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ON STEADY WATER WAVES AND THEIR PROPERTIES

MATS EHRNSTRÖM

Akademisk avhandling som för avläggande av filosofie doktorsexamen vid naturvetenskapliga fakulteten vid Lunds universitet kommer att offentligens försvaras på Matematikcentrum, Sölvegatan 18, sal MH:C, fredagen den 12 september 2008 kl. 13.00. Fakultetsopponent är professor M. Groves, Universität des Saarlandes, Saarbrücken, Tyskland.

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Table of Contents

Acknowledgements	i
Table of Contents	iii
Preface	v
References	x
Paper I: Deep-water waves with vorticity	
1 Introduction	3
2 Preliminaries	5
3 Main results	8
References	13
Paper II: Linear water waves with vorticity	
1 Introduction	19
2 Preliminaries	21
3 The linearization	24
4 The phase portraits for right-going waves	29
5 The physical particle paths	37
6 Summary and discussion	42

References	43
-------------------	-----------

Paper III: On streamlines and particle paths

1 Introduction	49
2 Mathematical formulation	51
3 Streamlines and the horizontal velocity	54
4 The forward drift	61
5 The particle trajectories	65
References	67

Paper IV: Travelling Waves for the Whitham Equation

1 Introduction	73
2 Preliminaries	77
3 Existence of periodic travelling waves	79
4 Continuity and compactness of bounded solutions	84
5 Nonexistence of a class of solitary waves	87
6 Numerical Approximation	88
7 Conclusion	93
References	94

Preface

Preface

The mathematical study of water waves has an old and impressive history [9]. Still, several centuries after Leonard Euler derived the equations of hydrodynamics, those continue to pose a great challenge to mathematics. Not only are there numerous physical settings—including assumptions concerning depth, dimension, density, surface tension, wave breaking, viscosity, periodicity, and asymptotic behaviour, to name a few—but also a wide range of mathematical models and techniques have been developed to attack the problem. One of the main reasons for this is the extreme difficulty involved in handling the exact governing equations analytically. As a result, mathematicians also study a variety of so called model equations, the solutions of which in different ways approximate the exact solutions of the original problem [22, 23, 30]. In addition, the advent of modern computers and numerical analysis have brought complementing tools into the field: by means of numerical schemes the time-dependent evolution of water waves can be studied both as discrete approximations of smoother solutions, but also from the more heuristic point of view of images (see e.g. [2]).

One important class of water waves are the periodic *steady waves* [20]. Those are approximately two-dimensional wave trains travelling with constant speed and shape. Generated by wind at sea, they arise as a result of dispersion: larger waves move faster than smaller ones, and eventually sort themselves out [24]. This phenomenon is known as *swell*. The mathematical study of such waves is classically known as the *water-wave problem*, and for a large class of waves it involves the study of a harmonic function [28]. The problem is overdetermined in the sense that, within an arbitrary domain, the boundary conditions at the surface excludes a solution. Instead, the solution of the problem can be reduced to finding an *a priori* unknown, free surface.

Until recently, the study of steady water waves focused primarily, or even almost exclusively, on waves propagating on irrotational currents. Such a current is one for which there is no local rotation, or curl, within the fluid. For waves entering, say, a region of still water, that is an appropriate model. On the opposite side, there are also rotational currents. One then says that *vorticity* is present. An example of this is tidal flow, for which constant vorticity is considered a realistic assumption [27]. Except from constant vorticity, the findings on steady waves with vorticity had for a long time been limited to an explicit solution by Gerst-

ner [3, 19], discovered already in 1809 (and for a long time overlooked [9]). A break-through in this area was the announcement by Constantin and Strauss in [7], that for an arbitrary vorticity distribution there exist steady periodic gravity-waves propagating over a flat bed. Since then, several other results on existence, uniqueness, stability, and other properties of different kinds of waves with vorticity have been accomplished. Among those are the three first papers contained in this thesis.

The first paper [11],

Deep-water waves with vorticity: symmetry and rotational behaviour,

deals with the mathematical theory behind an every-day observation: travelling waves are essentially symmetric with respect to their crestline. The first result in this direction was given by Garabedian in [18], and it concerns irrotational waves for which all the streamlines have a single maximum and minimum within a period. But it was not until much later, with the arrival of [25], that a proof without assumptions on the interior of the fluid was presented. A full history can be found in [11]. Shortly said, the main difficulty with extending the ideas from [25] is that the study of rotational waves is not the study of a harmonic function, but of a function satisfying a nonlinear elliptic partial differential equation. One thus needs to find sharp maximum principles for a general class of vorticity distributions. Specifically, in order to use the Serrin–Aleksandrov method of the moving plane, we rely on an observation presented in [10]. As a result, in [11] we may consider an arbitrary vorticity distribution, and unite the cases of finite and infinite depth. This paper also presents a short proof for that deep-water waves restrict the possible classes of vorticity distributions. As described in the introduction of [11], the paper is closely related to [5] and [6].

The second paper [15],

Linear water waves with vorticity: rotational features and particle paths,

is the result of a joint project with Prof. Gabriele Villari, Florence, and it deals with a subject simultaneously classical and novel. The study of linear water waves is by no means new [9], and already Stokes knew that there was a forward mass drift of the fluid particles within such waves [26]. Considering waves with vorticity is, however, a different issue, and the investigation [15] is the first of its kind. It was triggered by a series of papers—the first of which was [8]—that revisited, and spread new light upon, the problem of understanding and describing

the particle trajectories within travelling waves. In various ways for different kinds of waves, those investigations all show that the classical first approximation [1], depicting the particle trajectories as closed ellipses, needs to be complemented (even in the linear case): in steady periodic waves without vorticity the particles traverse *non-closed* oval orbits with a mean forward drift. In our deduction and investigation of linear waves with constant vorticity, it very soon became evident that vorticity may fundamentally change the interior fluid motion, even though the linearized surface is exactly the same as for irrotational waves. We find in [15] waves with interior vortices, and show that there are particle paths deviating from those of irrotational waves. At the time of publication, the existence of some of the corresponding waves in the setting of the full governing equations was not yet established. Very recently, however, the existence and properties of those have been confirmed for the Euler equations [29], as small perturbations of the ones presented in [15].

For the regular waves investigated in [15]—i.e. waves without stagnation points—there was the question as to what one could analytically deduce for the particle paths of the corresponding exact steady waves with vorticity. That eventually led to the third paper [13]:

On the streamlines and particle paths of gravitational water waves.

As mentioned above, [8] paved the way for new findings on particle trajectories within steady waves. Among those there were also results for Stokes waves [4, 21], i.e. the symmetric and periodic solutions of the irrotational water-wave problem. Since [15] indicated that vorticity has a major impact on the streamlines and the particle trajectories of waves with vorticity, the next step was to see what could be said by combining the ideas from [4] and [15]. The result turned out to be in line with the findings in [15]: for some classes of vorticity distributions, the picture resembles that of irrotational periodic waves; but a general statement for rotational waves could not be achieved. On the other hand, some new results concerning irrotational waves were obtained. In particular, in [13] it is shown that for Stokes waves and small waves with negative vorticity, the mean forward drift of the particles is strictly increasing from bottom to surface. The paper contains several other results, and is a first step towards an understanding of the interior fluid motion within exact steady waves with vorticity. The mathematical techniques are to a large extent *ad hoc* combinations of standard techniques and inequalities, although the basis is sharp maximum principles for elliptic equations [17]. To apply those we use the hodograph transformation reintroduced in [7],

and then exploit the properties of the equations. Some results are also based on an idea developed in [12].

The fourth paper [14] covers a slightly different topic, namely steady waves for an equation modelling irrotational waves. It is the fruit of a collaboration with Dr. Henrik Kalisch, Bergen, and is entitled

Travelling waves for the Whitham equation.

In [30] Whitham proposed a new model for the surface evolution of shallow water waves, the reason for this being that the dispersion relation for the widely used Korteweg–de Vries (KdV) equation does not allow for wave breaking. Whitham thus modified the KdV equation by replacing its dispersion term with the linear dispersion relation obtained from the full Euler equations. The result is no longer a differential equation, but a non-local integro-differential equation with a discontinuous kernel that blows up at the origin. Maybe because of this, the original kernel was in many investigations soon replaced by a continuous approximation matching the decay, but not the local behaviour, of the original one. That approximation yields a differential equation, sometimes referred to as the Burgers-Poisson equation [16], but at times also named the Whitham equation.

In [14] we return to the original equation, considering steady waves from an analytic as well as from a numerical perspective. By means of the Crandall–Rabinowitz bifurcation theorem we show existence for periodic waves in a suitable subalgebra of the Wiener algebra. The paper also contains a non-existence result for waves of large speeds, and a compactness theorem that yields convergence of bounded periodic waves. Because of the definition of the Whitham kernel, much of the work in [14] rely on Fourier integrals and Fourier theory in general. For the numerical results—which also indicate convergence to a highest wave—we use spectral methods, closely related to Fourier series. The article has recently been submitted.

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Preface

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Paper I

Deep-water waves with vorticity: symmetry and rotational behaviour

Abstract

We show that for steady, periodic, and rotational gravity deep-water waves, a monotone surface profile between troughs and crests implies symmetry. It is observed that if the vorticity function has a bounded derivative, then it vanishes as one approaches great depths.

1 Introduction

The study of symmetries in partial differential equations is well established, and the recent decades have seen some well-known contributions to the literature of maximum principles, notably [2, 18, 25]. In the field of fluid mechanics these methods have drawn interest because of the symmetries present in the governing equations. In particular, there are several results concerning the symmetry of steady gravity water-waves. These results date back at least to [16]. The author considers wave trains propagating with a steady speed across the open sea. These are two-dimensional periodic waves with gravity as the dominant restoring force. In this steady context, when the observer travels along with the wave, a streamline is the path of a fluid particle (though from a fixed reference point this is not the case, cf. [4, 8]). The author shows that if every streamline has one minimum and one maximum per period, then the wave is symmetric around the crest. The proof given in [16] was later simplified in [26]. In between, the authors of [10] took on the problem of proving symmetry for steady solitary waves. That investigation showed that for irrotational solitary waves the surface profile has to be monotonically decreasing from the crest. The basic tools of the symmetry argument—being maximum principles and the connected Aleksandrov method of moving planes—were used by the same authors also in [11]. Related papers are [1, 23].

In [24] the authors present a proof of symmetry based on the assumption that the surface profile has a positive derivative from trough to crest. As in earlier papers, the method of moving planes is used in a clever way. This idea is sometimes attributed also to Serrin for his famous paper [25]. The methods were recently

generalized to include water waves with vorticity, i.e. waves propagating on a rotational current. The main difference between the governing equations in this case and in the irrotational setting is that the refined methods of complex analysis cannot be used in the same way, since the velocity potential is not a harmonic function when vorticity is present. In the papers [6, 7], the authors prove that if the surface profile is strictly monotone between crests and troughs, then the wave is symmetric. This is done for periodic gravity waves on finite and infinite depth, respectively. The class of vorticity distributions allowed for is the one where the vorticity is a non-increasing function of the depth.

From a physical point of view this assumption is reasonable, but it is of interest to investigate the case of a general vorticity. The recent paper [13] shows that it is possible to apply similar methods to the general case. That discovery was utilized in [5], where the results from [7] were extended to include general vorticity distributions. A complementing view was offered in [21]: it is assumed that every streamline attains its minimum below the trough, and that the trough is the single lowest point in a period; if the surface profile is also locally strictly monotone around the trough, it is proved that the wave is symmetric and the surface profile strictly monotone from trough to crest. Though based on maximum principles, the method of proof in [21] is different from that of [5]. Moreover, while the assumption of [5] is a requirement only on the surface profile, the conditions in [21] are mainly “vertical” and require precise knowledge of the streamlines below the surface.

The aim of the present paper is to generalize the corresponding deep-water investigation [6] to an arbitrary vorticity distribution. In doing so we unify the cases of finite and infinite depth, thereby extending the results of [5] to deep-water waves. We also prove that for any steady deep-water wave—including solitary waves and waves with capillarity—if the vorticity function has a bounded derivative, then it tends to zero as it reaches great depths. That result relaxes the assumptions of a similar proposition in [6] concerning vorticity distributions for which the vorticity decreases with greater depths. In particular, it yields non-existence for deep-water waves of constant vorticity.

The methods used for the proof of symmetry are essentially the same as in [6, 7, 5]. To handle the larger class of vorticity distributions we use a partial hodograph transform, reintroduced in [9], and dating back to [12]. The main differences from [6, 7] lies in the use of a highly nonlinear operator defined in [13], and the fact that the proof is reduced to handle only two cases instead of

three. This however was observed also in [27]. Note also that our approach unifies the cases of finite and infinite depth. The proof of the vanishing vorticity is totally different from that in [6]. It is very simple, in the sense that it is based upon an application of the Green's identity.

2 Preliminaries

Let $d > 0$ be the depth below the mean water level, $y = 0$, so that the flat bottom can be described by $y = -d$. The *free surface* is then represented by $y = \eta(x) \in C^3(\mathbb{R}, \mathbb{R})$, and

$$\Omega_\eta := \{(x, y) \in \mathbb{R}^2: -d < y < \eta(x)\}$$

denotes the *fluid domain*. A point $(x, \eta(x))$ for which $\eta(x) = \min_{x \in \mathbb{R}} \eta(x)$ is called a *trough*, and if $\eta(x) = \max_{x \in \mathbb{R}} \eta(x)$ it is said to be a *crest*. In the following, we shall assume that we have a solution $\psi \in C^2(\overline{\Omega}_\eta)$ to the system

$$\begin{aligned} \Delta\psi &= -\gamma(\psi), & (x, y) &\in \Omega_\eta \\ |\nabla\psi|^2 + 2gy &= C, & y &= \eta(x) \\ \psi &= 0, & y &= \eta(x) \\ \psi &= -p_0, & y &= -d, \end{aligned} \tag{2.1}$$

that is L -periodic in the x -variable, $0 < L < \infty$. To deal also with deep-water waves, we accept $d = \infty$, and in that case the last line of (2.1) is exchanged for

$$\nabla\psi \rightarrow (0, -c) \quad \text{uniformly for } x \in \mathbb{R} \quad \text{as } y \rightarrow -d.$$

The system (2.1) can be deduced from the Euler equations (see e.g. [7, 9, 20, 28] for a more detailed discussion). Here, p_0 is the *relative mass flux* (cf. Remark 2.1 below), the *vorticity function* $\gamma: [0, -p_0] \rightarrow \mathbb{R}$ is continuously differentiable, $g > 0$ is the gravitational constant of acceleration, and C is a constant related to the energy. The second equation of (2.1) is sometimes referred to as the *Bernoulli surface condition*. The setting is that of gravitational water waves, hence the influence of capillarity is neglected in (2.1).

In the deduction of the system (2.1), the *stream function* ψ is defined (up to a constant) by

$$\psi_x = -v, \quad \psi_y = u - c < 0,$$

where u and v are the horizontal and vertical velocity, respectively, and $c > 0$ is the constant horizontal speed of propagation. The requirement that $\psi_y < 0$ in the fluid domain is used for the transformation of the Euler equations to the system (2.1). This assumption is supported by physical measurements [22]: for a wave not near breaking or spilling, the speed of an individual fluid particle is far less than that of the wave itself. In this paper we consequently demand that

$$\psi_y \leq -\delta < 0 \quad \text{in the closure } \overline{\Omega_\eta},$$

for some $\delta \in \mathbb{R}$.

When $\psi_y < 0$ in Ω_η , the system (2.1) is equivalent to yet another system (cf. [9]). Since for every fixed $x \in \mathbb{R}$, the mapping $\psi \leftrightarrow y$ is a bijection in Ω_η , we can use a partial hodograph transform, setting

$$(q, p) := (x, -\psi),$$

in the domain

$$\Omega := \{(q, p) \in \mathbb{R}^2 : p_0 < p < 0\},$$

where $p_0 := -\lim_{y \rightarrow -d} \psi(x, y) < 0$. In the case of finite depth, if one lets $h(q, p) := y(q, p) + d$ be the height above the flat bottom, then

$$h_q = -\frac{\psi_x}{\psi_y} = \frac{v}{u - c}, \quad h_p = -\frac{1}{\psi_y} = \frac{1}{c - u},$$

and the following system emerges as an equivalent of (2.1):

$$\begin{aligned} (1 + h_q^2) h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \gamma(-p) h_p^3 &= 0, & p \in (p_0, 0), \\ 1 + h_q^2 + (2gh - Q) h_p^2 &= 0, & p = 0, \\ h &= 0, & p = p_0. \end{aligned} \tag{2.2}$$

Here $Q > 0$ is determined by the constant C in (2.1), and the *height function* $h \in C^2(\overline{\Omega})$ is periodic with period $L > 0$. Knowing $h(q, 0)$ is equivalent to knowing the free surface $y = \eta(x)$; indeed $h(q, 0) = \eta(q) + d$. For deep-water waves one instead lets

$$h(q, p) := y(q, p) - Q/(2g),$$

and the condition that $h = 0$ on the bottom $p = p_0$ is exchanged for

$$\nabla h(q, p) \rightarrow (0, 1/c) \quad \text{uniformly in } q \in \mathbb{R} \quad \text{as } p \rightarrow p_0 = -\infty. \tag{2.3}$$

Remark 2.1.

- (i) The exact values of the quantities g, C, Q do not matter in our investigation.
- (ii) The notion of *relative mass flux* [9] captures the physical fact that the amount of water passing any vertical line is constant throughout the fluid domain:

$$\int_{-d}^{\eta(x)} (u(x, y) - c) dy = p_0, \quad x \in \mathbb{R},$$

holds since $u - c = \psi_y$, and ψ is constant on the surface $y = \eta(x)$ as well as on the bottom $y = -d$. Thus, determining the relative mass flux is equivalent to determining the constant value of ψ at the bottom, which is equivalent to determining p_0 in the definition of the domain Ω .

Before presenting the main results we state here two lemmas used in the proof of the main theorem. Lemma 2.2 is adapted specifically to the problem at hand, and Lemma 2.4 contains three simplified versions of classical maximum principles. While i) and ii) are due to Hopf [19], iii) originates from [25]. A good exposition over maximum principles, and the full details of the proofs, can be found in [15].

Lemma 2.2. [13] *Let $h, \tilde{h} \in C^2(\overline{\Omega})$ be solutions of the water-wave problem (2.2), with $h_p > 0$ bounded away from zero. Then there is a uniformly elliptic operator with continuous coefficients,*

$$\begin{aligned} \mathcal{L} = & (1 + h_q^2) \partial_p^2 + h_p^2 \partial_q^2 - 2h_p h_q \partial_p \partial_q \\ & + \left[\tilde{h}_{qq} (h_p + \tilde{h}_p) - 2\tilde{h}_q \tilde{h}_{pq} + \gamma(-p) (h_p^2 + h_p \tilde{h}_p + \tilde{h}_p^2) \right] \partial_p \\ & + \left[\tilde{h}_{pp} (h_q + \tilde{h}_q) - 2h_p \tilde{h}_{pq} \right] \partial_q, \end{aligned} \quad (2.4)$$

which satisfies

$$\mathcal{L} (h - \tilde{h}) = 0 \quad \text{in } \Omega.$$

Remark 2.3. Note that, for any $\lambda \in \mathbb{R}$, $\tilde{h}(q, p) := h(2\lambda - q, p)$ is also a solution to (2.1). This follows since all the q -derivatives go in pairs, making the minus signs cancel each other.

Lemma 2.4. [15] *Let $\Omega \subset \mathbb{R}^2$ be a rectangle, possibly extending infinitely in one direction, $w \in C^2(\overline{\Omega})$, and suppose that $\mathcal{L}w = 0$ for some uniformly elliptic operator $\mathcal{L} = \sum_{i,j} a_{ij}\partial_{ij} + \sum_i b_i\partial_i$ with continuous coefficients in $\overline{\Omega}$. Then the following hold:*

- i) [The maximum principle] *If $\min_{\overline{\Omega}} w$ or $\max_{\overline{\Omega}} w$ is attained in the interior of Ω , then w is a constant in $\overline{\Omega}$.*
- ii) [The Hopf boundary-point lemma] *Let p be a point on the smooth part of the boundary $\partial\Omega$ such that $w(p) < w(x)$ or $w(p) > w(x)$ for all $x \in \Omega$. Then $\nabla w(p) \neq (0, 0)$.*
- iii) [The Serrin edge-point lemma] *Let p be a corner point on $\partial\Omega$ such that $w(p) < w(x)$ or $w(p) > w(x)$ for all $x \in \Omega$. Suppose also that $a_{12}(p) = 0 = a_{21}(p)$. Then at least one of the first or second partial derivatives of w is non-vanishing at p .*

Remark 2.5. Substituting \mathcal{L} for the explicit operator presented in Lemma 2.2 we see that the condition $a_{12}(p) = 0 = a_{21}(p)$ in Lemma 2.4 (iii) is satisfied exactly when $h_q = 0$ (remember that $h_p > 0$ by assumption). This will be used in the proof of Theorem 3.1.

3 Main results

Theorem 3.1. *Steady periodic gravity water waves with a monotone surface profile between troughs and crests are symmetric.*

Remark 3.2. In [6, 7] monotonicity means strictly increasing from trough to crest, and vice versa. We require only that the surface be non-decreasing (non-increasing) on the same intervals. Since “trough” and “crest” then become somewhat ambiguous, they should be used for the middle points of possible flat troughs or crests. If the vorticity decreases with depth—which is assumed in [6, 7]—it is shown in [14] that such flat parts are impossible at the surface of symmetric waves.

Remark 3.3. In [24]—whose authors proposed the result in the irrotational setting—it is suggested that the same result is valid for waves with capillarity. So far, we have been unable to extend our result to such a general setting.

Wind-generated vorticity is initially a surface process, therefrom penetrating the fluid downwards. It is therefore physically reasonable to assume that the rotational motion decreases with depth. Assuming this, the authors of [6] showed that the vorticity vanishes as one approaches great depths. The following observation vindicates that proposition with a different type of argument, and extends the setting to a larger class of vorticity distributions. The result is consistent with the only known explicit solution of steady gravity deep-water waves, the Gerstner wave (see [3, 17]).

