On Vertex Operator Algebras of Affine Type at Admissible Levels

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Abstract

The main purpose of this thesis is the study of the structure and representation theory of simple vertex operator algebras $L(k\Lambda_0)$ of affine type at admissible levels k. To do this, it is crucial to obtain knowledge of the singular vectors which generate the maximal submodules, with respect to which these vertex operator algebras appear as irreducible quotients, and therefore a substantial part of the text is devoted to this matter. We study in particular the simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ with half-integer admissible levels, and especially the one with the minimal admissible level $-\frac{3}{2}$.

We tackle the problem of describing singular vectors in Verma modules for affine Lie algebras by providing a novel way of realizing the ideas presented in an article by F. G. Malikov, B. L. Feigin and D. B. Fuchs. Our approach is based on the rigorous construction of a broader algebraic framework by means of Ore localization in the universal enveloping algebra and via the introduction of certain conjugation automorphisms. We are able to express operators corresponding to those of Malikov et al. and to partially extend to our setting their main result regarding whether or not these operators represent elements of the enveloping algebra.

Using this knowledge about singular vectors we deal with the problem of finding the irreducible modules in the category $\mathcal{O}$ for vertex operator algebras $L(k\Lambda_0)$, when the level k is admissible. Applying the theory of Zhu's algebra, the highest weights of these modules are characterized as the zeros of a polynomial ideal determined by the single singular vector generating the maximal proper submodule of the generalized Verma module $N(k\Lambda_0)$. For the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$, we prove that these highest weights are precisely the four admissible weights of level $-\frac{3}{2}$, and moreover that any $L(-\frac{3}{2}\Lambda_0)$-module in the category $\mathcal{O}$ is completely reducible. We also show that there are no nontrivial intertwining operators between these irreducible modules, except those deriving from the module structures. Furthermore, we demonstrate how the Šapovalov form can be employed to gain insight into the polynomial ideal, if merely the weight of the corresponding singular vector is known.

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Populärvetenskaplig sammanfattning

Idén om ett särskilt slag av algebraiska strukturer, kallade "vertexoperator-algebror", uppstod på 1980-talet. Dessa algebror används för att formulera fenomen inom fysikens strängteori. Närmare bestämt beskrivs partiklar inom denna teori som vibrerande linjer (eller slutna öglor) snarare än punkter, och algebrornas operatorer avser att visa hur två sådana strängar kan slås ihop till en, vilket sker i en "vertex". Oberoende av den fysikaliska teoribildningen uppkom och vidareutvecklades dessa algebraiska strukturer också inom den rena matematiken i skärningspunkten mellan oändlig-dimensionella Lie-algebror, teorin om ändliga grupper och talteori.

I avhandlingen studeras strukturen och representationsteorin för enkla vertexoperator-algebror av affin typ på så kallade "tillåtliga" nivåer. Särskilt intresse ägnas åt halvtalsnivåer, i synnerhet nivån $-\frac{3}{2}$, för en viss sorts vertexoperator-algebror.

Tillvägagångssättet inbegriper en rigorös konstruktion som för avhandlingens syften omformulerar tidigare resultat till ett utvidgat algebraiskt ramverk. Utvidgningen medger att man, när det gäller de ursprungliga elementen, kan arbeta med kvoter och i viss mån allmänna exponenter.

Ett huvudproblem är att finna irreducibla representationer för den typ av vertexoperator-algebror som är centrala för avhandlingen. Dessa representationer karakteriseras av "vikter" som bestäms av polynomekvationer. För den ovan specificerade vertexoperator-algebran på nivån $-\frac{3}{2}$ visas att det finns fyra irreducibla representationer som samtliga är tillåtliga. Sammanflätningsoperatorerna mellan dessa representationer bestäms också. Det bevisas även hur man kan härleda ett allmänt villkor för polynomekvationerna för alla de i avhandlingen aktuella vertexoperator-algebrorna.

Contents

Introduction

The concept of vertex operator algebras arose as a new kind of algebraic structure in the beginning of the 1980s. Its main inspiration came from the development of string theory in physics, but it also emerged in pure mathematics in the contexts of infinite dimensional Lie algebras, finite groups, and number theory. Vertex operator algebras, although immensely more complicated, are to some extent analogous to Lie algebras. The principal subject of this thesis is the study of vertex operator algebras, but a substantial part of it deals exclusively with infinite dimensional Lie algebras.

We will start by giving a brief, and by necessity very selective, overview of the field of study at the core of this thesis, focusing on notions of basic importance. This is followed by a detailed account of the contents in each chapter.

Lie theory was originally conceived in the 1870s by the eponymous Norwegian mathematician, as a means to study geometric objects by "linearizing" the action of transformation groups. In this way, the local behavior of these continuous groups could be entirely translated into an algebraic structure, the Lie algebra. The ideas of Lie proved very fruitful, and over the following decades they were further elaborated and applied, giving rise to novel thoughts in mathematics as well as in physics. The study of Lie algebras from a purely algebraic point of view also contributed to this development.

At the foundation of the study of Lie algebras are the (finite dimensional) semisimple Lie algebras, a fact that can be partially explained by the rich representation theory they afford. Their relevance is further made clear by the Levi decomposition, which demonstrates that every finite dimensional real Lie algebra can be decomposed as a semi-direct product of a solvable ideal and a semisimple Lie algebra. The complex semisimple Lie algebras were completely classified by W. Killing and É. Cartan by the end of the 19th century into four infinite series and five exceptional cases. At the core of this classification and to the representation theory of these algebras as a whole is the notion of weights, i.e. one-dimensional representations of a maximal commutative subalgebra (a Cartan subalgebra), under which modules split up into simultaneous eigenspaces. Another fundamental aspect of the study of semisimple Lie algebras is that their finite dimensional representations are completely reducible, which means that to every submodule there is a complementary submodule; hence it suffices to consider simple representations.

In the mid-1960s, J.-P. Serre showed that a complex semisimple Lie algebra is uniquely determined by its Cartan matrix through the Chevalley–Serre relations. A couple of years later, and independently of each other, V. G. Kac, R. V. Moody and I. L. Kantor studied the Lie algebras that are determined by the relations of a suitably generalized Cartan matrix. The resulting, in general infinite dimensional, Lie algebras, now known as Kac–Moody algebras, turned out to provide a natural generalization of the semisimple Lie algebras. While retaining many important properties of the semisimple theory, Kac–Moody algebras also give rise to new phenomena. A certain class of these algebras, called the affine Lie algebras, have been extensively studied since they admit of explicit constructions from corresponding semisimple Lie algebras.

Due to its construction, a Kac–Moody algebra $\mathfrak{g}$ comes equipped with a natural triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (where $\mathfrak{h}$ denotes the Cartan subalgebra) and a corresponding finer gradation by weights (or roots) with respect to $\mathfrak{h}$. Hence it is natural to consider weight space representations of $\mathfrak{g}$, and in particular highest weight representations, i.e. modules generated by a weight vector that is annihilated by the upper triangular subalgebra $\mathfrak{n}_+$. Submodules of a highest weight module are likewise generated by vectors that lie in the kernel of the action of $\mathfrak{n}_+$. Finding these so-called singular vectors is a crucial and often challenging task. For a weight $\lambda \in \mathfrak{h}^*$, two highest weight modules of highest weight λ are particularly important: the universal highest weight module, or Verma module, $M(\lambda)$, and its irreducible quotient $L(\lambda)$.

By considering arbitrary quotients and submodules, as well as finite direct sums and tensor products, of highest weight modules, one arrives at the suitable class of modules for $\mathfrak{g}$ known as the category $\mathcal{O}$. A notable feature in the research on the modules in this category is the study of their formal characters, i.e. the power series expressing the graded dimensions of the modules. Arguably, the most important highest weight modules are those with a dominant integral highest weight. In the classical semisimple theory, it was proved by H. Weyl that the character of the module $L(\lambda)$, where λ is dominant integral, can be expressed in a simple and appealing way. Later this character formula was generalized by V. G. Kac to the case of symmetrizable Kac–Moody algebras.

In 1982, V. V. Deodhar, O. Gabber and V. G. Kac introduced the so-called admissible weights [DGK82]. By construction, these weights are devised so that highest weight modules with admissible highest weights should share structurally important properties with their counterparts with dominant integral highest weight. In [KW88], V. G. Kac and M. Wakimoto managed to extend the character formula for symmetrizable Kac–Moody algebras to include admissible highest weights. Thereby they were also able to establish essential information about the singular vectors of the corresponding Verma modules.

Next to be presented is the concept of vertex operator algebras. The first time "vertex operators" appeared in a purely mathematical context, was in a paper by J. Lepowsky and R. L. Wilson [LW78]. They presented an explicit realization of the affine Lie algebra $\mathfrak{sl}(2,\mathbb{C})^{\sim}$ as an algebra of differential operators on the polynomial ring (or Fock space) $\mathbb{C}[x_1, x_3, \ldots]$. More specifically, this algebra is spanned by the following operators: the identity operator, all left multiplication and partial differentiation operators $L(x_k)$ and $\partial/\partial x_k$ (which are defined to be of degree -1 and 1, respectively), and finally the operators Y_j, obtained as the homogeneous components of the expression

$$Y = -\frac{1}{2}\exp\Big(\sum L(4x_k/k)\Big)\exp\Big(-\sum(\partial/\partial x_k)\Big)$$

(where the sums run over all positive odd integers). Notice that although Y_j consists of an infinite expression, it gives rise to a well-defined operator. This is a characteristic feature of vertex operator algebra theory, as is the appearance of the intricate operators Y_j as components in "generating functions".

It was noticed by H. Garland that the above formula for Y displays a striking similarity to the concept of vertex operators as used by theoretical physicists. These vertex operators are designed to represent interactions of elementary particles in quantum mechanics. More specifically, in string theory, where particles appear as lines (or closed loops) rather than as points, these operators realize the merging of two such strings into one at a "vertex". This line of thought gives rise to various quantum field theories. An essential feature of these theories is the so-called operator product expansion, which determines the composition of vertex operators, and serves as the basis for the introduction in physics of an algebraic structure known as chiral algebras.

Within pure mathematics, the article [LW78] was followed by further representations of affine algebras involving "vertex operators". These expressions also appeared in the final stages of the classification of finite simple groups, during the study of the largest sporadic finite simple group, the Monster group. This group had been realized by R. Griess as a group of automorphisms of a certain intricate algebra. In works by I. Frenkel, J. Lepowsky and A. Meurman, and R. E. Borcherds, this algebra was extended to a module for vertex operators, and this module was in turn tied to advanced number theoretic results on modular-invariant functions (cf. monstrous moonshine and the Conway–Norton conjectures). The knowledge of vertex operators from mathematics and physics was eventually summarized in [Bor86] so as to define the structure of vertex algebras; the slightly different notion of vertex operator algebras was then introduced in [FLM88].

Essentially, a vertex (operator) algebra consists of a vector space V equipped with an infinite number of "multiplication" operators. For $u \in V$, the corresponding operators $(u_n)_{n\in\mathbb{Z}}$ are expressed in the form of a formal Laurent series $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$, where $Y\colon V \to \mathrm{End}(V)[[z,z^{-1}]]$ is called the vertex operator map. The multiplication components generate V from a distinguished vector $\mathbf{1} \in V$, and they are subject to an infinite set of involved relations summarized in the so-called Jacobi identity, which declares that

$$\begin{aligned} &z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)Y(v,z_2)Y(u,z_1) \\ &\quad = z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y(Y(u,z_0)v,z_2), \end{aligned}$$

for all $u, v \in V$. From the point of view of physics, this identity encapsulates the operator product expansion as well as other phenomena.

For an affine Lie algebra, the generalized Verma module $N(k\Lambda_0)$, which is a quotient of the Verma module $M(k\Lambda_0)$, can be given the structure of a vertex operator algebra. Here, Λ_0 denotes a specific weight, and $k \in \mathbb{C}$ is called the level of the vertex operator algebra. Any (nonzero) quotient module of $N(k\Lambda_0)$, in particular the irreducible quotient $L(k\Lambda_0)$, is endowed with an induced vertex operator algebra structure. While any highest weight module with a highest weight of level k has a natural structure as a vertex operator algebra module for $N(k\Lambda_0)$, this is no longer true when passing to a quotient of $N(k\Lambda_0)$. In for instance [FZ92] and [MP99] it is shown that if the level k is a non-negative integer, then $L(\lambda)$ is a module for $L(k\Lambda_0)$ only when λ belongs to the finite set of dominant integral weights at this level. Moreover, these vertex operator algebras $L(k\Lambda_0)$ are rational, which means that their $\mathbb{N}$-gradable weak modules are completely reducible.

In [Ada94], D. Adamović studied the module structure of the vertex operator algebras $L(k\Lambda_0)$ associated to affine Lie algebras of type $C_n^{(1)}$ and at admissible levels with denominator 2. It is proved that the irreducible modules, which belong to the category $\mathcal{O}$ as affine Lie algebra modules, are precisely those $L(\lambda)$ for which λ is an admissible weight at the given level, as well as that the $L(k\Lambda_0)$-modules in this category are completely reducible. The analogous statement was then proved to be true in [AM95] and [DLM97] for all admissible levels k in the case of the vertex operator algebra $L(k\Lambda_0)$ associated to $\mathfrak{sl}(2,\mathbb{C})^{\sim}$. From this basis, in [AM95] D. Adamović and A. Milas advanced the conjecture that for admissible levels k and all affine Lie algebras, the modules in the category $\mathcal{O}$ for the vertex operator algebra $L(k\Lambda_0)$ are completely reducible or, shortly put, that $L(k\Lambda_0)$ is rational in the category $\mathcal{O}$. It was proved by O. Perše in [Per07] and [Per08] that this conjecture is true at certain half-integer levels k for vertex operator algebras $L(k\Lambda_0)$ of type $B_n^{(1)}$ and $A_n^{(1)}$.

Chapter 1 is preliminary and is devoted to a general discussion of Kac–Moody algebras, introducing related concepts and settling notations to be used in the rest of the thesis. Section 1.1, presents the construction of Kac–Moody algebras from generalized Cartan matrices, and defines the modules in the category $\mathcal{O}$. In Section 1.2, the explicit realization of affine Lie algebras is explained, including a detailed exposition of the case of $\mathfrak{sl}(3,\mathbb{C})^{\sim}$.

Chapter 2 is concerned with singular vectors in Verma modules. The point of departure is the article [MFF86] by F. G. Malikov, B. L. Feigin and D. B. Fuchs, which introduces a way to interpret monomials with complex exponents in the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In the context of symmetrizable Kac–Moody algebras, they show that if the exponents are suitably chosen, the resulting expressions give rise to singular vectors when applied to the highest weight vector of a Verma module. Considering only expressions which they can prove to lie in $\mathcal{U}(\mathfrak{g})$, the authors manage to avoid defining a broader algebraic framework for the calculations.

The present thesis provides a rigorous setting in which these ideas are realized in a different manner, and partially extends the results of [MFF86] to this context. The construction includes Ore localization in $\mathcal{U}(\mathfrak{g})$ and the introduction of certain conjugation automorphisms. Section 2.1 deals specifically with Ore localization, and extends a theorem in [RCW84] by proving that, for an affine Lie algebra $\mathfrak{g}$, the set $S = \mathcal{U}(\mathfrak{n}_-) \setminus \{0\}$ is not only an Ore set in $\mathcal{U}(\mathfrak{n}_-)$, but in the whole of $\mathcal{U}(\mathfrak{g})$. In Section 2.2, we construct automorphisms of $\mathcal{U}(\mathfrak{g})S^{-1}$ representing complex powers of the inner automorphisms induced by the Chevalley generators

f_i of $\mathfrak{n}_-$. The following section is a digression, in which it is demonstrated that it is possible to extend $\mathcal{U}(\mathfrak{g})S^{-1}$ to an algebra containing half-integer powers of the elements f_i. In Section 2.4, we analyse the precise arrangement of powers in the Malikov–Feigin–Fuchs-operators, and show that corresponding expressions can be defined in $\mathcal{U}(\mathfrak{g})S^{-1}$ with the help of the automorphisms described in Section 2.2. It is verified that these operators give rise to "singular vectors" in Verma modules localized with respect to S. Theorem 2.19, finally, establishes that the main theorem of [MFF86], which proves that their expressions produce elements of $\mathcal{U}(\mathfrak{g})$, also holds in our setting in certain nontrivial cases. Section 2.5 is dedicated to the admittedly long and intricate proof of Theorem 2.19.

Chapter 3, about vertex operator algebra theory, is to be regarded as the core of the text, and this subject matter was also the original motivation for the thesis. Its main objective is to gain insight into simple vertex operator algebras of affine type at admissible levels, with special regard to those associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ and whose admissible level have denominator 2, and in particular the one with the minimal admissible level $-\frac{3}{2}$. The principal aim is to study the module structure of these vertex operator algebras, and to examine if the results are in line with the above-mentioned conjecture by D. Adamović and A. Milas. We prove that the irreducible modules of the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ are indeed the very $L(\Lambda)$, where Λ is one of the four admissible weights of level $-\frac{3}{2}$. The chapter concludes with a discussion about fusion rules.

In Section 3.1, we give the definition of a vertex operator algebra, along with some corresponding notions of modules. The next section explains the structure of vertex operator algebras of affine type. Section 3.3 contains basic definitions and results regarding the theory of Zhu's algebra. In Section 3.4, we introduce the concept of admissible weights and recount some associated results concerning the structure of highest weight modules and complete reducibility. Using the comprehensive classification of admissible weights in [KW89], we also provide a description of the admissible weights related to the affine algebras of type $A_n^{(1)}$ $(n \geq 2)$, and give a detailed listing of the admissible weights with denominator 2 associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$.

In Section 3.5 we use the theory as developed in the previous sections to study the module structure of simple vertex operator algebras of affine type at admissible levels. Using the theory of Zhu's algebra, the highest weights of irreducible $L(k\Lambda_0)$-modules in the category $\mathcal{O}$ are characterized as the zeros of a polynomial ideal $\pi(R)$ determined by the single generator $v^{(0)}$ of the maximal proper submodule of $N(k\Lambda_0)$. Where possible, the account is kept in general terms, but is otherwise focused on the series of simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ at admissible level $k \in -\frac{3}{2}+\mathbb{N}$. For the level $-\frac{3}{2}$, the singular vector $v^{(0)}$ is calculated in accordance with the procedures advanced in Chapter 2, and the weights determined by the ideal $\pi(R)$ are shown to coincide with the entire set of admissible weights at this level. It is concluded that the corresponding simple vertex operator algebra is rational in the category $\mathcal{O}$. Section 3.6 describes how the Šapovalov form can be employed to gain insight into the ideal $\pi(R)$, if merely the weight of $v^{(0)}$ is known. In this way we are able to calculate one polynomial in $\pi(R)$ for each of the simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ at admissible level $k \in -\frac{3}{2}+\mathbb{N}$. It is observed that the resulting constraints on the weights of irreducible modules in the category $\mathcal{O}$ agree with the Adamović–Milas conjecture.

Section 3.7, finally, deals with intertwining operators and fusion rules. We describe the theory for computing fusion rules via bimodules for Zhu's algebra, as put forth in [FZ92]. The results of this article are transcribed to the context of vertex operator algebras of affine type at admissible levels. We are then able to show that, excepting those intertwining operators deriving from the module structures, there are no nontrivial intertwining operators between the irreducible modules in the category $\mathcal{O}$ for the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$.

Unfortunately, at a late stage of the work on the thesis, we learned that O. Perše had already shown in [Per08] that the conjecture of Adamović–Milas was true for the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$. Consequently, in the present thesis this case should not be viewed as a new result, but just as an exemplification of the theory.

In a very recent result, appearing in a preprint in August 2012, T. Arakawa shows that the Adamović–Milas conjecture is generally valid [Ara12]. Regrettably, this information arrived too late to be discussed in the present work.

Chapter 1

Some graded infinite dimensional Lie algebras

In this chapter we will introduce some basic definitions regarding Kac–Moody algebras in general and affine Lie algebras in particular. In the process we will establish notations that will be employed in the rest of the thesis. This exposition is in no way intended to give a self-contained introduction to the subject matter, but merely to settle the terminology and make sure that the objects to be treated later on are unambiguously defined. For a thorough account of the topic of Kac–Moody algebras, the reader is referred to [Kac85] or to [MP95].

