transient.*

To address the case where $(a_n)_{n \geq 1}$ is bounded, we prove the following lemma, which extends a theorem of Bhattacharya and Volkov [3, Theorem 2].

Lemma 10. *Let $(a_n)_{n \geq 1}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2 = \infty$, and let $X = (X_n)_{n \geq 0}$ be the associated Rademacher random walk. Then, almost surely, X is unbounded below and unbounded above, and in particular, X changes sign infinitely often.*

In [3] it was shown that X changes sign infinitely often almost surely under the stronger assumption that (a_n) is a non-decreasing sequence of positive reals. Note that the condition $\sum_{n=1}^{\infty} a_n^2 = \infty$ is necessary, since otherwise X_n almost surely converges to a random limit. When X converges, it changes sign only finitely many times.

Lemma 10 implies that if the sequence (a_n) is bounded, say $a_n \leq C$ for all n , and $\sum_{n=1}^{\infty} a_n^2 = \infty$, then $\mathbb{P}(|X_n| < C \text{ i.o.}) = 1$, so the Rademacher random walk is weakly recurrent. Recall that if $\sum_{n=1}^{\infty} a_n^2$ converges, then the Rademacher random walk converges to a random limit and the walk is also weakly recurrent. Putting these two cases together gives the following corollary.

Corollary 11. *If the sequence $(a_n)_{n \geq 1}$ is bounded, the associated Rademacher random walk is weakly recurrent.*

We also have another immediate corollary of Lemma 10:

Corollary 12. *If the sequence $(a_n)_{n \geq 1}$ satisfies $\sum_{n=1}^{\infty} a_n^2 = \infty$ and $a_n \rightarrow 0$ as $n \rightarrow \infty$, then the associated Rademacher random walk is topologically recurrent.*

This is a special case of the well-known fact (which it easily implies by conditioning on the step sizes) that any symmetric random walk with bounded but not necessarily identically distributed steps is weakly recurrent.

Although we know that there is a weakly recurrent Rademacher walk with $a_n = \Theta(n^{1/2})$, we have not determined the behaviour in the natural case where $a_n = n^\alpha$ for any α in the range $0 < \alpha \leq 1/2$. We can, however, say something about the possible behaviours in this range by applying the following theorem.

Theorem 13. *Suppose the sequence (a_n) is unbounded and $a_n - a_{n-1} \rightarrow 0$. Then the associated Rademacher random walk is either transient or topologically recurrent.*

Theorem 1 shows that the transient case may occur, and the recurrence result of [3] concerning the sequence $a_n = \log n$ shows that the topologically recurrent case may occur.

The rest of the paper is organised as follows. In §2 we discuss a number of known anti-concentration results and prove a simple but useful lemma about combining anti-concentration at different scales. In §3 we prove some anti-concentration estimates for Rademacher sums, including a (mod m) analogue of the Erdős–Littlewood–Offord inequality (Lemma 15). In §4 we prove the anti-concentration bound Theorem 7, from which we have already deduced Theorem 1, and the

concentration bound Proposition 8. In §5 we prove Theorem 2 by exhibiting an explicit sequence $(a_n)_{n \geq 1}$ such that $a_n = \Theta(n^{1/2})$ as $n \rightarrow \infty$ and the associated Rademacher walk is weakly recurrent. In §6 we prove Lemma 9 and Lemma 10. In §7 we prove Theorem 5 and deduce Corollary 6. In §8 we prove Theorem 13. Finally, in §9 we prove Theorem 3 and Theorem 4.

2 Anti-concentration inequalities

In this section we recall several well-known anti-concentration inequalities that will be used later on, and we give a simple lemma which allows us to combine the anti-concentration of two random variables at different scales.

Recall that the *concentration function* Q of a real-valued random variable A is given by

$$Q_r(A) = \sup_{x \in \mathbb{R}} \mathbb{P}(x < A \leq x + r).$$

Note that for any integer $m \geq 1$, the union bound gives

$$Q_{mr}(A) \leq mQ_r(A).$$

We start with the general case of a one-dimensional Rademacher random walk X whose step sizes are general real numbers, not necessarily integers and not necessarily separated.

Lemma 14 (Erdős–Littlewood–Offord inequality). *If the step sizes $(a_n)_{n \geq 1}$ of a Rademacher walk $X = (X_n)_{n \geq 1}$ are all greater than or equal to a constant $c > 0$, then*

$$Q_{2c}(X_n) \leq \binom{n}{\lfloor n/2 \rfloor} 2^{-n} \sim \sqrt{\frac{2}{\pi n}} \text{ as } n \rightarrow \infty. \quad (3)$$

Erdős’ simple proof of this in [7] (improving a slightly worse bound by Littlewood and Offord) was to note that for any $x \in \mathbb{R}$, the set of assignments of $(\epsilon_1, \dots, \epsilon_n)$ for which $\sum_{i=1}^n \epsilon_i a_i \in (x, x + c]$ form an anti-chain in the hypercube $\{-1, 1\}^n$ with the coordinatewise partial order, and then to apply Sperner’s theorem [21] about the size of the largest anti-chain. In §3 we will prove the following (mod m) analogue of Lemma 14 for the case of integer step sizes.

Lemma 15. *Let m be a positive integer and let $b_1, \dots, b_n$ be positive integers coprime to m . Let $\epsilon_1, \dots, \epsilon_n$ be independent Rademacher random variables, and let $X_n = \sum_{i=1}^n \epsilon_i b_i$. Then*

$$\max_{r \in \mathbb{Z}/m\mathbb{Z}} \mathbb{P}(X_n \equiv r \pmod{m}) \leq \begin{cases} \frac{1}{m} + \sqrt{\frac{2}{\pi n}} & \text{if } m \text{ is odd,} \\ \frac{2}{m} + \sqrt{\frac{2}{\pi n}} & \text{if } m \text{ is even.} \end{cases}$$

Lemma 15 is a slight improvement of a very special case of the following very general result of Juškevičius and Šemetulskis [11, Cor. 1] about anti-concentration of random products in arbitrary groups:

Theorem 16. *Let G be a group and $m \geq 2$ an integer. Let $g_1, \dots, g_n$ be elements of order at least m , and consider the G -valued random variable $X = g_1^{\epsilon_1} * \dots * g_n^{\epsilon_n}$, where the ϵ_i are independent Rademacher(1/2) random variables. Then*

$$\sup_{g \in G} \mathbb{P}(X = g) \leq \frac{1}{\lceil m/2 \rceil} + \sqrt{\frac{2}{\pi n}}.$$

We include a self-contained proof of Lemma 15 that is somewhat different from the method in [11].

The Erdős–Littlewood–Offord inequality is a special case of a more general result, known as the Kolmogorov–Rogozin inequality, which gives anti-concentration bounds on a sum of independent random variables using bounds on the anti-concentration of the summands. The first results of this form were shown by Doeblin and Levy, before being improved by Kolmogorov, then Rogozin and then Kesten. The sharpest possible bounds were obtained recently by Juškevičius in [10], which summarizes the history of the problem.

For many applications, the following form of the Kolmogorov–Rogozin inequality suffices.

Theorem 17 (Rogozin, [18]). *There is a $C > 0$ such that for any independent random variables $X_1, \dots, X_n$ and real numbers $0 < \lambda_1, \dots, \lambda_n \leq 2r$,*

$$Q_r(X_1 + \dots + X_n) \leq C \cdot r \cdot \left(\sum_{i=1}^n \lambda_i^2 (1 - Q_{\lambda_i}(X_i)) \right)^{-1/2}.$$

Although we will not use it directly, we mention for completeness a key tool in proving many anti-concentration inequalities:

Lemma 18 (Esseen’s inequality [8]). *If X is a real-valued random variable with characteristic function ψ , then $Q_r(X) \leq t \int_{-2\pi/r}^{2\pi/r} |\psi(\lambda)| d\lambda$.*

Let us now consider the case where the step sizes are separated. Let $Y = (Y_n)_{n \geq 0}$ be the Rademacher random walk with step sizes 1, 2, 3, etc. It is known that for any Rademacher random walk $(X_n)_{n \geq 0}$ with *distinct positive integer* step-sizes, we have the anti-concentration bound

$$Q_1(X_n) = \sup_{x \in \mathbb{Z}} \mathbb{P}(X_n = x) \leq \mathbb{P}(Y_n = 0) \sim \frac{\sqrt{6/\pi}}{n^{3/2}} \text{ as } n \rightarrow \infty. \quad (4)$$

The inequality in (4) was proven by Richard Stanley [22], using enumerative algebraic geometry and answering a question of Erdős and Moser. Another wonderful proof using Lie algebras was given shortly afterwards by Proctor [17]. The asymptotic in (4) is due to Sullivan [24]. Before Stanley’s result, Sárközi and Szemerédi [19] had shown that $\sup_{x \in \mathbb{Z}} \mathbb{P}(X_n = x) = O(n^{-3/2})$. From their bound it already follows by a simple Borel–Cantelli argument that any Rademacher random walk whose step sizes are distinct positive integers must be transient (see [3, Theorem 4]).

Halász extended the result of Sárközi and Szemerédi as follows.

Lemma 19 (Halász [9, Theorem 2]). *Consider n vectors $a_1, \dots, a_n \in \mathbb{R}^d$ such that for any unit vector e , we have $|\langle e, a_i \rangle| \geq 1$ for at least δn values of i , and also $\|a_i - a_j\| \geq 1$ whenever $i \neq j$. Then*

$$\mathbb{P}\left(\sum_{i=1}^n \epsilon_i a_i \in B(x, 1)\right) \leq c(\delta, d) n^{-1-d/2}, \quad (5)$$

where $c(\delta, d)$ is a constant that does not depend on n .

In the case $d = 1$, the inner product condition involving δ may be dropped since it is implied by the separation condition for large enough n . We remark that (5) already gives sufficient anti-concentration to show that the Rademacher random walk with step sizes $a_n = n^\alpha$ is transient for any $\alpha > 2/3$. Indeed, if $\alpha \geq 1$, then $|a_i - a_j| \geq 1$ for all $i \neq j$, and we can immediately apply Lemma 19 in the case $d = 1$ to get an anti-concentration bound of $O(n^{-3/2})$ (and then we can finish by applying the Borel–Cantelli lemma). On the other hand, if $\alpha < 1$, then $a_{n+1} - a_n < 1$ and there is some a_i in each interval $[k, k+1)$. Hence, we can find a subsequence $a_{i_1}, a_{i_2}, \dots$ of length $\Theta(n^\alpha)$ such that $a_{i_j} \in [2j, 2j+1)$, and the theorem gives an anti-concentration bound of $O(n^{-3\alpha/2})$, which suffices to finish using Borel–Cantelli when $\alpha > 2/3$.

All of the bounds above deal with step sizes which are of a somewhat similar scale. If the step sizes grow exponentially with base at least 2, then $Q_1(X_n) = 1/2^n$ and there is much better anti-concentration than given by any of the results above. While we will not be dealing with scales quite as different as this, combining anti-concentration at different scales and (nearly) multiplying the anti-concentration bounds of each is the key to our proof. For this we use the following lemma.

Lemma 20 (Combining anti-concentration bounds at different scales).

Let $0 < r < s$ and let A and B be independent real-valued random variables. Then

$$Q_r(A + B) \leq \mathbb{P}(|A| \geq s) + 3Q_r(A)Q_s(B).$$

Hence,

$$Q_r(A + B) \leq (1 - Q_{2s}(A)) + 3Q_r(A)Q_s(B).$$

Proof. For any $x \in \mathbb{R}$ we have

$$\begin{aligned} \mathbb{P}(x < A + B \leq x + r) &\leq \mathbb{P}(|A| \geq s) + \mathbb{P}(|A| \leq s \text{ and } x < A + B \leq x + r) \\ &\leq \mathbb{P}(|A| \geq s) + \mathbb{P}(x - s < B \leq x + r + s) \cdot \sup_{b \in \mathbb{R}} \mathbb{P}(x - b < A \leq x - b + r) \\ &\leq \mathbb{P}(|A| \geq s) + 3Q_s(B)Q_r(A). \end{aligned}$$

Since we may replace A by $A + c$ for any constant c without changing $Q_r(A)$ or $Q_r(A + B)$, the final statement follows. $\square$

3 Modular anti-concentration of Rademacher sums

In the course of several of our proofs, we will need to control the probability that a sum of Rademacher random variables takes a particular value, or lies in

a particular congruence class modulo some positive integer. In this section we establish some useful results of these kinds.

For a real random variable X , the *characteristic function* of X is the function $\psi_X : \mathbb{R} \rightarrow \mathbb{C}$ given by $\psi_X(t) = \mathbb{E}(e^{itX})$. We say that a real random variable X is *monotone* if $|\psi_X(t)|$ is decreasing on $[0, \pi]$.

Theorem 21 ([1, Theorem 1.1]). *Let $X_1, \dots, X_k$ be independent integer-valued random variables with $\mathbb{E}(X_i) = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2 < \infty$. Suppose that their sum $X = X_1 + \dots + X_k$ is a monotone random variable with mean μ and variance σ^2 . Then, for every t for which $\mu + t\sigma$ is an integer, we have*

$$\left| \mathbb{P}(X = \mu + t\sigma) - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2} \right| \leq c \left(\frac{\sum_{i=1}^k \mathbb{E}(|X_i - \mu_i|^3)}{\sigma^3} \right)^2,$$

where c is a universal constant.

Note that the sum of independent monotone random variables is monotone and that Bernoulli(1/2) random variables are monotone. Hence, by applying the above theorem to a suitably shifted and rescaled sum, we obtain the following local limit theorem for sums of Rademacher random variables.

Corollary 22. *Let X be the sum of n Rademacher random variables. Then, for any $x \equiv n \pmod{2}$, we have*

$$\left| \mathbb{P}(X = x) - \frac{1}{\sqrt{\pi n/2}} e^{-\frac{x^2}{2n}} \right| \leq \frac{c}{n},$$

where c is an absolute constant.

In the remainder of this section we establish Lemma 15, which we restate here.

Lemma 15. *Let m be a positive integer and let $b_1, \dots, b_n$ be positive integers coprime to m . Let $\epsilon_1, \dots, \epsilon_n$ be independent Rademacher random variables, and let $X_n = \sum_{i=1}^n \epsilon_i b_i$. Then*

$$\max_{r \in \mathbb{Z}/m\mathbb{Z}} \mathbb{P}(X_n \equiv r \pmod{m}) \leq \begin{cases} \frac{1}{m} + \sqrt{\frac{2}{\pi n}} & \text{if } m \text{ is odd,} \\ \frac{2}{m} + \sqrt{\frac{2}{\pi n}} & \text{if } m \text{ is even.} \end{cases}$$

Before giving the proof, let us make a few observations about this result.

Fixing arbitrary positive integers $b_1, \dots, b_n$ and letting $m \rightarrow \infty$ along the primes gives an alternative proof of the integer case of the Erdős–Littlewood–Offord inequality, Lemma 14.

On the other hand, taking each $b_i = 1$ in gives a simpler modular anti-concentration result that is still useful.

Corollary 23. *Let $m \geq 2$ be an integer and let $X_n = \sum_{i=1}^n \epsilon_i$ be the sum of n independent Rademacher random variables. For every $\varepsilon > 0$, if $n \geq \varepsilon m^2$, we have*

$$\max_{0 \leq r < m} \mathbb{P}(X_n \equiv r \pmod{m}) < c(\varepsilon)/m,$$

where $c(\varepsilon) = 1 + \sqrt{\frac{2}{\pi\varepsilon}}$.

Remark 24. *For the case where $n \leq m^2$ and the obvious parity condition is satisfied, a complementary lower bound for $\mathbb{P}(X \equiv r \pmod{m})$, also of order $1/m$, may be found in [3, Corollary 3.2].*

If p is an odd prime, then a p -adic formal analogue of the Erdős–Littlewood–Offord inequality follows from Lemma 15, as follows. If $b_1, \dots, b_n \in \mathbb{Q}_p$ all satisfy $\|b_i\|_p \geq c$, then for any $z \in \mathbb{Q}_p$ we have

$$\mathbb{P}(X_n \in B(z, c)) \leq \frac{1}{p} + \sqrt{\frac{2}{\pi n}},$$

where $B(z, c)$ here means the open ball in $\mathbb{Q}_p$ of p -adic radius c about z .

In the proof of Lemma 15, we will need the following easy consequence of the rearrangement inequality.

Lemma 25. *Let $n, m \geq 1$ and let $y_1, \dots, y_m$ be given positive numbers. Consider the set $\mathcal{A}$ of n by m matrices in which the elements of each row are some permutation of $(y_1, \dots, y_m)$. Then*

$$\max_{A \in \mathcal{A}} \sum_{j=1}^m \prod_{i=1}^m A_{ij} = \sum_{j=1}^m y_j^n.$$

Proof. The set $\mathcal{A}$ is finite, so there exists a matrix $A \in \mathcal{A}$ which maximizes $\sum_{j=1}^m \prod_{i=1}^m A_{ij}$. For this A , the terms in each row must be sorted in the same as order as the products over the columns after that row is deleted, by the rearrangement inequality. Hence, they are sorted in the same way as the overall column products, which means that they are all sorted in the same way. $\square$

We now turn to the proof of Lemma 15.

Proof of Lemma 15. For each $x \in \mathbb{Z}/m\mathbb{Z}$, let $\delta_x : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{C}$ be the indicator function of x , and for each $\lambda \in \mathbb{Z}/m\mathbb{Z}$, let $e_\lambda : \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{C}$ be the function $e_\lambda(x) = \exp(2\pi i \lambda x / m) / \sqrt{m}$. Note that the functions e_λ form an orthonormal basis of $\ell^2(\mathbb{Z}/m\mathbb{Z})$. We have

$$\delta_0 = \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} \frac{1}{\sqrt{m}} e_\lambda.$$

A calculation shows that the effect on e_λ of convolution with $\frac{1}{2}(\delta_b + \delta_{-b})$ is to multiply it by $\cos(2\pi b \lambda / m)$. Hence, the probability mass function of X_n is given by

$$\mathbb{P}(X_n \equiv r \pmod{m}) = \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} \frac{1}{\sqrt{m}} \prod_{i=1}^n \cos(2\pi b_i \lambda / m) e_\lambda(r)$$

Since $|e_\lambda(r)| = \frac{1}{\sqrt{m}}$, the triangle inequality gives that

$$\mathbb{P}(X_n \equiv r \pmod{m}) \leq \frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} \prod_{i=1}^n |\cos(2\pi b_i \lambda/m)|. \quad (6)$$

The multi-set of values $(\cos(2\pi b \lambda/m) : \lambda \in \mathbb{Z}/m\mathbb{Z})$ is the same for every integer b coprime to m . Therefore, by Lemma 25, the right-hand side of (6) is maximized when all b_i are congruent (mod m) to any constant value b that is coprime to m , for example $b = 1$. In this case it takes the value $\frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} |\cos(2\pi \lambda/m)|^n$. In the case where m is even,

$$\frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} |\cos(2\pi \lambda/m)|^n = \frac{1}{m} \left(2 + 4 \sum_{\ell=1}^{\lfloor m/4 \rfloor} \cos(2\pi \ell/m)^n \right).$$

In the case where m is odd,

$$\frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} |\cos(2\pi \lambda/m)|^n = \frac{1}{m} \left(1 + 2 \sum_{\ell=1}^{\lfloor m/2 \rfloor} \cos(\pi \ell/m)^n \right).$$

Using the inequality $\cos x \leq e^{-x^2/2}$, which holds for $x \in [-\pi/2, \pi/2]$, for even m we obtain

$$\begin{aligned} \frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} |\cos(2\pi \lambda/m)|^n &\leq \frac{1}{m} \left(2 + 4 \sum_{\ell=1}^{\infty} e^{-2\pi^2 \ell^2 n/m^2} \right) \\ &\leq \frac{1}{m} \left(2 + 4 \int_0^{\infty} e^{-(4\pi^2 n/m^2) \cdot x^2/2} dx \right) \\ &= \frac{2}{m} + \sqrt{\frac{2}{\pi n}}, \end{aligned}$$

and similarly for odd m we obtain

$$\begin{aligned} \frac{1}{m} \sum_{\lambda \in \mathbb{Z}/m\mathbb{Z}} |\cos(2\pi \lambda/m)|^n &\leq \frac{1}{m} \left(1 + 2 \sum_{\ell=1}^{\infty} e^{-\pi^2 \ell^2 n/(2m^2)} \right) \\ &\leq \frac{1}{m} + \sqrt{\frac{2}{\pi n}}. \end{aligned}$$

□

4 Anti-concentration of Rademacher random walks

The main aims of this section are to prove the anti-concentration bound Theorem 7 and the concentration bound Proposition 8.

Our main tool in proving Theorem 7 is Lemma 20, for which we need both anti-concentration and concentration bounds on A . For the concentration bound, we will use the following well-known bound (see e.g. [15]), which is a straightforward application of Hoeffding's inequality.