Theorem 3.4. *For any steady deep-water wave, if $\|\dot{\gamma}\|_\infty < \infty$, then the vorticity function satisfies $\gamma(\psi(x, y)) \rightarrow 0$, uniformly for $x \in \mathbb{R}$, as $y \rightarrow -\infty$.*

Proof of Theorem 3.1. Since the system (2.2) is symmetric in the q -variable—one may as well consider $h(-q, p)$ —we can always assume that the horizontal position of the crest is in $[0, L/2)$. For a reflection parameter, $\lambda \in (-L/2, 0)$, let us introduce the reflection of q around λ ,

$$q^\lambda := 2\lambda - q,$$

and the associated *reflection function*,

$$\begin{aligned} w(q, p; \lambda) &:= h(q, p) - h(2\lambda - q, p), \\ (q, p) &\in [\lambda, 2\lambda + L/2) \times [p_0, 0]. \end{aligned}$$

In the setting of infinite depth, w is defined only for $p \in (p_0, 0]$. The reflection function satisfies the boundary conditions

$$\begin{aligned} w(\lambda, p; \lambda) &= 0, \\ \lim_{p \rightarrow p_0} w(q, p; \lambda) &= 0, \quad \text{uniformly for } q \in \mathbb{R}. \end{aligned} \tag{3.1}$$

The first property is immediate from the definition of $w(q, p; \lambda)$, and the second follows from the bottom boundary condition of (2.2) and (2.3); in the deep-water case by an application of the mean value theorem. Since the surface profile is non-decreasing from trough to crest by assumption, the reflection function satisfies $w(q, 0; \lambda) \geq 0$ for λ close enough to the trough. Hence exists

$$\lambda_0 := \sup\{\lambda : w(q, 0; \lambda^*) \geq 0 \text{ for all } \lambda^* \in (-L/2, \lambda)\},$$

and it suffices to consider the following two cases:

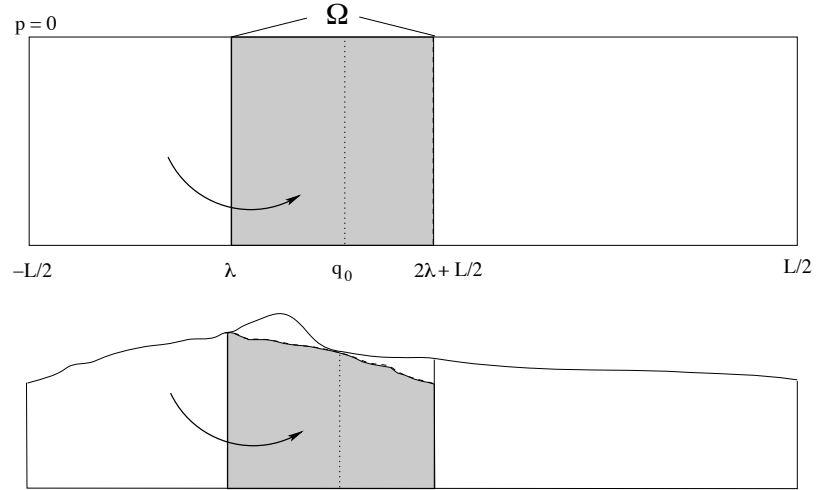


Figure 1: The (q,p) -reflection (top), steady variables below.

i) $\lambda_0 = 0$.

ii) $\lambda_0 \in (-L/2, 0)$, and there exists $q_0 > \lambda_0$ for which $w(q_0, 0; \lambda_0) = 0$.

In the case of i) the domain of reflection is maximal. The second situation is what occurs if the surface “collides” with its reflection, so that there is a point of tangency at $q = q_0$ (see Figure 1).

Assuming i), and keeping periodicity in mind, we get the additional boundary conditions

$$w(L/2, p; \lambda_0) = 0, \quad w(q, 0; \lambda_0) \geq 0. \quad (3.2)$$

Let us restrict the fluid domain to a half-period,

$$\Omega := (0, L/2) \times (p_0, 0).$$

Lemma 2.2 implies that maximum principles are available for the reflection function in Ω (see Remark 2.3 and note that $w \in C^2(\overline{\Omega})$). By Lemma 2.4 [i] there cannot exist an interior point for which $w(q, p; \lambda_0) \leq 0$, unless w vanishes everywhere in Ω . This follows from the boundary conditions (3.1) and (3.2). If w vanishes completely, we have symmetry, so suppose on the contrary that

$$w(q, p; \lambda) > 0 \quad \text{in } \Omega.$$

At the trough, where $(q, p) = (L/2, 0)$, both w and w_p vanishes by (3.2). Since $h_q(\pm L/2, 0) = 0$, also

$$w_q(L/2, 0; \lambda_0) = 0.$$

Differentiating the Bernoulli surface condition of (2.2) with respect to q , we then get

$$0 = 2h_q h_{qq} + (2gh_q)h_p^2 + 2(2gh - Q)h_p h_{qp} = 2(2gh - Q)h_p h_{qp},$$

forcing $h_{qp}(L/2, 0) = 0$, in view of that $(2gh - Q)h_p$ never vanishes according to the same surface condition. A similar argument holds for the λ_0 -reflection $h(-q, p)$, whence

$$w_{qp}(L/2, 0) = 0.$$

Keeping the definition of $w(q, p; \lambda_0)$ in mind we thus see that at the trough, $(q, p) = (L/2, 0)$, we have that

$$w = w_q = w_p = w_{qq} = w_{qp} = w_{pp} = 0.$$

At the trough it is moreover so that $h_q = 0$ by assumption. According to Lemma 2.4 [iii] and Remark 2.5 this contradicts the earlier conclusion that $w(q, p; \lambda_0) > 0$ in Ω , and we infer that if i) is the case then

$$h(q, p) = h(-q, p),$$

meaning that the wave is symmetric around the crest, located at $q = 0$.

Now for the case of ii). It is clear that $w(q, 0; \lambda_0) > 0$ for all $q > \lambda_0$ is impossible, for this would contradict the maximality of λ_0 , in view of that the crest is located to the right of $q = 0$. We shall increase the domain of definition of w by the following extension:

$$\begin{aligned} w(q, p; \lambda_0) &:= h(q, p) - h(2\lambda_0 + L - q, p), \\ (q, p) &\in (2\lambda_0 + L/2, \lambda_0 + L/2] \times [p_0, 0]. \end{aligned}$$

As above, in the setting of infinite depth we require $p \in (p_0, 0]$. If we redefine

$$\Omega := (\lambda_0, \lambda_0 + L/2) \times (p_0, 0),$$

it turns out that periodicity guarantees $w \in C^2(\overline{\Omega})$. Furthermore, note that as long as $2\lambda + L/2$ lies to the left of the crest, $w(q, 0; \lambda) \geq 0$ always holds for

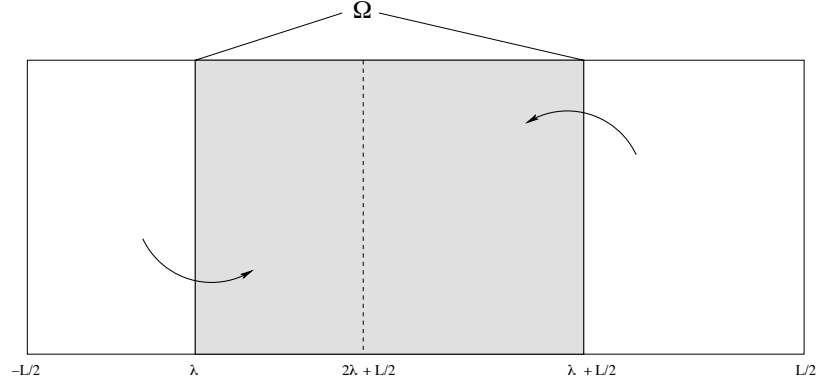


Figure 2: Symmetrization in the (q,p) -variables.

$\lambda < q \leq 2\lambda + L/2$, which follows from monotonicity of the surface profile. In the case of ii), we thus have that $2\lambda_0 + L/2$ lies to right of—or at least in line with—the crest. Consequently ii) implies that $h(q, p)$ is non-increasing for $q \in (2\lambda_0 + L/2, L/2)$, yielding the useful boundary condition

$$w(q, 0; \lambda_0) \geq 0 \quad \text{for all } q \in [\lambda_0, \lambda_0 + L/2].$$

Summing up the situation, we have

$$\begin{aligned} w(\lambda_0, p; \lambda_0) &= 0, & w(\lambda_0 + L/2, p; \lambda_0) &= 0 \\ w(q, 0; \lambda_0) &\geq 0, & \lim_{p \rightarrow p_0} \sup_{q \in \mathbb{R}} |w(q, p; \lambda_0)| &= 0, \end{aligned}$$

for $(q, p) \in \overline{\Omega}$. As in dealing with i), we shall make use of Lemma 2.2. By that, Remark 2.3, and the regularity of w , we may apply Lemma 2.4 [i] to conclude that

$$w(q, p; \lambda_0) > 0 \quad \text{in } \Omega, \quad \text{unless } w \text{ vanishes identically.}$$

Suppose first that w vanishes identically. Then $h(q, 0) = h(L/2, 0)$ whenever $2\lambda_0 + L/2 \leq q \leq L/2$, as a consequence of that $h(q, p)$ is non-increasing on that interval. Moreover $h(q, 0)$ is symmetric around λ_0 for $q \in [-L/2, 2\lambda_0 + L/2]$, so it is fact symmetric for all q . Then $h(q, p) - h(2\lambda - q, p)$ vanishes whenever $p = 0$ or $p \rightarrow p_0$. Once more applying Lemma 2.4 then yields that $h(q, p)$ is symmetric around $q = \lambda_0$, which then must be the location of the crest.

Now assume that $w(q, p; \lambda_0) > 0$ in Ω . At $(q_0, 0)$, the only possibility is $w_q(q_0, 0) = 0$ meaning that $h_q(q_0, 0) = -h_q(2\lambda_0 - q_0, 0)$. Since $h(q_0, 0) = h(2\lambda_0 - q_0, 0)$, the Bernoulli surface condition of (2.2) forces $h_p(q_0, 0) = h_p(2\lambda_0 - q_0, 0)$. Consequently

$$\nabla w(q_0, 0) = 0,$$

contradicting Lemma 2.4 [ii], and in extension the assumption that $w(q, p; \lambda_0) > 0$ in Ω . \square

Proof of Theorem 3.4. For a contradiction, suppose there exists a sequence $(x_n, y_n)_{n \geq 1}$ with $y_n \rightarrow -\infty$ as $n \rightarrow \infty$, such that $|\gamma(\psi(x_n, y_n))| \geq \alpha > 0$. Let $\varepsilon > 0$, and let R_n be squares containing (x_n, y_n) , all of fixed horizontal length

$$|R_n| = \delta = \min \left\{ \frac{1}{4}, \frac{\alpha}{2\sqrt{2}(c + \varepsilon)\|\dot{\gamma}\|_\infty} \right\}.$$

By the governing equations for deep-water waves, there exists y_ε such that $|\nabla\psi - (0, -c)| < \varepsilon$ whenever $y < y_\varepsilon$. Consequently Green's identity implies that

$$\left| \int_{R_n} \Delta\psi(x, y) \, dx \, dy \right| < 4\delta\varepsilon \leq \varepsilon,$$

whenever R_n lies underneath $y = y_\varepsilon$. For any two points $(x, y), (x_0, y_0) \in R_n$ we have that

$$\begin{aligned} & |\gamma(\psi(x, y)) - \gamma(\psi(x_0, y_0))| \\ & \leq \|\dot{\gamma}\|_\infty |\psi(x, y) - \psi(x_0, y_0)| \\ & \leq \|\dot{\gamma}\|_\infty \sup_R |\nabla\psi| |(x, y) - (x_0, y_0)| \\ & \leq \|\dot{\gamma}\|_\infty (c + \varepsilon) \sqrt{2}\delta, \end{aligned}$$

whence

$$\inf_{R_n} |\Delta\psi| \geq \alpha - \sqrt{2}(c + \varepsilon)\|\dot{\gamma}\|_\infty\delta \geq \alpha/2.$$

It then follows that

$$\frac{\delta^2\alpha}{2} \leq \int_{R_n} |\Delta\psi(x, y)| \, dx \, dy = \left| \int_{R_n} \Delta\psi(x, y) \, dx \, dy \right| < \varepsilon,$$

for all n with $y_n + \delta < y_\varepsilon$. Since δ is uniformly bounded from below for all $\varepsilon < c$, we may let $\varepsilon \rightarrow 0$, yielding $\alpha = 0$, and reaching the desired contradiction. \square

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Paper II

Linear water waves with vorticity: rotational features and particle paths

Abstract

Steady linear gravity waves of small amplitude travelling on a current of constant vorticity are found. For negative vorticity we show the appearance of internal waves and vortices, wherein the particle trajectories are not any more closed ellipses. For positive vorticity the situation resembles that of Stokes waves, but for large positive vorticity the trajectories are affected.

1 Introduction

The subject of this paper are periodic gravity water waves travelling with constant shape and speed. Such wave trains are an everyday observation and, typically, one gets the impression that the water is moving along with the wave. In general, this is not so. Rather, the individual fluid particles display a motion quite different from that of the wave itself. While for irrotational waves, recent studies (see below) have enlightened the situation, we investigate the situation of waves propagating on a rotational current, so that there is a non-vanishing curl within the velocity field.

For irrotational waves, a first approximation shows that the fluid particles move in ellipses, back and forth as the wave propagates above them. This can be found in classical [2, 27, 34] as well as modern [13, 23] text books, and it is consistent with the only known explicit solutions for gravity water waves: the Gerstner wave [4, 18] for deep water, and the edge wave solution for a flat beach [3], both with a depth-varying vorticity. A formal physical argument involving a balance between opposing forces was used in [24] to get a similar result without the use of irrotationality. There are also experimental evidence supporting this picture. Those include photographs [13, 33, 34] and movie films [1].

However, as anyone having used bottle post would guess, there are other findings. Even in [24], where it is asserted that the orbits are elliptic—and where the photographs and movie films are referenced—the author notes that “*I am not aware of any measurements that show that the particle orbits of shallow water waves*

are indeed ellipses.” In fact it was observed already in the 19th century that there seems to be a forward mass drift [35], so that the average motion of an average fluid particle is along with the wave. This phenomenon can be seen by making a second approximation of the governing equations, and it is known as *Stokes drift* (see also [39, 43]). In [28, 29] it was deduced that for steep waves the orbits deviate from simple ellipses. There is also mathematical evidence uniformly showing that a more thorough study of the equations yields non-closed orbits with a slight forward drift. Those include investigations of the precise orbits of the linearized system [6, 11, 20], and two recent papers on exact Stokes waves [5, 19] (steady irrotational and periodic gravity waves which are symmetric and monotone between trough and crest). The relation between such results and experimental data is discussed in [5], where it is argued that the ellipses, at least near the bottom, are approximations of the exact trajectories.

While many situations are adequately modelled by irrotational flows—for example waves propagating into still water—there are situations when such a mathematical model is insufficient. Tidal flow is a well-known example when constant vorticity is an appropriate model [37], a fact confirmed by experimental studies [36]. This is one reason why, recently, the interest for exact water waves with vorticity has increased. At this point existence [10], variational characterization [9], uniqueness [14, 17], symmetry [8], and a unique continuation principle [16] for finite depth steady gravity waves with vorticity are established. There is also a theory for deep-water waves [7, 21], as well as for capillary and capillary-gravity waves [41, 42]. However, due to the intricacy of the problem, studies of the governing equations for water waves are extremely difficult. In-depth analyses are very rare. To gain insight into qualitative features of flows with vorticity, Ko and Strauss recently performed a numerical study [26], extending earlier work by da Silva and Peregrine [37]. We will pursue a different approach. Notice that the intuitive notion of vorticity is captured in what happens when one pulls the tap out of a bath tub. It should therefore come as no surprise that the particle paths of waves travelling upon a rotational current deviate from those in the case of waves without vorticity. That is the main result of this paper. More precisely, we make a first attempt at understanding the particle trajectories by deducing a linear system for constant vorticity. Here, linearity means that the waves are small perturbations of shear flows, hence of small amplitude. The system obtained is solvable in the sense of closed expressions, and thus it is possible to make a phase portrait study of the steady wave.

It is found that for positive vorticity, the steady wave resembles that of the irrotational situation [6, 11], though the particles behave differently if the size of the vorticity is large enough. For negative vorticity, however, we show the existence of a steady periodic surface wave containing an internal wave as well as a vortex, or so called cats-eye (cf. [22] and [30, Ex. 2.4]). For unit depth this situation occurs if the absolute size of the negative vorticity exceeds the wave speed, while in the opposite situation both the steady wave and the physical particle trajectories resemble the irrotational case. When the size of the negative vorticity exceeds the wave speed the particle trajectories of the internal wave behave in the same manner as in the irrotational case – nearly closed ellipses with a forward drift – but within the vortex and the surface wave the particles are moving mainly forward. This indicates that such a wave may be uncommon or unstable, since measurements show that for waves not near breaking or spilling the speed of an individual particle is generally considerably less than that of the wave itself [27]. Such a situation is excluded in [5, 19], and our results are therefore not in contrast to those investigations.

An interesting feature of the phase portrait for negative vorticity is that it captures the almost ideal picture of what vorticity is. It furthermore indicates that in the case of large negative vorticity the governing equations allow for travelling waves very different from the classical Stokes waves (see [38] for a good reference of that subject). Finding those waves with analytic tools requires a novel approach; so far the existence results [10, 21, 40] for steady waves with vorticity rely on the assumption that no particle moves as fast as the wave itself. *This study suggests that the presence of vorticity—even when it is constant—changes the particle trajectories in a qualitative way, that this change depends on the size of the vorticity, and that it applies less to particles near the bottom.*

The disposition is as follows. Section 2 gives the mathematical background for the water-wave problem, while in Section 3 we deduce the linearization and its solution. The main findings are presented in Section 4, and the implications for the particle trajectories in Section 5. In Section 6 we give a brief summary and discussion of our results.

2 Preliminaries

The waves that one typically sees propagating on the surface of the sea are locally approximately periodic and two-dimensional (that is, the motion is identical in

any direction parallel to the crest line). Therefore, for a description of these waves propagating over a flat bed, it suffices to consider a cross section of the flow that is perpendicular to the crest line. Choose Cartesian coordinates (x, y) with the y -axis pointing vertically upwards, and the x -axis pointing rightwards from the point of view of some observer, while the origin lies on the flat bed below the crest. Let $(u(t, x, y), v(t, x, y))$ be the velocity field of the flow, let $h > 0$ be the depth below the mean water level $y = h$, and let $y = h + \eta(t, x)$ be the water's free surface. We assume that gravity is the restoring force once a disturbance was created, neglecting the effects of surface tension. Homogeneity (constant density) is a physically reasonable assumption for gravity waves [27], and it implies the equation of mass conservation

$$u_x + v_y = 0 \tag{2.1a}$$

throughout the fluid. Appropriate for gravity waves is the assumption of inviscid flow [27], so that the equation of motion is Euler's equation

$$\begin{cases} u_t + uu_x + vv_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases} \tag{2.1b}$$

where $P(t, x, y)$ denotes the pressure and g is the gravitational constant of acceleration. The free surface decouples the motion of the water from that of the air so that, ignoring surface tension, the dynamic boundary condition

$$P = P_0 \quad \text{on} \quad y = h + \eta(t, x), \tag{2.1c}$$

must hold, where P_0 is the constant atmospheric pressure [23]. Moreover, since the same particles always form the free surface, we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \quad \text{on} \quad y = \eta(t, x). \tag{2.1d}$$

The fact that water cannot penetrate the rigid bed at $y = 0$ yields the kinematic boundary condition

$$v = 0 \quad \text{on} \quad y = 0. \tag{2.1e}$$

The vorticity, ω , of the flow is captured by the curl,

$$v_x - u_y = \omega. \tag{2.1f}$$

We now introduce a non-dimensionalization of the variables. As above, h is the average height above the bottom, and we let a denote the typical *amplitude*, and λ the typical *wavelength*. It is reasonable, and fruitful, to take \sqrt{gh} as the scale of the horizontal velocity. That is the approximate speed of irrotational long waves [23]. We shall use c to denote the wave speed, and we let

$$c \mapsto \frac{c}{\sqrt{gh}}$$

be the starting point of the non-dimensionalization. We then make the transformations

$$x \mapsto \frac{x}{\lambda}, \quad y \mapsto \frac{y}{h}, \quad t \mapsto \frac{\sqrt{gh}t}{\lambda}, \quad u \mapsto \frac{u}{\sqrt{gh}}, \quad v \mapsto \frac{\lambda v}{h\sqrt{gh}}, \quad \eta \mapsto \frac{\eta}{a}.$$

Having made these transformations, define furthermore a new pressure function $p = p(t, x, y)$ by the equality

$$P := P_0 + gh(1 - y) + ghp.$$

Here P_0 is the constant atmospheric pressure, and $gh(1 - y)$ is the hydrostatic pressure distribution, describing the pressure change within a stationary fluid. The new variable p thus measures the pressure perturbation induced by a passing wave. It turns out that the natural scale for the vorticity is $\sqrt{h/g}$ and we thus map

$$\omega \mapsto \sqrt{\frac{h}{g}} \omega.$$

The water-wave problem (2.1) then transforms into the equations

$$u_x + v_y = 0, \tag{2.2a}$$

$$u_t + uu_x + vv_y = -p_x, \tag{2.2b}$$

$$v_t + uv_x + vv_y = -\frac{\lambda^2}{h^2} p_y, \tag{2.2c}$$

$$\frac{h^2}{\lambda^2} v_x - u_y = \omega, \tag{2.2d}$$

valid in the fluid domain $0 < y < 1 + \frac{a}{h}\eta$, and

$$v = \frac{a}{h} (\eta_t + u\eta_x), \tag{2.2e}$$

$$p = \frac{a}{h} \eta, \tag{2.2f}$$

valid at the surface $y = 1 + \frac{a}{h}\eta$, in conjunction with the boundary condition (2.1e) on the flat bed $y = 0$. Here appear naturally the parameters

$$\varepsilon := \frac{a}{h}, \quad \delta := \frac{h}{\lambda},$$

called the *amplitude parameter*, and the *shallowness parameter*, respectively. Since the shallowness parameter is a measure of the length of the wave compared to the depth, small δ models long waves or, alternatively, shallow water waves. The amplitude parameter measures the relative size of the wave, so small ε is customarily used to model a small disturbance of the underlying flow. We now set out to study steady (travelling) waves, and will therefore assume that the equations (2.1) have a space-time dependence of the form $x - ct$ in the original variables, corresponding to $\lambda(x - ct)$ in the equations (2.2). The change of variables

$$(x, y) \mapsto (x - ct, y)$$

yields the problem

$$u_x + v_y = 0, \tag{2.3a}$$

$$(u - c)u_x + vu_y = -p_x, \tag{2.3b}$$

$$(u - c)v_x + vv_y = -\frac{p_y}{\delta^2}, \tag{2.3c}$$

$$\delta^2 v_x - u_y = \omega, \tag{2.3d}$$

valid in the fluid domain $0 < y < 1 + \varepsilon\eta$,

$$v = \varepsilon(u - c)\eta_x, \tag{2.3e}$$

$$p = \varepsilon\eta, \tag{2.3f}$$

valid at the surface $y = 1 + \varepsilon\eta$, and

$$v = 0 \tag{2.3g}$$

along the flat bed $y = 0$.