Throughout this chapter and the entire thesis, we will let $\mathbb{N}$ and $\mathbb{Z}^+$ denote the non-negative and positive integers, respectively.

1.1 Kac–Moody algebras

To be able to define what we mean by a Kac–Moody algebra, we proceed as follows. Let $\mathbb{K}$ be a field of characteristic 0, let $\mathcal{J}$ be an arbitrary nonempty index set, and define A to be the $\mathcal{J} \times \mathcal{J}$-matrix with entries in $\mathbb{K}$ given by

$$A = (a_{ij})_{i,j\in\mathcal{J}} \in \mathbb{K}^{\mathcal{J}\times\mathcal{J}}.$$

A realization of A is a triple $R = (\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a vector space over $\mathbb{K}$, where the sets

$$\Pi = \{\, \alpha_i : i \in \mathcal{J} \,\} \quad \text{and} \quad \Pi^\vee = \{\, \alpha_i^\vee : i \in \mathcal{J} \,\}$$

are linearly independent subsets of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, and where

$$\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}.$$

In case that $\mathcal{J}$ is a finite set, then the realization is said to be minimal if $\mathfrak{h}$ is of minimal dimension; if $\operatorname{card}(\mathcal{J}) = n \in \mathbb{Z}^+$, this implies that $\dim \mathfrak{h} = n + \operatorname{codim}(A)$.

We regard $\mathfrak{h}$ as an abelian Lie algebra, and let $\mathfrak{a}$ be the free Lie algebra on the set of generators given by

$$\{ e_i, f_i : i \in \mathcal{J} \}.$$

Consider the free product $\mathfrak{d} = \mathfrak{a} * \mathfrak{h}$ of the Lie algebras $\mathfrak{a}$ and $\mathfrak{h}$. Let $\mathfrak{m}$ be the ideal of $\mathfrak{d}$ determined by the relations

$$\begin{cases} [h, e_i] &= \langle \alpha_i, h \rangle e_i \\ [h, f_i] &= -\langle \alpha_i, h \rangle f_i \\ [e_i, f_j] &= \delta_{ij} \alpha_i^\vee \end{cases},$$

for $h \in \mathfrak{h}$ and $i, j \in \mathcal{J}$. We call the quotient algebra

$$\mathfrak{u} = \mathfrak{u}(A, R) = \mathfrak{d}/\mathfrak{m}$$

the universal Lie algebra corresponding to the realization R of the matrix A. If $\mathfrak{S}$ denotes the subspace of $\mathfrak{d}$ defined by

$$\mathfrak{S} = \sum_{i \in \mathcal{J}} \mathbb{K} f_i \oplus \mathfrak{h} \oplus \sum_{i \in \mathcal{J}} \mathbb{K} e_i,$$

then $\mathfrak{S}$ is mapped isomorphically, as a vector space, by the quotient map $\mathfrak{d} \to \mathfrak{d}/\mathfrak{m}$; we identify $\mathfrak{S}$ with its image under this map. There is a natural triangular decomposition of $\mathfrak{u}$ given by

$$\mathfrak{u} = \mathfrak{u}_- \oplus \mathfrak{h} \oplus \mathfrak{u}_+,$$

where $\mathfrak{u}_-$ and $\mathfrak{u}_+$ are subalgebras that are freely generated by $\{ f_i : i \in \mathcal{J} \}$ and $\{ e_i : i \in \mathcal{J} \}$, respectively.

We define the radical of $\mathfrak{u}$, denoted by $\operatorname{rad} \mathfrak{u}$, to be the maximal ideal of $\mathfrak{u}$ whose intersection with $\mathfrak{h}$ is trivial. The quotient algebra

$$\mathfrak{g} = \mathfrak{g}(A, R) = \mathfrak{u}/\operatorname{rad} \mathfrak{u}$$

is then called the radical-free Lie algebra corresponding to the realization R of the matrix A. Again, the subspace $\mathfrak{S}$ is mapped isomorphically to the quotient, and we consider $\mathfrak{S}$ to also be a subspace of $\mathfrak{g}$. The elements e_i, f_i ($i \in \mathcal{J}$) generate the derived algebra $[\mathfrak{g}, \mathfrak{g}]$ (cf. (1.2)), and are called the Chevalley generators. The triangular decomposition of $\mathfrak{u}$ naturally carries over to $\mathfrak{g}$, and we write

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

The subalgebra $\mathfrak{h}$ is called the Cartan subalgebra whereas $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are called the upper and lower triangular subalgebras, respectively.

There are a few aspects of the structure of $\mathfrak{g}$ that we want to point out. Let Q and Q_+ denote, respectively, the free abelian group and semigroup generated by $\{\, \alpha_i : i \in \mathcal{J} \,\}$ in $\mathfrak{h}^*$, i.e.

$$Q = \sum_{i \in \mathcal{J}} \mathbb{Z}\alpha_i \quad \text{and} \quad Q_+ = \Big(\sum_{i \in \mathcal{J}} \mathbb{N}\alpha_i \Big) \setminus \{0\}.$$

Under the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ we obtain a weight space decomposition of $\mathfrak{g}$, given by

$$\mathfrak{g} = \Big(\bigoplus_{\alpha \in -Q_+} \mathfrak{g}_\alpha \Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in Q_+} \mathfrak{g}_\alpha \Big),$$

where

$$\mathfrak{g}_\alpha = \{\, x \in \mathfrak{g} : [h, x] = \langle \alpha, h \rangle x, \text{ for all } h \in \mathfrak{h} \,\}.$$

and $\mathfrak{h} = \mathfrak{g}_0$. There exists a natural anti-involution σ on $\mathfrak{g}$, which is determined by the conditions

$$\sigma(e_i) = f_i, \quad \text{for all } i \in \mathcal{J}, \quad \text{and} \quad \sigma|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\,. \tag{1.1}$$

This shows, for instance, that $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$ for all $\alpha \in Q$. The set of roots Δ of $\mathfrak{g}$ is defined as

$$\Delta = \big\{\, \alpha \in Q \setminus \{0\} : \dim \mathfrak{g}_\alpha \neq 0 \,\big\},$$

and the set of positive roots is given by $\Delta_+ = \Delta \cap Q_+$. We call the group Q the root lattice of $\mathfrak{g}$. Similarly the coroot lattice $Q^\vee$ of $\mathfrak{g}$ and the subset $Q_+^\vee \subset Q^\vee$ are defined as

$$Q^\vee = \sum_{i \in \mathcal{J}} \mathbb{Z}\alpha_i^\vee \quad \text{and} \quad Q_+^\vee = \Big(\sum_{i \in \mathcal{J}} \mathbb{N}\alpha_i^\vee \Big) \setminus \{0\}.$$

The elements of Π are called the simple roots of $\mathfrak{g}$, and those of $\Pi^\vee$ are called the simple coroots of $\mathfrak{g}$. We denote the $\mathbb{K}$-spans of $Q \subset \mathfrak{h}^*$ and $Q^\vee \subset \mathfrak{h}$ by $Q_\mathbb{K}$ and $Q_\mathbb{K}^\vee$, respectively. The derived algebra is then given by

$$[\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}_- \oplus Q_\mathbb{K}^\vee \oplus \mathfrak{n}_+. \tag{1.2}$$

If $a_{ii} \neq 0$, the subalgebra of $\mathfrak{g}$ generated by $\{f_i, \alpha_i^\vee, e_i\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{K})$, and if, in particular, $a_{ii} = 2$, an explicit isomorphism is given by

$$f_i \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \alpha_i^\vee \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_i \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Assuming that $a_{ii} = 2$, it turns out that $\operatorname{ad} e_i$ (or equivalently $\operatorname{ad} f_i$) is locally nilpotent in $\mathfrak{g}$ if and only if

$$a_{ji} \in -\mathbb{N} \quad \text{and} \quad a_{ji} = 0 \Longrightarrow a_{ij} = 0,$$

for all $j \in \mathcal{J}$ with $j \neq i$. This motivates the following definition.

Definition 1.1. The matrix $A = (a_{ij})_{i,j \in \mathcal{J}} \in \mathbb{K}^{\mathcal{J} \times \mathcal{J}}$ is called a generalized Cartan matrix if, for all i, $j \in \mathcal{J}$ with $i \neq j$, the following three conditions are satisfied:

- ◇ $a_{ii} = 2$;
- ◇ $a_{ij} \in -\mathbb{N}$;
- ◇ $a_{ij} = 0 \Longrightarrow a_{ji} = 0$.

The contents of a generalized Cartan matrix may alternatively be described by its Coxeter–Dynkin diagram. This is defined as the directed multigraph (i.e. a graph which is permitted to have multiple directed edges) with vertices indexed by the set $\mathcal{J}$ and $-a_{ij}$ edges from vertex i to vertex j for $i \neq j$. The matrix A is said to be decomposable or indecomposable according to whether the Coxeter–Dynkin diagram is disconnected or connected.

We are now in a position to define what we mean by a Kac–Moody algebra.

Definition 1.2. Let A be a generalized Cartan matrix of finite dimension, and let R be a minimal realization of A. The Lie algebra $\mathfrak{g}(A, R)$ is called the Kac–Moody algebra corresponding to the realization R of the matrix A.

Remark 1.3. Our definition of a Kac–Moody algebra coincides with the one in [Kac85]. In the language of [MP95], what we have defined corresponds to a minimally realized radical-free Kac–Moody algebra. In [MP95], any quotient algebra $\mathfrak{u}(A, R)/\mathfrak{s}$, where $\mathfrak{s} \subseteq \operatorname{rad} \mathfrak{u}$ is an ideal satisfying $\sigma(\mathfrak{s}) = \mathfrak{s}$, and where

A is a generalized Cartan matrix and R is any realization of A, is called a Kac–Moody algebra if only $\operatorname{ad} e_i$ is locally nilpotent for all $i \in \mathcal{J}$. For this to be the case it is necessary that the ideal $\mathfrak{t}$ given by

$$\mathfrak{t} = \langle (\operatorname{ad} e_i)^{-a_{ji}+1}(e_j), (\operatorname{ad} f_i)^{-a_{ji}+1}(f_j) : i, j \in \mathcal{J}, i \neq j \rangle$$

is contained in $\mathfrak{s}$. If A is symmetrizable, we actually have that $\mathfrak{t} = \operatorname{rad} \mathfrak{u}$, and hence, in this case, this definition of Kac–Moody algebras essentially agrees with the one we have adopted.

Now, let $\mathfrak{g} = \mathfrak{g}(A, R)$ be a Kac–Moody algebra, where A is of dimension $n \times n$ ($n \in \mathbb{Z}^+$). If the matrix A is decomposable, then $\mathfrak{g}$ naturally breaks up into a direct product of the Kac–Moody algebras corresponding to the connected components of the Coxeter–Dynkin diagram. Hence, in any classification of Kac–Moody algebras, one may restrict attention to those related to indecomposable generalized Cartan matrices.

The matrix A (and also the algebra $\mathfrak{g}$) is said to be symmetrizable if there exists an invertible diagonal matrix $D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$ such that DA is a symmetric matrix. Without loss of generality, we may as well require that $\epsilon_i \in \mathbb{Q}^+$, for $i = 1, \ldots, n$. The condition that A is symmetrizable is equivalent to the existence of a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$. If A is symmetrizable, we let $(\cdot, \cdot)$ denote a specific bilinear form of this sort, which we will now briefly characterize (in terms of D). Let

$$\nu \colon Q^\vee_{\mathbb{K}} \longrightarrow Q_{\mathbb{K}}$$

denote the linear isomorphism determined by the condition that

$$\nu(\alpha_i^\vee) = \epsilon_i \alpha_i,$$

for $i = 1, \ldots, n$. On $Q^\vee_{\mathbb{K}} \times \mathfrak{h}$ the form $(\cdot, \cdot)$ is defined by

$$(h_1, h_2) = \langle \nu(h_1), h_2 \rangle,$$

and by symmetry this also determines the form on $\mathfrak{h} \times Q^\vee_{\mathbb{K}}$. To extend $(\cdot, \cdot)$ to the whole of $\mathfrak{h} \times \mathfrak{h}$, we let the form be defined by an arbitrary symmetric bilinear form on a complementary subspace of $Q^\vee_{\mathbb{K}}$ in $\mathfrak{h}$. To see how the form is defined on the rest of $\mathfrak{g}$, let $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ where α and β are nonzero roots. Then $(\cdot, \cdot)$ is defined by the equations

$$(x, y) = 0, \quad \text{if } \alpha + \beta \neq 0,$$

whereas

$$[x, y] = (x, y)\nu^{-1}(\alpha), \quad \text{if } \alpha + \beta = 0.$$

In particular, the form is nondegenerate on $\mathfrak{h}$, which means that it can be carried over to $\mathfrak{h}^*$ by means of the linear isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by

$$h \longmapsto (h, \cdot).$$

We let the corresponding form on $\mathfrak{h}^*$ also be denoted by $(\cdot, \cdot)$.

Let r_i be the reflection of $\mathfrak{h}^*$ in $\alpha_i \in \Pi$ given by

$$r_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i.$$

We call this reflection a simple reflection, and the subgroup $\mathscr{W}$ of $GL(\mathfrak{h}^*)$ generated by all the r_i, for $i = 1, \ldots, n$, is called the Weyl group of $\mathfrak{g}$. The dual Weyl group $\mathscr{W}^\vee \subset GL(\mathfrak{h})$ is analogously defined in terms of the reflections $r_i^\vee$ in $\alpha_i^\vee$. The mapping $r_i \mapsto r_i^\vee$, for $i = 1, \ldots, n$, gives rise to an isomorphism from $\mathscr{W}$ to $\mathscr{W}^\vee$, by which we identify these groups. If $\mathfrak{g}$ is symmetrizable and $\alpha \in \mathfrak{h}^*$ is nonisotropic with respect to $(\cdot, \cdot)$, we let r_α denote the orthogonal reflection in α, i.e.

$$r_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha.$$

In this case it follows that $r_i = r_{\alpha_i}$.

We fix a particular weight $\rho \in \mathfrak{h}^*$ such that $\rho(\alpha_i^\vee) = 1$, for $i = 1, \ldots, n$. For any $w \in GL(\mathfrak{h}^*)$, we define its ρ-shift $w^\rho \colon \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$w^\rho(\beta) = w(\beta + \rho) - \rho.$$

This determines a group action of $GL(\mathfrak{h}^*)$ on $\mathfrak{h}^*$ in the sense that $(w_1 w_2)^\rho = w_1^\rho w_2^\rho$, for $w_1, w_2 \in GL(\mathfrak{h}^*)$.

Turning to modules, let M be a $\mathfrak{g}$-module. If M has a weight space decomposition with respect to the action of $\mathfrak{h}$, we denote it by

$$M = \bigoplus_{\mu \in P(M)} M_\mu,$$

where $P(M) \subseteq \mathfrak{h}^*$ is the set of weights of M, and

$$M_\mu = \{\, v \in M : hv = \mu(h)v, \text{ for all } h \in \mathfrak{h} \,\}$$

is the weight space corresponding to the weight μ. A general class of modules (which do not necessarily have a weight space decomposition) are the restricted modules, which are defined as follows.

Definition 1.4. Let M be a module for $\mathfrak{g}$ (or its derived algebra). Then M is called a restricted module if for every $v \in M$, it holds that $\mathfrak{g}_\alpha v = 0$ for all but finitely many positive roots α.

We will be chiefly concerned with modules in the category $\mathcal{O}$, which we define next.

Definition 1.5. Let M be a $\mathfrak{g}$-module which decomposes into finite dimensional weight spaces under the action of $\mathfrak{h}$. Then M belongs to the category $\mathcal{O}$ if there exists a finite number of weights $\lambda_i \in \mathfrak{h}^*$, $i = 1, \ldots, \ell$, such that

$$P(M) \subseteq \bigcup_{i=1}^{\ell} \{\, \lambda_i - \beta : \beta \in Q_+ \cup \{0\} \,\}.$$

Remark 1.6. In the language of category theory, the objects of the category $\mathcal{O}$ are the modules defined in Definition 1.5, and the morphisms are the $\mathfrak{g}$-module homomorphisms between these modules.

We call a weight vector $v \in M_\mu$ a singular vector if it is annihilated by $\mathfrak{n}_+$; if, furthermore, M is generated by v, then the module M will be called a highest weight module of highest weight μ. Given $\lambda \in \mathfrak{h}^*$, we let $M(\lambda)$ denote the Verma module corresponding to λ, and we let v_λ be a fixed generator of $M(\lambda)$. This module is the universal highest weight module of highest weight λ, in the sense that it is characterized (up to isomorphism) by the property that every other highest-weight module of the same highest weight is a homomorphic image of the Verma module. The Verma module $M(\lambda)$ has a unique maximal submodule $M^1(\lambda)$, and the corresponding irreducible quotient will be denoted by

$$L(\lambda) = M(\lambda)/M^1(\lambda).$$

As a left $\mathfrak{n}_-$-module, the Verma module $M(\lambda)$ is naturally isomorphic to $\mathcal{U}(\mathfrak{n}_-)$. Here and further on we let $\mathcal{U}$ denote the functor associating a Lie algebra with its universal enveloping algebra.

1.2 Untwisted affine Lie algebras and in particular $\mathfrak{sl}(3,\mathbb{C})^\sim$

In this section we will establish some notations for untwisted affine Lie algebras, as well as for the particular untwisted affine Lie algebra $\mathfrak{sl}(3,\mathbb{C})^\sim$, which will figure prominently later on in the thesis. The affine Lie algebras (of which there are two kinds, twisted and untwisted) are a class of Kac–Moody algebras, which, apart from their specification as Kac–Moody algebras in accordance with the previous section, can also be realized more explicitly. We will show how these two descriptions correspond for untwisted affine Lie algebras in general, and then review the case of $\mathfrak{sl}(3,\mathbb{C})^\sim$ in more detail.

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra defined by an indecomposable generalized Cartan matrix A, and let D be a diagonal matrix with positive rational diagonal entries, such that DA is a symmetric matrix. Since there are positive numbers on the diagonal of DA, we see that one of the following three cases will occur:

(i) DA is positive definite;

(ii) DA is positive semidefinite;

(iii) DA is indefinite.

The generalized Cartan matrix A is, in turn, referred to as being of finite, affine and indefinite type, respectively. Generalized Cartan matrices of finite type are called just Cartan matrices, and the Kac–Moody algebras related to these matrices are precisely the finite-dimensional simple Lie algebras. When the generalized Cartan matrix is of affine or indefinite type, the corresponding algebras are infinite dimensional. The generalized Cartan matrices of affine type have codimension 1, and the algebras they give rise to are known as the affine Lie algebras.

We will now describe how the untwisted affine Lie algebras are constructed from finite-dimensional simple Lie algebras. (The twisted affine Lie algebras can be realized in a similar fashion, but in this case the construction involves a certain "twisting" by an outer automorphism of the simple Lie algebra.)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let the notation of the previous section be applied to $\mathfrak{g}$, i.e. $\mathfrak{g}$ is viewed as a Kac–Moody algebra $\mathfrak{g}(A,R)$ corresponding to a generalized Cartan matrix A of finite type. We start by forming the loop algebra $L(\mathfrak{g})$ corresponding to $\mathfrak{g}$, which is defined as the tensor product algebra given by

$$L(\mathfrak{g}) = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g},$$

where $\mathbb{C}[t,t^{-1}]$ denotes the algebra of Laurent polynomials in the indeterminate t. For every $x \in \mathfrak{g}$ and $k \in \mathbb{Z}$, we will denote the element $t^k \otimes x$ in the loop algebra by $x(k)$.