Lemma 26.

$$\mathbb{P}\left(\sum_{i=1}^n a_i \epsilon_i \geq t \|a\|_2\right) \leq e^{-t^2/2}.$$

The result in Theorem 7 only requires the bounds on a_n to hold for large enough n , and there is no control over the initial terms. The following result shows that modifying a prefix of length m of the sequence (a_n) can only change the concentration by a factor of at most 2^{m+1} , and in particular, this will allow us to assume that the bounds on a_n hold for all n when proving Theorem 7.

Lemma 27. *Let $(a_n)_{n \geq 1}$ be a sequence and let $(a'_n)_{n \geq 1}$ be a sequence which differs from (a_n) only in the first m terms. Let $(X_n)_{n \geq 0}$ and $(X'_n)_{n \geq 0}$ be the corresponding Rademacher random walks. Then, for every n ,*

$$2^{-(m+1)} Q_r(X'_n) \leq Q_r(X_n) \leq 2^{m+1} Q_r(X'_n).$$

Proof. Suppose that A and B are discrete independent random variables. We claim that

$$\frac{Q_r(A)Q_r(B)}{2} \leq Q_r(A+B) \leq Q_r(A). \quad (7)$$

For the upper bound we have

$$\begin{aligned} \mathbb{P}(x < A+B \leq x+r) &= \sum_b \mathbb{P}(x-b < A \leq x-b+r) \mathbb{P}(B=b) \\ &\leq \sup_b \mathbb{P}(x-b < A \leq x-b+r) \cdot \sum_b \mathbb{P}(B=b) \\ &\leq \sup_x \mathbb{P}(x < A \leq x+r) \\ &= Q_r(A), \end{aligned}$$

which implies $Q_r(A+B) \leq Q_r(A)$.

For the lower bound, let $\varepsilon > 0$ be given and choose x, y such that

$$\begin{aligned} \mathbb{P}(x < A \leq x+r) &\geq (1-\varepsilon)Q_r(A), \\ \mathbb{P}(y < B \leq y+r) &\geq (1-\varepsilon)Q_r(B). \end{aligned}$$

Then the events

$$\{A+B \in (x+y, x+y+r]\} \quad \text{and} \quad \{A+B \in (x+y+r, x+y+2r]\}$$

are disjoint and cover the event

$$\{x < A \leq x+r, y < B \leq y+r\}.$$

The latter event has probability at least $(1-\varepsilon)^2 Q_r(A)Q_r(B)$ and so one of the first two events must have probability at least $(1-\varepsilon)^2 Q_r(A)Q_r(B)/2$. Letting $\varepsilon \rightarrow 0$ we obtain the lower bound in (7).

Now let $A = \sum_{i=1}^m \epsilon_i a_i$ and $B = \sum_{i=m+1}^n \epsilon_i a_i$. Define A' by $A' = \sum_{i=1}^m \epsilon_i a'_i$, so that $X_n = A+B$ and $X'_n = A'+B$. Note that $Q_1(A), Q_1(A') \geq 2^{-m}$. We have

$$Q_r(A+B) \leq Q_r(B) \leq 2^{m+1} \frac{Q_r(A')Q_r(B)}{2} \leq 2^{m+1} Q_r(A'+B)$$

and

$$Q_r(A+B) \geq \frac{Q_r(A)Q_r(B)}{2} \geq \frac{Q_r(A)Q_r(A'+B)}{2} \geq 2^{-(m+1)}Q_r(A'+B).$$

□

We are now armed with all the tools we need to prove Theorem 7, which we restate here for convenience.

Theorem 7. *Let $(a_n)_{n \geq 0}$ be a sequence and suppose that there are constants $c, C, \alpha > 0$ and $\delta \geq 0$ such that, for all large enough n ,*

$$cn^\alpha \leq a_n \leq Cn^{\alpha+\delta}.$$

Then, for any $\gamma > 0$, we have the anti-concentration bound

$$Q_1\left(\sum_{i=1}^n \epsilon_i a_i\right) = O\left(n^{-\left(\frac{1}{2} + \alpha f(\alpha, \delta) - \gamma\right)}\right),$$

where

$$f(\alpha, \delta) = \begin{cases} \frac{\alpha^2}{(\alpha+\delta)(\alpha+2\delta+2\sqrt{\delta^2+\alpha\delta})} & \text{if } \delta \leq \frac{\sqrt{\alpha^2+1}-\alpha}{2}, \\ \frac{\alpha^2}{(\alpha+\delta)(1+2\delta)(\alpha+1/2+\delta)} & \text{if } \delta \geq \frac{\sqrt{\alpha^2+1}-\alpha}{2}. \end{cases}$$

Before we give the full details, let us briefly sketch the proof in the case where $a_n = n^\alpha$. The key idea is to group some of the terms in the sum $\sum_{i=1}^n \epsilon_i a_i$ into subsets which are at different scales. In the interval $[2^k, 2^{k+\alpha})$ we expect around $2^{k/\alpha}$ of the a_n , and let us set A to be the Rademacher sum of any such terms. Using the Erdős–Littlewood–Offord inequality, the anti-concentration of A at the scale 2^{k+1} is around $2^{-k/(2\alpha)}$. On the other hand, if k is much larger than α , we expect the magnitude of A to be around $2^k \cdot 2^{k/(2\alpha)}$, and there should be concentration around this scale. Hence, if we set $k' = (1 + 1/(2\alpha) + \varepsilon)k$ and let A' be the Rademacher sum of the terms in the interval $[2^{k'}, 2^{k'+\alpha})$, then the sum A' is anti-concentrated at the scale $2^{k'}$ while the sum A_k is concentrated at the $2^{k'}$. This means we can use Lemma 20 to combine the anti-concentration.

This can clearly be repeated and we will take a sequence $k_1, k_2, \dots$ where $k_i = (1 + 1/(2\alpha) + \varepsilon)^i$ and try to combine the anti-concentration of the random variables $A_1, A_2, \dots$, where we think of A_i as the sum of the terms $\epsilon_i a_i$ for which $a_i \in [2^{k_i}, 2^{k_i+\alpha})$. In actuality, we will have to widen the interval in which we take the a_n and also limit the number that we take from each interval.

To get the required anti-concentration for all large enough n , the anti-concentration from sums of the form $A_1 + \dots + A_m$ is not enough. Indeed, while we get the required anti-concentration of $n^{-(\alpha+1/2-\gamma)}$ when n is not much bigger than $2^{k_m/\alpha+1}$, the anti-concentration drops to around $n^{-\alpha}$ by the time n is close to $2^{k_{m+1}/\alpha}$. In order to apply Lemma 20 we need $A_1 + \dots + A_{i-1}$ to be concentrated at the scale of A_i for each i . We can achieve this by ensuring that none of the terms in A_j are too big for each $j < m$, but observe that we don't need A_m to be concentrated as it is the last summand. To get the required concentration for all n between $2^{k_m/\alpha+1}$ and $2^{k_{m+1}/\alpha+1}$, we will replace A_m by the sum of the terms $\epsilon_i a_i$ for which

$a_i \in [2^{k_m}, \infty)$ and $i \leq n$, which we can do as we do not need concentration for the last summand.

We make this argument rigorous below. We also weaken the conditions on a_n . This means we must make our intervals wider to be able to guarantee that we can find enough a_n which fall in the intervals. This in turn means that the intervals must be further apart and we pay a penalty in the anti-concentration that we obtain. However, this is unavoidable; there must be some penalty to pay (for large δ) as shown by Proposition 8.

Proof. Pick a constant λ such that $\delta/\alpha < \lambda$. We will later take λ to be arbitrarily close to δ/α .

By Lemma 27, we can modify the first m terms in the sequence and only change the anti-concentration by at worst a factor of 2^{m+1} . We will therefore assume that

$$cn^\alpha \leq a_n \leq Cn^{\alpha+\delta}$$

holds for all $n \geq 1$ and not just n large enough. Using our assumptions on a_n , if n is such that

$$(2^k/c)^{1/\alpha} \leq n \leq (2^{(1+\lambda)k}/C)^{1/(\alpha+\delta)}$$

then $2^k \leq a_n \leq 2^{(1+\lambda)k}$. Hence, the number of n for which a_n is in the interval $[2^k, 2^{(1+\lambda)k}]$ is at least

$$\left(\frac{2^{(1+\lambda)k}}{C}\right)^{\frac{1}{\alpha+\delta}} - 1 - \left(\frac{2^k}{c}\right)^{1/\alpha} = \left(\frac{2^{\frac{\alpha\lambda-\delta}{\alpha(\alpha+\delta)}k}}{C^{1/(\alpha+\delta)}} - \frac{1}{c^{1/\alpha}}\right)2^{k/\alpha} - 1.$$

Since we have taken $\alpha\lambda > \delta$, this certainly is at least $2^{k/\alpha}$ for large enough k .

Now fix $\beta \geq \alpha$ and $\varepsilon > \lambda$, and define k_i by

$$k_i = \left(1 + \frac{1}{2\beta} + \varepsilon\right)^i.$$

We will later take ε arbitrarily close to λ and choose the value of $\beta \geq \alpha$ to optimise the anti-concentration bound. Let A_i be the random sum corresponding to the first $\lceil 2^{k_i/\beta} \rceil$ of the a_n which lie in the interval $[2^{k_i}, 2^{(1+\lambda)k_i}]$, noting that this is possible when k_i is large by the above argument and our assumption that $\beta \geq \alpha$.

We now claim that the anti-concentration of the sum of the A_i is roughly equal to the product of their individual anti-concentrations.

Claim 28. *There is $M > 0$ such that*

$$Q_1(A_1 + \dots + A_m) \leq M \cdot 3^m \cdot 2^{-\frac{(1+1/(2\beta)+\varepsilon)^{m+1}}{1+2\beta\varepsilon}}$$

for all m .

Proof. First, note that by the Erdős–Littlewood–Offord inequality (Lemma 14), we have

$$Q_{2^{k_i+1}}(A_i) \leq \sqrt{\frac{2}{\pi}} \cdot 2^{-\frac{k_i}{2\beta}}$$

for large enough i . We will induct on m , taking $s = 2^{k_m+1}$ and $r = 1$ in Lemma 20 to get anti-concentration at different scales.

To apply Lemma 20, we first need to show that the sum $A_1 + \dots + A_{m-1}$ is concentrated at the scale s . Using Lemma 26, we have that

$$\begin{aligned} \mathbb{P}(|A_1 + \dots + A_{m-1}| \geq s) &\leq \sum_{i=1}^{m-1} \mathbb{P}\left(|A_i| \geq \frac{s}{(m-1)}\right) \\ &\leq 2(m-1) \exp\left(-\frac{s^2}{4(m-1)^2 2^{(2+2\lambda+1/\beta)k_{m-1}}}\right). \end{aligned}$$

Substituting in $s = 2^{k_m+1} = 2 \cdot 2^{(1+\frac{1}{2\beta}+\varepsilon)k_{m-1}}$, we have

$$\mathbb{P}(|A_1 + \dots + A_{m-1}| \geq 2^{k_m+1}) \leq 2(m-1) \exp\left(-\frac{2^{2(\varepsilon-\lambda)k_{m-1}}}{(m-1)^2}\right).$$

Pick m_0 such that for all $m \geq m_0$, the quantity given above is at most $2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}}$, and choose $M \geq 1$ such that

$$Q_1(A_1 + \dots + A_m) \leq M \cdot 3^m \cdot 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}}$$

holds for all $m \leq m_0$.

Now suppose that the claimed bound holds for $m-1 \geq m_0$. Then, using the above, we have

$$\begin{aligned} Q_1(A_1 + \dots + A_m) &\leq 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}} + 3 \cdot M 3^{m-1} 2^{-\frac{(1+1/(2\beta)+\varepsilon)m}{1+2\beta\varepsilon}} \cdot \sqrt{\frac{2}{\pi}} 2^{-\frac{k_m}{2\beta}} \\ &= 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}} + M \cdot 3^m \cdot \sqrt{\frac{2}{\pi}} \cdot 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}} \\ &= \left(1 + M \cdot 3^m \cdot \sqrt{\frac{2}{\pi}}\right) 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}} \\ &\leq M \cdot 3^m \cdot 2^{-\frac{(1+1/(2\beta)+\varepsilon)m+1}{1+2\beta\varepsilon}}. \end{aligned}$$

□

Pick N and suppose that $2(2^{(1+\lambda)k_m}/c)^{1/\alpha} \leq N < 2(2^{(1+\lambda)k_{m+1}}/c)^{1/\alpha}$. Let us assume that N is large enough that $m \geq m_0$, where m_0 is defined as in the claim above. Note that $a_n > 2^{(1+\lambda)k_m}$ for all $n \geq N$ so, in particular, the sum X_N contains all of the terms in the sum $A_1 + \dots + A_m$. Let B be the Rademacher sum of the a_n for which $n \leq N$ and $a_n \geq 2^{k_m}$, so that the terms in this sum are a superset of the terms in A_m . We will bound the anti-concentration of $A_1 + \dots + A_{m-1} + B$ using Lemma 20.

First, note that B contains at least

$$N - \left(\frac{2^{k_m}}{c}\right)^{1/\alpha} \geq \frac{N}{2}$$

terms, all of which are at least 2^{k_m} . By the Erdős–Littlewood–Offord inequality, Lemma 14, we have

$$Q_{2^{k_m+1}}(B) \leq \sqrt{\frac{4}{\pi N}}.$$

We again have the concentration bound

$$\mathbb{P}(|A_1 + \dots + A_{m-1}| \geq 2^{k_m+1}) \leq 2(m-1) \exp\left(-\frac{2^{2(\varepsilon-\lambda)k_{m-1}}}{(m-1)^2}\right),$$

but this time we bound it directly. Since $m = \Theta(\log \log(N))$ and this term is $2^{-2^{2\Theta(m)}}$, it decays quicker than any polynomial. The term 3^m is also at most polylogarithmic in N , so is certainly in $O(N^{\gamma/2})$ for any fixed $\gamma > 0$.

We also have that

$$\begin{aligned} 2^{-\frac{(1+1/(2\beta)+\varepsilon)^m}{1+2\beta\varepsilon}} &= 2^{-\frac{(1+\lambda)(1+1/(2\beta)+\varepsilon)^{m+1}}{\alpha} \cdot \frac{\alpha}{(1+2\beta\varepsilon)(1+\lambda)(1+1/(2\beta)+\varepsilon)}} \\ &\leq \left(\frac{c^{1/\alpha}N}{2}\right)^{-\frac{\alpha}{(1+2\beta\varepsilon)(1+\lambda)(1+1/(2\beta)+\varepsilon)}}. \end{aligned}$$

Using Lemma 20, we find that $Q_1(A_1 + \dots + A_{m-1} + B)$ is bounded by

$$M \cdot O(N^{\gamma/2}) \cdot \left(\frac{c^{1/\alpha}N}{2}\right)^{-\frac{\alpha}{(1+2\beta\varepsilon)(1+\lambda)(1+1/(2\beta)+\varepsilon)}} \cdot \sqrt{\frac{4}{\pi N}} + O\left(2^{-2^{2\Theta(\log \log(N))}}\right)$$

which is

$$O\left(N^{-1/2 - \frac{\alpha}{(1+2\beta\varepsilon)(1+\lambda)(1+1/(2\beta)+\varepsilon)} + \gamma/2}\right).$$

It remains to choose the values λ , ε and β , and we recall that we require $\alpha\lambda > \delta$, $\varepsilon > \lambda$ and $\beta \geq \alpha$. If $\delta < \frac{\sqrt{\alpha^2+1}-\alpha}{2}$, then we take $\beta = 1/(2\sqrt{\lambda^2+\lambda})$, else we take $\beta = \alpha$. We then take λ sufficiently close to δ/α and ε sufficiently close to λ so that $\beta \geq \alpha$ and

$$\frac{\alpha}{(1+2\beta\varepsilon)(1+\lambda)(1+1/(2\beta)+\varepsilon)} \geq \alpha f(\alpha, \delta) - \frac{\gamma}{2}.$$

□

From this it is easy to deduce the following corollary.

Corollary 29. *Let $(a_n)_{n \geq 1}$ be a sequence such that $a_n = n^{\alpha+o(1)}$ for some constant $\alpha > 0$, and let $(X_n)_{n \geq 0}$ be the associated Rademacher random walk. Then $Q_1(X_n) = O(n^{-(\alpha+1/2+o(1))})$.*

We end this section by proving Proposition 8, which we restate here for convenience.

Proposition 8. *Fix $\alpha > 0$. Then*

$$Q_1\left(\sum_{i=1}^n \epsilon_i n^\alpha\right) = \Omega(n^{-(1/2+\alpha)}).$$

Moreover, for any $\delta \geq 1/2$, there exists a sequence $(a_n)_{n \geq 1}$ of step sizes and a sequence of times $(n_i)_{i \geq 1}$ such that

$$Q_1 \left(\sum_{i=1}^{n_i} \epsilon_i a_i \right) \geq n_i^{-(1/2 + \frac{\alpha}{2\delta} + o(1))}.$$

Proof. First, let us consider the sequence $(a_n)_{n \geq 1}$ where $a_n = n^\alpha$, and let $(X_n)_{n \geq 1}$ be the associated Rademacher random walk. Chebyshev's inequality gives that

$$\mathbb{P}(|X_n| < 2\sqrt{\text{Var}(X_n)}) \geq \frac{3}{4}.$$

The interval $(-2\sqrt{\text{Var}(X_n)}, 2\sqrt{\text{Var}(X_n)})$ may be covered using no more than $\lceil 4\sqrt{\text{Var}(X_n)} \rceil$ intervals of the form $[x, x+1)$ so there exists $x \in \mathbb{R}$ such that

$$\mathbb{P}(X_n \in [x, x+1)) \geq \frac{3}{16\lceil \sqrt{\text{Var}(X_n)} \rceil}.$$

The bound now follows by substituting in

$$\text{Var}(X_n) = \sum_{k=1}^n (k^\alpha)^2 = \frac{n^{2\alpha+1}}{2\alpha+1} + O(n^{2\alpha}).$$

Now set $r = \delta/\alpha$ and consider the sequence where $a_n = 2^{(1+r)^k}$ for $2^{\frac{(1+r)^{k-1}}{\alpha}} \leq n < 2^{\frac{(1+r)^k}{\alpha}}$. Note that we have $n^\alpha \leq a_n \leq n^{\alpha+\delta}$.

Let A_k be the Rademacher sum of the a_n which are equal to $2^{(1+r)^k}$. The most likely value for A_k has probability

$$\binom{N}{\lfloor N/2 \rfloor} 2^{-N} \geq \sqrt{\frac{1}{\pi N}}$$

where $N \geq 2$ is the number of terms equal to $2^{(1+r)^k}$. We have that $N \leq 2^{\frac{(1+r)^k}{\alpha}}$, so the most likely value for A_k has probability at least $2^{-\frac{(1+r)^k}{2\alpha}}/\sqrt{\pi}$. Hence, the most likely value for $A_1 + \dots + A_k$ has probability at least

$$\frac{2^{-\frac{((1+r)^k - 1)(r+1)}{2r\alpha}}}{\pi^{k/2}} \geq \frac{2^{-\frac{(1+r)^{k+1}}{2r\alpha}}}{\pi^{k/2}}.$$

Hence, if we take $n_k = \lceil 2^{\frac{(1+r)^k}{\alpha}} \rceil - 1$ to the last step with step size $2^{(1+r)^k}$, then the most likely value has probability at least $n^{-(1/2 + 1/(2r))}/\pi^{k/2}$, as required. $\square$

5 A recurrent example with $a_n = \Theta(n^{1/2})$

Consider the sequence

$$(a_n)_{n \geq 1} = (3, 1, 5, 3, 5, 3, 5, 3, 5, 3, 9, 7, 9, 7, 9, 7, 9, 7, \dots)$$

which is made up of consecutive blocks, where the k^{th} block has length $4^k/2$ and the steps in the k^{th} block alternate between $2^k + 1$ and $2^k - 1$, starting with $2^k + 1$. Denote the index of the beginning of the $(k)^{\text{th}}$ block by n_k . That is,

$$n_1 = 1, n_2 = 3, n_3 = 11, \dots, n_k = 1 + \sum_{i=1}^{k-1} 4^i/2 = (4^k + 2)/6.$$

It is not hard to see that $\sqrt{n/2} \leq a_n \leq 3\sqrt{n}$ for all n , and so $a_n = \Theta(\sqrt{n})$. We will prove that the Rademacher random walk associated with this sequence is weakly recurrent (and hence prove Theorem 2). In fact we will show that the walk visits every even integer infinitely often.

Let $X = (X_n)_{n \geq 0}$ be the associated Rademacher random walk. Note that all the steps of X are odd integers, so to study the return times of X to 0 we may focus our attention on the random walk Y defined by $Y_n = X_{2n}$, which only visits even integers. We will use hitting probability estimates for the simple symmetric random walk on $\mathbb{Z}^2$ and the Kochen–Stone theorem to show that almost surely Y visits every even integer infinitely often. The Kochen–Stone theorem has been used before to prove recurrence of random walks; see e.g. [2, 16].