3 The linearization

To enable the study of explicit solutions, we shall linearize around a laminar, though rotational, flow. Such shear flows are characterized by the flat surface,

$y = 1$, corresponding to $\eta = 0$, so insertion of this into (2.3) yields the one-parameter family of solutions,

$$U(y) := U(y; s) := s - \int_0^y \omega(y) dy,$$

with $\eta = 0, p = 0, v = 0$. We now write a general solution as a perturbation of such a solution U , i.e.

$$u = U + \varepsilon \tilde{u}, \quad v = \varepsilon \tilde{v}, \quad p = \varepsilon \tilde{p}. \quad (3.1)$$

We know from the exact theory of water waves that such solutions exist at the points where the non-trivial solutions bifurcate from the curve of trivial flows [10]. Remember that small ε corresponds to waves whose amplitude is small in comparison with the depth. Since the surface is described by $1 + \varepsilon\eta$, η should thus be of unit size. Dropping the tildes, we obtain

$$u_x + v_y = 0, \quad (3.2a)$$

$$(U - c)u_x + vU_y + \varepsilon(vu_y + uu_x) = -p_x, \quad (3.2b)$$

$$(U - c)v_x + \varepsilon(vv_y + uv_x) = -\frac{p_y}{\delta^2}, \quad (3.2c)$$

valid in the fluid domain $0 < y < 1 + \varepsilon\eta$;

$$v = (U - c + \varepsilon u)\eta_x, \quad (3.2d)$$

$$p = \eta, \quad (3.2e)$$

valid at the surface $y = 1 + \varepsilon\eta$; and

$$v = 0, \quad (3.2f)$$

on the flat bed $y = 0$. The corresponding linearized problem is valid in the sense that its solution satisfies the exact equations except for an error whose size can be expressed as a square of the size of the linear solution. The linearization is attained by formally letting $\varepsilon \rightarrow 0$, and it is given by

$$u_x + v_y = 0, \quad (3.3a)$$

$$(U - c)u_x + vU_y = -p_x, \quad (3.3b)$$

$$(U - c)v_x = -\frac{p_y}{\delta^2}, \quad (3.3c)$$

valid for $0 < y < 1$; and

$$v = (U - c)\eta_x, \quad (3.3d)$$

$$p = \eta, \quad (3.3e)$$

valid for $y = 1$. In order to explicitly solve this problem we restrict ourselves to the simplest possible class of vorticity distributions, i.e. when $\omega(y) = \omega \in \mathbb{R}$ is constant. It then follows that

$$U(y; s) = -\omega y + s.$$

Looking for separable solutions we make the *Ansatz* $\eta(x) = \cos(2\pi x)$ (note that the original wavelength λ and the original amplitude a have both been non-dimensionalized to unit length). The solution of (3.3) is then given by

$$\begin{aligned} u(x, y) &= 2\delta\pi C \cos(2\pi x) \cosh(2\pi\delta y), \\ v(x, y) &= 2\pi C \sin(2\pi x) \sinh(2\pi\delta y), \\ p(x, y) &= C \cos(2\pi x) (2\pi\delta(c - s + \omega y) \cosh(2\pi\delta y) - \omega \sinh(2\pi\delta y)), \end{aligned} \quad (3.4)$$

where $C := (c - s + \omega) / \sinh(2\pi\delta)$, and c, δ, h, s, ω must satisfy the relation

$$(c - s + \omega)(2\pi\delta(c - s + \omega) \coth(2\pi\delta) - \omega) = 1 \quad (3.5)$$

This indicates that the properties of the wave are adjusted to fit the rotational character of the underlying flow. Note in (3.4) that while the horizontal and vertical velocities are given by straightforward expressions, the complexity of the pressure has drastically increased compared to the irrotational case [6, 11]. Remember that this solution is a small disturbance of the original shear flow, according to (3.1). For small ε , we thus have an approximate solution to (2.3).

To normalize the reference frame Stokes made a now commonly accepted proposal. In the case of irrotational flow he required that the horizontal velocity should have a vanishing mean over a period. Stokes' definition of the wave speed unfortunately cannot be directly translated to waves with vorticity (see [10]). In the setting of waves with vorticity we propose the requirement

$$\int_0^1 u(x, 0) dx = 0, \quad (3.6)$$

a “Stokes’ condition” at the bottom. This is consistent with deep-water waves (cf. [15]), and for $U(y; s)$ it results in $s = 0$. As we shall see in subsection 3.1 this indeed seems to be the natural choice of s , since this and only this choice recovers the well established bound \sqrt{gh} for the wave speed. This is also the choice made in [37]. We emphasise that (3.6) is only a convention for fixing the reference frame; without such a reference it is however meaningless to for example discuss whether physical particle paths are closed or not.

The corresponding approximation to the original system (2.1) is

$$\begin{cases} u(t, x, y) &= -\omega y + \frac{a(f+kh\omega)}{\sinh(kh)} \cos(kx - ft) \cosh(ky), \\ v(t, x, y) &= \frac{a(f+kh\omega)}{\sinh(kh)} \sin(kx - ft) \sinh(ky), \\ P(t, x, y) &= P_0 + g(h - y) + \frac{a(f+kh\omega)}{k \sinh(kh)} \cos(kx - ft) \\ &\quad \times \left((f + k\omega y) \cosh(ky) - \omega \sinh(ky) \right), \\ \eta(t, x) &= h + a \cos(kx - ft). \end{cases} \quad (3.7)$$

Here

$$k := \frac{2\pi}{\lambda} \quad \text{and} \quad f := \frac{2\pi c}{\lambda}$$

are the *wave number* and the *frequency*, respectively. The size of the disturbance is proportional to a in the whole quadruple (η, u, v, p) , so this solution satisfies the exact equation with an error which is $O(a^2)$ as $a \rightarrow 0$. Concerning the uniform validity of the approximation procedure, leading to the linear system, a closer look at the asymptotic expression indicates that this solution is uniformly valid for

$$-\infty < x - ct < \infty \quad \text{as} \quad \varepsilon \rightarrow 0,$$

while for the vorticity we have uniform validity in the region

$$\varepsilon\omega = o(1) \quad \text{as} \quad \varepsilon \rightarrow 0.$$

A rigorous confirmation of this requires a detailed analysis similar to that presented in [12, 32], but is outside the scope of our paper.

3.1 The dispersion relation

The identity (3.5) can be stated in the physical variables as the dispersion relation

$$c - s\sqrt{gh} + h\omega = \frac{1}{2k} \left(\omega \tanh(kh) \pm \sqrt{4gk \tanh(kh) + \omega^2 \tanh^2(kh)} \right), \quad (3.8)$$

valid for linearized small amplitude gravity waves on a sheared current of constant vorticity. Note that $s\sqrt{gh} - h\omega$ is the surface velocity of the trivial solution $U(y; s)$ stated in the physical variables. The equation (3.8) is the general version of the dispersion relation presented in [10, Section 3.3]. They found the dispersion relation

$$c - u_0^* = \frac{1}{2} \left(\omega \tanh(h) + \sqrt{4g \tanh(h) + \omega^2 \tanh^2(h)} \right), \quad (3.9)$$

where u_0^* is the surface velocity of the trivial solution. The authors consider waves of wavelength 2π , whence $k = 1$. They also require that $u < c$, and that the relative mass flux is held constant along the bifurcation curve for which the linearization is the first approximation. In the case of (3.8) the problem to uniquely determine c from k , h , and ω is related to the fact that the requirement $u < c$ is necessary for the theory developed in [10], while in our linear theory, ω and s can be chosen as to violate that assumption. For example, when $s = 0$ and $h\omega < -c$ it is easy to see from (3.7) that for waves of small amplitude, $a \ll 1$, the horizontal velocity u exceeds the wave speed, at least at the surface where $u \approx -h\omega > c$. The sign in front of the square root depends on the sign of $c - s\sqrt{gh} + h\omega$. It is immediate from (3.8) that this expression is bounded away from 0. Positivity corresponds to the case dealt with in [10], and in that case the existence of exact solutions is well established. Our investigation indicates that there might also be branches of exact solutions fulfilling the opposite requirement $u > c$, and as shall be seen below, in that case it is possible that c is negative so that there are left-going waves on a right-going current. In [10] it is assumed that $c > 0$.

If $c - s\sqrt{gh} + h\omega$ is positive, and the vorticity is positive as well, we get a uniform bound for the wave speed. Let

$$\alpha := \frac{\tanh hk}{hk} \in (0, 1).$$

Then

$$\begin{aligned} \frac{c}{h} - s\sqrt{\frac{g}{h}} &= \frac{1}{2} \left(\omega(\alpha - 2) + \sqrt{\frac{4g\alpha}{h} + \omega^2\alpha^2} \right) \\ &= \frac{2(\omega^2(\alpha - 1) + \frac{g\alpha}{h})}{(2 - \alpha)\omega + \sqrt{\frac{4g\alpha}{h} + \omega^2\alpha^2}} < \sqrt{\frac{g\alpha}{h}}, \end{aligned}$$

meaning that

$$c < \left(\frac{\tanh kh}{kh} + s \right) \sqrt{gh} < (1 + s) \sqrt{gh}$$

If instead $c - s\sqrt{gh} + h\omega < 0$ and $\omega < 0$, the same argument gives that

$$c > -(1 + s) \sqrt{gh}.$$

These calculations vindicate the choice of $s = 0$, since in that case we recover the classical critical speed \sqrt{gh} . To summarize, we have proved

Theorem 3.1. *For a linear gravity wave on a linear current $U(y; 0) = -\omega y$ we have*

$$c \neq -h\omega,$$

and the dispersion relation is given by (3.8) with $s = 0$, where the square root is positive (negative) according as $c + h\omega$ is positive (negative). In particular, if the speed and the vorticity are of the same sign, then

$$|c| < \sqrt{gh}.$$

Remark 3.2. Another comment is here in place. In [10, Section 3.3] the authors show that for positive vorticity, local bifurcation from shear flows requires additional restrictions on the relative mass flux. This is related to the requirement that $u < c$, and the reason can be seen directly from (3.9), according to which

$$c - s\sqrt{gh} + h\omega > 0. \tag{3.10}$$

If $\omega \rightarrow \infty$, the inequality (3.10) admits both large negative and large positive s . But for $s > 0$ large enough, we find that $U(0; s) = s\sqrt{gh} > c$, which violates the assumptions made in [10]. Since in this paper $u > c$ is permitted, there is no corresponding restriction for positive ω .

4 The phase portraits for right-going waves

In this section we study a cross section of the steady solution for a right-going wave. This corresponds to a phase-portrait analysis of the ODE-system in steady variables with $c > 0$. Since

$$(\dot{x}(t), \dot{y}(t)) = (u(x(t), y(t), t), v(x(t), y(t), t))$$

we find that the particle paths are described by the system

$$\begin{cases} \dot{x}(t) &= -\omega y + A \cos(kx - ft) \cosh(ky) \\ \dot{y}(t) &= A \sin(kx - ft) \sinh(ky) \end{cases} \quad (4.1)$$

where

$$A := \frac{a(f + kh\omega)}{\sinh(kh)} \quad (4.2)$$

is proportional to the amplitude parameter a . In order to study the exact linearized system, let us rewrite (4.1) once more via the transformation

$$x(t) \mapsto X(t) := kx(t) - ft, \quad y(t) \mapsto Y(t) := ky(t), \quad (4.3)$$

yielding

$$\begin{cases} \dot{X}(t) &= Ak \cos(X) \cosh(Y) - \omega Y - f \\ \dot{Y}(t) &= Ak \sin(X) \sinh(Y) \end{cases} \quad (4.4)$$

Recall that the obtained wave is a perturbation of amplitude size, and thus the constant A (which includes a) should always be considered very small in relation to ω and f . Changing sign of A corresponds to the mapping $X \mapsto X + \pi$, so we might as well consider $A > 0$. Since we now study only right-going waves for which $c > 0$, when the vorticity is positive, A too will be positive by (4.2). For large enough negative vorticity, $-\omega > c/h$, the original A is however negative, meaning that the phase portrait will be translated by π in the horizontal direction. This is important for the following reason: the presumed surface

$$h + a \cos(X)$$

attains its maximum at $X = 0$. Thus the crest for $c + h\omega > 0$ is at $X = 0$ in our phase portraits, but at $X = \pi$ for $c + h\omega < 0$.

4.1 The case of positive vorticity

Lemma 4.1. *The phase portrait for the irrotational case is given by Figure 1, where the physically realistic wave corresponds to the area of bounded trajectories.*

Remark 4.2. The details of this are given in [11] and a similar investigation is pursued in [6]. We therefore give only the main phase plane arguments for Figure 1. Analytic details can be found in the just mentioned papers.

Proof. Symmetry and periodicity of (4.4) allow for considering only $\Omega := [0, \pi] \times [0, \infty)$. In this region the 0-isocline, $\dot{Y} = 0$, is given by the boundary $\partial\Omega$, i.e. $X = 0$, $X = \pi$, and $Y = 0$. Within Ω holds $\dot{Y} > 0$. The ∞ -isocline, $\dot{X} = 0$, is the graph of a smooth and convex function

$$\gamma(X) = \cosh^{-1} \left(\frac{f}{Ak \cos X} \right), \quad X \in [0, \pi/2),$$

with $\gamma(X) \rightarrow \infty$ as $X \nearrow \pi/2$. Here and elsewhere in this paper \cosh^{-1} denotes the positive branch of the preimage of \cosh . We have

$$\dot{X}(X, Y) > 0 \quad \text{exactly when} \quad Y > \gamma(X),$$

whence $\dot{X} < 0$ for $X \in [\pi/2, \pi]$ as well as below $\gamma(X)$.

The only critical point is thus given by $P := (0, \cosh^{-1}(f/Ak))$. Any trajectory intersecting $X = 0$ below P can be followed backwards in time below $\gamma(X)$ until it reaches $X = \pi$. For any trajectory intersecting $\gamma(X)$ the same argument holds. Hence there exists a separatrix separating those two different types of trajectories, and connecting $X = \pi$ with P .

Any trajectory intersecting $X = 0$ above P can be followed forward in time above $\gamma(X)$, and is thus unbounded. Any trajectory intersecting $\gamma(X)$ can in the same way be followed forward in time above $\gamma(X)$ and is likewise unbounded. There thus exists a second separatrix, unbounded as well, going out from P above $\gamma(X)$ and separating the trajectories intersecting $X = 0$ from those intersecting $\gamma(X)$. By mirror symmetry around $X = 0$ the critical point P must be a saddle point, and the phase portrait is complete. The last proposition of Lemma 4.1 is the only reasonable physical interpretation. \square

Theorem 4.3. *For positive vorticity, $\omega > 0$, the qualitative properties of the phase portrait are the same as for the irrotational case, $\omega = 0$.*

Proof. The proof is based on what we call the *comparison principle*, i.e. by comparing the phase portrait for $\omega > 0$ with that for $\omega = 0$. Now, changing ω does not affect the 0-isoclines. The change of \dot{X} induced by adding the term ωY is

$$\dot{X}_{\omega>0} < \dot{X}_{\omega=0}, \tag{4.5}$$

at any fixed point in the phase plane with $Y > 0$ (where the subscripts denote the two different phase-portraits). Hence the velocity field is conserved wherever

$\dot{X} < 0$ in the portrait for $\omega = 0$, and we need only check what happens with the ∞ -isocline (which encloses all the points where $\dot{X} > 0$).

For any fixed $X \in (-\pi/2, \pi/2)$ and $\omega \geq 0$, the function

$$\varphi(Y) := Ak \cos X \cosh Y - \omega Y - f, \quad Y > 0, \quad (4.6)$$

is convex, satisfying $\varphi(0) < 0$ and $\varphi(Y) \rightarrow \infty$ as $Y \rightarrow \infty$, whence it has a exactly one zero in $(0, \infty)$. It is moreover decreasing in ω , so that if ω increases the solution Y of $\varphi(Y) = 0$ increases. This means that the ∞ -isocline for $\omega > 0$ remains practically the same as in the irrotational case: it is a convex graph lying above the one for $\omega = 0$. Just as before there is no ∞ -isocline for $X \in (\pi/2, \pi)$, since there $\varphi(Y) < 0$. \square

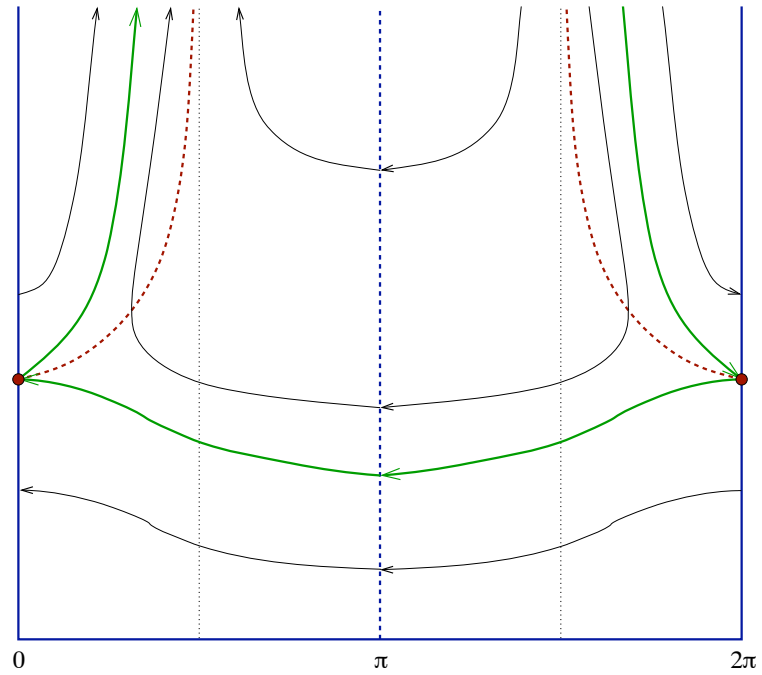


Figure 1. The phase portrait for positive and zero vorticity.

4.2 The case of negative vorticity

Theorem 4.4. *For negative vorticity and small amplitude, $a \ll 1$, the qualitative properties of the phase portrait are given by Figure 2. For $h\omega > -c$ the crest is at*

$X = 0$, while for $h\omega < -c$ the crest is at $X = \pi$. In particular, the steady wave for $h\omega < -c$ contains from bottom and up: an internal wave propagating leftwards, a vortex enclosed by two critical layers, and a surface wave propagating rightwards.

Remark 4.5. In all essential parts this resembles the Kelvin–Stuart cat’s-eye flow, which is a particular steady solution of the two-dimensional Euler equations [30, Ex 2.4]. It arises when studying strong shear layers (which in our case means large constant negative vorticity).

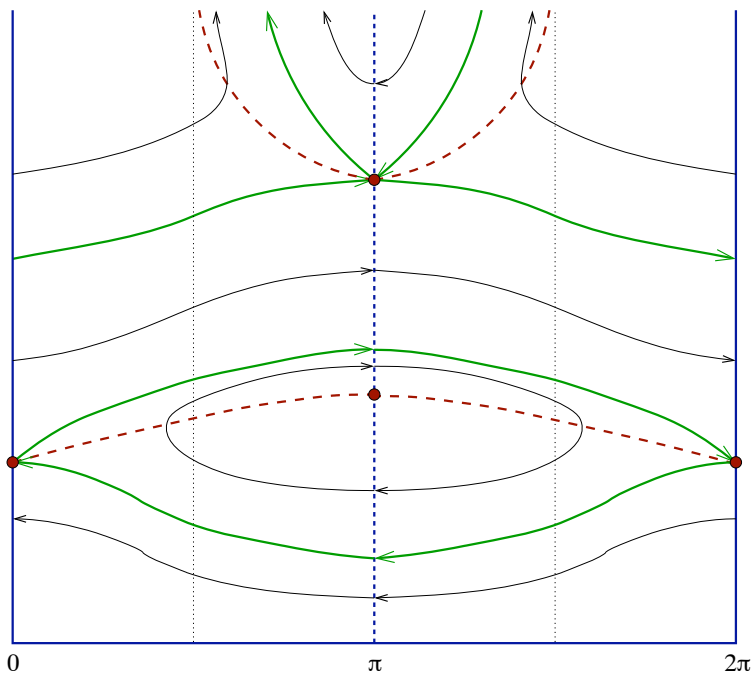


Figure 2. The phase portrait for negative vorticity.

In order to handle this we need to investigate the ∞ -isocline for $\omega > 0$. Recall that $A = A(a)$ depends linearly on the amplitude (see (4.2)).

Lemma 4.6. *For negative vorticity $\omega < 0$, if the amplitude, $a > 0$, is small enough so that*

$$\frac{\omega}{\alpha} \sinh^{-1} \left(\frac{\omega}{\alpha} \right) - \sqrt{1 + \left(\frac{\omega}{\alpha} \right)^2} - \frac{f}{\alpha} \quad (4.7)$$

is positive for $\alpha := Ak$, then the ∞ -isocline of (4.4) for $X \in [0, \pi]$ consists of two disjoint parts:

- i) the graph of an increasing function $Y_1(X)$ defined for $X \in [0, \pi]$, and
- ii) the graph of a decreasing function $Y_2(X)$ defined in $(\pi/2, \pi]$.