Next, we will make a central extension of $L(\mathfrak{g})$. Let ψ be the bilinear form on $L(\mathfrak{g})$ which satisfies

$$\psi(x(k), y(\ell)) = k\delta_{k+\ell,0}(x,y), \quad \text{for } x,\, y \in \mathfrak{g} \text{ and } k,\, \ell \in \mathbb{Z},$$

where $(\cdot,\cdot)$ is a fixed nondegenerate symmetric invariant bilinear form determined by a diagonal matrix $D = \operatorname{diag}(\epsilon_1, \dots, \epsilon_n)$, as described in the previous section. The form $(\cdot,\cdot)$ is uniquely defined up to a nonzero constant, and we will specify our choice of this form in a moment. (From the theory of semisimple Lie algebras, it follows that $(\cdot,\cdot)$ is a multiple of the well-known Killing form.) The form ψ is a 2-cocycle on $L(\mathfrak{g})$, and hence it gives rise to a one-dimensional central extension of the loop algebra which we denote by

$$\hat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}c.$$

This central extension turns out to be universal in the category of central extensions of $\mathfrak{g}$, and furthermore it is a covering of $L(\mathfrak{g})$ in the sense that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \hat{\mathfrak{g}}$.

To arrive at the affine Lie algebra which we are aiming for, the construction needs one final component. Let $\bar{d}$ be the derivation of $\mathbb{C}[t,t^{-1}]$ given by

$$\bar{d} = t\frac{d}{dt}.$$

Identifying $\bar{d}$ with $\bar{d} \otimes \mathrm{id}_{\mathfrak{g}}$, we get a derivation of $L(\mathfrak{g})$, which in turn is extended to a derivation of $\hat{\mathfrak{g}}$ by the condition that $\bar{d}(c) = 0$. We then adjoin the derivation $\bar{d}$ to $\hat{\mathfrak{g}}$, i.e. we form the semidirect product of the one-dimensional Lie algebra $\mathbb{C}\bar{d}$ and $\hat{\mathfrak{g}}$, and obtain

$$\tilde{\mathfrak{g}} = \mathbb{C}\bar{d} \ltimes \hat{\mathfrak{g}}.$$

This Lie algebra is called the untwisted affine Lie algebra associated with the finite-dimensional simple Lie algebra $\mathfrak{g}$. In short, it is defined as

$$\tilde{\mathfrak{g}} = \big(\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}\big) \oplus \mathbb{C}c \oplus \mathbb{C}\bar{d},$$

with Lie bracket determined by the equations

$$[x(k), y(\ell)] = [x,y](k+\ell) + k\delta_{k+\ell,0}(x,y)c \quad \text{and} \quad [\bar{d}, x(k)] = kx(k),$$

for $x, y \in \mathfrak{g}$ and $k, \ell \in \mathbb{Z}$, and

$$[c, \tilde{\mathfrak{g}}] = 0.$$

We now proceed to show how $\tilde{\mathfrak{g}}$ appears as a Kac–Moody algebra. Let the associated simple Lie algebra $\mathfrak{g}$ be identified as a subalgebra of $\tilde{\mathfrak{g}}$ by means of the mapping ι determined by

$$\begin{aligned} \iota \colon \mathfrak{g} &\longrightarrow \tilde{\mathfrak{g}}, \\ x &\stackrel{\iota}{\longmapsto} x(0). \end{aligned}$$

Define $\tilde{\mathfrak{h}}$ to be the abelian subalgebra of $\tilde{\mathfrak{g}}$ given by

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}\bar{d}. \tag{1.3}$$

The set of roots of $\mathfrak{g}$ are then identified as elements of $\tilde{\mathfrak{h}}^*$ by the requirement that $\alpha \in \Delta$ satisfies $\alpha(c) = \alpha(\bar{d}) = 0$. Furthermore, let $\delta \in \tilde{\mathfrak{h}}^*$ be defined by $\delta|_{\mathfrak{h}\oplus\mathbb{C}c} = 0$ and $\delta(\bar{d}) = 1$. We then obtain a root space decomposition of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$, where the set of roots $\tilde{\Delta} \subset \tilde{\mathfrak{h}}^*$ are given by

$$\tilde{\Delta} = \{\, \alpha + n\delta : \alpha \in \Delta, n \in \mathbb{Z} \,\} \cup \{\, n\delta : n \in \mathbb{Z} \setminus \{0\} \,\}.$$

The corresponding root spaces are given by

$$\tilde{\mathfrak{g}}_{\alpha+n\delta} = \mathbb{C}t^n \otimes_{\mathbb{C}} \mathfrak{g}_\alpha, \quad \text{for } \alpha \in \Delta \text{ and } n \in \mathbb{Z},$$

and

$$\tilde{\mathfrak{g}}_{n\delta} = \mathbb{C}t^n \otimes_{\mathbb{C}} \mathfrak{h}, \quad \text{for } n \in \mathbb{Z} \setminus \{0\}.$$

Together with $\tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{h}}$ these subspaces clearly span all of $\tilde{\mathfrak{g}}$.

Let θ denote the unique root of $\mathfrak{g}$ of maximal height, let $\theta^\vee$ be the corresponding coroot, and let $x_\theta \in \mathfrak{g}_\theta$ and $x_{-\theta} \in \mathfrak{g}_{-\theta}$ be root vectors such that $[x_\theta, x_{-\theta}] = \theta^\vee$. It then follows that

$$(x_\theta, x_{-\theta}) = \frac{2}{(\theta, \theta)}.$$

Let α_0 be the root of $\tilde{\mathfrak{g}}$ given by

$$\alpha_0 = \delta - \theta,$$

and let $e_0 \in \tilde{\mathfrak{g}}_{\alpha_0}$ and $f_0 \in \tilde{\mathfrak{g}}_{-\alpha_0}$ be defined by

$$e_0 = x_{-\theta}(1) \quad \text{and} \quad f_0 = x_\theta(-1).$$

Letting $\alpha_0^\vee$ denote the element

$$\alpha_0^\vee = [e_0, f_0] = \frac{2}{(\theta,\theta)}c - \theta^\vee,$$

we then obtain that $\langle \alpha_0, \alpha_0^\vee \rangle = 2$.

We now augment the matrix A by adding a row and a column indexed by 0, and thus creating an $(n+1)\times(n+1)$-matrix $\tilde{A} = (a_{ij})_{0\le i,j\le n}$. This is done by letting

$$a_{i0} = \langle \alpha_i, \alpha_0^\vee \rangle = -\langle \alpha_i, \theta^\vee \rangle \quad \text{and} \quad a_{0j} = \langle \alpha_0, \alpha_j^\vee \rangle = -\langle \theta, \alpha_j^\vee \rangle,$$

for i, $j \in \{0,1,\ldots,n\}$. From the fact that θ is the highest root of $\mathfrak{g}$, and from the positive definiteness of the form $(\cdot,\cdot)$ on $\mathfrak{g}$, it readily follows that $\tilde{A}$ is a generalized Cartan matrix.

We are now ready to describe $\tilde{\mathfrak{g}}$ as a Kac–Moody algebra. The triple $\tilde{R} = (\tilde{\mathfrak{h}}, \tilde{\Pi}, \tilde{\Pi}^\vee)$, where $\tilde{\Pi} = \Pi \cup \{\alpha_0\}$ and $\tilde{\Pi}^\vee = \Pi^\vee \cup \{\alpha_0^\vee\}$, is clearly a realization of the matrix $\tilde{A}$. It can then be shown that the untwisted affine Lie algebra $\tilde{\mathfrak{g}}$ is the Kac–Moody algebra corresponding to the realization $\tilde{R}$ of $\tilde{A}$ (cf. Theorem 7.4 in [Kac85]). In particular, the Chevalley generators for $\tilde{\mathfrak{g}}$ are given by $e_0, e_1, \ldots, e_n$ and $f_0, f_1, \ldots, f_n$. (Recall that, for $i = 1, \ldots, n$, we have that $e_i = e_i(0)$ and $f_i = f_i(0)$ are the Chevalley generators of $\mathfrak{g}$.)

We express the triangular decomposition of $\tilde{\mathfrak{g}}$ as

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}_+,$$

where the upper and lower triangular subalgebras are given by

$$\tilde{\mathfrak{n}}_+ = \big(t\mathbb{C}[t] \otimes_{\mathbb{C}} (\mathfrak{n}_- \oplus \mathfrak{h})\big) \oplus \big(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathfrak{n}_+\big) \tag{1.4}$$

and

$$\tilde{\mathfrak{n}}_- = \big(t^{-1}\mathbb{C}[t^{-1}] \otimes_{\mathbb{C}} (\mathfrak{n}_+ \oplus \mathfrak{h})\big) \oplus \big(\mathbb{C}[t^{-1}] \otimes_{\mathbb{C}} \mathfrak{n}_-\big). \tag{1.5}$$

Remark 1.7. Notice that the meaning of a tilde symbol written above the notation for a Lie algebra depends on the Lie algebra in question. Only when $\mathfrak{g}$ is a simple finite dimensional Lie algebra, is $\tilde{\mathfrak{g}}$ to be interpreted as the result of the "affinization" described above. The effect of the tilde symbol applied to the constituent Lie algebras of a triangular decomposition is explained by (1.3), (1.4) and (1.5).

We now fix the symmetric invariant bilinear form on $\tilde{\mathfrak{g}}$. From the definition of $\tilde{A}$, we see that if we let

$$\tilde{D} = \operatorname{diag}(\epsilon_0, \epsilon_1, \ldots, \epsilon_n), \quad \text{where} \quad \epsilon_0 = \frac{(\theta, \theta)}{2},$$

then $\tilde{D}\tilde{A}$ becomes symmetric. Moreover, the resulting form (uniquely defined as in the previous section except on $\mathbb{C}\bar{d} \times \mathbb{C}\bar{d}$) is the only extension to $\tilde{\mathfrak{g}}$ of the form $(\cdot, \cdot)$ on $\mathfrak{g}$, which has the required properties. Thus, we extend the notation $(\cdot, \cdot)$ to also denote the form on $\tilde{\mathfrak{g}}$ defined by the diagonal matrix $\tilde{D}$ and the additional condition that $(\bar{d}, \bar{d}) = 0$. This form on $\tilde{\mathfrak{g}}$ is then determined by the original form on $\mathfrak{g}$, which we fix by requiring that

$$(\theta, \theta) = 2.$$

The resulting form on $\tilde{\mathfrak{g}}$ is called the normalized standard form. In view of the definition of the Lie bracket on $\tilde{\mathfrak{g}}$, we see that changing the form $(\cdot, \cdot)$ on $\mathfrak{g}$ may be interpreted as rescaling the central element c. Thus, the condition that $(\theta, \theta) = 2$ may alternatively be expressed by specifying that $\Lambda_0(c) = 1$, where Λ_0 is the weight in $\tilde{\mathfrak{h}}^*$ dual to $\alpha_0^\vee$.

Define the integers a_i and $a_i^\vee$, for $i = 0, \ldots, n$, by the condition that

$$\delta = \sum_{i=0}^{n} a_i \alpha_i \quad \text{and} \quad c = \sum_{i=0}^{n} a_i^\vee \alpha_i^\vee.$$

If we let v and $v^\vee$ be the column vectors given by $v = (a_0, \ldots, a_n)^\intercal$ and $v^\vee = (a_0^\vee, \ldots, a_n^\vee)^\intercal$, then $v^\intercal \tilde{A} = 0$ and $\tilde{A} v^\vee = 0$. In particular, since A is an invertible matrix, it follows that $\tilde{A}$ is singular with codimension 1. The integers g and h given by

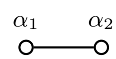
$$h = \sum_{i=0}^{n} a_i \quad \text{and} \quad g = \sum_{i=0}^{n} a_i^\vee$$

are called the Coxeter number and the dual Coxeter number of $\tilde{\mathfrak{g}}$, respectively.

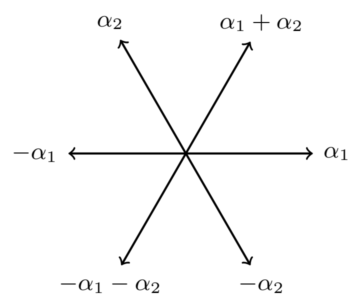
We conclude our general discussion of untwisted affine Lie algebras by making some remarks on their representations. Let M be a module for $\tilde{\mathfrak{g}}$ or $\hat{\mathfrak{g}}$. If c acts on M by the scalar ℓ, we say that M is of level ℓ. Let λ, $\mu \in \tilde{\mathfrak{h}}^*$ and consider the $\tilde{\mathfrak{g}}$-modules $L(\lambda)$ and $L(\mu)$. Regarding these modules as $\hat{\mathfrak{g}}$-modules, it is clear that they are isomorphic if and only if $\lambda - \mu \in \mathbb{C}\delta$. On the other hand, for any restricted $\hat{\mathfrak{g}}$-module N of level ℓ, where $\ell \neq -g$, there exists an operator on N through which the representation can be extended to $\tilde{\mathfrak{g}}$. In the study of vertex operator algebras, this operator (which will be denoted by $-L(0)$) gives a natural action of the element $\bar{d} \in \tilde{\mathfrak{g}}$, and hence in this context it suffices to consider $\hat{\mathfrak{g}}$-modules.

Now we apply the above exposition to the special case of the Kac–Moody algebra $\mathfrak{sl}(3,\mathbb{C})^{\sim}$. This is the untwisted affine Lie algebra associated with the simple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ consisting of traceless 3×3-matrices over $\mathbb{C}$. The Cartan matrix and affiliated Dynkin diagram for $\mathfrak{sl}(3,\mathbb{C})$ are given by

$$A_2 : \qquad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



and the corresponding root system is expressed by



As above, we write the maximal root $\alpha_1 + \alpha_2$ as

$$\theta = \alpha_1 + \alpha_2.$$

We fix the following basis for $\mathfrak{sl}(3,\mathbb{C})$:

$$x_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad x_{\theta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$x_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad x_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad x_{-\theta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$h_{\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In addition, we let

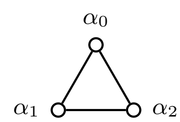
$$h_\theta = h_{\alpha_1} + h_{\alpha_2}.$$

This notation is chosen so that x_β belongs to the root space $\mathfrak{sl}(3, \mathbb{C})_\beta$ for every root β, and the Chevalley generators are given by $e_i = x_{\alpha_i}$ and $f_i = x_{-\alpha_i}$, for $i = 1, 2$. Furthermore the subalgebra generated by $\{x_\beta, h_\beta, x_{-\beta}\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, with the generators mapped in accordance with

$$x_\beta \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_\beta \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad x_{-\beta} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Following the general case discussed above, we see that $\mathfrak{sl}(3, \mathbb{C})^\sim$ is the Kac–Moody algebra over $\mathbb{C}$ corresponding to the affine matrix and Coxeter–Dynkin diagram of type $A_2^{(1)}$ given by

$$A_2^{(1)}: \qquad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



The normalized standard form is obtained by letting the symmetrizing diagonal matrix be the identity 3×3-matrix (and by defining $(\bar{d}, \bar{d})$ to be equal to 0). Alternatively, this is the form obtained from the specification above, by starting with the form on $\mathfrak{sl}(3, \mathbb{C})$ given by

$$(x, y) = \operatorname{tr}(xy), \quad \text{for } x, y \in \mathfrak{sl}(3, \mathbb{C}),$$

where tr is the trace function on 3×3-matrices.

To conclude this section we specify how the Chevalley generators and simple coroots are expressed in the notation for $\mathfrak{sl}(3, \mathbb{C})^\sim$ just introduced. The identification is given by

$$e_0 \leftrightarrow x_{-\theta}(1), \quad \alpha_0^\vee \leftrightarrow c - h_\theta(0), \quad f_0 \leftrightarrow x_\theta(-1),$$

and

$$e_i \leftrightarrow x_{\alpha_i}(0), \quad \alpha_i^\vee \leftrightarrow h_{\alpha_i}(0), \quad f_i \leftrightarrow x_{-\alpha_i}(0), \quad \text{for } i = 1, 2.$$

Chapter 2

Malikov–Feigin–Fuchs-operators

In line with [MFF86] we make the following observations. Let $\mathfrak{f}$ be the free Lie algebra on the set of generators $\{g_1, g_2, \dots, g_n\}$. By the Poincaré–Birkhoff–Witt theorem it is clear that for $\gamma_1, \dots, \gamma_N \in \mathbb{N}$ we have that

$$g_{i_1}^{\gamma_1} \cdot \dots \cdot g_{i_N}^{\gamma_N} = \sum_{j_1,\dots,j_n=0}^{\infty} P_{j_1,\dots,j_n}(g_1, \dots, g_n) g_1^{\gamma^{(1)}-j_1} \cdot \dots \cdot g_n^{\gamma^{(n)}-j_n}, \quad (2.1)$$

where

$$\gamma^{(k)} = \sum_{i_j=k} \gamma_j,$$

for $k = 1, \dots, n$, and where the expressions $P_{j_1,\dots,j_n} = P_{j_1,\dots,j_n}(g_1, \dots, g_n)$ (for $j_1, \dots, j_n \in \mathbb{N}$) denote uniquely determined elements in $\mathcal{U}([\mathfrak{f}, \mathfrak{f}])$ (of which all but finitely many are equal to zero). It turns out that these $P_{j_1,\dots,j_n}$ actually depend polynomially on $\gamma_1, \dots, \gamma_N$. Furthermore, using that the elements $P_{j_1,\dots,j_n}$ are defined for a free Lie algebra, we may apply the universal mapping property of free Lie algebras to extend the notation (2.1) by substitution to any Lie algebra $\mathfrak{g}$. Thus, we let $\mathfrak{g}$ be a Lie algebra containing elements denoted by $g_1, \dots, g_n$.

Since $P_{j_1,\dots,j_n} \in \mathcal{U}([\mathfrak{g}, \mathfrak{g}])[\gamma_1, \dots, \gamma_N]$, we can formally consider equation (2.1) for arbitrary complex numbers $\gamma_1, \dots, \gamma_N$. In [MFF86] the authors then regard this formula to "make sense" if the following two conditions are satisfied:

(i) $\gamma^{(k)} \in \mathbb{N}$, for $k = 1, \dots, n$;

(ii) if $j_k > \gamma^{(k)}$ then $P_{j_1,\dots,j_n}(g_1, \dots, g_n) = 0$.

When (i) and (ii) hold, the resulting (finite) sum is naturally interpreted as an element of $\mathcal{U}(\mathfrak{g})$. These considerations are then exploited for symmetrizable Kac–Moody algebras in [MFF86] to produce singular vectors of the form

$$f_{i_1}^{\gamma_1} \cdot \ldots \cdot f_{i_N}^{\gamma_N} v_\lambda \tag{2.2}$$

in the Verma module $M(\lambda)$.

In this chapter, we will give meaning to expressions of the form (2.1) in a different manner. One reason for doing this is to obtain a setting in which these objects can be manipulated more freely. We restrict the scope to the case where $\mathfrak{g}$ is an affine Kac–Moody algebra.

To begin with, we will extend $\mathcal{U}(\mathfrak{g})$ to include elements of the form f_i^{-1} (where f_i refers to the generators of the lower triangular subalgebra $\mathfrak{n}_-$). This is accomplished in Section 2.1. To give meaning to more general expressions, we will make use of the observation that the sequence of exponents γ_i in (2.2) is characterized by a certain kind of symmetry. This will lead to the introduction of a class of "conjugation automorphisms" in Section 2.2. When we employ the developed theory in the study of a specific vertex operator algebra later on, all the exponents γ_i will belong to $\frac{1}{2}\mathbb{Z}$. In Section 2.3 we therefore show that it is possible to use our results to construct an algebra which functions as an extension of $\mathcal{U}(\mathfrak{g})$ with elements of the form $f_i^{\pm 1/2}$.

Throughout this chapter we let $\mathfrak{g} = \mathfrak{g}(A, R)$ denote a Kac–Moody algebra over $\mathbb{C}$, and adopt the notation of Section 1.1 to $\mathfrak{g}$. Excepting the initial segments of Section 2.1 and Section 2.4, and Example 2.24, we will furthermore assume that $\mathfrak{g}$ is a Lie algebra of finite or affine type, and we will then let the index set $\mathcal{J}$ be equal to $\{1, \ldots, n\}$. Moreover, we let S stand for the set given by

$$S = \mathcal{U}(\mathfrak{n}_-) \setminus \{0\}.$$

2.1 Localization in the enveloping algebra

Adjoining the inverses f_i^{-1} to $\mathcal{U}(\mathfrak{g})$ is achieved by localizing. We start by describing how localization in a noncommutative ring, so called Ore localization, is defined (cf. [Lam98, pp. 299–308]). Throughout this section we assume that all rings and all homomorphisms between them are unital.

