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Structured Stability Margin and the Finite Argument Principle

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Abstract: If the structure of the uncertainty in a linear model is known, it is natural to use this information in robustness analysis. In particular, when the model depends on a number of uncertain parameters one sometimes defines a "structured stability margin" measuring the smallest parameter deviation giving instability. There are different definitions of the structured stability margin. They differ in the way the structure of the uncertainty is prescribed. In this article we suggest a new definition that use the probabilistic distribution of the parameters. We will define and calculate a 'structured stability margin' which is tailor made to make use of covariance information on parametric uncertainty. Such information is typically obtained from a parametric identification.

In the calculation of stability margin it is natural to evaluate the characteristic polynomial along the boundary of the stability region. The 'finite argument principle' is a tool, which can be used to reduce the number of such evaluations. The frequencies will also automatically concentrate to critical regions. We show explicitly, how the finite argument principle can be used to compute the structured stability margin. An example from robustness analysis of a mechanical system is presented.

1. Introduction

Different measures of the robustness of a linear control system design has been suggested in the literature. Most of these measures concern stability robustness although some measures of performance robustness exists. Stability robustness measures how well the controller can maintain closed loop stability when the open loop system changes around a nominal case. This change can be due to either erroneous modeling or to a real change in the process dynamics, e.g. changed operating conditions.

Different robustness measures differ in the way the process uncertainty is introduced. For the classical **gain margin** the open loop system $G(s)$ is perturbed to $kG(s)$. The gain margin is then given by the interval $[k_{min}, k_{max}]$ around the nominal value $k = 1$ for which the closed loop system is stable. For the **phase margin** the system is instead perturbed to $e^{-sT}G(s)$. There are also different robustness measures using the H_∞-norm. These all have in common that the uncertainty, $\Delta(i\omega)$, is assumed to be unstructured. In the complex plane the uncertainty is then restricted to have the form of circles $|\Delta(i\omega)| \leq \rho(i\omega)$. Very often some structure of the unmodeled dynamics is known. It is then conservative to use an unstructured description of the uncertainty, since that includes cases that will not occur in reality.

One way to overcome the conservatism is to use the μ-synthesis, see (Doyle 1982, 1984). In the μ-synthesis different structures can be imposed on the uncertainty. The theory is rather general and allows for the following structure on the uncertain dynamics:

$$\Delta = \text{diag}(\overbrace{\underbrace{\delta_1^r I, \ldots, \delta_R^r I}_{\text{diagonal}}}^{\text{real}}, \overbrace{\underbrace{\delta_1^c I, \ldots, \delta_C^c I}_{\text{diagonal}}, \underbrace{\Delta_1, \ldots \Delta_F}_{\text{fullblocks}}}^{\text{complex}}) \quad (1)$$

Here δ_i^r model parametric uncertainty, δ_i^c is used for uncertainty in the frequency domain and Δ_i^r represent unmodeled dynamics.

DEFINITION 1—(Structured Singular Value)

$$\mu^{-1}(G) = \min_{\Delta}\{\bar{\sigma}(\Delta) \mid \det(I - G\Delta) = 0\}$$

where Δ should vary over all matrices of the form (1). □

A severe drawback with μ is that in it's general form it is very hard to compute.

The choice of appropriate stability robustness measure is affected by things as:

- The more structure or probabilistic information of the uncertainty that is (correctly) used the more useful the measure will be.
- The measure should be computable with a reasonable amount of work.
- The measure should answer questions as: what uncertainty is most critical, what frequency is most critical ?

In section 2 we present a robustness measure ρ_{stab} that is based on parameter covariances. This measure gives very useful information about robust stability in connection with parametric identification. In section 3 we present a fast algorithm for calculating ρ_{stab} using the 'finite argument principle'. Explicit formulas in the frequency domain make the computation time rather insensitive to the number of uncertain parameters. In section 4 the theory is illustrated on an example from mechanics.

