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SMOOTH TIME-FREQUENCY ESTIMATION USING COVARIANCE FITTING

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ABSTRACT

In this paper, we introduce a time-frequency spectral estimator for smooth spectra, allowing for irregularly sampled measurements. A non-parametric representation of the time dependent (TD) covariance matrix is formed by assuming that the spectrum is piecewise linear. Using this representation, the time-frequency spectrum is then estimated by solving a convex covariance fitting problem, which also, as a byproduct, provides an enhanced estimation of the TD covariance matrix. Numerical examples using simulated non-stationary processes show the preferable performance of the proposed method as compared to the classical Wigner-Ville distribution and a smoothed spectrogram.

1. INTRODUCTION

Estimating the spectral content of time-varying, non-stationary, and possibly non-uniformly sampled sequences is a topic common for a wide range of fields, and one that has attracted notable attention in the recent decades [1–8]. Much of the recent efforts have focused on the estimation of the various kinds of sparse spectra, with the estimation of smooth spectra attracting notably less attention (see, e.g., [9] and the references therein). Often, smooth spectra are modeled using ARMA models, which allow for reliable estimates given reasonable *a priori* information of the model orders, although it is a non-trivial problem to find decent estimates of such model orders. Recently, there has also been efforts to formulate alternative smooth spectral estimators, such as the so-called LIMES method introduced in [9], wherein a piecewise smooth non-parametric spectral estimator was introduced. In spite of the frequent reoccurrence of non-uniformly sampled data sets, most of the available methods assume uniformly sampled data sets, and only limited work has been done on finding the time-frequency (TF) distribution of non-uniformly sampled measurements, with the notably exception of [10], wherein a method is presented that uses a set of signal dependent kernels, based on the minimum variance filter formulation, allowing for irregularly sampled measurement. In this work, we seek to alleviate this shortcoming by extending the aforementioned LIMES estimator [9] to the case of time
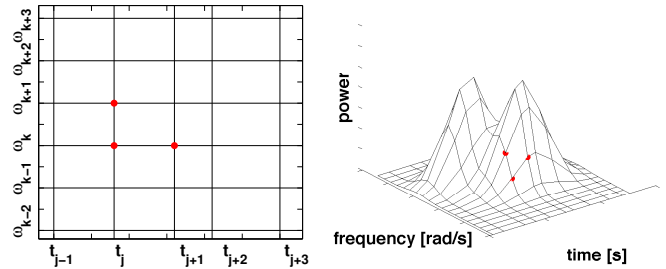
Fig. 1: An example of the assumed grid structure. The left plot shows the division of the TF plane into rectangles where any point inside the rectangle is assumed to lie on the plane defined by the lower three corners. In the right figure, an example of a TF distribution is shown with the corresponding grid structure.

dependent and non-uniformly sampled data sets. Numerical examples illustrates the achievable performance of the proposed estimator as compared to the classical Wigner-Ville distribution (WVD) and a smoothed spectrogram, clearly indicating the preferable performance of the proposed method. The remainder of the paper is organized as follows: in the next section, we present the data model used to form the TF distribution, followed in Section 3 by the derivation of the proposed estimator. Finally, Section 4 examines the achievable performance of the proposed estimator.