Let R be a ring and let T be a multiplicative subset of R, i.e. $T \cdot T \subseteq T$, $1 \in T$ and $0 \notin T$. Furthermore, let us say that a ring homomorphism $\psi\colon R \to L$ is T-inverting if $\psi(T)$ is contained in the multiplicative group of units of L. Then a ring R' is called a right ring of fractions with respect to T if there exists a ring homomorphism $\phi\colon R \to R'$ such that

(i) ϕ is T-inverting;

(ii) $R' = \{\, \phi(a)\phi(t)^{-1} : a \in R, t \in T \,\}$;

(iii) $\ker \phi = \{\, a \in R : at = 0 \text{ for some } t \in T \,\}$.

(The notion of a left ring of fractions is analogously defined *mutatis mutandis*.) If R' exists as above, it is determined up to isomorphism by the following universal mapping property: For any T-inverting ring homomorphism $\psi\colon R \to L$ there exists a unique ring homomorphism $\chi\colon R' \to L$ such that $\psi = \chi \circ \phi$. The diagram below illustrates this.

$$\begin{tikzcd} R \arrow[d, "\phi"'] \arrow[dr, "\psi"] & \\ R' \arrow[r, "\chi"'] & L \end{tikzcd}$$

It is easily seen that a necessary condition for the existence of R' is that

$$aT \cap tR \neq \varnothing, \quad \text{for all } a \in R \text{ and all } t \in T. \tag{2.3}$$

Now assume that T contains no left zero divisors. It then turns out that (2.3) is actually sufficient for R' to exist, and when (2.3) holds we let RT^{-1} denote a canonical right ring of fractions. If, furthermore, there are no right zero divisors in T either, then by (iii) ϕ is injective, and we may consider R to be a subset of RT^{-1}. In this case, i.e. if T contains neither left nor right zero divisors and (2.3) holds, the set T is called a right Ore set in R. Similarly we have the notion of a left Ore set and the corresponding left ring of fractions $T^{-1}R$. A set which is both left and right Ore is simply called an Ore set.

Remark 2.1. In our applications of Ore localization, we will work with sets that are both left and right Ore. By universal mapping properties it is clear that the two choices of localization that then arises yield rings that are canonically isomorphic. Thus we will identify the left and right rings of fractions. This identification is described in the diagram below, where $\epsilon_\ell\colon R \to T^{-1}R$ and $\epsilon_r\colon R \to RT^{-1}$

denote the T-inverting inclusion homomorphisms.

$$\begin{tikzcd}
 & R \arrow[dl, "\epsilon_\ell"'] \arrow[dr, "\epsilon_r"] & \\
T^{-1}R \arrow[rr, leftrightarrow, "\cong"] & & RT^{-1}
\end{tikzcd}$$

The following simple lemma will be referred to later.

Lemma 2.2. *Let R be a domain and let T_1 and T_2 be right Ore sets in R such that $T_1 \subseteq T_2 \subset R$. Then T_2 is a right Ore set in RT_1^{-1} and $(RT_1^{-1})T_2^{-1} \cong RT_2^{-1}$. (An analogous result holds with "left" replaced by "right".)*

Proof. Let $a \in R$, $t_1 \in T_1$ and $t_2 \in T_2$. Since T_2 is a right Ore set in R, there exist $t_2' \in T_2$ and $r \in R$ such that $at_2' = t_2 r$. Hence $(at_1^{-1})(t_1 t_2') = at_2' = t_2 r \in (at_1^{-1})T_2 \cap t_2 RT_1^{-1}$, which proves that T_2 is a right Ore set in RT_1^{-1}.

By the universal mapping property of right rings of fractions, it is clear that any T_2-inverting homomorphism on R can be uniquely extended (in two steps) to $(RT_1^{-1})T_2^{-1}$. Thus by the uniqueness of a right ring of fractions, we obtain the wanted isomorphism. □

If T is a right Ore set in R and B is a right R-module, we define the module of fractions BT^{-1}, of B with respect to T, by extension of scalars as

$$BT^{-1} = B \otimes_R RT^{-1}. \tag{2.4}$$

For basic results about this construction, see e.g. [Ste75, pp. 57–59]. Retaining the notation from this paragraph, we have the following proposition.

Proposition 2.3 ([Ste75], Corollary 3.3)**.** *Let $\mu\colon B \to BT^{-1}$ denote the natural R-module homomorphism. Then*

$$\ker \mu = \{\, x \in B : xt = 0 \textit{ for some } t \in T \,\}.$$

The corresponding notions and results with "left" in place of "right" are introduced analogously.

We now return to the consideration of a Kac–Moody algebra $\mathfrak{g}$. When $\mathfrak{g}$ is finite-dimensional, it is well known that the set of all nonzero elements of $\mathcal{U}(\mathfrak{g})$ constitutes an Ore set in $\mathcal{U}(\mathfrak{g})$, which means that we can form the skew field of fractions $\mathcal{U}(\mathfrak{g})\big(\mathcal{U}(\mathfrak{g}) \setminus \{0\}\big)^{-1}$. This can be proved by exploiting the fact that

$\mathcal{U}(\mathfrak{g})$ in this case is a (left and right) Noetherian ring. For a presentation of this construction, see e.g. [Dix77, pp. 117–125] or [Jac62, pp. 163–167]. Turning to a general Kac–Moody algebra, let M_i ($i \in \mathcal{J}$) denote the multiplicative monoid generated by f_i, and let M be the monoid generated by all the M_i. In other words,

$$M_i = \{ f_i^{\gamma} : \gamma \in \mathbb{N} \}, \tag{2.5}$$

for $i \in \mathcal{J}$, and

$$M = \{ f_{i_1}^{\gamma_1} \cdot \ldots \cdot f_{i_N}^{\gamma_N} : \gamma_k \in \mathbb{N} \text{ and } i_k \in \mathcal{J}, \text{for } k = 1, \ldots, N \}. \tag{2.6}$$

By elementary arguments we obtain the following proposition.

Proposition 2.4. *The monoids M_i ($i \in \mathcal{J}$) and M are Ore sets in $\mathcal{U}(\mathfrak{g})$.*

Proof. Let $x \in \mathcal{U}(\mathfrak{g})$ and $i \in \mathcal{J}$. By induction we get for any $\ell \in \mathbb{Z}^+$ that

$$x f_i^{\ell} = f_i x' + (-\operatorname{ad} f_i)^{\ell}(x),$$

for some $x' \in \mathcal{U}(\mathfrak{g})$. Since $\operatorname{ad} f_i$ is locally nilpotent on $\mathcal{U}(\mathfrak{g})$ we thus have that $x f_i^k = f_i x''$, for some $k \in \mathbb{Z}^+$ and some $x'' \in \mathcal{U}(\mathfrak{g})$, which implies that $xM_i \cap f_i\mathcal{U}(\mathfrak{g}) \neq \varnothing$.

Now assume that $f \in M$ is such that $xM \cap f\mathcal{U}(\mathfrak{g}) \neq \varnothing$, say $xm = fy$, where $m \in M$ and $y \in \mathcal{U}(\mathfrak{g})$. Let $j \in \mathcal{J}$. By what we have just proved, there exist $s \in \mathbb{Z}^+$ and $y'' \in \mathcal{U}(\mathfrak{g})$ such that $y f_j^s = f_j y''$. Hence

$$xmf_j^s = fyf_j^s = ff_jy'' \in xM \cap (ff_j)\mathcal{U}(\mathfrak{g}),$$

which shows that $xM \cap (ff_j)\mathcal{U}(\mathfrak{g}) \neq \varnothing$. If, in the above reasoning, we have that $f, m \in M_i$ and $i = j$, then this also proves that $xM_i \cap (ff_i)\mathcal{U}(\mathfrak{g}) \neq \varnothing$. By induction on the "length" of a monomial in M or M_i, we thus get that M and the M_i are right Ore sets in $\mathcal{U}(\mathfrak{g})$. Analogously it is obtained that these sets are also left Ore sets in $\mathcal{U}(\mathfrak{g})$. □

Since the elements of M are homogeneous with respect to the root lattice grading of $\mathcal{U}(\mathfrak{g})$, we can extend this grading to $\mathcal{U}(\mathfrak{g})M^{-1}$ by letting

$$\deg(x/m) = \deg x - \deg m,$$

for $x \in \mathcal{U}(\mathfrak{g})$ and $m \in M$.

By Proposition 2.4 we see that we may introduce the f_i^{-1} by localizing with respect to M. This localization is however not satisfactory for our purposes. For instance it does not allow us to commutate f_i^{-1} and f_j^{-1} in the denominator $(f_i f_j)^{-1} = f_j^{-1} f_i^{-1}$, if $i \neq j$. For a more essential reason why it will not be enough to localize with respect to M, see Remark 2.12 below.

From A. Rocha-Caridi and N. R. Wallach we have the following result.

Theorem 2.5 ([RCW84], Theorem 1.10)**.** *Let $\mathfrak{g}$ be an affine Lie algebra over $\mathbb{C}$. Then the set $S = \mathcal{U}(\mathfrak{n}_-) \setminus \{0\}$ is an Ore set in $\mathcal{U}(\mathfrak{n}_-)$.*

Remark 2.6. The proof of this theorem depends on the fact that affine Lie algebras admit a $\mathbb{Z}$-gradation where the dimensions of the graded subspaces are polynomially bounded. Thus, it does not seem possible to use the idea of this proof to gain insight into whether or not the conclusion of the theorem holds for Kac–Moody algebras of indefinite type. On the other hand, the line of proof of the theorem applies without changes to the case when $\mathfrak{g}$ is a finite-dimensional Kac–Moody algebra. However, in this situation, the conclusion of the theorem is also easily obtained by using the construction of the full skew field of fractions of $\mathcal{U}(\mathfrak{g})$ in conjunction with the Poincaré–Birkhoff–Witt theorem.

For the rest of this section we will let $\mathfrak{g}$ denote a Kac–Moody algebra of finite or affine type. According to the theorem we can define the following division ring:

$$K(\mathfrak{n}_-) = \mathcal{U}(\mathfrak{n}_-)S^{-1}.$$

The next proposition gives a means of extending an Ore set to a bigger subring.

Proposition 2.7 ([BR75], Lemma 4.2)**.** *Let R be a ring and let U be a subring of R. Assume that T is a right Ore set in U. Then the set*

$$R^T = \{\, r \in R : rT \cap tR \neq \varnothing \text{ for all } t \in T \,\}$$

is a subring of R. (*An analogous result holds for "left" instead of "right".*)

Retaining the assumptions and notation from this proposition, we get the following lemma.

Lemma 2.8. *Let $r \in R$ be such that for any $t \in T$ there exists $y \in R^T$ such that $rt = tr + y$. Then $r \in R^T$.*

Proof. For $t \in T$, let $y \in R^T$ be such that $rt = tr + y$. Since $yT \cap tR \neq \varnothing$, there exist $t' \in T$ and $r' \in R$ such that $yt' = tr'$. Hence

$$rtt' = trt' + yt' = trt' + tr' = t(rt' + r'),$$

which implies that $rT \cap tR \neq \varnothing$. Thus $r \in R^T$. □

The following theorem combines the previous three results.

Theorem 2.9. *Let $\mathfrak{g}$ be a Kac–Moody algebra of finite or affine type. Then the set $S = \mathcal{U}(\mathfrak{n}_-) \setminus \{0\}$ is an Ore set in $\mathcal{U}(\mathfrak{g})$.*

Proof. By Theorem 2.5, S is an Ore set in $\mathcal{U}(\mathfrak{n}_-)$. For any $h \in \mathfrak{h}$ and any $s \in S$ it is clear that $hs = sh + y$ for some $y \in \mathcal{U}(\mathfrak{n}_-)$. Hence Lemma 2.8 implies that $\mathfrak{h} \subseteq \mathcal{U}(\mathfrak{g})^S$. Furthermore, for any generator e_i of $\mathfrak{n}_+$ and any $s \in S$ we have that

$$e_i s = s e_i + w,$$

where $w \in \mathcal{U}(\mathfrak{n}_-)\alpha_i^\vee \mathcal{U}(\mathfrak{n}_-)$. Since $\alpha_i^\vee \in \mathcal{U}(\mathfrak{g})^S$, Proposition 2.7 implies that $\mathcal{U}(\mathfrak{n}_-)\alpha_i^\vee \mathcal{U}(\mathfrak{n}_-) \subseteq \mathcal{U}(\mathfrak{g})^S$. Hence we get from Lemma 2.8 that $e_i \in \mathcal{U}(\mathfrak{g})^S$. Thus we have shown that $\mathfrak{h}$ as well as all the f_i, e_i, for $i = 1, \dots, n$, are contained in $\mathcal{U}(\mathfrak{g})^S$. Hence it follows from Proposition 2.7 that $\mathcal{U}(\mathfrak{g})^S = \mathcal{U}(\mathfrak{g})$. □

Now we have shown that $\mathcal{U}(\mathfrak{g})S^{-1}$ exists, and have thereby created our desired extension of $\mathcal{U}(\mathfrak{g})$. It is easily seen that a $\mathbb{C}$-basis for $\mathcal{U}(\mathfrak{h} \oplus \mathfrak{n}_+)$ becomes a (left or right) $K(\mathfrak{n}_-)$-basis for $\mathcal{U}(\mathfrak{g})S^{-1}$. We introduce a Q-graded subalgebra $\mathrm{gr}\left(\mathcal{U}(\mathfrak{g})S^{-1}\right)$ of $\mathcal{U}(\mathfrak{g})S^{-1}$, which extends $\mathcal{U}(\mathfrak{g})$ as a graded algebra, as follows. Given $\alpha \in Q$, let

$$\left(\mathcal{U}(\mathfrak{g})S^{-1}\right)_\alpha = \{\, x \in \mathcal{U}(\mathfrak{g})S^{-1} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \,\} \tag{2.7}$$

be the homogeneous component of degree α, and let

$$\mathrm{gr}\left(\mathcal{U}(\mathfrak{g})S^{-1}\right) = \bigoplus_{\beta \in Q} \left(\mathcal{U}(\mathfrak{g})S^{-1}\right)_\beta \subseteq \mathcal{U}(\mathfrak{g})S^{-1}. \tag{2.8}$$

If $x \in \mathcal{U}(\mathfrak{g})_{\beta_1}$ and $y \in \mathcal{U}(\mathfrak{g})_{\beta_2} \cap S$, it is straightforward to check that $xy^{-1} \in \left(\mathcal{U}(\mathfrak{g})S^{-1}\right)_{\beta_1 - \beta_2}$.

Given a left $\mathcal{U}(\mathfrak{g})$-module W, we may introduce the module of fractions $S^{-1}W$ as

$$S^{-1}W = \mathcal{U}(\mathfrak{g})S^{-1} \otimes_{\mathcal{U}(\mathfrak{g})} W,$$

where we, in accordance with Remark 2.1, have identified $\mathcal{U}(\mathfrak{g})S^{-1}$ with $S^{-1}\mathcal{U}(\mathfrak{g})$. Now assume that $W = M(\lambda)$ is a Verma module. Then W is isomorphic to $\mathcal{U}(\mathfrak{n}_-)$ as a left $\mathcal{U}(\mathfrak{n}_-)$-module, and there are no zero divisors in $\mathcal{U}(\mathfrak{n}_-)$. Hence it is clear from Proposition 2.3 that $M(\lambda)$ is naturally embedded as a $\mathcal{U}(\mathfrak{g})$-submodule in $S^{-1}M(\lambda)$:

$$M(\lambda) \subset S^{-1}M(\lambda). \tag{2.9}$$

2.2 Conjugation automorphisms

We now proceed by giving meaning to expressions of the form $f_i^{\gamma} x f_i^{-\gamma}$ for $\gamma \in \mathbb{C}$ and $x \in \mathcal{U}(\mathfrak{g})S^{-1}$. This is accomplished in an indirect manner by introducing a sort of γ^{th} power of inner automorphisms of the form $x \mapsto f_i x f_i^{-1}$. By doing this we obtain automorphisms which can be thought of as $x \mapsto f_i^{\gamma} x f_i^{-\gamma}$.

To this end, let ψ_i ($i = 1, \ldots, n$) denote the inner automorphism of $\mathcal{U}(\mathfrak{g})S^{-1}$ induced by f_i, i.e.

$$\psi_i(x) = f_i x f_i^{-1},$$

for $x \in \mathcal{U}(\mathfrak{g})S^{-1}$ and $i = 1, \ldots, n$. Then we get that

$$\psi_i^{-1}(x) = f_i^{-1} x f_i = f_i^{-1}([x, f_i] + f_i x) = \big(1 - \mathcal{L}(f_i^{-1}) \operatorname{ad} f_i\big)(x). \tag{2.10}$$

Here and further on we use the following notation: for a ring R, and $x \in R$, we let $\mathcal{L}(x)$ and $\mathcal{R}(x)$ be the operators denoting left and right multiplication by x in R, respectively. From (2.10) we observe that, if $\mathcal{L}(f_i^{-1}) \operatorname{ad}(f_i)$ (or, equivalently, $\operatorname{ad}(f_i)$) were a locally nilpotent operator on $\mathcal{U}(\mathfrak{g})S^{-1}$, we would have had that

$$\psi_i = \big(1 - \mathcal{L}(f_i^{-1}) \operatorname{ad}(f_i)\big)^{-1} = \sum_{k=0}^{\infty} \mathcal{L}(f_i^{-k}) \operatorname{ad}(f_i)^k,$$

and thus

$$\psi_i = \exp\Big(\log\big(1 - \mathcal{L}(f_i^{-1}) \operatorname{ad}(f_i)\big)^{-1}\Big) = \exp\Big(-\log\big(1 - \mathcal{L}(f_i^{-1}) \operatorname{ad}(f_i)\big)\Big),$$

where the exponential and logarithm functions are defined by their usual power series. The operator $\mathcal{L}(f_i^{-1}) \operatorname{ad}(f_i)$ is however not in general locally nilpotent on the whole of $\mathcal{U}(\mathfrak{g})S^{-1}$, as the following example shows.

Example 2.10. Assume that $a_{ij} = a_{ji} = -1$ for some $j \neq i$. Then

$$[f_i, [f_i, f_j]] = [f_j, [f_j, f_i]] = 0,$$

while $[f_i, f_j] \neq 0$. It follows that

$$\mathrm{ad}(f_i)^n (f_j^{-1}) = (-1)^n n! f_j^{-n-1} [f_i, f_j]^n \neq 0,$$

for all $n \in \mathbb{N}$. Hence $\mathcal{L}(f_i^{-1})\,\mathrm{ad}(f_i)$ is not locally nilpotent on $\mathcal{U}(\mathfrak{g})S^{-1}$.

To get around this complication, we therefore introduce, for $i = 1, \ldots, n$, the subring $\mathcal{U}(\mathfrak{g})^{(i)}$ of $\mathcal{U}(\mathfrak{g})S^{-1}$ generated by $\mathcal{U}(\mathfrak{g})$ and f_i^{-1}:

$$\mathcal{U}(\mathfrak{g})^{(i)} = \mathcal{U}(\mathfrak{g})(f_i^{-1}) \subseteq \mathcal{U}(\mathfrak{g})S^{-1}.$$

Notice that $\mathcal{U}(\mathfrak{g})^{(i)}$ is equal to the ring $\mathcal{U}(\mathfrak{g})M_i^{-1}$ defined in Lemma 2.2. On $\mathcal{U}(\mathfrak{g})^{(i)}$ it is clear that $\mathcal{L}(f_i^{-1})\,\mathrm{ad}(f_i)$ is locally nilpotent, and we may define the operator $D^{(i)}$ given by

$$D^{(i)} = -\log\big(1 - \mathcal{L}(f_i^{-1})\,\mathrm{ad}(f_i)\big) = \sum_{k=1}^{\infty} \frac{1}{k} \mathcal{L}(f_i^{-k})\,\mathrm{ad}(f_i)^k$$

on $\mathcal{U}(\mathfrak{g})^{(i)}$. The following proposition, which is stated in a general context for clarity, shows that $D^{(i)}$ is a derivation of $\mathcal{U}(\mathfrak{g})^{(i)}$.

