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THE ENDOGENOUS ARCHITECTURE
FOR TWO-SIDED MATCHING MARKETS

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We propose an endogenous architecture where a market designer can impose arbitrary restrictions on the price space, the capacity structure, the entitlement structure, and the rationing system (the architectural components). This not only generates a specific matching model, but it also endogenously generates an equilibrium concept, a domain of preference profiles, and a matching mechanism (the architectural structure). For almost any imposed restrictions, the endogenously generated mechanism is non-manipulable on the endogenously generated domain of preference profiles. The proposed endogenous architecture resolve the market design puzzle by demonstrating that seemingly disparate matching models, including many of the classic applications, are part of the same unifying architecture.

KEYWORDS: endogenous architecture, architectural components, architectural structure, general matching framework, weak equilibrium, non-manipulability.

1. INTRODUCTION

In recent decades, market designers have taken increasingly active roles in (re)designing a wide range of matching markets. Prominent examples include auctions for electromagnetic spectrum (Ausubel and Milgrom, 2002, Milgrom, 2004), cadet branching (Sönmez and Switzer, 2013), college admissions (Gale and Shapley, 1962, Roth, 1985a), internet advertising (Edelman et al., 2007), refugee resettlement (Ahani et al., 2021, Delacrétaz et al., 2023), residency matching (Dubins and Freedman, 1981, Roth, 1982a), and school admissions (Abdulkadiroğlu and Sönmez, 2003).[1] The standard approach is to first identify relevant real-life problems and then to formulate realistic and analytically tractable models, equilibrium concepts, and mechanisms that guide mechanism designers in implementing theoretically grounded

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[1] These examples illustrate two-sided matching markets, that is, markets with two disjoint sets of agents, such as buyers and sellers or students and schools, where the objective is to match agents from opposite sides of the market, possibly using price mechanisms. These types of markets are also the focus of the present paper.

matching policies (see Sönmez, 2026, for a detailed description of this approach). Because these components are tailored towards specific applications, they are naturally based on strong assumptions intended to capture the relevant aspects of the investigated markets.

Our approach is fundamentally different. Instead of focusing on specific applications and tailor matching models to them, we introduce an endogenous architecture consisting of two building blocks: the architectural components and the architectural structure. The former includes a price space, a capacity structure, an entitlement structure, and a rationing system, while the latter encompasses an equilibrium concept (called weak equilibrium), a domain of preference profiles, and a matching mechanism. Within this framework, a market designer can impose arbitrary restrictions on the architectural components. This not only generates a specific matching model, but also the architectural structure. See Figure 1 for a graphical illustration. The novelty with this approach is thus that the market designer needs only to ensure that the restrictions on the architectural components are suitable for the intended real-life application since the architectural structure is endogenously created based on these choices. Our main result demonstrates that, for any restrictions imposed on the architectural components, the endogenously generated mechanism is non-manipulable on the endogenously generated domain of preference profiles, and the resulting outcome is a weak equilibrium.

We next argue that when analyzing general matching frameworks, like the one considered in the present paper, the architectural structure must necessarily depend on the restrictions imposed on the architectural components. In particular, equilibrium concepts often need to have different properties depending on whether, for example, monetary transactions are allowed, or if the matching problem can be solved using prices alone, or if rationing is required. Therefore, it is natural that the equilibrium concept depends on the architectural components, such as whether monetary transfers are allowed, not allowed, or restricted. Furthermore, the domain of preference profiles must, in most matching models, be restricted to avoid impossibilities or conflicts between desiderata that the market designer might have. This is well-known, for instance, in exchange markets where monetary transfers are not allowed (Ehlers, 2002), in standard school choice models (Alcalde and Barberá, 1994, Erdil and Ergin, 2008), and in two-sided matching markets with price restrictions (Andersson and Svensson, 2014). Therefore, the domain of preference profiles often depends on the architectural components, say, if agents are entitled to objects or not. These observations motivate our approach to endogenously generate the architectural structure based on the restrictions that the market designer imposes on the architectural components.

We also argue that our particular endogenous architecture is well-motivated. Specifically, for certain restrictions on the architectural components, seminal matching models, such as those introduced by Gale and Shapley (1962) and Vickrey (1961), can be recovered together with their corresponding architectural structure. This includes their main equilibrium concepts, such as stability or minimum price equilibrium, as well as the preference domains and mechanisms commonly considered in these models, such as the strict preference domain with the (man/woman/student-proposing) deferred acceptance algorithm, and the classic preference domain with the second-price auction mechanism. Our endogenously generated equilibrium concept also incorporates features that normally are part of specific matching models, including individual rationality, non-wastefulness, and various "fairness concepts." Thus, even if the proposed architectural structure initially may appear arbitrary, it is well-motivated, as it, under appropriate restrictions on the architectural components, reduces to axioms, concepts, and definitions familiar to researchers in the field. In other words, the mechanism that endogenously generates the architectural structure does the "right thing" and provides established architectures from the existing matching literature.

The main advantage of adopting our endogenous architecture is the flexibility it provides, enabling the analysis to extend beyond much of the existing literature. For instance, restrictions on the price space allow market designers to study markets with price ceilings, rent control, minimum wages, and resource constraints. Restrictions on the entitlement structure enable the analysis of markets with no initial ownership, personalized endowments, and price-specific endowments. Almost regardless of which restrictions, or which combination of them, that are imposed on the architectural components, the endogenously generated matching mechanism remains non-manipulable on the endogenously generated domain of preference profiles. Importantly, the non-manipulability result generally only holds in one-to-one settings (but in some special cases, it also holds in the many-to-one settings). All other results in the paper hold in many-to-one settings. See the further discussions in Section 3.

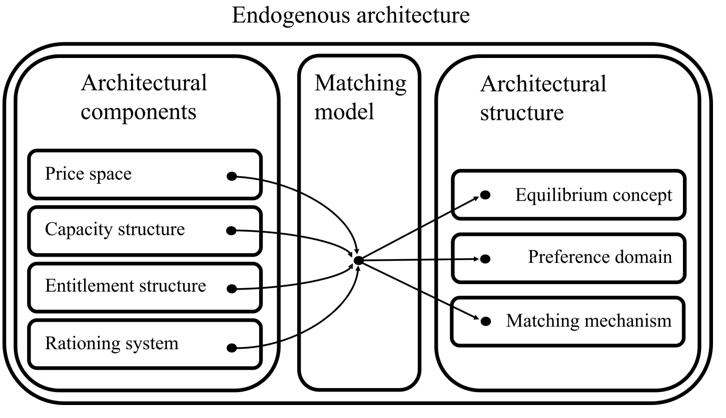

FIGURE 1.—The endogenous architecture consists of architectural components and an architectural structure. A market designer defines the matching model of interest by selecting specific architectural components. These selections will endogenously generate the architectural structure.

The considered endogenous architecture also allows us to solve a puzzle in the market design literature. As is well-known among researchers in the field, even if existing matching models diverge in terms of their specific architectural components and architectural structure, their mathematical structures, such as lattice and strategy properties, are often remarkably similar. In a recent article, Roth (2026) observes that results in the matching literature are often presented in a common format: *In [model X] defined on [domain D], when [mechanism Y] is employed, it is a dominant strategy for players [in the set Z] to submit their true preferences, and the resulting outcome is an [equilibrium].*[2] Key examples of such results include:

(a) In [the marriage market/school choice model] defined on [the strict preference domain], when the [man/woman/student-proposing deferred acceptance algorithm] is employed, it is a dominant strategy for [the proposing side] to submit their true preferences, and the resulting outcome is [stable] (Abdulkadiroğlu and Sönmez, 2003, Bergstrom and Manning, 1982, Gale and Shapley, 1962, Dubins and Freedman, 1981, Roth, 1982a, 1984, 1985a, Svensson, 1994).

(b) In [a private-value single/multi-item sealed-bid auction] defined on [the classic preference domain], when [the second-price auction] is employed, it is a dominant strategy for

[2]The [domain D] is not included in Roth (2026). We added it here since all parts of the architectural structure, including the preference domain, play a key role in the proposed endogenous architecture.

[the bidders] to submit their true preferences, and the resulting outcome is [a price equilibrium] (Vickrey, 1961, Clarke, 1971, Groves, 1973, Demange and Gale, 1985).

These results are not exclusive to matching and auction models but are also present in, for example, fairness models (Leonard, 1983, Sun and Yang, 2003, Svensson, 2009) and rationing models (Andersson and Svensson, 2014).[3]

As we demonstrate, the underlying reason for the observation in Roth (2026) and, consequently, the solution to the puzzle, is that the specific architectural components and architectural structure in the matching models of type (a) and (b) are special cases of our more general and endogenous counterparts, respectively. In other words, these specific matching models are part of a unifying matching framework. This may come as a surprise as there are large differences, for instance, in architectural structure between these models, that is, in the [domain D], the [mechanism Y] and the [equilibrium]. For example, in the models of type (a) the mechanisms rely on rationing systems and the preference domain only contains strict preference profiles, whereas in models of type (b) the mechanisms are based on price equilibrium and the preference domain contains all weak preference profiles. The solution to the puzzle is simply that matching models that appear to be quite different, in fact, are part of the same endogenous architecture. Given this insight, it is not at all surprising that the models of type (a) and (b) share, for example, strategy and lattice structures, and that non-manipulability results in the matching literature often have been stated in the same format.

1.1. *Related Literature*

There has been some previous attempts in the matching literature to find unifying matching frameworks. In settings without monetary transfers, the seminal you-request-my-house-I-get-your-turn mechanism, introduced by Abdulkadiroğlu and Sönmez (1999), integrated the serial dictatorship mechanism (Hylland and Zeckhauser, 1979, Svensson, 1994) and the top-trading cycles mechanism (Shapley and Scarf, 1974). Another example is Pápai (2000), where a large class of mechanisms that solve the house allocation problem is proposed. These rules are called hierarchical exchange rules, and they can be regarded as a generalization of the top-trading cycles mechanism, even if no initial ownership is assumed. A more recent contribution is Pycia and Ünver (2017) where a class of strategy-proof and Pareto efficient mechanisms is defined through entitlement structures. This class contains many well-known mechanisms, including the top-trading cycles mechanism, the serial dictatorship mechanism, and hierarchical exchange rules. An important difference between the aforementioned mechanisms and the endogenous mechanism considered in the present paper is that the former mechanisms only work in environments with strict preferences and in the absence of monetary transfers. The architectural components considered in the present paper allow for any restrictions on a price space, so monetary transfers may be allowed, not allowed, or restricted, and the endogenously generated domain of preferences includes the strict domain as a special case.

Somewhat closer to the endogenous architecture considered in the present paper is the matching with contracts framework, introduced by Hatfield and Milgrom (2005). They consider a matching model with bilateral contracts that encompasses and extends two-sided matching

[3]The matching and fairness models, discussed in relation to Roth (2026), share two important assumptions: all agents [in the set Z] have unit-demand, explicitly or implicitly, and agents [not in the set Z] are non-strategic, i.e., the two-sided market has one-sided preferences. This assumption is maintained throughout the present paper. It is well-known that no mechanism can generally prevent manipulation by agents on both sides of the market, as proved by Roth (1982a) and Demange and Gale (1985) for models of types (a) and (b), respectively.

models with and without monetary transfers.[4] Our endogenous architecture has some special characteristics compared to the matching with contracts framework (and papers such as Echenique, 2012, Hatfield and Kojima, 2010, Hatfield and Milgrom, 2005, Herings, 2018). For example, they analyze a model with a finite number of contracts where agents are assumed to have strict preferences and prices cannot take arbitrary values (Herings, 2018, does not assume strict preferences but indifference relations play no significant role in his adjustment process as agents are assumed to choose an arbitrary contract from their choice sets in the case when the cardinality of the choice sets is greater than one). By imposing specific restrictions on our architectural components, these features turn out to be special cases of our matching framework. In particular, the endogenous architecture considered in the present paper is not restricted to a finite set of contracts, and prices can take any value in a finite union of closed intervals (this set includes isolated points as well as the extended real line as special cases). This aside, one of the main novelties of the present paper is the endogenous link between the architectural components and the architectural structure. Importantly, these elements are exogenously given in standard matching models, whereas we consider an endogenous architecture.

Andersson and Svensson (2014) introduced a less flexible version of the domain of preference profiles than the one considered in the present paper. Their main focus is to identify a non-manipulable matching mechanism for markets with price restrictions on their considered domain. Also this model turns out to be a special case of our endogenous architecture, for example, because they lack an entitlement structure and consider a (much) less general price space. Because they do not partition the two main building blocks in the present paper, that is, the architectural components and the architectural structure, it is difficult to generalize their model further and to understand the endogenous link between these blocks. In particular, the proposed endogenous architecture clarifies and explains how well-known matching models, axioms, concepts, and definitions in the existing literature relate to each other in a unified matching framework. This feature is not only missing in Andersson and Svensson (2014), but also in all the literature cited in this section.

1.2. *Outline*

Section 2 defines the endogenous architecture. The main theoretical findings are stated in Section 3. Section 4 relates the endogenous architecture to some seminal matching models and results in the literature. It also discusses a few new and potentially important applications. Finally, Section 5 concludes the paper. The more extensive and technical proofs as well as some additional results and examples are delegated to Appendices A–C.

2. THE ENDOGENOUS ARCHITECTURE

The agents are gathered in the finite set $N = \{1, \dots, |N|\}$ and one can think about them as buyers or students. Each agent would like to acquire at most one indivisible object, say a license or a school seat, from the finite set of objects $S = \{1, \dots, |S|\}$. Without loss of generality and for notational simplicity, it is assumed that the set S contains $|N|$ objects with no value to any agent, called null-objects. Practically, this means that each agent always can be assigned some object from S. The null-objects also represent an outside option that always is available for the agents (see Section 2.1.3 for some further remarks). Each object $s \in S$ is associated with a real-number, or a numeraire good, r representing, for example, a price or a resource. For simplicity,

[4]Their considered mechanism later turned out to be essentially identical to the one proposed by Kelso and Crawford (1982) for two-sided labor markets. See Echenique (2012) for detailed arguments and discussions.

we will often refer to this good as a "price" (even if the numeraire good can be interpreted more broadly). It will also interchangeably be denoted by $\rho(s)$ when attached to some object s. For the null-objects, this real number is fixed and set to zero.

Each agent $i \in N$ has preferences over consumption bundles, that is, pairs of objects and real numbers $(s,r) \in S \times \mathbb{R} \equiv \mathcal{X}$, represented by a complete and transitive preference relation R_i with corresponding strict and indifference relations P_i and I_i, respectively. Preferences are monotonically decreasing in r, that is, for any $r, r' \in \mathbb{R}$ with $r' < r$ and each $s \in S$, $(s,r')P_i(s,r)$.[5] Objects are assumed to be boundedly desirable, meaning that for any pair $(s,r) \in \mathcal{X}$ and object $s' \in S$, there is a number $r' \in \mathbb{R}$ such that $(s',r')P_i(s,r)$. A preference relation satisfying these properties is said to be classic.

A preference profile is a list $R = (R_1, \ldots, R_{|N|})$. It belongs to the classic domain $\mathcal{R}$ if all preference relations in R are classic. An often studied subset of this domain is the strict preference domain $\mathcal{P}$, where all agents have strict preferences over all bundles.

An assignment is a mapping $\sigma : N \to S$. For any object $s \in S$, let $\sigma[s]$ denote the set of agents who are assigned object s. Given a subset $S' \subseteq S$, let $\sigma[S']$ be the set of agents that are assigned an object in S' at assignment σ. A matching $x = (\sigma, \rho)$ is a pair containing an assignment σ and a vector or real numbers ρ such that each agent i's consumption bundle is $(\sigma(i), \rho_{\sigma(i)})$, which we often will abbreviate as $(\sigma(i), \rho(i))$ or $(\sigma(i), \rho)$ for notational simplicity. Matching $x = (\sigma, \rho)$ is feasible if the capacities are respected at the given prices, that is, for any object $s \in S$ it holds that $|\sigma[s]|$ is smaller than or equal to the number of available objects at ρ_s.[6]

2.1. *The Architectural Components*

The architectural components consists of a price space, a capacity structure, an entitlement structure, and a rationing system, denoted by $\mathbb{P}$, c, $\mathbb{E}$, and $\succsim$, respectively. Together with the set of agents, the set of objects, and a preference profile, these components define the general matching problem $(N, S, R, \mathbb{P}, c, \mathbb{E}, \succsim)$. As will be explained in Section 4, for specific restrictions on the architectural components, the matching problem reduces to seminal matching problems in the literature.

2.1.1. *The General Price Space:* $\mathbb{P}$

In many real-life applications, the market designer needs only to impose mild restrictions on the real-number that is associated with an object. In an auction, for example, prices are normally only bounded from the below by the sellers' reservation prices. In other situations, the restrictions may be more complicated because of legislated price restrictions, resource scarcity, and similar types of constraints. To capture a large variety of settings, each object $s \in S$ is associated with a subset $\mathbb{P}_s \subseteq \mathbb{R}$ meaning that bundle $(s,r) \in \mathcal{X}$ is admissible only if $r \in \mathbb{P}_s$. Let:

$$\mathbb{X} = \{(s,r) \in \mathcal{X} : r \in \mathbb{P}_s\}.$$

To obtain maximal generality, we assume that each $\mathbb{P}_s$ is represented by a finite union of closed intervals. We allow for isolated points as trivial intervals, and closed intervals of the form $(-\infty, x]$ or $[x, +\infty)$. This construction permits, for example, prices to be integer numbers, resources to be consumed in specific price intervals, or objects to be subject to price controls.

[5]This is a modeling choice. With appropriate modifications, all results presented in the present paper also hold for the case when preferences are monotonically increasing in r.

[6]The capacity structure is formally defined and discussed in Section 2.1.2. See also Section 2.1.3 for some further remarks in the case when agents are entitled to objects.

Throughout it is assumed that $\mathbb{P}_s$ is bounded from below.[7] In case the objects are, say, licences or items to be sold, these lower bounds are naturally represented by the reservation prices of the sellers. In case the objects are public school seats, the lower bound is typically zero as students do not have to pay for their seats. Let $\mathbb{P} = \Pi_{s \in S}\mathbb{P}_s$, $\underline{r}_s = \min \mathbb{P}_s$, and $\overline{r}_s = \sup \mathbb{P}_s$.

2.1.2. *The General Capacity Structure: c*

In many market design applications, capacities are fixed and independent of their associated real values. This is not only true in the standard school choice problem where each school is assumed to have a fixed capacity, but also in environments where the real numbers represent prices and each object can be sold to (at most) one of the agents. We consider a more general capacity structure where the number of agents that can consume an object may depend on the real value attached to it, for example, if it represents some type of monetary resource. In a school choice environment, the real number may be a school voucher and in this case it is natural that school capacities increase as a function of the size of the voucher (see also Section 4.4.2).

To capture this type of general capacity structure, we assume that there is a non-decreasing function $c_s : \mathbb{P}_s \to \mathbb{N}$ for each object s indicating the number of agents that can consume it as a function of the real-number associated to it.[8] For higher values, the capacity of an object weakly increases, i.e., $c_s(r) \leq c_s(r')$ for all $r, r' \in \mathbb{P}_s$ such that $r < r'$. Furthermore, for the smallest feasible price $\underline{r}_s \in \mathbb{P}_s$, the capacities are greater than or equal to one, i.e., $c_s(\underline{r}_s) \geq 1$.[9] This function is assumed to be right-continuous,[10] possibly constant, or always equal to one. These functions are gathered in the vector $c = (c_1, \ldots, c_{|S|})$.

All real values $r \in \mathbb{P}_s$ at which the capacity of object s experiences a "discrete increase" are gathered in the vector ∂c_s. Let $\partial c = (\partial c_1, \ldots, \partial c_{|S|})$.

2.1.3. *The General Entitlement Structure: $\mathbb{E}$*

In many real-life applications, agents are entitled to specific objects under all circumstances, for example, when agents "own" the objects, say the houses they currently live in and always have the right to stay in. Even if we allow for this type of entitlement structure, we consider a more general setting where entitlements may depend on the real values that are associated with the objects. Consider, for instance, a situation where a governmental agency decides to sell their existing housing stock. Then the government may decide that no agent is entitled to any of the houses if the prices are "low," but that some low-income agent may be entitled to a specific house if the price becomes "sufficiently high." The general entitlement structure specifies if some agent is entitled to bundles of type $(s, r) \in \mathbb{X}$ and if so, which agent that is entitled to it.

Because we think about entitlements as being specific to certain agents, we assume unit capacity of each object that some agent is entitled to at some price.[11] Formally, for each agent

[7]Since the objects in S are boundedly desirable, all results hold under the weaker assumption that, for some $s \in S$, $\mathbb{P}_s$ is bounded from below.

[8]Here, $\mathbb{N}$ denotes the set of natural numbers, including zero.

[9]Otherwise, real numbers with capacity zero can be deleted from the matching problem.

[10]This means if $c_s(r) = k$, then for $\epsilon > 0$ "sufficiently small" such that $r + \epsilon \in \mathbb{P}_s$, we have $c_s(r) = c_s(r + \epsilon)$. If this were not the case, i.e., if $c_s(r) < c_s(r + \epsilon) = k$ for all $\epsilon > 0$, and there are k agents who demand (s, r), then no equilibrium (later defined in Definition 1) exists as no price can be set arbitrarily close to r. This is similar to Bertrand competition with asymmetric marginal costs (see Blume, 2003).

[11]All results are not only valid, but also proved under a weaker assumption where capacities are allowed to exceed one for prices at which no agent is entitled to an object. This more general assumption may, for example, reflect a

$i \in N$, the entitlements are gathered in the set e_i, i.e., there exist $S_i \subseteq S$ and $r \in \mathbb{P}_s$ for each $s \in S_i$ such that $e_i = \{(s,r) : s \in S_i\}$. If agent $i \in N$ has no entitlement, then $e_i = \emptyset$ and the participation constraint is defined by the null-item.[12] Let, for each object $s \in S$:

$$\mathbb{E}_s = \{r \in \mathbb{P}_s : \text{ there exists an agent } i \in N \text{ such that } (s,r) \in e_i\}.$$

Given $r \in \mathbb{E}_s$, let $\mathbb{E}_s[r]$ denote the singleton set containing the agent that is entitled to object s at price r, i.e., $\mathbb{E}_s[r] = \{i \in N : (s,r) \in e_i\}$. Note also that, by monotonicity of preferences, if an agent would be entitled to several bundles containing object s, individual rationality is defined based on the bundle with the lowest price (or the highest compensation, depending on the interpretation of the price space). For simplicity, it will therefore also be assumed that whenever $\mathbb{E}_s \neq \emptyset$, then $|\mathbb{E}_s| = 1$. Thus, $|\mathbb{E}_s[r]| = |\mathbb{E}_s| = c_s(r) = 1$ by our assumptions. Let $\mathbb{E} = (\mathbb{E}_1, \ldots, \mathbb{E}_{|S|})$.

2.1.4. *The General Rationing System:* $\succsim$

By restricting prices to belong to $\mathbb{P}_s$, it may not be possible to rely on market mechanisms, such as competitive mechanisms or equilibrium pricing rules, when solving the matching problem. Consequently, some type of rationing may be needed when the number of agents who demand a specific object exceeds its capacity. A rationing system guarantees that agents are assigned the objects that are in excess demand either because they are entitled to them or because they (for some reason) "have the best right" to them. Therefore, rationing is only used when prices alone cannot solve the matching problem.

In standard market design applications, say, markets for house allocation and school choice, rationing is normally represented by fixed (object-specific) priority structures, meaning that there exists some agent who always has the "highest priority" for any given object, say a student that has the highest priority at a specific school because of sibling priority and/or walking distance.[13] Our general rationing system allows for more elaborate cases and different agents may have the highest priority for the same object depending on the real value that is attached to it. For example, on markets where prices are restricted to be integer values, different agents may have the highest priority at different prices.