2. DATA MODEL

Consider a real-valued signal, $y(t)$, where $t = t_0, \ldots, t_{N-1}$ denotes the (possibly non-uniform) sampling time, with TF spectrum, $\Phi(t, \omega)$, which is here assumed to be band-limited within $[-B, B]$, as well as being alias-free and smooth. Expressed differently, consider the TF plane evaluated over a 2-D grid over time and frequency, wherein the former is distributed according to t, whereas the latter is uniformly distributed over M frequency grid points. Any point inside a given grid rectangle is assumed to lie on the plane defined by the lower three corner of the rectangle, as illustrated in Figure 1. Hence, for any point (t, f) in the TF plane, for
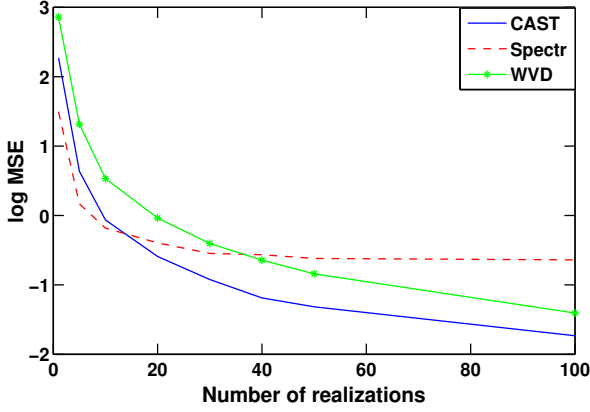
Fig. 2: The log MSE for the proposed algorithm as compared with the WVD and the smoothed spectrogram.
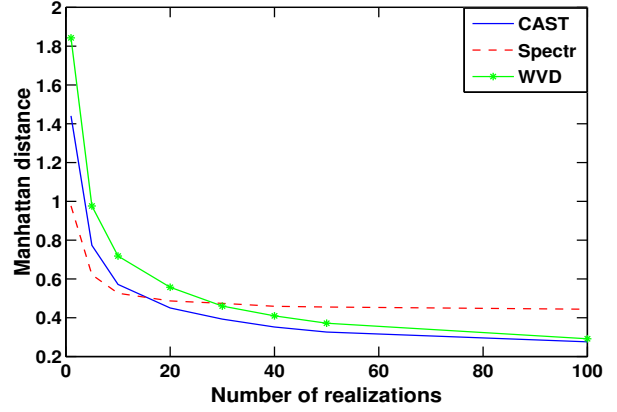

Fig. 3: The Manhattan distance for the proposed algorithm as compared with the WVD and the smoothed spectrogram.

$t \in [t_j, t_{j+1})$ and $\omega \in [\omega_k, \omega_{k+1})$, the TF spectrum satisfies

$$
\begin{aligned}
\Phi(t,\omega) =& \frac{\Phi(t_{j+1},\omega_k) - \Phi(t_j,\omega_k)}{t_{j+1}-t_j}(t-t_j)+ \\
& \frac{\Phi(t_j,\omega_{k+1}) - \Phi(t_j,\omega_k)}{\Delta_\omega}(\omega-\omega_k) + \Phi(t_j,\omega_k) \\
=& \frac{t-t_j}{t_{j+1}-t_j}\Phi(t_{j+1},\omega_k) + \frac{\omega-\omega_k}{\Delta_\omega}\Phi(t_j,\omega_{k+1})+ \\
& \left(1 - \frac{t-t_j}{t_{j+1}-t_j} - \frac{\omega-\omega_k}{\Delta_\omega}\right)\Phi(t_j,\omega_k) \\
\triangleq& \alpha_1(j,k)\Phi(t_{j+1},\omega_k) + \alpha_2(k)\Phi(t_j,\omega_k)+ \\
& (1-\alpha_1(j) - \alpha_2(k))\,\Phi(t_j,\omega_k)
\end{aligned}
\tag{1}
$$

where Δ_ω denotes the width of the frequency grid surface. Furthermore, the TD covariance function is defined as

$$
\mathbf{R}(p,s) = \int_{-B}^{B} \Phi\left(\frac{t_p+t_s}{2},\omega\right)e^{i\omega(t_p-t_s)}d\omega
\tag{2}
$$

where the limits of the integral are due to the assumption that the signal is band-limited, and t_p denotes the time of the p:th measurement.

