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Phase Transition in Vertex-Reinforced Random Walks on ℤ with Non-linear Reinforcement

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Vertex-reinforced random walk is a random process which visits a site with probability proportional to the weight w_k of the number k of previous visits. We show that if $w_k \sim k^\alpha$, then there is a large time T_0 such that after T_0 the walk visits 2, 5, or ∞ sites when $\alpha < 1$, $= 1$, or > 1, respectively. More general results are also proven.

1. INTRODUCTION

Consider nearest–neighbor stochastic process X_n, $n = 0, 1, 2, \ldots$, on $\mathbb{Z}$ with transition probabilities

$$\mathbb{P}(X_{n+1} = m+1 \mid X_n = m) = \frac{w_{L(n,m+1)}}{w_{L(n,m-1)} + w_{L(n,m+1)}},$$

$$\mathbb{P}(X_{n+1} = m-1 \mid X_n = m) = \frac{w_{L(n,m-1)}}{w_{L(n,m-1)} + w_{L(n,m+1)}},$$

where $L(n, m) := \sum_{i=1}^{n} I\{X_i = m\}$ is the number of visits to the site m by time n, and w_k, $k \in \mathbb{Z}_+$, is a fixed sequence of positive numbers, referred to as "weights." Note that the most commonly studied case is when $w_k = k+1$.

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In the above form, this process, called *Vertex-reinforced random walk*, or VRRW for short, has been introduced in Ref. 7, while the notion of the VRRW dates back to Ref. 6 and makes the contrast to (Edge) reinforced random walks defined in Copper smith and Diaconis. Especially after the publication of Ref. 7, VRRW has been drawing a lot of attention. In Ref. 11, VRRW on arbitrary graphs has been studied; in Ref. 1 and 2 some properties of more general nonhomogeneous VRRW on $\mathbb{Z}$ were investigated. Finally, 10 proves an important conjecture from Pemantle and Volkov[7] (1987, Unpublished manuscript) about the behavior of linearly-reinforced random walk, while Ref. 8 and references therein provide a broad review of various reinforced processes.

Now let $R = \{m \in \mathbb{Z} : X_n = m \text{ for some } n\}$ be the range and $R' = \{m \in \mathbb{Z} : X_n = m \text{ for infinitely many } n\}$ be the effective range of the VRRW. We say that the VRRW *gets stuck* if R is finite. It is also obvious that $R' \subseteq R$, and that if R' is not empty, then it must consist of a sequence of consecutive integers.

Throughout the paper, we will write $w_k \sim k^\alpha$ whenever there exists

$$0 < \lim_{k \to \infty} \frac{w_k}{k^\alpha} < \infty.$$

Theorem 1. Suppose that $w_k \sim k^\alpha$. Then

(a) if $\alpha < 1$ then $|R| = \infty$ and $|R'| \in \{0, \infty\}$;
(b) if $\alpha = 1$ and $w_k \equiv k + 1$ then $|R| < \infty$ and $|R'| = 5$;
(c) if $\alpha > 1$ then $|R| < \infty$ and $|R'| = 2$.

Further we will study three special cases.

2. SUBCRITICAL CASE

Throughout this section we assume that the sequence of weights w_k satisfies the following conditions:

(a) there exists $0 < \gamma \leq 1$ such that

$$w_n \geq \gamma w_k \text{ whenever } n > k \tag{2.1}$$

   ($\gamma = 1$ corresponds to increasing sequences);

(b) for any $r > 1$

$$\limsup_{n \to \infty} \frac{w_{\lfloor rn \rfloor}}{w_n} < \infty; \tag{2.2}$$

(c)

$$\sum_{k=1}^{\infty} \frac{1}{w_k} = \infty; \tag{2.3}$$

(d) for any integer $r > 1$ there is $z > 0$ such that

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{zn} \frac{1}{w_k}}{\sum_{k=1}^{n} \frac{1}{w_{rk}}} < \gamma, \tag{2.4}$$

where γ is the same as in condition (2.1).
Here and further in the text $\lfloor \cdot \rfloor$ denotes the integer part.

