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Abstract—This paper considers robust stability analysis of a large network of interconnected uncertain systems. To avoid analyzing the entire network as a single large, lumped system, we model the network interconnections with integral quadratic constraints. This approach yields a sparse linear matrix inequality which can be decomposed into a set of smaller, coupled linear matrix inequalities. This allows us to solve the analysis problem efficiently and in a distributed manner. We also show that the decomposed problem is equivalent to the original robustness analysis problem, and hence our method does not introduce additional conservativeness.

I. INTRODUCTION

Robust stability analysis with integral quadratic constraints (IQCs) provides a unified framework for analysis of uncertain systems with different kinds of uncertainties [1]. The main computational burden in IQC analysis lies in the solution of a semi-infinite linear matrix inequality (LMI) which is generally dense. One method for solving this semi-infinite LMI makes use of the KYP lemma to reformulate the frequency-dependent semi-infinite LMI as a single frequency-independent LMI. However, this reformulation yields a dense LMI, and hence the computational cost is high for large systems even if the underlying structure is exploited as in [2], [3], [4], [5], [6], [7], [8], [9], [10]. The semi-infinite LMI can also be solved approximately by discretizing the frequency variable to obtain a finite number of LMIs. The resulting LMIs are generally dense, and as a consequence, IQC analysis of large-scale systems is prohibitively expensive in most cases.

In this paper, we consider IQC-based robustness analysis of a large network of interconnected uncertain systems. We show that by representing the network interconnections as quadratic constraints, we obtain a semi-infinite LMI that is sparse. Reformulating this LMI using the KYP lemma destroys sparsity, but discretizing the frequency variable does not, and hence we limit our attention to frequency-gridding methods. We can exploit sparsity in each LMI in different ways: (i) we can use an SDP solver that exploits sparsity [11], [12], [6], or (ii) we can use chordal decomposition methods to split a sparse LMI into a set of smaller, coupled LMIs [7], [13]. The first method solves the robustness analysis problem in a centralized manner, and hence it requires complete information about all the subsystems in the network. The decomposition approach, on the other hand, allows us to solve the robustness analysis problem in a distributed manner

where a small cluster of subsystems only communicates with its neighboring clusters in the network.

A. Related Work

Control and robustness analysis of interconnected systems is an active area of research that has been considered in several papers in the past few decades; see e.g. [14], [15], [16], [17], [18]. Different methods for robustness analysis have been developed, e.g., μ analysis and IQC analysis [19], [20], [21], [1], [22]. While these analysis tools are effective for small and medium-sized interconnected systems, they fail to produce results for large-scale interconnected systems because of the high computational cost. To address this issue, [23] and [24] propose an efficient method for robust stability analysis of interconnected uncertain systems with an interconnection matrix that is normal, and [18] considers stability analysis and design methods for networks of certain systems with uncertain interconnections. A similar problem is also considered in [25]. In [26], the authors consider robust stability analysis of interconnected uncertain systems using IQC-based analysis, and they show that when the interconnection matrix is unitarily diagonalizable, the analysis problem can be decomposed into smaller problems that are easier to solve. Finally, [27] shows that by using Nyquist-like conditions and by considering the dynamics of individual subsystems and their neighbors, it is possible to relax the interconnection constraints and arrive at scalable analysis conditions for interconnected uncertain systems.

B. Outline

The paper is organized as follows. In Section II, we present an integral quadratic constraint for interconnected uncertain systems, and we show how this can be used to obtain a sparse analysis problem. In Section III, we show how the sparse analysis problem can be decomposed into smaller, coupled problems, and we discuss how the analysis problem can be solved distributedly. Finally, we conclude the paper in Section IV.

C. Notation

We denote with $\mathbb{R}$ the set of real numbers and with $\mathbb{R}^{m \times n}$ the set of real $m \times n$ matrices. The transpose and conjugate transpose of a matrix G are denoted by G^T and G^*, respectively. Given matrices G^i, $i = 1, \ldots, N$, $\text{diag}(G^1, \ldots, G^N)$ denotes a block diagonal matrix with

blocks G^i. Similarly, given a set of vectors $v_1, \ldots, v_N$, the column vector $(v_1, \ldots, v_N)$ is obtained by stacking these. The matrix inequality $G \succ H$ ($G \succeq H$) means that $G - H$ is positive (semi)definite. We denote with $\mathcal{L}_2$ the set of square integrable signals, and $\mathcal{RH}_\infty$ represents the set of real, rational transfer functions with no poles in the closed right half plane.