Theorem (Kochen and Stone [13, Theorem 1]). *Let $Z_1, Z_2, \dots$ be a sequence of random variables, each of which has nonzero mean and positive finite second moment. Suppose in addition that $\limsup_{n \rightarrow \infty} (\mathbb{E}(Z_n))^2 / \mathbb{E}(Z_n^2) > 0$. Then*

- (i) $\mathbb{P}(\liminf_{n \rightarrow \infty} Z_n / \mathbb{E}(Z_n) \leq 1) > 0$,
- (ii) $\mathbb{P}(\limsup_{n \rightarrow \infty} Z_n / \mathbb{E}(Z_n) \geq 1) > 0$, and
- (iii) $\mathbb{P}(\limsup_{n \rightarrow \infty} Z_n / \mathbb{E}(Z_n) > 0) \geq \limsup_{n \rightarrow \infty} (\mathbb{E}(Z_n))^2 / \mathbb{E}(Z_n^2)$.

Let E_k be the event that X visits 0 during the $(2k)^{\text{th}}$ block. We will show the following:

Lemma 30. *For E_k as defined above, we have $\mathbb{P}(E_k) = \Omega(1/k)$ as $k \rightarrow \infty$, so that $\sum_{k=1}^{\infty} \mathbb{P}(E_k) = \infty$. Moreover, there is a finite constant C such that for all $j < k$ we have*

$$\mathbb{P}(E_k \mid E_j) \leq C\mathbb{P}(E_k).$$

Before proving Lemma 30, let us explain how it implies our claim about the recurrence of the walk Y . Let $Z_n = \sum_{k=1}^n \mathbb{1}(E_k)$. From the first statement in Lemma 30 we deduce that $\mathbb{E}(Z_n) = \Omega(\log n)$, and in particular $\mathbb{E}(Z_n) \rightarrow \infty$ as $n \rightarrow \infty$. From the second statement in Lemma 30 we obtain for all $j \neq k$ that

$$\mathbb{P}(E_j \cap E_k) \leq C\mathbb{P}(E_j)\mathbb{P}(E_k),$$

so

$$\begin{aligned}
\mathbb{E}[Z_n^2] &= \mathbb{E}\left(\left(\sum_{k=1}^n \mathbb{1}_{E_k}\right)^2\right) \\
&= \sum_{k=1}^n \mathbb{P}(E_k) + \sum_{j=1}^n \sum_{k=1}^n \mathbb{1}_{\{j \neq k\}} \mathbb{P}(E_j \cap E_k) \\
&\leq \mathbb{E}(Z_n) + C \sum_{j=1}^n \sum_{k=1}^n \mathbb{1}_{\{j \neq k\}} \mathbb{P}(E_j) \mathbb{P}(E_k) \\
&\leq \mathbb{E}(Z_n) + C \left(\sum_{j=1}^n \mathbb{P}(E_j)\right)^2 \\
&= \mathbb{E}(Z_n) + C(\mathbb{E}(Z_n))^2
\end{aligned}$$

Since $\mathbb{E}(Z_n) \rightarrow \infty$ as $n \rightarrow \infty$ we obtain

$$\frac{(\mathbb{E}(Z_n))^2}{\mathbb{E}(Z_n^2)} \geq \frac{1}{C} - o(1).$$

This allows us to apply part (iii) of the Kochen–Stone theorem to deduce that

$$\mathbb{P}\left(\limsup_{n \rightarrow \infty} \frac{Z_n}{\mathbb{E}(Z_n)} > 0\right) \geq \frac{1}{C}.$$

Since $\mathbb{E}(Z_n) \rightarrow \infty$ as $n \rightarrow \infty$, this implies $\mathbb{P}(E_k \text{ occurs i.o.}) > 0$ and in particular $\mathbb{P}(X_{2n} = 0 \text{ i.o.}) > 0$. It then follows from Kolmogorov’s zero-one law that

$$\mathbb{P}(\exists r \in \mathbb{Z} : Y \text{ visits } r \text{ i.o.}) = 1,$$

since this is a tail event which occurs with positive probability.

From this it is easy to see that the probability that Y (and therefore X) visits every even integer is 1. Indeed, for every time t we have

$$\mathbb{P}(Y_{t+1} = Y_t + 2 \mid Y_t) = 1/4 = \mathbb{P}(Y_{t+1} = Y_t - 2 \mid Y_t),$$

and hence, for all $m, n \in 2\mathbb{Z}$,

$$\mathbb{P}(Y \text{ visits } m \text{ i.o. but does not visit } n \text{ i.o.}) = 0.$$

The result now follows by summing over the choices for m .

We are now ready to give a proof of Lemma 30.

Proof of Lemma 30. For any event E such that $\mathbb{P}(E) > 0$ and any integrable random variable $T \geq 0$ on the same probability space such that $\mathbb{E}(T) > 0$ and $T = 0$ on the complement of E , we have

$$\mathbb{P}(E) = \frac{\mathbb{E}(T \mathbb{1}_E)}{\mathbb{E}(T \mid E)} = \frac{\mathbb{E}(T)}{\mathbb{E}(T \mid E)}.$$

Take $E = E_k$, the event that X visits 0 during the $(2k)^{th}$ block, i.e. during the times $(n_{2k}, \dots, n_{2k+1} - 1)$, and let $T = T_k$, where T_k is the number of visits of X to 0 during the $(2k)^{th}$ block.

We will prove the following four estimates, where c_1 and c_2 are positive constants:

- (i) $\mathbb{E}(T_k) = \Omega(1)$ as $k \rightarrow \infty$,
- (ii) $\mathbb{E}(T_k \mid E_k) = O(k)$ as $k \rightarrow \infty$,
- (iii) $\mathbb{E}(T_k \mid E_j) < c_1$, for all j, k such that $j < k$,
- (iv) $\mathbb{E}(T_k \mid E_j \cap E_k) > c_2 k$ for all j, k such that $j < k$.

Observe that (i) and (ii) imply that

$$\mathbb{P}(E_k) = \frac{\mathbb{E}(T_k)}{\mathbb{E}(T_k \mid E_k)} = \Omega(1/k) \text{ as } k \rightarrow \infty.$$

On the other hand, (iii) and (iv) imply in the same way that

$$\mathbb{P}(E_k \mid E_j) = \frac{\mathbb{E}(T_k \mid E_j)}{\mathbb{E}(T_k \mid E_j \cap E_k)} < \frac{c_1}{c_2 k} \text{ for all } j, k \text{ with } j < k.$$

It follows that there is a finite constant C such that for all $j < k$

$$\mathbb{P}(E_k \mid E_j) \leq C \mathbb{P}(E_k).$$

We make the estimates (i)-(iv) by relating the portion of the walk Y during the $(2k)^{\text{th}}$ block to a simple symmetric random walk on $\mathbb{Z}^2$. Fix k for now. Let $m_0 = (n_{2k} - 1)/2$ and $m_1 = (n_{2k+1} - 1)/2$. Note that

$$m_1 - m_0 = \frac{n_{2k+1} - n_{2k}}{2} = 4^{2k}/2 = 2^{4k-2}.$$

We define a walk (a_m, b_m) for $m_0 \leq m \leq m_1$ by setting $(a_{m_0}, b_{m_0}) = (0, 0)$, and then, inductively for $m = m_0 + 1, \dots, m_1$,

$$(a_m, b_m) = (a_{m-1}, b_{m-1}) + \begin{cases} (1, 0) & \text{if } Y_m - Y_{m-1} = 2^{2k+1}, \\ (-1, 0) & \text{if } Y_m - Y_{m-1} = -2^{2k+1}, \\ (0, 1) & \text{if } Y_m - Y_{m-1} = 2, \\ (0, -1) & \text{if } Y_m - Y_{m-1} = -2. \end{cases}$$

Let L be the random arithmetic progression in $\mathbb{Z}^2$ defined by

$$L = \{(a, b) \in \mathbb{Z}^2 : 2^{2k+1}a + 2b + Y_{m_0} = 0\}.$$

For $m_0 \leq m \leq m_1$ we have $Y_m = 0$ if and only if $(a_m, b_m) \in L$. Thus, T_k is the number of times that (a_m, b_m) hits L .

Proof of (i) To get a lower bound on $\mathbb{E}(T_k)$, we first apply Chebyshev's inequality to show that Y_{m_0} is often not too large:

$$\begin{aligned} \text{Var}(Y_{m_0}) &= \text{Var}(X_{n_{2k-1}}) = \sum_{n=1}^{n_{2k}-1} a_n^2 \\ &= \sum_{n=1}^{2k-1} \frac{4^k}{4} ((2^n + 1)^2 + (2^n - 1)^2) \\ &\leq 4^{2k}/12 \cdot 2(1 + 2^{2k})^2 \\ &\leq 2^{8k}. \end{aligned}$$

Hence, using that $\mathbb{E}(Y_{m_0}) = 0$, we have that

$$\mathbb{P}(|Y_{m_0}| < 2^{1+4k}) \geq 1/4.$$

When $|Y_{m_0}| < 2^{1+4k}$, we can express Y_{m_0} as $2^{2k+1}\alpha + 2\beta$, where $|\alpha| \leq 2^{2k}$ and $|\beta| \leq 2^{2k}$. The expected number of visits of (a_m, b_m) to $(-\alpha, -\beta)$ is

$$\sum_{m=m_0+1}^{m_1} \mathbb{P}((a_m, b_m) = (-\alpha, -\beta)).$$

We now use the well-known observation about the simple symmetric two-dimensional random walk that $(a_t + b_t)_{t=m_0}^{m_1}$ and $(a_t - b_t)_{t=m_0}^{m_1}$ are independent simple symmetric random walks on $\mathbb{Z}$, started at 0 at $t = m_0$. By Corollary 22, whenever $m - m_0 > (m_1 - m_0)/2 = 2^{4k-3}$ and $m - m_0$ has the same parity as $\alpha + \beta$, we have

$$\begin{aligned} \mathbb{P}((a_m, b_m) = (-\alpha, -\beta)) &= \mathbb{P}(a_m + b_m = -\alpha - \beta) \mathbb{P}(a_m - b_m = -\alpha + \beta) \\ &\geq \left(\frac{1}{\sqrt{\pi(m - m_0)/2}} e^{-\frac{(\alpha+\beta)^2}{2(m-m_0)}} - \frac{c}{m - m_0} \right) \\ &\quad \cdot \left(\frac{1}{\sqrt{\pi(m - m_0)/2}} e^{-\frac{(\alpha-\beta)^2}{2(m-m_0)}} - \frac{c}{m - m_0} \right) \\ &\geq \left(\frac{e^{-16}}{\sqrt{\pi} 2^{4k-3}} - \frac{c}{2^{4k-3}} \right)^2 \\ &= \Omega(2^{-4k}). \end{aligned}$$

Since the number of values of m to which this applies is 2^{4k-4} , we obtain the asymptotic lower bound (i), i.e. $\mathbb{E}(T_k) = \Omega(1)$ as $k \rightarrow \infty$.

Proof of (ii) By conditioning on the time of the first visit to 0 during the $(2k)^{th}$ block, we find that

$$\mathbb{E}(T_k \mid E_k) \leq \mathbb{E}(T_k \mid X_{n_{2k}} = 0).$$

Hence, to prove (ii) it suffices to show that $\mathbb{E}(T_k \mid X_{n_{2k}} = 0) = O(k)$ as $k \rightarrow \infty$.

The expected number of returns to $(0, 0)$ in the first $2N$ steps of simple symmetric random walk on $\mathbb{Z}^2$ started at $(0, 0)$ is

$$\sum_{i=1}^N 2^{-2i} \binom{2i}{i}^2 = \Theta(\log(N)).$$

Hence, the expected number of visits of $(a_t, b_t)_{t=m_0}^{m_1}$ to $(0, 0)$ is $\Theta(\log(\frac{m_1 - m_0}{2})) = \Theta(k)$. We now show that the expected number of visits of $(a_t, b_t)_{t=m_0}^{m_1}$ to all the nonzero points in the arithmetic progression L_0 is $O(1)$, where

$$L_0 = \{(a, b) \in \mathbb{Z}^2 : 2^{2k+1}a + 2b = 0\} = \langle (1, -2^{2k}) \rangle.$$

$$\begin{aligned}
& \sum_{m=m_0+1}^{m_1} \mathbb{P}((a_m, b_m) = (n, -2^{2k}n)) \\
&= \sum_{m=m_0+1}^{m_1} \mathbb{P}(a_m + b_m = n(1 - 2^{2k})) \mathbb{P}(a_m - b_m = n(1 + 2^{2k})) \\
&\leq 2^{4k-2} \sup_{m \geq 2^{4k}n} \mathbb{P}(a_m + b_m = n(1 - 2^{2k})) \mathbb{P}(a_m - b_m = n(1 + 2^{2k})) \\
&\leq 2^{4k-2} \sup_{m \geq 2^{4k}n} \left(\frac{1}{\sqrt{\pi m/2}} e^{-\frac{(n(2^{2k}-1))^2}{2m}} + \frac{c}{m} \right)^2 \\
&\leq 2^{4k-2} \left(\sqrt{\frac{2}{\pi e(n(2^{2k}-1))^2}} + \frac{c}{2^{4k}n} \right)^2 \\
&\leq c/n^2,
\end{aligned}$$

Summing over all $n \in \mathbb{Z} \setminus \{0\}$ gives an upper bound of $c\pi^2/3$, and so we have proved estimate (ii).

Proof of (iii) For estimate (iii), we have

$$\begin{aligned}
\mathbb{E}(T_k \mid E_j) &\leq \max_{r \in 2\mathbb{Z}} \mathbb{E}(T_k \mid X_{n_{2j+1}} = r) \\
&\leq \sum_{n=n_{2k}}^{n_{2k+1}} Q_1(X_n - X_{n_{2j+1}}) \\
&\leq \frac{4^{2k}}{2} \cdot Q_1(X_{n_{2k}} - X_{n_{2k-1}}).
\end{aligned}$$

Let m_k be the number of pairs $(2i, 2i+1)$ with $n_{2k-1} \leq 2i < n_{2k}$ for which $\epsilon_{2i} = -\epsilon_{2i+1}$. Then m_k is a binomial random variable with $4^{2k}/4$ trials and success probability $1/2$. A Chernoff bound immediately implies that the probability that m_k lies in the interval $[4^{2k}/12, 4^{2k}/6]$ is $1 - o(4^{-2k})$. Conditional on the value of m_k , the increment $X_{n_{2k}} - X_{n_{2k-1}}$ is expressible as a sum $2A_1 + 2^{2k+1}A_2$, where A_1 is a sum of m_k Rademacher random variables, A_2 is a sum of $4^{2k}/4 - m_k$ Rademacher random variables, and A_1 and A_2 are independent.

Now condition on m_k , and assume that $m_k \in [4^{2k}/12, 4^{2k}/6]$. Consider the digits of $A_1 + 2^k A_2$ in base 4^k . The units digit is determined by A_1 alone, and by Corollary 23 there is a constant c not depending on k or on m_k such that $\sup_r \mathbb{P}(A_1 \equiv r \pmod{4^k}) \leq c/4^k$.

Conditional on m_k and A_1 , the next digit is determined by A_2 and again $\sup_s \mathbb{P}(A_2 \equiv s \pmod{4^k}) \leq c/4^k$. Hence, each possible value of the last two digits occurs with probability at most $c^2/4^{2k}$. It follows that

$$Q_1(X_{n_{2k}} - X_{n_{2k-1}}) \leq c^2/4^{2k} + o(4^{-2k}).$$

This proves estimate (iii).

Proof of (iv) For estimate (iv), we begin by noting that the arguments used for estimates (i) and (ii) still work when we further condition on E_j . Indeed, the

estimate for (ii) is unchanged (as we immediately condition on $X_{n_{2k}} = 0$) and the estimate for (i) is only improved by conditioning on E_j (as this reduces the variance of Y_{m_0}). Hence, we have

$$\mathbb{P}(E_k \mid E_j) \geq c/k,$$

for a constant $c > 0$ that does not depend on j . Let A_k be the event that X visits 0 between times n_{2k} and $n_{2k+1} - 4^k$. Let T'_k be the number of visits during this interval, so $A_k = \{T'_k \geq 1\}$ and $E_k \setminus A_k \subseteq \{T_k - T'_k \geq 1\}$. This implies that $\mathbb{P}(A_k^c \mid E_k \cap E_j) \leq \mathbb{E}(T_k - T'_k \mid E_k \cap E_j)$, and hence

$$\mathbb{E}(T_k - T'_k \mid E_k \cap E_j) \leq \frac{\mathbb{E}(T_k - T'_k \mid E_j)}{\mathbb{P}(E_k \mid E_j)} \leq \frac{k}{c} \mathbb{E}(T_k - T'_k \mid E_j).$$

To estimate $\mathbb{E}(T_k - T'_k \mid E_j)$ we repeat the method that we used for estimate (iii). Recall that $Q_1(X_n - X_{n_{2j+1}}) \leq Q_1(X_{n_{2k}} - X_{n_{2k-1}}) = O(4^{-2k})$, and so

$$\begin{aligned} \mathbb{E}(T_k - T'_k \mid E_j) &\leq \sum_{n=n_{2k+1}-4^k}^{n_{2k+1}} \mathbb{P}(X_n = 0 \mid E_j) \\ &\leq \sum_{n=n_{2k+1}-4^k}^{n_{2k+1}} Q_1(X_n - X_{n_{2j+1}}) \\ &= O(4^{-k}). \end{aligned}$$

Hence,

$$\mathbb{P}(A_k \mid E_k \cap E_j) = 1 - O(k4^{-k}) = 1 - o(1) \quad \text{as } k \rightarrow \infty,$$

uniformly in j . Now

$$\begin{aligned} \mathbb{E}(T_k \mid E_k \cap E_j) &\geq \mathbb{P}(A_k \mid E_k \cap E_j) \mathbb{E}(T_k \mid A_k \cap E_j) \\ &\geq (1 - o(1)) \min_{n \in \{n_{2k}, \dots, n_{2k+1}-4^k\}} \mathbb{E}(T_k \mid X_n = 0) \\ &= \Omega(\log(4^k)) \\ &= \Omega(k), \end{aligned}$$

where we have again used that the expected number of returns to 0 of a two-dimensional simple symmetric random walk in its first $2N$ steps is $\Omega(\log(N))$. $\square$

6 Recurrence and transience for slowly growing step sizes

We start by showing that a slowly growing non-decreasing integer sequence $(a_n)_{n \geq 1}$ gives a transient Rademacher random walk if the set of values it takes is a little sparse.

Lemma 31. *Suppose $(a_n)_{n \geq 1}$ is a non-decreasing sequence of positive integers which takes values in a set S . Suppose that the value s appears L_s times in (a_n) and suppose further that $\sum_{s \in S} 1/s < \infty$ and, for some $\varepsilon > 0$ and all large enough s , there is some $s' < s$ for which $L_{s'} \geq \varepsilon n^2$. Then the Rademacher random walk X associated to $(a_n)_{n \geq 1}$ is transient.*

Proof. Fix any finite set F . Using the hypothesis about s' , we can apply Corollary 23 to see that for all large enough $s \in S$, the probability that X_n is congruent to any element of $F \pmod{s}$ at the beginning of the block of steps of size s is at most $|F|c(\varepsilon)/s$. The walk X can only visit F during the s -block if it is congruent to an element of F modulo s and, since $\sum_{s \in S} 1/s < \infty$, the first Borel–Cantelli lemma shows that almost surely this happens for only finitely many $s \in S$. Hence, X is transient. $\square$

Now we can easily prove Lemma 9, which we restate here for convenience.

Lemma 9. *Let $f : \mathbb{N} \rightarrow [1, \infty)$ be any function such that $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then there exists a non-decreasing sequence $(a_n)_{n \geq 1}$ of integer step sizes for which $a_n \leq f(n)$ for all n and the associated Rademacher random walk is both irreducible and transient.*

Proof. Since we could choose to start with any finite number of steps with a step size of 0, we may assume without loss of generality that $f(n) \geq 9$ for all n . Let $p_i = (2i+1)^2$ for each $i \geq 1$, and choose a sequence $\ell_1, \ell_2, \dots$ of integers such that for each $i \geq 1$ we have

1. $\ell_{2i} \equiv i+1 \pmod{2}$,
2. $\ell_{2i+1} \equiv i \pmod{2}$,
3. $\ell_i \geq p_{i+1}^2$,
4. $f(n) \geq p_{i+1}$ for all $n > \sum_{k=1}^i \ell_k$.