Remark 1. If $w_k \sim k^\alpha$, $\alpha < 1$, then the conditions (2.1)–(2.4) are fulfilled.

Note that the regularity conditions (2.1) and (2.2) are more of a technical nature and ensure that the sequence of weights does not oscillate wildly, while (2.4) and especially (2.3) are more crucial. Also, provided (2.2) holds, both (2.3) and (2.4) follow from a stronger but simpler requirement on w_k's:

$$(\text{c-d}') \;\; \limsup_{n \to \infty} \frac{W(zn)}{W(n)} \to 0 \text{ as } z \downarrow 0, \tag{2.5}$$

where $W(a) = \sum_{k=1}^{\lfloor a \rfloor} 1/w_k$. Indeed, if the sum in (2.3) were finite, then the limit in (2.5) would be always 1. Next, for a fixed integer $r > 1$ it follows from (2.2) that there is $C = C(r) > 0$ such that $w_{rk} \le C w_k$ for all k, consequently

$$\limsup_{n \to \infty} \frac{\sum_{k=1}^{zn} \frac{1}{w_k}}{\sum_{k=1}^{n} \frac{1}{w_{rk}}} \le \limsup_{n \to \infty} \frac{\sum_{k=1}^{zn} \frac{1}{w_k}}{\sum_{k=1}^{n} \frac{1}{Cw_k}} = C \limsup_{n \to \infty} \frac{W(zn)}{W(n)},$$

which can be made arbitrary small by choosing small enough $z > 0$, thus ensuring (2.4).

Theorem 2. Suppose that the sequence of weights satisfy (2.1)–(2.4). Then the range R of VRRW is infinite a.s., that is the VRRW does not get stuck.

We start with auxiliary statements first.

Proposition 1. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent exponential random variables with rates $\lambda_1, \lambda_2, \ldots$, respectively, such that $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$, and $\lambda_n \geq \gamma \lambda_k$ for any $n > k \geq 1$. Then the sum $S_n := \sum_{k=1}^{n} \xi_k$ satisfies

$$\lim_{n \to \infty} \frac{S_n}{\mathbb{E}\, S_n} = 1 \text{ a.s.}$$

Proof of Proposition 1. Since $\mathbb{E}\, S_n = \sum_{k=1}^{\infty} 1/\lambda_k \to \infty$, Kolmogorov's law of large numbers for independent random variables (see, e.g. Ref. [9], p. 389) implies that it is sufficient to check that

$$\sum_{n=1}^{\infty} \frac{\mathbb{V}\text{ar}\, \xi_n}{(\mathbb{E}\, S_n)^2} < \infty. \tag{2.6}$$

However,

$$\frac{\mathbb{V}\text{ar}\, \xi_n}{(\mathbb{E}\, S_n)^2} = \frac{\lambda_n^{-2}}{(\lambda_1^{-1} + \cdots + \lambda_n^{-1})^2} \leq \frac{\lambda_n^{-2}}{(\gamma \lambda_n^{-1} + \cdots + \gamma \lambda_n^{-1})^2} = \frac{1}{\gamma^2 n^2}$$

and therefore (2.6) is verified. $\qquad\square$

Lemma 1. Consider a Pólya-type urn with green and red balls, with the probability to choose a ball of certain color being proportional to w_k, where k is the number of the balls of that color. After a green ball is chosen for the kth time, we add 1 green ball to the urn. After a red ball is chosen for the kth time, we add η_k red balls, where η_k's are i.i.d. geometric random variables independent of the state of the urn, with $\mathbb{P}(\eta_k = m) = (1-\theta)\theta^{m-1}$, $m = 1, 2, \ldots$, for some $0 < \theta < 1$. Then

$$\limsup_{n \to \infty} \frac{R_n}{G_n} < \infty \quad \text{a.s.}, \tag{2.7}$$

where R_n (G_n resp.) is the number of red (green resp.) balls in the urn after the nth ball of either color is chosen, and $n = 0, 1, 2, \ldots$