II. STABILITY ANALYSIS OF INTERCONNECTED SYSTEMS

We begin this section with a brief review of some key results from IQC analysis.

A. IQC Analysis

Let $\Delta : \mathbb{R}^d \to \mathbb{R}^d$ be a bounded and causal operator. The mapping $q = \Delta(p)$ can be characterized using integral quadratic constraints defined as follow.

Definition 1 ([1], [22]): Let Π be a bounded, self-adjoint operator. The operator Δ is said to satisfy the IQC defined by Π, i.e., $\Delta \in \mathrm{IQC}(\Pi)$, if

$$\int_0^\infty \begin{bmatrix} q \\ \Delta(q) \end{bmatrix}^T \Pi \begin{bmatrix} q \\ \Delta(q) \end{bmatrix} dt \geq 0, \quad \forall q \in \mathcal{L}_2. \tag{1}$$

This can also be written in frequency domain as

$$\int_{-\infty}^\infty \begin{bmatrix} \widehat{q}(j\omega) \\ \widehat{\Delta(q)}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{q}(j\omega) \\ \widehat{\Delta(q)}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{2}$$

where $\widehat{q}$ and $\widehat{\Delta(q)}$ are Fourier transforms of the signals q and $\Delta(q)$, respectively, and $\Pi(j\omega) = \Pi(j\omega)^*$ is a transfer function matrix.

IQCs can be used to describe different classes of operators and hence different uncertainty sets, e.g., operators with bounded gain, sector bounded uncertainty, and static nonlinearities [1]. We now consider a system described by the equations

$$p = Gq, \tag{3a}$$
$$q = \Delta(p), \tag{3b}$$

where $G \in \mathcal{RH}_\infty^{m \times m}$ is a transfer function matrix and Δ represents the uncertainty in the system. Recall that Δ is bounded and causal. Robust stability of the system (3) can be established using the following theorem.

Theorem 1 ([1], [22]): The uncertain system in (3) is robustly stable, if

1) for all $\tau \in [0, 1]$ the interconnection described in (3), with $\tau\Delta$, is well-posed.
2) for all $\tau \in [0, 1]$, $\tau\Delta \in \mathrm{IQC}(\Pi)$.
3) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \preceq -\epsilon I, \quad \forall \omega \in \mathbb{R}. \tag{4}$$

It follows from Theorem 1 that, given a multiplier Π, IQC analysis requires the solution of the semi-infinite LMI (4). As
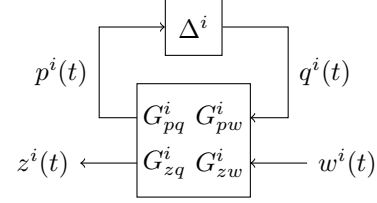

Fig. 1.   The ith uncertain subsystem with structured uncertainty.

mentioned in the introduction, we consider an approximate solution to the IQC analysis problem where the feasibility of the LMI in (4) is checked only for a finite number of frequency points. We will see later in this section that this allows us to reformulate the robust stability problem as a sparse LMI when the system of interest is a network of uncertain interconnected systems.

Remark 1 ([22], [26]): Suppose $\Delta^i \in \mathrm{IQC}(\Pi^i)$ where

$$\Pi^i = \begin{bmatrix} \Pi_{11}^i & \Pi_{12}^i \\ \Pi_{21}^i & \Pi_{22}^i \end{bmatrix}. \tag{5}$$

The block-diagonal operator $\mathrm{diag}(\Delta^1, \ldots, \Delta^N)$ then satisfies the IQC defined by

$$\tilde{\Pi} = \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} \end{bmatrix} \tag{6}$$

where $\bar{\Pi}_{ij} = \mathrm{diag}(\Pi_{ij}^1, \ldots, \Pi_{ij}^N)$.