We have $Y_1(X) < Y_2(X) \rightarrow \infty$ as $X \searrow \pi/2$, and for any $\delta > 0$ there exist $Y^* > 0$ and a possible smaller a such that the slope satisfies

$$0 < \frac{\partial Y}{\partial X} < \frac{\delta}{\pi} \quad \text{in} \quad R := [0, \pi] \times [Y^*, Y^* + \delta].$$

Proof. Just as before $\varphi(Y)$ as in (4.6) is convex for $X \in (-\pi/2, \pi/2)$ with $\varphi(0) < 0$. However, as

$$X \rightarrow \pi/2, \quad \text{we now have} \quad Y \rightarrow -f/\omega,$$

along the ∞ -isocline $\{(X, Y) : \varphi(Y) = 0\}$. According to the Implicit Function Theorem [25, Theorem I.1.1] the curve can be continued across this point into $X \in (\pi/2, \pi]$. There $\cos X < 0$, and consequently $\varphi(Y)$ is now concave with $\varphi(0) < 0$, $\dot{\varphi}(0) > 0$, and $\varphi(Y) \rightarrow -\infty$ as $Y \rightarrow \infty$. The function $\varphi(Y)$ attains its global maximum when

$$Y = \sinh^{-1} \left(\frac{\omega}{Ak \cos X} \right) > 0, \quad X \in (\pi/2, \pi].$$

Substituting this expression into (4.6), and dividing by $-Ak \cos X > 0$ yields (4.7), for $\alpha := -Ak \cos X$. Thus the equation $\varphi(Y) = 0$ has none, one, or two solutions according as (4.7) is negative, vanishing, or positive, for that α .

It is easy to see that if $\alpha > 0$ is small enough this expression is positive, while it becomes negative for large α . In view of that α vanishes as $X \searrow \pi/2$, we see that at $X = \pi/2$ a new branch of the ∞ -isocline appears from $Y = +\infty$. Keeping in mind that $\varphi(Y)$ is concave for $X \in (\pi/2, \pi)$, where $\cos X$ is decreasing, it follows that the upper branch of $\varphi(Y; X) = 0$ is decreasing as a parameterization $Y(X)$, while the lower branch is increasing in the same manner.

Depending on the relation between A , ω , and f , it may be that the two branches both reach $X = \pi$ separately, that they unite exactly there, or that they unite for some $X < \pi$, where they cease to exist. However, if A is small enough in relation to $|\omega|$ and f , (4.7) guarantees that both branches of the ∞ -isocline exist as individual curves throughout $X \in (\pi/2, \pi]$.

For the final assertion, remember that the slope is given by

$$\frac{\partial Y}{\partial X} = \frac{Ak \sin X \sinh Y}{Ak \cos X \cosh Y - \omega Y - f}. \quad (4.8)$$

Fix Y^* with $-\omega Y^* > (1 + f + \delta)$. Since $A \rightarrow 0$ as $a \rightarrow 0$ there exists a_0 such that for any $a < a_0$ the inequality $Ak \cosh(Y^* + \delta) < \delta/\pi$ holds. In view of that $\sinh \xi < \cosh \xi$ this proves the lemma. \square

We are now ready to give the proof of Theorem 4.4.

Proof. By periodicity and horizontal mirror symmetry it is enough to consider $\Omega := [0, \pi] \times [0, \infty)$ (remember that $Y = 0$ is the bed). The first critical point is

$$P_0 := (0, Y_0), \quad \text{where} \quad Ak \cosh Y_0 - \omega Y_0 - f = 0;$$

while the second and third critical points are

$$P_1 := (\pi, Y_1) \quad \text{and} \quad P_2 := (\pi, Y_2),$$

where Y_1 and Y_2 are, in order of appearance, the smallest and largest solutions of

$$Ak \cosh Y + \omega Y + f = 0.$$

In Ω holds $\dot{Y} > 0$, while the sign of \dot{X} is negative below $Y_1(X)$ and above $Y_2(X)$; elsewhere in Ω the sign of \dot{X} is positive (except for the ∞ -isoclines $Y_1(X)$ and $Y_2(X)$). This follows from Lemma 4.6, and can be confirmed by considering $\partial_Y \dot{X}$ for a fixed X .

The system (4.4) admits a Hamiltonian,

$$H(X, Y) := Ak \cos X \sinh Y - \frac{1}{2} \omega Y^2 - fY, \quad (4.9)$$

with

$$\begin{aligned} \dot{X} &= \partial_Y H, \\ \dot{Y} &= -\partial_X H, \end{aligned}$$

for which the trajectories of (4.4) are level curves. To determine the nature of the critical points we study the Hessian,

$$D^2 H = -Ak \begin{pmatrix} \cos X \sinh Y & \sin X \cosh Y \\ \sin X \cosh Y & -\cos X \sinh Y + \frac{\omega}{Ak} \end{pmatrix}.$$

At P_0 , where $X = 0$, we immediately get that there exist one positive and one negative eigenvalue, whence the Morse lemma [31] guarantees that P_0 is a *saddle point*. Insertion of $P_2 = (\pi, Y_2(\pi))$ yields

$$D^2H(P_2) = Ak \begin{pmatrix} \sinh Y_2(\pi) & 0 \\ 0 & -\sinh Y_2(\pi) - \frac{\omega}{Ak} \end{pmatrix}. \quad (4.10)$$

Now remember that P_2 is the point where the function $\varphi(Y)$, for $X = \pi$, attains its second zero. This happens when the derivative $\varphi'(Y) = -Ak \sinh Y - \omega < 0$, and consequently also P_2 is a *saddle point*.

That P_1 is a *centre* can be seen in the following way: $D^2H(P_1)$ differs from (4.10) only in that $Y_2(\pi)$ is substituted for $Y_1(\pi)$. Since P_1 is the point of the first zero of $\varphi(Y)$ for $X = \pi$, it follows that there $\varphi'(Y) = -Ak \sinh Y - \omega > 0$, whence the Hessian is a diagonal positive definite matrix. According to the Morse lemma there exists a chart $(x, y): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$H(X, Y) = H(P_1) + x^2(X, Y) + y^2(X, Y)$$

in a neighbourhood of P_1 . (An alternative and efficient way is a phase-portrait argument using the symmetry, which ensures that any trajectory that intersects $X = \pi$ twice is closed.)

P_0 being a saddle point, there is a separatrix γ_1^- —i.e. a trajectory separating two qualitatively different trajectory behaviours¹—which can be followed backwards in time from P_0 below $Y_1(X)$, and a separatrix γ_2^+ which can be followed forward in time from P_0 above $Y_1(X)$. By the direction of the velocity field, γ_1^- connects P_0 with $X = \pi$ below P_1 .

In the same manner there are separatrices γ_3^- and γ_4^+ leaving the saddle point P_2 , and since γ_4^+ lies above $Y_2(X)$ it is unbounded and encloses a family of unbounded trajectories starting from $X = \pi$ above P_2 . The separatrix γ_3^- can be followed backward from P_2 below $Y_2(X)$.

Now, according to the last part of Lemma 4.6, there is a family of trajectories, $\{F\}$, starting from $X = 0$ above P_0 and reaching $X = \pi$ in finite time in between P_1 and P_2 . Following γ_3^- we therefore must intersect $X = 0$ above

¹The use of the word *separatrix* is somewhat ambiguous. In our case, however, the geometrical definition corresponds to the analytic notion of the stable and unstable manifolds which are defined by $H(X, Y) = H(P_i)$, $i \in \{0, 2\}$, and whose existence follow from the Implicit Function Theorem [25, Theorem I.1.1].

$\{F\}$. In general, starting from $Y_2(X)$, following any trajectory backwards, we find that it must intersect $X = 0$ above $\{F\}$. Since $\dot{X}|_{Y_2(X)} = 0$, the same trajectory starting from $Y_2(X)$ is contained above $Y_2(X)$ and is unbounded. The family $\{F\}$ also guarantees that γ_2^+ connects P_0 with $X = \pi$ above P_1 , but below P_2 . The phase portrait is thus complete. \square

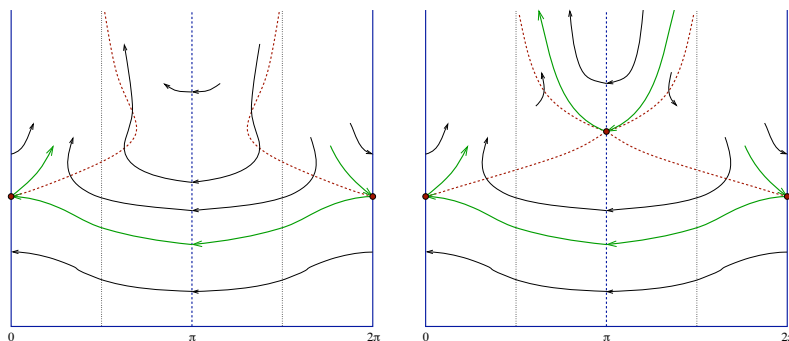


Figure 3. The bifurcation of the phase portrait.

Remark 4.7. In case (4.7) does not hold to be positive one might still pursue the analysis, finding a fluid region where the situation is the same as for positive and vanishing vorticity. Above the surface the situation is, however, radically different and, for fixed amplitude and other parameters, indicates a transition between negative and positive vorticity. In particular, there exists an $\omega < 0$ for which a bifurcation takes place: a second critical point appears, and as ω decreases it immediately gives birth to a third critical point. Since our linearized model presupposes that the amplitude is small, we do not investigate this transition further. Some of the main features are given, without proof, in Figure 3.

5 The physical particle paths

In this section we shall investigate how from the behaviour of the trajectories $(X(t), Y(t))$ we might infer the motion of the physical particles $(x(t), y(t))$. Remember that the relation between those two pairs are given by (4.3).

Before moving on we recall from [6, 11] that in the case of irrotational linear waves, the particles were found to move in *almost* closed orbits, with a slight but positive forward drift. This is in line with the classical *Stokes drift* [35] according to which there is a forward mass drift. It is also consistent with results for the

exact equations [5, 19], showing that within regular Stokes waves—which are irrotational—no orbits are closed.

We emphasize that *the appearance of vorticity radically changes the picture*. In particular, it will be shown that both when the vorticity is negative and satisfies $c + h\omega < 0$, and in the case of large positive vorticity, there does not exist a single pattern for all the fluid particles. Rather, different layers of the fluid behave in qualitatively different ways, with some layers moving constantly in one direction. For the same reason, it is hard to formulate any transparent results other than Theorem 5.3, stating that for small amplitude waves on a current of large positive vorticity there are indeed closed orbits; and Theorem 5.5 which asserts that for negative vorticity all fluid particles display a forward drift. Figure 4 however shows the main features for negative vorticity, as well as a possible situation when the vorticity is positive and very large.

Lemma 5.1. *Particles near the flat bed display a forward (rightward) drift.*

Proof. Consider the time τ that it takes for a particle $(X(t), 0)$, with $X(0) = \pi$, to reach $X(\tau) = -\pi$. We have

$$\begin{aligned} \tau &= \int_{\pi}^{-\pi} \frac{dt(X)}{dX} dX = \int_{-\pi}^{\pi} \frac{dX}{f - Ak \cos X} \\ &= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{f - Ak \cos X} + \frac{1}{f + Ak \cos X} \right) dX \\ &= 2f \int_{-\pi/2}^{\pi/2} \frac{dX}{f^2 - (Ak \cos X)^2} > \frac{2\pi}{f}. \end{aligned}$$

We assert that this holds also near the bed: for any fixed X we may differentiate $\dot{X}(X, Y)$ with respect to Y , obtaining

$$\partial_Y \dot{X} = Ak \cos X \sinh Y - \omega.$$

By continuity, there exists $\delta(\varepsilon) > 0$ such that $|\dot{X}(X, Y) - \dot{X}(X, 0)| < \varepsilon$ whenever $0 < Y < \delta$, uniformly for $X \in \mathbb{R}$. If we thus consider τ for a trajectory intersecting $X = 0$ at level $Y \in (0, \delta)$, we may choose ε arbitrarily small to obtain that $\tau > 2\pi/f$.

Now a closed physical trajectory implies $y(T) = y(0)$ for some $T > 0$ so that $Y(T) = Y(0)$ in view of (4.3). It follows from the phase portraits that for

trajectories close enough to the bed, this forces

$$X(T) - X(0) = -2\pi n, \quad \text{meaning} \quad 0 = x(T) - x(0) = fT - 2\pi n, \quad (5.1)$$

for some $n \in \mathbb{N}$. By periodicity, $T/n = \tau$ is the time it takes the trajectory $X(t)$ to pass from $X = \pi$ to $X = -\pi$ (which any trajectory near the bottom does). From (5.1) we infer that $\tau = 2\pi/f$. \square

Remark 5.2. We also see from this reasoning that if $\tau > 2\pi/f$, then the particle will be to right of its original position, and contrariwise.

5.1 The case of positive vorticity

Theorem 5.3. *If the vorticity $\omega > 0$ is large enough, and the amplitude small enough, then there are particles moving constantly forward, as well as particles moving constantly backward. In particular, there are closed orbits.*

Proof. By Lemma 5.1 the particles nearby the flat bed $y = 0$ display a slight forward drift. In principle, they behave as in the irrotational case (see [6, 11]).

By continuity $\dot{X}(X, Y)$ can be made arbitrarily small, uniformly for all (X, Y) , close enough to the critical point P_0 . Thus for the trajectories near P_0 the time τ as in Lemma 5.1 can be made arbitrarily large, and hence there is a forward drift $\dot{x} > 0$ for the corresponding physical particles.

In between those two layers something different might happen. Fix Y^* , $\delta > 0$, and choose $0 < a \ll 1$ small enough such that

$$Ak \cosh(Y^* + \delta) < \delta.$$

Then choose $\omega \gg 1$ such that $\omega Y^* - \delta > \pi$, and such that the solution Y_0 of $Ak \cosh Y_0 - \omega Y_0 - f = 0$ satisfies $Y_0 > Y^* + \delta$ (cf. (4.6) and the paragraph following it). Then the slope given by (4.8) satisfies

$$|\partial Y / \partial X| < \delta / \pi,$$

so that the trajectory for which $Y(0) = Y^* + \delta$ remains in $[Y^*, Y^* + \delta]$ where $\dot{X} < -\pi - f$. Hence

$$\dot{x}(t) < 0,$$

for all t for the physical particle and there is a constant backward drift. Since the physical surface is given by $Y = k(h + a \cos X)$ we can adjust our choices so

that $Y^* + \delta < k(h - a)$ guarantees that the orbits we consider are indeed within the fluid domain.

Using continuity once more we find that for large positive vorticity and small amplitude, there do exist closed physical orbits. \square

Remark 5.4. Since the physical crest might lie below the critical point P_0 we can only be sure that there is at least one (infinitesimally thin) layer of closed orbits. However, this appears to happen only for small a and large ω , and even so it does not have to affect more than one single trajectory in the (X, Y) -plane. The particle paths are depicted to the left in Figure 4.

5.2 The case of negative vorticity

In the case of irrotational linear waves, it was found in [6, 11] that all fluid particles display a forward drift. This is confirmed for negative vorticity, and the proof of Theorem 5.5 also shows the situation in the three different layers of the fluid. A schematic picture of this can be found to the right in Figure 4. Recall that, for $c + h\omega > 0$, all of the fluid domain lies beneath the lowest separatrix, so that the situation is the same as for zero vorticity. For $c + h\omega < 0$ the fluid domain stretches above the vortex so that the picture is quite different from irrotational waves.

Theorem 5.5. *For negative vorticity, all the fluid particles have a forward drift.*

Proof. For reference, consider Figure 2. We treat separately

- i) the interior wave (beneath the lowest separatrix),
- ii) the vortex (between the first and second separatrices from bottom and up),
and
- iii) the surface wave (between the second and third separatrices from bottom and up).

i) Any trajectory $(X(t), Y(t))$ in this region passes $X = k\pi$, $k \in \mathbb{Z}$. We may thus consider $\tau := \int_{\pi}^{-\pi} \frac{dt}{dX}$ as in Lemma 5.1, τ being the time it takes for the particle to travel from $X(0) = \pi$ to $X(\tau) = -\pi$. Again, for any fixed X ,

$$\partial_Y \dot{X} = Ak \cos X \sinh Y - \omega > -\omega - Ak \sinh Y = \varphi'(Y)$$

with the notation of (4.6). Since φ' has its only zero at the level of P_1 , it follows that $\partial_Y \dot{X} > 0$ in the interior wave. Since $\dot{X} < 0$ at $Y = 0$, we deduce that

$$\tau = \int_{-\pi}^{\pi} \frac{dX}{-\dot{X}} > \int_{-\pi}^{\pi} \frac{dX}{-\dot{X}|_{Y=0}} > \frac{2\pi}{f},$$

so that, according to Remark 5.2, the physical particle path describes a forward motion.

ii) Any trajectory $(X(t), Y(t))$ within the vortex is bounded and passes $X(0) = \pi$, whence

$$x(t) = x(0) + \frac{ft + B(t)}{k}, \quad \text{where } |B(t)| \leq \pi.$$

In particular, at P_1 we have $\dot{X} = 0$, so that the physical particle moves straight forward according to $x(t) = (\pi + ft)/k$, for all $t > 0$.

iii) We need only observe that whenever \dot{X} is positive so is $\dot{x} = (\dot{X} + f)/k$, whence all trajectories above the 0-isocline connecting P_0 and P_1 correspond to fluid particles moving *constantly* forward. \square

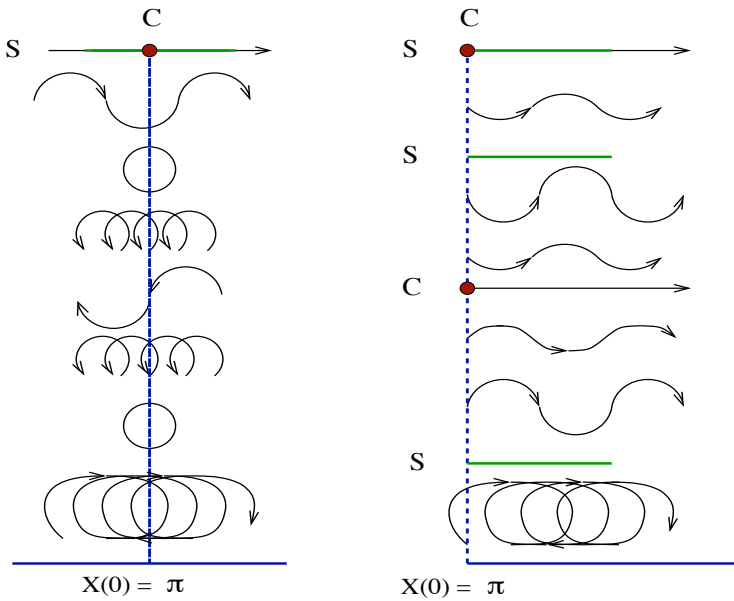


Figure 4. To the left the physical particle paths for small-amplitude waves with large positive vorticity is depicted. The arrows describe what happens to some typical particles in time, while the axis marked S correspond to the separatrix of Figure 1, and the C is the critical point in the same figure. Note that depending on the amplitude and the vorticity, the surface of the physical wave need not correspond to the uppermost arrow. Theorem 5.3 however guarantees that for large enough vorticity and small enough amplitude the surface lies strictly above the closed particle path (the first circle from bottom and up), so that there are particles with a mean backward drift as well as the opposite. To the right we see the particle paths when the vorticity is negative such that $c + h\omega < 0$. This corresponds to Figure 2 with the same notation as above. Near the bottom we have a mean forward drift with nearly closed ellipses, but within the vortex of Figure 2 we see a drastic change of behaviour with a constant forward drift. This is retained even above the separatrix separating the vortex from the surface wave.

6 Summary and discussion

We have deduced and investigated the closed solutions of linear gravity water waves on a linearly sheared current (constant vorticity). Such linear waves satisfy the exact governing equations with an error of magnitude a^2 , where a is the amplitude of the wave. The main purpose has been to understand how the presence of vorticity influences the particle paths. While in the irrotational case all the particles describe nearly closed ellipses with a slight forward drift, we have found that vorticity might change the picture. For positive vorticity the situation is very much the same as in the irrotational case, but for large enough vorticity and small enough waves, there are closed orbits within the fluid domain. For negative vorticity exceeding the wave speed sufficiently much all the particles describe a mean forward drift, but the nearly closed ellipses can be found only in an interior wave near the flat bed.

It seems that all waves of constant vorticity are qualitatively (though not quantitatively) the same, unless we accept the speed of individual particle to exceed the speed of the wave. Then appears waves with interior vortices. So far there is no corresponding exact theory of such rotational waves, since all work has focused on regular waves not near breaking and without stagnation points.

When discussing particle paths it is important to remember that the question of closed orbits is valid in relation to some reference speed. For irrotational waves Stokes required that the average horizontal velocity should vanish. For waves with

vorticity we propose that the same requirement at the bottom is the most sensible counterpart of Stokes' definition. This is supported by the fact that only for that choice we recover the classical critical wave speed \sqrt{gh} .

While interesting in its own right, the investigation pursued here might have further implications for the numerous and well-known model equations for water waves, e.g. the Korteweg–deVries, Camassa–Holm, and Benjamin–Bona–Mahony equations. They all describe the surface – or nearly so – of the wave. Though reasonable for irrotational waves, findings on uniqueness for rotational waves indicate the same as our investigation: beneath two identical surfaces there might be considerable different fluid motions (see Figures 1 and 2). Apart from the trivial case of a flat surface there are so far no known exact examples of this possible phenomenon, but if true it might motivate a new understanding of in what sense the established model equations model the fluid behaviour. Indeed vorticity, even when constant, is a major determining factor of the fluid motion, and it should as such be considered highly important in the study of water waves.

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Paper III

On the streamlines and particle paths of gravitational water waves

Abstract

We investigate steady symmetric gravity water waves on finite depth. For non-positive vorticity it is shown that the particles display a mean forward drift, and for a class of waves we prove that the size of this drift is strictly increasing from bottom to surface. This includes the case of particles within irrotational waves. We also provide detailed information concerning the streamlines and the particle trajectories.

1 Introduction

This paper is concerned with the streamlines and the particle trajectories of steady gravity water waves on finite depth. Such water waves are one of the most common wave formations at sea. As a result of dispersion, wind-generated gravity waves eventually sort themselves out [22, 24]. Larger waves move faster than smaller ones and swell is generated: approximately two-dimensional wave trains of periodic and symmetric waves moving with constant speed across the sea. The exact mathematical theory for such waves is well established, in particular for irrotational flows [18, 31]. Those model very well the situation when the waves propagate into a region of still water. There are however experimental evidence that for some situations such a model is inadequate [29]. One example is tidal flow, which is more correctly modelled by waves entering a rotational current of constant vorticity [30]. Therefore the importance of water motion with a non-vanishing curl—i.e. in the presence of vorticity—has recently come to draw a lot of attention (see e.g. [8, 12, 21, 35]). For us it is of relevance that for arbitrary vorticity distributions there exist symmetric waves [9], and that any wave for which the surface profile is monotone from crest to trough necessarily is symmetric [6].