This is always possible as $f(n) \rightarrow \infty$ as $n \rightarrow \infty$. Construct the sequence $(a_n)_{n \geq 1}$ by letting the first ℓ_1 terms be p_1 , the next ℓ_2 terms be p_2 , and so on. That is, $a_n = p_i$ whenever $c_i < n \leq c_{i+1}$, where $c_i = \sum_{j=1}^{i-1} \ell_j$. The final condition in the list above ensures that $a_n \leq f(n)$ for all n . Let $(X_n)_{n \geq 0}$ be a Rademacher walk with step sizes given by the sequence $(a_n)_{n \geq 1}$. Since $\gcd(p_{i-1}, p_i) = 1$, the construction ensures that $(a_n)_{n \geq 1}$ satisfies the hypotheses of Lemma 31, so X is transient.

It remains to show that X is irreducible. Note that for any $i \geq 1$ we have $(i+1)p_{2i} - ip_{2i+1} = 1$. By our assumptions on the parity of the ℓ_i , we have that in two consecutive blocks where the step sizes are p_{2i} in the first and p_{2i+1} in the second, it occurs with positive probability that the total increment in block $2i$ is $(i+1)p_{2i}$ and the total increment in block $(2i+1)$ is $-ip_{2i+1}$, in which case the total increment from these two blocks is 1. Likewise, it occurs with positive probability that the total increment in block i is $-(i+1)p_{2i}$ and the total increment in block $(i+1)$ is ip_{2i+1} , so that the total increment from the two blocks is -1 . Hence, X is irreducible. $\square$

We now turn to the case of bounded step sizes, and prove Lemma 10, which states that if $\sum_{n=1}^{\infty} a_n^2 = \infty$ then the Rademacher random walk X associated to $(a_n)_{n \geq 1}$ is almost surely unbounded both above and below.

Proof of Lemma 10. To show that (X_n) is almost surely unbounded both below and above, it suffices to show for any constant C that almost surely $X_n \leq C$ i.o. and $X_n \geq C$ i.o. as well. To prove this, we will show that whenever $\mathbb{P}(X_m = x) > 0$, we have

$$\mathbb{P}(\exists n > m : (X_n - C)(x - C) \leq 0 \mid X_m = x) = 1. \quad (8)$$

For any $n > m$, the increment $X_n - X_m$ is independent of X_m and has a symmetric distribution with variance $\sum_{k=m+1}^n a_k^2$. Its fourth moment satisfies

$$\begin{aligned} \mathbb{E}((X_n - X_m)^4) &= \mathbb{E}\left(\left(\sum_{k=m+1}^n \epsilon_k a_k\right)^4\right) \\ &= \sum_{k=m+1}^n a_k^4 + 3 \sum_{k=m+1}^n \sum_{j=m+1}^n \mathbb{1}_{\{j \neq k\}} a_k^2 a_j^2 \\ &= 3 \sum_{k=m+1}^n \sum_{j=m+1}^n a_k^2 a_j^2 - 2 \sum_{k=m+1}^n a_k^4 \\ &\leq 3 \left(\sum_{k=m+1}^n a_k^2\right)^2. \end{aligned}$$

We now apply the Paley–Zygmund inequality to the random variable Z defined by $Z = (X_n - X_m)^2$.

$$\begin{aligned} \mathbb{P}\left(|X_n - X_m| \geq \frac{1}{2} \left(\sum_{k=m+1}^n a_k^2\right)^{1/2}\right) &= \mathbb{P}\left(Z > \frac{1}{4} \mathbb{E}(Z)\right) \\ &\geq \left(\frac{3}{4}\right)^2 \frac{\mathbb{E}(Z)^2}{\mathbb{E}(Z^2)} \\ &\geq \frac{3}{16}. \end{aligned}$$

We remark that the above inequality complements the statement of Tomaszewski's conjecture, recently proved by Keller and Klein [12], which tells us that

$$\mathbb{P}\left(|X_n - X_m| \leq \left(\sum_{k=m+1}^n a_k^2\right)^{1/2}\right) \geq \frac{1}{2}.$$

We can now prove (8). Define a sequence of stopping times $\tau_0 = m < \tau_1 < \tau_2 < \dots$ inductively by

$$\tau_i = \min \left(\left\{ n > \tau_{i-1} : \frac{1}{2} \left(\sum_{k=1+\tau_{i-1}}^n a_k^2\right)^{1/2} > |X_{\tau_{i-1}} - C| \right\} \right).$$

Since $\sum_{k=1}^{\infty} a_k^2 = \infty$, we have $\tau_i < \infty$ a.s. for every integer $i \geq 0$. Let $\mathcal{G}_i$ denote the σ -algebra generated by $X_1, \dots, X_{\tau_i}$. For each $i \geq 1$ we have

$$\mathbb{P}((X_{\tau_i} - C)(X_{\tau_{i-1}} - C) \leq 0 \mid \mathcal{G}_{i-1}) \geq \frac{3}{32}.$$

By the conditional Borel–Cantelli lemma, we find that almost surely there exists a random $i < \infty$ such that $(X_{\tau_i} - C)(X_{\tau_{i-1}} - C) \leq 0$ and, taking $n = \tau_i$ for the least such i , we have $(X_n - C)(X_m - C) \leq 0$. $\square$

7 Transience for sequences that cover all natural numbers

The main aim of this section is to prove Theorem 5, and then to deduce Corollary 6.

Theorem 5. *Let $(a_n)_{n \geq 1}$ be a non-decreasing sequence of integers, and let L_i be the number of times i appears in the sequence. Suppose that, for all large enough n ,*

$$\sum_{\substack{i \leq n \\ \gcd(i, n) = 1}} L_i \geq 2n^2 \quad (1)$$

and

$$\sum_{i=1}^{n-1} i^2 L_i \geq 4n^2 \log^3(n) \cdot L_n. \quad (2)$$

Then the Rademacher random walk X associated with $(a_n)_{n \geq 1}$ is transient.

Proof. Fix $C > 0$, and let E_n be the event that the walk is within C of the origin after one of the steps of size n , that is, $E_n = \{\exists i : a_i = n, |X_i| \leq C\}$. We will show that the probability of these events is summable, and therefore only finitely of the E_n occur almost surely. Since the probability that the walk is C -recurrent is zero for all C , the walk must be transient.

Clearly, the probability of E_n is 0 if $L_n = 0$, so suppose that $L_n \geq 1$ and that n is large enough for (1) and (2) to hold. Let $N = \sum_{i=1}^{n-1} L_i$ so that $a_{N+1} = \dots = a_{N+L_n} = n$. We split the event E_n into two cases based on the size of $|X_N|$. When $|X_N|$ is large, the probability that L_n steps of size n will travel far enough to be within C of the origin is small enough to be summable. When $|X_N|$ is small, we use the fact that X_N is well-distributed over the equivalence classes modulo n , and the probability that steps of size n could possibly get within C of the origin is $O(C/n)$. By combining this with the probability that $|X_N|$ is small, we find that $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$.

First, consider the case where $|X_N|$ is large, by which we mean $|X_N|^2 \geq \sum_{i=1}^{n-1} i^2 L_i / \log^2(n)$. Define $M := \sum_{i=1}^{n-1} i^2 L_i / \log^2(n)$. By the reflection princi-

ple we have

$$\begin{aligned}\mathbb{P}\left(\max_{0 \leq t \leq L_n} \sum_{i=1}^t \epsilon_{(N+i)} n \geq \sqrt{M}\right) &= 2\mathbb{P}\left(\sum_{i=1}^{L_n} \epsilon_i n \geq \sqrt{M}\right) - \mathbb{P}\left(\sum_{i=1}^{L_n} \epsilon_i n = \sqrt{M}\right) \\ &\leq 2 \exp\left(-\frac{M}{2n^2 L_n}\right).\end{aligned}$$

By our assumption on $\sum_{i=1}^{n-1} i^2 L_i$, we have that the probability of this event is at most $2n^{-2}$.

Now consider the case where $|X_N|^2 \leq M$. We split the sum X_N into two parts, a small part which ensures that X_N is well distributed over the equivalence classes modulo n , and another part that we know must be large. Pick a set $A \subseteq \{1, \dots, N\}$ of size $|A| = n^2$ such that $\gcd(a_i, n) = 1$ for all $i \in A$ and $\sum_{i \in A} a_i^2$ is as small as possible. Let $A^c = [N] \setminus A$, and note that $\sum_{i \in A^c} a_i^2 \geq 2n^2 \log^3(n) L_n$. Define S_A and S_{A^c} by $S_A = \sum_{i \in A} a_i \epsilon_i$ and $S_{A^c} = \sum_{i \in A^c} a_i \epsilon_i$.

Note that the event $\{|X_N| \leq M\}$ is contained in the event

$$\{|S_A| \leq 2 \log(n) n^2\} \cap \{|S_{A^c}| \leq 2n^2 \log(n) L_n\},$$

and that if E_n is to occur, there must be some $c \in [-C, C]$ for which $X_N \equiv c \pmod n$. By our choice of A , for every x , the probability that $S_A \equiv x \pmod n$ is at most $3/n$ by Lemma 15, and this implies that, even given the value of S_{A^c} , the probability that there is some $c \in [-C, C]$ for which $X_N \equiv c \pmod n$ is at most $3/n$. To finish this case, we will show that the probability of $\{|S_{A^c}| \leq 2n^2 \log(n) L_n\}$ is at most $C'/\log^2(n)$ and that the probability of $\{|S_A| \leq 2 \log(n) n^2\}$ is at least $1 - n^{-2}$. These imply that the probability of $E_n \cap \{|X_N| \leq M\}$ is at most

$$\frac{C'}{\log^2(n)} \left(\frac{1}{n^2} + \frac{3}{n} \right)$$

Since the probability of $E_n \cap \{|X_N| \geq M\}$ is at most $2n^{-2}$, the probability of E_n is $O(n^{-1} \log^{-2}(n))$. Hence, the sum $\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$ and almost surely only finitely many E_n occur, which means the walk is transient.

Let us bound the probability that $|S_{A^c}| \leq 2n^2 \log(n) L_n$. By the Berry-Esseen Theorem, we have that

$$\begin{aligned}\mathbb{P}(|S_{A^c}| \leq 2n^2 \log(n) L_n) &= \mathbb{P}\left(\frac{|S_{A^c}|}{\sum_{i \in A^c} a_i^2} \leq \frac{2n^2 \log(n) L_n}{\sum_{i \in A^c} a_i^2}\right) \\ &\leq \mathbb{P}\left(\frac{|S_{A^c}|}{\sum_{i \in A^c} a_i^2} \leq \frac{1}{2 \log^2(n)}\right) \\ &\leq \Phi\left(\frac{1}{2 \log^2(n)}\right) - \Phi\left(-\frac{1}{2 \log^2(n)}\right) + C' \frac{n-1}{\sum_{i \in A^c} a_i^2} \\ &\leq \frac{1}{\sqrt{2\pi} \log^2(n)} + \frac{C'}{n}.\end{aligned}$$

Finally, we need to bound the probability that $|S_A| \leq 2 \log(n) n^2$, for which we use Lemma 26. Trivially, $\sum_{i \in A} a_i^2 \leq n^4$ and,

$$\mathbb{P}(S_A \geq 2 \log(n) n^2) \leq e^{-2 \log(n)}.$$

□

Given the theorem above, Corollary 6 follows easily. Indeed, we only need to show that $a_n = \lfloor \log^{1+\varepsilon}(n) \rfloor$ satisfies the appropriate conditions.

Corollary 6. *For any $\alpha > 1$, the Rademacher random walk with step sizes $a_n = \lfloor \log^\alpha(n) \rfloor$ is transient*

Proof. Clearly, (a_n) is a non-decreasing sequence of integers, so we can apply Theorem 5 provided the L_n satisfy the necessary conditions. The i th step size a_i equals n exactly when $e^{n^{1/\alpha}} \leq i < e^{(n+1)^{1/\alpha}}$, so $L_n = e^{(n+1)^{1/\alpha}} - e^{n^{1/\alpha}} + O(1)$. By the Mean Value Theorem, there is $x \in (n, n+1)$ such that

$$e^{(n+1)^{1/\alpha}} - e^{n^{1/\alpha}} = \frac{e^{x^{1/\alpha}}}{\alpha x^{1-1/\alpha}}.$$

In particular, $L_{n-1} \geq 2n^2$ for large enough n and (1) is satisfied. We also have

$$\begin{aligned} \sum_{i=1}^{n-1} i^2 L_i &\geq \sum_{i=2}^{n-1} i^2 \left(\frac{e^{i^{1/\alpha}}}{\alpha i^{1-1/\alpha}} + O(1) \right) \\ &\geq O(n^3) + \frac{1}{\alpha} \int_1^{n-1} x^{1+1/\alpha} e^{x^{1/\alpha}} dx \\ &= (n-1)^2 e^{(n-1)^{1/\alpha}} + O(e^{n^{1/\alpha}}/n^2). \end{aligned}$$

Comparing this with $4n^2 \log^3(n) \left(\frac{e^{(n+1)^{1/\alpha}}}{\alpha(n+1)^{1-1/\alpha}} + O(1) \right)$, we see that (2) is also satisfied, and we can apply Theorem 5 to finish the proof. $\square$

8 Unbounded step sequences whose gaps tend to zero

In this section we prove Theorem 13, which asserts that if the step sizes of a Rademacher random walk are unbounded with gaps converging to zero, the walk is either transient or topologically recurrent.

Lemma 32. *Suppose $(a_n)_{n \geq 1}$ is a sequence such that $\limsup_{n \rightarrow \infty} a_n = \infty$ and $|a_n - a_{n-1}| \rightarrow 0$ as $n \rightarrow \infty$. Let X and X' be Rademacher random walks with step sizes given by (a_n) started at time k at locations $X_k = N$ and $X'_k = N + d$ for some $d \neq 0$. Then, for any $\varepsilon > 0$, the walks X and X' can be coupled so that a.s. $\lim_{n \rightarrow \infty} (X_n - X'_n)$ exists and lies in $[0, \varepsilon]$.*

Proof. Think of the problem of coupling X and X' as a game as follows. Just before time i , we know $\epsilon_1, \dots, \epsilon_{i-1}$ and $\epsilon'_1, \dots, \epsilon'_{i-1}$ and we must choose either to couple ϵ'_i and ϵ_i so that $\epsilon'_i = \epsilon_i$, or to couple them so that $\epsilon'_i = -\epsilon_i$. Once we have made our choice, the value of ϵ_i is revealed, and ϵ'_i is either ϵ_i or $-\epsilon_i$ depending on the choice that we made. We win the game if for some $i \geq k$ we achieve $X_i - X'_i \in [0, \varepsilon]$. The proof consists of a strategy for winning this game eventually with probability 1. Once we have won the game we may couple all subsequent signs to be equal, so that $X_i - X'_i$ stabilizes.

Our strategy is organised as a sequence of episodes. In each episode we will win with probability at least $1/4$, conditional on all the outcomes in previous episodes. For $i \geq 1$, episode i will begin at time $m_{i-1} + 1$ and end at time m_i , where $m_0 = k$. To describe episode i for any $i \geq 1$, assume we know m_{i-1} , $X_{m_{i-1}}$ and $X'_{m_{i-1}}$, and assume we have not yet won the game at time m_{i-1} . Let $d_i = X_{m_{i-1}} - X'_{m_{i-1}}$. In particular, $d_1 = d$. Let $\delta_i = \min(\epsilon, |d_i|)$. Note that $\delta_i > 0$ since if $d_i = 0$ then we have already won the game. Choose $n_i \geq m_{i-1}$ sufficiently large that for all $n \geq n_i$ we have $|a_n - a_{n-1}| < \delta_i/2$. Let

$$x_i = \begin{cases} d_i/2 & \text{if } d_i > 0, \\ -d_i/2 + \delta_i/2, & \text{if } d_i \leq 0. \end{cases}$$

Since $\limsup_{n \rightarrow \infty} a_n = \infty$, we may find $m_i > n_i$ such that

$$a_{m_i} - a_{n_i} \in [x - \delta_i/2, x].$$

Now choose to couple the signs ϵ_i and ϵ'_i driving the movements of the walks X and X' to be equal for each i in the range $m_{i-1} \leq i \leq n_i - 1$ and for $n_i + 1 \leq i \leq m_i - 1$. Choose to couple $\epsilon_{n_i} = -\epsilon'_{n_i}$ and $\epsilon_{m_i} = -\epsilon'_{m_i}$.

There are four possible options for $X'_{m_i} - X_{m_i}$, each having probability $1/4$ conditional on the outcomes preceding episode i :

$$X'_{m_i} - X_{m_i} = \begin{cases} d_i + 2(a_{m_i} + a_{n_i}) & \epsilon_{n_i} = -1, \epsilon_{m_i} = -1, \\ d_i + 2(a_{m_i} - a_{n_i}) & \epsilon_{n_i} = +1, \epsilon_{m_i} = -1, \\ d_i - 2(a_{m_i} - a_{n_i}) & \epsilon_{n_i} = -1, \epsilon_{m_i} = +1, \\ d_i - 2(a_{m_i} + a_{n_i}) & \epsilon_{n_i} = +1, \epsilon_{m_i} = +1. \end{cases}$$

If $d_i > 0$ then $d_i - 2(a_{m_i} - a_{n_i}) \in [0, \varepsilon]$. If $d_i < 0$ then $d_i + 2(a_{m_i} - a_{n_i}) \in [0, \varepsilon]$. Hence, with probability at least $1/4$, we have $X'_{m_i} - X_{m_i} \in [0, \varepsilon]$, in which case we win the game no later than time m_i .

Carrying out the procedure above repeatedly until success, we will almost surely win after finitely many episodes, since in each episode we win with probability at least $1/4$, conditional on the outcomes in previous episodes. $\square$

Using this coupling, we show that if the walk hits the interval $[a, b]$ i.o. with positive probability, then the probability that it hits a second interval i.o. is also positive.

Corollary 33. *Suppose $(a_n)_{n \geq 1}$ is a sequence such that $\limsup_{n \rightarrow \infty} a_n = \infty$ and $|a_n - a_{n-1}| \rightarrow 0$ as $n \rightarrow \infty$. Let $X = (X_n)_{n \geq 1}$ be the Rademacher random walk with step sizes (a_n) . Let $a < b$ and $e < f$ and take $m \in \mathbb{N}$ such that $m > (b-a)/(f-e)$. If $\mathbb{P}(X_n \in [a, b] \text{ i.o.}) \geq p$, then $\mathbb{P}(X_n \in [e, f] \text{ i.o.}) \geq p/m$.*

Proof. Divide the interval $[a, b]$ into m equal intervals. By a union bound, at least one such interval $[a', b']$ is visited infinitely often with probability at least p/m . We have $[a' + t, b' + t] \subset [e, f]$, where $t = e - a'$. If $t = 0$ there is nothing to do, so let us assume that $t \neq 0$ and further that $t > 0$. The case $t < 0$ is similar. Apply the coupling of Lemma 32 starting at time 0 at $X_0 = 0$ and $X'_0 = t$, taking

$\varepsilon = f - (b' + t)$. Then the standard Rademacher random walk X'' defined by $X''_n = X'_n - t$ visits $[e, f]$ infinitely often if X visits $[a', b']$ infinitely often, and this occurs with probability at least p/m . $\square$

The proof of Theorem 13 follows.

Proof. Suppose X is not transient. Then there exists $C < \infty$ such that

$$\mathbb{P}(|X_n| < C \text{ i.o.}) > 0.$$

Apply Corollary 33 taking $[a, b] = [-C, C]$ and $p = \mathbb{P}(|X_n| < C \text{ i.o.})$, to see that whenever $e < f$ we have

$$\mathbb{P}(X_n \in [e, f] \text{ i.o.}) \geq \frac{p}{\lceil 2C/(f - e) \rceil} > 0.$$

Now suppose (for a contradiction) that for some interval $[g, h]$, we have $\mathbb{P}(X_n \in [g, h] \text{ i.o.}) < 1$. Then (by a standard martingale argument) there exists a finite k' and two sequences of signs $\beta_1, \dots, \beta_{k'}$ and $\gamma_1, \dots, \gamma_{k'}$ such that

$$\mathbb{P}(X_n \in [g, h] \text{ i.o.} \mid \epsilon_1 = \beta_1, \dots, \epsilon_k = \beta_{k'}) > 2/3$$

and

$$\mathbb{P}(X_n \in [g, h] \text{ i.o.} \mid \epsilon_1 = \gamma_1, \dots, \epsilon_k = \gamma_{k'}) < 1/3.$$

Take $m = 2$ in Corollary 33, applied to the walk with step sizes $a_{k'+1}, a_{k'+2}, \dots$, and with

$$[a, b] = [g - (\beta_1 a_1 + \dots + \beta_{k'} a_{k'}), h - (\beta_1 a_1 + \dots + \beta_{k'} a_{k'})]$$

and

$$[e, f] = [g - (\gamma_1 a_1 + \dots + \gamma_{k'} a_{k'}), h - (\gamma_1 a_1 + \dots + \gamma_{k'} a_{k'})]$$

to obtain a contradiction. $\square$

9 Recurrent Rademacher walks where the step sizes grow arbitrarily quickly

In this section we prove Theorem 3 and Theorem 4, both of which show the existence of sequences of step sizes which grow arbitrarily quickly yet give recurrent Rademacher random walks. The constructions used in the proofs of Theorem 3 and Theorem 4 both work by considering a suitable two-dimensional random walk and understanding the range of the second coordinate at the times when the first coordinate is zero, but the proof of Theorem 3 is much simpler.