B. Network of Uncertain Systems

Consider a network of N uncertain subsystems of the form

$$\begin{aligned} p^i &= G_{pq}^i q^i + G_{pw}^i w^i \\ z^i &= G_{zq}^i q^i + G_{zw}^i w^i \\ q^i &= \Delta^i(p^i), \end{aligned} \tag{7}$$

where $G_{pq}^i \in \mathcal{RH}_\infty^{d_i \times d_i}$, $G_{pw}^i \in \mathcal{RH}_\infty^{d_i \times m_i}$, $G_{zq}^i \in \mathcal{RH}_\infty^{l_i \times d_i}$, $G_{zw}^i \in \mathcal{RH}_\infty^{l_i \times m_i}$, and $\Delta^i : \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$. The ith subsystem is shown in Fig. 1. The network interconnections are defined by the equation

$$\underbrace{\begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^N \end{bmatrix}}_{w} = \underbrace{\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1N} \\ \Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{N1} & \Gamma_{N2} & \cdots & \Gamma_{NN} \end{bmatrix}}_{\Gamma} \underbrace{\begin{bmatrix} z^1 \\ z^2 \\ \vdots \\ z^N \end{bmatrix}}_{z}, \tag{8}$$

where the ijth block Γ_{ij} is a 0-1 matrix that defines the connections from system j to system i, and $w = (w^1, \ldots, w^N)$ and $z = (z^1, \ldots, z^N)$ are the stacked inputs and outputs, respectively. Similarly, we define $q = (q^1, \ldots, q^N)$ and $p = (p^1, \ldots, p^N)$. Using the interconnection matrix, the
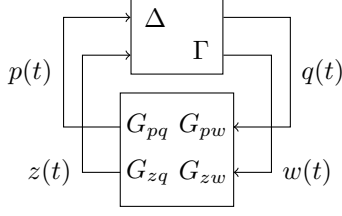
Fig. 2. Compact representation of the network of interconnected systems.

entire system can be expressed as

$$p = G_{pq}q + G_{pw}w \tag{9a}$$

$$z = G_{zq}q + G_{zw}w \tag{9b}$$

$$q = \Delta(p) \tag{9c}$$

$$w = \Gamma z, \tag{9d}$$

where the matrices $G_{pq,}$, G_{pw}, G_{zq}, and G_{zw} are block-diagonal and of the form $G_{ij} = \mathrm{diag}(G_{ij}^1, \ldots, G_{ij}^N)$, and $\Delta = \mathrm{diag}(\Delta^1, \ldots, \Delta^N)$ is block-diagonal. The full system is shown in Fig. 2. One approach to robustness analysis of interconnected systems of the form (9) is to eliminate w and z from the system equations. This yields the lumped system

$$p = \bar{G}q, \tag{10a}$$

$$q = \Delta(p), \tag{10b}$$

where

$$\bar{G} = G_{pq} + G_{pw}(I - \Gamma G_{zw})^{-1} \Gamma G_{zq}$$

is the lumped system matrix. Note that the matrix $I - \Gamma G_{zw}$ must have a bounded inverse for all frequencies in order for the interconnection to be well-posed. The lumped system matrix $\bar{G}$ is unfortunately dense in general, even if the matrix $I - \Gamma G_{zw}$ is sparse. This follows from the Cayley–Hamilton theorem. As a result, IQC analysis based on the lumped system description (10) is prohibitively expensive for large networks of uncertain systems.

C. IQCs for Interconnections

Robust stability of the interconnected uncertain system (9) can be investigated using the IQC framework. To this end, we now define an IQC for the interconnection equation $w = \Gamma z$. The equation $w = \Gamma z$ can be expressed as the following quadratic constraint

$$-\|w - \Gamma z\|_X^2 = -\begin{bmatrix} z \\ w \end{bmatrix}^* \begin{bmatrix} -\Gamma^T \\ I \end{bmatrix} X \begin{bmatrix} -\Gamma & I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} \geq 0 \tag{11}$$

where $\| \cdot \|_X$ denotes the norm induced by the inner product $\langle \alpha, X\beta \rangle$ for some $X \succ 0$ of order $\bar{m} = \sum_{i=1}^N m_i$. As a

result, we can define an IQC for the interconnections using the following multiplier

$$\hat{\Pi} = \begin{bmatrix} -\Gamma^T X \Gamma & \Gamma^T X \\ X\Gamma & -X \end{bmatrix} \tag{12}$$

where X is positive definite. Note that the framework described in this paper can be extended to incorporate dynamic and/or uncertain interconnections by defining suitable IQCs for such interconnections.