Proposition 2.11. *Let R be a unital ring and let x be a unit in R. Assume that $\mathcal{L}(x^{-1})\,\mathrm{ad}(x)$ is a locally nilpotent operator on R. Then the operator D defined by*

$$D = -\log\big(1 - \mathcal{L}(x^{-1})\,\mathrm{ad}(x)\big) = \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{L}(x^{-n})\,\mathrm{ad}(x)^n$$

is a derivation of R.

Proof. Let ψ be the inner automorphism of R induced by x. As was noted in the beginning of this section, this means that ψ is the operator on R given by

$$\psi = \big(1 - \mathcal{L}(x^{-1})\,\mathrm{ad}(x)\big)^{-1} = \sum_{j=0}^{\infty} \mathcal{L}(x^{-j})\,\mathrm{ad}(x)^j.$$

It follows that

$$\psi^n = \sum_{j=0}^{\infty}(-1)^j \binom{-n}{j} \mathcal{L}(x^{-j}) \operatorname{ad}(x)^j = \sum_{j=0}^{\infty} \binom{n+j-1}{j} \mathcal{L}(x^{-j}) \operatorname{ad}(x)^j.$$

Thus, for y, $z \in R$ we get that

$$\begin{aligned}
D(y)z + yD(z) &= D(y)z + \sum_{n=1}^{\infty} \frac{1}{n} yx^{-n} \operatorname{ad}(x)^n(z) \\
&= D(y)z + \sum_{n=1}^{\infty} \frac{1}{n} x^{-n}\psi^n(y) \operatorname{ad}(x)^n(z) \\
&= D(y)z + \sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{1}{n}\binom{n+j-1}{j} x^{-(n+j)} \operatorname{ad}(x)^j(y) \operatorname{ad}(x)^n(z) \\
&= D(y)z + \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \frac{1}{m-k}\binom{m-1}{k} x^{-m} \operatorname{ad}(x)^k(y) \operatorname{ad}(x)^{m-k}(z) \\
&= \sum_{m=1}^{\infty} \frac{1}{m} x^{-m} \operatorname{ad}(x)^m(y) z \\
&\quad + \sum_{m=1}^{\infty}\sum_{k=0}^{m-1} \frac{1}{m}\binom{m}{k} x^{-m} \operatorname{ad}(x)^k(y) \operatorname{ad}(x)^{m-k}(z) \\
&= \sum_{m=1}^{\infty}\sum_{k=0}^{m} \frac{1}{m}\binom{m}{k} x^{-m} \operatorname{ad}(x)^k(y) \operatorname{ad}(x)^{m-k}(z) \\
&= \sum_{m=1}^{\infty} \frac{1}{m} x^{-m} \operatorname{ad}(x)^m(yz) = D(yz),
\end{aligned}$$

where we have changed the summation variables according to

$$\begin{cases} m = n+j \\ k = j \end{cases}$$

to obtain the fourth equality, and thereafter used the identity

$$\frac{1}{m-k}\binom{m-1}{k} = \frac{1}{m}\binom{m}{k}$$

in the next equality. □

Since the exponential of a locally nilpotent derivation is an automorphism, we now get that for any $\gamma \in \mathbb{C}$, the map $A_\gamma^{(i)}$ given by

$$A_\gamma^{(i)} = \exp(\gamma D^{(i)}) = \exp\big(-\gamma \log(\psi_i^{-1})\big) \tag{2.11}$$

is an automorphism of $\mathcal{U}(\mathfrak{g})^{(i)}$. From the general properties of the exponential function we also obtain that

$$A_{\gamma_1+\gamma_2}^{(i)} = A_{\gamma_1}^{(i)} A_{\gamma_2}^{(i)}, \text{ and } (A_\gamma^{(i)})^{-1} = A_{-\gamma}^{(i)}. \tag{2.12}$$

By Lemma 2.2 we see that we can identify $\mathcal{U}(\mathfrak{g})^{(i)}S^{-1}$ with $\mathcal{U}(\mathfrak{g})S^{-1}$. Under this identification, let $\epsilon_i : \mathcal{U}(\mathfrak{g})^{(i)} \to \mathcal{U}(\mathfrak{g})S^{-1}$ denote the natural inclusion map. Since $\epsilon_i \circ A_\gamma^{(i)} : \mathcal{U}(\mathfrak{g})^{(i)} \to \mathcal{U}(\mathfrak{g})S^{-1}$ is S-inverting, it follows from the universal mapping property of a right ring of fractions that $A_\gamma^{(i)}$ extends to an automorphism of the whole of $\mathcal{U}(\mathfrak{g})S^{-1}$. This is illustrated in the diagram below, where the bottom arrow denotes the extension of $A_\gamma^{(i)}$ to $\mathcal{U}(\mathfrak{g})S^{-1}$.

$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g})^{(i)} \arrow[d, "\epsilon_i"'] \arrow[dr, "\epsilon_i \circ A_\gamma^{(i)}"] & \\
\mathcal{U}(\mathfrak{g})S^{-1} \arrow[r] & \mathcal{U}(\mathfrak{g})S^{-1}
\end{tikzcd}$$

From now on we will consider $A_\gamma^{(i)}$ as an automorphism on the whole of $\mathcal{U}(\mathfrak{g})S^{-1}$. Clearly, the properties expressed in (2.12) still hold for the extended automorphisms.

Remark 2.12. Notice that, in order to define the automorphisms $A_\gamma^{(i)}$ on the localized ring, it was crucial to localize with respect to the set S and not just to the monoid M (see (2.6)), because, in general, M is not closed under $A_\gamma^{(i)}$.

For $\ell \in \mathbb{Z}$, the automorphism $A_\ell^{(i)}$ of $\mathcal{U}(\mathfrak{g})S^{-1}$ is expressed by

$$A_\ell^{(i)}(x) = f_i^\ell x f_i^{-\ell}, \tag{2.13}$$

for all $x \in \mathcal{U}(\mathfrak{g})S^{-1}$. Equation (2.13) immediately follows from the construction if $x \in \mathcal{U}(\mathfrak{g})^{(i)}$, and is quickly verified for general elements of $\mathcal{U}(\mathfrak{g})S^{-1}$. For all

$\gamma \in \mathbb{C}$, we have that $A_\gamma^{(i)}$, as an automorphism of $\mathcal{U}(\mathfrak{g})^{(i)}$, is given by

$$\begin{aligned} A_\gamma^{(i)} &= \exp\big(-\gamma \log\big(1 - \mathcal{L}(f_i^{-1})\operatorname{ad}(f_i)\big)\big) = \big(1 - \mathcal{L}(f_i^{-1})\operatorname{ad}(f_i)\big)^{-\gamma} \\ &= \sum_{k=0}^{\infty}(-1)^k\binom{-\gamma}{k}\mathcal{L}(f_i^{-k})\operatorname{ad}(f_i)^k \\ &= \sum_{k=0}^{\infty}\binom{\gamma+k-1}{k}\mathcal{L}(f_i^{-k})\operatorname{ad}(f_i)^k. \end{aligned} \tag{2.14}$$

In subsequent sections, when we need to make explicit calculations involving the automorphisms $A_\gamma^{(i)}$, we will mostly make use of the formula in (2.14). However, there is also an equivalent way of writing these automorphisms in terms of "right operators", which we will find useful in the proof of Theorem 2.19 in Section 2.5. To obtain this second expression for $A_\gamma^{(i)}$, we start out with the opposite algebra of $\mathcal{U}(\mathfrak{g})S^{-1}$ instead, and then review the corresponding steps in this framework. The following proposition records these two parallel ways of expressing $A_\gamma^{(i)}$ as a series of operators.

Proposition 2.13. *For $i = 1, \dots, n$ and $\gamma \in \mathbb{C}$, the automorphism $A_\gamma^{(i)}$ satisfies*

$$A_\gamma^{(i)}\big|_{\mathcal{U}(\mathfrak{g})^{(i)}} = \sum_{k=0}^{\infty}\binom{\gamma+k-1}{k}\mathcal{L}(f_i^{-k})\operatorname{ad}(f_i)^k \tag{2.15}$$

and

$$\begin{aligned} \big(A_\gamma^{(i)}\big)^{-1}\big|_{\mathcal{U}(\mathfrak{g})^{(i)}} &= A_{-\gamma}^{(i)}\big|_{\mathcal{U}(\mathfrak{g})^{(i)}} \\ &= \sum_{k=0}^{\infty}\binom{\gamma+k-1}{k}\mathcal{R}(f_i^{-k})(-\operatorname{ad} f_i)^k. \end{aligned} \tag{2.16}$$

Proof. The first formula has already been proved in (2.14). To prove the second formula, let $B_\gamma^{(i)}$ denote the automorphism which is analogous to $A_\gamma^{(i)}$, but is obtained in the context of the opposite algebra of $\mathcal{U}(\mathfrak{g})S^{-1}$. The inner automorphism induced by f_i in the opposite algebra is given by ψ_i^{-1}. In view of (2.11), it follows that

$$B_\gamma^{(i)}\big|_{\mathcal{U}(\mathfrak{g})^{(i)}} = \exp\big(-\gamma\log(\psi_i)\big),$$

and hence we get that

$$B_\gamma^{(i)} = \left(A_\gamma^{(i)}\right)^{-1} = A_{-\gamma}^{(i)}.$$

It is now clear that equation (2.16) is obtained in the same way as (2.15) by applying the corresponding arguments to $B_\gamma^{(i)}$ in the opposite algebra. □

We end this section with a simple lemma, which collects some properties about the automorphisms $A_\gamma^{(i)}$.

Lemma 2.14. *For $i = 1, \ldots, n$ and $\gamma \in \mathbb{C}$, the automorphisms $A_\gamma^{(i)}$ of $\mathcal{U}(\mathfrak{g})S^{-1}$ satisfy the following properties:*

(i) *the restriction of $A_\gamma^{(i)}$ to $\operatorname{gr}\left(\mathcal{U}(\mathfrak{g})S^{-1}\right)$ is homogeneous of degree 0;*

(ii) *if $x \in \mathcal{U}(\mathfrak{g})$ is such that $(\operatorname{ad} f_i)^{k+1}(x) = 0$, then $A_\gamma^{(i)}(x) \in f_i^{-k}\mathcal{U}(\mathfrak{g})$;*

(iii) $A_\gamma^{(i)}(e_j) = e_j - \delta_{ij}\gamma f_i^{-1}(\alpha_i^\vee + \gamma + 1)$*, for $j = 1, \ldots, n$;*

(iv) $A_\gamma^{(i)}(h) = h + \gamma\alpha_i(h)$*, for any $h \in \mathfrak{h}$.*

Proof. All these results are direct consequences of formula (2.15) in the previous proposition. □

2.3 Extending $\mathcal{U}(\mathfrak{g})$ by half-integer powers of f_i

In this section we will use the theory developed in the previous two sections to extend $\mathcal{U}(\mathfrak{g})S^{-1}$ with elements of the form $f_i^{\pm 1/2}$ ($i = 1, \ldots, n$). The resulting algebra, to be denoted by $\mathcal{U}(\mathfrak{g})^e$, is a freely generated right module over $\mathcal{U}(\mathfrak{g})S^{-1}$. The study of the specific case in which f_i have half-integer exponents was originally motivated by the consideration of vertex operator algebras associated to affine Lie algebras at admissible level with denominator 2. The algebra $\mathcal{U}(\mathfrak{g})^e$ will, however, not be used in the rest of the thesis. In the general exposition in the following sections, we do not want the range of exponents to be limited to $\frac{1}{2} + \mathbb{N}$, and hence we must anyway write expressions in terms of the automorphisms $A_\gamma^{(i)}$. Nonetheless, the construction of $\mathcal{U}(\mathfrak{g})^e$ provides an explicit application of these automorphisms, and indicates that they have been properly defined.

To define $\mathcal{U}(\mathfrak{g})^e$ we proceed as follows. First, we introduce the automorphisms β_i of $\mathcal{U}(\mathfrak{g})S^{-1}$, for $i = 1, \ldots, n$, by letting

$$\beta_i = A_{1/2}^{(i)}.$$

This means that $\beta_i^2 = \psi_i$ and hence β_i is a "square root" of the inner automorphism ψ_i. We then define V to be the free right $\mathcal{U}(\mathfrak{g})S^{-1}$-module with basis

$$\{1\} \cup \left\{ F_{i_1}^{1/2} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} : k \in \mathbb{Z}^+, i_j \in \{1, \dots, n\}, i_j \neq i_{j+1} \right\}.$$

The idea is to introduce a left module structure on V that will correspond to the left regular representation of the ring we intend to construct. To achieve this, we define a representation $\pi \colon \mathcal{U}(\mathfrak{g})S^{-1} \to \mathrm{End}_{\mathcal{U}(\mathfrak{g})S^{-1}}(V)$, with action on the basis given by

$$\pi(x) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\bigl(\dots \bigl(\beta_{i_1}^{-1}(x)\bigr) \dots \bigr).$$

Here $\mathrm{End}_{\mathcal{U}(\mathfrak{g})S^{-1}}(V)$ denotes the ring of endomorphisms of V, considered as a right $\mathcal{U}(\mathfrak{g})S^{-1}$-module. Furthermore, for $i = 1, \dots, n$, we specify operators $f_i^{1/2}, f_i^{-1/2} \in \mathrm{End}_{\mathcal{U}(\mathfrak{g})S^{-1}}(V)$, acting on basis elements according to

$$f_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = \begin{cases} \pi(f_i) F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} & \text{if } i = i_1 \\ F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} & \text{if } i \neq i_1 \end{cases}$$

and

$$f_i^{-1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = \begin{cases} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} & \text{if } i = i_1 \\ \pi(f_i^{-1}) F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} & \text{if } i \neq i_1 \end{cases}.$$

The interaction of these operators is described in the next lemma.

Lemma 2.15. *For $i = 1, \dots, n$, the following identities among operators in $\mathrm{End}_{\mathcal{U}(\mathfrak{g})S^{-1}}(V)$ are satisfied:*

(i) $(f_i^{1/2})^2 = \pi(f_i)$;

(ii) $(f_i^{-1/2})^2 = \pi(f_i^{-1})$;

(iii) $f_i^{1/2} f_i^{-1/2} = \mathrm{id}_V$;

(iv) $f_i^{-1/2} f_i^{1/2} = \mathrm{id}_V$;

(v) $f_i^{1/2} \pi(x) f_i^{-1/2} = \pi\bigl(\beta_i(x)\bigr)$.

Proof.
(i), $i \neq i_1$:

$$\begin{aligned}
&(f_i^{1/2})^2 F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{1/2} F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \\
&= \pi(f_i) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2},
\end{aligned}$$

$i = i_1$:

$$\begin{aligned}
&(f_i^{1/2})^2 F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{1/2} \pi(f_i) F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \\
&= f_i^{1/2} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_2}^{-1}(f_i)\big) \dots \big) \\
&= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_2}^{-1}(f_i)\big) \dots \big) \\
&= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_2}^{-1}\big(\beta_{i_1}^{-1}(f_i)\big)\big) \dots \big) \\
&= \pi(f_i) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2}.
\end{aligned}$$

Hence $(f_i^{1/2})^2 = \pi(f_i)$.
(ii), $i \neq i_1$:

$$\begin{aligned}
&(f_i^{-1/2})^2 F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{-1/2} \pi(f_i^{-1}) F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \\
&= f_i^{-1/2} F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_1}^{-1}\big(\beta_i^{-1}(f_i^{-1})\big)\big) \dots \big) \\
&= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_1}^{-1}(f_i^{-1})\big) \dots \big) \\
&= \pi(f_i^{-1}) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2},
\end{aligned}$$

$i = i_1$:

$$\begin{aligned}
&(f_i^{-1/2})^2 F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{-1/2} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \\
&= \pi(f_i^{-1}) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2}.
\end{aligned}$$

Hence $(f_i^{-1/2})^2 = \pi(f_i^{-1})$.
(iii), $i \neq i_1$:

$$\begin{aligned}
&f_i^{1/2} f_i^{-1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{1/2} \pi(f_i^{-1}) F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \\
&= f_i^{1/2} F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_1}^{-1}\big(\beta_i^{-1}(f_i^{-1})\big)\big) \dots \big) \\
&= \pi(f_i) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\dots \big(\beta_{i_1}^{-1}(f_i^{-1})\big) \dots \big) \\
&= \pi(f_i) \pi(f_i^{-1}) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = F_{i_1}^{1/2} \cdots F_{i_k}^{1/2},
\end{aligned}$$

$i = i_1$:

$$\begin{aligned}&f_i^{1/2} f_i^{-1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{1/2} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2}.\end{aligned}$$

Hence $f_i^{1/2} f_i^{-1/2} = \mathrm{id}_V$.
(iv), $i \neq i_1$:

$$\begin{aligned}&f_i^{-1/2} f_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{-1/2} F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2},\end{aligned}$$

$i = i_1$:

$$\begin{aligned}&f_i^{-1/2} f_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{-1/2} \pi(f_i) F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \\ &= f_i^{-1/2} F_{i_2}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_2}^{-1}(f_i)\big)\ldots\big) \\ &= \pi(f_i^{-1}) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_2}^{-1}\big(\beta_{i_1}^{-1}(f_i)\big)\big)\ldots\big) \\ &= \pi(f_i^{-1}) \pi(f_i) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = F_{i_1}^{1/2} \cdots F_{i_k}^{1/2}.\end{aligned}$$

Hence $f_i^{-1/2} f_i^{1/2} = \mathrm{id}_V$.
(v), $i \neq i_1$:

$$\begin{aligned}&f_i^{1/2} \pi(x) f_i^{-1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} = f_i^{1/2} \pi(x f_i^{-1}) F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \\ &= f_i^{1/2} F_i^{1/2} F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(\beta_i^{-1}(x f_i^{-1})\big)\big)\ldots\big) \\ &= \pi(f_i) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(\beta_i^{-1}(x f_i^{-1})\big)\big)\ldots\big) \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}(f_i)\big)\ldots\big) \cdot \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(\beta_i^{-1}(x) f_i^{-1}\big)\big)\ldots\big) \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(f_i \beta_i^{-1}(x) f_i^{-1}\big)\big)\ldots\big) \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(\psi_i\big(\beta_i^{-1}(x)\big)\big)\big)\ldots\big) \\ &= F_{i_1}^{1/2} \cdots F_{i_k}^{1/2} \beta_{i_k}^{-1}\big(\ldots\big(\beta_{i_1}^{-1}\big(\beta_i(x)\big)\big)\ldots\big) \\ &= \pi\big(\beta_i(x)\big) F_{i_1}^{1/2} \cdots F_{i_k}^{1/2},\end{aligned}$$

$i = i_1$:

$$\begin{aligned}
&f_i^{1/2}\pi(x)f_i^{-1/2}F_{i_1}^{1/2}\cdots F_{i_k}^{1/2} = f_i^{1/2}\pi(x)F_{i_2}^{1/2}\cdots F_{i_k}^{1/2}\\
&= f_i^{1/2}F_{i_2}^{1/2}\cdots F_{i_k}^{1/2}\beta_{i_k}^{-1}\big(\dots\big(\beta_{i_2}^{-1}(x)\big)\dots\big)\\
&= F_{i_1}^{1/2}\cdots F_{i_k}^{1/2}\beta_{i_k}^{-1}\big(\dots\big(\beta_{i_2}^{-1}(x)\big)\dots\big)\\
&= F_{i_1}^{1/2}\cdots F_{i_k}^{1/2}\beta_{i_k}^{-1}\big(\dots\big(\beta_{i_2}^{-1}\big(\beta_{i_1}^{-1}\big(\beta_{i_1}(x)\big)\big)\big)\dots\big)\\
&= \pi\big(\beta_i(x)\big)F_{i_1}^{1/2}\cdots F_{i_k}^{1/2}.
\end{aligned}$$