2. The Structured Stability Margin

The most basic performance criteria of a control system is stability. We will study stability of a family of characteristic polynomials of the form

$$p(s,\theta) = p_0(s) + \sum_{k=1}^{m} \theta_k \, p_k(s) \qquad (2)$$

Here $p_0(s)$ is the nominal polynomial of degree n, $p_k(s)$ are polynomials of degree less or equal to n and $\theta = \begin{pmatrix} \theta_1 & \ldots & \theta_m \end{pmatrix}$ are uncertain parameters.

EXAMPLE 1—Uncertain Open System
Assume that the linear system

$$y = \frac{B(s)}{A(s)}u = \frac{\sum_{k=0}^{n} b_k s^k}{\sum_{k=0}^{n} a_k s^k}u$$

is controlled with the linear controller

$$R(s)u = -S(s)y + T(s)u_c$$

Assume that the parameters a_k and b_k are unknown. After identification their nominal values are a_k^0 and b_k^0. The closed loop characteristic polynomial can then be written in the form (2) :

$$AR + BS = A_0R + B_0S + \sum(a_k - a_k^0)s^k R + \sum(b_k - b_k^0)s^k S = p_0(s) + \sum \theta_k p_k(s)$$

□

We now exploit the uncertainty information provided by identification to form a realistic robustness measure :

DEFINITION 2—Structured Stability Margin
Consider the family of polynomials of the form (2). Let P be a positive definite matrix. The robustness measure ρ_{stab} is the largest scalar such that all polynomials of the form (2) with

$$\theta^T P^{-1} \theta < \rho_{\text{stab}}^2$$

are stable. □

Relation to other robustness measures

The stability measure we have defined is not a standard μ-measure. It uses an uncertainty structure which in μ can be described by one nonsquare block

$$\Delta = \begin{pmatrix} \delta_1^r & \ldots & \delta_m^r \end{pmatrix}$$

or alternatively one single full real $m \times m$ block and the restriction that G must have rank one.

During the final preparation of this paper we noticed that our stability measure can be seen to be equivalent to a special case of the real stability radius as defined in (Hinrichsen and Pritchard 1988). Our algorithm however has the advantage that we have a simple method for optimization over ω and that the discontinuous case need no separate treatment.

Ellipsoidal uncertainty

The ellipsoidal type of uncertainty is motivated by some general results from identification theory. For prediction error methods the asymptotic (in the number of data) uncertainty is given by :

THEOREM 1—Asymptotic Parameter Variance
Consider the Prediction Error Method

$$\widehat{\theta}_N = \arg\min_\theta V_N(\theta, Z^N)$$

$$V_N(\theta, Z^N) = \frac{1}{N}\sum_{t=1}^{N}\frac{1}{2}\epsilon^2(t,\theta)$$

where Z^N is the data set, ϵ is the prediction errors, and θ the parameters. Assume that the model structure is linear and that there is a unique value θ^* giving perfect model fit such that

$$\widehat{\theta}_N \rightarrow \theta^*, \quad \text{with prob. 1 as } N \rightarrow \infty$$

Then, under some technical conditions we have

$$\sqrt{N}(\theta_N - \theta^*) \in \text{As}\,\mathcal{N}(0, P_\theta)$$

Proof: See (Ljung 1987, pp. 241). □

The covariance matrix P_θ will depend on the data set and different variables in the identification algorithm. P_θ is influenced by e.g. signal excitation, choice of prefilters and choice of control law. Good estimates of P_θ can be obtained based on a finite number of data points. As an example one can for quadratic criteria use

$$\widehat{P}_N = \widehat{\sigma}_N \left[\frac{1}{N}\sum_{t=1}^{N}\psi(t,\widehat{\theta}_N)\psi^T(t,\widehat{\theta}_N)\right]^{-1}$$

$$\widehat{\sigma}_N = \frac{1}{N}\sum_{t=1}^{N}\epsilon^2(t,\widehat{\theta}_N)$$

where ψ are the regressors.