Proof of Lemma 1. We will use Rubin's construction for decoupling of urns. This construction was introduced in Ref. 3, further examples of its application can be found in, e.g. Refs. 4 and 5. The construction runs as follows. Consider two weakly increasing right-continuous processes $X(t)$ and $Y(t)$. Let $X(t)$ be a pure birth process with $X(0) = G_0$ and

$$\mathbb{P}(X(t+dt) = k+1 \mid X(t) = k) = w_k\, dt$$

and $Y(t)$ be a compound birth process with $Y(0) = R_0$ and

$$\mathbb{P}(Y(t + dt) = k + n \mid Y(t) = k) = (1 - \theta)\theta^{n-1} \times w_k dt, \qquad n = 1, 2, \ldots$$

Then the pair $(X(t), Y(t))$ considered jointly at the times of the jumps of either of the processes has the distribution of the urn described in the lemma, with X (Y, resp.) corresponding to the number of green (red resp.) balls.

Next, set $t_0 = s_0 = 0$ and for $k \geq 1$ let

$$t_k = \inf\{t : X(t) > X(t_{k-1})\} \equiv \inf\{t : X(t) = G_0 + k\}$$

and

$$s_k = \inf\{s : Y(s) > Y(s_{k-1})\}$$

be the times of those jumps; the increments $\eta_k = Y(s_k) - Y(s_{k-1})$ correspond to the numbers of added red balls. Let event A_ν be

$$A_\nu = \left\{ \sum_{k=1}^{m} \eta_k \leq \nu m \text{ for all positive integers } m \right\}.$$

Then by the strong law of large numbers, $\mathbb{P}(A_\nu \text{ for some } \nu > 0) = 1$. From now on we will condition on the event A_ν.

Let r be the smallest positive integer exceeding $R_0 + \nu > 1$. Since $Y(s_m) = R_0 + \sum_{k=1}^{m} \eta_k$, the rate at which the process Y would jump, after having jumped exactly m times, $m \geq 1$, is $w_{f(m)}$ where $f(m) = Y(s_m) \leq rm$. Consequently, for each m, $\Delta_{m+1} := s_{m+1} - s_m$, having an exponential distribution with parameter $w_{f(m)}$, is by (2.1) stochastically larger than an exponential random variable Δ'_m with parameter w_{rm}/γ.

Now we apply Proposition 1 to the sequence of independent exponential random variables Δ'_k with $\lambda_k = w_{rk}/\gamma$ which by (2.1), (2.3), and (2.4) satisfies its conditions. Together with the fact that Δ_k's are stochastically larger than Δ'_k's, this yields

$$\liminf_{n \to \infty} \frac{\sum_{k=1}^{m} \Delta_k}{\sum_{k=1}^{m} 1/w_{rk}} \geq \gamma \quad \text{a.s.} \tag{2.8}$$

On the other hand (2.4) implies that there exists $z = z(\nu) > 0$ and a small $\varepsilon > 0$ such that for m sufficiently large $\sum_{k=1}^{zm} 1/w_k \leq \gamma(1 - \varepsilon) \sum_{k=1}^{m} 1/w_{rk}$. Combining this with (2.8) we have for large m

$$s_m = \sum_{k=1}^{m} \Delta_m \geq \gamma(1 - \varepsilon/2) \sum_{k=1}^{m} 1/w_{rk} \geq (1 + \varepsilon/2) \sum_{k=1}^{zm} 1/w_k.$$

Applying similar arguments to the process $X(n)$, we conclude that since the differences $n_{k+1} - n_k$ have independent exponential distributions with rates w_{k+G_0}, respectively, by Proposition 1

$$\lim_{i \to \infty} \frac{n_i}{\sum_{k=G_0}^{i+G_0-1} 1/w_k} = 1.$$

Therefore, for m large,

$$n_{zm-G_0} \le (1+\varepsilon/2) \sum_{k=1}^{zm} 1/w_k \le s_m.$$

Hence for large m we have $X(s_m) \ge zm - G_0$, yielding

$$\liminf_{t \to \infty} \frac{X(t)}{Y(t)} = \liminf_{m \to \infty} \frac{X(s_m)}{Y(s_m)} \ge \liminf_{m \to \infty} \frac{zm - G_0}{rm} = z/r > 0.$$