We now show that the IQC analysis problem can be expressed as a sparse LMI by using (12) to model the interconnections. Consider the interconnected uncertain system defined in (9), and suppose that

$$\Delta^i \in \mathrm{IQC}(\Pi^i), \quad i = 1, \ldots, N$$

such that $\Delta \in \mathrm{IQC}(\bar{\Pi})$ where $\bar{\Pi}$ is defined in (6). We will henceforth assume that Δ satisfies conditions 1 and 2 in Theorem 1. Then, by Remark 1, the network of uncertain systems is robustly stable if there exists $\bar{\Pi}$ and $X \succ 0$ such that

$$\begin{bmatrix} G_{pq} & G_{pw} \\ G_{zq} & G_{zw} \\ I & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{\Pi}_{11} & 0 & \bar{\Pi}_{12} & 0 \\ 0 & \hat{\Pi}_{11} & 0 & \hat{\Pi}_{12} \\ \bar{\Pi}_{21} & 0 & \bar{\Pi}_{22} & 0 \\ 0 & \hat{\Pi}_{21} & 0 & \hat{\Pi}_{22} \end{bmatrix} \begin{bmatrix} G_{pq} & G_{pw} \\ G_{zq} & G_{zw} \\ I & 0 \\ 0 & I \end{bmatrix} \preceq -\epsilon I, \tag{13}$$

for $\epsilon > 0$ and for all $\omega \in [0, \infty]$. This can also be written as

$$\begin{bmatrix} G_{pq} & G_{pw} \\ I & 0 \\ G_{zq} & G_{zw} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} & 0 & 0 \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} & 0 & 0 \\ 0 & 0 & \hat{\Pi}_{11} & \hat{\Pi}_{12} \\ 0 & 0 & \hat{\Pi}_{21} & \hat{\Pi}_{22} \end{bmatrix} \begin{bmatrix} G_{pq} & G_{pw} \\ I & 0 \\ G_{zq} & G_{zw} \\ 0 & I \end{bmatrix} \preceq -\epsilon I, \tag{14}$$

or equivalently, as

$$\begin{bmatrix} G_{pq} & G_{pw} \\ I & 0 \end{bmatrix}^* \begin{bmatrix} \bar{\Pi}_{11} & \bar{\Pi}_{12} \\ \bar{\Pi}_{21} & \bar{\Pi}_{22} \end{bmatrix} \begin{bmatrix} G_{pq} & G_{pw} \\ I & 0 \end{bmatrix} - \\ \begin{bmatrix} -G_{zq}^* \Gamma^T \\ I - G_{zw}^* \Gamma^T \end{bmatrix} X \begin{bmatrix} -\Gamma G_{zq} & I - \Gamma G_{zw} \end{bmatrix} \preceq -\epsilon I. \tag{15}$$

The following theorem establishes the equivalence between robustness analysis of the lumped system (10) via (4) and robustness analysis of (9) via (15).

Theorem 2: The LMI (15) is feasible if and only if

$$\tilde{G}_{11} = \begin{bmatrix} \bar{G} \\ I \end{bmatrix}^* \bar{\Pi} \begin{bmatrix} \bar{G} \\ I \end{bmatrix} \preceq -\epsilon I \tag{16}$$

is feasible.

Proof: We start by applying a congruence transformation $A \mapsto T^* A T$ to the left-hand side of (15) where T is nonsingular and defined as

$$T = \begin{bmatrix} I & 0 \\ (I - \Gamma G_{zw})^{-1} \Gamma G_{zq} & I \end{bmatrix}.$$

The result is a 2-by-2 block matrix

$$\tilde{G} = \begin{bmatrix} \tilde{G}_{11} & \tilde{G}_{12} \\ \tilde{G}_{21} & \tilde{G}_{22} \end{bmatrix} \tag{17}$$

where $\tilde{G}_{11}$ is defined as in (16), and

$$\tilde{G}_{12} = \tilde{G}_{21}^* = \begin{bmatrix} \bar{G} \\ I \end{bmatrix}^* \bar{\Pi} \begin{bmatrix} G_{pw} \\ 0 \end{bmatrix},$$

$$\tilde{G}_{22} = \begin{bmatrix} G_{pw} \\ 0 \end{bmatrix}^* \bar{\Pi} \begin{bmatrix} G_{pw} \\ 0 \end{bmatrix} - (I - \Gamma G_{zw})^* X (I - \Gamma G_{zw}).$$