To formally define the general rationing system, we need to introduce some notation. For a given $r \in \mathbb{P}_s$, let $r^+ = \inf\{r' \in \mathbb{P}_s : r' > r\}$, where the usual convention gives that, for each $s \in S$, $\overline{r}_s^+ = \infty$. Since each $\mathbb{P}_s$ is the finite union of closed intervals, $r^+ = r$ whenever r is not the greatest element in one of these intervals, and r^+ is the least element of the next higher interval when it is. Let $\mathbb{T}_s = \overline{\partial}\mathbb{P}_s \cup \mathbb{E}_s$ where $\overline{\partial}\mathbb{P}_s$ denotes the upper thresholds of the set $\mathbb{P}_s$, i.e., $r \in \mathbb{P}_s$ such that $r < r^+$. Because rationing will only be used when prices alone cannot solve the matching problem, rationing occurs only for the thresholds $\mathbb{T}_s$. Let $\mathbb{T} = \Pi_{s \in S}\mathbb{T}_s$.

For each object $s \in S$ and any given $r \in \mathbb{T}_s$, an exogenously given weak priority order $\succsim_{(s,r)}$ over the agents in N, which defines the general rationing system:

$$\succsim = ((\succsim_{(s,r)})_{r \in \mathbb{T}_s})_{s \in S}.$$

situation where an agent that is entitled to an object must keep the object "intact," say a piece of land, a property, or a building, even if the object can be divided into multiple smaller units. See Appendices A.1 and C for further definitions, discussions, and results.

[12] Of course, it may be the case that an agent with an entitlement prefers a null-object over her entitlement. To avoid introducing additional notation and without loss of generality, it will be assumed that an agent with an entitlement always (weakly) prefers it to the null-object.

[13] For a general treatment of object-specific priorities in a matching with contracts framework, see Kominers and Sönmez (2016).

For any distinct agents $i, j \in N$, $i \succ_{(s,r)} j$ and $i \sim_{(s,r)} j$ means that agent i has a higher and identical priority, respectively, than agent j to bundle (s,r). The priority order is applied on both thresholds $\overline{\partial}\mathbb{P}_s$ and $\mathbb{E}_s$. To make the general rationing system $\succsim$ manageable, we will impose the following additional assumptions which all relate to specific situations at the thresholds $\mathbb{T}_s$:

(i) If $r \in \overline{\partial}\mathbb{P}_s \backslash \mathbb{E}_s$, the priority order $\succsim_{(s,r)}$ is strict among all agents, i.e., for all distinct agents i and j, either $i \succ_{(s,r)} j$ or $j \succ_{(s,r)} i$.

(ii) If $r \in \mathbb{E}_s \backslash \overline{\partial}\mathbb{P}_s$ and agent i is entitled to the bundle (s,r), then agent i has the highest priority to that bundle and all other agents have equal priority, i.e., for all agents j, k distinct from i, $i \succ_{(s,r)} j \sim_{(s,r)} k$.

(iii) If $r \in \mathbb{E}_s \cap \overline{\partial}\mathbb{P}_s$ and agent i is entitled to the bundle (s,r), then agent i has the higher priority to that bundle than all agents who are not entitled to (s,r), i.e., $\succsim_{(s,r)}$ is strict and for all $i \in \mathbb{E}_s[r]$ and all $j, k \in N \backslash \mathbb{E}_s[r]$, either $i \succ_{(s,r)} j \succ_{(s,r)} k$ or $i \succ_{(s,r)} k \succ_{(s,r)} j$.

2.2. *The Architectural Structure*

A market designer may impose arbitrary restrictions on the architectural components. This will not only generate a specific matching application (or model), the restrictions will also endogenously determine the architectural structure (see Figure 1). The latter consists of an equilibrium concept (called weak equilibrium), a domain of preference profiles, and a matching mechanism.

2.2.1. *The Equilibrium Concept*

A weak equilibrium is a feasible matching (σ, ρ) satisfying four properties. The first property is the standard participation constraint that guarantees all agents to weakly prefer their consumption bundle to their outside options. The second property is a notion of efficiency (non-wastefulness) ensuring that no object remains unassigned if an agent prefers a consumption bundle containing the unassigned object to the one that she has been assigned. The third property states that if no agent is entitled to object s or if the real number ρ_s associated to it is not on the upper threshold of $\mathbb{P}_s$, no agent can envy the agent that is assigned the bundle that contains object s. The fourth property captures the idea that envy is only justified when the agent that is assigned a bundle containing object s is entitled to it or has higher priority than all envious agents, and if the real number ρ_s attached to object s is on the upper threshold of $\mathbb{P}_s$.

Intuitively, one can think about the third and the fourth property as follows. When several agents would like to be assigned a specific bundle at a given matching, an agent is assigned the bundle either because she is entitled to it or because she has higher priority than all envious agents. In this case, the only possibility to eliminate envy is to increase ρ_s, but the upper restrictions on $\mathbb{P}_s$ may disqualify increases that are sufficiently high to eliminate all envy, and the closest one can get in eliminating all envy is to hit the upper threshold $\overline{\partial}\mathbb{P}_s$. In this sense, these two properties capture the idea to eliminate envy whenever possible, but to minimize envy whenever impossible.

DEFINITION 1: *For a given profile $R \in \mathcal{R}$, a weak equilibrium (σ, ρ) is a vector $\rho \in \mathbb{P}$ and an assignment $\sigma : N \to S$ with the following properties:*

(i) *Individual rationality: For each $i \in N$, if $(s,r) \in e_i$ then $(\sigma(i), \rho) R_i (s,r)$, and if $e_i = \emptyset$ then $(\sigma(i), \rho) R_i (s', 0)$ where s' is a null-object.*

(ii) *Non-wastefulness: for each $i \in N$ and all $s \in S$, $(s, \rho) P_i (\sigma(i), \rho)$ implies $|\sigma[s]| = c_s(\rho_s)$.*

(iii) *Interior no-envy: If* $\rho_s \notin \mathbb{T}_s$, *then for each* $i \in N$, $(\sigma(i), \rho) R_i (s, \rho)$.
(iv) *Threshold no-justified-envy: If* $\rho_s \in \mathbb{T}_s$ *and* $(s, \rho) P_i (\sigma(i), \rho)$, *then each* $j \in \sigma[s]$ *has* $j \succ_{(s,\rho_s)} i$. *Moreover, either each* $j \in \sigma[s]$ *has* $(s, \rho_s) \in e_j$ *or* $(\sigma(i), \rho) P_i (s, \rho_s^+)$.

The proposed equilibrium notion is endogenously determined by the restrictions that the market designer imposes on the architectural components since conditions (i)–(iv) are directly affected by them. For example, individual rationality depends on the considered entitlement structure whereas non-wastefulness depends on which restrictions that are imposed on the general capacity structure. Moreover, both interior no-envy and threshold no-justified-envy are contingent on the chosen thresholds $\mathbb{T}$ that, in turn, are decided by the restrictions that the market designer imposes on the general price space and the general entitlement structure.

Let $\mathcal{W}(R)$ denote the set of weak equilibria at profile R. As we prove in Appendix A, given our assumptions, this set is non-empty for any profile R. More specifically, Theorem 5 shows that a necessary and sufficient condition for the existence of a weak equilibrium is that unit entitlement at non-maximal prices is satisfied. This means that if multiple agents are entitled to object s at certain prices, then this price must be the maximal price of object s and if one agent is entitled to object s at a non-maximal price, then the capacity of the object s must be equal to one at all non-maximal prices.

2.2.2. *The Domain of Preference Profiles:* $\tilde{\mathcal{R}}$

It is well-known that it is often impossible to find a non-manipulable matching mechanism on the classic preference domain $\mathcal{R}$. See, for example, Alcalde and Barberá (1994) Andersson and Svensson (2014), Ehlers (2002), and Erdil and Ergin (2008) for discussions and examples in different contexts. Because one of our main focuses is to identify non-manipulable mechanisms, we will exclusively focus on a reduced domain of preference profiles $\tilde{\mathcal{R}} \subset \mathcal{R}$ where no two objects are "connected by indifference" at any vector $\rho \in \mathbb{P}$. This type of domain was first introduced by Andersson and Svensson (2014). Their domain has been demonstrated to contain "almost all" (in a topological sense) preference profiles of the classic domain $\mathcal{R}$ in, for example, matching markets without entitlements and price restrictions (Andersson and Svensson, 2016), matching markets with entitlements but no upper price restrictions (Andersson et al., 2016), and school choice with crowding (Phan et al., 2024). However, our considered domain is more flexible since all thresholds that arise when the market designer imposes specific restrictions on the architectural components endogenously determine the domain of preference profiles.[14]

Consider now the thresholds $\partial\mathbb{P}_s$, ∂c_s and $\mathbb{E}_s$, where $\partial\mathbb{P}_s = \underline{\partial}\mathbb{P}_s \cup \overline{\partial}\mathbb{P}_s$.[15] Let $\mathbb{V}_s = \partial\mathbb{P}_s \cup \partial c_s \cup \mathbb{E}_s$ and $\mathbb{V} = \Pi_{s \in S} \mathbb{V}_s$.

DEFINITION 2: *Given a preference profile* $R \in \mathcal{R}$, *two objects* s *and* t *are connected by indifference if there is a vector* $\rho \in \mathbb{P}$, *with* $\rho_s \in \mathbb{V}_s$ *and* $\rho_t \in \mathbb{V}_t$, *a sequence of distinct agents* $\{i, \ldots, k\}$, *relabeled as* $\{1, \ldots, n\}$ *for convenience, and a sequence of objects, also relabeled as* $\{1, \ldots, n+1\}$ *for convenience (where objects* $1, \ldots, n$ *are distinct, and objects 1 and 2 are distinct if* $n = 1$*), such that* $(s, \rho_s) = (1, \rho_1)$, $(i, \rho_i) I_i (i+1, \rho_{i+1})$ *for each* $1 \leq i \leq n$, *and* $(n+1, \rho_{n+1}) = (t, \rho_t)$. *Profile* R *satisfies no connection by indifference (NCBI, henceforth) if no two objects are connected by indifference. Let* $\tilde{\mathcal{R}}$ *the set of all profiles where no two objects are connected by indifference.*

[14] Andersson and Svensson (2014) considered a less general matching model, for example, because they considered the simplest possible capacity structure and didn't allow for entitlements, so their considered domain only depends on the thresholds $\overline{\partial}\mathbb{P}_s$.

[15] Here, $\underline{\partial}\mathbb{P}$ is defined analogously as $\overline{\partial}\mathbb{P}$. That is, for a given $r \in \mathbb{P}_s$, let $r^- = \sup\{r' \in \mathbb{P}_s : r' < r\}$ and $\underline{\partial}\mathbb{P}_s$ denotes the lower threshold of the set $\mathbb{P}_s$, i.e., $r \in \underline{\partial}\mathbb{P}_s$ if and only if $r^- < r$.

Note that it is sometimes more convenient or practical for market designers to ask agents to report their preferences on a domain of preference profiles that is "smaller" than $\tilde{\mathcal{R}}$, say, the strict preference domain $\mathcal{P}$. Because our non-manipulability result holds for any subset $\dot{\mathcal{R}} \subset \tilde{\mathcal{R}}$, this is not an argument against our endogenous approach. In other words, our approach offers market designers, interested in implementing a non-manipulable mechanism, the opportunity to operate on a "large" domain of preference profiles, but designers need obviously not take advantage of this opportunity.

2.2.3. *The Matching Mechanism*

We focus on non-manipulable matching mechanisms that make selections from the set of weak equilibria $\mathcal{W}(R)$. However, the structure of this set not only depends on the given preference profile $R \in \tilde{\mathcal{R}}$, but also on the restrictions on the architectural components. So, the set $\mathcal{W}(R)$ may be very large for some restrictions, but the mechanism needs to make a unique selection independently of the size of the set (or at least, an essentially unique selection, see Footnote 21). Consequently, we must impose additional requirements on the mechanism, called agent-optimality and maximality, to generally ensure uniqueness.

A matching mechanism φ is a rule that for each given profile R in the domain of preference profiles $\dot{\mathcal{R}} \subseteq \mathcal{R}$ selects a matching x, that is, $x = \varphi(R)$.

DEFINITION 3: *A matching mechanism φ is manipulable at a profile R on the domain of preference profiles $\dot{\mathcal{R}} \subseteq \mathcal{R}$ by an agent $i \in N$ if there exists a preference R_i' such that $R' = (R_i', R_{-i}) \in \dot{\mathcal{R}}$, and for the two matchings $\varphi(R) = (\sigma, \rho)$ and $\varphi(R') = (\sigma', \rho')$ we have $(\sigma'(i), \rho')P_i(\sigma(i), \rho)$. If the matching mechanism is not manipulable by any agent $i \in N$, at any profile $R \in \dot{\mathcal{R}}$, it is said to be non-manipulable.*

Informally, a matching is an agent-optimal weak equilibrium if it is the most preferred weak equilibrium for each agent at a given preference profile. Maximality refines this notion by, in addition, requiring that the number of agents, who are assigned to their entitled objects, is minimized, thereby ensuring that no additional gains can be achieved without violating the underlying constraints. In this sense, maximality serves both as a second-best efficiency concept and as an equilibrium selection principle. Moreover, maximality aligns with common policy objectives, such as maximizing the number of "trades" in school switch or house exchange programmes.

DEFINITION 4: *For a given profile $R \in \dot{\mathcal{R}}$, a weak equilibrium (σ, ρ) is an agent-optimal weak equilibrium if, for all agents $i \in N$, it holds that $(\sigma(i), \rho)R_i(\sigma', \rho')$ for any $(\sigma', \rho') \in \mathcal{W}(R)$.*

DEFINITION 5: *For a given profile $R \in \dot{\mathcal{R}}$, an agent-optimal weak equilibrium (σ, ρ) is maximal if there exists no agent-optimal weak equilibrium (σ', ρ') such that $(\sigma'(i), \rho')I_i(\sigma(i), \rho)$ for all $i \in N$, and:*

$$|\{i \in N : (\sigma'(i), \rho'_{\sigma'(i)}) \in e_i\}| < |\{i \in N : (\sigma(i), \rho_{\sigma(i)}) \in e_i\}|.$$

A mechanism φ defined on the domain of preference profiles $\dot{\mathcal{R}}$ is a maximal agent-optimal weak equilibrium mechanism if $\varphi(R) = (\sigma, \rho)$ is a maximal agent-optimal weak equilibrium. Of course, the mechanism is only well-defined if there exists such an equilibrium for all preference profiles $R \in \dot{\mathcal{R}}$. This is established in the following section for the case when $\dot{\mathcal{R}} = \tilde{\mathcal{R}}$, that is, for the considered endogenous domain.

3. THEORETICAL RESULTS

The proofs of the results is this section are delegated to Appendices A and B.

THEOREM 1—Existence: *For any profile $R \in \tilde{\mathcal{R}}$, there exists a maximal agent-optimal weak equilibrium.*

Our first main result shows the existence of a maximal weak equilibrium which all agents (weakly) prefer to any other weak equilibrium. The deeper insight from Theorem 1 fundamentally differs from the corresponding results in the literature on two-sided matching markets, later discussed in Section 4. Namely, even if there exists an agent-optimal weak equilibrium, the equilibrium need not have the structure of a lattice (see Example 1 in Appendix C). Among other things, this implies that a vector ρ that is part of an agent-optimal weak equilibrium need not be a minimum price vector. Formally, a price vector $\rho^{\min}$ is a minimum price vector if, for any two weak equilibria $x' = (\sigma', \rho')$ and $x'' = (\sigma'', \rho'')$, there is a weak equilibrium $(\sigma, \rho^{\min})$ where $\rho_j^{\min} = \min\{\rho_j', \rho_j''\}$ for all $j \in S$. However, as demonstrated next, by considering maximal agent-optimal weak equilibria, a minimum price vector exists. Note that this is not an issue in standard two-sided matching models since entitlements are not part of those models, exactly as this is not an issue in for the endogenous matching model considered in the present paper if $e_i = \emptyset$ for all $i \in N$.

THEOREM 2—Uniqueness: *For any profile $R \in \tilde{\mathcal{R}}$, there exists a minimum price vector $p^{\min}$ such that:*

(i) *if (σ, ρ) is a maximal agent-optimal weak equilibrium, then $\rho \geq p^{\min}$ and $(\sigma, p^{\min})$ is a maximal agent-optimal weak equilibrium,*

(ii) *for any weak equilibrium (σ, ρ), for all $s \in S$ such that $\sigma[s] \neq \mathbb{E}_s[\rho_s]$, it holds that $\rho_s \geq p_s^{\min}$.*

Consequently, there exists a unique minimum price that is compatible with maximal agent-optimal weak equilibria. As a result, the maximal agent-optimal weak equilibrium mechanism is uniquely defined in terms of agents' welfare and can choose a matching $(\sigma, p^{\min})$ for each preference profile $R \in \tilde{\mathcal{R}}$.[16] This insight is key in the proof of the main non-manipulability result.

THEOREM 3—Non-Manipulability: *Let φ be a maximal agent-optimal weak equilibrium mechanism defined on the domain of preference profiles $\tilde{\mathcal{R}}$ and suppose that $c_s(\rho_s) = 1$ for all $s \in S$ and all $\rho_s \in \mathbb{P}_s$. Then φ is non-manipulable.*

All results in the present paper except Theorem 3 are valid in many-to-one models. The reason for restricting the non-manipulability result to one-to-one models is that it does not generally hold in many-to-one models. This is illustrated by means of a counter example in Appendix C. It is important to point out that this insight does not mean that it is impossible to

[16]By requiring maximality, another standard property in the mechanism design literature may be violated. Namely that the real number attached to an object always is at the lower bound whenever the object remains unassigned. That is, if $\sigma[s] = \emptyset$, then $\rho_s = \underline{r}_s$. For example, if the seller's reservation price is set to $\underline{r}_s$ in a standard auction model and the object remains unsold, then the price of the object is $\rho_s = \underline{r}_s$. In Example 2 in Appendix C, it is demonstrated that this requirement is too strong when restricting attention to maximal agent-optimal weak equilibria. One reason for this is that the capacity of an object is allowed to be a function of the real number attached to (in, say, standard auction models, capacities are fixed).

generate non-manipulable mechanisms in many-to-one matching models using the endogenous approach in the present paper. It only means that it is impossible to obtain non-manipulable mechanisms regardless of which restrictions, or which combination of them, that are imposed on the architectural components. For example, the classic school choice problem where the matching is identified using the (student-optimal) deferred acceptance algorithm is a special case of our endogenous architecture (see Section 4.1) and it is well-known that no student can gain by misrepresenting her preferences in this many-to-one context.

For one-to-one matching markets, Theorem 3 holds independently of which restrictions or which combination of them that are imposed on the architectural components that not are related to the capacity structure. It thus gives market designers that are interested in implementing non-manipulable matching mechanisms an immense freedom. They only need to make sure that the restrictions on the architectural components are suitable for the real-life application in mind, as the rest of the architectural structure is endogenously created based on these choices. In particular, market designers need not spend any time figuring out what a non-manipulable matching mechanism may look like, the maximal agent-optimal weak equilibrium mechanism always works on the domain of preference profiles $\tilde{\mathcal{R}}$.

However, our proposed endogenous architecture should be treated with some caution. It is obviously important that markets designers verify that the endogenous equilibrium concept is reasonable for the specific real-life applications in mind. We argue that our approach, at least, not is unreasonable since it is based on axioms that are part of characterizations of commonly used equilibrium concepts, including, for example, stability and price equilibrium. Furthermore, it is well-known that frequently adopted equilibrium concepts often are incompatible with non-manipulability in the classic domain (recall the discussions in Section 2.2.2), demonstrating the need to operationalize a reduction of the domain of preference profiles. The present paper suggests one such reduction without claiming that this is the unique way to do it. In the matching with contracts literature for example, including the work by Echenique (2012), Hatfield and Kojima (2010) and Hatfield and Milgrom (2005), the standard reduction is to only consider profiles in the strict domain $\mathcal{P}$ (this domain is a subset of our endogenous domain of preference profiles $\tilde{\mathcal{R}}$). Another argument that supports our endogenous approach is that seminal contributions in the matching literature reduce to canonical cases of our endogenous approach. In this sense, our endogenous matching architecture unifies much of the existing literature. This insight is discussed next in somewhat greater detail.

4. CANONICAL CASES AND APPLICATIONS

The main advantage of the proposed endogenous architecture is the flexibility it offers. Restrictions on the general price space allows the market designer to study, say, markets with price ceilings and resource constraints, restrictions on the general entitlement structure enables the market designer to study markets with price-specific endowments, restrictions on the rationing structure allows for price specific rationing schemes, and so on. The main objective of the present paper is not to provide new and potentially important applications, but rather to propose an endogenous architecture. Nevertheless, we will informally discuss some new market design applications further down. However, before elaborating more on this is somewhat detail, we demonstrate how the considered architecture can be reduced to some well-known canonical cases.[17]

[17] All proofs in this section are based on seminal results in the matching literature. These results are stated without any detailed comments or remarks. The interested reader is referred to the original articles for an in-depth analysis.

4.1. *Two-sided Matching Markets: Fixed Prices and Strict Preferences*

Consider the classic one-to-one two-sided marriage market with one-sided preferences (Abdulkadiroğlu and Sönmez, 2003, Bergstrom and Manning, 1982, Gale and Shapley, 1962, Dubins and Freedman, 1981, Roth, 1982a, 1984, Svensson, 1994) and the many-to-one two-sided school choice problem with one-sided preferences (Abdulkadiroğlu and Sönmez, 2003, Gale and Shapley, 1962, Roth, 1985a). These models can be obtained from the endogenous architecture by imposing the following restrictions on the architectural components:

(1) the numeraire good is fixed and normalized to zero for each object, i.e., $\mathbb{P}_s = \{0\} = \partial\mathbb{P}_s$ for each $s \in S$,

(2) each object can be assigned to at most one agent, i.e., $c_s(0) = 1$ for all $s \in S$ (in the school choice problem, c_s is defined by the capacity of school s),

(3) no agent is entitled to any object, i.e., $\mathbb{E}_s = \emptyset$ for each $s \in S$,

(4) the rationing system is given by a strict priority-order $\succ$ where, for each $s \in S$, $\succ_{(s,\rho)} = \succ_{(s,\rho')}$ for all $\rho, \rho' \in \partial\mathbb{P}_s$.

To see how these specific restrictions endogenously generate the architectural structure for the marriage market and the school choice problem, recall that stability is the central equilibrium concept and that the matching model generated by conditions (1)–(4) normally is analyzed on the strict preference domain, $\mathcal{P}$. Formally, a matching (σ, ρ) where $\rho_s = 0$ for all $s \in S$ is stable if and only if it satisfies, no-justified envy, non-wastefulness, and individual rationality (Balinski and Sönmez, 1999, Lemma 2). Furthermore, the set of stable matchings has the structure of a lattice, meaning that there is a unique stable matching x^* that is weakly preferred by all agents in the set N to any other stable matching x (Gale and Shapley, 1962, Roth, 1985b). The matching x^* is called the agent-optimal stable matching. It is also useful to recall that the deferred acceptance algorithm (Abdulkadiroğlu and Sönmez, 2003, Gale and Shapley, 1962) can be adopted to solve the matching problem for the marriage market and the school choice problem. It is well-known that this algorithm produces an agent-optimal stable matching and that no agent in N can gain by strategically misrepresenting her preferences (see Abdulkadiroğlu and Sönmez, 2003, Dubins and Freedman, 1981, Gale and Shapley, 1962, Roth, 1982b).

The following proposition demonstrates that the architectural structure that is the endogenously generated when imposing restrictions (1)–(4) on the architectural components coincides with the architectural structure discussed in the preceding paragraph.

PROPOSITION 1: *Consider the special case of the endogenous architecture where the architectural components are restricted by conditions (1)–(4) from the above.*

(a) *A weak equilibrium coincides with a stable matching. In particular, an maximal agent-optimal weak equilibrium coincides with an agent-optimal stable matching.*

(b) $\tilde{\mathcal{R}} = \mathcal{P}$.