3. THE PROPOSED TF SPECTRAL ESTIMATOR

Exploiting the expression in (1), one may form the covariance matrix defined in (2) as

$$
\mathbf{R}(p,s) = \int_{-B}^{B} \Phi(t,\omega)e^{i\omega\tau}d\omega
\tag{3}
$$

$$
= \sum_{k=1}^{M} \int_{\omega_k}^{\omega_{k+1}} \big(\alpha_1\Phi(t_{j+1},\omega_k) + \alpha_2\Phi(t_j,\omega_{k+1})+ \\
(1-\alpha_1-\alpha_2)\,\Phi(t_j,\omega_k)\big)e^{i\omega\tau}d\omega
\tag{4}
$$

where $\tau = t_p - t_s$ and $t = \frac{t_p+t_s}{2}$. Forming the tensors

$$
\mathcal{F}_{j,k} = \int_{\omega_k}^{\omega_{k+1}} \alpha_1(j)e^{i\omega\tau}d\omega
\tag{5}
$$

$$
\mathcal{G}_{k} = \int_{\omega_k}^{\omega_{k+1}} \alpha_2(k)e^{i\omega\tau}d\omega
\tag{6}
$$

$$
\mathcal{H}_{k} = \int_{\omega_k}^{\omega_{k+1}} e^{i\omega\tau}d\omega
\tag{7}
$$

one may rewrite (3) as

$$
\begin{aligned}
R(p,s) =& \sum_{k=1}^{M} \mathcal{F}_{j,k}(t,\tau)\Phi(t_{j+1},\omega_k)+ \\
& (\mathcal{H}_1 - \mathcal{F}_{j,1} - \mathcal{G}_1)\,(t,\tau)\Phi(t_j,\omega_1)+ \\
& \sum_{k=2}^{M} \{(\mathcal{G}_{k-1} + \mathcal{H}_k - \mathcal{F}_{j,k} - \mathcal{G}_k)\,(t,\tau)\times \\
& \Phi(t_j,\omega_k)\} + \mathcal{G}_M(t,\tau)\Phi(t_j,\omega_{M+1})
\end{aligned}
\tag{8}
$$

where $\mathcal{X}_{z,y}$ denotes the (z,y):element of the tensor $\mathcal{X}$, and for all different t_p and t_s,

$$
\mathcal{F}_{j,k} = \begin{cases} 0 & \text{if } \tau = 0 \\ \frac{t-t_j}{t_{j+1}-t_j}\frac{e^{i\omega_{k+1}\tau}}{i\tau}\left(1 - e^{i\Delta_\omega\tau}\right) & \text{if } \tau \neq 0 \end{cases}
$$

$$
\mathcal{G}_{k} = \begin{cases} \frac{1}{2}\Delta_\omega & \text{if } \tau = 0 \\ e^{i\omega_{k+1}\tau}\left(\frac{1}{i\tau} + \frac{1}{\tau^2\Delta_\omega}\left(1 - e^{-i\Delta_\omega\tau}\right)\right) & \text{if } \tau \neq 0 \end{cases}
$$

$$
\mathcal{H}_{k} = \begin{cases} \Delta_\omega & \text{if } \tau = 0 \\ \frac{e^{i\omega_{k+1}\tau}}{i\tau}\left(1 - e^{-i\Delta_\omega\tau}\right) & \text{if } \tau \neq 0. \end{cases}
$$

Then, the TF spectral estimate may be found by solving the convex covariance fitting problem

$$
\min_{\boldsymbol{\Phi}} \quad \left\|\hat{\mathbf{R}} - \mathbf{R}(\boldsymbol{\Phi})\right\|_F \quad \text{subj. to} \quad \boldsymbol{\Phi}^T = \boldsymbol{\Gamma}\boldsymbol{\Phi}^T
\tag{9}
$$

Fig. 4: The MSE for the proposed algorithm as compared with the WVD and the smoothed spectrogram.