Now recall that the event A_ν on which we have conditioned, occurs a.s. for some ν, whence $\limsup_{n \to \infty} Y(n)/X(n) < \infty$ a.s., and hence (2.7) follows.
$\square$

Proposition 2. Suppose that the site i is visited by the VRRW infinitely often. Let a_n, b_n, c_n, and d_n $(n=1,2,\dots,)$ be the local times at the sites $\{i-1, i, i+1, i+2\}$, respectively. Then, on the event

$$\limsup_{n \to \infty} \frac{d_n}{b_n} = M < \infty$$

we also have a.s.

$$\limsup_{n \to \infty} \frac{c_n}{a_n} < \infty.$$

Proof. Consider the VRRW at the times n when $X_n = i$. Then, for sufficiently large times, between two such consecutive times either a_n increases *at least* by one, or c_n increases by a number, which is stochastically smaller than a geometric random variable with some fixed parameter θ, depending on M only. Indeed, every time[a] the walk visits the site $i+1$ at time n, the probability to jump right is smaller than $w_{d_n}/1 w_{b_n}$. And since for all large n the ratio $d_n/b_n \le 2M =: r$, condition (2.2) implies that w_{d_n}/w_{b_n} for large n is bounded by $\theta := 2 \limsup_{m \to \infty} w_{\lfloor rm \rfloor}/w_m < \infty$. Therefore, we can make a stochastic comparison between the pair (a_n, c_n) and the urn

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[a] Note that if the site $i+1$ is visited finitely many times then the statement of the proposition follows immediately.

described in Lemma 1, where a_n is coupled with the number of green balls and c_n is coupled with the number of red balls. Then the number of green balls will be always smaller than a_n, while the number of red balls will be larger than c_n. Therefore, by Lemma 1, the ratio c_n/a_n remains smaller than some finite number as $n \to \infty$. ☐

Lemma 2. The effective range R' is either empty or infinite.

Proof. Indeed, suppose that $R' \neq \emptyset$ and yet R' is finite, then there are two integers j and k, $j < k$, such that $R' = [j, j+1, \ldots, k]$. First, apply Proposition 2 to the segment $[k-2, k-1, k, k+1]$ to obtain that

$$\limsup_{n \to \infty} \frac{L(n, k)}{L(n, k-2)} < \infty$$

since $L(n, k+1)$ is bounded as $n \to \infty$. Next, apply Proposition 2 recursively to $[k-3-i, k-2-i, k-1-i, k-i]$ for $i = 0, 1, 2, \ldots, k-j-2$ to get eventually that

$$\limsup_{n \to \infty} \frac{L(n, j+1)}{L(n, j-1)} < \infty,$$

which contradicts the assumption that $\{j+1\} \in R'$ but $\{j-1\} \notin R'$.

Proof of Theorem 2. If R is finite, then the effective range $R' \subseteq R$ is also finite and nonempty, which contradicts Lemma 2. ☐

3. CRITICAL CASE

Here we suppose that $w_k = k+1$. Then (2.1)–(2.3) are fulfilled, but (2.4) is not. The VRRW with this sequence of weights has been introduced in Ref.[7], where it was proven that:

(a) $\mathbb{P}(5 \leq |R'| < \infty) = 1$;
(b) $\mathbb{P}(|R'| = 5) > 0$.

In the same paper, it was conjectured (partly based on simulations) that the event in (b) has the probability equal to one; this seemingly "obvious" but in fact very hard to prove conjecture was finally proven in Ref. 10.

4. SUPERCRITICAL CASE

In this section, we consider any sequence of weights satisfying

$$\sum_{k=1}^{\infty} \frac{1}{w_k} < \infty. \tag{4.1}$$

This is a natural counterpart of condition (2.3) of the sub-critical case.

Remark 2. If $w_k \sim k^{\alpha}$, $\alpha > 1$, then (4.1) is fulfilled.

Lemma 3. Suppose that w_k satisfy (4.1). Consider an urn model, in which the probability to choose a ball of certain color (red or green) is proportional to w_k, where k is the number of balls of that color. If a green ball is chosen, 1 green ball is added to the urn. If a red ball is chosen, a positive integer number of red balls is put to the urn, this number being independent of the number of green balls. Then the number of balls of one of the colors remains bounded as the time goes to infinity.