(c) *The maximal agent-optimal weak equilibrium mechanism coincides with (a direct version of) the (agent-proposing) deferred acceptance algorithm.*

PROOF: To prove part (a), note that since the numeraire good is fixed and normalized to zero and no agent is entitled to any object, the set $\mathbb{T}_s$ contains a single element, i.e., $\mathbb{T}_s = \mathbb{P}_s = \{0\}$. Consequently, part (iii) of Definition 1 is not relevant for the specific restriction on the architectural components. Now, in the case when $\mathbb{P}_s = \{0\}$ for each $s \in S$, parts (i), (ii) and (iv) of the Definition 1 constitute the classic definitions of individual rationality, non-wastefulness, and no-justified envy, respectively (see, e.g., Abdulkadiroğlu and Sönmez, 2003). The conclusion that a weak equilibrium is a stable matching then follows from Lemma 2 in Balinski and Sönmez (1999). To prove that an maximal agent-optimal weak equilibrium coincides with the agent-optimal stable matching, we use a result in Gale and Shapley (1962) that states that there is a unique stable matching x^* that is weakly preferred by all agents in the set N to any other stable matching x. In other words, the unique agent-optimal stable matching x^* Pareto dominates[18] any other stable matching. Because a weak equilibrium coincides with a stable matching, by our previous conclusions, and since an agent-optimal weak equilibrium Pareto dominates any other weak equilibrium by Theorem 2, the conclusion follows immediately.

To prove part (b), we note that conditions (1)–(4) imply that $\mathbb{V}_s = \{0\}$. The proof that $\widetilde{\mathcal{R}} = \mathcal{P}$ then follows from the same arguments as in Andersson and Svensson (2014, Example 4). The proof of part (c) follows directly from part (a) and the observation that the outcome of the deferred acceptance algorithm is single-valued when preferences are strict. *Q.E.D.*

4.2. *Two-sided Matching Markets: Prices Bounded from Below and Weak Preferences*

Consider the standard single-item and multi-item auctions with unit-demand bidders (e.g., Clarke, 1971, Groves, 1973, Demange and Gale, 1985, Vickrey, 1961). These auction models can be obtained from the endogenous architecture by imposing the following restrictions on the architectural components:[19]

(5) the prices of each object $s \in S$ are bounded from below by some $\kappa_s \in \mathbb{R}$, i.e., $\mathbb{P}_s = [\kappa_s, \infty)$ for each $s \in S$,

(6) each object can be assigned to at most one agent, i.e., $c_s(r) = 1$ for all $s \in S$ and all $r \in \mathbb{P}_s$,

(7) no agent is entitled to any object, i.e., $\mathbb{E}_s = \emptyset$ for each $s \in S$,

(8) the rationing system $\succsim$ is empty, i.e., no agent is given priority to any object $s \in S$ at any $r \in \mathbb{P}_s$.

To demonstrate that these restrictions on the architectural components endogenously generate the architectural structure for the standard single-item and multi-item auctions, recall that price equilibrium[20] often is adopted as an equilibrium notion in these models and that the matching model generated by conditions (5)–(8) normally is analyzed on the classic preference domain $\mathcal{R}$. Formally, for a given profile $R \in \mathcal{R}$, a matching $x = (\sigma, \rho)$ is a price equilibrium if $(\sigma(i), \rho(i)) R_i (\sigma(j), \rho(j))$ for all $i \in N$ and all $j \in S$ where prices are bounded from below by the reservation prices of the sellers. Moreover, the set of equilibrium price vectors have the structure of a lattice, meaning that there always exists a unique minimum equilibrium price vector $\rho^{\min}$ (Shapley and Shubik, 1971, Demange and Gale, 1985). That is, for any two price equilibria $x = (\sigma, \rho)$ and $x' = (\sigma', \rho')$, it always holds that the vector $\rho^{\min}$ is an equilibrium price vector when $\rho_j^{\min} = \min\{\rho_j, \rho'_j\}$ for all $j \in S$. A price equilibrium containing the price vector $p^{\min}$ is called a minimum price equilibrium.[21] It is also useful to recall that if the minimum equilibrium price vector $p^{\min}$ is used as a (direct) mechanism to assign the objects in S to the agents in N, the resulting matching is Pareto efficient (from the viewpoint of the agents

[18]To define the axiom formally, consider matchings $x = (\sigma, \rho)$ and $x' = (\sigma', \rho')$. Then matching x Pareto dominates matching x' if $(\sigma(i), \rho(i)) R_i (\sigma'(i), \rho'(i))$ for all $i \in N$ and $(\sigma(i), \rho(i)) P_i (\sigma'(i), \rho'(i))$ for some $i \in N$. A matching that is not Pareto dominated by any matching at a given profile is said to be Pareto efficient.

[19]The fair allocation problem with monetary compensations (e.g., Leonard, 1983, Sun and Yang, 2003, Svensson, 2009) can be obtained in a similar fashion by a simple sign convention, but this will not be discussed in the present paper.

[20]In Demange and Gale (1985), this condition is referred to as stability. In fact, price equilibrium and stability are often used interchangeably.

[21]The fact that the equilibrium price vector $\rho^{\min}$ is unique does not imply that a minimum price equilibrium is unique as there may be several equilibrium assignments that are compatible with $\rho^{\min}$ (agents may be indifferent between several bundles at these prices). However, if this is the case, agents are utility indifferent between all minimum price equilibria. In this sense, a minimum price equilibrium is essentially single-valued.

in N) and no agent can gain by strategically misrepresenting her preferences (Clarke, 1971, Demange and Gale, 1985, Groves, 1973, Shapley and Shubik, 1971, Vickrey, 1961).

The following proposition demonstrates that by imposing restrictions (5)–(8) on the architectural components, the architectural structure discussed in the preceding paragraph will be obtained.

PROPOSITION 2: *Consider the special case of the endogenous architecture where the architectural components are restricted by conditions (5)–(8) from the above.*

(a) *A weak equilibrium coincides with a price equilibrium. In particular, an maximal agent-optimal weak equilibrium coincides with a minimum price equilibrium.*
(b) $\tilde{\mathcal{R}} = \mathcal{R}$.
(c) *The maximal agent-optimal weak equilibrium mechanism coincides with the minimum equilibrium price mechanism.*

PROOF: To prove part (a), note that no agent is entitled to any object and prices are allowed to be infinitely high, so $\mathbb{T} = \emptyset$. This means that part (iv) of Definition 1 is redundant. That a weak equilibrium coincides with a price equilibrium then follows since definition of a price equilibrium is identical to the interior no-envy condition when $\mathbb{P}_s = [\kappa_s, \infty)$ for each $s \in S$, and the observation that interior no-envy implies both non-wastefulness and individual rationality. It remains to prove that an maximal agent-optimal weak equilibrium is a minimum price equilibrium. Because the set of equilibrium prices has the structure of a lattice (Shapley and Shubik, 1971, Demange and Gale, 1985), it follows that if $(\sigma^{\min}, \rho^{\min})$ is a minimum price equilibrium, then $(\sigma^{\min}(i), \rho^{\min}(i)) R_i (\sigma(j), \rho(j))$ for any price equilibrium (σ, ρ) at profile R. Hence, the minimum price equilibrium is the only Pareto efficient equilibrium at profile R. The conclusion then follows since an agent-optimal weak equilibrium Pareto dominates any other weak equilibrium.

The proof of part (b) follows from the same arguments as in Andersson and Svensson (2014, Example 5). The proof of part (c) follows directly from part (a) and the observation that the outcome of the minimum equilibrium price mechanisms is single-valued (more precisely, essentially single-valued, see footnote 21 for further remarks). *Q.E.D.*

4.3. *Matching with Contracts and Matching Markets with Salaries*

This section discusses the standard matching with contracts framework (Hatfield and Milgrom, 2005) and matching markets with salaries (Kelso and Crawford, 1982). Under a substitutability condition, Echenique (2012) demonstrated that the former framework can be embedded into the latter using a one-to-one function that maps each market with contracts to a corresponding market with salaries.[22] This embedding will be discussed further down, but we first discuss a crucial difference between these frameworks and the endogenous architecture considered in the present paper.[23]

Both Hatfield and Milgrom (2005) and Kelso and Crawford (1982) consider many-to-one worker-firm matching markets where firms have strict preferences over sets of contracts and worker-salary pairs, respectively. Even if the endogenous architecture considered in the present

[22]For in-depth discussions of various substitutability notions and embedding techniques, see, e.g., Echenique (2012), Hatfield and Kojima (2010), Hatfield and Milgrom (2005), Kominers (2012) and Schlegel (2015).

[23]Another minor difference is that the numeraire good is interpreted as a compensation and not a price. This will not play any significant role as all results in the present paper hold for this interpretation by reversing all relevant monotonicity assumptions.

paper only is specified for one-sided preferences (the "workers" in N have preferences over the "firms" in S), the general rationing system $\succsim$ can be modified to represent firm preferences by imposing additional assumptions. By construction of the rationing system, individual workers can only be ranked by firms at given wages. That is, for any given wage r, firm s assigns priorities to the workers in N via the relation $\succsim_{(s,r)}$. In the matching with contracts framework, firms must also be able to strictly rank sets of contracts. More precisely, let $x \in X$ be a contract between one worker and one firm. Each firms assigns a utility number to each set of contracts $X' \subset X$ such that the ranking is strict. That is, for each $s \in S$, the utility function $u_s : 2^X \mapsto \mathbb{R}$ is one-to-one. In matching markets with salaries, firms strictly rank bundles of type (s, i, r) using corresponding one-to-one utility functions. See Echenique (2012) for a thorough analysis.

The key to integrate the standard matching with contracts framework and matching markets with salaries in our endogenous architecture is to define such utility functions based on $\succsim$. We will next propose the simplest possible way in which this can be achieved. Suppose that all prices in $\mathbb{P}$ are (non-negative) integers, implying that all priorities are strict since all prices are on a threshold. Let also $\succ_{(s,r)}=\succ_{(s,r')}$ for all $r, r' \in \mathbb{P}_s$ and assign each contract x or bundle of type (s, i, r), for any given $\succ_{(s,r)}$, a utility number based on the rank-score (say that the agent with the highest priority gets score $|N|$, the agent with the second highest priority gets score $|N-1|$, and so on) minus the wage r (the minus sign reflects that firms prefer lower wages to higher). The utility for a set of contracts $X' \subset X$ or a set of bundles of type (s, i, r) can then be defined by summing individual utilities that are included in the contract of the set of bundles. After imposing some tie-breaking, we obtain one-to-one utility functions of type u_s. Denote this strict preference profile by $\rhd = (\rhd_1, \ldots, \rhd_{|S|})$.

As will be explained later, because it is easier to see the connection between matching markets with salaries and the considered endogenous framework, we will for now focus on the standard matching with contracts framework. It can be obtained from the endogenous architecture by imposing the following restrictions on the architectural components:

(9) wages r are non-negative integers that belong to the finite set $\mathbb{P}_s = \mathbb{P} \subset \mathbb{N}_{++}$ for all $s \in S$,

(10) firm s have $\kappa_s \in \mathbb{N}_{++}$ identical positions, i.e., $c_s(r) = \kappa_s$ for all $r \in \mathbb{P}$ and each $s \in S$,

(11) no agent is entitled to any object, i.e., $\mathbb{E}_s = \emptyset$ for each $s \in S$,

(12) the "rationing system," i.e., the preferences of the firms, is defined by $\rhd$.

To see how these specific restrictions endogenously generate the architectural structure for the classic matching with contracts framework, we must introduce some further notation. Given a set of contracts $X' \subset X$, the chosen set of worker i, $C_i(X')$, is either the empty set (if there are no acceptable contracts in X') or the singleton set containing the most preferred contract (recall that workers have strict preferences by construction of $\mathcal{R}$ since all wages are located at thresholds). Let $C_N(X') = \cup_{i \in N} C_i(X')$ denote the set of contracts that are chosen by some worker in N, and let the remaining contracts in X', that is, $R_N(X') = X' \setminus C_N(X')$ represent the set of rejected contracts. The corresponding sets for the firms, $C_s(X')$, $C_S(X')$ and $R_S(X')$, are defined analogously. A set of contracts $X' \subset X$ is a stable allocation if it is individually rational and if there are no blocking pairs. The latter axiom means that there does not exist an alternative contract $x' \in X$ and a pair (i, s) such that worker i and firm s would both strictly prefer x' over their current match in X', and x' is feasible within the contractual terms available to i and s.

As established by Hatfield and Milgrom (2005, Theorem 3), if the preferences of the firms imply that contracts are substitutes[24], the set of stable allocations is a non-empty complete finite

[24]Formally, the preferences for firm s satisfies substitutability if, for any subset of contracts $X' \subset X$, and any two contracts x and x', $x \notin C_s(X' \cup \{x\})$ implies $x \notin C_s(X' \cup \{x, x'\})$. See, for example, Echenique (2012), Hatfield

lattice, implying that it contains a unique worker-optimal and a unique firm-optimal allocation. Recall that the firm-optimal allocation is the least preferred allocation for the workers (and vice versa). Hatfield and Milgrom (2005) also introduced the so-called cumulative offer algorithm that identifies the worker-optimal or firm-optimal matching, depending on the "direction" of the algorithm. We will only focus on the latter of these matchings and we will also not describe or define the algorithm in detail. Informally, and given our focus on the worker-optimal allocation, the workers in N start by offering contracts to the firms in S at the highest possible wage level. Each firm then considers all received offers and tentatively accepts their most preferred contract (if any) and rejects all other offers. Rejected workers then lower their offers by the minimal amount (if wages can be decreased) and resubmit them to their most preferred firm that has not rejected them permanently. Firms again tentatively hold their most preferred offer (if any) and reject the ones they prefer (strictly) less. This process continues until there are no more rejections, that is, until all offers are either accepted or permanently rejected. This algorithm terminates to the worker-optimal stable allocation.

The following proposition demonstrates that by imposing restrictions (9)–(12) on the architectural components, the architectural structure discussed in the preceding paragraph will be obtained.

PROPOSITION 3: *Consider the special case of the endogenous architecture where the architectural components are restricted by conditions (9)–(12) from the above.*

(a) *A weak equilibrium coincides with a stable allocation. In particular, an maximal agent-optimal weak equilibrium coincides with a worker-optimal stable allocation.*
(b) $\tilde{\mathcal{R}} = \mathcal{P}$.
(c) *The maximal agent-optimal weak equilibrium mechanism coincides with (a direct version of) the (worker-optimal) cumulative offer algorithm.*

PROOF: To prove part (a), note that since wages are integers and no agent is entitled to any object, all prices are located on thresholds, i.e., $\mathbb{T} = \mathbb{P}$. Consequently, part (iii) of Definition 1 is not relevant for the specific restriction on the architectural components. Because $\mathbb{T} = \mathbb{P}$, part (i) of the Definition 1 constitutes the classic definitions of individual rationality, whereas parts (ii) and (iii) jointly means that there are no blocking pairs, the conclusion follows using the same arguments as the proof of Proposition 1(i). To prove that an maximal agent-optimal weak equilibrium coincides with the worker-optimal stable allocation, we note that the following holds for $\rhd$ by construction: for any subset of contracts $X' \subset X$, and any two contracts x and x', $x \notin C_s(X' \cup \{x\})$ implies $x \notin C_s(X' \cup \{x, x'\})$. But then $\rhd$ satisfies the substitutability condition, so there exists a worker-optimal stable allocation (Hatfield and Milgrom, 2005, Theorem 3). Because a weak equilibrium coincides with a stable matching, by our previous conclusions, and since an agent-optimal weak equilibrium Pareto dominates any other weak equilibrium by Theorem 2, the conclusion follows immediately.

To prove part (b), recall that $\mathbb{V} = \mathbb{P}$. But then $\tilde{\mathcal{R}} = \mathcal{P}$ since no object in S is connected by indifference on the domain $\tilde{\mathcal{R}}$ by construction. The proof of part (c) follows directly from part (a), Theorem 2, and the observation that the outcome of the (worker-optimal) cumulative offer algorithm is single-valued and converges to the worker-optimal stable allocation. *Q.E.D.*

We end this section with a few remarks on the relation between the matching with contracts framework (Hatfield and Milgrom, 2005) and matching markets with salaries (Kelso and

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and Milgrom (2005), or Kelso and Crawford (1982). The considered preference relation $\rhd$ satisfies this condition by construction.

Crawford, 1982). Using the same substitutability assumption as in the proof of Proposition 3, Echenique (2012) demonstrated that there exists a one-to-one function that maps each contract $x \in X$ into a triple of type $(s, i, r) \in S \times N \times \mathbb{P}$. In other words, that there is an embedding that maps each market with contracts to a corresponding market with salaries (see Hatfield and Kojima, 2010, Kominers, 2012, Schlegel, 2015, for further generalizations). Furthermore, the set of stable matchings is demonstrated to be invariant under this embedding, and the algorithms proposed by Hatfield and Milgrom (2005) and Kelso and Crawford (1982) both converge to the worker-optimal stable matching. Proposition 3 demonstrated that by imposing specific restrictions on the architectural components, the endogenous mechanism considered in the present paper identifies the worker-optimal stable matching on the strict preference domain, exactly as in the corresponding results by Hatfield and Milgrom (2005) and Kelso and Crawford (1982). However, this is a canonical case of our endogenous framework.

A final remark is that these observations together with Proposition 3 and the fact that $\triangleright$ satisfy substitutability under our additional assumptions demonstrate that the considered endogenous architecture not only includes a matching with contracts framework, but also that it can be interpreted in terms of a matching market with salaries. The latter should come as no surprise since we, in similarity with Kelso and Crawford (1982), define consumption bundles of type $(\sigma(i), \rho)$ or, equivalently, of type (s, i, r). In fact, the embedding technique in Echenique (2012) centers around the idea to construct wages in such a way that contracts can be transformed into consumption bundles of exactly this type. In this sense, it is not necessary in our engogenous architecture to go via the matching with contracts framework to obtain matching markets with salaries. We can obtain it directly with appropriate assumptions on $\succsim$.

4.4. *New Market Design Applications*

The present paper does not aspire to work out the details for new market design applications, but we will nevertheless elaborate in some detail on a few new and potentially important applications. That said, it is not difficult to imagine that there are many additional novel applications that require complex endogenous architectures.

4.4.1. *Housing Markets with Means-Tested Priorities*

In most countries, there is a scarcity of available apartments and public housing authorities use a wide range of mechanisms to allocate vacant apartments among applicants. These types of matching markets can obviously be organized in many different ways (see, e.g., Arnosti and Peng, 2020, Waldinger, 2021). Our endogenous architecture gives public housing authorities many degrees of freedom when designing the matching mechanism. First, priorities may be means-tested, meaning that the income of potential tenants may determine the priority when, for example, the rents become "too high." Note that his does not exclude that the priorities for some houses are based on, say lotteries or queueing time, or that lotteries or queues are used only for specific rent levels. Second, a subset of the houses may be subject to rent control, whereas the rents of some houses are determined by the market. Third, the housing authority may also use entitlements, for example, by entitling targeted groups of tenants specific apartments. These three policy instruments can all help in designing public housing programs (or social housing programs), that is, programs with subsidized or affordable housing in buildings owned by public housing authorities or non-profit organizations. These types of programs exist all over the world, for example, in Australia, Brazil, Canada, China, Denmark, France, Japan, South Africa, Taiwan, and the U.S.[25]

[25]See https://en.wikipedia.org/wiki/Public_housing for additional examples and detailed descriptions of country specific public (or social) housing systems.

By imposing the following restrictions on the architectural components, the endogenous architecture proposed in the present paper can solve this type of general house allocation problem:

(i) the prices of each object $s \in S$ are bounded from below and above by some $\underline{\kappa}_s \in \mathbb{R}$ and $\overline{\kappa}_s \in \mathbb{R}$, respectively, i.e., $\mathbb{P}_s = [\underline{\kappa}_s, \overline{\kappa}_s]$ for each $s \in S$ where $\overline{\kappa}_s = \infty$ is allowed,
(ii) each object can be assigned to at most one agent, i.e., $c_s(r) = 1$ for all $s \in S$ and all $r \in \mathbb{P}_s$,
(iii) agent $i \in N$ may be entitled to objects in S at specific prices, i.e., it is allowed that $e_i \neq \emptyset$ for $i \in N$,
(iv) the rationing system is given by $\succsim$ or some restriction of it.

We will not provide a complete literature review here, but rather note that some seminal mechanisms for house allocation are covered by the maximal agent-optimal weak equilibrium mechanism considered in the present paper. This includes the serial dictatorship mechanism with weak priorities (Svensson, 1994) and the generalization of it where agents are allowed to have diverse priorities for different houses (Ergin, 2002). Furthermore, it covers the classic market based house allocation problem (Shapley and Shubik, 1971) as well as house allocation with rent control (Andersson and Svensson, 2014) and rental housing markets with options to buy and trade (Andersson et al., 2016).

4.4.2. *School Choice with Vouchers*

Many countries, including Chile, Colombia, Sweden, Japan, Canada, and Poland, have both educational voucher programs (see Zimmer and Bettinger, 2010, for an overview) and sophisticated school choice programs (see Abdulkadiroğlu and Andersson, 2023, for a recent survey). Determining the size of the vouchers and solving the student-school matching problem simultaneously may be an effective tool to reduce differences in school quality. It is an empirical fact that some schools are more demanded than others, for example, because the students in those schools historically have performed better than the average student in the population, because the schools are located on better neighborhoods, or because parents in the neighborhoods where the schools are located have a higher educational background than the average parent in the population. Now, by reversing the model and by interpreting the "price" as a "compensation" (recall the discussions in Section 4.3) and by applying the maximal agent-optimal weak equilibrium mechanism, the mechanism will by construction reduce the monetary compensation for schools in excess demand while maintaining a high compensation for less popular schools. Consequently, more money will be transferred to less popular schools via student vouchers and this may, at least in the long run, help to reduce differences in school quality, e.g., because the teacher per student ratio can increase in less popular schools (as they are allocated more money per student).

By imposing the following restrictions on the architectural components, the endogenous architecture proposed in the present paper can solve both the student-school matching problem and determine the size of the vouchers simultaneously:

(i) The voucher for school $s \in S$ is restricted to belong to the set $\mathbb{P}_s = [\kappa_s, \kappa] \subset \mathbb{R}_+$ where κ is a school-common upper limit whereas κ_s represents a school-specific lower limit,
(ii) each school have a fixed capacity, i.e., $c_s(r) \in \mathbb{N}_{++}$ for each $s \in S$ and all $r \in \mathbb{P}_s$,
(iii) no agent is entitled to any object, i.e., $\mathbb{E}_s = \emptyset$ for each $s \in S$,
(iv) the rationing system is given by a strict priority-order $\succ$ where, for each $s \in S$, $\succ_{(s,r)} = \succ_{(s,r')}$ for all $r, r' \in \partial\mathbb{P}_s$.

Apart from the allowing for flexible school-specific vouchers, the matching model is identical to the classic school choice model by Abdulkadiroğlu and Sönmez (2003). Namely, there is a set of students and schools, and each school have a fixed capacity together with strict priorities over students. Exactly as for the deferred acceptance algorithm, our proposed mechanism will generate a stable outcome is stable in the sense that it satisfies, no-justified envy, non-wastefulness, and individual rationality when taking the vouchers into account.

4.4.3. *Auctions with Discrete and Continuous Price Increments*

Spectrum for wireless communication is normally allocated via auctions (Milgrom, 2004). A common feature in many auctions, including spectrum auctions, is that they sometimes are "slow," meaning that it takes many bidding rounds before the auction converges. A well-known example is the auction for British telecom licenses in 2000 that took two months to complete (Binmore and Klemperer, 2002). As a result, "fast" auction formats have been suggested in the literature (Andersson et al., 2013, Klemperer, 2010, Sankaran, 1994). Some of them propose various "proxies" to increase speed further (see, e.g., Ausubel and Milgrom, 2002, Parkes and Ungar, 2000). Another way to increase speed and yet keep the precision in finding a price equilibrium is to initiate the auction with larger discrete price increments and then to use continuous pricing for higher prices. A problem with such approach may arise if the auctioneer overestimates the market value of the objects or licences, resulting in that the auction terminates before reaching the "continuous price stage." If so, rationing may have to be used, for example, by prioritizing bidders' based on their expected social and economic value, say, in terms of network coverage (priority based on fast and/or most widespread geographic rollout), technological standards (priority based on deploying advanced, secure, and/or energy-efficient infrastructure), or consumer pricing (priority based on subscription rates for the public).