In a number of recent articles the exact behaviour of fluid particles within such waves have been investigated [4, 5, 11, 15, 19, 20]. The background is the following. For over a century it has been known that the very first approximation of steady irrotational gravity water waves display closed elliptic particle trajectories [28]. However, as was first noted in [11], a thorough study of the linearized

system shows that the particle paths indeed have the shape of an oval, but are not closed. This is so for other types of waves too, as has been shown in [5, 20]. Since the linearized problem can be solved explicitly, details of the particle paths can be more easily studied. In particular, it can be seen that all particles traverse oval orbits. When vorticity is present things are not as transparent, not even for linear waves on a current of constant vorticity. For example, in [15] it is shown that when the size of the vorticity is large, the particle paths of linear waves need not all be oval; some particles may move constantly forward along with the wave. Near the flat bed, though, the particles always behave like the classical first approximation: they move slightly forward in non-closed oval shapes. As will be discussed below this is all in relation to some reference speed, i.e. the generalized Stokes condition (2.4).

For exact water waves the details are far more elusive since closed expressions are not available. The investigations [4, 19] show, however, that even for exact irrotational water waves the particles display a mean forward drift. The notion of *mean forward drift* means that every new time a particle reaches its highest point it has moved some distance along with the wave. The papers [4, 19] also assert that for irrotational Stokes waves on finite, as well as on infinite, depth all particles move in oval orbits. In this paper we show that the mean forward drift is preserved for all negative vorticity distributions. For irrotational waves and small enough rotational waves, we are able to show that this forward drift is strictly increasing from bottom to surface. A proof of this for linear waves without vorticity was given in [5]. In addition we establish some surprisingly nice properties of the velocity field and the particle paths, in particular so for irrotational waves. We have not been able to confirm the oval orbital shapes for rotational waves.

The novelty of our approach lies in the fact that we establish precise point-wise information about the velocity field and its derivatives within the entire fluid domain. In this way we extend the propositions in [4, 19], providing further understanding even for the irrotational case. The proof techniques rely heavily on sharp maximum principles, for which we refer the reader to the excellent sources [16, 26]. For steady rotational waves, to our knowledge this is the first investigation of its kind apart from [15]. Some of the results here obtained can be extended to deep-water waves and solitary waves. However, it should be noted that there are important differences between those types of waves. Notably, the investigation [7] shows that the particle trajectories within irrotational solitary waves differ in fundamental ways from those in periodic waves; and in [13] it is proved that

the class of vorticity distributions allowed for in deep-water waves is much more restrictive than for finite depth.

The paper is organized as follows. Section 2 gives the mathematical background, and the main results are proved in Sections 3 and 4. A synthesis and an analysis of the particle paths are given in Section 5, presented as two, hopefully illustrative, examples.

2 Mathematical formulation

Let $d > 0$ be the depth below the mean water level, $y = 0$, so that the flat bottom can be described by $y = -d$. The *free surface* can be represented by a function

$$\eta \in C^3(\mathbb{R}, \mathbb{R}),$$

and we require that $\eta(0) = \max_{x \in \mathbb{R}} \{\eta(x)\}$ be the vertical coordinate of the crest, unique within a period. Naturally $\min_{x \in \mathbb{R}} \{\eta(x)\} > -d$, so that the trough is above the flat bed, $y = -d$. We shall be concerned with the nontrivial case when $\max \eta > \min \eta$. The wave is steady of period $L > 0$ —without loss of generality we may take $L = 2\pi$ —and we require the surface profile to be monotone between crests and troughs. It is therefore symmetric around the crest [6], and we have that

$$\eta(x + 2\pi) = \eta(x), \quad \eta(x) = \eta(-x), \quad \text{and} \quad \eta'(x) < 0 \text{ for } x \in (0, \pi).$$

We let Ω_η denote the *fluid domain* and define it as the interior of its boundary

$$\partial\Omega_\eta := \{y = -d\} \cup \{(x, \eta(x))\}_{x \in \mathbb{R}}.$$

A solution to the water-wave problem is then defined as a function $\psi \in C^2(\overline{\Omega}_\eta)$, such that

$$\begin{cases} \Delta\psi = -\gamma(\psi), & (x, y) \in \Omega_\eta, \\ |\nabla\psi|^2 + 2gy = C, & y = \eta(x), \\ \psi = 0, & y = \eta(x), \\ \psi = -p_0, & y = -d, \end{cases} \quad (2.1)$$

that is even and 2π -periodic in the x -variable. In (2.1) p_0 is called the *relative mass flux*, the *vorticity function* $\gamma: [0, -p_0] \rightarrow \mathbb{R}$ is continuously differentiable,

$g > 0$ is the gravitational constant, and C is a constant related to the energy. The setting is that of gravitational water waves, meaning that the influence of capillarity is neglected in (2.1), and the water is assumed to be inviscid. The *stream function* ψ is defined (up to a constant) by

$$\psi_x = -v, \quad \psi_y = u - c < 0,$$

where u and v are the horizontal and the vertical velocity, respectively, and $c > 0$ is the constant horizontal speed of propagation. The notion of relative mass flux introduced in [9] captures the physical fact that the amount of water passing any vertical line is constant throughout the fluid domain:

$$\int_{-d}^{\eta(x)} (u(x, y) - c) dy = p_0, \quad x \in \mathbb{R},$$

holds since $u - c = \psi_y$, and ψ is constant on the surface $y = \eta(x)$, as well as on the bottom $y = -d$.

Provided that $u - c = \psi_y < 0$, the system (2.1) can be deduced from the Euler equations (see e.g. [9, 34] for a more detailed discussion). This assumption is supported by physical measurements [24]: for a wave not near breaking or spilling, the speed of an individual fluid particle is far less than that of the wave itself. For irrotational waves it is known, however, that there exist so called highest waves for which the crest is a stagnation point (see e.g. [1]), i.e. $\nabla\psi = 0$. While the exact problem is still open for waves with vorticity [10, 32], there are indications that for some classes of vorticity there do exist steady waves with particle layers not satisfying $\psi_y < 0$ [15, 23]. In this paper we shall consider only waves that are not near breaking or stagnation, so that $\psi_y < 0$ in $\overline{\Omega}_\eta$.

A hodograph transform converts the free boundary problem (9) into a problem with a fixed boundary. Let us express the height,

$$h := y + d,$$

above the flat bed in terms of the new space variables

$$q := x, \quad \text{and} \quad p := -\psi. \tag{2.2}$$

Notice that $\psi_y < 0$ so that (2.2) is a local change of variables, with

$$h_q := -\frac{\psi_x}{\psi_y} = \frac{v}{u - c}, \quad h_p := -\frac{1}{\psi_y} = \frac{1}{c - u}.$$

The above local coordinate transform is actually a global change of variables (see [9]) so that we can transform the problem (2.1) into these variables to obtain

$$\begin{aligned} (1 + h_q^2) h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \gamma(-p)h_p^3 &= 0, & p \in (p_0, 0), \\ 1 + h_q^2 + (2gh - Q)h_p^2 &= 0, & p = 0, \\ h &= 0, & p = p_0. \end{aligned} \tag{2.3}$$

with h even and of period 2π in the q variable. This is an elliptic equation (since $h_p > 0$) with a nonlinear boundary condition. Instead of studying (2.1) in the domain Ω_η , which depends on η , we investigate (2.3) in the fixed rectangle $R := (-\pi, \pi) \times (p_0, 0)$, looking for functions $h \in C^2(\overline{R})$ that are 2π -periodic in q . Notice that knowing $h(q, 0)$ is equivalent to knowing the free surface $y = \eta(x)$, as $h(q, 0) = \eta(q) + d$.

Let $(x, \sigma(x))$ denote the parameterization of a (general) streamline

$$\{(x, y): \psi(x, y) = -p\}.$$

Notice that since $\psi_y < 0$, the above definition is sensible, and we have that

$$\sigma'(x) = -\frac{\psi_x(x, \sigma(x))}{\psi_y(x, \sigma(x))} = h_q(q, p).$$

To normalize the reference frame Stokes made a now commonly accepted proposal. In the setting of irrotational flow he required that the horizontal velocity should have a vanishing mean over a period. Stokes' definition of the wave speed unfortunately cannot be directly translated to waves with vorticity, a consequence of the fact that if $\text{div } \nabla \psi \neq 0$, then ψ_y has different means at different depths in view of the Divergence theorem. In the setting of periodic waves with vorticity we propose the requirement

$$\int_{-\pi}^{\pi} u(x, -d) dx = 0, \tag{2.4}$$

a ‘‘Stokes' condition’’ at the bottom. This is consistent with deep-water waves (cf. [13]), and is also the choice made in [30] (In [25], however, the normalization is done at the surface of the underlying flow). Calculations performed on linear water waves with constant vorticity indicate that this is the natural choice, since that and only that choice recovers the well-established bound \sqrt{gh} for the wave

speed [15]. We emphasize that (2.4) is only a convention for fixing the reference frame; except from the assertion of forward drift it does not change the results of this paper. Without such a reference it is however meaningless to discuss whether physical particle paths are closed or not.

3 Streamlines and the horizontal velocity

In this section we establish the main results for the steady reference frame.

Lemma 3.1.

- i) *Every streamline satisfies $\sigma' < 0$ for $x \in (0, \pi)$, and the maximal steepness of the streamlines is a strictly monotone function of depth.*
- ii) *For $\gamma' \geq 0$ and $\gamma \leq 0$, the maximal horizontal velocity, $\max_{x \in \mathbb{R}} u(x, \sigma(x))$, is strictly increasing from bottom to surface.*
- iii) *The vertical velocity is strictly positive for $x \in (0, \pi)$, and if $\gamma' \leq 0$, then $\max_{x \in \mathbb{R}} |v(x, \sigma(x))|$ is strictly increasing from bottom to surface.*

Lemma 3.2 (The horizontal velocity).

- i) *If $\gamma \leq 0$, then the horizontal velocity u is non-increasing from crest to trough, i.e.*

$$D_x u(x, \eta(x)) \leq 0 \quad \text{for } x \in (0, \pi). \quad (3.1)$$

- ii) *If $\gamma = 0$ and $|\eta'| \leq 1/\sqrt{3}$, then along any streamline $(x, \sigma(x))$ holds*

$$D_x u(x, \sigma(x)) < 0 \quad \text{for } x \in (0, \pi),$$

and the pointwise steepness of the streamlines is everywhere decreasing with depth.

- iii) *If $\gamma(0) \geq 0$ and $\gamma', \gamma'' \leq 0$ then*

$$\partial_x u(x, y) < 0 \quad \text{in } (0, \pi)$$

for waves in a neighbourhood of the bifurcation point in [9].

iv) *If the horizontal velocity attains its maximum at the surface, then it does so either at the crest, or at the concave part of the surface where*

$$(c - u)\gamma \leq g = -\eta''(c - u)^2.$$

Corollary 3.3.

If (3.1) holds, then for $x \in (0, \pi)$ we have the uniform bound $\eta'' \geq -\frac{g}{C-2g\eta(0)}$, together with

$$\eta'(x) \geq -\frac{gx}{C - 2g\eta(0)}, \quad \text{and} \quad \eta(x) \geq \eta(0) - \frac{gx^2}{2(C - 2g\eta(0))}.$$

Remark 3.4. Part iii) of Lemma 3.2 equivalently states that $\partial_y v > 0$ within the half-period $(0, \pi)$. Note also that the equality of part iv) in Lemma 3.2, as well as Corollary 3.3, is based solely on the surface conditions of 2.1, and hence unrelated for example to periodicity, symmetry, and depth.

Proof of Lemma 3.1.

i) For details see [9, Eq. (5.18)]. The key idea is that h_q is annihilated by

$$\begin{aligned} (1 + h_q^2) \partial_p^2 - 2h_p h_q \partial_p \partial_q + h_p^2 \partial_q^2 \\ + 2h_q h_{pp} \partial_q + [3\gamma(-p)h_p^2 - 2h_q h_{pq}] \partial_p. \end{aligned} \quad (3.2)$$

The second statement follows from applying the strong maximum principle to subdomains $(0, \pi) \times (-d, p)$ of this half-period.

ii) Consider $h_p = 1/(c - u) > 0$. Since h_p belongs to the kernel of the uniformly elliptic operator

$$\begin{aligned} (1 + h_q^2) \partial_p^2 - 2h_q h_p \partial_p \partial_q + h_p^2 \partial_q^2 \\ - 2h_p h_{qp} \partial_q + h_p (2h_{qq} + 3\gamma h_p) \partial_p - \gamma' h_p^2, \end{aligned} \quad (3.3)$$

the strong maximum principle implies that $\max u$ is never attained in the interior of any C^2 -subdomain of the fluid. In view of that

$$u_y = \psi_{yy} = -\gamma(-p_0) \geq 0$$

at the flat bed, it is a consequence of the Hopf boundary point lemma that u does not attain its maximum on the bottom. The proposition then follows by periodicity.

- iii) Since $\sigma' = -\psi_x/\psi_y$ and $\psi_y < 0$, the positivity of v is immediate from i). Note also that v vanishes on the flat bed. Since $v \in \text{Ker}\{\Delta + \gamma'(\psi)\}$, we may apply the strong maximum principle to any subdomain

$$\{(x, y): 0 < x < \pi, -d < y < \sigma(x)\}.$$

Symmetry then yields the assertion. □

Proof of Lemma 3.2.

- i) See [33, Thm 2.2]
 ii) We first note that

$$\partial_q h_p = \left(\frac{\partial x}{\partial q} \partial_x + \frac{\partial y}{\partial q} \partial_y \right) \frac{1}{c - u(x, \sigma(x))} = \frac{D_x u(x, \sigma(x))}{[c - u(x, \sigma(x))]^2}.$$

It thus suffices to show that $h_{qp} < 0$ everywhere in $q \in (0, \pi)$. In view of the above calculation, Lemma 3.2 i) shows that $h_{qp} \leq 0$ along the surface for $q \in [0, \pi]$. At the bottom we have $h_{qp} < 0$ for $q \in (0, \pi)$. This follows from Lemma 3.1 i) and the Hopf boundary point lemma: since $\sigma' = h_q$ vanishes on the flat bed, and h_q satisfies the strong maximum principle according to (3.2), it must have a negative derivative in the direction inwards the fluid domain. Along the vertical sides, where $q = 0$ and $q = \pi$, symmetry implies that $h_{qp} = 0$. On the boundary containing a half-period we thus conclude that $h_{qp} \leq 0$. The aim now is to establish that the strong maximum principle holds for h_{qp} in $q \in [0, \pi]$, according to which $h_{qp} \geq 0$ at an interior point of the half-period would force $h_{qp} = \text{const}$ everywhere.

We begin by differentiating (3.2) with respect to p . That gives

$$\begin{aligned} & \left[(1 + h_q^2) \partial_p^2 - 2h_q h_p \partial_q \partial_p + h_p^2 \partial_q^2 - 2h_p h_{qp} \partial_q \right. \\ & \quad \left. + (3\gamma(-p)h_p^2 - 2h_q h_{qp}) \partial_p \right. \\ & \quad \left. + 2h_{pp} h_{qq} + 6\gamma(-p)h_p h_{pp} - 3\gamma'(-p)h_p^2 - 2h_{qp}^2 \right] h_{qp} \\ & = -2h_p h_{pp} h_{qqq} - 2h_q h_{qq} h_{ppp}. \end{aligned} \tag{3.4}$$

Now, rewrite (3.2) and (3.3) as

$$h_{qqq} = \frac{1}{h_p^2} (2h_p h_q (h_{qp})_q - 2h_q h_{pp} h_{qq} - (1 + h_q^2) (h_{qp})_p - (3\gamma(-p)h_p^2 - 2h_q h_{qp}) (h_{qp})),$$

and

$$h_{ppp} = \frac{1}{(1 + h_q^2)} (2h_q h_p (h_{qp})_p - h_p^2 (h_{qp})_q + 2h_p (h_{qp})^2 - h_p (2h_{qq} + 3\gamma(-p)h_p) h_{pp} + \gamma'(-p)h_p^3).$$

To proceed, substitute those expressions into (3.4), to eliminate the third order expressions h_{qqq} and h_{ppp} . We consider then the result as an operator acting on h_{qp} , and deal with the coefficients one at a time.

First, let us leave the highest derivatives untouched in (3.4), i.e. $(1 + h_q^2) \partial_p^2 - 2h_q h_p \partial_q \partial_p + h_p^2 \partial_q^2$. The terms involving ∂_q are

$$\left(4h_q h_{pp} - 2h_q h_{qp} - \frac{2h_q h_p^2 h_{qq}}{1 + h_q^2} \right) \partial_q,$$

and those involving ∂_p are

$$\left(3\gamma(-p)h_p^2 - 2h_q h_{qp} + \frac{4h_q^2 h_p^2 h_{qq}}{h_p (1 + h_q^2)} - \frac{2h_{pp} (1 + h_q^2)}{h_p} \right) \partial_p.$$

Collecting the zero order terms we obtain

$$\begin{aligned} & 2h_{pp} h_{qq} h_{qp} + 6\gamma h_p h_{pp} h_{qp} - 3\gamma' h_p^2 h_{qp} - 2(h_{qp})^3 \\ & + \frac{2h_{pp}}{h_p} (2h_q (h_{qp})^2 - 2h_q h_{pp} h_{qq} - 3\gamma h_p^2 h_{qp}) \\ & + \frac{2h_q h_{qq}}{1 + h_q^2} (2h_p (h_{qp})^2 - 2h_p h_{qq} h_{pp} - 3\gamma h_p^2 h_{pp} + \gamma' h_p^3) \\ & = \frac{1}{h_p (1 + h_q^2)} \left(h_p (1 + h_q^2) [2(h_{pp} h_{qq} - (h_{qp})^2) - 3\gamma' h_p^2] h_{qp} \right. \\ & \quad \left. + 4h_q ((h_{qp})^2 - h_{qq} h_{pp}) [(1 + h_q^2) h_{pp} + h_p^2 h_{qq}] \right) \end{aligned}$$

$$\begin{aligned}
& + 2h_q h_p^3 h_{qq} (\gamma' h_p - 3\gamma h_{pp}) \Big) \\
= & \frac{1}{(1+h_q^2)} \left((1+h_q^2) [2(h_{pp} h_{qq} - (h_{qp})^2) - 3\gamma' h_p^2] h_{qp} \right. \\
& + 4h_q ((h_{qp})^2 - h_{qq} h_{pp}) [2h_q h_{qp} - \gamma h_p^2] \\
& \left. + 2h_q h_p^2 h_{qq} (\gamma' h_p - 3\gamma h_{pp}) \right),
\end{aligned}$$

where we have made repeated use of the elliptic equality in (2.3). Bringing everything together, a last simplification yields the identity

$$\begin{aligned}
& \left[(1+h_q^2) \partial_p^2 - 2h_p h_q \partial_p \partial_q + h_p^2 \partial_q^2 \right. \\
& + \left(4h_q h_{pp} - 2h_p h_{qp} - \frac{2h_q h_p^2 h_{qq}}{1+h_q^2} \right) \partial_q \\
& + \left(3\gamma h_p^2 - 2h_q h_{qp} + \frac{4h_q^2 h_{qq} h_p}{1+h_q^2} - \frac{2h_{pp}(1+h_q^2)}{h_p} \right) \partial_p \quad (3.5) \\
& \left. + \frac{1}{1+h_q^2} \left(2(h_{qq} h_{pp} - h_{qp}^2)(1-3h_q^2) - 3\gamma' h_p^2 \right) \right] h_{qp} \\
= & \frac{2h_q h_p^2}{1+h_q^2} (\gamma(h_{qq} h_{pp} + 2h_{qp}^2) - \gamma' h_p h_{qq}).
\end{aligned}$$

By multiplying the elliptic equation in (2.3) with h_{pp} and then completing the squares, we obtain for $\gamma = 0$ that

$$0 = (h_q h_{pp} - h_p h_{qp})^2 + h_{pp}^2 + h_p^2 (h_{qq} h_{pp} - h_{qp}^2), \quad (3.6)$$

which forces $h_{qp}^2 \geq h_{qq} h_{pp}$ everywhere in the fluid. Finally, Lemma 3.1 i) and the assumption guarantees that $h_q^2 \leq 1/3$ everywhere. This means that the coefficient in front of h_{qp} in (3.5) is non-positive, so that h_{qp} satisfies the strong maximum principle.