Theorem 1 is based on the hypothesis that there exists a parameter in the parameter set giving a perfect model fit. If this is not true, the model error will contain both a bias term and a variance term. A preliminary result in estimating the variance term for this case is given in (Hjalmarsson 1990).

THEOREM 2—Probabilistic Stability
Assume that the parameters are normally distributed with

$$\theta \in \mathcal{N}(\theta_0, P) \tag{3}$$

and that ρ_{stab} is calculated as in Definition 2. The closed loop system will then be stable with probability

$$\text{prob}_{stab} \geq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\rho_{stab}} e^{-x^2/2}dx \tag{4}$$

Proof: All parameters in the ellipsoid

$$(\theta-\theta_0)^T P^{-1}(\theta-\theta_0) < \rho_{stab}^2 \tag{5}$$

will by definition give a stable closed loop system. Integration over this ellipsoid gives (4). □

Remark. There might be parameters outside the ellipsoid (5) that also give a stable system and the value is therefore only a lower bound.

In the rest of the paper we will work in continuous time and the stability region will be the left half plane. The generalization to other stability regions, for instance the unit circle, is direct and present little difficulty.

3. Calculation of Stability Margin

The structured stability margin can be calculated based on the following result

Proposition.– Imaginary axis sweep

$$\rho_{stab} = \sup\{\rho :\ 0 \notin p(j\omega, \rho\Theta) \text{ for } \omega \in \mathcal{R}\}$$
$$\Theta = \{theta :\ \theta^T P^{-1}\theta \leq 1\}$$

Proof: No Hurwitz polynomial has a zero on the imaginary axis, so $0 \notin p(j\omega, \rho_{stab}\Theta)$ for $\omega \in \mathcal{R}$. On the other hand, by continuity any family that contains both stable and unstable polynomials, must also contain a polynomial with a zero on the imaginary axis. This proves the statement. □

A difficulty in using the proposition is that an infinite number of frequencies must be considered. Instead we will use the finite argument principle below, to reduce the calculation to a small, finite number of frequencies. In the following we use the principle branch of arguments, i.e. $-\pi < \arg \leq \pi$

THEOREM 3—Finite Argument Principle
Suppose p is a polynomial of degree n with complex coefficients. If there are frequencies $-\infty = \omega_1 < \omega_2 < \ldots < \omega_N = +\infty$ such that the equality

$$\sum_{l=2}^{N}\arg\frac{p(j\omega_l)}{p(j\omega_{l-1})} = 2\pi n$$

is well defined and true, then p is Hurwitz. Conversely, if p is a Hurwitz polynomial of degree n, then there are $\{\omega_l\}_{l=1}^{2n+1}$ such that the equality holds.

Proof: Suppose $p(s) = p_0(s-\beta_1)\cdots(s-\beta_n)$. Since $\arg(u_1\cdots u_n) \le |\arg u_1| + \cdots + |\arg u_n|$ with strict inequality if $\arg u_k < 0$ for some k, the equality implies that

$$
\begin{aligned}
2\pi n &= \sum_{l=2}^{N} \arg \frac{p(j\omega_l)}{p(j\omega_{l-1})} \\
&= \sum_{l=2}^{N} \arg \prod_{l=1}^{n} \frac{(j\omega_l - \beta_k)}{(j\omega_{l-1} - \beta_k)} \\
&\le \sum_{l=2}^{N}\sum_{l=1}^{n} \left| \arg \frac{(j\omega_l - \beta_k)}{(j\omega_{l-1} - \beta_k)} \right| \\
&= \sum_{k=1}^{n} \left| \sum_{l=2}^{N} \arg \frac{(j\omega_l - \beta_k)}{(j\omega_{l-1} - \beta_k)} \right| = 2\pi n
\end{aligned}
$$

Strict inequality is impossible, so all factors in the products must have positive argument and p is Hurwitz.