This type of multi-unit auction suits the proposed endogenous architecture well since the general price spaces and the proposed matching mechanism not only allow for a combination of discrete price increments and continuous clock increments, but also for sophisticated rationing systems. This can be achieved by imposing the following restrictions on the general architectural components:

(i) the prices of each object $s \in S$ belong to the price space $\mathbb{P}_s$ where the price space allow for isolated points as wells as closed intervals of the form $[\kappa_s, \infty)$, or any combination of the two,
(ii) each object can be assigned to at most one agent, i.e., $c_s(r) = 1$ for all $s \in S$ and all $r \in \mathbb{P}_s$,
(iii) no agent is entitled to any object, i.e., $\mathbb{E}_s = \emptyset$ for each $s \in S$,
(iv) the rationing system $\succsim$ is based on social and economic value.

It should be noted that the maximal agent-optimal weak equilibrium mechanism, as defined in the present paper, is a direct mechanism (i.e., a sealed bid mechanism) whereas iterative auction mechanisms are overwhelmingly more prevalent in real-life applications (Ausubel, 2004, Perry and Reny, 2005). We leave it for future research to design an iterative version of the endogenous mechanism proposed in the present paper.

5. CONCLUSIONS

We have introduced a common endogenous architecture for market design. The main advantage of adopting our endogenous approach is the flexibility that enables the analysis to stretch beyond much of the existing literature. The market designer needs to worry only about restricting the architectural components in such a way that the resulting matching model gives an accurate description of the real-life problem of interest since the architectural structure is endogenously generated in the process. Of course, one may question if our proposed endogenous approach always generates a "reasonable" architectural structure. We have argued that our approach at least not is unreasonable since it, with appropriate restrictions on the architectural components, reduces to definitions and concepts that have proven to be important for applications. This is not only true for the domain of preference profiles and the matching mechanism, but also for the proposed equilibrium concept that contains ingredients that normally are part of more specific matching models, say, individual rationality, non-wastefulness, and various "envy concepts."

Our approach also enables us to solve a puzzle in the market design literature. As described in the introduction section, Roth (2026) observed that even if existing matching models diverge in terms of architectural components and architectural structure, their mathematical structures and properties are remarkably similar. As demonstrated in the present paper, the solution to this puzzle is simply that these models all are all part of the same unifying general matching model. Therefore, all of them are covered by the following theorem:

THEOREM 4: *In [all one-to-one matching models covered by the architectural components considered in the present paper] defined on [the domain of preference profiles $\tilde{\mathcal{R}}$], when [the maximal agent-optimal weak equilibrium mechanism] is employed, it is a dominant strategy for players [in the set N] to submit their true preferences and the resulting outcome is [a weak equilibrium].*

Even if the considered endogenous architecture extends the existing literature, it should be noted that it focuses exclusively on "stability type models" and does not include "core type models." The latter class of matching models includes, for example, markets for house exchange (Abdulkadiroğlu and Sönmez, 1999, Ma, 1994, Sönmez, 1999, Shapley and Scarf, 1974, Roth, 1982b), priority swapping in school choice (Abdulkadiroğlu and Sönmez, 2003), and organ exchange (Ergin et al., 2020, Roth et al., 2004). Future research includes an even more general endogenous architecture that also contains these types of models as special cases. A good starting point for such research program is to investigate if the endogenous architecture considered in the present paper can be married to the general exchange models that already have been developed in the literature, for example, Abdulkadiroğlu and Sönmez (1999), Pápai (2000), and Pycia and Ünver (2017).

A final remark relates to an observation in Section 4.4.3, namely that our main results are valid only for direct mechanisms and we have, in contrast to many seminal papers in the market design literature, including, e.g., Abdulkadiroğlu and Sönmez (1999, 2003), Gale and Shapley (1962), Shapley and Scarf (1974) and Vickrey (1961), not provided an iterative procedure for how to generally identify an maximal agent-optimal weak equilibria. Also this task is left for future research, but a natural starting point is Andersson and Svensson (2018) where an auction model for matching markets with price restrictions is developed.

APPENDIX A: EQUILIBRIUM EXISTENCE AND STRUCTURE

A.1. *Existence of weak equilibrium*

Unless otherwise specified, we assume unit entitlement for non-maximal prices meaning that whenever $\mathbb{E}_s \backslash \{\overline{r}_s\} \neq \emptyset$, then $|\mathbb{E}_s| = 1$, say $\mathbb{E}_s = \{r\}$, and both exactly one agent is entitled to bundle (s, r) and the capacity of object s is equal to one at price r, i.e., $|\mathbb{E}_s| = c_s(r) = 1$. Hence, it is only possible for multiple agents to have an entitlement of object s at its maximal price and $\overline{r}_s < \infty$. Furthermore, we suppose $c_s(r) < |N|$ for $r < \overline{r}_s$ as otherwise we can lower the maximal price of s to the minimal price where we have full capacity, and also $c_s(r) < |N| - 1$ for all $r \in \mathbb{E}_s \backslash \{\overline{r}_s\}$ as otherwise we have the following: if $c_s(r) = |N| - 1 = \mathbb{E}_s[r]$ and $r < \overline{r}_s$, then for any weak equilibrium (σ, ρ) with $\sigma[s] \cap \mathbb{E}_s[r] \neq \emptyset$ individual rationality yields $\rho_s \leq r$ and otherwise in case $\rho_s > r$ we may lower ρ_s to r and still obtain a weak equilibrium (keeping other prices and the assignment σ unchanged), i.e. any agent-optimal weak equilibrium (σ, ρ) satisfies $\rho_s \leq r$ and again we may lower the maximal price of s to the price r.

The following theorem demonstrates that unit entitlement for non-maximal prices is necessary and sufficient for the existence of a weak equilibrium. The NCBI condition does not play any role in the result, that is, imposing NCBI does not change the conclusion.

THEOREM 5: *Weak equilibrium exists on the entire domain $\mathcal{R}$ if and only if unit entitlement for non-maximal prices is satisfied.*

PROOF: For the "only if part," suppose that there always exists a weak equilibrium, but that unit entitlement for non-maximal prices is violated.

We consider different cases. Suppose that $r \in \mathbb{E}_s$ is minimal such that $r < \overline{r}_s$ and $|\mathbb{E}_s[r]| < c_s(r) < |N|$ where the latter condition follows from $r < \overline{r}_s$. Then $r^+ \leq \overline{r}_s$. Let R be such that for all $i \in N$, (s, r^+) is preferred over the null-object and no other objects are better than the null-object. Suppose that (σ, ρ) is a weak equilibrium for R. By individual rationality of the agent that is entitled to (s, r), we have $\rho_s \leq r$. By non-wastefulness and our choice of R, it follows that $c_s(r) = |\sigma[s]| < |N|$. But now, since there is $i \in \sigma[s]$ who is not entitled to (s, r), for any unassigned agent $j \in N \backslash \sigma[s]$, we have $(s, \rho_s^+) P_j (\sigma(j), \rho)$, i.e., either interior envy (if $\rho_s \notin \partial \mathbb{P}_s$) or threshold no justified envy (if $\rho_s \in \partial \mathbb{P}_s$) is violated.

Now, the same argument remains valid if $1 < |\mathbb{E}_s[r]| = c_s(r) < |N|$ as for one agent in $\mathbb{E}_s[r]$, we may suppose that the null-object is better than $(s, \min \mathbb{P}_s)$ and the assumption $c_s(r) < |N| - 1$ yields the result.

Hence, for the minimal $r \in \mathbb{E}_s \backslash \{\overline{r}_s\}$, we have $1 = c_s(r) = |\mathbb{E}_s[r]|$. It remains to show that $\mathbb{E}_s = \{r\}$. Suppose that this is not the case, and let r' be minimal in $\mathbb{E}_s \backslash \{r\}$. Suppose that $(s, r') \in e_j$ with $j \neq k$ where $(s, r) \in e_k$. Now, consider again the profile R as above and suppose that (σ, ρ) is a weak equilibrium. By individual rationality for agent k and $(s, r) \in e_k$, we have $\rho_s \leq r$ and $k \in \sigma[s]$. But again by individual rationality for agent j and $(s, r') \in e_j$, we have $j \in \sigma[s]$ and $\rho_s \leq r' < r$. As r' is the minimal element of $\mathbb{E}_s \backslash \{\overline{r}_s\}$, and since the capacity function is monotone, $c_s(\rho_s) \leq c_s(r') = 1$. But $j \neq k$ and $\{j, k\} \subseteq \sigma[s]$ gives a contradiction.

For the "if part," suppose that unit entitlement for non-maximal prices is satisfied. Let R be a profile. For s and i, let $t_s(i)$ be such that $(0, 0) P_i (s, t_s(i))$, i.e. the null object with zero price is preferred over s with price $t_s(i)$, and let $\overline{t}_s = \max_{i \in N} t_i(s)$ (where we choose $\overline{t}_s \in \mathbb{P}_s$ if $\overline{r}_s = +\infty$ and otherwise we have $\overline{r}_s < \overline{t}_s$). We construct a weak equilibrium with a simple algorithm. Set initial allocation (σ^0, ρ^0) as follows: for each $i \in N$, let $(\sigma^0(i), r_i) \in \max_{R_i} \{(s, r) \in e_i : (s, r) \; P_i \; (0, 0)\}$, if this is non-empty, and $(s_i, r_i) = (0, 0)$ otherwise. Note that $|\mathbb{E}_s[r]| \leq c_s(r)$ makes this assignment feasible. For $s \in S$ for which there is no $\sigma^0(i) = s$, set $\rho_s^0 = \min\{\overline{r}_s, \overline{t}_s\}$, and otherwise, unit entitlement for non-maximal prices ensures that setting $\rho^0_{\sigma^0(i)} = r_i$ yields a well-defined price vector ρ^0. We next describe a procedure for eliminating waste that will also be important in later proofs.

VACANCY CHAIN DYNAMICS: Given $s \in S$ and allocation (σ^k, ρ^k) of step k in the procedure, define f_s^k as

$$f_s^k = \max_{\succ_{(s, \rho_s^k)}} \left\{ i \in N : (s, \rho_s^k) \; P_i \; (\sigma^k(i), \rho^k) \right\} \in N$$

whenever $|\sigma^k[s]| < c_s(\rho_s^k)$ and $(s, \rho_s^k) \; P_i \; (\sigma^k(i), \rho^k)$ for at least one $i \in N$, and otherwise set $f_s^k = 0$. The only way for the maximum to *not* be (uniquely) well-defined is for all agents i and j who prefer (s, ρ_s^k) as above to have $i \sim_{(s, \rho_s^k)} j$. This would imply that $\rho_s^k \in \mathbb{E}_s \setminus \partial \mathbb{P}_s$ and that $\sigma[s] \neq \emptyset$. But then, given unit entitlement for non-maximal prices, s would be at capacity and so f_s^k would be set to 0 anyway; thus, f_s^k is well-defined on S.

Now choose s for which $f_s^k \in N$. Set $\sigma^{k+1}(f_s^k) = s$, and for all $j \neq f_s^k$, $\sigma^{k+1}(j) = \sigma^k(j)$. For all $t \in S$, set $\rho_t^{k+1} = \rho_t^k$ except if both $\sigma^k(f_s^k) = t$ and $(t, \rho_t^k) \in e_{f_s^k}$, in which case set $\rho_t^k = \min\{\overline{r}_s, \overline{t}_s\}$. In words, when an agent leaves her entitlement, the price is set as high possible to reduce demand for s as much as possible. Suppose that $(s, \rho_s^{k+1})\ P_j\ (\sigma^{k+1}(j), \rho^{k+1})$ and this violates one of the weak-equilibrium conditions. As prices and welfare have weakly increased from k to $k+1$, $(s, \rho_s^k)\ P_j\ (\sigma^k(j), \rho^k)$, so if the violation is of individual rationality or interior no-envy, then this violation existed at (σ^k, ρ^k). Suppose then that there is $i \in \sigma^{k+1}[s]$ with $j \succ_{(s,\rho_s^{k+1})} i$. If $\rho_s^{k+1} = \rho_s^k$, then, because f_s^k maximizes $\succ_{(s,\rho_s^k)}$ among those who desire (s, ρ_s^k), $i \neq f_s^k$. It follows that $i \in \sigma^k[s]$. On the other hand if $\rho_s^{k+1} \neq \rho_s^k$ then it must be that $\rho_s^k < \rho_s^{k+1} = \min\{\overline{r}_s, \overline{t}_s\}$ and f_s^k left s empty, a contradiction. Thus it remains to consider $(s, \rho_s^{k+1,+})\ R_j\ (\sigma^{k+1}(j), \rho^{k+1})$. Then, again recalling that prices and welfare are weakly higher at $k+1$ than at k, we have $(s, \rho_s^{k+})\ R_j\ (\sigma^k(j), \rho^k)$ and the violation existed at (σ^k, ρ^k).

In sum, vacancy chain dynamics do not introduce any violations of weak equilibrium, and clearly they can be repeated until all waste is eliminated. As (σ^0, ρ^0) had no violations of weak equilibrium except for waste, we conclude that there is $K \in \mathbb{N}$ such that (σ^K, ρ^K) is a weak equilibrium. *Q.E.D.*

A.2. *Equilibrium Structure: Preliminaries*

We first need to introduce some notation and some basic lemmas. Consider two weak equilibria, denoted by (σ, ρ) and (σ', ρ'), and construct a labeled and directed transfer graph T on the vertices S such that $s \xrightarrow{i} t \in T$ if $\sigma(i) = s$, $\sigma'(i) = t$ and $s \neq t$. Define now the following sets:

$$\begin{aligned} S^+ &= \{s \in S : \rho'_s < \rho_s\}, & N^+ &= \{i \in N : (\sigma'(i), \rho')P_i(\sigma(i), \rho)\}, \\ S^= &= \{s \in S : \rho'_s = \rho_s\}, & N^= &= \{i \in N : (\sigma'(i), \rho')I_i(\sigma(i), \rho)\}, \\ S^- &= \{s \in S : \rho'_s > \rho_s\}, & N^- &= \{i \in N : (\sigma(i), \rho)P_i(\sigma'(i), \rho')\}. \end{aligned}$$

Denote by $s \rightsquigarrow t \subseteq T$ a simple path in the transfer graph, i.e., a path with no repeated vertices or arcs. Note that since the graph T may contain several arcs with the same orientation between a given pair of vertices, there may be many distinct paths from s to t even on the same ordered list of vertices. A path is positive if it contains an arc for which agent welfare increases in the direction of the arc. Thus, for graph T, a path is positive if it contains a N^+-labeled arc. A path is negative if it contains a N^--labeled arc. Consequently, abusing the language, a path can be both positive and negative. A positive path is positive-definite if it contains only labels from the set $N^+ \cup N^=$. Negative-definiteness is defined analogously.

LEMMA 1: *Let (σ, ρ) and (σ', ρ') be two weak equilibria at profile $R \in \mathcal{R}$. Then:*
(i) *$u \xrightarrow{i} s \in T$ with $s \in S^-$ implies either $i \in N^-$ or $\sigma[s] = \mathbb{E}_s[\rho_s] \neq \emptyset$, and;*
(ii) *if $s \xrightarrow{i} t, t \xrightarrow{j} u \in T$, then we cannot have $i \in N^+$ and $j \in N^-$.*

PROOF: Note that, by construction of the graph T, $(\sigma(i), \rho) = (u, \rho_u)$ and $(\sigma'(i), \rho') = (s, \rho'_s)$.

To prove part (i), assume first that there exists $j \in \sigma[s]$ who is not entitled to (s, ρ_s), i.e., $(s, \rho_s) \notin e_j$. Then, it needs to be shown that $i \in N^-$. Suppose that $\rho_s \notin \mathbb{T}_s$. Now, interior no-envy gives $(\sigma(i), \rho)R_i(s, \rho_s)$, and monotonicity together with $s \in S^-$ and $\rho_s < \rho'_s$ give $(s, \rho_s)P_i(s, \rho'_s)$. Consequently, $(\sigma(i), \rho)P_i(\sigma'(i), \rho')$ by transitivity. Hence, $i \in N^-$. Suppose now that $\rho_s \in \mathbb{T}_s$. As agent $j \in \sigma[s]$ is not entitled to the bundle (s, ρ_s), we must have

$\rho_s \in \partial \mathbb{P}_s$ (as otherwise $j \sim_{(s,\rho_s)} i$ so $(\sigma(i),\rho)\ R_i\ (\sigma(j),\rho)$ and we use the same argument as above to conclude $i \in N^-$) and $\rho_s < \rho_s^+$. Since $s \in S^-$, i.e., $\rho'_s > \rho_s$, it follows that $\rho_s \neq \overline{r}_s$ and $\rho_s < \rho_s^+ \leq \rho'_s$. Now, threshold no-justified-envy (as $j \in \sigma[s]$ and $(s,\rho_s) \notin e_j$) gives $(\sigma(i),\rho)P_i(s,\rho_s^+)$ and monotonicity together with $\rho_s^+ \leq \rho'_s$ give $(s,\rho_s^+)R_i(s,\rho'_s)$. Consequently, $(\sigma(i),\rho)P_i(\sigma'(i),\rho')$ by transitivity. Hence, $i \in N^-$.

We therefore have that if $\sigma[s] \setminus \mathbb{E}_s[\rho_s] \neq \emptyset$, then $i \in N^-$. Consider next the case that $\sigma[s] = \emptyset$. If $i \notin N^-$, then because $s \in S^-$,

$$(s,\rho_s)\ P_i\ (s,\rho'_s) = (\sigma'(i),\rho')\ R_i\ (\sigma(i),\rho),$$

contradicting non-wastefulness at (σ,ρ). Thus $\sigma[s]$ being empty implies $i \in N^-$ as well. Taking the contrapositive of these implications yields $i \notin N^- \implies \emptyset \neq \sigma[s] \subseteq \mathbb{E}_s[\rho_s]$. Since $s \in S^-$, $\overline{r}_s \geq \rho'_s > \rho_s$, and so $|\mathbb{E}_s[\rho_s]| \leq 1$. We therefore finally conclude that $i \notin N^- \implies (\mathbb{E}_s[\rho_s] \neq \emptyset \text{ and } \sigma[s] = \mathbb{E}_s[\rho_s])$, yielding our desired either/or statement.

In showing (ii), suppose not, i.e., that $s \xrightarrow{i} t, t \xrightarrow{j} u \in T$, $i \in N^+$ and $j \in N^-$. For $k \in \{i,j\}$, let $(t,r_k)\ I_k\ (\sigma(k),\rho)$ and $(t,r'_k)\ I_k\ (\sigma'(k),\rho')$. Then the contradiction hypothesis yields $r'_i < r_i$ and $r_j < r'_j$. Since $r'_i = \rho'_t < r_i$, individual rationality gives $(t,\rho'_t) \notin e_i$. Since (σ',ρ') is a weak equilibrium, this then implies that $r'_j < \rho'^+_t = r'^+_i$. Since $r_j = \rho_t < r'_j$, we make the symmetric arguments to get $r_i < \rho_t^+ = r_j^+$. Concatenating these inequalities gives

$$r'_i < r_i < \rho_t^+ = r_j^+ \leq r'_j < \rho'^+_t = r'^+_i.$$

It need not be that $r'_i \in \mathbb{P}_t$, but $r'^+_i \in \mathbb{P}_t$, and so we have that all quantities on the above line are in the closed interval $[r'^-_i, r'^+_i]$. Moreover, $\mathbb{P}_t \cap [r'^-_i, r'^+_i] = \{r'^-_i, r'^+_i\}$. This yields a contradiction, however as ρ_t^+ and ρ'^+_t are elements of $\mathbb{P}_t$. *Q.E.D.*

LEMMA 2: *Let (σ,ρ) and (σ',ρ') be two weak equilibria at profile $R \in \tilde{\mathcal{R}}$. Let $s \rightsquigarrow t \subseteq T$ be a path where the initial arc is labeled by an agent in N^+. Then $s \rightsquigarrow t$ is positive-definite.*

PROOF: To simplify notation, agents and objects are relabeled as follows. Let $s = 1$ and $t = n+1$, and all objects in between are labeled according to the order in which they are met in the path. Then label agents so that $\sigma(i) = i$ and $\sigma'(i) = i+1$. To avoid confusion, agent indices are decorated with an underline, so it will actually be written $\sigma(\underline{i}) = i$.

Then because the initial arc in the path is labeled by an agent in N^+, it follows that $(\sigma'(\underline{1}),\rho')P_{\underline{1}}(\sigma(\underline{1}),\rho)$. To arrive at a contradiction, suppose that there is an agent in the path who is made worse off in going from (σ,ρ) and (σ',ρ'). Suppose, without loss of generality (by taking a sub-path), that this agent is $\underline{n}$, i.e., that $(\sigma(\underline{n}),\rho)P_{\underline{n}}(\sigma'(\underline{n}),\rho')$. Assume also, again without loss of generality, that $(\sigma(\underline{i}),\rho)I_{\underline{i}}(\sigma'(\underline{i}),\rho')$ for all agents $\underline{i} \in \{\underline{2},\ldots,\underline{n-1}\}$ on the path. By part (ii) of Lemma 1 we have $n > 2$. Using part (i) of Lemma 1 the foregoing give:

$$\rho'_i \leq \rho_i \text{ or } \underline{i} \in \sigma[i] = \mathbb{E}_i[\rho_i] \text{ for all } i \in \{2,\ldots,n\}. \tag{1}$$

The remaining part of the proof is decomposed into four parts. The first three parts establish some facts related to object 2, object n, and objects $\{2,\ldots,n-1\}$, respectively, at matchings (σ,ρ) and (σ',ρ'). Given these facts, the last part demonstrates that the assumption that agent $\underline{n}$ is worse off implies $R \notin \tilde{\mathcal{R}}$. The latter finding contradicts NCBI and, thus, concludes the proof.