- iii) We shall determine ψ_{xy} in a manner similar to that used in [14]. Since $\psi = 0$ along the surface, we have $\psi_x = -\eta' \psi_y$, and insertion into the

Bernoulli surface condition of (2.1) yields that

$$\psi_y^2 = \frac{C - 2g\eta}{1 + \eta'^2}. \quad (3.7)$$

By differentiation along the surface, $(x, \eta(x))$, it follows that

$$\psi_{xy} + \eta' \psi_{yy} = -\partial_x \sqrt{(C - 2g\eta)/(1 + \eta'^2)}. \quad (3.8)$$

On the other hand, we may differentiate $\psi_x = \eta' \psi_y$ once more along the surface, obtaining

$$\psi_{xx} + \eta'^2 \psi_{yy} + 2\eta' \psi_{xy} + \eta'' \psi_y = 0. \quad (3.9)$$

A third equality is supplied by

$$\psi_{xx} + \psi_{yy} = -\gamma(0). \quad (3.10)$$

We now combine (3.8), (3.9), and (3.10) to isolate ψ_{xy} . The calculation—cf. [14] for details—yields that at the surface ψ_{xy} can be determined as

$$\begin{aligned} \psi_{xy} = \eta' & \left(1 + \eta'^2\right)^{-5/2} (C - 2g\eta)^{-1/2} \left(2\eta''(C - 2g\eta) \right. \\ & \left. + (1 - \eta'^4)g + \gamma(0)\sqrt{C - 2g\eta} \left(1 + \eta'^2\right)^{3/2}\right). \end{aligned} \quad (3.11)$$

Since $C - 2g\eta$, g , and $\gamma(0)$ are all positive, and $\eta' < 0$ in $(0, \pi)$, we have that for η' and η'' small enough, $\psi_{xy} \leq 0$ at the surface. Moreover, $\psi_{xy} = 0$ for $x = k\pi$, $k \in \mathbb{Z}$, by symmetry. And on the bottom holds $\psi_{xy} \leq 0$, according to Lemma 3.1 iii) and the boundary condition $\psi_x(x, -d) = 0$. Since ψ_{xy} obeys the maximum principle,

$$(\Delta + \gamma') \psi_{xy} = -\gamma'' \psi_x \psi_y \geq 0,$$

a non-negative maximum thus cannot be attained in the interior of the half-period $0 < x < \pi$.

iv) According to (3.7), differentiation along the surface gives

$$D_x \psi_y^2(x, \eta(x)) = \frac{2\eta' \left[(2g\eta - C)\eta'' - g(1 + \eta'^2) \right]}{(1 + \eta'^2)^2}. \quad (3.12)$$

Thus a maximum of ψ_y along the surface implies that either $\eta' = 0$ or $(2g\eta - C)\eta'' = g(1 + \eta'^2)$. The equality $g = -\eta''(c - u)^2$ is obtained by substituting the second expression into the Bernoulli surface condition of (2.1). To obtain the inequality, first note that if a maximum is attained at the surface, then $\psi_{yy} \geq 0$ at that point. Since (3.8), (3.9), and (3.10) can be used to show that (cf. [14] for details)

$$\begin{aligned} \psi_{yy} = & \left(1 + \eta'^2\right)^{-5/2} (C - 2g\eta)^{-1/2} \left(\eta''(C - 2g\eta) (\eta'^2 - 1) \right. \\ & \left. + 2g\eta'^2 \left(1 + \eta'^2\right) - \gamma(0)\sqrt{C - 2g\eta} \left(1 + \eta'^2\right)^{3/2} \right), \end{aligned} \quad (3.13)$$

we need only substitute $(2g\eta - C)\eta'' = g(1 + \eta'^2)$ into that expression to see that

$$0 \leq \psi_{yy} \Big|_{\eta'' = \frac{g(1+\eta'^2)}{(2g\eta-C)}} = \frac{g\sqrt{1 + \eta'^2} - \gamma(0)\sqrt{C - 2g\eta}}{(1 + \eta'^2)\sqrt{C - 2g\eta}}. \quad (3.14)$$

□

Proof of Corollary 3.3. Recall (3.12). It follows from the assumption that

$$\frac{2\eta'\eta''}{1 + \eta'^2} \leq \frac{-2g\eta'}{C - 2g\eta}, \quad \text{meaning} \quad \frac{d}{dx} \log \left(\frac{1 + \eta'^2}{C - 2g\eta} \right) \leq 0. \quad (3.15)$$

This can be integrated to

$$\frac{1 + \eta'^2}{C - 2g\eta} \leq \frac{1}{C - 2g\eta(0)}. \quad (3.16)$$

Since $\eta' < 0$ in $(0, \pi)$, we may rearrange to obtain that

$$\frac{-\eta'}{\sqrt{2g(\eta(0) - \eta(x))}} \leq \frac{1}{\sqrt{C - 2g\eta(0)}}. \quad (3.17)$$

The assertion concerning η is established by integrating (3.17). The bound on η' follows from employing the lower bound on η to (3.16). Finally, the uniform bound on η'' is immediate from combining the left-hand side of (3.15) with (3.16). □

4 The forward drift

Theorem 4.1.

- i) For $\gamma \leq 0$ there are no closed particle trajectories. In particular, all fluid particles display a mean forward drift.
- ii) If $\gamma = 0$ with $|\eta'| \leq 1/\sqrt{3}$, then the mean forward drift is strictly increasing from bed to surface.
- iii) If $\gamma < 0$ then for all waves in a neighbourhood of the bifurcation point found in [9], the mean forward drift is strictly increasing from bed to surface.

Remark 4.2. In the proof of Theorem 4.1 i) we describe a relationship between the closedness of paths and a certain time, called τ . That idea comes from [11]. It is crucial in all recent investigations concerning particle trajectories in periodic water waves.

Lemma 4.3. For $(x, \sigma(x))$ a non-trivial streamline, and $\gamma \leq 0$, the quantity

$$\int_0^\pi |\psi_y(x, \sigma(x))| (1 + \sigma'^2(x)) dx$$

is non-decreasing as a function of depth, and

$$\int_0^\pi |\psi_y(x, \sigma(x))| dx < c\pi. \quad (4.1)$$

Proof of Lemma 4.3. For two streamlines $(x, \sigma_1(x))$ and $(x, \sigma_2(x))$ with $\sigma_1(x) < \sigma_2(x)$, let

$$\Sigma := \{(x, y) : 0 < x < \pi, \sigma_1(x) < y < \sigma_2(x)\}.$$

According to the Divergence theorem we have that

$$\begin{aligned} - \int_\Sigma \gamma dA &= \int_\Sigma \nabla \cdot \nabla \psi dA = \int_{\sigma_1} \nabla \psi \cdot \frac{\nabla \psi}{|\nabla \psi|} ds - \int_{\sigma_2} \nabla \psi \cdot \frac{\nabla \psi}{|\nabla \psi|} ds \\ &= \int_0^\pi \left(|\nabla \psi(x, \sigma_1(x))| \sqrt{1 + \sigma_1'^2(x)} - |\nabla \psi(x, \sigma_2(x))| \sqrt{1 + \sigma_2'^2(x)} \right) dx \\ &= \int_0^\pi \left(|\psi_y(x, \sigma_1(x))| (1 + \sigma_1'^2(x)) - |\psi_y(x, \sigma_2(x))| (1 + \sigma_2'^2(x)) \right) dx. \end{aligned}$$

This implies that for any nontrivial streamline $(x, \sigma(x))$,

$$\int_0^\pi |\psi_y(x, \sigma(x))| (1 + \sigma'^2(x)) dx = c\pi + \int_0^\pi \int_{-d}^{\sigma(x)} \gamma dA,$$

in view of that $\int_0^\pi \psi_y(x, -d) dx = -c\pi$ by the normalization (2.4). The lemma follows. \square

Proof of Theorem 4.1.

i) Let $(\mathbb{X}(t), \mathbb{Y}(t))$ be any physical trajectory, so that

$$(x(t), y(t)) = (\mathbb{X}(t) - ct, \mathbb{Y}(t))$$

is the corresponding path in the steady variables. We have that $\dot{x}(t) = u - c \leq -\delta < 0$. Thus $(x(t), y(t))$ passes any $x \in \mathbb{R}$, and there is no loss of generality in choosing $x(0) = \pi$. We may also define τ through

$$x(\tau) := -\pi.$$

Since

$$\sigma' = \frac{v}{u - c} = \frac{\dot{y}}{\dot{x}},$$

the streamlines describe the flow of the particles *in the steady reference frame*. By symmetry we thus have $y(\tau) = y(0)$. It moreover follows from Lemma 3.1 that $y(0)$ is the lowest point of the trajectory, attained below the trough, and in view of symmetry $y(\tau/2)$ is the highest, attained below the crest. In between, $\dot{y}(t) \neq 0$. Hence

$$y(T) = y(0) \quad \text{implies that} \quad x(T) - x(0) = 2\pi n,$$

for some $n \in \mathbb{Z}$. In particular, $n = -1$ for $T = \tau$.

Returning to the physical variables, this means that any new time a particle $(\mathbb{X}(t), \mathbb{Y}(t))$ attains its lowest (or highest) position it has moved a distance of

$$\mathbb{X}(\tau) - \mathbb{X}(0) = c\tau - 2\pi$$

in the horizontal direction. We infer that a physical particle trajectory is closed if and only if $\tau = 2\pi/c$. We also see from this reasoning that if $\tau > 2\pi/c$, then the particle displays a mean forward drift, and contrariwise.

The good thing is that τ can be evaluated:

$$\begin{aligned}\tau/2 &= t(0) - t(\pi) = - \int_0^\pi \frac{dt}{dx} dx \\ &= - \int_0^\pi \frac{dx}{\dot{x}(x, \sigma(x))} = \int_0^\pi \frac{dx}{c - u(x, \sigma(x))} = \int_0^\pi \frac{dx}{|\psi_y(x, \sigma(x))|}.\end{aligned}$$

According to Hölders inequality,

$$\begin{aligned}\pi^2 &= \left(\int_0^\pi dx \right)^2 \leq \int_0^\pi |\psi_y(x, \sigma(x))| dx \\ &\quad \times \int_0^\pi \frac{dx}{|\psi_y(x, \sigma(x))|} \leq c\pi \int_0^\pi \frac{dx}{|\psi_y(x, \sigma(x))|},\end{aligned}$$

so that $\tau \geq \frac{2\pi}{c}$. Notice that Lemma 4.3 guarantees that there is strict inequality above the flat bed. So the only way we could have equality is if there is equality in Hölder on the flat bed, implying $\psi_y(x, -d) = -1$. But if ψ_y is constant at the bottom, then so is h_p , and hence

$$h_{qp}(q, p_0) = 0.$$

On the other hand, h_q satisfies the strong maximum principle (cf. (3.2)). And since $h_q = \sigma'$, it is strictly negative within the half-period $(0, \pi)$ (cf. Lemma 3.1), and it vanishes at the flat bed. The Hopf boundary point lemma then forces

$$h_{qp}(q, p_0) < 0, \quad \text{for } 0 < q < \pi.$$

Thus the Hölder inequality must be strict, and we conclude that

$$\tau > \frac{2\pi}{c}.$$

- ii) For two streamlines $(x, \sigma_1(x))$ and $(x, \sigma_2(x))$ with $\sigma_1(x) < \sigma_2(x)$ we are interested in the difference

$$\int_0^\pi \left(\frac{1}{|\psi_y(x, \sigma_2(x))|} - \frac{1}{|\psi_y(x, \sigma_1(x))|} \right) dx,$$

the positivity of which we want to prove. Let p_1 and p_2 correspond to σ_1 and σ_2 , respectively. Then we need to prove exactly that

$$\int_0^\pi (h_p(q, p_2) - h_p(q, p_1)) dq = \int_\Sigma h_{pp} dq dp > 0,$$

for $\Sigma := (0, \pi) \times (p_1, p_2)$. As follows from (2.3),

$$h_{pp} = \frac{2h_q h_p h_{qp}}{1 + h_q^2} - \frac{h_{qq} h_p^2}{1 + h_q^2}.$$

Recall that $\sigma' = h_q$ so that, according to Lemma 3.1 i), we have $h_q < 0$ in Σ . Furthermore, as explained in its proof, Lemma 3.2 ii) asserts that $h_{qp} < 0$ in Σ . In view of that $h_p > 0$, we see that the first term is positive everywhere in Σ . As for the second, integration by parts in the q -variable yields that

$$- \int_\Sigma \frac{h_{qq} h_p^2}{1 + h_q^2} dq dp = 2 \int_\Sigma \arctan(h_q) h_p h_{qp} dq dp,$$

since h_q vanishes on the vertical sides, $q = 0$ and $q = \pi$. Summing up, it follows from the oddness of \arctan that

$$\int_\Sigma h_{pp} dq dp > 0.$$

iii) Consider the quotient

$$\left(\int_0^\pi \frac{dx}{|\psi_y(x, \sigma_2(x))|} - \int_0^\pi \frac{dx}{|\psi_y(x, \sigma_1(x))|} \right) / (\sigma_2(x) - \sigma_1(x)),$$

the sign of which determines the change of τ and thus of the mean drift. The Lebesgue dominated convergence theorem can be applied to consider the limit $\sigma_2 \rightarrow \sigma_1$, being

$$\frac{d}{d\sigma} \int_0^\pi \frac{dx}{|\psi_y(x, \sigma(x))|} := \int_0^\pi \frac{\psi_{yy}(x, \sigma(x)) dx}{\psi_y^2(x, \sigma(x))}. \quad (4.2)$$

Since $\psi_{yy} = -\gamma > 0$ at the bifurcation point, it follows by continuity that the expression in (4.2) is positive in a neighbourhood of the trivial flow from which the nontrivial waves bifurcate.

□

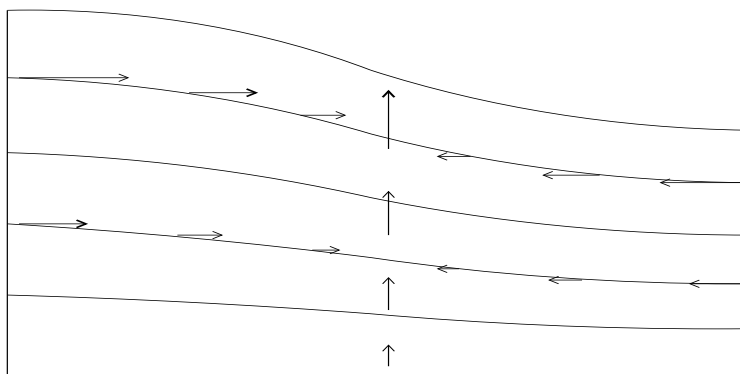


Figure 1: The streamlines and the velocity field for irrotational gravity waves of small amplitude. The steepness of the streamlines and the vertical velocity is pointwise increasing from bottom to surface. The horizontal velocity is everywhere decreasing along the streamlines, and it is increasing from bottom and up beneath the crest, whilst decreasing beneath the trough.

5 The particle trajectories

We are now ready to discuss what our results mean for the streamlines and particle trajectories. We shall do so with the aid of two examples.

5.1 Irrotational waves

Irrotational waves display extraordinary regular features (see Figure 1). From bottom and up the angles between the streamlines and the horizontal plane are pointwise increasing, and for small enough waves this is true also for the vertical velocity. So is the maximal horizontal velocity, which for every streamline is attained below the crest, wherefrom it strictly decreases towards the trough. The surface is bounded below by a concave parabola, the curvature of which is determined by gravity and the maximal horizontal velocity.

In the language of particle paths, every particle traverses a non-closed oval orbit as the wave passes above. This forward drift is strictly increasing from bottom to surface. As the wave propagates above, the particle moves upwards starting from the time a trough passes until the next crest passes (see Figure 2). At the

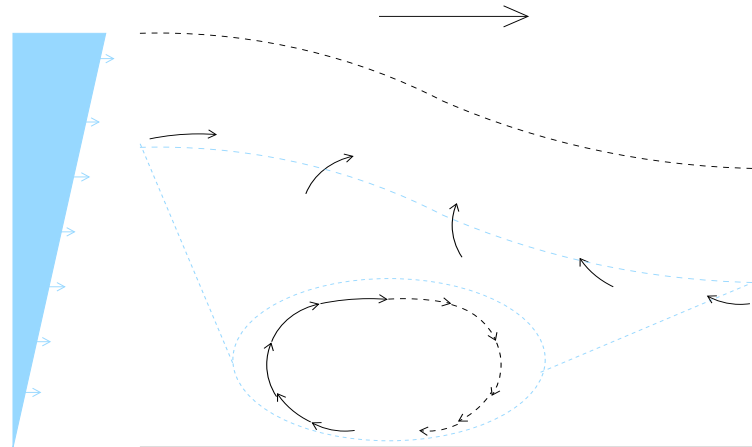


Figure 2: Left: For irrotational waves and small waves of negative vorticity the forward drift is strictly increasing from bottom to surface. Centre: To an observer standing still as the waves passes the particles in irrotational waves traverse non-closed oval orbits, corresponding to what an observer travelling along with the wave understands as a streamline. Not only the forward drift, but also the vertical size of these orbits is increasing from bottom to surface, and the horizontal velocity of each particle is strictly increasing from the bottom of the orbit to its top.

top of its orbit the particle attains its maximal horizontal velocity. The movement then continues in a symmetric way, and the particle begins its descent with the horizontal speed strictly decreasing until it reaches its minimal value as the next trough passes.

5.2 Waves of negative vorticity

Some of the above features persist for waves of negative vorticity. The maximal steepness of the streamlines is strictly increasing from bed to surface, but we lack a proof of this property holding along any vertical line. At the surface the horizontal velocity is non-increasing from crest to trough, and so is it at the bottom; in between we do not know.

The surface is bounded below by the same parabola as are irrotational waves.

The fluid particles show a mean forward drift. The forward drift is always strictly increasing from bed to surface in some neighbourhood of the bifurcation point from laminar flows (cf. [9]). We have not been able to verify the oval shape of the trajectories throughout the fluid since there is no general control of the horizontal velocity in the interior of the fluid. At present the possibility of particles moving constantly forward cannot be ruled out.

Remark 5.1. We remark that for water of infinite depth there is an explicit solution due to Gerstner [17], and re-discovered by Rankine [27]. For that solution, with a particular non-zero vorticity, all paths are circular. We refer to the discussion in [3]. There is also an extension to three-dimensional edge waves in [2].

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Paper IV

Travelling Waves for the Whitham Equation

Abstract

The existence of travelling waves for the original Whitham equation is investigated. This equation combines a generic nonlinear quadratic term with the exact linear dispersion relation of surface water waves on finite depth. It is found that there exist small-amplitude periodic travelling waves with sub-critical speeds. As the period of these travelling waves tends to infinity, their velocities approach the limiting long-wave speed c_0 , and the waves approach a solitary wave. It is also shown that there can be no solitary waves with velocities much greater than c_0 . Finally, numerical approximations of some periodic travelling waves are presented.

1 Introduction

The study of waves on the surface of a fluid has been a source of intriguing mathematical problems for a long time. When studying such waves, viscosity is often neglected, so that the governing equations are the nonlinear Euler equations, supplemented by a set of nonlinear boundary conditions at the unknown fluid surface. This set of equations is commonly known as the water-wave problem. Of special interest is the study of permanent progressive waves, such as solitary or travelling periodic waves. These waves which are also called steady waves propagate without changing their shape over time.

An early highlight in the study of such steady waves was the discovery by Gerstner [16] of a family of exact solutions of the two-dimensional Euler equations in the form of periodic travelling waves. A special feature of this family of solutions is that it includes surface profiles that are not smooth, but have a cusp [11, 9]. While Gerstner's wave has non-zero vorticity, most studies of steady surface waves have been pursued in the case when the flow is irrotational. Starting with the seminal work of Stokes [30] in the mid 1800s, periodic wave trains on the surface of a fluid have attracted a great deal of attention. Stokes made the conjecture that the highest wave has a sharp crest [31], and a great deal of work has been directed towards understanding this phenomenon, including the mathematical proof of the fact that this highest wave exists. For an overview of results in this direction, the reader may consult the surveys by Toland [33] and Groves [17], and the book

by Okamoto and Shoji [25]. While Gerstner's wave is an exact solution only for infinite depth, Stokes waves have been shown to exist for any depth.

A different line of research was initiated by the discovery of the solitary wave by John Scott Russell [28]. His observations and experiments gave an impetus to finding a mathematical formulation capable of describing such waves. The Korteweg-de Vries (KdV) equation

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + \frac{1}{6} c_0 h_0^2 \eta_{xxx} = 0 \quad (1.1)$$

is a simplified model equation for waves at the water surface which includes the essential effects of nonlinearity and dispersion [7, 22]. Balancing these two effects is the basic mechanism behind the existence of both solitary-wave solutions and periodic travelling waves. Equation (1.1) is given in dimensional form, and $c_0 := \sqrt{gh_0}$ is the limiting long-wave speed, h_0 denotes the undisturbed water depth (assuming a flat bottom), and g is the gravitational constant of acceleration. The function $\eta(t, x)$ describes the deflection of the fluid surface from the rest position at a point x at time t . The equation is a valid approximation describing the evolution of surface water waves in the case when the waves are long compared to the undisturbed depth h_0 of the fluid, and the average amplitude of the waves is small when compared to h_0 [18]. In addition, transverse effects are assumed to be weak.

The success of the KdV equation in describing steady waves and the discovery of its completely integrable Hamiltonian structure has led to an intense study of this equation for the last four decades. The mathematical theory for the KdV equation has reached a very advanced level, with a solid theory of well-posedness in place, and a sound understanding of the stability properties of solitary and travelling waves [1, 2, 4, 5, 20, 26]. However, as a model for water waves, the KdV equation may not be the best choice for a number of reasons. Most importantly, it has some shortcomings concerning the propagation of shorter waves. The linear wave speed in the KdV equation is given by

$$c(\xi) = c_0 - \frac{1}{6} c_0 h_0^2 \xi^2, \quad (1.2)$$

where $\xi = \frac{2\pi}{\lambda}$ is the wave number, and λ is the wavelength. This is a second-order approximation to the wave speed

$$c(\xi) = \frac{\omega}{\xi} = \sqrt{\frac{g \tanh \xi h_0}{\xi}}, \quad (1.3)$$

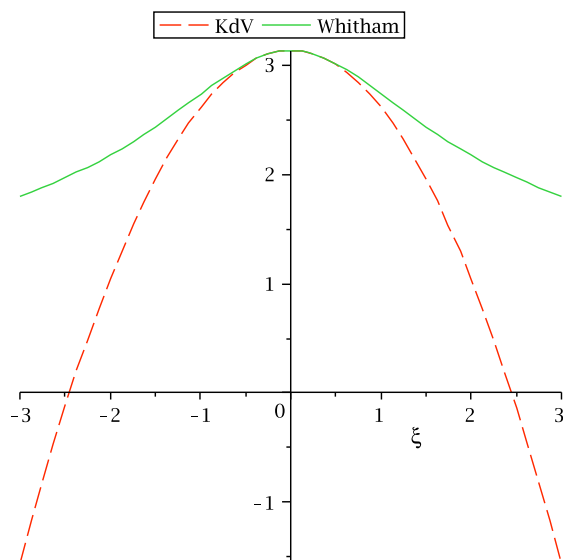


Figure 1: Comparison of linear wave speeds $c = \omega/\xi$ for the KdV and Whitham equations. Here $g = 9.81$ and $h_0 = 1$. The maximum of both graphs is at $c_0 = \sqrt{gh_0}$.

of the linearized water-wave problem. The latter expression for $c(\xi)$ appears when the full water-wave problem is linearized around the vanishing (irrotational) solution, and solutions of the form $\exp(ix\xi - i\omega t)$ are sought [18, 35]. However, as noted in [12, 13], the dispersion relation takes a different form in the presence of vorticity. A comparison of the two expressions (1.2) and (1.3) for $c(\xi)$ is presented in Figure 1. As can be seen, the linearized KdV equation does not give a faithful representation of the full dispersion relation even for intermediate values of the wave number ξ . This problem with the KdV equation as a model for water waves was recognized early on, and has been remedied somewhat by the introduction of the regularized long-wave equation

$$\eta_t + c_0 \eta_x + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x - \frac{1}{6} h_0^2 \eta_{xxt} = 0, \quad (1.4)$$

by Peregrine [27] and Benjamin, Bona and Mahoney [3]. The linear wave speed of (1.4) is given by

$$c(\xi) = \frac{c_0}{1 + \frac{1}{6} h_0^2 \xi^2}, \quad (1.5)$$

which is qualitatively closer to (1.3) than (1.2). A comprehensive review of these modelling issues was given in [3].