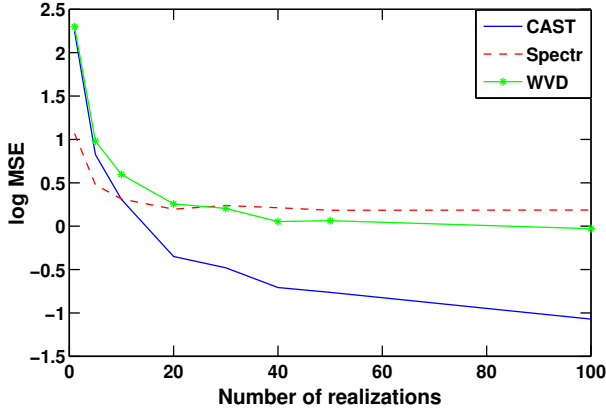**Fig. 5**: The Manhattan distance for the proposed algorithm as compared with the WVD and the smoothed spectrogram.

with $|| \cdot ||_F$ denoting the Frobenius norm, $(\cdot)^T$ the transpose, and where $\hat{\mathbf{R}}$ is an initial estimate of the TD covariance matrix, and the exchange matrix, $\mathbf{\Gamma}$, is formed as

$$\mathbf{\Gamma} = \begin{bmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{bmatrix} \qquad (10)$$

where all the empty indices of the matrix are zero. $\mathbf{\Gamma}$ ensures that $\mathbf{\Phi}(t, \omega)$ is symmetric in frequency, i.e., that $\mathbf{\Phi}(t, \omega)$ for the frequencies between $-B$ and 0 is mirrored at the frequencies between 0 and B. This constrain may, obviously, be dropped when the signal is not real-valued, thus allowing also for non-symmetric spectra. The optimization problem in (9) minimizes the distance between the estimated TD covariance matrix, $\hat{\mathbf{R}}$, and the TF spectrum, based on the transformation made up by the tensors $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$. In order to minimize the cost function in (9), we propose an Alternating Direction Method of Multipliers (ADMM) [11] scheme, wherein the cost function in (9) is divided into two parts

$$f(\boldsymbol{X}) + g(\boldsymbol{U}) = 1/2||\hat{\mathbf{R}} - \mathbf{R}(\boldsymbol{X})||_F^2 + 1/2||\hat{\mathbf{R}} - \mathbf{R}(\boldsymbol{U})||_F^2$$

with the constraint $\mathbf{u} = \mathbf{\Gamma}\mathbf{x}$ yielding the iterative updating

$$\boldsymbol{X}^{k+1} = \underset{\boldsymbol{X}}{\operatorname{argmin}} \left(f(\boldsymbol{X}) + (\rho/2)||\boldsymbol{X} - \mathbf{\Gamma}\boldsymbol{U}^k + \boldsymbol{D}^k||_2^2 \right)$$

$$\boldsymbol{U}^{k+1} = \underset{\boldsymbol{U}}{\operatorname{argmin}} \left(g(\boldsymbol{U}) + (\rho/2)||\boldsymbol{X}^{k+1} - \mathbf{\Gamma}\boldsymbol{U} + \boldsymbol{D}^k||_2^2 \right)$$

$$\boldsymbol{D}^{k+1} = \boldsymbol{D}^k + \boldsymbol{X}^{k+1} + \mathbf{\Gamma}\boldsymbol{U}^{k+1}$$

where ρ is the augmented Lagrangian parameter, $\mathbf{d}$ the dual variable, and k denotes the k:th iteration. It should be stressed that the initial estimation of $\mathbf{R}$ is important to the performance of the proposed method. Given the non-stationary nature of the assumed data, no temporal averaging may be exploited, and the TD covariance estimate thus instead needs to
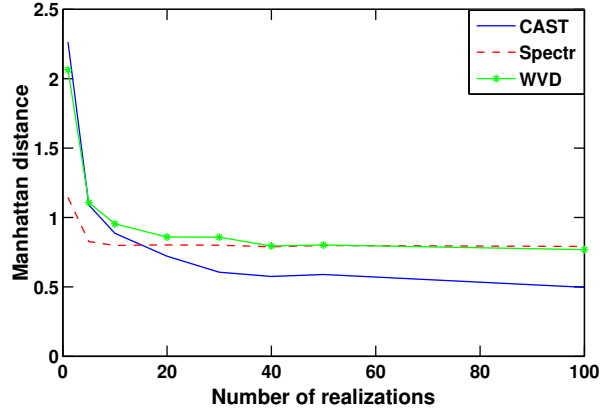
be formed over multiple realizations, say K. Here, we let