Proof of Lemma 2. Using Rubin's construction, we again consider two birth processes $X(t)$ and $Y(t)$, the latter being a compound birth process, with jump rates given by w_k. The only distinction from the construction presented in the proof of Lemma 1, is that the distribution of the increments η_k of the Y process is not specified anymore; nor are they assumed to be i.i.d. or even independent. Following the last section of Davis [3], it is easy to see that provided (4.1), both processes X and Y explode, that is there are two random stopping times $\tau_X, \tau_Y < \infty$ such that $X(t)$ ($Y(t)$ resp.) is defined for all $t < \tau_X$ (τ_Y resp.) but $\lim_{t \uparrow \tau_X} X(t) = \infty$ ($\lim_{t \uparrow \tau_Y} Y(t) = \infty$ resp.)

Now observe that τ_X is a continuous random variable which is independent of the explosion time of the other process τ_Y (the latter may or may be not continuous, depending on the distribution of increments η_k's). Therefore, $\mathbb{P}(\tau_X = \tau_Y) = 0$ and hence either $X(\cdot)$ or $Y(\cdot)$ reaches infinity while the other still remains finite, yielding that the number of balls of the corresponding color will be bounded by a random finite constant. $\quad\square$

Theorem 3. For a VRRW with the weights satisfying (4.1), $|R'| = 2$.

Proof. First we will show that $|R'| < \infty$. Indeed, if X_n starting from 0 reaches site $k > 0$ by time $T_k = \inf\{n > 0 : X_n = k\}$, the probability that it

will never reach $k+1$ after T_k (hence, ever) is at least

$$\nu := \inf \prod_{j=1}^{\infty} \frac{w_{k_j}}{w_{k_j} + w_0},$$

where the infimum is taken over all strictly increasing sequences of positive integers $(k_1, k_2, \dots,)$. However, it is easy to see that

$$\nu = \prod_{j=1}^{\infty} \frac{w_j}{w_j + w_0},$$

which is a positive constant, due to (4.1). Therefore, by Borel-Cantelli lemma, the range of VRRW is bounded from above. Similar arguments show that it is also bounded from below.

Now suppose that $|R'| \geqslant 3$ and let k be the right-most point of R', whence $(k-2) \in R'$. Consider the numbers of visits to $k-2$ and k from $k-1$, after $k+1$ has been visited for the last time. Let each visit to k correspond to adding of a green ball, and each visit to $k-2$ to adding a red ball. Then the urn process with the balls of these two colors satisfies the conditions of Lemma 3, and consequently only one of the sites k and $k-2$ is visited infinitely often, which contradicts the assumption that both k and $k-2$ lie in R'. □

5. REMAINING PROOF AND OPEN PROBLEMS

Proof of Theorem 1.   From Theorem 2 and Remark 1, it follows that when $\alpha < 1$, then the range of the walk is infinite, hence $|R| = \infty$, and Lemma 2 yields $|R'| \in \{0, \infty\}$.

Part (b) immediately follows from Theorem 1.4 of Tarrès[10], and part (c) from Theorem 3 and Remark 2. □

Now we present a few open problems.

Problem 1.   *In the subcritical case, is it possible that $|R'| = \infty$ but $R' \neq \mathbb{Z}$?*

Problem 2.   *The conditions (2.1)–(2.4) are sufficient to guarantee that the range of VRRW is infinite. Can this also be proven under milder conditions, especially can (2.1) and (2.2) be replaced by something like condition (5.1)?*

Problem 3.   *Prove that in the subcritical case the VRRW is, in fact, recurrent, that is it visits all sites of $\mathbb{Z}$ infinitely often.*

Problem 4. *We have shown that in most cases $|R'| \in \{0, 2, 5, \infty\}$. Is there a sequence of weights satisfying*

$$\inf_{k \in \mathbb{Z}_+} w_k > 0 \tag{5.1}$$

for which $|R'|$ can take a different value?

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