Also recognizing the problems of the KdV equation as a model equation for water waves, Whitham introduced what is now called the Whitham equation [34]. The idea was to use the exact form of the wave speed (1.3) instead of a second-order approximation like (1.2) or (1.5). The equation proposed by Whitham has the form

$$\eta_t + \frac{3}{2} \frac{c_0}{h_0} \eta \eta_x + K_{h_0} * \eta_x = 0, \quad (1.6)$$

where the convolution is in the x -variable. The equation is written in dimensional variables, with $\eta(t, x)$ representing the deflection of the surface from rest, just as in the KdV equation. The convolution kernel is given by

$$K_{h_0} := \mathcal{F}^{-1} \left(\sqrt{\frac{g \tanh h_0 \xi}{\xi}} \right), \quad (1.7)$$

where \mathcal{F}^{-1} is the inverse Fourier transform to be defined by (2.1) in Section 2. Often, instead of the kernel K_{h_0} , the kernel

$$\frac{\pi}{4} \exp\left(-\frac{\pi}{2}|x|\right) \quad (1.8)$$

is used. This kernel matches the asymptotic behaviour of K_{h_0} [35], and has certain mathematical advantages over (1.7), such as not having a singularity at the origin. Moreover, this approximation gives rise to a differential equation, the so-called Burgers-Poisson equation [14]. The properties of (1.8) were exploited by Seliger [29], who showed that for this simplified kernel wave breaking is possible.

Even though there does not exist a formal asymptotic expansion or a rigorous proof of convergence of solutions of (1.6) to solutions of the water-wave problem, the Whitham equation remains a source of intriguing problems. The monograph by Naumkin and Shishmarev [24] is devoted entirely to equations like (1.6). In particular, some questions of Whitham concerning breaking and peaking of waves described by generalizations of (1.6) are answered. However, the work of Naumkin and Shishmarev is mainly focused on problems of time evolution. Steady solutions of the equation (1.6) with the kernel (1.8) were studied in [36]. However, for the original Whitham equation the literature is rather sparse. The inherently non-local character of (1.6) makes things much more intricate. In particular, it is still not known whether the proper Whitham equation (with the kernel K_{h_0}) admits a nontrivial solitary-wave solution.

The present article is a study of steady waves for the non-local Whitham equation with its original kernel. In Section 3 we make use of the Crandall–Rabinowitz local bifurcation theorem to prove the existence of small-amplitude periodic travelling waves. A similar treatment was outlined by Gabov in [15], but for the exact kernel (1.7) no proof was given. In Section 4 we prove *a priori* continuity and compactness properties of bounded travelling-wave solutions. These properties imply convergence of periodic solutions to solitary-wave solutions. Section 5 is on non-existence. It is shown that for large velocities there can be no continuous solitary-wave solutions of the steady Whitham equation. In Section 6 we compute numerical approximations of both travelling and solitary waves. It is worth mentioning that the Whitham equation has excited interest precisely for the reason that it features wave breaking and peaking. This was indicated already by Whitham [34], and investigated at length by Naumkin and Shishmarev in the monograph [24]. According to this theory, there is a highest wave, which will have a cusp at the centre. Some computations in this direction are carried out in Section 6.

2 Preliminaries

In this article, the standard notation of mathematical analysis is used. For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the set of measurable real-valued functions of a real variable whose p^{th} powers are Lebesgue integrable over a subset $\Omega \subseteq \mathbb{R}$. If $f \in L^p(\Omega)$, its norm is given by $\|f\|_{L^p(\Omega)}^p := \int_{\Omega} |f|^p dx$. The space $L^\infty(\Omega)$ consists of all measurable, essentially bounded functions with norm $\|f\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} |f(x)|$. We define the Fourier transform \mathcal{F} of a function $f \in L^1(\mathbb{R})$ by

$$\mathcal{F}f(\xi) := \int_{-\infty}^{\infty} f(x) \exp(-ix\xi) dx,$$

and the inverse Fourier transform \mathcal{F}^{-1} by

$$\mathcal{F}^{-1}f(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) \exp(ix\xi) d\xi, \quad (2.1)$$

for any $\hat{f} \in L^1(\mathbb{R})$. We shall also use the notation $\hat{f} := \mathcal{F}f$. The Fourier coefficients of $2L$ -periodic functions on \mathbb{R} are defined by

$$\hat{f}_k := \int_{-L}^L f(x) \exp\left(-ix\frac{k\pi}{L}\right) dx.$$

We write

$$f(x) \sim \frac{1}{2L} \sum_{k \in \mathbb{Z}} \hat{f}_k \exp\left(ix \frac{k\pi}{L}\right)$$

to indicate that under certain conditions on f , this infinite trigonometric series converges to f pointwise, uniform, or in norm. For example, if $f \in L^p((-L, L))$, $p > 1$, then the Carleson–Hunt theorem [19] guarantees that the series converges to $f(x)$ almost everywhere. If in addition $f(x)$ is an even function, the series can be written as

$$f(x) \sim \frac{1}{2L} \hat{f}_0 + \frac{1}{L} \sum_{k=1}^{\infty} \hat{f}_k \cos\left(ix \frac{k\pi}{L}\right) = \frac{1}{L} \sum_{k=0}^{\infty} \hat{f}_k' \cos\left(ix \frac{k\pi}{L}\right),$$

where the prime indicates that the first term of the sum is multiplied by $1/2$.

Next we turn to recording some elementary properties of the Whitham kernel, K_{h_0} , and its Fourier transform. It is immediate that the function $\sqrt{g(\tanh h_0 \xi)}/\xi$ is even and strictly decreasing on $(0, \infty)$. It is in fact real analytic since in a neighbourhood of the origin

$$\frac{\tanh \xi}{\xi} = \sum_{n=1}^{\infty} \frac{2^{2n} (2^{2n} - 1) B_{2n} \xi^{2(n-1)}}{(2n)!} > 0,$$

by using the Taylor series expansion for \tanh (B_n are the Bernoulli numbers). Moreover, $\sqrt{g(\tanh h_0 \xi)}/\xi$ takes the following limits:

$$\lim_{\xi \rightarrow 0} \sqrt{\frac{g \tanh h_0 \xi}{\xi}} = \sqrt{gh_0}, \quad \lim_{\xi \rightarrow \infty} \sqrt{\frac{g \tanh h_0 \xi}{\xi}} = 0.$$

Consequently, $\int_{-\infty}^{\infty} K_{h_0}(x) dx = \sqrt{gh_0}$, and

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \sqrt{gh_0} \left\| \mathcal{F}^{-1} \left(\sqrt{\frac{\tanh \xi}{\xi}} \right) \right\|_{L^1(\mathbb{R})}. \quad (2.2)$$

Thus it can be shown that $K_{h_0} \in L^1(\mathbb{R})$ in the following way. The substitution of variables $y := x\xi$ and partial integration shows that the growth of $\mathcal{F}^{-1} \left(\sqrt{(\tanh \xi)/\xi} \right)$ is of order $x^{-1/2}$ as $x \rightarrow 0$ (for a rigorous proof of this fact, cf. Section 4). Since the function $\sqrt{(\tanh \xi)/\xi}$ is analytic, the inverse Fourier transform has rapid decay. Thus splitting the integral according to

$$\|K_{h_0}\|_{L^1(\mathbb{R})} = \int_{|x| \leq 1} |K_{h_0}(x)| dx + \int_{|x| \geq 1} |K_{h_0}(x)| dx,$$

it is plain that K_{h_0} has finite $L^1(\mathbb{R})$ -norm. In fact, this argument establishes more generally that $K_{h_0} \in L^p(\mathbb{R})$ for $1 \leq p < 2$.

Since the existence of travelling waves is in view, we make the usual ansatz $\eta(t, x) = \phi(x - ct)$, with $c > 0$ being the propagation speed of a right-going steady wave. Using this form, the equation (1.6) transforms into

$$-c\phi' + \frac{3}{2}\frac{c_0}{h_0}\phi\phi' + K_{h_0} * \phi' = 0,$$

which may be integrated to

$$-c\phi + \frac{3}{4}\frac{c_0}{h_0}\phi^2 + K_{h_0} * \phi = B, \tag{2.3}$$

for some real constant B . For solutions $\phi \in L^2(\mathbb{R})$, it appears that the convolution $K_{h_0} * \phi$ is in $L^2(\mathbb{R})$ since K_{h_0} is in $L^1(\mathbb{R})$. Therefore, the left-hand side must vanish as $|x| \rightarrow \infty$, and we shall consider here only the case when $B = 0$. The scaling

$$\phi \mapsto \frac{3}{4}\frac{c_0}{h_0 c}\phi$$

then yields the normalized problem

$$\phi = \phi^2 + \frac{1}{c}K_{h_0} * \phi. \tag{2.4}$$

3 Existence of periodic travelling waves

Theorem 3.1. *For a given $L > 0$ and a given depth $h_0 > 0$, there exists a local bifurcation curve of steady, $2L$ -periodic, even and continuous solutions of the Whitham equation. Those solutions are perturbations of $C \cos(\pi x/L)$, $C \in \mathbb{R}$, and their wave speed at the bifurcation point is determined by the full dispersion relation*

$$c^* = \sqrt{\frac{gL \tanh(h_0\pi/L)}{\pi}}. \tag{3.1}$$

In particular, as $L \rightarrow \infty$ we have $c^ \rightarrow \sqrt{gh_0}$.*

We shall make use of the Crandall–Rabinowitz bifurcation theorem [21, Section I.5], which we state in a form suitable for our purposes. Here and elsewhere D_c is the Fréchet derivative with respect to c .

Lemma 3.2. *Let W be a Banach space, $c \in I := (0, \sqrt{gh_0})$ a parameter, and let $\mathcal{L} : W \rightarrow W$ be the Fréchet derivative at 0 with respect to u of the Whitham map*

$$u \mapsto u - \frac{1}{c} K_{h_0} * u - u^2. \quad (3.2)$$

Suppose that \mathcal{L} and $D_c \mathcal{L}$ exist and are continuous $W \rightarrow W$, and that for some $c^ \in I$ the following conditions hold:*

- i) $\dim \ker(\mathcal{L}) = 1$,*
- ii) $W = \ker(\mathcal{L}) \oplus \text{ran}(\mathcal{L})$,*
- iii) $(D_c \mathcal{L}) \ker(\mathcal{L}) \cap \text{ran}(\mathcal{L}) = 0$.*

Then there exist some $\varepsilon > 0$, and a continuous bifurcation curve $\{(c_s, \phi_s) : |s| < \varepsilon\}$ with $c_s|_{s=0} = c^$, such that ϕ_0 is the vanishing solution of (2.4), and $\{\phi_s\}_s$ is a family of nontrivial solutions with corresponding wave speeds $\{c_s\}_s$. Moreover, we have*

$$\text{dist}(\phi_s, \ker(\mathcal{L})) = o(s) \quad \text{in } W.$$

Remark 3.3. We remark that our method works equally well for the generalized Whitham equation

$$\eta_t + \frac{3}{2} \frac{c_0}{h_0} \eta^p \eta_x + K_{h_0} * \eta_x = 0,$$

whenever $1 \leq p \in \mathbb{Z}$. In that case the Whitham map becomes $u \mapsto u - \frac{1}{c} K_{h_0} * u - u^p$. Since the linearization around the vanishing solution is the same for this map as for (3.2), all that is needed to check is the continuity of the full map in W . As we shall see in the proof of Theorem 3.1, our choice of W is an algebra, so that continuity is evident.

Remark 3.4. It can be seen from the proof of Theorem 3.1 that for wave speeds $c \neq \sqrt{gh_0}$ and different from (3.1), the linear Whitham map \mathcal{L} is a continuous bijection $W \rightarrow W$. It then follows from the implicit function theorem [21, Thm I.1.1] that in a neighbourhood of the trivial flows, there are no other solutions in W of the Whitham equation.

Before we turn to the proof, let us explain how the convolution operator $K_{h_0} *$ acts on periodic functions. Suppose then that $f \in L^\infty(\mathbb{R})$ is periodic. Since K_{h_0}

is in $L^1(\mathbb{R})$, we can write the integral

$$\begin{aligned} \int_{-\infty}^{\infty} K_{h_0}(x-y)f(y) dy &= \sum_{k=-\infty}^{\infty} \int_{-L}^L K_{h_0}(x-y+2kL)f(y) dy \\ &= \int_{-L}^L \left(\sum_{k=-\infty}^{\infty} K_{h_0}(x-y+2kL) \right) f(y) dy =: \int_{-L}^L A(x-y)f(y) dy. \end{aligned}$$

Inspection of the the definition of $A(x)$ shows that it is $2L$ -periodic, even, and continuous on $[-L, L] \setminus \{0\}$. Moreover, a straightforward proof using Minkowski's inequality shows that $A(x)$ belongs to $L^p(-L, L)$, for $1 \leq p < 2$. Therefore, according to the Carleson–Hunt theorem [19], $A(x)$ can be approximated pointwise by its Fourier series. Thus we have

$$A(x) = \sum_{k=0}^{\infty} \hat{A}_k \cos\left(\frac{k\pi x}{L}\right), \text{ a.e.,}$$

where the Fourier coefficients of A are given by

$$\begin{aligned} \hat{A}_k &= \int_{-L}^L \sum_{j=-\infty}^{\infty} K_{h_0}(x+2jL) \exp\left(-\frac{ixk\pi}{L}\right) dx \\ &= \sum_{j=-\infty}^{\infty} \int_{-L}^L K_{h_0}(x+2jL) \exp\left(-\frac{i(x+2jL)k\pi}{L}\right) dx \quad (3.3) \\ &= \int_{-\infty}^{\infty} K_{h_0}(x) \exp\left(-\frac{ixk\pi}{L}\right) dx = \hat{K}_{h_0}\left(\frac{k\pi}{L}\right). \end{aligned}$$

Thus it appears that the periodic problem is given by the same multiplier as the problem on the line, and we have the representation

$$K_{h_0} * f(x) = \frac{1}{L} \sum_{k=0}^{\infty} \hat{f}_k \hat{A}_k \cos\left(\frac{k\pi x}{L}\right) = \frac{1}{L} \sum_{k=0}^{\infty} \hat{f}_k \hat{K}_{h_0}\left(\frac{k\pi}{L}\right) \cos\left(\frac{k\pi x}{L}\right). \quad (3.4)$$

Proof of Theorem 3.1. Looking for a steady solution we consider first the linearized equation

$$\mathcal{L} \psi := \psi - \frac{1}{c} K_{h_0} * \psi = 0.$$

For $\psi \in L^\infty(\mathbb{R})$ we see that

$$\hat{\psi} \left(1 - \frac{1}{c} \sqrt{\frac{g \tanh h\xi}{\xi}} \right) = 0.$$

This makes sense in the setting of distributions. Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing functions (see [32]). Then $\frac{1}{c} \widehat{K}_{h_0} * \psi$, $\hat{\psi}$ and $\frac{1}{c} \widehat{K}_{h_0}$ all exist in $\mathcal{S}'(\mathbb{R})$. Since $1 - \sqrt{g \tanh(h\xi)/\xi}$ is in $L^\infty(\mathbb{R}) \cap C^\infty(\mathbb{R})$, the product of $\hat{\psi}$ and this function is well-defined acting on functions in $\mathcal{S}(\mathbb{R})$. The convolution theorem [32, Section 4.3] then implies that $\frac{1}{c} \widehat{K}_{h_0} * \psi(v) = \frac{1}{c} (\hat{\psi} \widehat{K}_{h_0})(v)$ for any $v \in \mathcal{S}(\mathbb{R})$. Now, if $c < \sqrt{gh_0}$ the support of $\hat{\psi}$ is contained in $\{\pm\xi_0\}$, where $\xi_0 := \xi_0(c, h_0)$ is the unique positive root of $g \tanh h_0 \xi = c^2 \xi$; if $c = \sqrt{gh_0}$ then $\text{supp}(\hat{\psi}) \subset \{0\}$; and if $c > \sqrt{gh_0}$ it follows that $\hat{\psi}(\xi) = 0$ for all ξ . The non-trivial solutions of the linear problem are thus given by

$$\begin{cases} \psi(x) = C, & c = \sqrt{gh_0}, \\ \psi(x) = C \cos(\xi_0 x), & c < \sqrt{gh_0}, \end{cases} \quad (3.5)$$

where $C \in \mathbb{R}$ can be any constant. Note that the constant solutions different from zero are non-physical, and therefore discarded in this analysis. We want to find even periodic small amplitude solutions by bifurcating from a curve of trivial flows. For this purpose, fix the depth h_0 and the half wavelength $L > 0$. The speed $c > 0$ shall be our bifurcation parameter. It is clear from (3.5) that, in any real linear space of $2L$ -periodic functions,

$$\dim \ker(\mathcal{L}) = 1,$$

if and only if $\xi_0 = k\pi/L$, $k \in \mathbb{Z}^+$. Settling for the lowest mode, $k = 1$, gives a unique c as in (3.1), which from now on will be presupposed as our candidate for c^* as in Lemma 3.2.

Looking for even, continuous, and periodic solutions, we introduce the commuting Banach algebra

$$W := \left\{ u(x) = \frac{1}{L} \sum_{k=0}^{\infty'} \hat{u}_k \cos\left(\frac{k\pi x}{L}\right) \mid \|u\| := \frac{1}{L} \sum_{k=0}^{\infty'} |\hat{u}_k| < \infty \right\},$$

which is a suitable subalgebra of the Wiener algebra (cf. [6]). This follows since for even functions the complex Fourier coefficients satisfy $\hat{u}_k = \hat{u}_{-k}$, so that

our norm is equivalent to the classical norm for the Wiener algebra. Note that each member of W is uniformly continuous on all of \mathbb{R} . We shall consider the Whitham equation as the map (3.2) from W , and it will be shown that it is a continuous map into W . As shown in (3.4) the periodic problem is given by the same multiplier as the problem on the line. In effect,

$$\mathcal{L}u \sim \frac{1}{L} \sum_{k=0}^{\infty} \hat{u}(k) \left(1 - \frac{1}{c} \hat{A}(k)\right) \cos\left(\frac{k\pi x}{L}\right) \quad (3.6)$$

holds *a.e.* on $[-L, L]$. By the Riemann-Lebesgue lemma [23, p.133] $\hat{A}(k) \rightarrow 0$ as $k \rightarrow \infty$, so the right-hand side is in W , hence continuous, and

$$\|\mathcal{L}u\| \leq \left(1 + \frac{1}{c} \max_k \{\hat{A}(k)\}\right) \|u\|,$$

so that $\mathcal{L}: W \rightarrow W$ is continuous. Since also the left-hand side is continuous, (3.6) is an equality, which in its turn implies that the full nonlinear Whitham map $u \mapsto \mathcal{L}u - u^2$ is a continuous endomorphism on W , since this is an algebra. The fact that $\ker(\mathcal{L}) = \text{span}_{\mathbb{R}}(\cos(\pi x/L))$ yields

$$\hat{A}(1) = c, \quad \text{and} \quad \hat{A}(k) \neq c, \quad k \neq 1. \quad (3.7)$$

To show that $\text{codim ran}(\mathcal{L})$ is one-dimensional, consider a given $u \in W$. Take $u^\perp \in W$ with $\widehat{u^\perp}(1) = 0$. Then the function

$$v(x) := \frac{1}{L} \sum_{k=0}^{\infty} \frac{\widehat{u^\perp}(k)}{1 - \frac{1}{c} \hat{A}(k)} \cos\left(\frac{k\pi x}{L}\right)$$

is well-defined and belongs to W (this can be seen from (3.3), but it also follows from the Riemann-Lebesgue lemma in combination with (3.7)). Indeed

$$v(x) = \mathcal{L}^{-1} u^\perp(x).$$

Consequently,

$$u(x) = \mathcal{L}v + \frac{\hat{u}(1)}{L} \cos\left(\frac{k\pi x}{L}\right),$$

so that $W = \ker(\mathcal{L}) \oplus \text{ran}(\mathcal{L})$. The derivative with respect to the bifurcation parameter c is

$$(D_c \mathcal{L})u = -(D_c \frac{1}{c} K_{h_0}) * u = \frac{1}{c^2} K_{h_0} * u.$$

Hence—by exactly the same arguments as above—we have that

$$(D_c \mathcal{L})u = \frac{1}{Lc^2} \sum_{k=0}^{\infty'} \hat{u}(k) \hat{A}(k) \cos\left(\frac{k\pi x}{L}\right)$$

is bounded as a map on W . In particular,

$$(D_c \mathcal{L}) \ker(\mathcal{L}) = \ker(\mathcal{L}) \cap \text{ran}(\mathcal{L}) = 0.$$

□

4 Continuity and compactness of bounded solutions

We present here a regularity and a compactness result for travelling solutions of the Whitham equation. This casts light on the relation between L -periodic solutions and solitary wave solutions.

Theorem 4.1. *Let ϕ be a solution of (2.4) such that $\|\phi\|_{\infty} < 1/2$. Then ϕ is continuous.*

Proof. Without loss of generality we pursue the analysis for $k(\xi) := \sqrt{\tanh(\xi)/\xi}$. In view of that $D_{\xi} \tanh \xi = 1 - \tanh^2 \xi \in \mathcal{S}(\mathbb{R})$, it follows from the Leibniz rule that

$$D_{\xi}^n k(\xi) \in \mathcal{O}\left(\xi^{-1/2-n}\right) \quad \text{as } |\xi| \rightarrow \infty.$$

Hence we use partial integration to rewrite

$$K(x) := \frac{1}{2\pi} \int k(\xi) \exp(ix\xi) d\xi = \frac{1}{(-ix)^n} \int k^{(n)}(\xi) \exp(ix\xi) d\xi,$$

for any $x \neq 0$, $n \in \mathbb{Z}^+$. Consequently, we have well-defined derivatives of all orders away from the origin,

$$D_x^j \int k(\xi) \exp(ix\xi) d\xi \in \mathcal{O}(x^{j-n}) \quad \text{as } |x| \rightarrow \infty.$$

For any fixed j , we may choose n as large as required to obtain that $K(x)$ is smooth away from the origin, and all its derivatives have rapid decay at infinity.