Theorem 3. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any non-decreasing function. There is an integer sequence $(a_n)_{n \geq 1}$ such that $a_n \geq f(n)$ for all n and the associated Rademacher random walk is recurrent.*

Proof. We define the sequence (a_n) in blocks, starting from the empty sequence. Suppose that $a_1, \dots, a_N$ have already been chosen, and let $M = \sum_{i=1}^N a_i$. Since the two-dimensional simple symmetric random walk is recurrent, we can choose some L such that the probability that the two-dimensional simple symmetric random walk (SSRW) has hit every point in $\{(0, y) : y \in [-M, M]\}$ by time L is at least $1/2$. Now we choose r such that $r \geq g(2L + N)$ and define the next $2L$ steps to alternate between $r+1$ and r . This sequence clearly satisfies the necessary growth condition and it remains to prove that the sequence is weakly recurrent.

Let E_k be the event that the walk hits 0 in the k th block. We claim that uniformly for any outcome on the preceding blocks, the probability of E_k is at least $1/2$. Suppose that k th block starts at a_{N+1} and that the walk is at m immediately before the k th block, i.e. $X_N = m$. Pair up consecutive steps in the k th block and consider the walk $Y = (Y_n)$ where $Y_n = X_{N+2n}$. This starts at m and takes steps of $\pm(2r+1), \pm 1$ each with probability $1/4$. As before, we define a two dimensional random walk (x_n, y_n) by setting $(x_0, y_0) = (0, 0)$, and then inductively defining (x_n, y_n) for $n = 1, \dots, L$ by

$$(x_n, y_n) = (x_{n-1}, y_{n-1}) + \begin{cases} (1, 0) & \text{if } Y_n - Y_{n-1} = 2r + 1, \\ (-1, 0) & \text{if } Y_n - Y_{n-1} = -(2r + 1), \\ (0, 1) & \text{if } Y_n - Y_{n-1} = 1, \\ (0, -1) & \text{if } Y_n - Y_{n-1} = -1. \end{cases}$$

Clearly, if the walk (x_n, y_n) hits the point $(0, -m)$, then the walk X has hit zero and by our choice of L this happens with probability at least $1/2$. To finish the proof that the walk is recurrent, we can apply the Kochen–Stone theorem to the random variable $Z_n = \sum_{k=1}^n \mathbb{1}_{E_k}$. $\square$

We now turn to the proof of Theorem 4. We will again consider the times when the first coordinate of a two-dimensional random walk is 0, but we will have to work with a more complicated two-dimensional random walk and we will need the following lemma.

Lemma 34. *Define for each $n \geq 1$*

$$c_n = \sum_{m=1}^{n-1} \frac{m^{-3/2}}{\sqrt{1 + \log m}}.$$

Consider the two-dimensional Rademacher random walk $(Y_n, Z_n)_{n \geq 0}$ with n^{th} step $\pm(1, c_n)$, starting at $(Y_0, Z_0) = (0, 0)$. Almost surely the set $\{Z_n : Y_n = 0\}$ is dense in $\mathbb{R}$.

From this lemma it is relatively straightforward to prove Theorem 4.

Theorem 4. *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be any non-decreasing function. There is a strictly increasing real sequence $(a_n)_{n \geq 1}$ such that $a_n \geq f(n)$ for all n and the associated Rademacher random walk is topologically recurrent.*

Proof. Define for each $n \geq 1$

$$c_n = \sum_{m=1}^{n-1} \frac{m^{-3/2}}{\sqrt{1 + \log m}}.$$

Note that $\sum_{m=1}^{\infty} \frac{m^{-3/2}}{\sqrt{1 + \log m}} < \infty$, so $c_n \nearrow c_{\infty}$ as $n \rightarrow \infty$ where $c_{\infty} < \infty$. Also,

$$c_{n+1} - c_n = \frac{n^{-3/2}}{\sqrt{1 + \log n}}.$$

We will define the step size sequence $(a_n)_{n \geq 1}$ as the concatenation of blocks of the form $(x_j + c_0, x_j + c_1, \dots, x_j + c_{n_j})$, for $j \geq 1$, where for each j we choose n_j and then x_j suitably large given the previous choices. Let $M_j = \sum_{n=1}^{n_1 + \dots + n_{j-1}} a_n$ be the sum of all the terms in the blocks preceding the j^{th} block. According to Lemma 34, we may choose n_j so that with probability at least $1/2$, the walk $(Y_n, Z_n)_{n=0}^{n_j}$ visits the $(1/j)$ -neighbourhood of each point in $\{0\} \times [-2M_j, 2M_j]$. Then choose x_j so large that $x_j \geq f(n_1 + \dots + n_j)$. This ensures $a_n \geq f(n)$ for all n . Note that $n_j \rightarrow \infty$ as $j \rightarrow \infty$ and hence $M_j \rightarrow \infty$ also. For any $t \in \mathbb{R}$, once $M_j + 1/j > t$, we have that conditional on $X_{n_1 + \dots + n_{j-1}}$, the Rademacher walk X_n visits the interval $(t - 1/j, t + 1/j)$ with probability at least $1/2$. Hence, X is topologically recurrent by the conditional Borel–Cantelli lemma. $\square$

It remains to prove Lemma 34.

Proof of Lemma 34. Consider the sequence of successive times $0 = \tau_1 < \tau_2 < \tau_3 < \dots$ at which $Y_n = 0$. This is almost surely an infinite increasing sequence tending to ∞ since the walk $Y = (Y_n)_{n \geq 0}$ on its own is a simple symmetric random walk on $\mathbb{Z}$, which is recurrent. The distribution of each increment $Z_{\tau_{i+1}} - Z_{\tau_i}$ conditional on the earlier increments has a symmetric distribution. Therefore we can condition on the increment sizes $|Z_{\tau_{i+1}} - Z_{\tau_i}|$ and obtain a *random* Rademacher walk $(Z_{\tau_i})_{i \geq 1}$ with random step sizes $b_i := |Z_{\tau_{i+1}} - Z_{\tau_i}|$ for $i \geq 1$. We claim that almost surely $b_i \rightarrow 0$ as $i \rightarrow \infty$ and $\sum_{i=1}^{\infty} b_i^2 = \infty$. Given this claim, we may apply Lemma 10 to see that almost surely the Rademacher walk $(Z_{\tau_i})_{i \geq 1}$ is topologically recurrent, which is to say that almost surely $\{Z_n : Y_n = 0\}$ is dense in $\mathbb{R}$.

The key observation is that since the sign of Y_n is constant for n in the interval $[1 + \tau_i, \tau_{i+1} - 1]$, and $Y_n = 0$ when $n \in \{\tau_i, \tau_{i+1}\}$, and $c_{n+1} - c_n > 0$ for all n , we have

$$\begin{aligned} |Z_{\tau_{i+1}} - Z_{\tau_i}| &= \left| \sum_{n=1+\tau_i}^{\tau_{i+1}} (Y_n - Y_{n-1})c_n \right| = \left| - \sum_{n=1+\tau_i}^{\tau_{i+1}} Y_n(c_{n+1} - c_n) \right| \\ &= \sum_{n=1+\tau_i}^{\tau_{i+1}-1} |Y_n|(c_{n+1} - c_n) = \sum_{n=1+\tau_i}^{\tau_{i+1}-1} |Y_n| \frac{n^{-3/2}}{\sqrt{1 + \log n}}. \end{aligned}$$

Since $Y_n = 0$ when $n \in \{\tau_i, \tau_{i+1}\}$, the final sum above is unchanged if we replace the lower limit by $n = \tau_i$ or the upper limit by τ_{i+1} .

Let us use the Komlós–Major–Tusnády coupling to couple the simple symmetric random walk Y_n to a standard Brownian motion $(B_s)_{s \geq 0}$. This is a coupling

with the property that for every $\alpha > 0$ there is a positive constant c_α such that for all n one has

$$\mathbb{P}\left(\max_{1 \leq j \leq n} \frac{|B_j - Y_j|}{\log n} > c_\alpha\right) < c_\alpha n^{-\alpha}.$$

(See [14, Theorem 7.1.1].) Taking $\alpha > 1$ and using Borel–Cantelli it follows that almost surely

$$\limsup_{s \rightarrow \infty} \frac{|B_s - Y_{[s]}|}{\log s} < \infty.$$

Since $|Y_{n+1} - Y_n| = 1$ for all n , we also have

$$\limsup_{s \rightarrow \infty} \frac{|B_s - Y_{[s]}|}{\log s} < \infty.$$

Let E be the (almost surely finite) random variable defined by

$$E = \sup_{s \geq 1} \frac{\max(|B_s - Y_{[s]}|, |B_s - Y_{[s]}|)}{1 + \log s}.$$

Then, for each $i \geq 2$, we have $\tau_i \geq 2$ so the function $x^{-3/2}/\sqrt{1 + \log x}$ is decreasing on the interval $[\tau_i, \tau_{i+1}]$ and hence

$$\sum_{n=\tau_i}^{\tau_{i+1}-1} |Y_n| \frac{n^{-3/2}}{\sqrt{1 + \log n}} > \int_{\tau_i}^{\tau_{i+1}} (|B_s| - (1 + \log s)E) \frac{s^{-3/2}}{\sqrt{1 + \log s}} ds$$

and

$$\sum_{n=1+\tau_i}^{\tau_{i+1}} |Y_n| \frac{n^{-3/2}}{\sqrt{1 + \log n}} < \int_{\tau_i}^{\tau_{i+1}} (|B_s| + (1 + \log s)E) \frac{s^{-3/2}}{\sqrt{1 + \log s}} ds.$$

Note that $\int_1^\infty (\sqrt{1 + \log s}) s^{-3/2} ds < \infty$, so the sequence of errors

$$\left(\int_{\tau_i}^{\tau_{i+1}} (\sqrt{1 + \log s}) s^{-3/2} E ds \right)_{i \geq 2}$$

is almost surely in $\ell^1(\mathbb{N})$ and hence also in $\ell^2(\mathbb{N})$.

Therefore, to show that the sequence $(b_i)_{i \geq 1} = (|Z_{\tau_{i+1}} - Z_{\tau_i}|)_{i \geq 1}$ almost surely tends to 0 but does not lie in $\ell^2(\mathbb{Z})$, it suffices to do the same for the sequence $(I_i)_{i \geq 1}$ defined by

$$I_i := \int_{\tau_i}^{\tau_{i+1}} |B_s| \frac{s^{-3/2}}{\sqrt{1 + \log s}} ds.$$

We make the change of variable $s = e^t$, to get

$$I_i = \int_{\log(\tau_i)}^{\log(\tau_{i+1})} |B_{e^t}| \frac{e^{-t/2}}{\sqrt{t+1}} dt = \int_{\log(\tau_i)}^{\log(\tau_{i+1})} \frac{|W_t|}{\sqrt{t+1}} dt,$$

where the process

$$W_t := B_{(e^t)} e^{-t/2}$$

is a stationary Ornstein–Uhlenbeck process whose stationary distribution π is Gaussian with mean 0 and variance 1. For this identity in law, see [23, §8.5.1], in which W is called an *ancient* Ornstein–Uhlenbeck process. W satisfies the SDE

$$dW_t = -\frac{1}{2}W_t dt + dB'_t,$$

where $(B'_t)_{t \in \mathbb{R}}$ is another (two-sided) standard Brownian motion.

The rough idea now is that large increments I_i , exceeding a positive constant size, correspond to increasingly large excursions of W from 0, which only occur finitely often, almost surely, but on the other hand the large excursions of W from 0 whose integral is at least a positive constant occur sufficiently regularly to give a subsequence of (I_i) whose sum diverges. Some care is needed to make this precise.

To show that $I_i \rightarrow 0$ almost surely as $i \rightarrow \infty$, we consider the excursions of W_t above -1 and the excursions of W_t below 1. The times $\log(\tau_i)$ are times at which $|Y_{e^t}| = 0$ and hence $|B_{e^t}| \leq (1+t)E$. There exists a random time $t_0 < \infty$ such that

$$(1+t)e^{-t/2}E < 1/2 \text{ for } t \geq t_0.$$

For $i \geq i_0 := \lceil e^{t_0} \rceil$, we have $\tau_i \geq i$ so for all $t > \log(\tau_i)$ we have $(1+t)e^{-t/2}E < 1/2$. For each $i \geq i_0$, on the interval $\tau_i < n < \tau_{i+1}$, either all $Y_n > 0$, in which case

$$W_t > -(1+t)e^{-t/2}E > -1/2 \text{ for all } t \in [\log(\tau_i), \log(\tau_{i+1})],$$

or all $Y_n < 0$, in which case

$$W_t < (1+t)e^{-t/2}E < 1/2 \text{ for all } t \in [\log(\tau_i), \log(\tau_{i+1})].$$

It follows that for each $i \geq i_0$, the integral I_i is dominated either by an integral of the form

$$\frac{1}{\sqrt{a_i+1}} \int_{a_i}^{b_i} (W_t + 1) dt,$$

where $[a_i, b_i]$ is an excursion interval of W above the level -1 which reaches the level $-1/2$, or by an integral of the form

$$\frac{1}{\sqrt{a_i+1}} \int_{a_i}^{b_i} (-W_t - 1) dt$$

where $[a_i, b_i]$ is an excursion interval of W below the level 1 which reaches the level $1/2$. Since $\log(\tau_i) \rightarrow \infty$ as $i \rightarrow \infty$, and W hits each of -1 and 1 at an unbounded set of times almost surely, we have that $a_i \rightarrow \infty$ as $i \rightarrow \infty$.

The law of the iterated logarithm for B_s as $s \rightarrow \infty$ corresponds to a simpler statement about the maximal growth of the stationary Ornstein–Uhlenbeck process (see [23, eq. (8.5.2)]):

$$\limsup_{t \rightarrow \infty} \frac{|W_t|}{\sqrt{\log t}} = \sqrt{2} \text{ a.s.}$$

Hence, there is a random time t_1 such that

$$|W_t| < 2\sqrt{\log t} \text{ for all } t \geq t_1. \tag{9}$$

It is known that the hitting times in one-dimensional Ornstein–Uhlenbeck processes have exponential tails. However, we could not find a simple single reference for this fact, and we prove it here. Denote by $\tau_x(y)$ the hitting time of level y starting from level x . First, from Sato [20] it is known that for $x \neq 0$, $\tau_x(0)$ has a probability density function on $(0, \infty)$ given by

$$\frac{|x|}{\sqrt{2\pi}}(e^t - 1)^{-3/2}e^t \exp\left(-\frac{x^2}{2(e^t - 1)}\right).$$

This indeed decays exponentially as $t \rightarrow \infty$, and immediately implies that the hitting time $\tau_x(y)$ has an exponential tail whenever y lies between x and 0. The hitting time of a general level y starting from x is bounded above by the hitting time of 0 starting from x plus an independent hitting time of y starting from 0, so it suffices to show that $\tau_0(y)$ has an exponential tail, and for this we may assume $y > 0$ without loss of generality since the Ornstein–Uhlenbeck process is symmetric about 0. One way to bound $\tau_0(y)$ is to run the process starting at 0 until it hits $\{-y, y\}$; if it has hit y then we are done; otherwise run from y until you first hit 0, then try again, repeating until success. We succeed after a Geometric(1/2) number of trials. A standard exercise using moment generating functions shows that the sum of a geometric number of i.i.d. random variables with exponential tails itself has an exponential tail. Hence it suffices to show that the hitting time of $\{-y, y\}$ starting from 0 has an exponential tail. This is done in Breiman [5, Thm 1] using the explicit form of the Laplace transform of this hitting time that had been obtained earlier by Darling and Siegert [6]. (Breiman claimed to bound the tail of the first hitting time of y from 0, but we believe his proof actually bounds the first hitting time of $\{-y, y\}$ from 0.)

From the exponential tails of hitting times, it follows that the successive excursions of W above -1 that reach $-1/2$ have lengths that form an i.i.d. sequence with exponential tails. Indeed, the length of each one is the sum of two independent hitting times: the hitting time of $-1/2$ starting at -1 , and the hitting time of -1 starting at $-1/2$. The same applies to successive excursions below 1 that reach $1/2$. Among both such types of excursion, we now consider only those whose length is at least 1, since by (9) excursions shorter than this after time t_1 can only produce increments I_i that tend to 0. For each type of excursion, the n^{th} instance of the remaining long excursions starts at a time that is at least n , since they all have length at least 1 and are disjoint (although excursions of the two different kinds can overlap). Hence, if L_t is the length of the longest excursion of either kind up to time t , then

$$\sup_{t \geq 2} L_t / \log t < \infty \text{ a.s.}$$

because of the exponential tail bound that we established above. Combining this bound on the excursion lengths with the bound (9) on their heights, we get a bound on the largest integral $\int_{a_i}^{b_i} |W_t + 1| dt$ where $b_i - a_i \geq 1$ and $a_i \leq t$: it is no more than

$$2\sqrt{\log t} \log t \sup_{t \geq 2} L_t / \log t,$$

which is $o(\sqrt{t+1})$ as $t \rightarrow \infty$. It follows that $|Z_{\tau_{i+1}} - Z_{\tau_i}| \rightarrow 0$ as $i \rightarrow \infty$.

It remains to show that the sequence $(|Z_{\tau_{i+1}} - Z_{\tau_i}|)_{i \geq 1}$ almost surely does not lie in $\ell^2(\mathbb{N})$.

Let us call an excursion of W above -1 a *good excursion* if it contains an excursion above 1 whose integral is at least 1 .

Consider a good excursion of W_t above -1 that starts at time s and finishes at time t , where $t_0 \leq s < t - 1$, with a subinterval $(s', t') \subset (s, t)$ such that $W_r \geq 1$ for all $r \in (s', t')$ and $\int_{s'}^{t'} W_r dr > 1$. Then there is an excursion of Y above 0 , say from time τ_i to time τ_{i+1} , where

$$s < \log \tau_i < s' < t' < \log \tau_{i+1} < t$$

and

$$I_i = \int_{\log(\tau_i)}^{\log(\tau_{i+1})} \frac{W_t}{\sqrt{t+1}} dt > \frac{1}{\sqrt{t'+1}} \int_{s'}^{t'} W_t dt > \frac{1}{\sqrt{t'+1}}.$$

The number of disjoint good excursions that lie entirely between times 4^{k-1} and 4^k grows as $\Theta(4^k)$, almost surely as $k \rightarrow \infty$. (Again, this follows from the fact that the hitting times of 1 starting from -1 and vice versa in the Ornstein–Uhlenbeck process have exponential tails, together with the Markov property of the Ornstein–Uhlenbeck process.) Hence, the sequence $(I_i)_{i \geq 1}$ almost surely does not lie in $\ell^2(\mathbb{N})$. This completes the proof of Lemma 34. $\square$

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Paper V



Forest Fire Model on $\mathbb{Z}_+$ with Delays

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Abstract

We consider a generalization of the forest fire model on $\mathbb{Z}_+$ with ignition at zero only, studied in [15]. Unlike that model, we allow delays in the spread of the fires and the non-zero burning time of individual “trees”. We obtain some general properties for this model, which cover, among others, the phenomena of an “infinite fire”, not present in the original model.

Mathematics Subject Classification: primary 60G55, 60K35; secondary 60F20

Keywords and phrases: forest fire model

1 Introduction

The following forest fire model was introduced in [3, 15]; an extension of this model was also studied in [9]. Each node of $\mathbb{Z}_+$ can be vacant or occupied by a tree. If a node is vacant, and only in this case, a single tree appears at this node after an exponentially distributed time with a rate 1; this time is independent of everything else. A constant source of fire is located at node 0, and whenever a tree catches fire, the fire spreads instantly to the next tree on the right and continues doing so until the whole cluster is burnt, which takes zero time. The fire stops only when it reaches a vacant node (i.e. without a tree).

Various forest fire models have been abundant, especially in the physics literature, partly for their practical applications and partly for the fact that they exhibit so-called *self-organized criticality*, which is a property of many dynamical systems that naturally evolve into a critical state, and where small changes can, surprisingly, trigger large-scale events. This phenomenon can be useful for explaining why complex systems (e.g., earthquakes, avalanches, stock markets, etc.) exhibit power-law behaviour without the necessity of external fine-tuning. Self-organized criticality is considered as a mechanism by which complexity arises in nature; for a more detailed analysis of this notion, we refer the reader to [2]. For examples of forest-fire

models in the physics literature see e.g. [11], where the authors compute the average number of trees destroyed by lightning, derive scaling laws and calculate all critical exponents; in [12] the analytic solution of the self-organized critical forest-fire model in one dimension is given. There are some interesting relations between percolation theory and forest fire models: in [7], the authors study the connection between frozen percolation and forest fire processes without recovery; in [1] it is shown that in higher dimensions a similar model introduced in [4] will eventually recover from fires. An extensive and detailed review of the forest models, including some practical ones, can be found in [8]. For other and more recent relevant models, please see [1, 5, 6, 8, 9, 10, 13] and references therein.