Part (i): object 2. Since $(2,\rho'_2) = (\sigma'(\underline{1}),\rho')$ and $(\sigma'(\underline{1}),\rho')P_{\underline{1}}(\sigma(\underline{1}),\rho)$, it follows that:

$$(2,\rho'_2)P_{\underline{1}}(\sigma(\underline{1}),\rho). \tag{2}$$

Consequently, $(2,\rho_2') \notin e_{\underline{1}}$ by individual rationality of (σ,ρ). Recall next that $(2,\rho_2) = (\sigma(\underline{2}),\rho)$ and $(\sigma(\underline{2}),\rho)I_{\underline{2}}(\sigma'(\underline{2}),\rho')$. Thus, if $\rho_2' < \rho_2$, then by monotonicity:

$$(2,\rho_2')P_{\underline{2}}(2,\rho_2)I_{\underline{2}}(\sigma'(\underline{2}),\rho'), \tag{3}$$

implying that $\rho_2' \in \mathbb{T}_2$. Now, if for all $\underline{j} \in \sigma'[2]$ we have $(2,\rho_2') \in e_{\underline{j}}$, then by $\underline{1} \in \sigma'[2]$, this contradicts the previous conclusion that $(2,\rho_2') \notin e_{\underline{1}}$. Therefore, $(\sigma'(\underline{2}),\rho')P_{\underline{2}}(2,\rho_2'^{+})$ by threshold no-justified-envy. This together with (3) give $(2,\rho_2')P_{\underline{2}}(2,\rho_2)P_{\underline{2}}(2,\rho_2'^{+})$, so $\rho_2' < \rho_2 < \rho_2'^{+}$ by monotonicity. But this contradicts the definition of $\rho_2'^{+}$ as $\rho_2' < \rho_2$ implies $\rho_2'^{+} \le \rho_2$. Hence, $\rho_2' \ge \rho_2$. But then monotonicity and (2) yield:

$$(\sigma(\underline{2}),\rho)R_{\underline{1}}(2,\rho_2')P_{\underline{1}}(\sigma(\underline{1}),\rho),$$

so it must be the case that $\rho_2 \in \mathbb{T}_2$. The following is the main take-away from this part of the proof:

$$\rho_2' \ge \rho_2, \rho_2 \in \mathbb{T}_2 \text{ and } \underline{1} \in \sigma'[2] \backslash \mathbb{E}_2[\rho_s']. \tag{4}$$

Part (ii): object n. Recall the assumption that $(\sigma(\underline{n}),\rho)P_{\underline{n}}(\sigma'(\underline{n}),\rho')$, so, by individual rationality, agent $\underline{n}$ is not entitled to bundle (n,ρ_n), i.e., $(n,\rho_n) \notin e_{\underline{n}}$. From (1), it thus follows that $\rho_n' \le \rho_n$. From monotonicity, and the construction that $(\sigma'(\underline{n-1}),\rho') = (n,\rho_n')$ and $(n,\rho) = (\sigma(\underline{n}),\rho)$, it follows:

$$(\sigma'(\underline{n-1}),\rho')R_{\underline{n}}(\sigma(\underline{n}),\rho)P_{\underline{n}}(\sigma'(\underline{n}),\rho').$$

This condition implies that the threshold no-justified-envy holds for agent $\underline{n}$ for object n at matching (σ',ρ'), i.e., $\rho_n' \in \mathbb{T}_n$. Suppose now that $\rho_n' < \rho_n$. If both $\sigma'[n] \neq \mathbb{E}_n[\rho_n']$ and $\rho_n' \in \partial\mathbb{P}_n$, then:

$$(\sigma'(\underline{n}),\rho')P_{\underline{n}}(\sigma(\underline{n}),\rho_n'^{+})R_{\underline{n}}(\sigma(\underline{n}),\rho),$$

contradicting the assumption that $(\sigma(\underline{n}),\rho)P_{\underline{n}}(\sigma'(\underline{n}),\rho')$. Thus, it must be the case that $\sigma'[n] = \mathbb{E}_n[\rho_n']$ or $\rho_n' \notin \partial\mathbb{P}_n$. Now, by $\rho_n' \in \mathbb{T}_n$, the latter implies $\rho_n' \in \mathbb{E}_n$ and, consequently, $\underline{n-1} \succ_{(n,\rho_n')} \underline{n}$. Recalling that for $\rho_n' \in \mathbb{E}_n \backslash \partial\mathbb{P}_n$, $\underline{j} \sim_{(n,\rho_n')} \underline{k}$ unless either $\underline{j}$ or $\underline{k}$ is entitled to (n,ρ_n'), it is concluded that $\sigma'[n] = \mathbb{E}_n[\rho_n']$. The following is the main take-away from this part of the proof:

$$\rho_n' \in \mathbb{T}_n \text{ and } \rho_n' \le \rho_n, \tag{5}$$

$$\rho_n' < \rho_n \text{ implies } \sigma'[n] = \mathbb{E}_n[\rho_n']. \tag{6}$$

Part (iii): objects $\{2,\dots,n-1\}$. Take any agent $\underline{i} \in \{\underline{2},\dots,\underline{n-1}\}$ and recall that $(i,\rho_i) = (\sigma(\underline{i}),\rho)$ by construction. If $\rho_i' < \rho_i$, then the following holds by monotonicity and the assumption that $(\sigma(\underline{i}),\rho)I_{\underline{i}}(\sigma'(\underline{i}),\rho')$:

$$(i,\rho_i')R_{\underline{i}}(i,\rho_i'^{+})R_{\underline{i}}(\sigma(\underline{i}),\rho)I_{\underline{i}}(\sigma'(\underline{i}),\rho'),$$

where at least one of the weak relations is strict by construction. Since $(i,\rho_i'^{+})R_{\underline{i}}(\sigma'(\underline{i}),\rho')$, it follows from threshold no-justified-envy that $\rho_i' \in \mathbb{E}_i$ and $\sigma'[i] = \mathbb{E}_i[\rho_i']$. Using the same arguments as in part (ii) of this proof and the conclusion $(i,\rho_i')P_{\underline{i}}(\sigma'(\underline{i}),\rho')$ from the above, we conclude that $\underline{i-1} \in \sigma'[i]$, $\underline{i-1} \succ_{(i,\rho_i')} \underline{i}$ and, consequently, that agent $\underline{i-1}$ is entitled to bundle (i,ρ'_i) and $\sigma'[i]=\mathbb{E}_i[\rho'_i]$. These conclusions together with (1) give the following main take-away from this part of the proof:

$$\rho'_i>\rho_i \text{ implies } \underline{i}\in\sigma[i]=\mathbb{E}_i[\rho_i] \text{ for all } \underline{i}\in\{\underline{2},\dots,\underline{n-1}\}, \tag{7}$$

$$\rho'_i<\rho_i \text{ implies } \underline{i-1}\in\sigma'[i]=\mathbb{E}_i[\rho'_i] \text{ for all } \underline{i}\in\{\underline{2},\dots,\underline{n-1}\}. \tag{8}$$

Part (iv): $R\notin\tilde{\mathcal{R}}$. Condition (4) established that $\rho'_2\geq\rho_2$ and $\rho_2\in\mathbb{T}_2$. Suppose now that there is some object $v\in\{3,\dots,n\}$ with $\rho'_v<\rho_v$ and choose v minimal in this property, so each object i with $2\leq i<v$ has $\rho'_i\geq\rho_i$. Let $u\rightsquigarrow v$ be a subpath such that each i with $u<i<v$ has $\rho'_i=\rho_i$ and that either (a) $u=2$ or (b) $\rho'_u>\rho_u$. Such a path always exists given our assumptions.

Consider first case (a). By (8), since $\rho'_v<\rho_v$, $(v,\rho'_v)\in e_{\underline{v-1}}$. This, $u=2$, and the assumptions that $(\sigma(\underline{i}),\rho)I_{\underline{i}}(\sigma'(\underline{i}),\rho')$ for all agents $\underline{i}\in\{\underline{2},\dots,\underline{n-1}\}$ and $\rho'_i=\rho_i$ for $i=u+1,\dots,v-1$ give:

$$\begin{aligned}
&(2,\rho_2)I_{\underline{2}}(3,\rho'_3),\\
&(j,\rho'_j)I_{\underline{j}}(j+1,\rho'_{j+1}) \text{ for all } j=3,\dots,v-2,\\
&(v-1,\rho'_{v-1})I_{\underline{v-1}}(v,\rho'_v)\in e_{\underline{v-1}}.
\end{aligned}$$

But then $R\notin\tilde{\mathcal{R}}$ since $\rho_2\in\mathbb{T}_2$ by (4).

Consider now case (b). When $\rho'_u>\rho_u$, it follows from (7) that $(u,\rho_u)\in e_{\underline{u}}$ and by using similar arguments as in the above, the following indifference chain can be obtained:

$$\begin{aligned}
&e_{\underline{u}}\ni(u,\rho_u)I_{\underline{u}}(u+1,\rho_{u+1}),\\
&(j,\rho_j)I_{\underline{j}}(j+1,\rho_{j+1}) \text{ for all } j=u+1,\dots,v-2,\\
&(v-1,\rho_{v-1})I_{\underline{v-1}}(v,\rho'_v)\in e_{\underline{v-1}},
\end{aligned}$$

which contradicts that $R\in\tilde{\mathcal{R}}$.

From cases (a) and (b), it is concluded that each $v\in\{2,\dots,n\}$ has $\rho'_v\geq\rho_v$. The above insights together with conditions (4) and (5) can therefore be summarized as:

$$\rho_2\in\mathbb{T}_2,\ \rho'_n\in\mathbb{T}_n,\text{ and } \rho'_i\geq\rho_i \text{ for } 2\leq i<n. \tag{9}$$

Let now $i^*\in\{2,\dots,n-1\}$ be the greatest integer such that $\rho'_{i^*}>\rho_{i^*}$ or let $i^*=2$. Then from (7), it follows that $\rho_{i^*}\in\mathbb{T}_{i^*}$ and $(i^*,\rho_{i^*})\in e_{\underline{i}^*}$. Now, identical identical arguments as in the above can be used to establish an indifference chain between bundles (i^*,ρ_{i^*}) and (n,ρ'_n). But this contradicts that $R\in\tilde{\mathcal{R}}$. *Q.E.D.*

Recall that T is a labeled and directed transfer graph on the vertices S where $s\xrightarrow{i}t\in T$ if $\sigma(i)=s$, $\sigma'(i)=t$ and $s\neq t$. Given an arc $s\xrightarrow{i}t\in T$, let:

$$D_{s\xrightarrow{i}t}=\{s\xrightarrow{i}t\}\cup\{u\xrightarrow{l}v\in T: u=t \text{ or there exists a simple path } t\rightsquigarrow u\subseteq T\}.$$

Note that by Lemma 2, if $i\in N^+$ then $D_{s\xrightarrow{i}t}$ contains no N^--labeled arcs. Let D be the union of all $D_{s\xrightarrow{i}t}$ induced by $s\xrightarrow{i}t\in T$ with $i\in N^+$, i.e.:

$$D=\bigcup_{i\in N^+:\sigma(i)\neq\sigma'(i)} D_{\sigma(i)\xrightarrow{i}\sigma'(i)}.$$

This construction is used to obtain a new assignment σ'' by "executing" all arcs in D as follows: for all $j \in N$,

$$\sigma''(j) = \begin{cases} \sigma'(j) & \text{if } \sigma(j) \xrightarrow{j} \sigma'(j) \in D, \\ \sigma(j) & \text{otherwise.} \end{cases}$$

Thus, assignment σ'' is the result of executing all arcs in D. Let D_S be the set of objects that are terminal vertices in D:

$$D_S = \{s \in S : \text{there exists } t \to s \in D\}.$$

In particular, if $D_{s \xrightarrow{i} t}$ is one of the constituent sets of D, it *need not* be the case that $s \in D_S$. Thus, one notes that by executing the arcs in D, it is possible that, for some $s \in S$, we have $\sigma[s] \neq \emptyset \neq \sigma'[s]$, but $\sigma''[s] = \emptyset$. When this occurs, then all outgoing arcs from s are labeled with agents belonging N^+ while all ingoing arcs of s are labeled with agents belonging to $N^- \cup N^=$ (and no arc labeled with an agent belonging to an agent in N^+ is connected via a simple path to any $u \to s$). Now, if $\sigma''[s] = \emptyset$, then we set $\rho''_s = \max\{\rho_s, \rho'_s\}$. For s and i, let $t_s(i)$ be such that both $(\sigma(i), \rho) R_i (s, t_s(i))$ and $(\sigma'(i), \rho') R_i (s, t_s(i))$, and let $\overline{t}_s = \max_{i \in N} t_i(s)$ (where we choose $\overline{t}_s \in \mathbb{P}_s$ if for all $i \in N$ we have $t_s(i) \leq \overline{r}_s$).

Consider now matching (σ'', ρ'') where σ'' is defined as in the above and ρ'' is given by:

$$\rho''_s = \begin{cases} \max\{\rho_s, \rho'_s\} & \text{if } \sigma''_s[s] = \emptyset, \\ \rho'_s & \text{if } s \in D_S \text{ or there exists } i \in N^+ \text{ such that } \sigma(i) = \sigma'(i) = s, \\ \rho_s & \text{otherwise.} \end{cases}$$

Note that if $\sigma''[s] = \emptyset$, then we cannot have both $\sigma[s] = \mathbb{E}_s[\rho_s] \neq \emptyset$ and $\sigma'[s] = \mathbb{E}_s[\rho'_s] \neq \emptyset$ as otherwise, by unit entitlement at non-maximal prices, we must have $\rho_s = \rho'_s$ and $\sigma[s] = \sigma'[s]$ where the latter implies $\sigma''[s] = \sigma[s] = \sigma'[s]$, a contradiction to $\sigma''[s] = \emptyset$.

A.3. *Proof of Theorem 1*

Theorem 1 will be proved through a number of lemmas, and it will build on the notation and results previously provided in this appendix. More specifically, it will first be established that the above defined matching (σ'', ρ'') is feasible (Lemma 3). It will then be demonstrated that $(\sigma''(i), \rho'') \in \max\{(\sigma(i), \rho), (\sigma'(i), \rho')\}$ for all $i \in N$ (Lemma 4). We will then show that rationing can be used whenever an object is wasted (Lemma 5). These insights are then applied to show that by eliminating waste of $(\sigma''(i), \rho'')$ via "vacancy chain dynamics," we obtain a weak equilibrium (i.e., an equilibrium satisfying individual rationality, non-wastefulness, interior no-envy, and threshold no-justified-envy), which all agents weakly prefer to both (σ, ρ) and (σ', ρ') (Lemma 6). This result is key in proving Theorem 1.

LEMMA 3: *Matching (σ'', ρ'') is feasible.*

PROOF: If D is incident to vertex s, then there is $t \xrightarrow{i} s \in D$. Thus, $t \xrightarrow{i} s$ and any $s \xrightarrow{l} u \in T$ are executed in going from σ to σ'', and it follows that, for each $s \in D_S$, $|\sigma''[s]| \leq |\sigma'[s]|$, i.e., every outgoing arc is executed but not necessarily every ingoing arc. Then since $\rho''_s = \rho'_s$ by construction and $|\sigma'[s]| \leq c_s(\rho'_s)$ as the matching (σ', ρ') is a weak equilibrium, it follows that $|\sigma''[s]| \leq c_s(\rho''_s)$.

It remains to check feasibility for $s \notin D_S$. If $\sigma''[s] = \emptyset$, then there is nothing to check. Thus, let $\sigma''[s] \neq \emptyset$. Note first that under σ'' object s has been assigned to the same agent(s) as under σ while possibly some outgoing arc has been executed. Hence, $|\sigma''[s]| \leq |\sigma[s]|$. Note next that either $\rho''_s = \rho_s$ or $\rho''_s = \rho'_s \neq \rho_s$. In the former case, feasibility follows since $|\sigma[s]| \leq c_s(\rho_s)$ as (σ, ρ) is a weak equilibrium and, consequently, $|\sigma''[s]| \leq c_s(\rho''_s)$. Consider instead the latter case where, by construction of ρ''_s, there is an agent $i \in N^+$ with $\sigma(i) = \sigma'(i) = s$. But then $\rho'_s < \rho_s$ otherwise it cannot be the case that $i \in N^+$. Suppose now that there is $s \xrightarrow{j} t \in T$. If $(\sigma(j), \rho) R_j (\sigma'(j), \rho')$, i.e., $j \notin N^+$, then by monotonicity and since $\sigma(j) = s$, $(s, \rho'^+_s) R_j (\sigma(j), \rho) R_j (\sigma'(j), \rho')$ and $(s, \rho'_s) P_j (\sigma'(j), \rho')$. Thus, by threshold no-justified-envy of (σ', ρ'), $\sigma'[s] = \mathbb{E}_s[\rho'_s]$. By $i \in \sigma'[s]$, we obtain $(s, \rho'_s) \in e_i$. However, $i \in N^+$ by assumption and then because $(s, \rho'_s) = (\sigma'(i), \rho')$, it follows that $(\sigma'(i), \rho') P_i (\sigma(i), \rho)$ which contradicts individual rationality for agent i at (σ, ρ). Hence, $j \in N^+$ and, more generally, every arc outgoing from s is labeled by an N^+-agent and so it is executed in going from σ to σ''. Using the same arguments as in the above, it again follows that $|\sigma''[s]| \leq c_s(\rho''_s)$. *Q.E.D.*

LEMMA 4: *The following holds for all $i \in N$:*

$$(\sigma''(i), \rho'') \in \max_{R_i} \{(\sigma(i), \rho), (\sigma'(i), \rho')\}. \tag{10}$$

PROOF: Let N' be the set of agents either in N^+ or labeling an arc in D. From Lemma 2, it follows that $N' \subseteq N^+ \cup N^=$.

Consider first any agent $i \in N'$. By construction of the matching (σ'', ρ''), i.e., $(\sigma''(i), \rho'') = (\sigma'(i), \rho')$ for all $i \in N'$, it follows that:

$$(\sigma''(i), \rho'') R_i (\sigma(i), \rho) \text{ for all } i \in N'. \tag{11}$$

Now, if there is an object $t \in D_S \cap S^-$, then by construction, $s \xrightarrow{i} t \in D$ for some $s \in S \backslash \{t\}$ and $i \in N$. By construction, $i \in N'$ and $\sigma''(i) = \sigma'(i) = t \neq \sigma(i)$. Since $N' \subseteq N^+ \cup N^=$ and $t \in S^-$, by Lemma 1, we have $\sigma[t] = \mathbb{E}_t[\rho_t]$ and $(t, \rho_t) \in e_j$ for all $j \in \sigma[t]$. Since $\rho'_t > \rho_t$, any $j \in \sigma[t]$ must leave object t by individual rationality, so $t \xrightarrow{j} u \in T$ for some $u \in S$. Then by definition of D, $t \xrightarrow{j} u \in D$. Consequently, $j \in N'$. In sum:

$$\sigma[D_S \cap S^-] \subseteq N'. \tag{12}$$

Consider now agent $i \notin N'$. Then, $\sigma''(i) = \sigma(i)$ by construction of the matching (σ'', ρ'').

First, suppose that $\sigma(i) \in D_S$. Then agent i does not label any arc in T. Hence, $\sigma'(i) = \sigma(i)$, and it follows that $(\sigma''(i), \rho'') = (\sigma'(i), \rho')$. From $i \notin N'$ and (12), it follows that $\sigma(i) \notin S^-$ and $\rho''_{\sigma(i)} = \rho'_{\sigma(i)} \leq \rho_{\sigma(i)}$. Then, by monotonicity:

$$(\sigma''(i), \rho'') R_i (\sigma(i), \rho) \text{ for all } i \in \sigma[D_S] \setminus N'. \tag{13}$$

Second, suppose that $\sigma(i) \notin D_S$ and let $s = \sigma(i)$. Suppose first that $\rho''_s = \rho'_s \neq \rho_s$. Then there is an agent $j \in N^+$ for whom $\sigma(j) = \sigma'(j) = s$. This implies via monotonicity that $\rho'_s < \rho_s$, and then by individual rationality that $(s, \rho'_s) \notin e_j$. This implies that $\sigma'[s] \neq \mathbb{E}_s[\rho_s]$. Since $\sigma(i) = \sigma'(j) = s$ and $\rho'_s < \rho_s$, it follows that if $(\sigma(i), \rho) R_i (\sigma'(i), \rho')$, then $(\sigma'(j), \rho') P_i (\sigma'(i), \rho')$ and $(s, \rho'^+_s) R_i (\sigma'(i), \rho')$. The latter contradicts threshold no-justified-envy as $\sigma'[s] \neq \mathbb{E}_s[\rho_s]$, so it must be the case that $i \in N^+ \subseteq N'$. But this case has already been covered in the above.

Second, suppose that $\rho''_s = \rho_s$. In this case, $(\sigma''(i), \rho'') = (\sigma(i), \rho)$. Then since $i \notin N'$, i.e., $i \notin N^+$, it follows that:

$$(\sigma''(i), \rho'') R_i (\sigma'(i), \rho') \text{ for all } i \in \sigma[S \setminus D_S] \setminus N'. \tag{14}$$

Now, since agents are consuming their bundle under matching (σ,ρ) or matching (σ',ρ') at matching $(\sigma''(i),\rho'')$, and since:

$$(\sigma[D_S]\setminus N')\cup(\sigma[S\setminus D_S]\setminus N')=(\sigma[D_S]\cup\sigma[S\setminus D_S])\setminus N'=N\setminus N',$$

conditions (11), (13) and (14) yield (10). *Q.E.D.*

Recall the *vacancy chain dynamics* defined in the proof of Theorem 5. Operate this procedure on (σ'',ρ'') to arrive at $(\overline{\sigma},\overline{\rho})$. As shown there, if we can prove that (σ'',ρ'') violates only the non-wastefulness condition of weak equilibrium, then we will have shown that $(\overline{\sigma},\overline{\rho})$ is a weak equilibrium. The following lemma rules out one class of violations.

LEMMA 5: *If object s is wasted under (σ'',ρ''), then $\rho''_s\in\partial\mathbb{P}_s$ and $(\sigma''(\ell),\rho'')P_\ell(s,\rho''^{+}_s)$ for all $\ell\in N$.*

PROOF: Suppose that for some $s\in S$ and $i\in N$, we have $(s,\rho''_s)P_i(\sigma''(i),\rho'')$ and $|\sigma''[s]|<c_s(\rho''_s)$. By construction, this implies $\rho''_s<\bar{t}_s$. Suppose to the contrary that $\rho''_s\notin\partial\mathbb{P}_s$. We consider three different cases: $\rho_s=\rho'_s$, $\rho_s<\rho'_s$ and $\rho_s>\rho'_s$.

For the first case, let $\rho_s=\rho'_s$. By non-wastefulness of (σ,ρ) and (σ',ρ'), we have $|\sigma[s]|=c_s(\rho_s)=c_s(\rho'_s)=|\sigma'[s]|$. If $\sigma[s]=\sigma'[s]$, then s is not incident to any T arcs, and by construction, $\sigma''[s]=\sigma[s]$, $\rho''_s=\rho_s$ and $|\sigma''[s]|=c_s(\rho''_s)$, a contradiction to the fact that s is wasted. Thus, $\sigma[s]\neq\sigma'[s]$ and since $|\mathbb{E}_s[\rho_s]|\le c_s(\rho_s)$, we obtain either $\sigma[s]\neq\mathbb{E}_s[\rho_s]$ or $\sigma'[s]\neq\mathbb{E}_s[\rho_s]$. But now by construction, $\rho''_s=\max\{\rho_s,\rho'_s\}$ if $\sigma''[s]=\emptyset$, and $\rho''_s=\rho_s=\rho'_s$ if $\sigma''[s]\neq\emptyset$. Hence, we obtain a contradiction to the interior no-envy of either (σ,ρ) or (σ',ρ'), depending on whether $\sigma[s]\neq\mathbb{E}_s[\rho_s]$ or $\sigma'[s]\neq\mathbb{E}_s[\rho_s]$ holds. Hence, $\rho''_s=\rho_s=\rho'_s\in\partial\mathbb{P}_s$, and by threshold no-justified-envy of (σ,ρ) and (σ',ρ'), $(\sigma''(\ell),\rho'')P_\ell(s,\rho''^{+}_s)$ for all $\ell\in N$.

For the second case, let $\rho_s<\rho'_s$. Note that, by unit entitlement at non-maximal prices, we cannot have both $\sigma[s]=\mathbb{E}_s[\rho_s]$ and $\sigma'[s]=\mathbb{E}_s[\rho'_s]$. Thus, either $\sigma[s]\neq\mathbb{E}_s[\rho_s]$ or $\sigma'[s]\neq\mathbb{E}_s[\rho'_s]$. By construction, $\rho_s\le\rho''_s$ and so $(s,\rho_s)P_i(\sigma''(i),\rho'')R_i(\sigma(i),\rho)$. By non-wastefulness of (σ,ρ), $|\sigma[s]|=c_s(\rho_s)$. Thus if $\sigma[s]\neq\mathbb{E}_s[\rho_s]$, then since (σ,ρ) satisfies interior no-envy, we must then have $\rho_s\in\partial\mathbb{P}_s$. Now, we must also have $\rho''_s=\rho_s$ as otherwise $\rho^+_s\le\rho''_s$, giving $(s,\rho^+_s)P_i(\sigma''(i),\rho'')R_i(\sigma(i),\rho)$ and violating threshold no-justified-envy at (σ,ρ). By construction, this implies $\sigma''[s]\neq\emptyset$ as otherwise $\rho''_s=\max\{\rho_s,\rho'_s\}\neq\rho_s$. As (σ,ρ) satisfies threshold no-justified-envy and $\rho''_s=\rho_s$, we then obtain $(\sigma''(\ell),\rho'')R_\ell(\sigma(\ell),\rho)P_\ell(s,\rho''^{+}_s)$ for all $\ell\in N$. Alternatively, if $\sigma[s]=\mathbb{E}_s[\rho_s]$, then $\rho_s<\rho'_s$ and unit entitlement at non-maximal prices give $|\sigma[s]|=c_s(\rho_s)=1$ and $\rho'_s\notin\mathbb{E}_s$. If it were the case that $\rho''_s=\rho_s$, then by definition of ρ'', $\sigma''[s]=\sigma[s]$, contradicting that s is wasted. Therefore, $\rho''_s=\rho'_s$, and since $\rho'_s\notin\mathbb{E}_s$, by invoking the threshold no-justified-envy of (σ',ρ') and Lemma 4, we obtain $\rho''_s=\rho'_s\in\partial\mathbb{P}_s$ and $(\sigma''(\ell),\rho'')P_\ell(s,\rho''^{+}_s)$ for all $\ell\in N$.