Consider then

$$\begin{aligned} K * \phi(x) &= I_1(x) + I_2(x) \\ &:= \int_{|x-z| \leq 1} K(x-z)\phi(z) dz + \int_{|x-z| \geq 1} K(x-z)\phi(z) dz. \end{aligned}$$

Since $|K(x-z)| \leq C|x-z|^{-2}$ and ϕ is bounded, it follows from an application of the dominated convergence theorem that $I_2(x)$ is continuous. By the change of variables $\xi \mapsto s := (x-z)\xi$ we have that

$$\begin{aligned} I_1(x) &= \frac{1}{2\pi} \iint_{|z-x| \leq 1} k(\xi) \exp(i(x-z)\xi) \phi(z) d\xi dz \\ &= \frac{1}{2\pi} \int_{|z-x| \leq 1} \frac{x-z}{|x-z|^{3/2}} \left(\int \sqrt{\frac{\tanh(s|x-z|^{-1})}{s}} \exp(is) ds \right) \phi(z) dz. \end{aligned}$$

Likewise, the inner integral can be divided into two parts,

$$\begin{aligned} \int \sqrt{\frac{\tanh sy}{s}} \exp(is) ds &= I_i(y) + I_{ii}(y) \\ &:= \int_{|s| \leq 1} \sqrt{\frac{\tanh sy}{s}} \exp(is) ds + \int_{|s| \geq 1} \sqrt{\frac{\tanh sy}{s}} \exp(is) ds, \end{aligned}$$

where we have used the shorthand $y := |x-z|^{-1}$. It is clear that $|\tanh sy| \leq 1$, so that $I_i(y)$ is well-defined. Its integrand is furthermore bounded by $|s|^{-1/2}$, uniformly for all y . Now to I_{ii} . Using partial integration, we obtain that

$$\begin{aligned} \int_{|s| \geq 1} \sqrt{\frac{\tanh(sy)}{s}} \exp(is) ds &= -2 \sin(1) \sqrt{\tanh(y)} \\ &+ \frac{1}{2i} \int_{|s| \geq 1} \frac{(sy \tanh^2(sy) - sy + \tanh(sy))}{s^{3/2} \sqrt{\tanh(|s|y)}} \exp(is) ds \\ &= -2 \sin(1) \sqrt{\tanh(y)} + \frac{1}{2i} \int_{|s| \geq 1} \frac{f(sy)}{s^{3/2}} \exp(is) ds, \end{aligned}$$

where $f(\tau) := (\tau \tanh^2 \tau - \tau + \tanh \tau) / \sqrt{|\tanh \tau|}$. It is immediate that the boundary term is bounded by 2, and it can be seen that f is uniformly bounded with $\|f\|_\infty = 1$. This implies that if $x_n \rightarrow x$, then there is a uniform integrable bound, $C|x - z|^{-1/2}(|s|^{1/2} + |s|^{3/2})^{-1}$, for the integrands of $I_1(x_n)$. Just as for $I_2(x)$ it follows from dominated convergence that $I_1(x)$ is continuous, and hence $K * \phi(x)$ is. Using (2.4), we see that

$$|\phi(x) - \phi(y)| = \frac{|K * \phi(x) - K * \phi(y)|}{1 - \phi(x) - \phi(y)} \leq \frac{|K * \phi(x) - K * \phi(y)|}{1 - 2\|\phi\|_\infty}, \quad (4.1)$$

and hence ϕ is continuous. Here we have used the assumption that $\|\phi\|_\infty < 1/2$. \square

Corollary 4.2. *Let $r < 1/2$. The set of solutions of the steady Whitham equation (2.4) contained in the closed ball $\|\phi\|_\infty \leq r$ is compact in $L^\infty_{loc}(\mathbb{R})$.*

Proof. Pick any sequence $(\phi_n)_n$ of solutions of (2.4) that fulfil $\|\phi_n\|_\infty \leq r$. By Theorem 4.1 those are continuous on \mathbb{R} . Moreover, it can be seen from the proof of Theorem 4.1 that the continuity of $\phi * K(x)$ is uniform with respect to $\|\phi\|_\infty$. It then follows from (4.1) that ϕ_n are equicontinuous. The Arzela-Ascoli theorem thus yields the existence of a subsequence $(\phi_{n_k})_k \subseteq (\phi_n)_n$ and a continuous function ϕ , such that ϕ_{n_k} converges to ϕ in $L^\infty_{loc}(\mathbb{R})$.

To prove that ϕ is a solution of the Whitham equation, let $v \in C_0(\mathbb{R})$ be any continuous function with compact support. Then

$$\int \left(\phi_n(x) - \phi_n^2(x) - \int \frac{1}{c} K_{h_0}(y-x) \phi_n(y) dy \right) v(x) dx = 0.$$

Since $\phi_n(x)$ converges pointwise to $\phi(x)$, the functions $v, \frac{1}{c} K_{h_0} \in L^1$, and $\|\phi_n\|_\infty \leq r$, it follows from the Lebesgue bounded convergence theorem that

$$\int (\phi - \phi^2 - \frac{1}{c} K_{h_0} * \phi) v dx = 0.$$

In view of that v is arbitrary this implies that ϕ fulfils (2.4) almost everywhere. The fact that ϕ is continuous implies that it is indeed a solution of the steady Whitham equation in the pointwise sense. \square

Remark 4.3. In Section 3 we find periodic solutions for any period $L > 0$. Under the conditions of Corollary 4.2, any such sequence of solutions converges to travelling-wave solution on the line as $L \rightarrow \infty$. This will be illustrated numerically in Section 6.

5 Nonexistence of a class of solitary waves

The Whitham equation was designed to incorporate both breaking and dispersion. However, if the depth $h_0 > 0$ is small when compared to the wave speed, then the dispersion term is small, and moreover, dispersion is very weak. As a result, for large velocities c , there are no travelling waves.

Theorem 5.1. *There are no steady and bounded continuous solutions of the Whitham equation with*

$$c > \kappa \sqrt{gh_0} \quad \text{and} \quad \inf \phi \leq 0 < \sup \phi, \quad (5.1)$$

where $\kappa = 2(\sqrt{2} + 1) \left\| \mathcal{F}^{-1} \left(\sqrt{(\tanh \xi)/\xi} \right) \right\|_{L^1(\mathbb{R})}$.

Remark 5.2. Note that the condition (5.1) means that there are no solitary waves with velocities much larger than the critical long wave speed $\sqrt{gh_0}$. Using the estimate

$$1 = \left\| \mathcal{F} \mathcal{F}^{-1} \left(\sqrt{\frac{\tanh \xi}{\xi}} \right) \right\|_{L^\infty(\mathbb{R})} \leq \left\| \mathcal{F}^{-1} \left(\sqrt{\frac{\tanh \xi}{\xi}} \right) \right\|_{L^1(\mathbb{R})},$$

the value of κ appearing in the statement of the theorem may be estimated below by $2(\sqrt{2} + 1)$.

Proof of Theorem 5.1. The proof proceeds by contradiction. Suppose that there exists a nontrivial bounded solution ϕ to (2.4). Then the following inequalities must hold.

$$\left(\|\phi\|_{L^\infty(\mathbb{R})} - \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})} \right) \|\phi\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})},$$

and

$$\|\phi\|_{L^\infty(\mathbb{R})} \leq \left(\|\phi\|_{L^\infty(\mathbb{R})} + \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})} \right) \|\phi\|_{L^\infty(\mathbb{R})},$$

so that

$$1 - \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})} \leq 1 + \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})}, \quad (5.2)$$

in view of that $\sup \phi > 0$. Note first that

$$\phi^2(x) \geq \phi(x) - \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})}$$

for all x . This is a simple consequence of (2.4). For the desired contradiction it is thus enough to show that there is some x , such that

$$\phi^2(x) < \phi(x) - \left\| \frac{1}{c} K_{h_0} \right\|_{L^1(\mathbb{R})} \|\phi\|_{L^\infty(\mathbb{R})},$$

or in other words

$$\phi^2(x) - \phi(x) + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})} < 0. \quad (5.3)$$

An application of (5.2) yields that

$$\begin{aligned} & \phi^2(x) - \phi(x) + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(\mathbb{R})} \\ & \leq \phi^2(x) - \phi(x) + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}(1 + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}), \end{aligned}$$

and we set out to examine the right hand side,

$$F(\phi) := \phi^2 - \phi + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}(1 + \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})}).$$

Observe that $F(\phi)$ is negative whenever

$$\begin{aligned} & \frac{1}{2} \left(1 - \sqrt{2 - (2\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2} \right) < \phi, \quad \text{and} \\ & \phi < \frac{1}{2} \left(1 + \sqrt{2 - (2\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2} \right). \end{aligned} \quad (5.4)$$

The left and right hand sides of (5.4) are real if $\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \frac{1}{2}(\sqrt{2} - 1)$. Taking the scaling and (2.2) into consideration, that follows from the requirement (5.1). Therefore, under that assumption, we have that

$$\begin{aligned} & \frac{1}{2} \left(1 - \sqrt{2 - (2\|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} + 1)^2} \right) \\ & < \frac{1}{2} < 1 - \|\frac{1}{c}K_{h_0}\|_{L^1(\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

in view of (5.2). Since ϕ is continuous with $\inf \phi \leq 0$, there is thus an x , such that both inequalities in (5.4) are satisfied. As a result, we have obtained that (5.3) holds, reaching the desired contradiction. \square

6 Numerical Approximation

For the numerical approximation of periodic travelling waves of the Whitham equation, a spectral projection is used. As above, the undisturbed depth h_0 and the wavelength L are fixed, and the speed c is used as the bifurcation parameter.

For the purpose of approximating periodic solutions of (2.3), a Fourier method is optimal. To define the Fourier-collocation projection, define the subspace

$$S_N = \text{span}_{\mathbb{C}} \left\{ \exp(ikx) \mid k \in \mathbb{Z}, -N/2 \leq k \leq N/2 - 1 \right\}$$

of $L^2((0, 2\pi))$. The collocation points are defined to be $x_j = \frac{2\pi j}{N}$ for $j = 0, 1, \dots, N-1$. Let I_N be the interpolation operator from $C_{per}^\infty([0, 2\pi])$ onto S_N . As explained in [10], this operator is defined in the following way. Given $u \in C_{per}^\infty([0, 2\pi])$, $I_N u$ is the unique element of S_N that coincides with u at the collocation points x_j . For the spectral projection, we use the equation (2.3) with $B = 0$. According to Theorem 3.1, the equation is defined on the interval $[-L, L]$, whereas the discrete Fourier transform to be used is most conveniently defined on $[-\pi, \pi]$. Therefore, the scaling $\phi(x) \rightarrow \phi(ax)$ is used, where $a = \frac{L}{\pi}$. Special attention has to be paid to the operator K_{h_0} . A straightforward calculation shows that

$$(K_{h_0} * u)(ax) = \sqrt{a} K_{h_0/a} * (u(a \cdot))(x). \quad (6.1)$$

Therefore, the rescaled equation for 2π -periodic solutions is

$$-c\phi + \frac{3}{4} \frac{c_0}{h_0} \phi^2 + \sqrt{a} K_{h_0/a} * \phi = 0.$$

The discretized form of this equation is

$$-c\phi_N + \frac{3}{4} \frac{c_0}{h_0} \phi_N^2 + \sqrt{a} [K_{h_0/a}]_N \phi_N = 0, \quad (6.2)$$

which is enforced at the collocation points x_j . If ϕ_N is written in terms of its discrete Fourier coefficients $\tilde{\phi}_N(k)$ as

$$\phi_N(x) = \sum_{-N/2 \leq k \leq N/2-1} \tilde{\phi}_N(k) \exp(ikx),$$

the operator $[K_{h_0/a}]_N$ can be evaluated using the formula

$$\begin{aligned} [K_{h_0/a}]_N \phi_N(x) &= \sqrt{\frac{gh_0}{a}} \tilde{\phi}_N(0) \\ &+ \sum_{\substack{1-N/2 \leq k \leq N/2-1 \\ k \neq 0}} \sqrt{\frac{g}{k} \tanh k(h_0/a)} \tilde{\phi}_N(k) \exp(ikx). \end{aligned}$$

Thus the operator $[K_{h_0/a}]_N$ is the truncation at the $N/2$ -st Fourier mode of the operator given by the periodic convolution with $K_{h_0/a}$. Note that this formulation includes the truncation of the Fourier mode $\tilde{\phi}_N(-N/2)$ which otherwise can lead to instabilities in the computation. The equation (6.2) is treated pseudospectrally. That is, multiplication is carried out in physical space, while the term involving $K_{h_0/a}$ is evaluated using the discrete Fourier transform.

The resulting system of equations can be solved using any standard nonlinear equation solver. We have chosen to use the Matlab routine *fsolve* which appears to work very efficiently. To make sure that the computed functions are approximate travelling waves for the Whitham equation, we have also used a dynamic integrator for the time-dependent Whitham equation. The equation (1.6) is translated to the interval $[0, 2\pi]$ by the scaling $\eta(x, t) \rightarrow \frac{1}{a} \eta(ax, t)$, where $a = \frac{L}{\pi}$ as before. The discretization is then defined by the following problem. Find a function $\eta_N : [0, T] \rightarrow S_N$, such that

$$\begin{cases} \partial_t \eta_N + \frac{3}{2} \frac{c_0}{h_0} \partial_x I_N(\eta_N^2) + \frac{1}{\sqrt{a}} [K_{h_0/a}]_N * \partial_x \eta_N = 0, & x \in [0, 2\pi], \\ \eta_N(\cdot, 0) = \phi_N. \end{cases} \quad (6.3)$$

In Figure 2, a branch of travelling-wave solutions is shown. Here the wavelength is chosen to be 2π , and the depth is $h_0 = 1$. Note that in this case, the wavenumber is $k = \frac{2\pi}{2L} = 1$, and therefore the phase velocity of a linear wave is given by $\sqrt{g \tanh(h_0)} \sim 2.7334$. In panel (c) shown in Figure 2, it appears that as the amplitude approaches zero, the velocity of the travelling wave approaches the linear wave speed. Note also that not the whole branch is shown in panels (a) and (b). Two periods of the highest wave we were able to compute is shown in panel (d). This solution seems to nearly have a cusp, a fact already noted by Whitham [34] using an asymptotic argument. Since a Fourier-collocation method is used, it is implicitly assumed that the solutions are smooth, and it is not possible to find the very highest wave predicted by Whitham. A possible method for finding the highest wave is outlined in [8], where a scheme based on Lagrange polynomials is used, and the highest point on the wave is treated as a boundary condition. However, the Whitham equation as it appears here was not treated in [8]. In Table 1, we record the numerical errors incurred by the time integration of an approximate travelling wave with velocity $c = 2.7$ propagating for 5 and 50 periods. To find the most advantageous combinations of the number of Fourier modes N and the time step h , we used a computation for one period. We then use this combina-

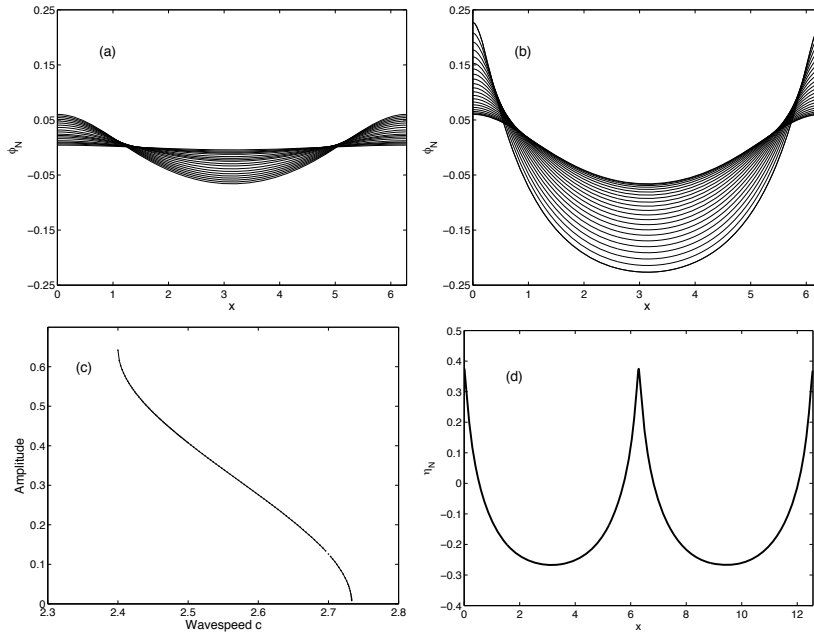


Figure 2: (a) and (b) Part of a branch of solutions of (6.2) with $h_0 = 1$ and $L = \pi$. Note that the highest wave is not shown here. (c) Amplitude vs. wave speed. (d) Two periods of the (nearly) highest wave.

N	h	L^2 -error	$ u _\infty - u_N _\infty$	L^2 -error	$ u _\infty - u_N _\infty$
2^5	1.0e-03	7.092e-04	9.927e-04	0.0078	0.0045
2^6	1.0e-03	3.821e-06	3.606e-06	3.316e-05	3.022e-06
2^7	1.0e-04	6.058e-06	1.208e-08	9.899e-07	6.675e-09
2^8	5.0e-06	1.217e-07	2.038e-11	2.198e-07	5.417e-11
5 periods			50 periods		

Table 1: Error in evolution code after 5 and 50 periods for the travelling wave shown in Figure 3.

tion, and integrated for 5 and 50 periods. The discrete L^2 -error, the difference in maximal height between the original wave, and the profile after 5 and 50 periods were computed. As can be seen, the error is decreasing for increasing N and decreasing h . Moreover, the fact that the difference in maximal height is generally smaller than the L^2 -error suggests that the error incurred during the time evolution is mostly due to a phase shift of the solution. This can also be observed in Figure 3, where the same travelling wave is shown after time integration for 10000 periods. These results also suggest that the travelling waves are orbitally stable, but no special investigation of this question has been carried out.

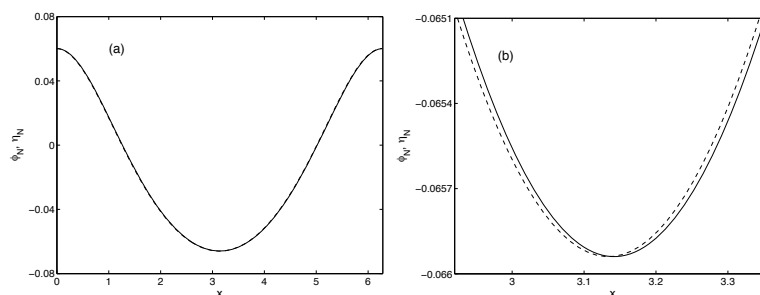


Figure 3: Solid line: approximate travelling wave ϕ_N for the Whitham equation with $h_0 = 1$, $L = \pi$, and $c = 2.7$. Dashed line: η_N after time integration using (6.3) for 10000 periods. In (a), the difference between ϕ_N and η_N is hardly visible. In this computation, $N = 512$ and $h = 0.0005$. The L^2 error was 0.0021, while the difference in height was $2.2385e - 06$. This and the magnification (b) suggests that the the error is mainly due to a phase shift.

In Section 4, a connection between travelling waves with finite period and solitary waves is given. In particular, it is shown that if the amplitude of a family of travelling waves with increasing wavelength L is bounded below $\frac{1}{2}$, then these travelling waves converge to a solitary wave. Here, we want to illustrate this result numerically. In Figure 4, a family of approximate travelling waves is shown in the case when the wavelength L is increasing, while h_0 and c are held constant. Note that amplitude is initially increasing, but seems to level off to an approximate value of 0.145. As Figure 5 shows, even though the wavelength L keeps increasing, the shape of the travelling waves does not change very much if a

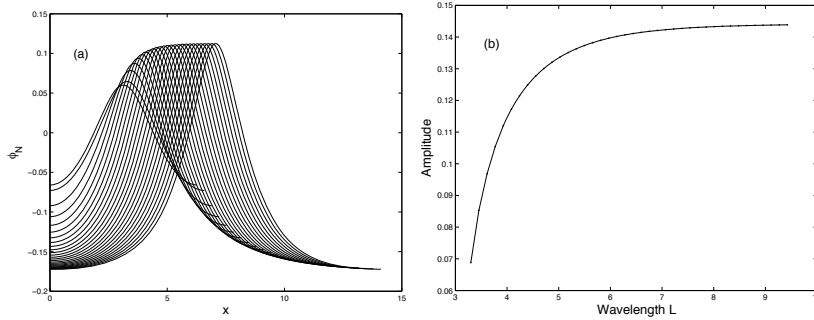


Figure 4: (a) Approximate travelling waves for the Whitham equation with $h_0 = 1$ and $c = 2.733$, and with increasing wavelength. (b) Amplitude as a function of wavelength.

certain threshold is passed. The numerical evidence suggests that these waves converge to a solitary wave, as was intimated by the proof in Section 4. Generally, a solitary wave is assumed to decay to zero at infinity. For the limiting solitary wave suggested in figures 4 and 5, this can be achieved by a Galilean transformation of the form

$$\phi \rightarrow \phi + \gamma \quad \text{and} \quad c \rightarrow c + 2\gamma.$$

This introduces a non-zero constant B in equation (2.3). However, it can be seen that the constant levels off to zero as the amplitudes of the sequence of travelling wave approaches the asymptotic value as shown in Figure 4 (b).

7 Conclusion

We have investigated the existence of travelling-wave solution of the Whitham equation, a nonlinear dispersive integro-differential equation capable of supporting breaking and peaking solutions. It has been found that small-amplitude travelling-wave solutions exist. Moreover, in the limit as the wavelength goes to infinity, these solutions converge to travelling-wave solutions on the real line. Nontrivial bounded and continuous solutions do not exist if the wave speed c is much larger than the limiting long-wave speed c_0 . Numerical approximations have been found of various travelling-wave solutions, including small-amplitude and finite-amplitude waves, as well as waves which are near the highest wave which

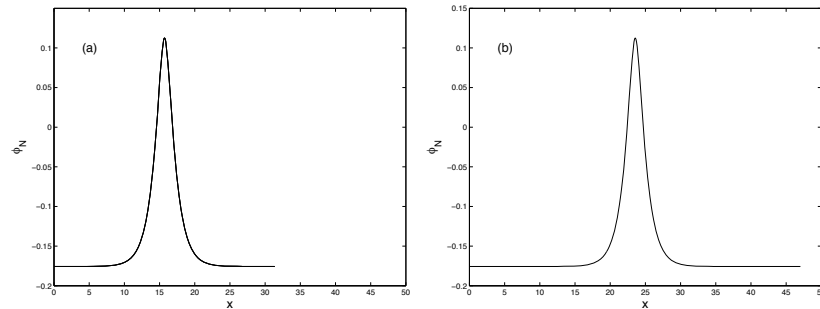


Figure 5: (a) Approximate travelling wave for the Whitham equation with $h_0 = 1$, $c = 2.733$, and $L = 5\pi$. (b) Approximate travelling wave for the Whitham equation with $h_0 = 1$, $c = 2.733$, and $L = 7.5\pi$.

is known to have a cusp. As the wavelength increases, the travelling waves appear to converge to a nontrivial solitary wave.

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