Hence $f_i^{1/2}\pi(x)f_i^{-1/2} = \pi\big(\beta_i(x)\big)$. □

Let $\mathcal{U}(\mathfrak{g})^e$ denote the subalgebra of $\mathrm{End}_{\mathcal{U}(\mathfrak{g})S^{-1}}(V)$ generated by $f_i^{\pm 1/2}$, $i = 1, \dots, n$, and $\pi(\mathcal{U}(\mathfrak{g})S^{-1})$. From Lemma 2.15 we see that every element of $\mathcal{U}(\mathfrak{g})^e$ can be written in the form $\sum f_{i_1}^{1/2} f_{i_2}^{1/2} \cdot \ldots \cdot f_{i_k}^{1/2}\pi(x)$. Letting $\mathcal{U}(\mathfrak{g})^e$ act on $1 \in V$, we see that we may identify $\pi(\mathcal{U}(\mathfrak{g})S^{-1})$ with $\mathcal{U}(\mathfrak{g})S^{-1}$, and that $\mathcal{U}(\mathfrak{g})^e$ then becomes a free right $\mathcal{U}(\mathfrak{g})S^{-1}$-module with basis

$$\{1\} \cup \big\{\, f_{i_1}^{1/2} f_{i_2}^{1/2} \cdot \ldots \cdot f_{i_k}^{1/2} : k \in \mathbb{Z}^+, i_j \in \{1, \dots, n\}, i_j \neq i_{j+1} \,\big\}. \tag{2.17}$$

We have now constructed our desired extension of $\mathcal{U}(\mathfrak{g})$ in two steps:

$$\mathcal{U}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})S^{-1} \subset \mathcal{U}(\mathfrak{g})^e.$$

Given a left $\mathcal{U}(\mathfrak{g})$-module W, we can define a left $\mathcal{U}(\mathfrak{g})^e$-module W^e as the induced module obtained from $S^{-1}W$, i.e.

$$W^e = \mathcal{U}(\mathfrak{g})^e \otimes_{\mathcal{U}(\mathfrak{g})S^{-1}} S^{-1}W.$$

Notice that $S^{-1}W$ may naturally be considered as the $\mathcal{U}(\mathfrak{g})S^{-1}$-submodule $\mathbb{C}1 \otimes S^{-1}W$ of W^e. In particular, if $W = M(\lambda)$ is a Verma module, we get from (2.9) an extension of modules in two steps:

$$M(\lambda) \subset S^{-1}M(\lambda) \subset M(\lambda)^e.$$

For the inner automorphism of $\mathcal{U}(\mathfrak{g})^e$ given by $x \mapsto f_i^{1/2} x f_i^{-1/2}$ we notice that its restriction to $\mathcal{U}(\mathfrak{g})S^{-1}$ coincides with β_i. Thus, using Lemma 2.14 (iii) and (iv), we get for any $\gamma \in \frac{1}{2}\mathbb{Z}$ and $h \in \mathfrak{h}$ that

$$[h, f_i^\gamma] = h f_i^\gamma - f_i^\gamma h = f_i^\gamma\big(A_{-\gamma}^{(i)}(h) - h\big) = -\gamma\alpha_i(h) f_i^\gamma, \tag{2.18}$$

and

$$[e_j, f_i^\gamma] = e_j f_i^\gamma - f_i^\gamma e_j = f_i^\gamma \big(A_{-\gamma}^{(i)}(e_j) - e_j\big) = \delta_{ij}\gamma f_i^{\gamma-1}(\alpha_i^\vee - \gamma + 1). \quad (2.19)$$

With definitions analogous to those in (2.7) and (2.8) we also introduce a graded subalgebra gr $\big(\mathcal{U}(\mathfrak{g})^e\big)$ of $\mathcal{U}(\mathfrak{g})^e$ such that

$$\mathcal{U}(\mathfrak{g}) \subset \operatorname{gr}\big(\mathcal{U}(\mathfrak{g})S^{-1}\big) \subset \operatorname{gr}\big(\mathcal{U}(\mathfrak{g})^e\big),$$

where gr $\big(\mathcal{U}(\mathfrak{g})^e\big)$ is graded by $\frac{1}{2}Q$.

2.4 Singular vectors in Verma modules

We begin this section by showing how the authors in [MFF86] use their theory to calculate singular vectors in Verma modules. Then we continue by explaining how these results can be translated and interpreted in the setting developed in this chapter.

Let $\mathfrak{g}$ denote a symmetrizable Kac–Moody algebra. Given $\lambda \in \mathfrak{h}^*$ and $w \in \mathscr{W}$, the authors of [MFF86] proceed in the following way. Write w as a product of simple reflections, say $w = r_{i_N} \cdot \ldots \cdot r_{i_1}$, and define the sequence $\lambda_0, \ldots, \lambda_N \in \mathfrak{h}^*$ recursively by letting

$$\lambda_0 = \lambda, \quad (2.20)$$

and

$$\begin{aligned}\lambda_k = r_{i_k}^\rho(\lambda_{k-1}) &= (\lambda_{k-1} + \rho) - \big(\lambda_{k-1} + \rho\big)(\alpha_{i_k}^\vee)\alpha_{i_k} - \rho \\ &= \lambda_{k-1} - \big(\lambda_{k-1}(\alpha_{i_k}^\vee) + 1\big)\alpha_{i_k} \end{aligned} \quad (2.21)$$

for $k = 1, \ldots, N$. Then let the numbers $\gamma_k \in \mathbb{C}$ be determined by the equation

$$\lambda_k - \lambda_{k-1} = \gamma_k \alpha_{i_k}, \quad (2.22)$$

which means that

$$-\gamma_k = \lambda_{k-1}(\alpha_{i_k}^\vee) + 1.$$

Finally, define $F(w; \lambda)$, construed in the setting of [MFF86] as explained in the introduction to this chapter, by

$$F(w; \lambda) = f_{i_N}^{-\gamma_N} \cdot \ldots \cdot f_{i_1}^{-\gamma_1}. \quad (2.23)$$

In view of the equations

$$[h, f_i^{\gamma}] = -\gamma\alpha_i(h) f_i^{\gamma}, \quad \text{for } h \in \mathfrak{h}, \tag{2.24}$$
$$[e_j, f_i^{\gamma}] = \delta_{ij}\gamma f_i^{\gamma-1}(\alpha_i^{\vee} - \gamma + 1), \quad \text{for } i, j \in \{1, \ldots, n\}, \tag{2.25}$$

we see that the exponents $-\gamma_k$ are chosen so that $F(w;\lambda)$, when applied to the highest weight vector $v_\lambda \in M(\lambda)$, may give rise to a singular vector in the Verma module. (Notice that, if $\gamma \notin \mathbb{N}$, then the equations in (2.24) and (2.25) must constitute parts of some larger expressions in order to have a meaning according to the definitions of [MFF86].) Consequently, we have the following lemma in [MFF86].

Lemma 2.16 ([MFF86], Lemma 4.1)**.** *Provided that $F(w;\lambda)$ "makes sense", the vector $F(w;\lambda)v_\lambda$ is a singular vector of the module $M(\lambda)$.*

Notice that if $F(w;\lambda)$ "makes sense", it is a homogeneous element of degree $w^\rho\lambda - \lambda \in Q$. In particular, if w is a reflection in a real root α, then this means that for some $m \in \mathbb{Z}$

$$(\lambda + \rho)(\alpha^{\vee}) = m,$$

or equivalently

$$2(\lambda + \rho)(\nu^{-1}(\alpha)) = m(\alpha, \alpha).$$

In the case when α is a positive real root and $m \in \mathbb{N}$, this is the condition obtained from the Šapovalov–Kac–Kazhdan determinant formula for symmetrizable Kac–Moody algebras ([KK79, Theorem 1]). The main theorem given in [MFF86] is the following.

Theorem 2.17 ([MFF86], Theorem 4.2)**.** *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, let α be a positive real root of $\mathfrak{g}$, and let $\lambda \in \mathfrak{h}^*$. If, for some $m \in \mathbb{N}$,*

$$2(\lambda + \rho)(\nu^{-1}(\alpha)) = m(\alpha, \alpha),$$

then $F(r_\alpha;\lambda)$ "makes sense" and $F(r_\alpha;\lambda)v_\lambda$ is a singular vector in $M(\lambda)$.

We will now show how these results can be expressed in our setting. We will not be able to prove Theorem 2.17 in its most general form. However, our line of proof is more explicit than that given in [MFF86], and it also suggests that a generalization of Theorem 2.17 to non-symmetrizable algebras is possible.

From now on, we assume as the scope of our setting that $\mathfrak{g}$ is a finite or affine Kac–Moody algebra over $\mathbb{C}$. First we introduce notations needed in order to describe the equations defining the numbers γ_k as a system of equations. For any generalized Cartan matrix A of dimension $n \times n$, and any K-tuple $J = (j_1, \ldots, j_K) \in \{1, 2, \ldots, n\}^K$, we let C_A^J be the $K \times K$-matrix with entries given by

$$C_A^J = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0 \\ -a_{j_1,j_2} & -1 & 0 & \ldots & 0 & 0 \\ -a_{j_1,j_3} & -a_{j_2,j_3} & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{j_1,j_{K-1}} & -a_{j_2,j_{K-1}} & -a_{j_3,j_{K-1}} & \ldots & -1 & 0 \\ -a_{j_1,j_K} & -a_{j_2,j_K} & -a_{j_3,j_K} & \ldots & -a_{j_{K-1},j_K} & -1 \end{pmatrix}.$$

From (2.21) and (2.22) we have that

$$\lambda_k = \lambda + \gamma_1 \alpha_{i_1} + \ldots + \gamma_k \alpha_{i_k},$$

and hence, for $k = 1, \ldots, N$, that

$$\begin{aligned} \gamma_k = -(\lambda_{k-1} + \rho)(\alpha_{i_k}^\vee) &= -(\lambda + \rho + \gamma_1 \alpha_{i_1} + \ldots + \gamma_{k-1} \alpha_{i_{k-1}})(\alpha_{i_k}^\vee) \\ \iff \quad (\lambda + \rho)(\alpha_{i_k}^\vee) &= -\gamma_1 a_{i_1,i_k} - \ldots - \gamma_{k-1} a_{i_{k-1},i_k} - \gamma_k \end{aligned}$$

Thus, we see that, with $I = (i_1, \ldots, i_N)$, the equations defining the γ_k ($k = 1, \ldots, N$) can be expressed as

$$C_A^I \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_N \end{pmatrix} = \begin{pmatrix} (\lambda + \rho)(\alpha_{i_1}^\vee) \\ (\lambda + \rho)(\alpha_{i_2}^\vee) \\ \vdots \\ (\lambda + \rho)(\alpha_{i_N}^\vee) \end{pmatrix}. \tag{2.26}$$

We now impose the condition that w is a reflection of the form r_α, where α is a real root of $\mathfrak{g}$. We write α as

$$\alpha = r_{i_1} \ldots r_{i_{M-1}}(\alpha_{i_M}). \tag{2.27}$$

The reflection r_α is then given by

$$r_\alpha = r_{r_{i_1} \ldots r_{i_{M-1}}(\alpha_{i_M})} = r_{i_1} \ldots r_{i_{M-1}} r_{i_M} r_{i_{M-1}} \ldots r_{i_1}.$$

This means that the N-tuple I now is of the form

$$I = (i_1, \ldots, i_{M-1}, i_M, i_{M-1}, \ldots, i_1),$$

where $N = 2M - 1$. Let us, for any set Ω, call an N-tuple $(\omega_1, \ldots, \omega_N) \in \Omega^N$ palindromic if it is fixed under the order reversing permutation, i.e. if $\omega_k = \omega_{N-k+1}$, for $k = 1, \ldots, N$. The fact that I is palindromic will be of great importance in what follows. We also introduce the two M-tuples I_1 and I_2 by letting

$$I_1 = (i_1, \ldots, i_M) \quad \text{and} \quad I_2 = (i_M, \ldots, i_1),$$

which means that I_1 and I_2 are approximately the first and second "half" of I, respectively. Notice that in accordance with (2.27), I_1 is a sequence describing how α is related to a simple root by simple reflections.

Next, we define the numbers $s_k \in \mathbb{Z}$, for $k = M, \ldots, 1$, in a way similar to how the γ_k were defined above. Given the expression for α in (2.27), we introduce $\alpha^{(M)}, \alpha^{(M-1)}, \ldots, \alpha^{(1)} \in \Delta$ by letting

$$\alpha^{(M)} = \alpha_{i_M},$$

and

$$\alpha^{(k)} = r_{i_k}(\alpha^{(k+1)}) = \alpha^{(k+1)} - \alpha^{(k+1)}(\alpha_{i_k}^\vee)\alpha_{i_k},$$

for $k = M - 1, \ldots, 1$. The numbers s_k are then defined by letting

$$s_M = 1, \tag{2.28}$$

and

$$s_k = -\alpha^{(k+1)}(\alpha_{i_k}^\vee), \tag{2.29}$$

for $k = M - 1, \ldots, 1$. (The introduction of $s_M = 1$ is admittedly somewhat redundant, but it serves to make the exposition more consistent.) This means that

$$\alpha^{(k+1)} = s_M \alpha_{i_M} + \ldots + s_{k+1}\alpha_{i_{k+1}},$$

and hence that

$$(-a_{i_M,i_k})s_M + \ldots + (-a_{i_{k+1},i_k})s_{k+1} - s_k = 0, \tag{2.30}$$

for $k = 1, \ldots, M-1$. This, in turn, can be written more succinctly as

$$C_A^{I_2} \begin{pmatrix} s_M \\ s_{M-1} \\ \vdots \\ s_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.31}$$

In the proof of Theorem 2.19, we will need to assume that the integers s_k are all non-negative. This is the same as requiring that the sequence of roots $\alpha^{(M)}, \ldots, \alpha^{(1)}$ is of increasing height. It is well known that this can always be accomplished by choosing the expression for α in (2.27) properly (see e.g. Proposition 14, Ch. 4.1, in [MP95]). Thus, this provision, which amounts to a property of the sequence I_1, does not limit the applicability of Theorem 2.19.

We now show how the $N \times N$-matrix C_A^I can be expressed in terms of the $M \times M$-matrices $C_A^{I_1}$ and $C_A^{I_2}$ (recall that $N = 2M-1$), and an auxiliary $N \times N$-matrix R given by

$$R = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & \iddots & & \ddots & \\ 1 & & & & 1 \end{pmatrix}.$$

That is to say that R has entries equal to 1 on the main diagonal and lower half of the counter-diagonal, while every other entry is zero. We notice that when we multiply an $N \times N$-matrix P by R on the left, the result is that the first $M-1$ rows in P are mirrored in the M^{th} row and added to the last $M-1$ rows. The last sentence also holds true if we change "left" to "right", "row" to "column", and interchange "first" and "last". We now use the matrices $C_A^{I_1}$ and $C_A^{I_2}$ as blocks to define the matrix E_A^I by

$$E_A^I = \begin{pmatrix} C_A^{I_1} & & 0 \\ & -1 & \\ 0 & & C_A^{I_2} \end{pmatrix},$$

where the two blocks have one entry (equal to -1) in position (M, M) in common. All entries of E_A^I which are outside the blocks defined by $C_A^{I_1}$ and $C_A^{I_2}$ are equal to 0. More precisely, if we write $C_A^{I_1}$ and $C_A^{I_2}$ as

$$C_A^{I_1} = \left(\begin{array}{c:c} P & 0 \\ \hdashline p & -1 \end{array}\right) \quad \text{and} \quad C_A^{I_2} = \left(\begin{array}{c:c} -1 & 0 \\ \hdashline q & Q \end{array}\right),$$

then E_A^I is given by

$$E_A^I = \left(\begin{array}{c:c:c} P & 0 & 0 \\ \hdashline p & -1 & 0 \\ \hdashline 0 & q & Q \end{array}\right).$$

It is now straightforward to check that

$$C_A^I = R E_A^I R.$$

The following lemma explains the structure of the solutions γ_k to (2.26).

Lemma 2.18. *If* $I = (i_1, \ldots, i_{M-1}, i_M, i_{M-1}, \ldots, i_1)$ *is a palindromic tuple then*

$$C_A^I \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{M-1} \\ \gamma_M \\ \gamma_M s_{M-1} - \gamma_{M-1} \\ \vdots \\ \gamma_M s_1 - \gamma_1 \end{pmatrix} = \begin{pmatrix} (\lambda+\rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ (\lambda+\rho)(\alpha_{i_M}^\vee) \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_1}^\vee) \end{pmatrix}.$$

In other words, $\gamma_{2M-k} = \gamma_M s_k - \gamma_k$, *for* $k = 1, \ldots, M-1$.

Proof. By definition we have that

$$C_A^{I_1} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{pmatrix} = \begin{pmatrix} (\lambda+\rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_M}^\vee) \end{pmatrix}. \tag{2.32}$$

Using this and (2.31), and retaining the notation introduced before the lemma, we obtain that

$$C_A^I \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{M-1} \\ \gamma_M \\ \gamma_M s_{M-1} - \gamma_{M-1} \\ \vdots \\ \gamma_M s_1 - \gamma_1 \end{pmatrix} = RE_A^I R \left[\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{M-1} \\ 0 \\ -\gamma_{M-1} \\ \vdots \\ -\gamma_1 \end{pmatrix} + \gamma_M \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_M \\ s_{M-1} \\ \vdots \\ s_1 \end{pmatrix} \right]$$

$$= RE_A^I \left[\begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{M-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \gamma_M \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_M \\ s_{M-1} \\ \vdots \\ s_1 \end{pmatrix} \right]$$

$$= R\left[\begin{pmatrix} (\lambda+\rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ (\lambda+\rho)(\alpha_{i_M}^\vee)+\gamma_M \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \gamma_M \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right]$$

$$= R\begin{pmatrix} (\lambda+\rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ (\lambda+\rho)(\alpha_{i_M}^\vee) \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} (\lambda+\rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ (\lambda+\rho)(\alpha_{i_M}^\vee) \\ (\lambda+\rho)(\alpha_{i_{M-1}}^\vee) \\ \vdots \\ (\lambda+\rho)(\alpha_{i_1}^\vee) \end{pmatrix},$$

which proves the lemma. □

This lemma shows that in the setting of [MFF86], the operator $F(r_\alpha;\lambda)$ (cf. (2.23)) is given by

$$F(r_\alpha;\lambda) = f_{i_1}^{-\gamma_M s_1+\gamma_1}\cdots f_{i_{M-1}}^{-\gamma_M s_{M-1}+\gamma_{M-1}} f_{i_M}^{-\gamma_M} f_{i_{M-1}}^{-\gamma_{M-1}}\cdots f_{i_1}^{-\gamma_1}. \quad (2.33)$$

If $F(r_\alpha;\lambda)$ "makes sense", it is homogeneous of degree

$$r_\alpha^\rho\lambda - \lambda = -(\lambda+\rho)(\alpha^\vee)\alpha \in -Q_+.$$

But we also have that

$$\begin{aligned} r_\alpha^\rho\lambda - \lambda &= \sum_{k=1}^N \gamma_k\alpha_{i_k} \\ &= \gamma_1\alpha_{i_1} + \ldots + \gamma_{M-1}\alpha_{i_{M-1}} + \gamma_M\alpha_{i_M} \\ &\quad + (\gamma_M s_{M-1} - \gamma_{M-1})\alpha_{i_{M-1}} + \ldots + (\gamma_M s_1 - \gamma_1)\alpha_{i_1} \\ &= \gamma_M(s_1\alpha_{i_1} + \ldots + s_{M-1}\alpha_{i_{M-1}} + \alpha_{i_M}) = \gamma_M\alpha. \end{aligned}$$