For the third case, let $\rho_s>\rho'_s$, and we have a nearly symmetric argument. Note that by unit entitlement at non-maximal prices we cannot have both $\sigma[s]=\mathbb{E}_s[\rho_s]$ and $\sigma'[s]=\mathbb{E}_s[\rho'_s]$. Thus, $\sigma[s]\neq\mathbb{E}_s[\rho_s]$ or $\sigma'[s]\neq\mathbb{E}_s[\rho'_s]$. By construction, $\rho'_s\le\rho''_s$ and $(s,\rho'_s)P_i(\sigma''(i),\rho'')R_i(\sigma'(i),\rho')$. By non-wastefulness of (σ',ρ'), $|\sigma'[s]|=c_s(\rho'_s)$. Thus if $\sigma'[s]\neq\mathbb{E}_s[\rho'_s]$, then since (σ',ρ') satisfies interior no-envy, we must then have $\rho'_s\in\partial\mathbb{P}_s$. But then $\rho''_s=\rho'_s$ as otherwise $\rho'^{+}_s\le\rho''_s$ and threshold no-justified-envy is violated at (σ',ρ'). By construction, this implies $\sigma''[s]\neq\emptyset$ as otherwise $\rho''_s=\max\{\rho_s,\rho'_s\}\neq\rho'_s$. As (σ',ρ') satisfies threshold no-justified-envy, we obtain $(\sigma''(l),\rho'')R_l(\sigma'(l),\rho')P_l(s,\rho''^{+}_s)$ for all $l\in N$. Alternatively, if $\sigma'[s]=\mathbb{E}_s[\rho'_s]$, then $\rho'_s<\rho_s$ and unit entitlement at non-maximal prices give $|\mathbb{E}_s[\rho'_s]|=c_s(\rho'_s)=1$ and $\rho_s\notin\mathbb{E}_s$. If $\rho''_s=\rho_s$, then again we invoke the threshold no-justified-envy of (σ,ρ) and Lemma 4 to obtain obtain $\rho''_s=\rho_s\in\partial\mathbb{P}_s$ and $(\sigma''(\ell),\rho'')P_\ell(s,\rho''^{+}_s)$ for all $\ell\in N$. Therefore $\rho''_s=\rho'_s$. By individual rationality of (σ,ρ) we have $\sigma'[s]\cap\sigma[s]=\emptyset$; in particular there is no agent $j\in N^+$

with $\sigma(j) = \sigma'(j) = s$. Thus, by definition of ρ''_s, $s \in D_S$ which further implies that $\sigma''[s]$ is not empty. However, as $|\mathbb{E}_s[\rho'_s]| = c_s(\rho'_s) = 1$, plugging ρ''_s for ρ'_s contradicts that s is wasted. *Q.E.D.*

The following lemma establishes that the set of weak equilibria form a directed set in the welfare ordering. It subsequently remains only to make some limit arguments to complete the proof of Theorem 1.

LEMMA 6: *Matching $(\overline{\sigma}, \overline{\rho})$ is a weak equilibrium which all agents weakly prefer to both (σ, ρ) and (σ', ρ').*

PROOF: Given the work done in the proof of Theorem 5, we need only show that (σ'', ρ'') satisfies all the requirements of weak equilibrium except non-wastefulness.

In showing individual rationality of (σ'', ρ''), as both (σ, ρ) and (σ', ρ') are individually rational, this follows from Lemma 4.

In showing interior no-envy, suppose that for some $s \in S$ and $i \in N$, we have both ${\rho''}_s \notin \mathbb{T}_s$ and $(s, {\rho''}_s) P_i(\sigma''(i), \rho'')$. But then $\rho''_s = \rho'_s$ or $\rho''_s = \rho_s$ and by Lemma 4, interior no-envy is violated at (σ, ρ) or (σ', ρ'), a contradiction.

It thus only remains to show threshold no-justified-envy. Suppose that $(s, {\rho''}_s) P_i(\sigma''(i), \rho'')$ for some $i \in N$ and $s \in S$ such that ${\rho''}_s \in \mathbb{T}_s$.

If $\sigma''[s] = \emptyset$, then s is wasted under (σ'', ρ''), as i desires it, and so by Lemma 5 we have that $(\sigma''(i), \rho'') P_i(s, \rho''^{+}_s)$. As there is no $j \in \sigma[s]$ for i to envy, and as i was chosen arbitrarily among agents desiring s, threshold no-justified-envy is satisfied at s. Assume henceforth that $\sigma''[s] \neq \emptyset$.

Case i: $\rho''_s = \rho'_s \neq \rho_s$. The first subcase to check is when $s \in D_s$. Then $\sigma''[s] \subseteq \sigma'[s]$, recalling that all outgoing arcs from s in T are executed. By the case assumption and Lemma 4, for each $j \in \sigma''[s]$, we have

$$(\sigma'(j), \rho') = (\sigma''(j), \rho'') = (s, \rho''_s)\ P_i\ (\sigma''(i), \rho'')\ R_i\ (\sigma'(i), \rho').$$

Thus as (σ', ρ') is a weak-equilibrium, $j \succ_{(s,\rho'_s)} i$, and again recalling the case assumption $j \succ_{(s,\rho''_s)} i$. Suppose now that $(s, \rho''^{+}_s)\ R_i\ (\sigma''(i), \rho'')$. Invoking the case assumption and Lemma 4 again yields $(s, \rho'^{+}_s)\ R_i\ (\sigma'(i), \rho')$. As $\rho''_s \in \mathbb{T}_s$, we also have $\rho'_s < \rho'^{+}_s \leq \overline{r}_s$, and so $\sigma'[s] = \mathbb{E}_s[\rho'_s]$ (as (σ', ρ') is a weak equilibrium) is a singleton. Then $\emptyset \neq \sigma''[s] \subseteq \sigma'[s]$ implies that these are in fact equal. Thus, $(s, \rho''^{+}_s)\ R_i\ (\sigma''(i), \rho'')$ implies that $\sigma''[s] = \mathbb{E}_s[\rho''_s]$. As i was arbitrary among agents desiring s, if there is one satisfying this last condition, then we have again verified threshold no-justified-envy at s, and if there are none, then of course we have $(\sigma''(i), \rho'')\ P_i\ (s, \rho''^{+}_s)$ for all i desiring s, and the required priority relations have already been verified.

Thus it remains to consider the case that there is $j \in N$ with $\sigma(j) = \sigma'(j) = s$ and $j \in N^+$. Clearly $\rho'_s < \rho_s$, and by individual rationality of (σ, ρ), $(s, \rho'_s) \notin e_j$. As above, we invoke the case assumption and Lemma 4 to get $(\sigma'(j), \rho')\ P_i\ (\sigma'(i), \rho')$, except now, since j is not entitled to (s, ρ'_s), we can add $(\sigma'(i), \rho')\ P_i\ (s, \rho'^{+}_s)$ to the list. Since $\rho'_s = \rho''_s$, $\rho'^{+}_s = \rho''^{+}_s$, and we again invoke Lemma 4 to get $(\sigma''(i), \rho'')\ P_i\ (s, \rho''^{+}_s)$.

Case ii: $\rho''_s = \rho_s$. In this case we *cannot* have $j \in N$ with $\sigma(j) = \sigma'(j) = s$ and $j \in N^+$. Thus, either $s \in D_S$ and $\sigma''[s] \subseteq \sigma'[s]$, or else $\sigma''[s] = \sigma[s]$. In the former case, ρ''_s was set at ρ'_s, and so we have $\rho'_s = \rho_s$. In both cases, there is $(\hat{\sigma}, \hat{\rho}) \in \{(\sigma, \rho), (\sigma', \rho')\}$ such that, for each $j \in \sigma''[s]$, $(\sigma''(j), \rho'') = (\hat{\sigma}(j), \hat{\rho})$. Since both $\{(\sigma, \rho), (\sigma', \rho')\}$ are weak equilibria, we deduce via the same arguments as the previous case that $j \succ_{(s,\rho''_s)} i$. Again we must suppose that $(s, \rho''^{+}_s)\ R_i\ (\sigma''(i), \rho'')$. As with the previous case, we find that $\hat{\sigma}[s] = \mathbb{E}_s[\hat{\rho}_s]$ is a singleton and so $\sigma''[s] = \mathbb{E}_s[\rho''_s]$ via the non-emptiness of $\sigma''[s]$ and its inclusion $\hat{\sigma}[s]$. *Q.E.D.*

The remaining part of this section concludes the proof of Theorem 1, i.e., that there exists an agent-optimal weak equilibrium. Without loss of generality, let $\underline{r}_s = 0$ for all $s \in S$ (recall that prices are bounded from below, by assumption). Recall that $\mathcal{W}$ denotes the set of weak equilibria (as proved in Appendix A.1, this set is always non-empty).

For any agent i, fix a representation u_i of R_i. Then, as prices are bounded from below by zero, agent i's utility is bounded from above by $\max_{s\in S} u_i(s,0)$. Thus, $\sup_{(\sigma,\rho)\in\mathcal{W}} u_i(\sigma(i),\rho) \equiv \overline{u}_i$ exists. Let $(\sigma^k,\rho^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{W}$ such that $\lim_{k\to\infty} u_i(\sigma^k(i),\rho^k) = \overline{u}_i$. Now, let $j \neq i$, $\sup_{(\sigma,\rho)\in\mathcal{W}} u_j(\sigma(j),\rho) \equiv \overline{u}_j$, and $(\hat{\sigma}^k,\hat{\rho}^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{W}$ such that $\lim_{k\to\infty} u_j(\hat{\sigma}^k(j),\hat{\rho}^k) = \overline{u}_j$. But then by the above construction and Lemma 6, for any k, for (σ^k,ρ^k) and $(\hat{\sigma}^k,\hat{\rho}^k)$, we find a weak equilibrium $(\overline{\sigma}^k,\overline{\rho}^k)$ which all agents weakly prefer to both (σ^k,ρ^k) and $(\hat{\sigma}^k,\hat{\rho}^k)$.

By iterating, hence, without loss of generality, let $(\sigma^k,\rho^k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{W}$ such that $\lim_{k\to\infty} u_i(\sigma^k(i),\rho^k) = \overline{u}_i$ for all $i \in N$. Again, by the same construction as above and Lemma 6, for any $k > 1$, for $(\sigma^{k-1},\rho^{k-1})$ and (σ^k,ρ^k), we find a weak equilibrium $(\overline{\sigma}^k,\overline{\rho}^k)$ which all agents weakly prefer to both $(\sigma^{k-1},\rho^{k-1})$ and (σ^k,ρ^k). But now again by the above argument and Lemma 6, for $(\overline{\sigma}^k,\overline{\rho}^k)$ and $(\sigma^{k+1},\rho^{k+1})$ we find a weak equilibrium $(\overline{\sigma}^{k+1},\overline{\rho}^{k+1})$ which all agents weakly prefer to both $(\overline{\sigma}^k,\overline{\rho}^k)$ and $(\sigma^{k+1},\rho^{k+1})$. Hence, by iterating this argument, without loss of generality, we may suppose that $(\sigma^k,\rho^k)_{k\in\mathbb{N}}$ is a monotonically increasing sequence in terms of agents' utilities, i.e., for any k, we have $u_i(\sigma^k(i),\rho^k) \leq u_i(\sigma^{k+1}(i),\rho^{k+1})$ for all $i \in N$.

As the set of assignments is finite, without loss of generality, again we may take a sequence where the assignment remains identical, i.e., for all k, $\sigma^k = \sigma^{k+1} \equiv \sigma$. But then as the sequence is monotonically increasing in terms of agents' utilities, we obtain for any k and $i \in N$, $\rho^k_{\sigma(i)} \geq \rho^{k+1}_{\sigma(i)}$ and $(\rho^k_{\sigma(i)})_{k\in\mathbb{N}}$ is a weakly decreasing sequence. But then as price spaces are closed and bounded from below by zero, $(\rho^k_{\sigma(i)})_{k\in\mathbb{N}}$ must converge, say to $\lim_{k\to\infty} \rho^k_{\sigma(i)} \equiv \rho_{\sigma(i)}$.

Now, by construction and as preferences are continuous, we have $u_i(\sigma(i),\rho_{\sigma(i)}) = \overline{u}_i$ for all $i \in N$. Again, for unassigned objects, we set prices as high as possible, i.e., for $s \in S$ with $\sigma[s] = \emptyset$, let $\rho_s = \min\{\overline{r}_s, \overline{t}_s\}$ where if $\overline{r}_s = \infty$ or $(\sigma(i),\rho) R_i (s,\overline{r}_s)$ for all $i \in N$, then we choose $\overline{t}_s \in \mathbb{P}_s$ minimal such that $(\sigma(i),\rho) R_i (s,\overline{t}_s)$ for all $i \in N$.

We show that (σ,ρ) is a weak equilibrium, i.e., (σ,ρ) is an agent-optimal weak equilibrium, which is the desired conclusion.

In showing individual rationality, if $(s,r) P_i (\sigma(i),\rho)$ for some $i \in N$ and $(s,r) \in e_i$, then by continuity of preferences, the assignment remains identical and $(\rho^k_{\sigma(i)})_{k\in\mathbb{N}}$ converges to $\rho_{\sigma(i)}$, we have for k large enough, $(s,r) P_i (\sigma(i),\rho^k) = (\sigma^k(i),\rho^k)$, which contradicts individual rationality of (σ^k,ρ^k).

In showing non-wastefulness, suppose that for some $s \in S$ and $i \in N$, $(s,\rho_s) P_i (\sigma(i),\rho)$ and $c_s(\rho_s) < |\sigma[s]|$. If $\sigma[s] = \emptyset$, then we must have $\rho_s = \overline{r}_s$. Hence, as the assignment remains identical, $\sigma^k[s] = \emptyset$, and by feasibility, $\rho^k_s \leq \overline{r}_s$, which implies for k large enough that:

$$(s,\rho^k_s) R_i (s,\overline{r}_s) = (s,\rho_s) P_i (\sigma(i),\rho) R_i (\sigma^k(i),\rho^k),$$

which is a contradiction to non-wastefulness of (σ^k,ρ^k). If $\sigma[s] \neq \emptyset$, then as the assignment remains identical, for all k, $\sigma^k[s] \neq \emptyset$ and $(\rho^k_s)_{k\in\mathbb{N}}$ converges to ρ_s. As c_s is right-continuous, $(\rho^k_s)_{k\in\mathbb{N}}$ is a weakly decreasing sequence converging to ρ_s, and the assignment remains identical, we have for k large enough, $c_s(\rho_s) = c_s(\rho^k_s) < |\sigma[s]| = |\sigma^k[s]|$. Hence, by continuity of preferences, the assignment remains identical and $(\rho^k_{\sigma(i)})_{k\in\mathbb{N}}$ converges to $\rho_{\sigma(i)}$, we obtain $(s,\rho^k_s) P_i (\sigma^k(i),\rho^k)$ and $c_s(\rho^k_s) < |\sigma^k[s]|$ for k large enough, which contradicts non-wastefulness of (σ^k,ρ^k).

In showing interior no-envy, suppose that for some $s \in S$ and $i \in N$, $(s,\rho_s)P_i(\sigma(i),\rho)$ and $\rho_s \notin \mathbb{T}_s$. By non-wastefulness of (σ,ρ), we must have $\sigma[s] \neq \emptyset$. As the assignment remains identical, we obtain for all k, $\sigma^k[s] \neq \emptyset$ and $(\rho_s^k)_{k\in\mathbb{N}}$ converges to ρ_s. Then, by continuity of preferences and as the assignment remains identical, for k large enough, we have $(s,\rho_s^k)P_i(\sigma^k(i),\rho^k)$. As $\mathbb{T}_s$ is a discrete set and $\mathbb{P}_s$ is the union of closed intervals, and $\rho_s \notin \mathbb{T}_s$ and $(\rho_s^k)_{k\in\mathbb{N}}$ converges to ρ_s^k, it follows that there exists k large enough such that $\rho_s^k \notin \mathbb{T}_s$. But then interior no-envy is violated for (σ^k,ρ^k), a contradiction.

In showing threshold no-justified-envy, suppose that for some $s \in S$ and $i \in N$, $(s,\rho_s)P_i(\sigma(i),\rho)$ and $\rho_s \in \mathbb{T}_s$. By non-wastefulness of (σ,ρ), $|\sigma[s]| = c_s(\rho_s)$ and $\sigma[s] \neq \emptyset$. If for some $j \in \sigma[s]$, $(s,\rho_s) \in e_j$, then by unit entitlement, $\sigma[s] = \{j\}$ and there is nothing to show. Thus, suppose for some $j \in \sigma[s]$, we have $(s,\rho_s) \notin e_j$. As (σ,ρ) satisfies interior no-envy and $\rho_s \in \mathbb{T}_s$, we must have $\rho_s \in \partial\mathbb{P}_s$ and $\rho_s < \rho_s^+$. Now, as $(\rho_s^k)_{k\in\mathbb{N}}$ is a decreasing sequence converging to ρ_s and $\rho_s < \rho_s^+$, we must have $\rho_s^k = \rho_s$ for k large enough. As (σ^k,ρ^k) satisfies threshold no-justified-envy and the assignment remains identical, for k large enough we obtain:

$$(s,\rho_s) = (s,\rho_s^k)P_i(\sigma(i),\rho)R_i(\sigma^k(i),\rho^k)P_i(s,\rho_s^{k+}) = (s,\rho_s^+).$$

Hence, by transitivity of R_i, $(s,\rho_s)P_i(\sigma(i),\rho)P_i(s,\rho_s^+)$. Furthermore, as (σ^k,ρ^k) satisfies threshold no-justified-envy, for k large enough, we have $j \succ_{(s,\rho_s^k)} i$ for all $j \in \sigma^k[s]$. As the assignment remains identical and $\rho_s^k = \rho_s$ for k large enough, we now obtain $j \succ_{(s,\rho_s)} i$ for all $j \in \sigma[s]$, the desired conclusion.

Note that by construction, (σ,ρ) is an agent-optimal weak equilibrium as for all $i \in N$, we have $u_i(\sigma(i),\rho) = \overline{u}_i \geq u_i(\sigma',\rho')$ for any $(\sigma',\rho') \in \mathcal{W}$.

In showing the existence of a maximal agent-optimal weak equilibrium, let as above, any R_i be represented by u_i. Note that R_i is initially defined over all bundles (s,r) with $r \in \mathbb{R}$. For any $s \in S$, let p_s^i be such that $u_i(s,p_s^i) = \overline{u}_i$. Hence, if (σ,ρ) is an agent-optimal weak equilibrium and $\sigma(i) = s$, then $\rho_s = p_s^i$. Without loss of generality, we may suppose again that for any s with $\sigma[s] = \emptyset$, $\rho_s = \min\{\overline{t}_s, \overline{r}_s\}$ where if $\overline{r}_s = \infty$ or $(\sigma(i),\rho)R_i(s,\overline{r}_s)$ for all $i \in N$, then we choose $\overline{t}_s \in \mathbb{P}_s$ minimal such that $(\sigma(i),\rho)R_i(s,\overline{t}_s)$ for all $i \in N$. With this convention, the set of agent-optimal weak equilibria becomes finite as the set of assignments is finite. Hence, there exists a maximal agent-optimal weak equilibrium.

A.4. *Proof of Theorem 2*

In showing part (i) of Theorem 2, as in the above, let any R_i be represented by u_i. Note that R_i is initially defined over all bundles (s,r) with $r \in \mathbb{R}$. For any $s \in S$, let p_s^i be such that $u_i(s,p_s^i) = \overline{u}_i$.

Hence, if (σ,ρ) is an agent-optimal weak equilibrium and $\sigma(i) = s$, then $\rho_s = p_s^i$. Without loss of generality, we may suppose again that for any s with $\sigma[s] = \emptyset$, $\rho_s = \min\{\overline{t}_s, \overline{r}_s\}$ where if $\overline{r}_s = \infty$ or $(\sigma(i),\rho)R_i(s,\overline{r}_s)$ for all $i \in N$, then we choose $\overline{t}_s \in \mathbb{P}_s$ minimal such that $(\sigma(i),\rho)R_i(s,\overline{t}_s)$ for all $i \in N$. With this convention, the set of agent-optimal weak equilibria becomes finite. Now, if for some $j \in N$, we have $u_j(s,r) = \overline{u}_j$ and for some agent-optimal weak equilibrium (σ',ρ') we have $(\sigma'(j), \rho'_{\sigma'(j)}) \notin e_j$, then for any maximal agent-optimal weak equilibrium (σ,ρ) the latter holds for agent j, i.e., $(\sigma(j),\rho_{\sigma(j)}) \notin e_j$. Otherwise, suppose $(\sigma(j),\rho_{\sigma(j)}) \in e_j$ and consider the transfer graph T in going from (σ,ρ) to (σ',ρ'). Now, if $s \xrightarrow{i} t \in T$, then as (σ,ρ) and (σ',ρ') are agent-optimal, we have $(\sigma(i),\rho)I_i(\sigma'(i),\rho')$, i.e., all arcs in T are indifference arcs. Now, starting with $\sigma(j) \xrightarrow{j} \sigma'(j)$ take a maximal simple path $\sigma(j) \rightsquigarrow t$ ending in some $t \in S$. This is either a cycle (if $t = \sigma(i)$) or a chain (if $t \neq \sigma(i)$). Consider $t_1 \xrightarrow{\ell} t_2 \xrightarrow{\ell'} t_3$ in this path and assume $\rho_{t_1} \in \mathbb{T}_s$. By NCBI and ℓ's indifference, $\rho'_{t_2} \notin \mathbb{T}_{t_2}$.

Thus,

$$(t_2, \rho_{t_2}) = (\sigma(\ell'), \rho)\ I_{\ell'}\ (\sigma'(\ell'), \rho')\ R_{\ell'}\ (t_2, \rho'_{t_2})$$

yields $\rho_{t_2} \leq \rho'_{t_2}$. If $\rho_{t_2} < \rho'_{t_2}$, then $\rho^+_{t_2} \leq \rho'_{t_2}$ so

$$(\sigma(\ell'), \rho) = (t_2, \rho_{t_2})\ R_\ell\ (t_2, \rho^+_{t_2})\ R_\ell\ (t_2, \rho'_{t_2}) = (\sigma'(\ell), \rho')\ I_\ell\ (\sigma(\ell), \rho)$$

and we must then have $(t_2, \rho_{t_2}) \in e_{\ell'}$. Thus, as $(\sigma(j), \rho_{\sigma(j)}) \in e_j$, $\sigma(j) \rightsquigarrow t$ is composed of subpaths of the form $t_0 \rightsquigarrow t_k$ where $(t_0, \rho_{t_0}) \in e_{i_0}$, and $\rho'_{t_\ell} = \rho_{t_\ell} \notin \mathbb{T}_{t_\ell}$ for $\ell \in \{1, \ldots, k\}$. Thus, by executing $\sigma(j) \rightsquigarrow t$, welfare remains constant and agents only *leave* their entitlements. We then contradict that (σ, ρ) was a maximal agent-optimal weak equilibrium.