$$\mathbf{y}_k = \begin{bmatrix} y_k(1) & y_k(2) & \dots & y_k(N) \end{bmatrix}^T \qquad (11)$$

denote the k:th such realization, and form the initial estimate

$$\hat{\mathbf{R}} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{y}_k^* \mathbf{y}_k \qquad (12)$$

We term the resulting method the Covariance-fitting Approach for Smooth Time-frequency (CAST) estimator.

4. NUMERICAL RESULTS

We proceed to examine the performance of the proposed algorithm as compared to the classical WVD , defined as

$$WVD(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y\left(t + \frac{\tau}{2}\right) y^*\left(t - \frac{\tau}{2}\right) e^{-i\omega\tau} d\tau$$

as well as the smoothed periodogram, given by

$$\Psi(t, \omega) = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} h^*(s - t) y(s) e^{-i\omega s} ds \right|^2$$

where $h(t)$ denotes some smooth window function (see also, e.g., [8]). We here consider non-stationary data sequences with TD covariance function

$$\mathbf{R}(p, s) = q\left(\frac{t_p + t_s}{2}\right) r(t_p - t_s) \qquad (13)$$

where $q(t)$ may be any positive valued function such that $\int |q(t)|^2 dt < \infty$ (or any positive constant), and $r(\tau)$ must fulfill the properties of a wide sense stationary covariance function, i.e., so-called locally stationary processes (LSP) in
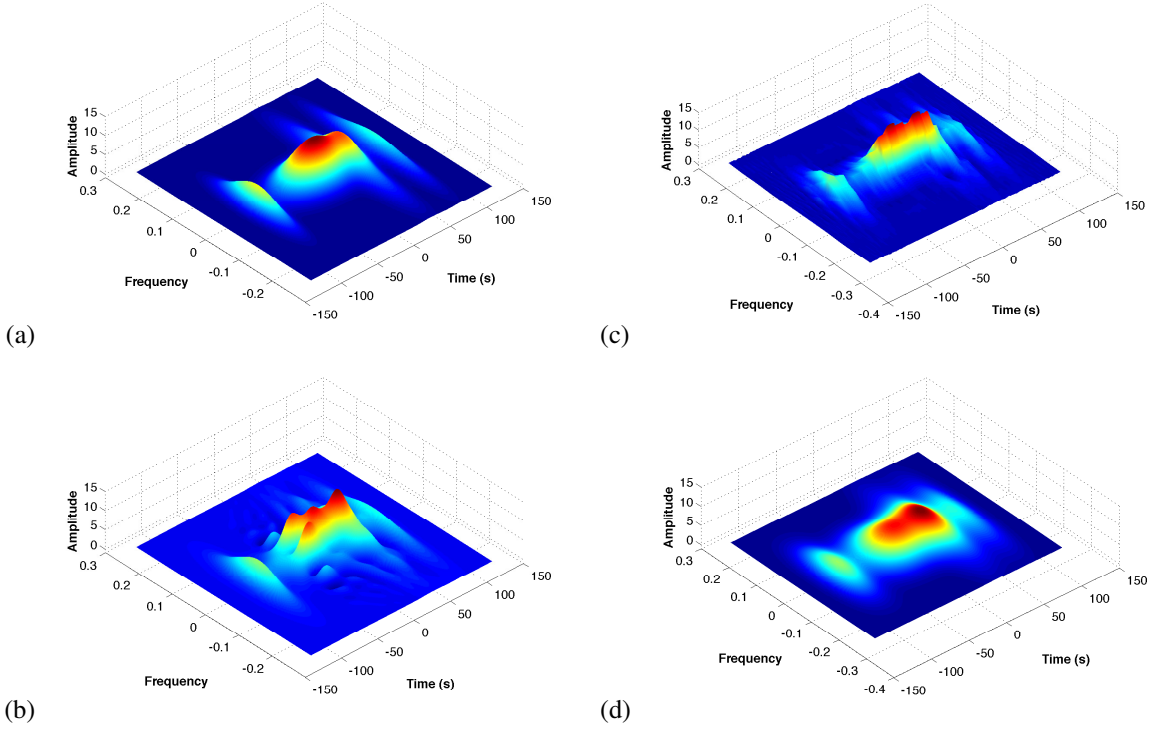
Fig. 6: The figure shows the (a) true, and estimated TF distributions using (b) the WVD, (c) the proposed CAST estimator, and (d) the smoothed spectrogram.