Since $\tilde{G}$ is homogeneous in $\bar{\Pi}$ and X, it follows that (15) is equivalent to $\tilde{G} \prec 0$. Now if (16) is feasible, then (15) is feasible if

$$S = \tilde{G}_{22} - \tilde{G}_{21} \tilde{G}_{11}^{-1} \tilde{G}_{21}^* \prec 0. \tag{18}$$

Since $(I - \Gamma G_{zw})$ is nonsingular for any $\bar{\Pi}$ such that $\tilde{G}_{11} \prec 0$, we can scale $X \succ 0$ so that $S \prec 0$. Conversely, if (15) is feasible, then $\tilde{G}_{11} \prec 0$ and hence we can scale $\bar{\Pi}$ and X so that $\tilde{G}_{11} \preceq -\epsilon I$. ∎

Theorem 2 implies that our approach does not introduce conservativeness. However, the LMI (15) is generally dense if the matrix X is dense. We have the following corollary to Theorem 2.

Corollary 1: In (15), it is sufficient to consider a diagonal scaling matrix of the form $X = xI$ with $x > 0$.

Proof: Suppose $\bar{\Pi}$ is feasible in (16). Then we can choose $X = xI$ with $x > 0$ such that the Schur complement S in (18) is negative definite. ∎

Corollary 1 implies that we can choose X to be any positive diagonal matrix. This mean that the LMI (15) becomes sparse when the interconnection matrix Γ is sufficiently sparse. In the next section, we discuss how we can solve (15) efficiently when the LMI is sparse.

III. DECOMPOSING THE ANALYSIS PROBLEM

In networks where each subsystem is connected only to a small number of neighboring subsystems, the LMI (15) is generally quite sparse. As mentioned in the introduction, we can solve the LMI (15) in a centralized manner using a sparse SDP solver such as DSDP, or alternatively, we can make use of decomposition techniques to facilitate parallel and/or distributed computation. In this section, we discuss chordal decomposition, and we show how this decomposition can be used to formulate an efficient distributed algorithm for robust stability analysis.

A. *Chordal Decomposition*

Chordal sparsity plays a fundamental role in many sparse matrix algorithms [28]. We say that a matrix is chordal if the corresponding sparsity graph is chordal, and a graph is called chordal if all its cycles of length at least four have a chord; see e.g. [29, Ch. 4]. A clique is a maximal set of vertices that induce a complete subgraph. The cliques of the sparsity graph correspond to the maximal dense principal submatrices in the matrix. We will use the following result from [30] to decompose the LMI (15).

Theorem 3 (Agler et al. [30]): Let A be negative semidefinite and chordal with cliques $J_1, J_2, \ldots, J_L$. Then there exists a decomposition

$$A = \sum_{i=1}^{L} A_i, \quad A_i \preceq 0, \ i = 1, \ldots, L \tag{19}$$

such that $[A_i]_{jk} = 0$ for all $(j, k) \notin J_i \times J_i$.

Proof: See [30] and [31]. ∎

Remark 2: It is also possible to apply Theorem 3 to negative semidefinite matrices with nonchordal sparsity by using a so-called chordal embedding. A chordal embedding can be computed efficiently with symbolic factorization techniques; see e.g. [32].

A band matrix is an example of a chordal matrix, i.e., if A has half-bandwidth w, then

$$A_{jk} = 0, \quad |j - k| > w,$$

and the cliques of the corresponding sparsity graph are given by the sets

$$J_i = \{i, i+1, \ldots, i+w\}, \quad i = 1, \ldots, n - w,$$

where n is the order of A.