Let:

$$S_0 = \{s \in S :\ \text{for any maximal agent-optimal weak equlibrium } (\sigma, \rho), \sigma[s] \supseteq \mathbb{E}_s[\rho_s] \neq \emptyset\}.$$

Note that by the above, S_0 is invariant to the choice of the maximal agent-optimal weak equilibrium and by unit-entitlement at non-maximal prices, if $\sigma[s] \supsetneq \mathbb{E}_s[\rho_s] \neq \emptyset$, then $\rho_s = \overline{r}_s$. Now, for any $s \in S_0$, let $p^{\min}_s = r$, where for some $j \in N$, $(s, r) \in e_j$; this is well-defined by unit entitlement for non-maximal prices.

For any other $s \in S \backslash S_0$, let N_s denote the set of agents who are assigned s in some agent-optimal weak equilibrium. If $N_s \neq \emptyset$, then let:

$$\underline{p}_s = \max_{i \in N_s} p^i_s,$$

and we show that for all $i \in N_s$, $p^i_s = \underline{p}_s$. Suppose that for some $i, j \in N_s$, we have $p^i_s \neq p^j_s$, say $p^i_s < p^j_s$, and let (σ, ρ) be a maximal agent-optimal weak equilibrium such that $\sigma(i) = s$. Then, $\rho_s = p^i_s$ and as $\overline{u}_j = u_j(s, p^j_s)$, $p^i_s < p^j_s$ and $s \notin S_0$ must have $\sigma(j) \neq s$ and threshold no-justified-envy holds, i.e., $p^i_s \in \partial\mathbb{P}_s$ and $(\sigma(j), \rho) P_j (s, p^{i+}_s)$, but the latter is a contradiction as $u_j(\sigma(j), \rho_{\sigma(j)}) = \overline{u}_j = u_j(s, p^j_s)$ and $p^{i+}_s \leq p^j_s$. For any $s \in S \backslash S_0$ with $N_s \neq \emptyset$, we then set $p^{\min}_s = \underline{p}_s$.

For any $s \in S \backslash S_0$ such that $N_s = \emptyset$, let:

$$\underline{p}_s = \min\{\overline{r}_s, \max_{i \in N}\{p^i_s\}\}.$$

If $\underline{p}_s \in \mathbb{P}_s$, then let $p^{\min}_s = \underline{p}_s$, and otherwise let $p^{\min}_s = \underline{p}^+_s = \inf\{r' \in \mathbb{P}_s : r' > \underline{p}_s\}$.

Let (σ, ρ) be a maximal agent-optimal weak equilibrium. Now, for all $s \in S_0$, we must have $\sigma[s] \supseteq \mathbb{E}_s[\rho_s] \neq \emptyset$ and $\rho_s = p^{\min}_s$. For any $s \in S \backslash S_0$ such that $N_s \neq \emptyset$, if $\sigma[s] \neq \emptyset$, then $\rho_s = p^{\min}_s$, and if $\sigma[s] = \emptyset$, then $\rho_s \geq p^{\min}_s$ (as otherwise we have a contradiction to non-wastefulness). For any $s \in S \backslash S_0$ such that $N_s = \emptyset$, we have $\sigma[s] = \emptyset$ and due to the same reason $\rho_s \geq p^{\min}_s$.

Now, $\rho \geq p^{\min}$ and it is easy to see that $(\sigma, p^{\min})$ is a maximal agent-optimal weak equilibrium, which concludes Part (i) of the proof.

Part (ii) follows directly from the definition of $p^{\min}$.

APPENDIX B: STRATEGY-PROOFNESS

This section proves the strategy proofness of maximal agent-optimal weak equilibrium (MAOWE) in the one-to-one case. Let φ be a rule defined on $\tilde{\mathcal{R}}$ that always chooses a MAOWE with minimal prices. Note that this last stipulation has no welfare consequences beyond MAOWE. The argument will proceed by contradiction: we assume there is an agent $i \in N$ who can successfully gain from misreporting at some profile. We will see easily that it is without loss of generality to assume that i's misreport preserves their *true* indifference set at the *truth-telling* outcome. Because of this, we analyze a slightly transformed weak envy graph at the truth-telling outcome, and its relation to the transfer graph that would take the agents to the assignment that prevails under i's misreport. In particular, the transfer graph will be a subgraph of the modified weak envy graph. We will further find that the prices after manipulation are the same as before. Thus, the benefit to manipulation cannot be via price decreases, and must come via reassignment. But then our lemmas will show that the foregoing subgraph property rules this out as well.

Before introducing the contradiction argument, let us develop some tools. Given any matching (σ, ρ), $s \in S$, and $i \notin \sigma[s]$, let r_i^s solve $(s, r_i^s)\ I_i\ (\sigma(i), \rho)$. Then define the *potential prices*, denoted by $\rho^\star$, component-wise as:

$$\rho_s^\star = \begin{cases} \max\left(\{r_i^s : i \in N \setminus \sigma[s]\}\right) & \text{if } \max\left(\{r_i^s : i \in N \setminus \sigma[s]\}\right) \in \mathbb{P}_s \\ \max\left(\{r_i^s : i \in N \setminus \sigma[s]\} \cup \{\min \mathbb{P}_s\}\right)^- & \text{otherwise.} \end{cases}$$

Note that $\rho_s^\star \in \mathbb{P}_s$ and if $\rho_s^\star < \rho_s^-$, it means that all $i \in N$ have $(\sigma(i), \rho)\ P_i\ (s, \rho_s^-)$, and therefore ρ_s could be lowered—to ρ_s^- if $\rho_s^- < \rho_s \in \mathbb{T}_s$ or to some $\rho_s - \epsilon$ otherwise—without contradicting weak equilibrium. We therefore have

FACT 1: If $(\sigma, \rho) = \varphi(R)$, then for each $s \in S$, $\rho_s^\star \geq \rho_s^-$.

Potential prices express the willingness of agents to take the place of another, allowing for the smallest possible price change that can be compatible with weak equilibrium. However, there may be several agents i for whom $(s, \rho_s^\star)\ P_i\ (\sigma(i), \rho)$ (which implies $\max\left(\{r_i^s : i \in N \setminus \sigma[s]\}\right) \in (\rho_s^\star, \rho_s^{\star +})$). We therefore study the *structure graph* given by:

$$\Gamma^* = \{s \xrightarrow{i} t : (t, \rho_t^\star)\ R_i\ (\sigma(i), \rho)\} \setminus \{s \xrightarrow{i} t : \exists j \succ_{(t,\rho_t^\star)} i \text{ such that } (t, \rho_t^\star)\ P_j\ (\sigma(j), \rho).\}$$

This is the modified weak-envy graph we alluded to in the first paragraph of this section. It tracks what agents want under potential prices, but also takes the rationing constraints into consideration. We will use the structure graph to find a Pareto-improving reallocation of the objects that results in a weak-equilibrium, with some object prices converted to their potential price. We *execute* a path $s \rightsquigarrow t \subseteq \Gamma^*$, by constructing (μ, ζ) so that:

$$\mu(i) = \begin{cases} v & u \xrightarrow{i} v \in s \rightsquigarrow t \\ \sigma(i) & \text{otherwise} \end{cases} \quad \text{and} \quad \zeta_v = \begin{cases} \rho_v^\star & u \to v \in s \rightsquigarrow t \\ \rho_v & \text{otherwise.} \end{cases}$$

Path $s \rightsquigarrow t \subseteq \Gamma^*$ is *feasible* if the resulting allocation is feasible. Execution of such a path results in an allocation that all agents find at least as good as that from which it was constructed. Thus we shall write $(\mu, \zeta)\ R\ (\sigma, \rho)$, where the lack of subscript on the relation indicates "for all agents."

LEMMA 7: *Assume* $(\sigma, \rho) = \varphi(R)$. *Let* Γ^* *be its structure graph. If* $s \rightsquigarrow t \subseteq \Gamma^*$ *is feasible, then it is an indifference path.*

PROOF: Suppose that Γ^* has a path or cycle that can feasibly be executed. Then it has a path $s^* \rightsquigarrow t^* \subseteq \Gamma^*$ that is maximal with this property. Let (μ, ζ) be the resulting matching;

we show that it is a weak equilibrium, and so by the agent-optimality of φ, $(\mu,\zeta)\ I\ (\sigma,\rho)$. As mentioned above, $(\mu,\zeta)\ R\ (\sigma,\rho)$, and since (σ,ρ) is a weak equilibrium, (μ,ζ) satisfies individual rationality. In showing non-wastefulness of (μ,ζ), suppose that $(s,\zeta_s)\ P_i\ (\mu(i),\zeta)$ and $\mu[s]=\emptyset$. Thus, s has no in-arcs on the path and so $\zeta_s=\rho_s$, yielding via the foregoing observations that $(s,\rho_s)P_i(\sigma(i),\rho)$. Since (σ,ρ) is non-wasteful, $\sigma[s]\neq\emptyset$, and so s has an out-arc on the path, and in fact, $s=s^*$. But since $(s,\zeta_s)\ P_i\ (\mu(i),\zeta)$, there is $\sigma(j)\xrightarrow{j} s^*\in\Gamma^*$, where j need not equal i, contradicting the maximality of $s^*\rightsquigarrow t^*$.

Now we show that (μ,ζ) satisfies the remaining conditions of weak equilibrium. Suppose $(s,\zeta_s)\ P_i\ (\mu(i),\zeta)$. If $\mu[s]\subseteq\sigma[s]$, then $\zeta_s=\rho_s$, yielding $(s,\rho_s)\ P_i\ (\mu(i),\zeta)\ R_i\ (\sigma(i),\rho)$. Thus, s has an in-arc in Γ^*, and since $s^*\rightsquigarrow t^*$ is a maximal path, s is not incident to it at all, and so $\mu[s]=\sigma[s]$. Thus, i's envy is not justified at (μ,ζ) because it is not justified at (σ,ρ). So we must consider the case that s has an in-arc $\sigma(j)\xrightarrow{j} s\in s^*\rightsquigarrow t^*\subseteq\Gamma^*$. Moreover, $\zeta_s=\rho^\star_s$ so $(s,\rho^\star_s)\ P_i\ (\mu(i),\zeta)\ R_i\ (\sigma(i),\rho)$. By the construction of Γ^*, since $\sigma(j)\xrightarrow{j} s\in\Gamma^*$ while i has strict preference into $(s,\rho^\star_s)$, $(s,\rho^\star_s)\ P_j\ (\sigma(j),\rho)$ and $j\succ_{(s,\rho^\star_s)} i$. As $\zeta_s=\rho^\star_s$, $j\succ_{(s,\zeta_s)} i$. By individual rationality, $(s,\rho^\star_s)\notin e_j$, so these strict preferences of i and j also imply that the r^s_i and r^s_j from the definition of potential prices are not in $\mathbb{P}_s$, and in particular, both lie in the open interval $(\rho^\star_s,\rho^{\star+}_s)$. Thus, $(\mu(i),\zeta)\ R_i\ (\sigma(i),\rho)\ P_i\ (s,\rho^{\star+}_s)=(s,\zeta^+_s)$, and threshold no-justified-envy is verified. *Q.E.D.*

We now formalize the contradiction hypothesis that will yield our proof. Fix $R\in\tilde{\mathcal{R}}$, $\mathfrak{i}\in N$, and $R'_\mathfrak{i}$, such that $R'=(R'_\mathfrak{i},R_{-\mathfrak{i}})\in\tilde{\mathcal{R}}$, and $\varphi_\mathfrak{i}(R')\ P_\mathfrak{i}\ \varphi_\mathfrak{i}(R)$. We use notation $\mathfrak{i}$ to indicate the manipulating agent because we shall not need to refer to this agent later—we will actually show $\varphi(R')\ I\ \varphi(R)$ as our contradiction—and so we can reserve i for arbitrary agents. First, without loss of generality, we add some structure to $R'_\mathfrak{i}$. In particular, we find $R''_\mathfrak{i}$ that has the same indifference set as $R_\mathfrak{i}$ at $\varphi_\mathfrak{i}(R)$ and which is a rather extreme Maskin monotonic transform of $R'_\mathfrak{i}$ at $\varphi_\mathfrak{i}(R')$. Let $(\sigma,\rho)=\varphi(R)$ and $(\sigma',\rho')=\varphi(R')$. For each $s\neq\sigma'(\mathfrak{i})$ there is $r'_s\in\mathbb{R}$ such that $(s,r'_s)\ P_\mathfrak{i}\ (\sigma'(\mathfrak{i}),\rho')$ and $(s,r'_s)\ P'_\mathfrak{i}\ (\sigma'(\mathfrak{i}),\rho')$. Let $r'<\min_s r'_s$. Then choosing $R''_\mathfrak{i}$ with $(\sigma'(\mathfrak{i}),\rho)\ I''_\mathfrak{i}\ (s,r')$ for all $s\neq\sigma'(\mathfrak{i})$ clearly yields the desired Maskin monotonic transform. By setting r' low enough we can guarantee via feasibility and the Maskin monotonicity of weak equilibrium, that if $(\sigma''(\mathfrak{i}),\rho'')=\varphi_\mathfrak{i}(R''_\mathfrak{i},R_{-\mathfrak{i}})$, then $\sigma''(\mathfrak{i})=\sigma'(\mathfrak{i})$ and $\rho''_{\sigma''(\mathfrak{i})}\leq\rho'_{\sigma(\mathfrak{i})}$. We can also perturb r to ensure it does not cause any violations of NCBI.

Since $(\sigma'(\mathfrak{i}),\rho')\ P_\mathfrak{i}\ (\sigma(\mathfrak{i}),\rho)$, the foregoing construction makes the upper contour set of $R''_\mathfrak{i}$ at $(\sigma'(\mathfrak{i}),\rho')$ a strict subset of the interior of the upper contour set of $R_\mathfrak{i}$ at $(\sigma(\mathfrak{i}),\rho)$. Thus we can also set $(\sigma(\mathfrak{i}),\rho)\ I''_\mathfrak{i}\ (s,r^s_\mathfrak{i})$ to get the desired agreement on the indifference set through (σ,ρ). This also does not violate NCBI, since $R\in\tilde{\mathcal{R}}$.

Finally, we can fill in the remaining indifference sets by linear interpolation, with further local perturbation to remove any violations of NCBI. So we have that $\varphi(R''_\mathfrak{i},R_{-\mathfrak{i}})\ P_\mathfrak{i}\ \varphi_\mathfrak{i}(R)$, and so with some abuse of notation, we henceforth assume that $R'_\mathfrak{i}=R''_\mathfrak{i}$. Hence, $\varphi(R)=(\sigma,\rho)$, $\varphi(R')=(\tau,\gamma)$ and both (σ,ρ) and (τ,γ) are weak equilibria under R'.

Denote by T the transfer graph from (σ,ρ) to (τ,γ); recall that this was defined in Section A.2. As $R'_\mathfrak{i}$ is a Maskin monotonic transform of $R_\mathfrak{i}$ at $(\sigma(\mathfrak{i}),\rho)$, by Lemma 4, $(\tau,\gamma)\ R\ (\sigma,\rho)$ and so all arcs in T are either $+$ or $=$-labeled. Moreover, recalling that each R'_i and R_i agree on the indifference set through $(\sigma(i),\rho)$, these labelings would not change if they were calculated with respect to R' instead of R; this will be a key feature in the contradiction.

Given directed graph Γ we denote by $\mathrm{verts}[\Gamma]$ the set of vertices incident to it, which is to say the vertices incident to some arc in Γ. A subgraph $\Gamma'\subseteq\Gamma$ is a *source* if it has no arcs incoming from $\Gamma\setminus\Gamma'$. It is a *minimal source* if no $\Gamma''\subsetneq\Gamma'$ is a source. With abuse of terminology, we call verts[Γ'] a source if Γ' is a source. Define

$$\Gamma_{\to s} = \left\{ u \xrightarrow{i} v \in \Gamma : \exists u \rightsquigarrow s \subseteq \Gamma \text{ s.t. } u \xrightarrow{i} v \in u \rightsquigarrow s \right\}.$$

This is clearly a source.

LEMMA 8: *Let Γ' be a minimal source in directed graph Γ. Then either* verts[Γ'] *is a singleton or Γ' is the (not necessarily disjoint) union of cycles.*

PROOF: Assume verts[Γ'] is not a singleton. If Γ' contains a path that is not part of a cycle, then there is $t \rightsquigarrow s \subseteq \Gamma'$ with t having no incoming arcs. But then $\{t\}$ would be a source. *Q.E.D.*

LEMMA 9: *For each $s \in S$,* verts[$\Gamma^*_{\to s}$]$\cup\{s\}$ *contains a singleton source $\widehat{s}$ with $\rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$. Moreover, $\widehat{s}$ is not incident to any arc in T.*

PROOF: By construction, $\Gamma^*_{\to s}$ is a source in Γ^*. Let S' be a minimal source in verts[$\Gamma^*_{\to s}$] or be $\{s\}$ if $\Gamma^*_{\to s}$ is empty. By Lemma 8, it is either a singleton or $\Gamma^*|_{S'}$ is the union of cycles. To arrive at a contradiction, we assume the latter. Since matchings are one-to-one, any cycle in Γ^* is feasible. Thus, by Lemma 7, each cycle in $\Gamma^*|_{S'}$ is an indifference cycle. By NCBI, it follows that for each $t \in S'$, $\rho_t \notin \mathbb{T}_t$. Then there is $\bar{\epsilon} > 0$ such that, for each $t \in S'$, $[\rho_t - \bar{\epsilon}, \rho_t) \subseteq \mathbb{P}_t$, and for each $j \notin \sigma[S']$ $(\sigma(j), \rho)\ P_j\ (t, \rho_t - \bar{\epsilon})$. For the moment, we view the sub-problem of matching $N' = \sigma[S']$ to S' as an assignment game with non-transferable utility, as studied in Demange and Gale (1985) and **?** Chapter 9. Each $i \in N'$ can match with $S' \cup \{i\}$ and symmetrically $s \in S'$ can match with $S' \cup \{s\}$. Each agent $i \in N' \cup S'$ has a utility function $u_{ij}(x)$ that measures their welfare for matching with $j \in N' \cup S'$ and getting transfer x. This is monotonically *increasing*, so for our N' agents we take the negative of their payment as their transfer. Being monotonic, we can invert it and say that $w_{ij}(u)$ is the transfer required by i to give them utility u when matching to j. For each $s \in S'$ and each $i \in N'$, set $w_{si}(u) = u$, so s prefers higher payments and does not care whom they match to. Set $u_{s,s}(0) = \rho_s - \bar{\epsilon}$, so that $w_{s,s}(\rho_s - \bar{\epsilon}) = 0$; this is the outside-option for s. For each $i \in N'$, set $u_{i,i}(0) = u_{i\sigma(i)}(-\rho_{\sigma(i)}) = -\rho_{\sigma(i)}$, so that otherwise $w_{is}(-p)$ solves $(\sigma(i), p)\ I_i\ (s, -w_{is}(-p))$. We have from Demange and Gale (1985) that there is a stable matching μ, with transfer vectors $x \in \mathbb{R}^{N'}$ and $z \in \mathbb{R}^{S'}$ with the further property that there is $s^* \in S'$ with $u_{s^*\mu[s^*]}(z_{s^*}) = \rho_{s^*} - \bar{\epsilon}$ (Lemma 9.14 in **?**). Since the matching is stable, for all $i \in N'$, $s \in S'$,

$$w_{is}(u_{i\mu(i)}(x_i)) + z_s = w_{is}(u_{i\mu(i)}(x_i)) + w_{si}(u_{s\mu[s]}(z_s)) \geq 0.$$

Thus, $w_{is}(u_{i\mu(i)}(x_i)) \geq -z_s$, and since u_{is} is the inverse of w_{is}, $u_{i\mu(i)}(x_i) \geq u_{is}(-z_s)$. We also have from **?** that, at stable outcomes, transfers are only between matched pairs, so $x_i = -z_{\mu(i)}$, and we get $u_{i\mu(i)}(-z_{\mu(i)}) \geq u_{is}(-z_s)$. Thus, $\zeta = z \in \mathbb{R}^{S'}$ is an equilibrium price vector for the matching of N' with S'.

By construction, $u_{s^*\mu[s^*]}(z_{s^*}) = \rho_{s^*} - \bar{\epsilon}$ implies $z_{s^*} = \rho_{s^*} - \bar{\epsilon}$. Letting $\mu[s^*] = \{i^*\}$, we also have from the decomposition lemma of Demange and Gale (1985) (Lemma 9.7 in **?**) that $u_{i^*s^*}(-\zeta_{s^*}) > u_{i^*,i^*}(0) = u_{i^*\sigma(i^*)}(-\rho_{\sigma(i^*)})$, so $(\mu(i^*), \zeta)\ P_{i^*}\ (\sigma(i^*), \rho)$, contradicting that (σ, ρ) was agent-optimal.

Hence, $\widehat{s}$ is a singleton source in $\Gamma^*_{\to s}$. Being a source, there are no arcs $t \to \widehat{s} \in \Gamma^*$, which by definition implies that $(\sigma(j), \rho)\ P_j\ (\widehat{s}, \rho_{\widehat{s}})$ for all $j \notin \sigma[\widehat{s}]$. If $\rho_s > \min \mathbb{P}_s$, then it could be lowered while maintaining weak equilibrium, contradicting that φ selects MAOWE with minimal prices. Furthermore, since all T-arcs are $+$ or $=$-labeled, an arc $t \to \widehat{s} \in T$ would imply that $\gamma_{\widehat{s}} < \rho_{\widehat{s}}$, contradicting the previous deduction. *Q.E.D.*

The lemma shows how each $s \in S$ can be "reached" via a path in Γ^* originating at an item whose price cannot be lowered. The proof will continue in part by developing the following intuition, based on the no-envy constraints in weak equilibrium: if $\gamma_s < \rho_s$, then either this is true for all s' on the path or some $s' \neq \widehat{s}$ is also incident to T. The former is ruled out by the fact that $\rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$.

For each $s \in S$, fix a singleton source $\widehat{s}$ from the previous lemma with $\rho_{\hat{s}} = \min \mathbb{P}_{\hat{s}}$ and a path $\widehat{s} \rightsquigarrow s \in \Gamma^*$. We decompose this path as follows:

$$\widehat{s} = s^0 \rightsquigarrow t^1 \xrightarrow{i_1} s^1 \rightsquigarrow t^2 \xrightarrow{i_2} s^2 \rightsquigarrow t^3 \ldots s^n \rightsquigarrow t^{n+1} = s.$$

Each $s^k \rightsquigarrow t^{k+1}$ segment is an indifference path and may be empty, in which case $t^{k+1} = s^k$. The $t^k \xrightarrow{i_k} s^k$ are arcs of strict preference, and recall that by the way the structure graph is created, these could represent

$$(s^k, \rho^-_{s^k})\ P_{i_k}\ (t^k, \rho_{t^k}) = (\sigma(i_k), \rho)\ P_{i_k}\ (s^k, \rho_{s^k})$$

preferences or

$$(s^k, \rho_{s^k})\ P_{i_k}\ (\sigma(i_k), \rho)$$

preferences. In either case, as (σ, ρ) is a weak equilibrium, $\rho_{s^k} \in \mathbb{T}_{s^k}$, and as the foregoing lemma gives $s^0 = \min \mathbb{P}_{s^0}$, we have that every s-type vertex in the path has its price at a threshold value. NCBI then implies that the t-type vertices are *not* priced at threshold values. Since s was chosen arbitrarily, we have nearly proven the following lemma:

LEMMA 10: *If $r \in \mathbb{T}_s$, then there is no agent i with $(s, r)\ I_i\ (\sigma(i), \rho)$ and $\sigma(i) = s' \in S \backslash \{s\}$.*

PROOF: To complete the foregoing reasoning, consider s' and the path $\widehat{s'} \rightsquigarrow s'$. The path $\widehat{s'} \rightsquigarrow s' \xrightarrow{i} s$ either ends at a threshold-priced vertex, or has a final subpath of indifferences $s^n \rightsquigarrow s'$ where s^n is threshold priced. Thus, by NCBI, there is no agent with $\sigma(i) = s'$ who is indifferent to (s, r). *Q.E.D.*

COROLLARY 1: *If $(\tau(i), \gamma) \in e_i$ then $(\tau(i), \gamma) = (\sigma(i), \rho)$.*

PROOF: Suppose $(\tau(i), \gamma) \in e_i$. Then

$$(\tau(i), \gamma)\ R_i\ (\sigma(i), \rho)\ R_i\ (\tau(i), \gamma),$$

where the first relation follows from $(\tau, \gamma)\ R\ (\sigma, \rho)$ and the second relation from individual rationality. Thus, $\tau(i) \neq \sigma(i)$ would violate Lemma 10. So $\tau(i) = \sigma(i)$ and individual rationality yields $\gamma_{\tau(i)} = \rho_{\sigma(i)}$. *Q.E.D.*

LEMMA 11: *The transfer graph T from (σ, ρ) to (τ, γ) is the disjoint union of cycles.*

PROOF: If $t \rightsquigarrow s \in T$ is a path, so that s is its terminal vertex, then by feasibility, $\sigma[s] = \emptyset$. Then $\widehat{s} \rightsquigarrow s \subseteq \Gamma^*$ is either a feasible path or $\Gamma^*_{\to s} = \emptyset$. In the latter case, since ρ is a minimal price, $\rho_s = \min \mathbb{P}_s$. Moreover, by Lemma 10 and non-wastefulness, all agents prefer their (σ, ρ) bundle to $(s, \min \mathbb{P}_s)$, so any agent i with $v \xrightarrow{i} s \in T$ becomes worse off, a contradiction.