In the current paper we consider a generalization of the model from [15] by allowing non-zero tree burning times, as well as by introducing possible delays, which affect how the fire spreads from a burning site to its right neighbour. Neither of these possibilities was considered in [9] or [15], which motivated us to write the current paper. In our model, each site can be in three possible states: (a) vacant (no tree); (b) having a (non-burning) tree; (c) having a burning tree. Formally, assume that all sites are vacant initially and

- (1) the trees appear at each vacant, non-burning site $x \in \mathbb{Z}_+$ at a rate 1 independently of anything, after which they can be burned by a fire arriving from the left, which will eventually return the site to the original *vacant* state;
- (2) there is a constant source of fire attached to site $x = 0$, so whenever a tree appears there, it starts burning immediately;
- (3) when a tree at position x catches fire for the i -th time ($x, i \in \mathbb{Z}_+$), denote this time as $f_{x,i}$, it takes $\theta_{x,i} \geq 0$ time to burn, where $\theta_{x,i}$ are drawn independently from some non-negative distribution θ , so the state x is “burning” during the time interval $[f_{x,i}, f_{x,i} + \theta_{x,i}]$;
- (4) when a tree at position x burns for the i -th time, the fire may **spread** to tree $(x + 1)$ in time $\Delta_{x,i} \geq 0$, where $\Delta_{x,i}$ are drawn independently from some non-negative distribution Δ , and the counting starts only after both the fire starts at x and the tree at $(x + 1)$ appears. Therefore, if the fire starts at position x at time $t := f_{x,i}$, then the fire at position $(x + 1)$ starts at time $f_{x+1,j} = t + \Delta_{x,i}$ if there is already a tree¹ at $(x + 1)$ by time $t + \Delta_{x,i}$. On the other hand, if the site $x + 1$ is vacant at time $t + \Delta_{x,i}$, assuming that the tree at $x + 1$ appears at time $t + \Delta_{x,i} + \nu$ for some $\nu > 0$, it will either start burning immediately at time $f_{x+1,j} = t + \Delta_{x,i} + \nu$ if $\nu < \theta_{x,i}$, or if $\nu \geq \theta_{x,i}$ the fire

¹which is *not burning* at this time

will not propagate beyond site x ; we include the possibility that the fire at position x goes off before the position $x + 1$ catches fire²;

- (5) while a tree is burning, another tree cannot grow at the same site, nor can the fire from the left neighbour of this site spread to it. So the next tree at that spot can appear only after it was burnt completely and became vacant again, and it will take again $\exp(1)$ time independently of anything.

Observe the following:

- we may have more than one fire running in the model at the same time (but in different locations);
- if the Δ s are truly random, it is conceivable that while site x has consecutive fires at times $f_{x,i}$ and $f_{x,i+1}$, respectively (such that $f_{x,i+1} > f_{x,i} + \theta_{x,i}$), the time $\tilde{t} = f_{x,i} + \theta_{x,i} + \Delta_{x,i}$ by which the i -th fire at x *stops affecting* site $x + 1$ exceed the time $f_{x,i+1} + \Delta_{x,i+1}$ by which the next $(i + 1)$ -st fire at x *starts affecting* $x + 1$. In this case, we postulate that the $(i + 1)$ -st fire at x can ignite site $x + 1$ *only after* the time $\tilde{t}$;
- it is conceivable that a fire never stops but continues to infinity; we will call this phenomenon *an infinite fire*.

Now we introduce some notations. Let $\eta_x(t) \in \{0, 1\}$ be the state of the site $x \in \mathbb{Z}_+$ at time $t \geq 0$, and we say that the site x is vacant (occupied, respectively) if $\eta_x = 0$ ($\eta_x = 1$ respectively). A vacant site $x \in \mathbb{Z}_+$ becomes occupied at rate 1; once x is occupied, it becomes vacant again after it catches fire that spreads from the site $x - 1$ and stops burning after a time distributed as θ . We assume that the process $\eta_x(t)$ is right-continuous.

Let ν_k be the time at which the k -th tree appears at the origin (and instantaneously begins burning); if $\theta \equiv 0$, the times $\nu_1, \nu_2, \dots$ form a Poisson point process with rate 1, otherwise

$$\nu_k = (\xi_1 + \theta_{0,1}) + (\xi_2 + \theta_{0,2}) + \dots + (\xi_{k-1} + \theta_{0,k-1}) + \xi_k \quad (1.1)$$

where ξ_k are i.i.d. $\exp(1)$ random variables.

Assume for the moment that $\theta \equiv 0$. Let n_k denote the right-most point burnt by fire number k ; this fire will eventually spread to the site n_k which will happen at time $\nu_k + \zeta_k$ where $\zeta_k = \Delta'_0 + \Delta'_1 + \dots + \Delta'_{n_k-1}$ with Δ 's being i.i.d. copies of Δ . We can interpret ζ_k as

²in real life, this can be interpreted as grassland between trees spreading fire, or that a burning tree emits sparks which are spread by the wind until they reach the next tree. Thus, the next tree might start burning after the tree on its left had already burnt

the duration of the k -th fire. Then

$$\begin{aligned} \eta_1(\nu_k + \Delta'_0 -) &= \eta_2(\nu_k + \Delta'_0 + \Delta'_1 -) = \cdots = \eta_{m_k}(\nu_k + \Delta'_0 + \Delta'_1 + \cdots + \Delta'_{n_k-1} -) = 1; \\ \text{but } \eta_{m_k+1}(\nu_k + \Delta'_0 + \Delta'_1 + \cdots + \Delta'_{n_k} -) &= 0; \end{aligned}$$

If $n_k = \infty$ and $\zeta_k = \infty$, we shall call the k -th fire *an infinite fire*. Finally, we can similarly define n_k and the duration of the k -th fire in the case when θ is not identically equal to zero, but we will not get such simple formulae.

1.1 Graphic representation

We can represent our model in the upper right quadrant of $\mathbb{R}^2$ as follows. For each $x \in \{0, 1, 2, \dots\}$ consider a vertical ray (x, t) , $t \geq 0$. Suppose a tree appears at site x for the i -th time at time t , and starts burning at time $t' \geq t$. Then colour the segment with the vertical coordinates $[t, t']$ **green**. The tree will burn between times t' and $t' + \theta_{x,i}$ so colour the interval $(t', t' + \theta_{x,i})$ **red**. The rest of the line remains uncoloured. As a result, consecutive segments on this ray alternate: the uncoloured segments are followed by green and then by red segments (it is conceivable that a green segment degenerates to a single point if the fire at x starts immediately after the tree is planted there; this can happen if $\theta_{x-1,j} > 0$). Please see Figure 1 for the case when the duration of fire is always zero, and Figure 2 for the general case. Interestingly, the graphic representation of the process is somewhat reminiscent of direct percolation models, though we could not find a direct connection which would be useful in obtaining additional insights.

If the k -th fire does not stop at x , and x started burning at time t , then we can draw an arrow from some point in the red segment with coordinates (x, s) with $s \in [t, t + \theta_{x,i})$ to the point $(x + 1, s + \Delta_{x,i})$ such that the latter point is green, and each fire can be represented by a sequence of such arrows.

Remark 1.1. *As it follows from the description of the model, distinct arrows from x to $x + 1$ do not cross.*

1.2 Structure of the paper

The rest of the paper is organized as follows. In Section 2 we prove the existence of the phenomenon called “the infinite fire” in case of non-trivial but identically distributed spreading times Δ . In Section 3 we consider the situation when the burning time θ is identically zero, yet the spreading time is either constant or asymptotically decreasing, and establish conditions for (non)-existence of an infinite fire. In Section 4, on the other hand, we show that if the

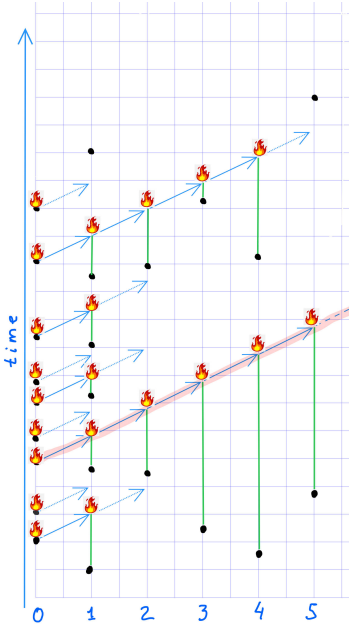


Figure 1: Graphical representation in the case $\theta \equiv 0$, and a constant $\Delta \equiv a > 0$. Solid arrows represent the spread of fire, while dotted arrows indicate unsuccessful attempts to spread. The black dots mark the appearance of trees, and green lines denote periods during which a tree occupies a site before it burns. The highlighted line of arrows represents the infinite fire.

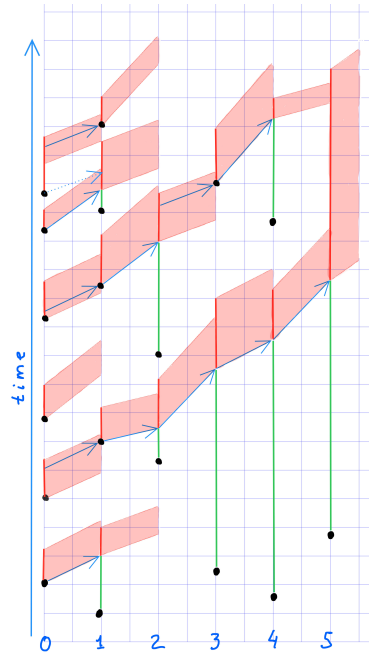


Figure 2: The black dots correspond to the appearance of trees; the green (red resp.) segments are the periods when a site is occupied by a “healthy” (burning resp.) tree. The shaded areas represent the periods when a burning tree at site x affects site $x + 1$. The solid arrows show the fire spreading to a neighbouring tree; the dotted arrows represent the situations when the fire *tried* to spread unsuccessfully.

spreading of the fire is immediate ($\Delta = 0$) then no infinite fire occurs, and we also obtain some results on how quickly the fire spreads.

2 Existence of infinite fire

The infinite fire is a phenomenon which cannot happen in the forest fire models studied in either [9] or [15]; it arises, however, when there is a delay in the spreading of fire.

Theorem 2.1. *Suppose that $\mathbb{P}(\Delta > 0) > 0$. Then a.s. there will be an infinite fire, i.e. $\mathbb{P}(\inf\{k : n_k = \infty\} < \infty) = 1$.*

The next auxiliary statement is quite obvious, yet, for the sake of completeness, we present its short proof.

Lemma 2.2. *Suppose that Δ is a non-negative random variable with $\mathbb{P}(\Delta > 0) > 0$. Then for some constants $\delta_1 > 0$ and $\delta_2 > 0$*

$$\mathbb{P}\left(\sum_{j=1}^n \Delta'_j \geq n\delta_1 \text{ for all } n \geq 1\right) \geq \delta_2$$

where Δ'_j are i.i.d. copies of Δ .

Proof. By the condition of the lemma, there exist $a_1 > 0$ and $p_1 > 0$ such that $\mathbb{P}(\Delta \geq a_1) \geq p_1$. Hence Δ'_j is stochastically larger than

$$Z_j = \begin{cases} a_1, & \text{with probability } p_1; \\ 0, & \text{with probability } 1 - p_1 \end{cases}$$

and we can assume that Z_j s are also i.i.d. Consequently, it suffices to show that

$$\mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \text{ for all } n = 1, 2, \dots\right) \geq \delta_2$$

where $\delta_1 := \mathbb{E}Z_j/2 = a_1p_1/2 > 0$.

By the strong law of large numbers $\frac{1}{n} \sum_{j=1}^n Z_j \rightarrow a_1p_1$ a.s. as $n \rightarrow \infty$, hence, since $\delta_1 < \mathbb{E}Z_j$, a.s. there exists a (random) $N = N(\omega)$ such that $\sum_{j=1}^n Z_j \geq n\delta_1$ for all $n \geq N$. Since N is finite, for some non-random $n_0 \geq 1$ we have $\mathbb{P}(N \leq n_0) \geq 1/2$. As $Z_j \in \{0, a_1\}$, we have

$$\begin{aligned} \frac{1}{2} &\leq \mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \quad \forall n > n_0\right) \leq \mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \quad \forall n > n_0 \mid Z_1 = Z_2 = \dots = Z_{n_0} = a_1\right) \\ &= \frac{\mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \quad \forall n > n_0 \text{ and } Z_1 = Z_2 = \dots = Z_{n_0} = a_1\right)}{\mathbb{P}(Z_1 = Z_2 = \dots = Z_{n_0} = a_1)} \\ &= \frac{\mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \quad \forall n \geq 1 \text{ and } Z_1 = Z_2 = \dots = Z_{n_0} = a_1\right)}{p_1^{n_0}} \leq \frac{\mathbb{P}\left(\sum_{j=1}^n Z_j \geq n\delta_1 \quad \forall n \geq 1\right)}{p_1^{n_0}} \end{aligned}$$

so the Lemma holds with e.g. $\delta_2 := p_1^{n_0}/2 > 0$. □

Proof of Theorem 2.1. First, we will show that for any $x \in \mathbb{Z}_+$ the fire will eventually reach x . In fact, by induction we will show a stronger statement: for each $x = 0, 1, 2, \dots$ there will be infinitely many fires that reach the site x and they will occur at arbitrary large times (i.e., all $f_{x,i}$ s are finite, and $f_{x,i} \nearrow \infty$ as i increases.) The proof is based on induction.

For site $x = 0$ this fact is trivial as $f_{0,k} \equiv \nu_k$, see (1.1). Now suppose that our statement is true for some $x \geq 0$. This means that there are infinitely many fires that start at times $f_{x,1} < f_{x,2} < \dots$. Since the probability of not having a single tree at $x+1$ by time s is $e^{-s} \downarrow 0$ and $\lim_{i \rightarrow \infty} f_{x,i} = \infty$, eventually for some time $f_{x,j}$ there will be a tree at $x+1$ so it will start burning at time $f_{x,j} + \Delta_{x,j} < \infty$ and burn during time $\theta_{x+1,1} \geq 0$; thus we ensured that there will be at least one fire at $x+1$. Now consider the process after the time $f_{x,j} + \Delta_{x,j} + \theta_{x+1,1}$. It takes an exponentially distributed time for a new tree to appear at $x+1$; after that, it will eventually be burnt by a fire started at site x at some time $f_{x,j'}$ with $j' > j$. This fire will start burning the tree at $x+1$ again at time $f_{x,j'} + \Delta_{x,j'}$. Similar arguments hold for the third fire, the fourth fire, etc. The exponential waiting time for the appearance of new trees also implies that $f_{x+1,j}$ is stochastically larger than the time of arrival of the j th point of a Poisson process with rate 1, yielding $f_{x+1,i} \rightarrow \infty$. The induction step is finished.

Let m_1 denote the location of the tree, farthest from the origin, burnt by the first fire; if the fire continues indefinitely, we let $m_1 = +\infty$. For $k \geq 2$, we recursively define m_k as the location of the tree, farthest from the origin, burnt by the first fire which reaches beyond m_{k-1} ; we can think of m_k s as the successive maxima reached by the fires. If $m_{k-1} = \infty$ for some k , we set $m_k = m_{k+1} = \dots = \infty$ as well. The successive maxima obviously satisfy $0 \leq m_1 < m_2 < m_3 < \dots$ (with the convention $\infty < \infty$).

Let $A_k = \{m_k < \infty\}$, i.e., the k -th fire was finite, and let $T_k = f_{m_k,1} + \theta_{m_k,1}$ be the time when the site m_k stops burning for the first time. Let $\mathcal{F}_k$ be the σ -algebra³ generated by the states of sites $\{0, 1, \dots, m_k + 1\}$ during the period $[0, T_k]$; *we do not include in it* any events related to appearances of the trees at sites $m_k + 2, m_k + 3, \dots$. We want to compute a lower bound on $\mathbb{P}(A_{k+1}^c \mid \mathcal{F}_k)$, assuming that A_k has occurred and $m_k = m$. Suppose that the fire has reached $m+1$ by some time t (note that $t > T_k$). It takes $\Delta_{m+1,1}$ units of time for the fire to reach $m+2$. Assuming that there is a tree at $m+2$ by that time (i.e., $t + \Delta_{m+1,1}$), the fire will continue to $m+3$ if there is a tree at $m+3$ by time $t + \Delta_{m+1,1} + \Delta_{m+2,1}$, and so on. Thus the fire will continue *ad infinitum*, provided that

$$\begin{aligned} \text{there is a tree at site } (m+i) \text{ by time } & t + \Delta_{m+1,1} + \Delta_{m+2,1} + \dots + \Delta_{m+i-1,1} \\ & \text{for each } i = 2, 3, \dots \end{aligned}$$

³note that this is a sigma-algebra generated by the stopping time rather than a discrete time

Since the probability that there is no tree at site x by time $s \geq 0$ is e^{-s} (assuming no previous fires at x), the probability of the above event given t and $\{\Delta_{m+i,1}\}_{i=1}^{\infty}$ is

$$\prod_{i=2}^{\infty} \left(1 - e^{-t - \sum_{k=1}^{i-1} \Delta_{m+k,1}}\right) \geq \prod_{i=2}^{\infty} \left(1 - e^{-\sum_{k=1}^{i-1} \Delta_{m+k,1}}\right). \quad (2.1)$$

At the same time, by Lemma 2.2

$$\mathbb{P}\left(e^{-\sum_{j=1}^n \Delta_j'} \leq e^{-n\delta_1} \text{ for all } n = 1, 2, \dots\right) \geq \delta_2$$

hence if we denote

$$\epsilon_0 := \prod_{n=1}^{\infty} (1 - e^{-n\delta_1}) > 0$$

then the probability that the RHS of (2.1) is larger than ϵ_0 is no less than δ_2 . As a result,

$$\mathbb{P}(A_{k+1}^c \mid \mathcal{F}_k) \geq \epsilon_0 \delta_2 > 0 \quad \text{for all } k \geq 1$$

and hence by the conditional Borel-Cantelli lemma there will be an infinite fire a.s. $\square$

3 Instant burning

In this Section, we will consider the case when $\theta = 0$, and $\mathbb{E}\Delta_x > 0$ that is, the trees are burnt instantly but the expected time for the fire to spread from site x to $x + 1$ is strictly positive.

Theorem 3.1. *Suppose $\theta \equiv 0$, $0 < \mathbb{E}\Delta < \infty$ and $\text{Var}(\Delta) > 0$. Then there is only one infinite fire.*

Remark 3.2. *The non-random case $\Delta \equiv \text{const} > 0$ will be covered later in Theorem 3.3.*

Proof. The existence of infinite fire follows from Theorem 2.1 since $\mathbb{E}\Delta > 0$ implies $\mathbb{P}(\Delta > 0) > 0$, so we only need to show that there cannot be a *second infinite fire*.

For each site $x \in \mathbb{Z}_+$ let i_x be the index of the fire for this site, which turned out to be the first infinite fire (hence there will be an x_1 such that for all $x > x_1$ we have $i_x = 1$).

Assume now that there is a second infinite fire. Since for any $t > 0$ the number of distinct fires at each site x by time t is a.s. finite, there will be a (random) site x_2 such that each site with $x > x_2$ was not burnt by any *finite fire* between the times when it was burnt by the first and the second infinite fire. Hence, for each $x > x^* = \max(x_1, x_2)$, the first (the second resp.) fire to burn x is, in fact, the first (the second resp.) infinite fire. Since burning is immediate ($\theta \equiv 0$), for $x > x^*$

$$f_{x+1,1} = f_{x,1} + \Delta_{x,1} \quad \text{and} \quad f_{x+1,2} = f_{x,2} + \Delta_{x,2}.$$

Let us define $S_x = \sum_{k=1}^x \xi_k$, where $\xi_k = \Delta_{k,2} - \Delta_{k,1}$ are i.i.d. mean zero random variables. Then $f_{x,i_x+1} - f_{x,i_x} = C(x^*) + S_x$ for $x \geq x^*$ where $C(x^*)$ is some quantity depending on x^* only. Indeed, for all sites at locations $x > x^*$ the first fire to burn x will be *the first infinite fire*, and the second fire to burn x will be the second infinite fire. Since S_x is a one-dimension zero-mean random walk, it is recurrent by [14, P8, Section I.2] and hence $\mathbb{P}(\liminf_{x \rightarrow \infty} S_x = -\infty \mid x^*) = 1$, which contradicts the fact that we must always have $f_{x,i_x+1} > f_{x,i_x}$. $\square$

3.1 Constant spreading time

Recall that the burning time θ is assumed to be zero, and assume additionally that Δ is the same non-random positive constant at all locations; in this Section we want to study this special case. An interesting fact, observed by Edward Crane, is that after the infinite fire, the model here can be coupled with that of [15]. We have already shown that there exists an infinite fire in this case; now we are interested in finding out the distribution of *when exactly* the infinite fire starts, and what the index κ of that fire can be.