To prove the second part note that if p is Hurwitz, we can choose $\{\omega_l\}_{l=2}^{2n}$ as the zeros of $\mathrm{Re}\ p(i\omega, 0)$ and $\mathrm{Im}\ p(i\omega, 0)$.

□

The stability margin at a fixed frequency

Suppose $p(s,\theta)$ is a polynomial of degree n in s depending linearly on θ. Let $\Theta \subset \mathcal{R}^m$ be convex and bounded. Define for any complex number s

$$
\begin{aligned}
\rho_\alpha(s) &= \sup\{\rho : 0 \notin \mathrm{Re}\ [p(s,\rho\Theta)/\alpha]\} \\
\rho(s) &= \sup\{\rho :\ 0 \notin p(s,\rho\Theta)\}
\end{aligned}
$$

See Figures 1 and 2.

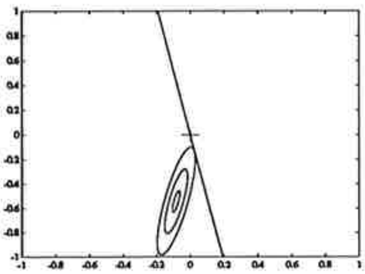

Figure 1. $\rho_\alpha(s)$ **is the size of the smallest ellipsoid touching the the line with normal direction** α

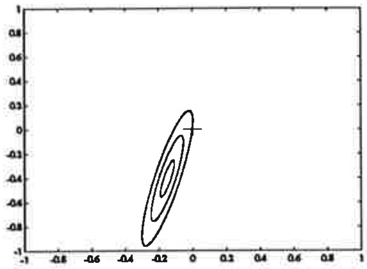

Figure 2. $\rho(s)$ **is the size of the smallest ellipsoid touching the origin**

It is easy to find explicit formulae for $\rho_\alpha(s)$ and $\rho(s)$ in the ellipsoid case. To each complex vector $u \in \mathbf{C}^{m\times 1}$, assign a real $m \times 2$-matrix $\tilde{u} = [\mathrm{Re}\ u\ \mathrm{Im}\ u]$. We then have the following result, which motivates the new notation.

THEOREM 4—Ellipsoidal Θ
Suppose

$$
\begin{aligned}
p(s,\theta) &= p_0(s) + \theta^T p_K(s) \\
\Theta &= \{\theta :\ \theta^T P^{-1}\theta \le 1\}
\end{aligned}
$$

If $\tilde{p}_K(s)$ has full rank, then

$$
\rho_\alpha(s) = \frac{\mathrm{Re}\ (p_0(s)/\alpha)}{\left(\mathrm{Re}\ (p_K(s)/\alpha)^T P\, \mathrm{Re}\ (p_K(s)/\alpha)\right)^{1/2}}
$$

Let $\tilde{\alpha}(s) = \tilde{p}_0(s)[\tilde{p}_K(s)^T P \tilde{p}_K(s)]^{-1}$. Then

$$
\rho(s) = \sup_{\alpha\neq 0} \rho_\alpha(s) = \rho_{\alpha(s)}(s)
$$

Proof: First note that $\mathrm{Re}\ (u/\alpha) = \mathrm{Re}\ (u\bar{\alpha})/|\alpha|^2 = \tilde{u}\tilde{\alpha}^T/|\alpha|^2$. Let $P = L^T L$ and use Cauchy-Schwarz inequality :

$$
\begin{aligned}
(\theta^T \tilde{p}_K(s)\tilde{\alpha}^T)^2 &= (\theta^T L^{-1} L \tilde{p}_K(s)\tilde{\alpha}^T)^2 \ \le \\
&\le\ \theta^T (L^T L)^{-1}\theta \cdot \tilde{\alpha}\tilde{p}_K(s)^T L^T L \tilde{p}_K(s)\tilde{\alpha}^T
\end{aligned}
$$