Thus, we get that for some $m\in\mathbb{N}$,

$$\gamma_M = -(\lambda+\rho)(\alpha^\vee) = -m.$$

This means that

$$F(r_\alpha;\lambda) = f_{i_1}^{ms_1+\gamma_1}\cdots f_{i_{M-1}}^{ms_{M-1}+\gamma_{M-1}} f_{i_M}^{m} f_{i_{M-1}}^{-\gamma_{M-1}}\cdots f_{i_1}^{-\gamma_1},$$

and from this expression we are lead to introduce a corresponding operator in our setting, which is given by

$$\begin{aligned} F(I_1,\gamma) &= f_{i_1}^{ms_1} A_{\gamma_1}^{(i_1)}\big(\cdots f_{i_{M-1}}^{ms_{M-1}} A_{\gamma_{M-1}}^{(i_{M-1})}(f_{i_M}^{m})\big) \\ &= \big(\mathcal{L}(f_{i_1}^{ms_1})A_{\gamma_1}^{(i_1)}\cdots \mathcal{L}(f_{i_{M-1}}^{ms_{M-1}})A_{\gamma_{M-1}}^{(i_{M-1})}\big)(f_{i_M}^{m}) \in \mathcal{U}(\mathfrak{n}_-)S^{-1}. \end{aligned}$$

It will be convenient to express $F(I_1,\gamma)$ more uniformly as

$$F(I_1,\gamma) = \big(\mathcal{L}(f_{i_1}^{ms_1})A_{\gamma_1}^{(i_1)}\cdots \mathcal{L}(f_{i_M}^{ms_M})A_{\gamma_M}^{(i_M)}\big)(1).$$

The M-tuple $\gamma = (\gamma_1,\ldots,\gamma_{M-1},-m)$ is determined by (2.32). If, say $i_q = i_r$ for some q and r such that $1 \le q < r \le M$, then the q^{th} and r^{th} row of (2.32) gives that

$$(\lambda+\rho)(\alpha_{i_q}^\vee) = (-a_{i_1,i_q})\gamma_1 + \ldots + (-a_{i_{q-1},i_q})\gamma_{q-1} - \gamma_q,$$

and that

$$\begin{aligned} (\lambda+\rho)(\alpha_{i_r}^\vee) = (-a_{i_1,i_r})\gamma_1 + \ldots + (-a_{i_{q-1},i_r})\gamma_{q-1} - 2\gamma_q \\ + (-a_{i_{q+1},i_r})\gamma_{q+1} + \ldots + (-a_{i_{r-1},i_r})\gamma_{r-1} - \gamma_r. \end{aligned}$$

Together these equations show that

$$\gamma_q = (-a_{i_{q+1},i_r})\gamma_{q+1} + \ldots + (-a_{i_{r-1},i_r})\gamma_{r-1} - \gamma_r. \tag{2.34}$$

On the other hand, if $I_1 = (i_1,\ldots,i_M)$ and $\gamma = (\gamma_1,\ldots,\gamma_{M-1},-m)$ are such that for any q and r, with $1 \le q < r \le M$, we have that $i_q = i_r$ implies that (2.34) holds, then it is clear that there exists $\lambda \in \mathfrak{h}^*$ such that (2.32) is satisfied. For the moment we therefore consider $F(I_1,\gamma) \in \mathcal{U}(\mathfrak{n}_-)S^{-1}$ as an operator which only depends on the tuples

$$I_1 = (i_1,\ldots,i_M) \in \{1,2,\ldots,n\}^M \tag{2.35}$$

and

$$\gamma = (\gamma_1,\ldots,\gamma_M) = (\gamma_1,\ldots,\gamma_{M-1},-m) \in \mathbb{C}^{M-1}\times\mathbb{Z}^-, \tag{2.36}$$

without any reference to a positive real root α or weight $\lambda \in \mathfrak{h}^*$. We let (**C**) be the condition on the pair (I_1, γ) just described, i.e.

$$\text{(C)} \qquad i_q = i_r, \quad \text{for } 1 \leq q < r \leq M,$$
$$\Longrightarrow \qquad \gamma_q = (-a_{i_{q+1},i_q})\gamma_{q+1} + \ldots + (-a_{i_{r-1},i_q})\gamma_{r-1} - \gamma_r.$$

Our goal is now to show that if (**C**) is satisfied then $F(I_1, \gamma) \in \mathcal{U}(\mathfrak{n}_-)$. Although this is essentially proved by [MFF86] in their setting (see Theorem 2.17), we will not be able to prove this for general I_1 and γ. However, our line of proof can be directly generalized to any Kac–Moody algebra $\mathfrak{g}$, for which we can prove that $S = \mathcal{U}(\mathfrak{n}_-) \setminus \{0\}$ is an Ore set in $\mathcal{U}(\mathfrak{g})$. The theorem below states the further conditions on I_1 and γ which we require.

Theorem 2.19. *Assume that I_1 and γ are M-tuples as expressed by* (2.35) *and* (2.36)*, respectively, such that (I_1, γ) satisfies condition* (**C**). *Furthermore, assume that no three entries of I_1 are equal, and that the integers s_k (defined with reference to I_1 by* (2.28) *and* (2.29)*) are all positive. Let T be the set given by*

$$T = \{\, (q, r) : i_q = i_r \text{ where } 1 \leq q < r \leq M \,\},$$

and suppose that the following three restrictions on T hold:

- $(q, M) \in T \Longrightarrow m = 1$;
- $(q, r) \in T \Longrightarrow s_r = 1$;
- $(q, r), (q', r') \in T \Longrightarrow r < q'$ *or* $r' < q$ *or* $(q, r) = (q', r')$.

Then $F(I_1, \gamma) \in \mathcal{U}(\mathfrak{n}_-)$.

Since the proof of Theorem 2.19 is rather long and intricate, we postpone it to the next section.

Our next result shows that there exists a factorization of $F(I_1, \gamma)$ into m factors, where $-m$ is the last entry of γ. As a consequence of this factorization it follows that, if the last entry of I_1 is distinct from all other entries of I_1, we may in the proof of Theorem 2.19 assume that $m = 1$ without loss of generality.

Proposition 2.20. *$F(I_1, \gamma)$ factorizes in $\mathcal{U}(\mathfrak{n}_-)S^{-1}$ as*

$$F(I_1, \gamma) = F(I_1, \gamma^{(1)})F(I_1, \gamma^{(2)}) \cdots F(I_1, \gamma^{(m)}),$$

where

$$\gamma^{(k)} = (\gamma_1 + (m-k)s_1, \ldots, \gamma_{M-1} + (m-k)s_{M-1}, -1).$$

Furthermore, if $i_q \neq i_M$, for $q = 1, \ldots, M-1$, then condition (**C**) *holds for (I_1, γ) if and only if it holds for $(I_1, \gamma^{(k)})$, for some $k \in \{1, \ldots, m\}$.*

Proof. To exhibit the factorization, we argue by induction on the length of I_1. Assume that such factorizations exist for tuples of length less than M. Let $\bar{I}_1$, $\bar{\gamma}$ and $\bar{\gamma}^{(k)}$ be the $(M-1)$-tuples obtained from I_1, γ and $\gamma^{(k)}$, respectively, by deleting their first entries. We then have that

$$F(\bar{I}_1, \bar{\gamma}) = F(\bar{I}_1, \bar{\gamma}^{(1)})F(\bar{I}_1, \bar{\gamma}^{(2)}) \cdots F(\bar{I}_1, \bar{\gamma}^{(m)}).$$

Using the fact that $A_\gamma^{(i)}$ is an automorphism and that

$$f_i^\ell A_\gamma^{(i)}(x) = A_{\gamma+\ell}^{(i)}(x) f_i^\ell$$

for any $\ell \in \mathbb{Z}$ and $x \in \mathcal{U}(\mathfrak{g})S^{-1}$ (cf. (2.13)), we obtain that

$$\begin{aligned}
F(I_1, \gamma) &= \mathcal{L}(f_{i_1}^{ms_1}) A_{\gamma_1}^{(i_1)}\big(F(\bar{I}_1, \bar{\gamma})\big) = \mathcal{L}(f_{i_1}^{ms_1}) A_{\gamma_1}^{(i_1)}\Big(\prod_{k=1}^{m} F(\bar{I}_1, \bar{\gamma}^{(k)})\Big) \\
&= \mathcal{L}(f_{i_1}^{ms_1})\Big(\prod_{k=1}^{m} A_{\gamma_1}^{(i_1)}\big(F(\bar{I}_1, \bar{\gamma}^{(k)})\big)\Big) \\
&= \prod_{k=1}^{m} \mathcal{L}(f_{i_1}^{s_1}) A_{\gamma_1+(m-k)s_1}^{(i_1)}\big(F(\bar{I}_1, \bar{\gamma}^{(k)})\big) = \prod_{k=1}^{m} F(I_1, \gamma^{(k)}).
\end{aligned}$$

To prove the statement about condition (**C**), we notice that the last entries of the M-tuples I_1, γ and $\gamma^{(k)}$ do not affect this assertion. In view of the definition of $\gamma^{(k)}$, it follows from the linear nature of condition (**C**) that it is enough to verify that the pair of $(M-1)$-tuples

$$((i_1, \ldots, i_{M-1}), (s_1, \ldots, s_{M-1})) \tag{2.37}$$

satisfies this condition. Thus, assume that $i_q = i_r$, where $1 \leq q < r \leq M - 1$. Analogous to how condition (**C**) was derived in (2.34), we obtain from the definition of the s_k integers in (2.30) that

$$(-a_{i_M, i_r})s_M + \ldots + (-a_{i_{r+1}, i_r})s_{r+1} - s_r = 0$$

and that

$$(-a_{i_M,i_q})s_M + \ldots + (-a_{i_{r+1},i_q})s_{r+1} - 2s_r \\ + (-a_{i_{r-1},i_q})s_{r-1} + \ldots + (-a_{i_{q+1},i_q})s_{q+1} - s_q = 0.$$

Combining these equations yields that

$$s_q = (-a_{i_{q+1},i_q})s_{q+1} + \ldots + (-a_{i_{r-1},i_q})s_{r-1} - s_r,$$

which shows that condition (**C**) holds for the pair in (2.37), and hence implies the equivalence of condition (**C**) for (I_1, γ) and $(I_1, \gamma^{(k)})$. $\square$

Remark 2.21. If the pair (I_1, γ) satisfies the premises of Theorem 2.19, it follows that the factorization in the above proposition takes place in $\mathcal{U}(\mathfrak{n}_-)$. This is to say that $F(I_1, \gamma)$ as well as $F(I_1, \gamma^{(k)})$, for $k = 1, \ldots, m$, all belong to $\mathcal{U}(\mathfrak{n}_-)$.

We now return to the point of view that $F(I_1, \gamma)$ is defined with reference to a positive real root α and a weight $\lambda \in \mathfrak{h}^*$ such that

$$(\lambda + \rho)(\alpha^\vee) = m.$$

Recall that this means that

$$F(I_1, \gamma) = \big(\mathcal{L}(f_{i_1}^{ms_1})A_{\gamma_1}^{(i_1)} \cdots \mathcal{L}(f_{i_M}^{ms_M})A_{\gamma_M}^{(i_M)}\big)(1),$$

where $I_1 = (i_1, \ldots, i_M)$ is determined by a fixed representation of α of the form

$$\alpha = r_{i_1} \ldots r_{i_{M-1}}(\alpha_{i_M}),$$

and where $(\gamma_1, \ldots, \gamma_M) \in \mathbb{C}^M$ and $(s_1, \ldots, s_M) \in \mathbb{N}^M$ are, respectively, given by the systems of equations

$$C_A^{I_1} \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_M \end{pmatrix} = \begin{pmatrix} (\lambda + \rho)(\alpha_{i_1}^\vee) \\ \vdots \\ (\lambda + \rho)(\alpha_{i_M}^\vee) \end{pmatrix} \tag{2.38}$$

and

$$C_A^{I_2} \begin{pmatrix} s_M \\ s_{M-1} \\ \vdots \\ s_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{2.39}$$

The reason why $F(I_1, \gamma)$ is defined in this way is of course that it should produce a singular vector in the Verma module $M(\lambda)$. We have seen the heuristic argument for this to be true in the setting of [MFF86] for the corresponding operator $F(r_\alpha; \lambda)$, and their result was formulated in Lemma 2.16. We now set about to show that the analogous conclusion holds in our setting.

Proposition 2.22. *Let $\lambda \in \mathfrak{h}^*$ and assume that γ satisfies* (2.38). *Then, in the module of fractions $S^{-1}M(\lambda)$,*

$$e_j F(I_1, \gamma) v_\lambda = 0,$$

for $j = 1, \ldots, n$. In particular, if $F(I_1, \gamma) \in \mathcal{U}(\mathfrak{n}_-)$ then $F(I_1, \gamma) v_\lambda$ is a singular vector in $M(\lambda)$.

Before we proceed with the proof of this proposition, we introduce some additional notation and prove a simple lemma. We let

$$F_{M+1}(I_1, \gamma) = 1, \tag{2.40}$$

and, for $k = M, \ldots, 2, 1$, we put

$$F_k(I_1, \gamma) = \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(F_{k+1}(I_1, \gamma)\big). \tag{2.41}$$

Furthermore, when the reference to I_1 and γ is understood, we let

$$F_k = F_k(I_1, \gamma), \tag{2.42}$$

which in particular means that $F_1 = F(I_1, \gamma)$.

Lemma 2.23. *For every $k \in \{1, \ldots, M\}$ the following holds:*

(i) *F_k is homogeneous of degree*

$$-(ms_k \alpha_{i_k} + \ldots + ms_M \alpha_{i_M});$$

(ii) *if $F_{k+1} \in \mathcal{U}(\mathfrak{n}_-)$ and $i_k \neq i_\ell$ for $\ell > k$, then $F_k \in \mathcal{U}(\mathfrak{n}_-)$ and*

$$F_k = \Big[\sum_{j=0}^{ms_k} \binom{\gamma_k + j - 1}{j} \mathcal{L}(f_{i_k}^{ms_k - j}) (\operatorname{ad} f_{i_k})^j\Big](F_{k+1}). \tag{2.43}$$

Proof. (i) This is immediate from Lemma 2.14 (i).
(ii) Apparently this holds for $k = M$. If $k < M$ then, by definition, s_k is given by

$$s_k = (-a_{i_{k+1},i_k})s_{k+1} + \ldots + (-a_{i_M,i_k})s_M. \tag{2.44}$$

Considering the fact that $(\mathrm{ad}\, f_{i_k})^{-a_{i_\ell,i_k}+1}(f_{i_\ell}) = 0$, for $\ell \neq k$, the first part of the lemma together with (2.44) clearly imply that

$$(\mathrm{ad}\, f_{i_k})^{ms_k+1}(F_{k+1}) = 0.$$

From (2.15) we obtain (2.43), which in turn shows that $F_k \in \mathcal{U}(\mathfrak{n}_-)$. □

Proof of Proposition 2.22. To prove this we will commute e_j through the operators making up $F(I_1, \gamma)$. In this process we will repeatedly make use of the results of Lemma 2.14 (iii) and (iv), which state that

$$A_\gamma^{(i)}(e_j) = e_j - \delta_{ij}\gamma f_i^{-1}(\alpha_i^\vee + \gamma + 1)$$

and that

$$A_\gamma^{(i)}(\alpha_j^\vee) = \alpha_j^\vee + \gamma a_{ij}.$$

We will show by downward induction, for $k = M, \ldots, 2, 1$, that

$$e_j F_k = F_k e_j + \sum_{\ell=k}^{M} U_\ell^{(k)}\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k}^{\ell-1} \gamma_r a_{i_r,i_\ell} + \gamma_\ell\Big), \tag{2.45}$$

for some elements $U_\ell^{(k)} \in \mathcal{U}(\mathfrak{g})S^{-1}$ ($\ell = k, \ldots, M$). For $k = M$, we we get that

$$\begin{aligned} e_j F_M &= e_j f_{i_M}^m = f_{i_M}^m\big(e_j - \delta_{i_M,j}(-m) f_{i_M}^{-1}(\alpha_{i_M}^\vee - m + 1)\big) \\ &= F_M e_j + \delta_{i_M,j} m f_{i_M}^{m-1}(\alpha_{i_M}^\vee + \gamma_M + 1), \end{aligned}$$

whence (2.45) holds in this case. Now assume that $1 \le k < M$. Then

$$\begin{aligned} &e_j F_k = e_j \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(F_{k+1}\big) \\ &= \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(\big(e_j - \delta_{i_k,j}(-ms_k - \gamma_k) f_{i_k}^{-1}(\alpha_{i_k}^\vee - ms_k - \gamma_k + 1)\big)F_{k+1}\big) \\ &= \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(e_j F_{k+1}\big) \\ &\quad + \delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1}) A_{\gamma_k}^{(i_k)}\big((\alpha_{i_k}^\vee - ms_k - \gamma_k + 1)F_{k+1}\big). \end{aligned} \tag{2.46}$$

We consider the two terms making up the expression in (2.46) separately. By induction, the first of these terms can be written as

$$
\begin{aligned}
&\mathcal{L}(f_{i_k}^{ms_k})A_{\gamma_k}^{(i_k)}\big(e_j F_{k+1}\big) \\
&= \mathcal{L}(f_{i_k}^{ms_k})A_{\gamma_k}^{(i_k)}\Big(F_{k+1}e_j + \sum_{\ell=k+1}^{M} U_\ell^{(k+1)}\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k+1}^{\ell-1}\gamma_r a_{i_r,i_\ell} + \gamma_\ell\Big)\Big) \\
&= \mathcal{L}(f_{i_k}^{ms_k})A_{\gamma_k}^{(i_k)}\big(F_{k+1}\big)(e_j - \delta_{i_k,j}\gamma_k f_{i_k}^{-1}(\alpha_{i_k}^\vee + \gamma_k + 1)) \\
&\quad + \sum_{\ell=k+1}^{M} \mathcal{L}(f_{i_k}^{ms_k})A_{\gamma_k}^{(i_k)}\big(U_\ell^{(k+1)}\big)\cdot \\
&\quad \cdot\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k+1}^{\ell-1}\gamma_r a_{i_r,i_\ell} + \gamma_\ell + \gamma_k a_{i_k,i_\ell}\Big) \\
&= F_k e_j - \delta_{i_k,j}\gamma_k\mathcal{L}(f_{i_k}^{ms_k-1})A_{\gamma_k+1}^{(i_k)}\big(F_{k+1}\big)(\alpha_{i_k}^\vee + \gamma_k + 1) \\
&\quad + \sum_{\ell=k+1}^{M} \mathcal{L}(f_{i_k}^{ms_k})A_{\gamma_k}^{(i_k)}\big(U_\ell^{(k+1)}\big)\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k}^{\ell-1}\gamma_r a_{i_r,i_\ell} + \gamma_\ell\Big).
\end{aligned} \tag{2.47}
$$

The second term of (2.46) is given by

$$
\begin{aligned}
&\delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1})A_{\gamma_k}^{(i_k)}\big((\alpha_{i_k}^\vee - ms_k - \gamma_k + 1)F_{k+1}\big) \\
&= \delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1}) \\
&\quad A_{\gamma_k}^{(i_k)}\Big(F_{k+1}\Big(\alpha_{i_k}^\vee - ms_k - \gamma_k + 1 + \sum_{r=k+1}^{M} ms_r a_{i_r,i_k}\Big)\Big) \\
&= \delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1})A_{\gamma_k}^{(i_k)}\big(F_{k+1}(\alpha_{i_k}^\vee - \gamma_k + 1)\big) \\
&= \delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1})A_{\gamma_k}^{(i_k)}\big(F_{k+1}\big)(\alpha_{i_k}^\vee - \gamma_k + 1 + 2\gamma_k) \\
&= \delta_{i_k,j}(ms_k + \gamma_k)\mathcal{L}(f_{i_k}^{ms_k-1})A_{\gamma_k}^{(i_k)}\big(F_{k+1}\big)(\alpha_{i_k}^\vee + \gamma_k + 1),
\end{aligned} \tag{2.48}
$$

where the first equality follows from Lemma (2.23) (i), and the second equality is a consequence of the definition of the s_ℓ integers (cf. (2.39)). Adding together the results of (2.47) and (2.48), we obtain that

$$\begin{aligned}
e_j F_k &= F_k e_j + \sum_{\ell=k+1}^{M} \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(U_\ell^{(k+1)}\big)\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k}^{\ell-1} \gamma_r a_{i_r,i_\ell} + \gamma_\ell\Big) \\
&\quad + \delta_{i_k,j}\mathcal{L}(f_{i_k}^{ms_k-1})\big((ms_k+\gamma_k)A_{\gamma_k}^{(i_k)}(F_{k+1}) - \gamma_k A_{\gamma_k+1}^{(i_k)}(F_{k+1})\big)\cdot \\
&\quad \cdot (\alpha_{i_k}^\vee + \gamma_k + 1) \\
&= F_k e_j + \sum_{\ell=k}^{M} U_\ell^{(k)}\Big(\alpha_{i_\ell}^\vee + 1 + \sum_{r=k}^{\ell-1}\gamma_r a_{i_r,i_\ell} + \gamma_\ell\Big),
\end{aligned}$$

where

$$U_\ell^{(k)} = \mathcal{L}(f_{i_k}^{ms_k}) A_{\gamma_k}^{(i_k)}\big(U_\ell^{(k+1)}\big),$$

for $\ell = k+1, \ldots, M$, and

$$U_k^{(k)} = \delta_{i_k,j}\mathcal{L}(f_{i_k}^{ms_k-1})\big((ms_k+\gamma_k)A_{\gamma_k}^{(i_k)}(F_{k+1}) - \gamma_k A_{\gamma_k+1}^{(i_k)}(F_{k+1})\big).$$

Thus, equation (2.45) is true for all $k \in \{1, \ldots, M\}$. The defining equations of the parameters γ_ℓ (cf. (2.38)) state that

$$\begin{aligned}
& -\gamma_1 a_{i_1,i_\ell} - \ldots - \gamma_{\ell-1} a_{i_{\ell-1},i_\ell} - \gamma_\ell = (\lambda+\rho)(\alpha_{i_\ell}^\vee) \\
\Longleftrightarrow \qquad & \lambda(\alpha_{i_\ell}^\vee) + 1 + \sum_{r=1}^{\ell-1}\gamma_r a_{i_r,i_\ell} + \gamma_\ell = 0,
\end{aligned}$$

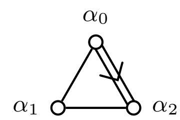
for $\ell = 1, \ldots, M$. Hence, with $k=1$, it follows from (2.45) that

$$e_j F(I_1,\gamma) v_\lambda = e_j F_1 v_\lambda = 0. \qquad \square$$

We conclude this section with an example, which suggests that it may be possible to generalize the method described in this section for obtaining singular vectors to the case of non-symmetrizable Kac–Moody algebras. Indeed, the only obstruction to such a generalization is the fact that, if $\mathfrak{g}$ is non-symmetrizable, then the ring of fractions $\mathcal{U}(\mathfrak{g})S^{-1}$ has not been proven to exist.