Theorem 3.3. *Assume that $\theta \equiv 0$ and $\Delta \equiv a > 0$. Suppose that the infinite fire starts at site 0 at some time $T \geq 0$ (which is, of course, random). Then this fire reaches point $x \geq 1$ precisely at time $T + ax$, and if we denote $\hat{\eta}_x(t) = \eta_x(t + T + ax)$, $t \geq 0$, then the process $\hat{\eta}$ is exactly the one introduced in [15], i.e., the one with $\theta = \Delta \equiv 0$.*

Proof. This fact follows immediately from the definition of the process $\hat{\eta}_x(t)$ and can be also seen from the graphical representation of the fire process (see Figure 1), where all the arrows have the same slope a , and the infinite fire corresponds to a straight line going from $(0, T)$ towards infinity at the constant slope a . $\square$

To get an estimate of when the infinite fire starts, observe the following. The times of consecutive fires at the origin form a Poisson process, and thus the k th fire starts at time $\nu_k = \xi_1 + \xi_2 + \dots + \xi_k$ where ξ_i are i.i.d. exponential(1) random variables (see (1.1)). Let $\kappa \geq 1$ be the index of the infinite fire, thus $T = \nu_\kappa$. In the statement below we estimate the probability that the first fire is infinite.

Proposition 3.4. *Assume the same as in Theorem 3.3. Then*

$$1 - \frac{1}{2\mu} \leq \mathbb{P}(\kappa = 1) \leq \mu(1 - e^{-1/\mu})$$

where $\mu := e^a - 1$. Hence, for large $a > 0$ we have $\mathbb{P}(\kappa = 1) = 1 - \frac{1}{2(e^a - 1)} + o(1)$.

Proof. The first fire burns 0 at time ξ_1 . For this fire to spread indefinitely, one needs to have at least one tree at position x at time $\xi_1 + ax$ for each $x \in \{1, 2, \dots\}$; given ξ_1 , this probability can be computed as $\prod_{x=1}^{\infty} (1 - e^{-\xi_1 - ax})$.

Since for any sequence $y_1, y_2, \dots \in [0, 1]$ we have

$$1 - \sum_{k=1}^{\infty} y_k \leq \prod_{k=1}^{\infty} (1 - y_k) \leq \exp \left(- \sum_{k=1}^{\infty} y_k \right), \quad (3.1)$$

and

$$\sum_{x=1}^{\infty} e^{-\xi_1 - ax} = \frac{e^{-\xi_1}}{e^a - 1}.$$

By taking the expectation over ξ_1 , we get

$$\begin{aligned} \mathbb{P}(\kappa = 1) &= \mathbb{E} \prod_{x=1}^{\infty} (1 - e^{-\xi_1 - ax}) \leq \int_0^{\infty} \exp \left(- \frac{e^{-u}}{e^a - 1} \right) e^{-u} du \\ &= \int_0^1 \exp \left(- \frac{v}{e^a - 1} \right) dv = (e^a - 1) \left(1 - e^{-\frac{1}{e^a - 1}} \right). \end{aligned}$$

On the other hand, by (3.1)

$$\mathbb{P}(\kappa = 1) \geq \int_0^{\infty} \left(1 - \frac{e^{-u}}{e^a - 1} \right) e^{-u} du = 1 - \frac{1}{2(e^a - 1)},$$

which completes the proof. $\square$

Remark 3.5. *A much better approximation*

$$\mathbb{P}(\kappa = 1) \approx 1 - \frac{e^{-a} + e^{-2a}}{2} - \frac{e^{-3a}}{6}$$

can be obtained by using a tighter lower bound of $\prod_{k=1}^{\infty} (1 - y_k)$. It gives the relative error of 0.01 or less for $a \geq 1/2$, and at most 0.07 for $1/2 \geq a \geq 1/3$.

Suppose the first fire did not become the infinite fire; let m_1 be the right-most point burnt by it. By the arguments similar to those in Theorem 3.3, we can couple our process after time ν_1 with the simple fire process $\hat{\eta}_x(t)$ from [15] such that

$$\hat{\eta}_x(t) = \eta_x(\nu_1 + ax + t) \text{ for } x \in \{0, 1, \dots, m_1, m_1 + 1\}.$$

Let us also observe that given ν_1 ,

$$\mathbb{P}(m_1 = k) = e^{-\nu_1 - a(k+1)} \prod_{x=1}^k (1 - e^{-\nu_1 - ax}), \quad k = 1, 2, \dots$$

Let $\delta_1 = 1$ and, assuming that the first fire was not infinite, let δ_2 be *the index* of the first fire which extends beyond m_1 , i.e., burns site $m_1 + 1$. Denote by $m_2 (\geq m_1 + 1)$ the right-most point burnt by this fire. Then, given the time when it started, that is ν_{δ_2} ,

$$\mathbb{P}(m_2 = m_1 + k \mid \nu_{\delta_2}) = e^{-\nu_{\delta_2} - a(m_1 + k + 1)} \prod_{x=m_1+2}^{m_1+k} (1 - e^{-\nu_{\delta_2} - ax}), \quad k = 1, 2, 3, \dots$$

Similarly, one can define pairs (δ_3, m_3) , (δ_4, m_4) , etc., until (δ_N, m_N) where $\delta_N = \kappa$ and $m_N = +\infty$.

Now notice that from coupling with the process in [15], we have $\nu_{\delta_{k+1}} - \nu_{\delta_k} = \hat{\tau}_{m_k+1}$ where $\hat{\tau}_z$ is the first time when the original forest fire process⁴ burns site $z \geq 1$. Hence $\nu_{\delta_k} = \psi_k(m)$ where

$$\psi_k(\mu) = \sum_{\ell=0}^k \hat{\tau}_{\mu_\ell+1} \quad (3.2)$$

where we set $\mu_0 := -1$ and thus $\hat{\tau}_0$ has the same $\exp(1)$ distribution as ν_1 . As a result, for any positive integer n and a sequence $0 \equiv \mu_0 < \mu_1 < \mu_2 < \dots < \mu_{n-1}$,

$$\begin{aligned} & \mathbb{P}(\kappa \geq n, m_1 = \mu_1, m_2 = \mu_2, \dots, m_{n-1} = \mu_{n-1}) \\ &= \mathbb{E} \left[\prod_{i=1}^{n-1} e^{-\nu_{\delta_i} - a(\mu_{i-1}+1)} \prod_{x=\mu_{i-1}+1}^{\mu_i} (1 - e^{-\nu_{\delta_i} - ax}) \right] \\ &= \mathbb{E} \left[\prod_{i=1}^{n-1} \left(e^{-(\psi_{i-1}(\mu) + a(\mu_{i-1}+1))} \prod_{x=\mu_{i-1}+1}^{\mu_i} (1 - e^{-(\psi_{i-1}(\mu) + ax)}) \right) \right] \\ &= \mathbb{E} \left[\left(\prod_{i=1}^{n-1} e^{-(\psi_{i-1}(\mu) + a(\mu_{i-1}+1))} \right) \prod_{x=1}^{\mu_{n-1}} (1 - e^{-(ax + \psi_{\min\{k: x \leq \mu_k\}}(\mu))}) \right]. \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{P}(\kappa > n) &= \mathbb{E} \left[\sum_{\mu_1=1}^{\infty} \sum_{\mu_2=\mu_1+1}^{\infty} \dots \sum_{\mu_{n-1}=\mu_{n-2}+1}^{\infty} \left(\prod_{i=1}^{n-1} e^{-(\psi_{i-1}(\mu) + a(\mu_{i-1}+1))} \right) \right. \\ &\quad \left. \times \prod_{x=1}^{\mu_{n-1}} (1 - e^{-(ax + \psi_{\min\{k: x \leq \mu_k\}}(\mu))}) \right] \end{aligned}$$

which expresses the probability in terms of random variables $\hat{\tau}_x$ whose distribution can be obtained explicitly from Lemma 1.1 in [15], and then by using (3.2).

3.2 Non-identically distributed Δ 's

In this section, we modify the model to allow Δ_x , $x = 0, 1, 2, \dots$, to have different distributions, which may depend on the location.

⁴that is, the one with $\Delta = \theta = 0$

Theorem 3.6. *Let $c \leq 1$. Assume that either*

(a) $\Delta_x \leq c/x$ *for all large x , or*

(b) *for some $C > 0$ we have $|\Delta_x| \leq C$ for all x , $\sum_x \text{Var}(\Delta_x) < \infty$, $|\sum_x [\mathbb{E}\Delta_x - \frac{c}{x}]| < \infty$.*

Then the infinite fire does not occur a.s. i.e., $\mathbb{P}(n_k < \infty \text{ for all } k) = 1$.

Conversely, if $c > 1$, and either

(c) $\Delta_x > c/x$ *for all large x , or*

(d) $\mathbb{E}\Delta_x = \frac{c+o(1)}{x}$ *and $\mathbb{P}\left(\lim_{x \rightarrow \infty} |\Delta_x| \sqrt{x} = 0\right) = 1$.*

Then the infinite fire occurs a.s.

Proof. Let $c \leq 1$. We will only consider case (b), as case (a) is similar (and easier). Suppose the fire reaches some site $x \geq 1$ for the first time at time t_x . Conditioned on this time t_x , the probability that this fire will continue indefinitely is

$$q_x := \prod_{n=x+1}^{\infty} (1 - e^{-(t_x + \Delta_x + \Delta_{x+1} + \dots + \Delta_{n-1})}) = \prod_{n=x+1}^{\infty} (1 - e^{-(t_x + S_{x,n})})$$

as it takes $S_{x,n} := \Delta_x + \Delta_{x+1} + \dots + \Delta_{n-1}$ units of time for the fire to reach site n and we need to ensure that already there is a tree at this location. Let $\xi_x = \Delta_x - \frac{c}{x}$. Then $\sum_x \xi_x$ converges a.s. by Kolmogorov's three-series theorem. We have

$$S_{x,n} = \sum_{i=x}^{n-1} \left(\xi_i + \frac{c}{i} \right) = c \log \frac{n}{x} + o_x(1) + \sum_{i=x}^{n-1} \xi_i = c \log \frac{n}{x} + o_x(1)$$

where $o_x(1) \rightarrow 0$ a.s. as $x \rightarrow \infty$. As a result,

$$e^{-t_x - S_{x,n}} = e^{-t_x - o_x(1)} \left(\frac{x}{n} \right)^c$$

which is not summable in $n \geq 1$ as $c \leq 1$, yielding $q_x = 0$.

Now assume $c > 1$. We will consider only the harder case (d). Let $\Delta_x^{(i)}$ be the (random) spread time from site x to site $x+1$, for the fire number $i = 1, 2, \dots$ started at the origin. Note that some of the $\Delta_x^{(i)}$ s will be irrelevant for the understanding of the process: for example, if the first fire reaches only site 1, and the second fire reaches site 3, then $\Delta_1^{(1)}, \Delta_2^{(1)}$ will be irrelevant for the behaviour of the process, but $\Delta_1^{(2)}, \Delta_2^{(2)}$ will be relevant.

Let $\zeta_n \in \{1, 2, \dots\}$ be the index of the fire that burns n for the very first time. Pick a small $\delta > 0$ such that $c - \delta > 1$, and let

$$D_{n,i} := \left\{ \sum_{x=0}^{n-1} \Delta_x^{(i)} < (c - \delta) \log n \right\},$$

$$A_n := \{\text{the first fire which reaches the } n\text{-th tree does not spread to the } (n+1)^{\text{st}} \text{ tree}\}$$

Note that in these notations $(D_{n,\zeta_n})^c = \{\text{the fire which reaches the } n\text{-th tree for the first time takes more than } (c - \delta) \log n \text{ units of time from its start}\}.$

We have $0 \leq \Delta_x \leq C_x$ a.s., where $C_x \leq \frac{\delta}{\sqrt{2x}}$ for all large x , yielding

$$\sum_{x=1}^{n-1} C_x^2 \leq \delta^2 \log n$$

for large n . Since also

$$\sum_{x=1}^n \mathbb{E} \Delta_x = (c + o(1)) \log(n)$$

by Hoeffding's inequality,

$$\mathbb{P}(D_{n,i}) \leq \exp\left(-\frac{2\delta^2 \log^2 n}{\delta^2 \log n}\right) = \frac{1}{n^2}.$$

To prove the existence of an infinite fire, it suffices to show that A_n happens only *finitely often*, almost surely. We have

$$\begin{aligned} \mathbb{P}(A_n) &= \mathbb{P}(A_n \cap D_{n,\zeta_n}^c) + \mathbb{P}(A_n \cap D_{n,\zeta_n}) \leq \mathbb{P}(A_n \mid D_{n,\zeta_n}^c) + \mathbb{P}(D_{n,\zeta_n}) \\ &= \sum_{i=1}^{\infty} [\mathbb{P}(A_n \mid D_{n,i}^c, \zeta_n = i) + \mathbb{P}(D_{n,i} \mid \zeta_n = i)] \times \mathbb{P}(\zeta_n = i) \\ &\leq \sum_{i=1}^{\infty} [\mathbb{P}(\text{no tree at } (n+1) \text{ upto time } (c - \delta) \log n) + \mathbb{P}(D_{n,i} \mid \zeta_n = i)] \times \mathbb{P}(\zeta_n = i) \\ &= \frac{1}{n^{c-\delta}} + \sum_{i=1}^{\infty} \mathbb{P}(D_{n,i} \mid \zeta_n = i) \times \mathbb{P}(\zeta_n = i) \leq \frac{1}{n^{c-\delta}} + \frac{1}{n^2} \end{aligned} \tag{3.3}$$

which holds as long as we can show that

$$\mathbb{P}(D_{n,i} \mid \zeta_n = i) \leq \mathbb{P}(D_{n,i}). \tag{3.4}$$

Let us show that this is indeed the case. Observe that $\{\zeta_n = i\} = X \cap Y$, where

$$\begin{aligned} X &:= \{\text{the first } i-1 \text{ fires do not reach site } n\}, \\ Y &:= \{\text{the } i\text{-th fire reaches site } n\}. \end{aligned}$$

Note also that X and $D_{n,i}$ are independent since the distributions of the first $(i-1)$ fires are not influenced by the $\{\Delta_0^{(i)}, \Delta_1^{(i)}, \dots, \Delta_{n-1}^{(i)}\}$, and $\mathbb{P}(Y) \geq \mathbb{P}(Y \mid D_{n,i})$, since the probability of the i -th fire reaching n increases whenever any of $\Delta_x^{(i)}$ in the above set increases. As a result,

$$\begin{aligned} \mathbb{P}(X \cap Y \mid D_{n,i}) &= \mathbb{P}(X \mid Y, D_{n,i}) \mathbb{P}(Y \mid D_{n,i}) = \mathbb{P}(X \mid Y) \mathbb{P}(Y \mid D_{n,i}) \\ &\leq \mathbb{P}(X \mid Y) \mathbb{P}(Y) = \mathbb{P}(X \cap Y) \end{aligned}$$

which together with the Bayes formula

$$\mathbb{P}(D_{n,i} \mid \zeta_n = i) = \frac{\mathbb{P}(\zeta_n = i \mid D_{n,i}) \times \mathbb{P}(D_{n,i})}{\mathbb{P}(\zeta_n = i)}$$

yields (3.4).

Finally, by (3.3), since $c - \delta > 1$, the probabilities $\mathbb{P}(A_n)$ are summable over n , and by the Borel-Cantelli lemma, A_n occurs finitely often, almost surely. As a result, there cannot be infinitely many successive maxima, implying that an infinite fire a.s. exists. $\square$

4 Instant spread, non-zero burning times.

Throughout this Section we assume that $\Delta \equiv 0$ but $\theta \geq 0$.

Theorem 4.1. *Suppose that $\Delta \equiv 0$. Then a.s. there are no infinite fires.*

Proof. Recall that $f_{x,i}$ denotes the time when the site $x \in \mathbb{Z}_+$ is burnt for the i th time. Then at time $t = f_{x,1}$ there may already be trees at sites $x+1, x+2, \dots$ and the fire will spread to those sites immediately (as $\Delta = 0$); however, since the probability of not having a single tree during time s is e^{-t} and these events are independent for each site $y > x$, with probability one there will be a site $y \geq x+1$ such that there are no trees there; we let y be the leftmost such site. This means that

$$f_{x,1} = f_{x+1,1} = \dots = f_{y-1,1} < f_{y,1}.$$

For this fire still to reach y , there must be a tree "planted" at y during the time interval $[t, t + \theta_{y-1,1}]$. Conditionally on $\theta_{y-1,1}$, this probability equals $1 - e^{-\theta_{y-1,1}}$, so every time the fire spreads immediately from some site x to $y-1$, with probability $p_* := \mathbb{E}e^{-\theta} > 0$, independently of the past, it will not spread further. Since this probability does not depend on x or s either, it means that a.s. the fire will eventually stop. $\square$

Now we want to study some further properties of the fire process. Recall that m_k denotes the location of the tree, farthest from the origin, burnt by the k -th fire.

Lemma 4.2. *Assume $\Delta \equiv 0$ and fix some $\epsilon > 0$. The time $f_{m_i,1}$ when the fire reaches m_i for the first time is less than $(1 + \epsilon) \log m_i$ for all but finitely many i , almost surely.*

Proof. Suppose that the fire reaches site n at time $t = f_{n,1}$. The probability that there is no tree at $n+1$ at this time is e^{-t} , so in case $t \geq (1 + \epsilon) \log n$, the probability of not spreading the fire further is less than

$$e^{-(1+\epsilon) \log n} = \frac{1}{n^{1+\epsilon}}$$

Hence $\mathbb{P}(A_n) \leq \frac{1}{n^{1+\epsilon}}$ where

$$A_n = \{n \text{ is a local maximum and } f_{n,1} > (1 + \epsilon) \log n\}$$

so by the Borel-Cantelli lemma, A_n occurs only for finitely many n s a.s. From this, the statement follows. $\square$

The next statement is similar to Proposition 2 in [9].

Lemma 4.3. *Assume $\Delta \equiv 0$ and fix some $\epsilon > 0$. Let*

$$F_n = F_n(\epsilon) = \{f_{n,1} \geq (1 - \epsilon) \log n\}.$$

Then

$$\mathbb{P}(F_n^c) < e^{-n^\epsilon}$$

and hence the event F_n^c a.s. occurs only finitely often.

Proof. In order to reach n by time t , there should be a tree at each location $x \in [0, n]$ by that time (or even earlier, because $\theta > 0$), so

$$\mathbb{P}(f_{n,1} < (1 - \epsilon) \log n) \leq (1 - e^{(1-\epsilon) \log n})^n = \left[\left(1 - \frac{1}{n^{1-\epsilon}} \right)^{n^{1-\epsilon}} \right]^{n^\epsilon} < e^{-n^\epsilon}.$$

The second statement follows from the Borel-Cantelli lemma. $\square$

Lemmas 4.2 and 4.3 give a tight bound on $f_{m_i,1}$, since by Lemma 4.3, that $f_{m_i,1} < (1 - \epsilon) \log m_i$, only finitely often almost surely.

Recall that $f_{x,2}$ is the time when the site x is burnt for the second time. We believe that the following holds, even though we cannot prove it.

Conjecture 4.4. *There is a $c \geq 1$, such that a.s. there exists a (possibly random) κ such*

$$f_{m_i,2} \leq c \log m_i \quad \text{for all } i \geq \kappa. \quad (4.1)$$

The motivation for this conjecture is as follows. First of all, we already know that $(1 - \epsilon) \log m_i < f_{m_i,1} < (1 + \epsilon) \log m_i$, for all large i almost surely, regardless of the value of θ . In the case where $\theta \equiv 0$, the difference $f_{m_i,2} - f_{m_i,1}$ has the same distribution as $f_{m_i,1}$. Since altering θ does not have any effect on the order of f_{m_i} , we suspect it does not have any substantial effect on $f_{m_i,2} - f_{m_i,1}$ either. Later, we show that the number of “jumps” (continuous stretches of fire, please see the proof of part (b) of Lemma 4.6 for the exact definition) in the fire

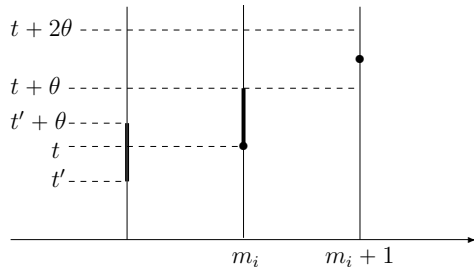


Figure 3: how the second fire reaching m_i can spread very far.

reaching m_i for the first time is of order $\log i$, which is quite low compared to the order of m_i that exceeds $\exp\{e^{ai}\}$, as we show later. This indicates that the path (using the graphic representation language) of the first fire reaching m_i is not “very bumpy”, and thus it might behave more like a flat path (instantaneous burn). Hence, while we do not claim $f_{m_i,2} - f_{m_i,1}$ equals $(1 + o(1)) \log m_i$, we have a strong reason to believe that it is *at least* of order $\log m_i$.