with equality when the vector $\theta^T L^{-1}$ is parallel to $\tilde{\alpha}\tilde{p}_K(s)^T L^T$. The definition of $\rho_\alpha(s)$ now gives

$$
\begin{aligned}
\rho_\alpha(s) &= \min_{\theta\in\Theta} \frac{\mathrm{Re}\ (p_0(s)/\alpha)}{\mathrm{Re}\ (\theta^T p_K(s)/\alpha)} = \\
&= \frac{\mathrm{Re}\ (p_0(s)/\alpha)}{\left(\mathrm{Re}\ (p_K(s)/\alpha)^T P\, \mathrm{Re}\ (p_K(s)/\alpha)\right)^{1/2}}
\end{aligned}
$$

Next, let $\tilde{p}_K(s)^T P \tilde{p}_K = M^T M$, $M \in \mathcal{R}^{2\times 2}$. Another application of Cauchy-Schwarz inequality gives

$$
\begin{aligned}
\tilde{p}_0(s)\tilde{\alpha}^T &= \tilde{p}_0(s) M^{-1} M \tilde{\alpha}^T \\
&\le \left(\tilde{p}_0(s)[\tilde{p}_K(s)^T P \tilde{p}_K(s)]^{-1}\tilde{p}_0(s)\right)^{1/2} \\
&\quad \cdot \left(\tilde{\alpha}\tilde{p}_K(s)^T P \tilde{p}_K(s)\tilde{\alpha}^T\right)^{1/2}
\end{aligned}
$$

with equality when $\tilde{p}_0(s)M^{-1}$ is parallel to $\tilde{\alpha}M^T$ and in particular when $\alpha = \alpha(s)$. □

Minimization over ω

The following theorem gives both lower and upper bounds on the stability margin and is the basis for the algorithm to calculate ρ_{stab}:

THEOREM 5—Lower and Upper Bounds
Let Θ be an arbitrary convex parameter set, then

$$\rho(j\omega) = \sup_{\alpha \neq 0} \rho_\alpha(j\omega) \qquad (6)$$

Furthermore, if

$$\sum_{2}^{N} \arg[p_0(j\omega_l)/p_0(j\omega_{l-1})] = 2n\pi \qquad (7)$$

then

$$\min_l \sup_{\alpha\neq 0} \min\{\rho_\alpha(j\omega_{l-1}), \rho_\alpha(j\omega_l)\} \leq \\ \leq \rho_{stab} \leq \min_l \rho(j\omega_l). \qquad (8)$$

Proof: Suppose that $\rho < \rho(j\omega)$. Then $0 \notin p(j\omega, \rho\Theta)$. Convexity of $p(j\omega,\rho\Theta)$ implies that $0 \notin$ Re $[p(j\omega,\rho\Theta)/\alpha]$ for some complex $\alpha \neq 0$, so $\rho < \rho_\alpha(j\omega)$. Hence $\rho_\alpha(j\omega) \leq \sup_{\alpha\neq 0}\rho_\alpha(\omega)$ and since the converse inequality is obvious from definitions, the desired equality follows.

Suppose for the second statement that

$$\rho < \min_l \sup_{\alpha\neq 0} \min\{\rho_\alpha(j\omega_{l-1}), \rho_\alpha(j\omega_l)\}$$

Then $0 \notin \mathrm{conv}\{p(j\omega_{l-1},\rho\Theta), p(j\omega_l,\rho\Theta)\}$ for all j, so the function

$$\theta \mapsto \sum_{l=2}^{N} \arg \frac{p(j\omega_{l-1},\theta)}{p(j\omega_l,\theta)}$$

is continuous in $\rho\Theta$. It takes only integer values, so it must be constant and by our assumption the value is $2\pi n$. Hence $p(\cdot,\rho\Theta)$ is Hurwitz and $\rho < \rho_{stab}$. This proves the lower bound on ρ_{stab}. The upper bound is obvious from definitions so the proof is complete. □

Discontinuities of $\rho(j\omega)$

In (Barmish et.al 1990) it was observed that the stability margin can be a discontinuous function of the problem data. The example in the next section is one example where $\rho(j\omega)$ is discontinuous ω.