Thus we have that $\widehat{s} \rightsquigarrow s \subseteq \Gamma^*$ is feasible. By Lemma 7, it is an indifference path, and by Lemma 9, $\rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$. By NCBI, no vertex on the path (except $\widehat{s}$) is threshold-priced, and in particular there are no endowment points. Let μ be the matching that results from execution of the path. Let Γ'^* be the structure graph for (μ, ρ) and observe that it contains the reversed path $s \rightsquigarrow \widehat{s}$. Now since $\rho_s \neq \min \mathbb{P}_s$, there is a *different* path $\widehat{s}' \rightsquigarrow s \subseteq \Gamma'^*$, with $\rho_{\widehat{s}'} = \min \mathbb{P}_s$. Thus we have $\widehat{s}' \rightsquigarrow s \rightsquigarrow \widehat{s} \subseteq \Gamma'^*$, which is feasible as $\mu[\widehat{s}] = \emptyset$, and so is an indifference path. This contradicts NCBI, recalling that $\rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$ as well.

Finally, the disjointness of the cycles follows from one-to-one matching. *Q.E.D.*

The next lemma is the last before our proof. It is the key tool in establishing that $T \subseteq \Gamma^*$. Then given the foregoing lemma, we have that $T \cap \Gamma^*$ is composed of Γ^* indifference cycles. If $\gamma = \rho$, then our argument is basically complete; the next lemma provides this as well.

LEMMA 12: *Assume* $t \xrightarrow{i} s \in \Gamma^*$ *and* $(\tau(i), \gamma)\ I_i\ (\sigma(i), \rho)$. *If* $v \xrightarrow{j} s \in T$, *then* $(\tau(j), \gamma)\ I_j\ (\sigma(j), \rho)$ *and* $v \xrightarrow{j} s \in \Gamma^*$. *If there is no* $v \xrightarrow{j} s \in T$, *then* $\gamma_s = \rho_s$.

PROOF: Assume $\gamma_s < \rho_s$. If $\emptyset \neq \tau[s] = \{j\}$, then $(\tau(j), \gamma) \neq (\sigma(j), \rho)$ and by Corollary 1, $(\tau(j), \gamma) = (s, \gamma_s) \notin e_j$. Let r_s^i have $(s, r_s^i)\ I_i\ (\sigma(i), \rho)\ I_i\ (\tau(i), \gamma)$. We have

$$r_s^i \geq \rho_s^\star \geq \rho_s^- \geq \gamma_s$$

where the first inequality follows from $t \xrightarrow{i} s \in \Gamma^*$, the second from Fact 1 and the third from $\rho_s > \gamma_s$.

If any of these inequalities are strict, then $r_i^s > \gamma_s$ and we have a violation of either non-wastefulness (if $\tau[s] = \emptyset$) or no-justified-envy (as $j \in \tau[s]$ is not entitled to (s, γ_s)). On the other hand, they cannot all be equal, as this would imply $\gamma_s = \rho_s^- < \rho_s$, $\rho_s \in \mathbb{T}_s$ and $(\sigma(i), \rho)\ I_i\ (s, \rho_s^-)$, in violation of Lemma 10. Hence, we conclude that $\rho_s \leq \gamma_s$.

Next we rule out the case $\rho_s^\star < \gamma_s$. By definition, for each $j \in N$, $r_s^j \leq \rho_s^{\star+}$. Thus, $\rho_s^\star < \gamma_s$ implies $r_s^j < \gamma_s$ by Lemma 10, and so since $(\tau, \gamma)\ R\ (\sigma, \rho)$, $\tau[s] = \emptyset$. As $\min \mathbb{P}_s \leq \rho_s^\star < \gamma_s$, this contradicts that γ is a minimal price. We now have that $\rho_s \leq \gamma_s \leq \rho_s^\star$.

Assume now that there is $v \xrightarrow{j} s \in T$. If j is not indifferent, then

$$(s, \rho_s)\ R_j\ (s, \gamma_s) = (\tau(j), \gamma)\ P_j\ (\sigma(j), \rho), \tag{15}$$

so $\rho_s \leq \gamma_s < r_s^j$ and $\rho_s \in \mathbb{T}_s$. As $t \xrightarrow{i} s \in \Gamma^*$, $\rho_s^\star \leq r_s^i$. If $\gamma_s < \rho_s^\star$, then $\gamma_s^+ \leq r_s^i$. As ((15)) implies that $(s, \gamma_s) \notin e_j$, and recalling that $(s, r_s^i)\ I_i\ (\tau(i), \gamma)\ I_i\ (\sigma(i), \rho)$, we see that i has justified envy of j at (τ, γ). Therefore, we consider $\gamma_s = \rho_s^\star$, and by ((15)), $\gamma_s = \rho_s^\star < r_s^j$. This implies that $\rho_s^\star \in \mathbb{T}_s$, and by definition of Γ^*, either $j = i$ or $i \succ_{(s, \rho_s^\star)} j$. In the former case, the lemma hypothesis $(\tau(i), \gamma)\ I_i\ (\sigma(i), \rho)$ and ((15)) are contradictory, and in the latter case i again has justified envy of j in priority at (τ, γ). Thus, $v \xrightarrow{j} s \in T$ implies j is indifferent between (τ, γ) and (σ, ρ).

Now consider the case $\gamma_s > \rho_s$. We have

$$(s, \rho_s)\ P_i\ (s, \gamma_s)\ R_i\ (s, \rho_s^\star)\ R_i\ (\sigma(i), \rho)\ I_i\ (\tau(i), \gamma). \tag{16}$$

Thus $\rho_s \in \mathbb{T}_s$, and since $\rho_s^+ \geq \gamma_s$, it must be that $\{k\} = \sigma[s] = \mathbb{E}_s[\rho_s]$. Moreover, by individual rationality of (τ, γ), $s \xrightarrow{k} \tau(k) \in T$ (and note we have shown that if there is no $v \xrightarrow{j} s \in T$, then $\gamma_s = \rho_s$). By Lemma 11, there is $v \xrightarrow{j} s \in T$, and so by the previous paragraph, $(s, \gamma_s) = (\tau(j), \gamma)\ I_j\ (\sigma(j), \rho)$. By Lemma 10, $\gamma_s \notin \mathbb{T}_s$, so since (τ, γ) is a weak equilibrium, the weak relations in (16) are all indifferences. This then yields $\gamma_s = \rho_s^\star$ and $\sigma(j) \xrightarrow{j} s \in \Gamma^*$, as desired.

It remains to consider the case that $\gamma_s = \rho_s$ and there is $v \xrightarrow{j} s \in T$. We again have that i is indifferent, so $(\sigma(i), \rho)\ I_i\ (\tau(i), \gamma) = (s, \gamma_s) = (s, \rho_s)$. By Lemma 10, $\rho_s \notin \mathbb{T}_s$ and so by construction of Γ^*, $v \xrightarrow{j} s \in \Gamma^*$. *Q.E.D.*

We now complete the proof. Choose $s \in S$ arbitrarily. We have $\widehat{s} \rightsquigarrow s \in \Gamma^*$, with $\rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$. Our proof uses a nested induction argument, with the outer argument proceeding along $\widehat{s} \rightsquigarrow s$ and the inner finding that any s_k incident to $\widehat{s} \rightsquigarrow s$ and incident to T will be on a T cycle that is also in Γ^*.

Outer Induction Step: Step $n-1$ Hypothesis: Let $s_{n-1} \xrightarrow{i_{n-1}} s_n \in \Gamma^*$ have $\gamma_{n-1} = \rho_{n-1}$ and $(\tau(i_{n-1}), \gamma)\ I_{i_{n-1}}\ (\sigma(i_{n-1}), \rho)$. Now if s_n is not incident to any T-arc, then by Lemma 12, $\gamma_{s_n} = \rho_{s_n}$. To complete the induction hypothesis for step n, observe that as $\{i_n\} = \sigma[s_n] = \tau[s_n]$, the equality of prices imply the indifference of i_n.

Inner Induction Step: Step $n-1$ Hypotheses: (i) $t_n \xrightarrow{j_{n-1}} t_{n-1} \in \Gamma^* \cap T$ and (ii) $(\tau(j_{n-1}), \gamma)\ I_{j_{n-1}}\ (\sigma(j_{n-1}), \rho)$. Now recall that $R'_{j_{n-1}}$ and $R_{j_{n-1}}$ agree on the indifference set through $(\sigma(j_{n-1}), \rho)$. Thus, if it were the case that $\rho_{t_n} \in \mathbb{T}_{t_n}$, then since $\tau(j_{n-1}) = t_{n-1}$ and $\sigma(j_{n-1}) = t_n$, we would have $(\tau(j_{n-1}), \gamma)\ I'_{j_{n-1}}\ (\sigma(j_{n-1}), \rho) = (t_n, \rho_{t_n})$ being a violation of Lemma 10. Thus, $\rho_{t_n} \notin \mathbb{T}_{t_n}$. By Lemma 11, there is $t_{n+1} \xrightarrow{j_n} t_n \in T$. By Corollary 1, $(t_n, \gamma_n) \notin e_{j_n}$. Thus if $\gamma_{t_n} < \rho_{t_n}$, then by the interiority of ρ_{t_n}, $\gamma^+_{t_n} < \rho_{t_n}$ and we would have $(t_n, \gamma^+_{t_n})\ P'_{j_{n-1}}\ (t_n, \rho_{t_n}) = (\sigma(j_{n-1}), \rho)\ I'_{j_{n-1}}\ (\tau(j_{n-1}), \gamma)$, contradicting that (τ, γ) is a weak equilibrium. Thus, $\rho_{t_n} \leq \gamma_{t_n}$, and we calculate

$$(\sigma(j_n), \rho)\ R_{j_n}\ (t_n, \rho_{t_n})\ R_{j_n}\ (t_n, \gamma_{t_n}) = (\tau(j_n), \gamma)\ R_{j_n}\ (\sigma(j_n), \rho), \tag{17}$$

where the first relation follows from $\rho_{t_n} \notin \mathbb{T}_{t_n}$ and the last relation from $(\tau, \gamma)\ R\ (\sigma, \rho)$. This is step n hypothesis (ii), and since $\rho_{t_n} \notin \mathbb{T}_{t_n}$, hypothesis (i) is by construction of Γ^*.

Applying Induction: By Lemma 9, $\widehat{s}$ is not incident to any T arc. Moreover, since $(\tau, \gamma)\ R\ (\sigma, \rho)$ and $i_0 \in \sigma[\widehat{s}]$ stays where they are, we must have $\gamma_{\widehat{s}} \leq \rho_{\widehat{s}} = \min \mathbb{P}_{\widehat{s}}$, and so we have the base for the outer induction step. We can therefore proceed inductively along $\widehat{s} \rightsquigarrow s$, until we reach some t_0 that is incident to a T arc. By the inner induction step, we have $s_{n-1} \xrightarrow{\ell} t_0 \in \widehat{s} \rightsquigarrow s \subseteq \Gamma^*$, $\gamma_{s_{n-1}} = \rho_{s_{n-1}}$, and $(\tau(\ell), \gamma)\ I_\ell\ (\sigma(\ell), \rho)$. Since t_0 is incident to T, we cannot conclude that $\gamma_{t_0} = \rho_{t_0}$ yet. By Lemma 11, there is $t_1 \xrightarrow{j_1} t_0 \in T$. Since t_0 has the required in-arcs in Γ^* and T, we can invoke Lemma 12 to get both $(t_0, \gamma_{t_0}) = (\tau(j_1), \gamma)\ I_{j_1}\ (\sigma(j_1), \rho)$, which is hypothesis (ii) of the inner induction step, and $t_1 \xrightarrow{j_1} t_0 \in \Gamma^*$, which completes hypothesis (i) of the inner induction step.

Thus, we see that we can run the outer induction step along $\widehat{s} \rightsquigarrow s$ until we reach a vertex incident to T. Then we invoke the inner induction step to find that this vertex belongs to a cycle $C \subseteq T \cap \Gamma^*$, and so by Lemma 7 is a Γ^* indifference cycle. In fact, by (17), $\gamma_t = \rho_t$ for all t incident to C, making it a T indifference cycle as well. If $\widehat{s} \rightsquigarrow s$ exits C, it does so at $s' \in S$ with $\gamma_{s'} = \rho_{s'}$, which completes the induction hypothesis for the outer induction step again, and so we can continue. Note that the argument establishes $\gamma_s = \rho_s$ for all $s \in S$, as each s can be reached by a path of the form $\widehat{s} \rightsquigarrow s$. It also establishes that $(\sigma(i), \rho)\ I_i\ (\tau(i), \gamma)$ for all i who label a Γ^* arc, and that $T \subseteq \Gamma^*$. The remaining agents have $\tau(i) = \sigma(i)$ and their indifference follows from $\gamma = \rho$, completing the proof.

APPENDIX C: EXAMPLES

The following examples demonstrate that the set of weak equilibria does not possess a lattice structure (Example 1), that the real number attached to an object need not be at the lower bound when the object remains unassigned at maximal agent-optimal weak equilibria (Example 2), and that the non-manipulation result does not hold in many-to-one models (Example 3).

EXAMPLE 1: Let $N = \{1, \ldots, 7\}$ and $S = \{a, b, c, e, f\}$. Suppose that only agent 5 is entitled to an object and, more specifically, that $e_5 = \{(e, 1)\}$ and $e_i = \emptyset$ for all $i \neq 5$. The price space is defined by $\mathbb{P}_a = [0, 1] \cup [2, +\infty)$ and $\mathbb{P}_s = [0, +\infty)$ for all $s \neq a$. Because object a may have to be rationed, suppose that $1 \succ_{(a,1)} 2 \succ_{(a,1)} 3 \succ_{(a,1)} 4$. Regarding capacities, we assume that $c_a(r) = 2$ for all $r \in \mathbb{P}_a$, $c_b(r) = 2$ for all $r \in \mathbb{P}_b$, $c_e(r) = 1$ for all $r \in \mathbb{P}_e$, and all other capacities are always equal to two.

Consider now matchings (ρ, σ) and (ρ', σ') where (i) $\rho_s = 1$ for all $s \in S$ and $\sigma = (a, a, b, b, e, f, c)$ and (ii) $\rho'_a = 2$, $\rho'_b = 2$, $\rho'_f = \frac{1}{2}$, $\rho'_i = \rho_i$ for $i \in \{c, d, e\}$, and $\sigma' = (e, c, a, b, f, f, c)$.

Note that for any $s \in S$, we have $\sigma[s] \neq \emptyset \neq \sigma'[s]$, i.e., a condition saying that "if an object is unassigned, then its price should be equal to the lower bound" has no bite. Furthermore, $(\sigma(5), \rho) = (e, 1) \in e_5$, but $e_5 P_5 (s, \rho_s)$ for all $s \neq e$, i.e., we cannot increase the number of assigned objects by choosing a Pareto-indifferent matching.

Regarding agent preferences, suppose that:

- for agent 1, $(e, 1) P_1 (a, 1) P_1 (s, 0)$ for all $s \neq a, e$ (i.e., $1 \in N^+$),
- for agent 2, $(a, 1) P_2 (c, 1) P_2 (a, 2) P_2 (s, 0)$ for all $s \notin \{a, c\}$ (i.e., $2 \in N^-$), but agent 2 does not envy agent 3 at (σ', ρ'),
- for agent 3, $(a, 1) P_3 (b, 1) P_3 (a, 2) P_3 (b, 2) P_3 (s, 0)$ for all $s \neq a, b$ (i.e., $3 \in N^-$),
- for agent 4, $(b, 1) P_4 (b, 2) P_4 (s, 0)$ for all $s \neq b$,
- for agent 5, $(e, 1) \in e_5$ and $(e, 1) I_5 (f, \frac{1}{2}) P_5 (s, 0)$ for all $s \notin \{e, f\}$,
- for agent 6, $(f, 1) P_6 (s, 0)$ for all $s \neq f$, and;
- for agent 7, $(c, 1) P_7 (s, 0)$ for all $s \neq c$.

It can be checked that the profile R satisfies the NCBI condition.

Also, (σ, ρ) is a weak equilibrium because for agent 3 we have $(a, 1) P_3 (b, 1) P_3 (a, 2)$ and agents 1 and 2 have higher priority that agent 3 under $\succsim_{(a,1)}$. Moreover, $(e, 1) \in e_5$, so $(e, 1) P_1 (\sigma(1), \rho) = (a, 1)$ is allowed by threshold no-justified-envy. One can also check that (σ', ρ') is a weak equilibrium as for agent 2 we have $(c, 1) P_2 (a, 2)$ and for agent 3 we have $(a, 2) P_3 (b, 2)$, i.e., there is no envy by agents 2 and 3.

Now, if the set of weak equilibria would have the structure of a lattice, we obtain (σ'', ρ'') by letting each agent choose from his two most preferred bundles of (σ, ρ) and (σ', ρ'). By $(a, 1) P_2 (c, 1)$, agent 2 would choose $(a, 1)$ (i.e., $\sigma''(2) = a$ and $\rho''_a = 1$), by $(e, 1) P_1 (a, 1)$ agent 1 would choose $(e, 1)$ (i.e., $\sigma''(1) = e$), and by $(b, 1) P_3 (a, 2)$ agent 3 would choose $(b, 1)$ (i.e., $\sigma''(3) = b$ and $\rho''_b = 1$). Hence, $\sigma''[a] = \{2\}$ and $\rho''_a = 1$. Since $c_a(1) = 2$ and $(a, 1) P_3 (b, 1) = (\sigma''(3), \rho'')$, the matching (σ'', ρ'') violates non-wastefulness and cannot be a weak equilibrium. Hence, the set of weak equilibria does not possess a lattice structure. □

EXAMPLE 2: Let $S = \{a, b, c\}$ and $N = \{1, 2, 3, 4\}$. For any $s \in S$, let $\mathbb{P}_s = [0, +\infty)$, $c_s(r) = 1$ for $r \in [0, 1)$ and $c_s(r) = 2$ for $r \in [1, +\infty)$. For all $i \in N$, let $(a, 1) I_i (b, 1) I_i (c, 1)$ and $e_i = \emptyset$. Now, for any agent-optimal weak equilibrium (σ, ρ), we must have $\rho_s = 1$ for all $s \in S$. But setting $\sigma[a] = \{1, 2\}$ and $\sigma[c] = \{3, 4\}$, matching (σ, ρ) is a maximal agent-optimal weak equilibrium while $\sigma[b] = \emptyset$ and $p_b = 1 > 0 = \underline{r}_b$. □

EXAMPLE 3: Let $S = \{a, b, c\}$ and $N = \{1, 2, 3, 4\}$. Set $\mathbb{P}_a = \mathbb{P}_b = [0, \infty)$ and $\mathbb{P}_c = [0, \epsilon] \cup [1, \infty)$ for some $\epsilon > 0$ and $0 < \delta < \epsilon$. The capacity of objects a and b are both always 1 and the capacity of object c is always 2. Let the priority at (c, ϵ) be given by $1 \succ_{(c,\epsilon)} 2 \succ_{(c,\epsilon)} 3 \succ_{(c,\epsilon)} 4$.

Suppose that all agents have quasi-linear preferences $u_{is}(p) = v_{is} - \rho_s$, where the values v_{is} are represented by:

$$v = (v_{is})_{i \in N, s \in S} = \begin{pmatrix} 1 & 1-\delta & 0 \\ \frac{1}{2} & 1 & 1-\delta \\ 0 & 1-\delta & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Note that by setting δ "sufficiently small," we rule out any indifference chain containing bundle (c, ϵ). Now, $\rho = (0,0,0)$ with $\sigma = (a,b,c,c)$ is the unique maximal agent-optimal weakly equilibrium for v.

Suppose now that $v_1' = (1, \frac{3}{2}, 0)$, and consider $\rho' = (\frac{1}{2} - \delta, 1-\delta, 1)$ with $\sigma' = \sigma$. It is easy to check that this matching is a weak equilibrium for $v' = (v_1', v_{-1})$. We next argue that it must also be a maximal agent-optimal weak equilibrium for $v' = (v_1', v_{-1})$. Let ρ^* denote the minimum price under v'. Note first that the no-envy property implies that agent 4 must be assigned object c and $\rho_c^* \leq 1$. If $\rho_c^* \leq \epsilon$, then agents 2, 3, and 4 will only demand object c unless $\rho_b^* \leq \epsilon + \delta$. But then as long as $\epsilon + \delta < \frac{1}{2}$, all agents (including agent 1) will have their demand sets contained in b and c. In sum, if $\rho_c^* \leq \epsilon$, then the set $\{b, c\}$ is strictly over-demanded. Thus, it must be the case that $\rho_c^* = 1$.

Now, by feasibility, object c has be assigned to exactly two agents. By no-envy, agent 4 must be assigned one copy of object c. If agent 1 is assigned the other copy, by no-envy, it must be the case that $\rho_b^* \geq \frac{5}{2}$, which is not minimal as $\rho_b' < \frac{5}{2}$. The corresponding calculations for agents 2 and 3 give $1+\delta$ and $1-\delta$, respectively. Hence, the lowest possible weak equilibrium price for object b is $1-\delta$. In summary, $\rho_b^* = 1-\delta$ and $\rho_c^* = 1$ are the lowest possible weak equilibrium prices for $v' = (v_1', v_{-1})$, so it remains only to demonstrate that $\rho_a^* = \frac{1}{2} - \delta$ also is a lowest possible weak equilibrium price, because if this is the case it is easy to confirm that $\rho^* = \rho'$ with σ' is the maximal agent-optimal weak equilibrium for $v' = (v_1', v_{-1})$. Suppose now that $\rho_a^* = \frac{1}{2} - \delta$ and note that at prices ρ^*, both agents 1 and 2 demand objects a and b. Thus, if $\rho_a^* < \frac{1}{2} - \delta$, object a would be strictly over-demanded. Hence, $\rho_a^* = \frac{1}{2} - \delta$.

So, if agent 1's true preferences are given by $v_1 = (1, \frac{3}{2}, 0)$, she gets bundle $(a, \frac{1}{2} - \delta)$. By misrepresenting $v_1' = (1, 1-\delta, 0)$, she gets bundle $(a, 0)$. Thus, agent 1 strictly gains by misrepresenting her preferences as long as $\delta < \frac{1}{2}$. $\square$

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