Theorem 4.5. *Suppose that $\Delta = 0$ and $\theta > 0$ is constant. Assume that Conjecture 4.4 is true. Then $f_{n,1} = \mathcal{O}(\log n)$, i.e. there exists a non-random $M > 0$ such that $f_{n,1} \leq M \log n$ for large n .*

We start with the auxiliary statement.

Lemma 4.6. *Suppose that $\Delta = 0$ and $\theta > 0$ is constant.*

- (a) *Let $r > 0$. Then a.s. $m_i \geq i^r$ for all but finitely many i .*
- (b) *Assume that Conjecture 4.4 is true. Let $\epsilon > 0$ and c be the constant from (4.1). Then a.s. $m_{i+1} \leq m_i^{c+\epsilon}$ for all but finitely many i .*

Proof. (a) The statement is trivial for $r \leq 1$, since $m_i \geq i$, so from now on assume that $r > 1$. Let $0 < \epsilon < \frac{1}{r}$. Suppose that the fire has reached m_i at time $t := f_{m_i,1}$; w.l.o.g. we can assume that t is sufficiently large.

Since the fire has successfully spread from $m_i - 1$ to m_i , this fire had reached $m_i - 1$ at time $t' \in (t - \theta, t]$, and thus with probability at least $e^{-2\theta}$ there will be no new tree at $m_i - 1$ during the time interval $[t' + \theta, t + 2\theta]$ (as $t' + \theta$ is the time when the tree at $m_i - 1$ stops burning). Under this event, the *second* fire to reach m_i cannot happen before time $t + 2\theta$, which, in turn, provides the site $m_i + 1$ with at least θ units of time to grow a tree (there was none during the time $[t, t + \theta]$ since we know that m_i was the local maximum by that time); please see Figure 3. Hence, with probability at least

$$e^{-2\theta} \cdot (1 - e^{-\theta}) \tag{4.2}$$

$f_{m_i+1,1} \in [f_{m_i,2}, f_{m_i,2} + \theta)$, i.e., the second fire to reach m_i will also reach $m_i + 1$. However, by that time, there can already be the whole stretch of trees at sites $m_i + 2, m_i + 3, \dots$; in fact, the number L_i of consecutive sites which do have a tree by this time has a geometric distribution with parameter $p = e^{-f_{m_i+1,1}} < e^{-t}$. Let $\tilde{\mathcal{G}}^*$ be the sigma-algebra containing all the events which happened by time $f_{m_i+1,1}$ to the left of $m_i + 1$. Then

$$\mathbb{P}\left(L_i \geq \frac{1}{p} \mid \mathcal{G}^*\right) = (1-p)^{\lceil 1/p \rceil} = e^{-1} + o(1) \quad (4.3)$$

where $o(1) \rightarrow 0$ as $p \rightarrow 0$ (i.e., $t \rightarrow \infty$). Let

$$B_i := \{m_{i+1} \geq m_i + m_i^{1-\epsilon}\} = \{L_i \geq m_i^{1-\epsilon}\}. \quad (4.4)$$

If event F_{m_i} from Lemma 4.3 occurs, then $t = f_{m_i,1} \geq (1-\epsilon) \log m_i$ yielding $1/p > m_i^{1-\epsilon}$. Taking into account the bound of probability (4.2) and (4.3), we obtain that

$$\mathbb{P}(B_i \mid \mathcal{G}^*) \geq [e^{-2\theta-1} \cdot (1 - e^{-\theta}) + o(1)] \times 1_{F_{m_i}} \geq c' \times 1_{F_{m_i}} \quad (4.5)$$

for some $c' > 0$.

Let $\mu_i = m_i^\epsilon$. From Taylor expansion, we obtain that on B_i we have

$$\mu_{i+1} - \mu_i \geq \epsilon + O(m_i^{-\epsilon}) \geq \epsilon/2$$

for sufficiently large i ; also, trivially, $\mu_{i+1} \geq \mu_i$. Consequently, we can create a sequence of i.i.d. Bernoulli(c') random variables ξ_i , $i = 1, 2, \dots$, such that $\mu_{i+1} - \mu_i$ is stochastically larger than $\frac{\epsilon}{2} \xi_i 1_{F_{m_i}}$. By the strong law, $\frac{1}{n} \sum_{i=1}^n \xi_i \rightarrow c'$ a.s., and at the same time

$$0 \leq \frac{1}{n} \sum_{i=1}^n \xi_i - \frac{1}{n} \sum_{i=1}^n \xi_i 1_{F_{m_i}} \leq \frac{|\{i : F_{m_i} \text{ did not occur}\}|}{n}$$

By Lemma 4.3, the quantity in the numerator is a.s. finite and does not depend on n , hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \xi_i 1_{F_{m_i}} = c' \quad \text{a.s.}$$

This, in turn, leads to

$$\liminf_{i \rightarrow \infty} \frac{\mu_i}{i} \geq \frac{c' \epsilon}{2} \quad \text{a.s.}$$

yielding

$$m_i \geq \left(\frac{c' \epsilon}{3} i\right)^{1/\epsilon} \geq i^r \quad \text{a.s.}$$

for all sufficiently large i , since $1/\epsilon > r$.

(b) Consider the first fire that reaches beyond m_i (the one which occurred at time $f_{m_i+1,1}$).

This fire will extend further to the right in several so-called “jumps”, rigorously defined in the next two paragraphs, until it eventually stops somewhere. The number of these jumps will be denoted by $N_i \in \{1, 2, \dots\}$.

Let $L_{i,1}$ be the length of the maximum stretch of sites to the right of m_i burned at the same time as $m_i + 1$, so that $f_{m_i+y,1} = f_{m_i+1,1}$ for $y = 2, 3, \dots, L_{i,1}$. After the fire reaches site $m_i + L_{i,1}$, and $m_i + L_{i,1} + 1$ is empty at that time (since we know that the fire did not immediately go further) there are two possibilities: either with probability $e^{-\theta}$ within the next θ units of time no tree appears at the site $m_i + L_{i,1} + 1$, or such a tree does appear.

In the first case, we set $N_i = 1$ and stop. In the latter case, we say that the fire has a **jump** at position $m_i + L_{i,1} + 1$, meaning that the fire, whilst burning both $m_i + L_{i,1}$ and $m_i + L_{i,1} + 1$, reached the latter site with a delay (up to θ ; $0 < f_{m_i+L_{i,1}+1,1} - f_{m_i+L_{i,1},1} < \theta$). In case of such a jump, the fire will immediately spread to the additional $L_{i,2}$ sites to the right of $m_i + L_{i,1}$, and thus $f_{m_i+L_{i,1}+y,1} = f_{m_i+L_{i,1}+1,1}$ for $y = 2, 3, \dots, L_{i,2}$, but there is no tree at the site $m_i + L_{i,1} + L_{i,2} + 1$ at this time.

Similarly, we can define $L_{i,3}, L_{i,4}, \dots, L_{i,N_i}$, where N_i is the number of continuous stretches of sites that were burnt at the same time (before the fire eventually stopped completely); N_i is the number of such jumps. As a result, N_i is a geometric random variable with rate $e^{-\theta}$, and the site $(m_i + \sum_{j=1}^k L_{i,j}) + 1$ is burnt slightly later than the site $m_i + \sum_{j=1}^k L_{i,j}$ for each $k = 1, 2, \dots, N_i - 1$, with the delay of up to θ .

From the construction,

$$m_{i+1} - m_i = \sum_{j=1}^{N_i} L_{i,j}. \quad (4.6)$$

Note also that N_i is typically not very large, formally, for some sufficiently large $c_1 > 0$

$$\mathbb{P}(C_i^c) \leq (1 - e^{-\theta})^{\lfloor c_1 \log i \rfloor} \leq \frac{1}{i^2} \quad \text{where } C_i := \{N_i \leq c_1 \log i\} \quad (4.7)$$

which implies by the Borel-Cantelli lemma that a.s. $N_i \leq c_1 \log i$ for all but finitely many i s.

Our goal now is to find an upper bound on how far the fire, that passed beyond m_i for the first time, can travel (i.e., the first fire to burn $m_i + 1$). Conjecture 4.4 postulates that the time to reach m_i for the *second time* is at most $c \log m_i$ for all $i \geq \kappa$, and for these values of i each $L_{i,j}$ can be coupled with $\xi_{i,j}$, $j = 1, 2, \dots$, where $\xi_{i,j}$ are independent geometric random variables with rate $e^{-(\theta(j-1)+c \log m_i)} = m_i^{-c} e^{-\theta(j-1)}$, respectively, such that $L_{i,j} \leq \xi_{i,j}$. This coupling results from the fact that under this conjecture, the time to reach m_i for the second time $f_{m_i,2} \leq c \log m_i$ and then after $j - 1$ jumps the time is at most $\Delta := f_{m_i,2} + \theta(j - 1) \leq c \log m_i + \theta(j - 1)$, and the probability of not having a single tree by this time is $e^{-\Delta}$.

Together with (4.6) this yields

$$(m_{i+1} - m_i) 1_{i \geq \kappa} \leq \sum_{j=1}^{N_i} \xi_{i,j} \quad (4.8)$$

where $X \leq Y$ means that Y stochastically dominates X . Let

$$\mathcal{B}_i := \left\{ \sum_{j=1}^{N_i} \xi_{i,j} \geq m_i^{c+\epsilon/2} \times c_1 \log i \right\} \supseteq \left\{ \sum_{j=1}^{N_i} \xi_{i,j} \geq m_i^{c+\epsilon} - m_i \right\}$$

for large i (the inclusion follows from part (a)). Then, using (4.7), for $i \geq \kappa$

$$\begin{aligned} \mathbb{P}(\mathcal{B}_i \mid \mathcal{F}_{f_{m_{i+1},1}}) &\leq \mathbb{P}(\mathcal{B}_i \mid N_i \leq c_1 \log i \mid \mathcal{F}_{f_{m_{i+1},1}}) + \mathbb{P}(N_i > c_1 \log i \mid \mathcal{F}_{f_{m_{i+1},1}}) \\ &\leq \mathbb{P}\left(\xi_{i,j} \geq m_i^{c+\epsilon/2} \text{ for some } j = 1, 2, \dots, \lfloor c_1 \log i \rfloor\right) + \frac{1}{i^2} \\ &\leq c_1 \log i \times \max_{j \leq c_1 \log i} \mathbb{P}\left(\xi_{i,j} \geq m_i^{c+\epsilon/2}\right) + \frac{1}{i^2} \end{aligned} \quad (4.9)$$

Since

$$\mathbb{P}\left(\xi_{i,j} > m_i^{c+\epsilon/2}\right) \leq \left(1 - \frac{1}{m_i^c e^{\theta(j-1)}}\right)^{m_i^{c+\epsilon/2}} \leq \exp\left(-e^{-\theta(j-1)} m_i^{\epsilon/2}\right) \leq \exp\left(-\frac{m_i^{\epsilon/2}}{i^{c_1 \theta}}\right)$$

as $j \leq c_1 \log i$, and by part (a) of the lemma m_i grows faster than i^r for any $r > 0$, the RHS of (4.9) is summable over i , and thus the events $\left\{\sum_{j=1}^{N_i} \xi_{i,j} \geq m_i^{c+\epsilon} - m_i\right\}$ happen finitely often by the Borel-Cantelli lemma. As a result, from (4.8)

$$m_{i+1} \leq m_i^{c+\epsilon}$$

for all large i a.s. □

Proof of Theorem 4.5. Fix some $\epsilon > 0$. Lemma 4.6 implies that $\log m_{i+1} \leq (1 + \epsilon) \log m_i$ for all large i , a.s. On the other hand, by Lemma 4.2, $f_{m_{i+1},1} \leq (1 + \epsilon) \log m_{i+1}$. Hence, for all $n \in \{m_i, \dots, m_{i+1}\}$ we have

$$f_{n,1} \leq f_{m_{i+1},1} \leq (1 + \epsilon)^2 \log m_i \leq (1 + \epsilon)^2 \log n.$$

Together with Lemma 4.3, this implies $f_{n,1} = \mathcal{O}(\log n)$. □

The next statement strengthens part (a) of Lemma 4.6 (*without* using Conjecture 4.4).

Theorem 4.7. *There exists an $\alpha > 0$ such that $m_i \geq \exp(\exp(\alpha i))$ for all large i a.s.*

Proof. Fix some $\epsilon > 0$, and denote $\mathbb{P}_*(\cdot) = \mathbb{P}(\cdot \mid \mathcal{F}_{f_{m_i,1}})$, the conditional probability measure based on the information available until the first time m_i was burnt; thus $\mathcal{F}_{f_{m_i,1}}$ is a *stopped sigma-algebra*.

First, we want to estimate the time difference between the second and the first fires which burnt the next maximum site m_{i+1} . Define

$$A_i = \{f_{m_{i+1},2} - f_{m_{i+1},1} \geq 2\epsilon \log m_i\}$$

and recall the definitions of F_n from Lemma 4.3, B_i from (4.4) and C_i from (4.7); the event C_i is independent of $\mathcal{F}_t$, while the event F_{m_i} is $\mathcal{F}_t$ -measurable.

On F_{m_i} , we have $f_{m_{i+1},1} > f_{m_i,1} \geq (1 - \epsilon) \log m_i$. Let us now estimate $\mathbb{P}_*(A_i^c | F_{m_i})$. The fire which reached m_{i+1} for the first time is also the one which reached sites $x \in (m_i, m_{i+1}]$ for the first time. Recall that N_i denotes the number of jumps of this fire on these positions; from the proof of part (b) of Lemma 4.6 it follows that with probability $\geq 1 - 1/i^2$ the event $C_i = \{N_i \leq \lfloor c \log i \rfloor\} \subseteq \{e^{\theta N_i} \leq i^{c\theta}\}$ occurs (we are putting an upper bound on the number of jumps in a fire and this fact did not require Conjecture 4.4).

For $f_{m_{i+1},2} - f_{m_{i+1},1}$ to be smaller than $2\epsilon \log m_i$, at least one tree must appear during the time interval $[f_{m_{i+1},1} - \theta N_i, f_{m_{i+1},1} + 2\epsilon \log m_i]$ at each location $x \in [m_i, m_{i+1})$ (the term $-\theta N_i$) corresponds to the fact that there are up to N_i jumps among these positions), and the probability of this event is at most $\left(1 - \frac{1}{m_i^{2\epsilon} e^{N_i \theta}}\right)$ for each of these sites, independently of the others. Hence

$$\begin{aligned} \mathbb{P}_*(A_i^c | F_{m_i}) &\leq \mathbb{P}_*(\text{tree appears at all } x \text{ for } m_i < x \leq m_{i+1} | F_{m_i}) \\ &\leq \mathbb{P}_*(\text{tree appears at all } x \text{ for } m_i < x \leq m_{i+1} | B_i, F_{m_i}) + \mathbb{P}_*(B_i^c | F_{m_i}) \\ &\leq \mathbb{P}_*(\text{tree appears at all } x \text{ for } m_i < x \leq m_{i+1} | B_i, C_i, F_{m_i}) + \mathbb{P}(C_i^c | F_{m_i}) + \mathbb{P}_*(B_i^c | F_{m_i}) \\ &\leq \left(1 - \frac{1}{m_i^{2\epsilon} e^{\theta c \log i}}\right)^{m_i^{1-\epsilon}} + \frac{1}{i^2} + (1 - c') = \left(1 - \frac{1}{m_i^{2\epsilon} i^{c\theta}}\right)^{m_i^{1-\epsilon}} + \frac{1}{i^2} + 1 - c' \\ &\leq \exp\left\{-\frac{m_i^{1-3\epsilon}}{i^{c\theta}}\right\} + 1 - 2c'' \quad \text{for large } i \end{aligned}$$

for some $c'' > 0$, since on B_i we have $m_{i+1} - m_i \geq m_i^{1-\epsilon}$, and using (4.5).

The longer the time interval between the first and second fire at m_{i+1} (as implied by the event A_i), the greater the probability that it will burn at least $m_i^{1+\epsilon/2}$ additional sites. Hence, if both F_{m_i} and A_i occur, the probability that *the second fire to arrive at m_{i+1}* will also stop before immediately burning additional $m_i^{1+\epsilon/2}$ sites equals

$$\begin{aligned} &\mathbb{P}_*(f_{m_{i+1}+k,1} > f_{m_{i+1},2} \text{ for some } k \in [1, m_i^{1+\epsilon/2}] | F_{m_i}, A_i) \\ &\leq \mathbb{P}_*(f_{m_{i+1}+1,1} > f_{m_{i+1},2} | F_{m_i}, A_i) + \sum_{k=2}^{m_i^{1+\epsilon/2}} \mathbb{P}_*(f_{m_{i+1}+k,1} > f_{m_{i+1},2} | F_{m_i}, A_i) \\ &\leq \frac{e^{-\theta}}{m_i^{2\epsilon}} + \frac{m_i^{1+\epsilon/2}}{m_i^{1+\epsilon}} \leq \frac{2}{m_i^{\epsilon/2}} \end{aligned} \tag{4.10}$$

(when m_i is large). The first term in the above inequality corresponds to the probability that a tree appeared at position $m_{i+1} + 1$ during the time interval

$$[f_{m_{i+1},1} + \theta, f_{m_{i+1},2}] \supseteq [f_{m_{i+1},1} + \theta, f_{m_{i+1},1} + 2\epsilon \log m_i],$$

while the second term corresponds to the event that *at least one* of the locations $m_{i+1} + 2, m_{i+1} + 3, \dots, m_{i+1} + m_i^{1+\epsilon/2}$ remains vacant, and we know that the probability of such a site to remain vacant is $\exp\{-(1+\epsilon)\log m_i\} = 1/m_i^{1+\epsilon}$. Since F_{m_i} is $\mathbb{P}_*$ -measurable, using (4.10) we have

$$\begin{aligned} \mathbb{P}_*(m_{i+2} - m_i \geq m_i^{1+\epsilon/2}) &\geq \mathbb{P}_*(m_{i+2} - m_i \geq m_i^{1+\epsilon/2}, A_i, F_{m_i}) \\ &= \mathbb{P}_*(m_{i+2} - m_i \geq m_i^{1+\epsilon/2} \mid A_i, F_{m_i}) \mathbb{P}_*(A_i \mid F_{m_i}) \mathbb{P}_*(F_{m_i}) \\ &\geq \left(1 - \frac{2}{m_i^{\epsilon/2}}\right) \cdot \left(2c'' - \exp\left\{-\frac{m_i^{1-3\epsilon}}{2i^{c\theta}}\right\}\right) 1_{F_{m_i}} \geq c'' \cdot 1_{F_{m_i}} \cdot 1_{m_i > i^r} \end{aligned}$$

for some fixed non-random $r > 1$ and all large i . Now the inequality $m_{i+2} - m_i \geq m_i^{1+\epsilon/2}$ implies

$$\log m_{i+2} \geq \left(1 + \frac{\epsilon}{2}\right) \log m_i \implies \log \log m_{i+2} - \log \log m_i \geq \log \left(1 + \frac{\epsilon}{2}\right) =: \epsilon' > 0. \quad (4.11)$$

Setting $i = 2j$, $j = 1, 2, \dots$, we conclude that we can construct an i.i.d. sequence of Bernoulli(c'') random variables ξ_j such that

$$\log \log m_{2(j+1)} - \log \log m_{2j} \geq \epsilon' \xi_j \cdot 1_{F_{m_{2j}}} \cdot 1_{m_{2j} > (2j)^r}.$$

By Lemma 4.6 $m_i > i^r$ for any fixed r and all sufficiently large i ; we also know that the events $F_{m_i}^c$ occur only finitely often a.s. yielding

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{\log \log m_{2k}}{k} &= \liminf_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-1} (\log \log m_{2j+2} - \log \log m_{2j})}{k} \\ &\geq \liminf_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-1} \epsilon' \xi_j \cdot 1_{F_{m_{2j}}} \cdot 1_{m_{2j} > (2j)^r}}{k} = \epsilon' \lim_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-1} \xi_j}{k} = \epsilon' c'' > 0 \quad \text{a.s.} \end{aligned}$$

(please compare this with the proof of part (a) of Lemma 4.6) from which it follows that for some $\alpha > 0$ we have $m_i \geq \exp\{e^{\alpha i}\}$ for all sufficiently large i a.s. $\square$

Remark 4.8. *Inequality (4.11) shows that for some $\epsilon'' > 0$ we have $m_{i+1} \geq m_i^{1+\epsilon''}$ for all large i a.s., thus complementing part (b) of Lemma 4.6 (conditional on Conjecture 4.4).*

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