The discontinuity can however only arise when the polynomials p_k all have equal argument. We have the following result,

Corollary. If $\tilde{p}_K(j\omega)$ has full rank, then $\rho_\alpha(j\omega)$, $\rho(j\omega)$ and $\alpha(j\omega)$ are all continuous at ω.

Proof: This follows from the expressions in Theorem 4. □

Algorithm

We are now ready to formulate the algorithm for computing ρ_{stab}

- Choose as initial frequencies $\omega_2, \ldots \omega_{N-1}$ the zeros of Re $p_0(j\omega)$ and Im $p_0(j\omega)$. Then p_0 satisfies (7).
- At each iteration step, add the frequency $(\omega_l+\omega_{l-1})/2$ obtained from the minimizing l of the lower bound.
- Stop when lower and upper bounds are sufficiently close.

The algorithm has been implemented in Matlab. A typical calculation time on Sun 3/50 is ten seconds. The calculation time is relative insensitive to the number of uncertain parameters.

4. Example

As an example we will study the robustness of an inverted pendulum with uncertain length and mass, see figure 4.

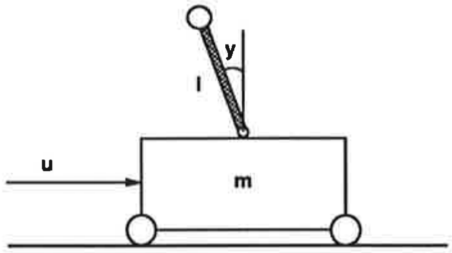

Figure 3. The inverted pendulum with uncertain mass and length.

If the system is linearized around the upright position the transfer function from force to pendulum angle is

$$y = \frac{b}{s^2 - a}u, \qquad b = 1/ml \\ a = g/l$$

The control objective is to stabilize the pendulum around the upright position. The following regulator gives good nominal performance :

$$u = -\frac{s+0.7}{(s+2)^2}y$$

The nominal values after identification are $a_0 = 1$ and $b_0 = 8$. The covariance of the estimates are

$$P = E(\begin{pmatrix} a-a_0 \\ b-b_0 \end{pmatrix} \begin{pmatrix} a-a_0 \\ b-b_0 \end{pmatrix}^T) = \begin{pmatrix} 0.2 & 0.2 \\ 0.2 & 10 \end{pmatrix}$$

$\rho(j\omega)$ is shown in Fig. 4. Note that because of the discontinuity the lower and upper bounds in theorem 5 will be unequal.

Figure 4. $\rho(j\omega)$ **for Example 2. Note that the function is discontinuous at the critical frequency. The lower and upper bounds from theorem 5 are also shown**

The Matlab algorithm gives

$$\rho_{stab} = 0.21,$$

the critical frequency is

$$\omega_c = 1.10,$$

For clarity this example only had two uncertain parameters. The computation time increases relatively slowly with the number of parameters, around 100 parameters are well within the limits of the existing software.

5. Conclusions

We have defined and calculated a structured stability margin that gives interesting information in connection with parameter identification. This stability margin uses information on parameter covariances.

It is not always necessary to be able to maximize a robustness measure over all controllers. It is very seldom that stability robustness is the only design criteria. A robustness measure is mostly helpful as a piece of information to judge a controller design. A good robustness measure can signal a potentially bad design but should not be used as the only design criteria. We have not studied how to minimize our robustness measure over all controllers. This is an area for future research.

We believe that identification based structured stability margins has a great potential as robustness measures for adaptive robust control. This is another area for future research.

The matlab code to calculate the structured stability margin is available by Email from the authors.

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