The cliques of a chordal graph can be represented using a clique tree which is a maximum-weight spanning tree of a weighted clique intersection graph; refer to [28] for further details. The clique tree can be used to parameterize all possible clique-based splittings of A. For example, consider an irreducible chordal matrix $A \preceq 0$ with two cliques J_1 and J_2, and let $A = \tilde{A}_1 + \tilde{A}_2$ be an any clique-based splitting of A such that $[\tilde{A}_i]_{jk} = 0$ for all $(j, k) \notin J_i \times J_i$. Then, by Theorem 3, there exists a symmetric matrix Z that satisfies

$$Z_{ij} = 0, \quad (i, j) \notin (J_1 \cap J_2) \times (J_1 \cap J_2)$$

such that

$$A_1 = \tilde{A}_1 + Z \preceq 0, \ A_2 = \tilde{A}_2 - Z \preceq 0.$$

It is easy to verify that the matrices A_1 and A_2 satisfy (19). We can use this decomposition technique to parameterize all possible splittings of the LMI (15). Suppose (15) is irreducible and chordal with cliques $J_1, \ldots, J_L$. Then if we choose $X = xI$, we can split (15) as

$$\tilde{A}_1 + \tilde{A}_2 + \cdots + \tilde{A}_L \tag{20}$$

such that the matrices $\tilde{A}_1, \ldots, \tilde{A}_L$ share only the variable x, and $[\tilde{A}_i]_{jk} = 0$ for all $(j, k) \notin J_i \times J_i$. This means that (15) is equivalent to L coupled LMIs

$$\tilde{A}_i + Z_{p_i i} - \sum_{j \in \mathrm{ch}(i)} Z_{ij} \preceq 0, \ i = 1, \ldots, L \tag{21}$$

where p_i is the parent of the ith clique in the clique tree and $\mathrm{ch}(i)$ is the set of children of clique i. The matrix Z_{ij} couples cliques i and j, and it satisfies

$$[Z_{ij}]_{kl} = 0, \quad (k, l) \notin (J_i \cap J_j) \times (J_i \cap J_j).$$
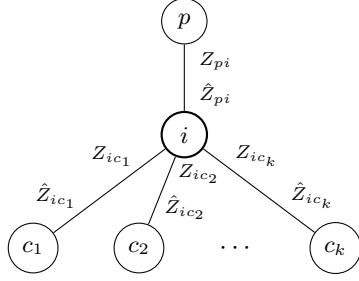
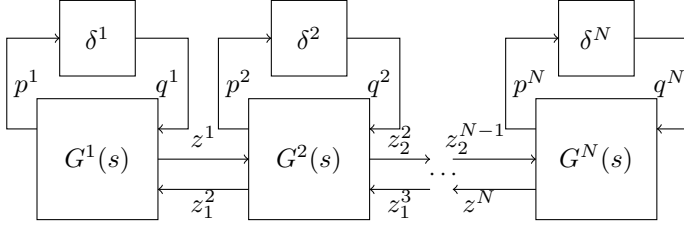Fig. 3. Subtree of clique tree: clique i, its parent, and its children.

Fig. 4. A chain of N uncertain system.

Note that the term $Z_{p_i i}$ in (21) disappears if clique i is the root of the clique tree, and the summation over the children of clique i disappears if i is a leaf clique. Furthermore, note that the sum of the left-hand sides of the LMIs in (21) is equal to (20).

B. Distributed Algorithm

The set of LMIs (21) can be expressed as

$$\tilde{A}_i + \hat{Z}_{p_i i} - \sum_{j \in \mathrm{ch}(i)} Z_{ij} \preceq 0, \ i = 1, \ldots, L \quad (22\mathrm{a})$$

$$x_k = x_l, \ \hat{Z}_{kl} = Z_{kl}, \ \forall (k,l) \in \mathcal{T} \quad (22\mathrm{b})$$

where $\hat{Z}_{p_i i}$ is the copy of $Z_{p_i i}$ associated with clique i, x_i is the copy of x associated with clique i, and $(k,l) \in \mathcal{T}$ means that (k,l) is an edge in the clique tree $\mathcal{T}$. The coupling between the LMIs (22a) is now described as a set of equality constraints (22b). Fig. 3 shows a subtree of the clique tree with the auxiliary variables associated with the subtree. The formulation (22) is a convex feasibility problem of the form

$$\begin{aligned} \text{find} \quad & x, s_1, s_2, \ldots, s_L \\ \text{subject to} \quad & s_i \in \mathcal{C}_i, \ i = 1, \ldots, L \quad (23) \\ & s_i = H_i(z), \ i = 1, \ldots, L \end{aligned}$$