Silverman's sense [12]. Here, $r(t)$ and $q(t)$ have been selected to be Gaussian functions on the form

$$q(t) = e^{-\frac{1}{2}\left(\frac{t}{f_s}\right)^2} \tag{14}$$

$$r(\tau) = e^{-\frac{c}{8}\left(\frac{\tau}{f_s}\right)^2} \tag{15}$$

with the parameter c dictating the level of stationarity, and f_s being a time scaling factor measuring the time-spread of the process. In the first example, we consider a uniformly sampled linear combination of two LSPs; the first with $c = 10$, $f_s = 30$, and centered at time -50 s, whereas the second use c_5, $f_s = 30$, and is centered at time 50 s. Figures 2 and 3 illustrate the log mean squared error (logMSE), defined as the logarithm of

$$D_{\text{MSE}} = \frac{1}{NM} \sum_{j=1}^{N} \sum_{k=1}^{M} \left(\hat{\Phi}(j,k) - \Phi_{\text{true}}(j,k)\right)^2 \tag{16}$$

with M denoting the number of considered frequencies, which here and in the following examples is set to $M = 1011$, and the Manhattan distance, defined as

$$D_{\text{Manhattan}} = \frac{1}{u_{\Phi}} \sum_{j=1}^{N} \sum_{k=1}^{M} \left|\hat{\Phi}(j,k) - \Phi_{\text{true}}(j,k)\right| \tag{17}$$

where $u_{\Phi} = \sum_{j=1}^{N} \sum_{k=1}^{M} |\Phi_{\text{true}}(j,k)|$. As may be seen in the figures, after about 10 realizations, the proposed method

yields preferable performance as compared to the other methods for both error measures. Next, we investigate the performance when the data is irregularly sampled. First, an irregularly sampling scheme is created by removing 50% of the samples at random positions, using the same sampling scheme in each iteration. Figures 4 and 5 show the resulting performance, clearly indicating the benefits of the proposed method also in this case. It may be noted that the proposed method is also notably more robust to the use of non-uniform sampling as compared to the reference methods. Finally, Figure 6 illustrates a typical result for the proposed method as compared to the WVD and the spectrogram, where the signal is a linear combination of 4 LSPs with the parameters as specified as $c = \{1.05, 5, 5, 2\}$, $f_s = \{8, 30, 10, 7\}$, and with center time $\{-80, 0, 40, 80\}$, using $K = 50$ realizations. As seen in the figures, the proposed method has good time and frequency localization as well as amplitude accuracy, whereas the spectrogram suffer from notable leakage, affecting not only the localization, but also the amplitude estimate. The WVD estimate has better amplitude estimates as compared with the spectrogram, but has the wrong shape for the middle component due to cross-terms. The proposed method has good localization and amplitude estimate. The pixeled appearance of the proposed method is due to the fact that the number of frequency points is set to only $M = 10$.

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