where $\mathcal{C}_1, \ldots, \mathcal{C}_L$ are convex sets, $s_1, \ldots, s_L$ are local variables, and the constraints $s_i = H_i(z)$ ensure global consensus. The problem (23) can be solved distributedly using e.g. the alternating projection (AP) method, Dykstra's AP method, or the alternating direction method of multipliers [33].
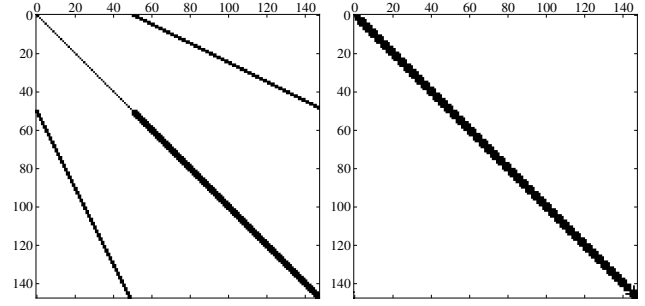

(a) Before reordering. (b) After reordering.

Fig. 5. Sparsity pattern associated with (15) for a chain of 50 systems.

C. Example: Chain of Uncertain Systems

We now consider as an example a chain of N uncertain systems where each system $G^i(s)$ is defined as in (7) with scalar uncertainties $\delta^1, \ldots, \delta^N$. Fig. 4 shows the chain of uncertain systems.

The inputs and outputs are defined as $w_i = (w_i^1, w_i^2)$ and $z_i = (z_i^1, z_i^2)$ for $i = 2, \ldots, N-1$, and moreover, $w_1 = w_1^1$, $w_N = w_N^1$, $z_1 = z_1^2$, and $z_N = z_N^2$. The interconnections are described by the equations $w_i^1 = z_{i-1}^2$ and $w_i^2 = z_{i+1}^1$ for $i = 2, \ldots, N-1$, and $w_1^2 = z_2^1$ and $w_N^1 = z_{N-1}^2$. We assume that the signals w_i^j and z_i^j are scalar-valued. As a result, the interconnection matrix Γ that describes this network has nonzero blocks $\Gamma_{i,i-1} = \Gamma_{i-1,i}^T$ for $i = 2, \ldots, N$, and these blocks are given by

$$\Gamma_{i,i-1} = \Gamma_{i-1,i}^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ i = 3, \ldots, N-1, \quad (24)$$

and

$$\Gamma_{21} = \Gamma_{12}^T = (1,0), \quad \Gamma_{N-1,N} = \Gamma_{N,N-1}^T = (0,1). \quad (25)$$

In this example, we will assume that the uncertainties $\delta^1, \ldots, \delta^N$ are scalar, and moreover, $\delta^i \in \mathrm{IQC}(\Pi_i)$ for all $i = 1, \ldots, N$. If we let X be diagonal, the sparsity pattern associated with the LMI (15) is then chordal with $2N-2$ cliques, and the largest clique is of order 4. The LMI (15) can therefore be decomposed into $2N-2$ coupled LMIs of order at most 4. The sparsity pattern associated with (15) for a chain of 50 uncertain systems is shown in Fig. 5. In this example, it is also possible to combine overlapping cliques such that we get a total of N cliques of order at most 5, and in general, there is a trade-off between the number of cliques and the order of the cliques. Note that analyzing the lumped system yields an LMI (16) of order N whereas the sparse LMI (15) is of order $3N-2$. For large networks, solving the sparse LMI can be much faster, but for small and medium-sized networks, the original dense LMI (16) may be cheaper to solve.

IV. Conclusions

IQC-based robustness analysis of a large network of interconnected systems involves the solution of a large, dense

LMI. By expressing the network interconnections in terms of IQCs, we have shown that it is possible to express the robustness analysis problem as a sparse LMI that can be decomposed into a set of smaller but coupled LMIs. The decomposed problem can be solved using distributed computations, and we have shown that it is equivalent to the original problem. These findings suggest that our method is applicable to robustness analysis of large networks of interconnected uncertain systems, but further work needs to be done to establish what types of network structure yield computationally efficient decompositions.

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