Example 2.24. Let $\mathfrak{g}$ be the non-symmetrizable Kac–Moody algebra with generalized Cartan matrix A and affiliated Coxeter–Dynkin diagram given by

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



We consider the situation in which the index tuple I is of the form

$$I = (i_1, \ldots, i_N) = (2, 1, 0, 1, 2),$$

which means that $I_1 = (2, 1, 0)$ consists of distinct entries. We get that

$$\alpha = r_{\alpha_2}(r_{\alpha_1}(\alpha_0)) = r_{\alpha_2}(\alpha_0 + \alpha_1) = \alpha_0 + \alpha_1 + 3\alpha_2,$$

and hence that

$$s_2 = 1 \quad \text{and} \quad s_1 = 3.$$

We let the middle exponent be equal to 1, i.e.

$$\gamma_M = \gamma_3 = -1.$$

The matrices C_A^I and $C_A^{I_1}$ are given by

$$C_A^I = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 \\ -2 & 1 & 2 & 1 & -1 \end{pmatrix} \quad \text{and} \quad C_A^{I_1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

By Lemma 2.18 the equations determining γ_1 and γ_2 are then expressed by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \\ 1 & -2 & 1 & -1 & 0 \\ -2 & 1 & 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -1 \\ -1 - \gamma_2 \\ -3 - \gamma_1 \end{pmatrix} = \begin{pmatrix} (\lambda + \rho)(\alpha_2^\vee) \\ (\lambda + \rho)(\alpha_1^\vee) \\ (\lambda + \rho)(\alpha_0^\vee) \\ (\lambda + \rho)(\alpha_1^\vee) \\ (\lambda + \rho)(\alpha_2^\vee) \end{pmatrix},$$

which can be written more succinctly as

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -1 \end{pmatrix} = \begin{pmatrix} (\lambda + \rho)(\alpha_2^\vee) \\ (\lambda + \rho)(\alpha_1^\vee) \\ (\lambda + \rho)(\alpha_0^\vee) \end{pmatrix}. \tag{2.49}$$

With $I_1 = (2,1,0)$ and $\gamma = (\gamma_1, \gamma_2, -1)$, we now define $F_1 = F(I_1, \gamma)$ as it would have been defined if we had shown that S is an Ore set $\mathcal{U}(\mathfrak{g})$. Since all the indices in I_1 are distinct, it is clear from the outset that defining F_1 in this way will yield an element in $\mathcal{U}(\mathfrak{g})$. Putting quotation marks around expressions which are not well-defined for the present Lie algebra $\mathfrak{g}$, but serve as motivations for the definition, we let F_1 be given by

$$\begin{aligned} F_1 &= \text{``}\mathcal{L}_{f_2^3} A_{\gamma_1}^{(2)} \mathcal{L}_{f_1} A_{\gamma_2}^{(1)}\text{''}(f_0) = \\ &= \Big(\sum_{j=0}^{3} \binom{\gamma_1 + j - 1}{j} \mathcal{L}_{f_2^{3-j}} (\mathrm{ad}\, f_2)^j \Big) \cdot \\ &\quad \cdot \Big(\sum_{j=0}^{1} \binom{\gamma_2 + j - 1}{j} \mathcal{L}_{f_1^{1-j}} (\mathrm{ad}\, f_1)^j \Big)(f_0). \end{aligned}$$

This means that

$$\begin{aligned} F_1 &= \text{``}\mathcal{L}_{f_2^3} A_{\gamma_1}^{(2)} \mathcal{L}_{f_1} A_{\gamma_2}^{(1)}\text{''}(f_0) = \text{``}\mathcal{L}_{f_2^3} A_{\gamma_1}^{(2)}\text{''}\Big(f_1 f_0 + \binom{\gamma_2}{1}[f_1, f_0]\Big) \\ &= f_2^3 f_1 f_0 + \binom{\gamma_2}{1} f_2^3 [f_1, f_0] + \binom{\gamma_1}{1} f_2^2 [f_2, f_1] f_0 + \binom{\gamma_1}{1} f_2^2 f_1 [f_2, f_0] \\ &\quad + \binom{\gamma_1}{1}\binom{\gamma_2}{1} f_2^2 [f_2, [f_1, f_0]] + 2\binom{\gamma_1 + 1}{2} f_2 [f_2, f_1][f_2, f_0] \\ &\quad + \binom{\gamma_1 + 1}{2} f_2 f_1 [f_2, [f_2, f_0]] + \binom{\gamma_1 + 1}{2}\binom{\gamma_2}{1} f_2 [f_2, [f_2, [f_1, f_0]]] \\ &\quad + 3\binom{\gamma_1 + 2}{3} [f_2, f_1][f_2, [f_2, f_0]] \\ &\quad + \binom{\gamma_1 + 2}{3}\binom{\gamma_2}{1} [f_2, [f_2, [f_2, [f_1, f_0]]]]. \end{aligned}$$

Now we show that, for $j = 0, 1, 2$, we have $e_j F_1 v_\lambda = 0$, if $\lambda \in \mathfrak{h}^*$ is such that (2.49) holds.

For $j = 0$ we get that

$$e_0 F_1 v_\lambda = c_1^{(0)} f_2^3 f_1 v_\lambda + c_2^{(0)} f_2^2 [f_2, f_1] v_\lambda,$$

where

$$c_1^{(0)} = \lambda(\alpha_0^\vee) - \gamma_2 - \gamma_1 = (\lambda + \rho)(\alpha_0^\vee) - \gamma_1 - \gamma_2 - 1,$$

$$\begin{aligned}
c_2^{(0)} &= \gamma_1\lambda(\alpha_0^\vee) + \gamma_1 - \gamma_1\gamma_2 - 2\binom{\gamma_1+1}{2} \\
&= \gamma_1(\lambda(\alpha_0^\vee) + 1 - \gamma_2 - (\gamma_1+1)) = \gamma_1((\lambda+\rho)(\alpha_0^\vee) - \gamma_1 - \gamma_2 - 1).
\end{aligned}$$

For $j = 1$ we obtain that

$$e_1F_1v_\lambda = c_1^{(1)}f_2^3f_0v_\lambda + c_2^{(1)}f_2^2[f_2,f_0]v_\lambda + c_3^{(1)}f_2[f_2,[f_2,f_0]]v_\lambda,$$

where

$$\begin{aligned}
c_1^{(1)} &= \lambda(\alpha_1^\vee) + 1 + \gamma_2 - \gamma_1 = (\lambda+\rho)(\alpha_1^\vee) - \gamma_1 + \gamma_2, \\
c_2^{(1)} &= \gamma_1\lambda(\alpha_1^\vee) + 2\gamma_1 + \gamma_1\gamma_2 - 2\binom{\gamma_1+1}{2} \\
&= \gamma_1(\lambda(\alpha_1^\vee) + 2 + \gamma_2 - (\gamma_1+1)) = \gamma_1((\lambda+\rho)(\alpha_1^\vee) - \gamma_1 + \gamma_2), \\
c_3^{(1)} &= \binom{\gamma_1+1}{2}\lambda(\alpha_1^\vee) + 3\binom{\gamma_1+1}{2} + \binom{\gamma_1+1}{2}\gamma_2 - 3\binom{\gamma_1+2}{3} \\
&= \binom{\gamma_1+1}{2}(\lambda(\alpha_1^\vee) + 3 + \gamma_2 - (\gamma_1+2)) \\
&= \binom{\gamma_1+1}{2}((\lambda+\rho)(\alpha_1^\vee) - \gamma_1 + \gamma_2).
\end{aligned}$$

Finally, for $j = 2$ we get that

$$\begin{aligned}
e_2F_1v_\lambda = {} & c_1^{(3)}f_2^2f_1f_0v_\lambda + c_2^{(3)}f_2^2[f_1,f_0]v_\lambda + c_3^{(3)}f_2[f_2,[f_1,f_0]]v_\lambda \\
&+ c_4^{(3)}[f_2,f_1][f_2,f_0]v_\lambda + c_5^{(3)}f_2f_1[f_2,f_0]v_\lambda + c_6^{(3)}f_2[f_2,f_1]f_0v_\lambda \\
&+ c_7^{(3)}f_1[f_2,[f_2,f_0]]v_\lambda + c_8^{(3)}[f_2,[f_2,[f_1,f_0]]]v_\lambda,
\end{aligned}$$

where

$$\begin{aligned}
c_1^{(3)} &= 3\lambda(\alpha_2^\vee) + (-1+1+3) + \gamma_1 + 2\gamma_1 = 3((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_2^{(3)} &= 3\gamma_2\lambda(\alpha_2^\vee) + \gamma_2(-1+1+3) + 3\gamma_1\gamma_2 = 3\gamma_2((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_3^{(3)} &= 2\gamma_1\gamma_2\lambda(\alpha_2^\vee) + \gamma_1\gamma_2(-1+1) + \binom{\gamma_1+1}{2}\gamma_2(1+3) \\
&= 2\gamma_1\gamma_2((\lambda+\rho)(\alpha_2^\vee) + \gamma_1),
\end{aligned}$$

$$
\begin{aligned}
c_4^{(3)} &= 2\binom{\gamma_1+1}{2}\lambda(\alpha_2^\vee) - 2\binom{\gamma_1+1}{2} + 3\binom{\gamma_1+2}{3}(0+2) \\
&= 2\binom{\gamma_1+1}{2}(\lambda(\alpha_2^\vee) - 1 + \gamma_1 + 2) = 2\binom{\gamma_1+1}{2}((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_5^{(3)} &= 2\gamma_1\lambda(\alpha_2^\vee) + \gamma_1(-1+1) + 2\binom{\gamma_1+1}{2} + \binom{\gamma_1+1}{2}(0+2) \\
&= 2\gamma_1((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_6^{(3)} &= 2\gamma_1\lambda(\alpha_2^\vee) + \gamma_1(-1+1) + 2\binom{\gamma_1+2}{3}\cdot 2 \\
&= 2\gamma_1((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_7^{(3)} &= \binom{\gamma_1+1}{2}\lambda(\alpha_2^\vee) + \binom{\gamma_1+1}{2}\cdot(-1) + 3\binom{\gamma_1+2}{3}\cdot 1 \\
&= \binom{\gamma_1+1}{2}(\lambda(\alpha_2^\vee) - 1 + \gamma_1 + 2) = \binom{\gamma_1+1}{2}((\lambda+\rho)(\alpha_2^\vee) + \gamma_1), \\
c_8^{(3)} &= \binom{\gamma_1+1}{2}\gamma_2\lambda(\alpha_2^\vee) + \binom{\gamma_1+1}{2}\gamma_2\cdot(-1) \\
&\quad + \binom{\gamma_1+2}{3}\gamma_2(-1+1+3) = \binom{\gamma_1+1}{2}\gamma_2(\lambda(\alpha_2^\vee) - 1 + \gamma_1 + 2) \\
&= \binom{\gamma_1+1}{2}\gamma_2((\lambda+\rho)(\alpha_2^\vee) + \gamma_1).
\end{aligned}
$$

In view of (2.49) we see that $c_i^{(j)} = 0$ for all i and j.

2.5 Proof of Theorem 2.19

We start this section by proving a preparatory lemma which describes an observation that is fundamental to the proof of Theorem 2.19. Since the formal statement of this lemma turns out to be a rather complicated way of expressing a simple principle, it is useful to start with a brief exemplification of its basic idea.

Let $\mathfrak{g}$ be the Kac–Moody algebra in Example 2.24 at the end of the previous section, and let $x \in \mathcal{U}(\mathfrak{n}_-)$ be given by

$$
\begin{aligned}
x &= [f_2, [f_0, f_1]] f_0^2 f_1 [[f_0, [f_0, f_2]], f_1] \\
&= [f_2, (\operatorname{ad} f_0)(f_1)] f_0^2 f_1 [(\operatorname{ad} f_0)^2(f_2), f_1].
\end{aligned}
$$

Notice that the occurrences of f_0 in the expression making up x are either written as $(\mathrm{ad}\, f_0)^k(f_j)$ (for $j = 1, 2$ and $k \in \mathbb{N}$) or in the form f_0^k ($k \in \mathbb{N}$). Consider the locally nilpotent action of $\mathrm{ad}\, f_0$ on $\mathcal{U}(\mathfrak{n}_-)$, and recall that $\mathrm{ad}\, f_0$ is a derivation of $\mathcal{U}(\mathfrak{n}_-)$, both with respect to associative multiplication and the Lie bracket. Letting $(\mathrm{ad}\, f_0)^4$ act on x, we thus obtain that

$$\begin{aligned}(\mathrm{ad}\, f_0)^4(x) = \frac{4!}{2! \cdot 1! \cdot 1!}&[(\mathrm{ad}\, f_0)^2(f_2), (\mathrm{ad}\, f_0)(f_1)] f_0^2 (\mathrm{ad}\, f_0)(f_1) \cdot \\ &\cdot [(\mathrm{ad}\, f_0)^2(f_2), (\mathrm{ad}\, f_0)(f_1)],\end{aligned}$$

whereas $(\mathrm{ad}\, f_0)^k(x) = 0$ for $k > 4$.

We will now specify this reasoning in more precise and general formulations. Let Z be the set given by

$$Z = \{1, \dots, n\} \times \mathbb{N},$$

and let G be the free abelian group on Z. Define $\mathfrak{f}$ to be the free Lie algebra on the set

$$\{\, f_j^{(k)} : (j,k) \in Z \,\}.$$

Given $i \in \{1, \dots, n\}$, we then let $\kappa_i \colon \mathcal{U}(\mathfrak{f}) \to \mathcal{U}(\mathfrak{n}_-)$ be the algebra homomorphism determined by

$$\kappa_i(f_j^{(k)}) = (\mathrm{ad}\, f_i)^k(f_j),$$

for $(j,k) \in Z$, which clarifies the idea behind the construction of $\mathfrak{f}$. We regard $\mathfrak{f}$ as a G-graded Lie algebra in the natural way. For $i \in \{1, \dots, n\}$, let $\tau_i \colon G \to \mathbb{Z}$ be the group homomorphism satisfying

$$\tau_i(j,k) = \begin{cases} \max\{-a_{ij} - k, 0\} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}.$$

Then τ_i induces a $\mathbb{Z}$-gradation of $\mathfrak{f}$, which, via the mapping κ_i, is evidently related to the nilpotency of $\mathrm{ad}\, f_i$.

Retaining the notation just introduced, we have the following lemma.

Lemma 2.25. *Let $i \in \{1, \dots, n\}$ and $p \in \mathbb{N}$, and consider the diagram*

$$\begin{tikzcd} \mathcal{U}(\mathfrak{f}) \arrow[r, "\phi_i"] \arrow[d, "\kappa_i"'] & \mathcal{U}(\mathfrak{f}) \arrow[d, "\kappa_i"] \\ \mathcal{U}(\mathfrak{n}_-) \arrow[r, "(\mathrm{ad}\, f_i)^p"'] & \mathcal{U}(\mathfrak{n}_-) \end{tikzcd}$$

where ϕ_i is the algebra homomorphism determined by

$$\phi_i(f_j^{(k)}) = f_j^{(\max\{-a_{ij},k\})},$$

for $(j,k) \in Z$. Assume that $y \in G$ is given by

$$y = \sum_{z \in Z} c_z z,$$

for some $c_z \in \mathbb{N}$, and let $\mathcal{U}(\mathfrak{f})_y$ denote the homogeneous subspace of $\mathcal{U}(\mathfrak{f})$ of degree y. If

(i) $p > \tau_i(y)$ *then* $\big((\mathrm{ad}\, f_i)^p \circ \kappa_i\big)|_{\mathcal{U}(\mathfrak{f})_y}$ *is the zero map;*

(ii) $p = \tau_i(y)$ *then* $\big((\mathrm{ad}\, f_i)^p \circ \kappa_i\big)|_{\mathcal{U}(\mathfrak{f})_y} = a_y(\kappa_i \circ \phi_i)|_{\mathcal{U}(\mathfrak{f})_y}$*, where $a_y \in \mathbb{Z}^+$ is the multinomial coefficient given by*

$$a_y = \frac{\tau_i(y)!}{\prod_{z \in Z} \big(\tau_i(z)!\big)^{c_z}}.$$

Proof. Since $\mathrm{ad}\, f_i$ is a derivation of $\mathcal{U}(\mathfrak{n}_-)$ (both regarded as an associative algebra and as a Lie algebra), this result follows immediately from the fact that $(\mathrm{ad}\, f_i)^{-a_{ij}+1}(f_j) = 0$, for $i \neq j$, and some elementary combinatorics. $\square$

We are now ready to prove our main theorem concerning the question of whether $F(I_1,\gamma)$ belongs to $\mathcal{U}(\mathfrak{n}_-)$. Throughout the proof we will employ the simplified notation for $F(I_1,\gamma)$ and its constituting components, which was introduced in (2.40), (2.41) and (2.42).

Proof of Theorem 2.19. We will prove this theorem by induction on $|T|$. If $|T| = 0$ the result follows immediately from repeated application of Lemma 2.23 (ii). Assume that $|T| > 0$, and let (q,r) be the pair belonging to T such that

$$q = \min\{\, q' : (q',r') \in T \text{ for some } r' \,\}.$$

If we prove that $F_q \in \mathcal{U}(\mathfrak{n}_-)$, then Lemma 2.23 (ii) shows that F_1 also belongs to $\mathcal{U}(\mathfrak{n}_-)$. Without loss of generality, we therefore suppose that $q = 1$. If $(q',M) \in T$ for some q', then $m = 1$ by assumption. On the other hand, if $(q',M) \notin T$ for any q', then Proposition 2.20 provides a way to reduce to the case when $m = 1$.

Thus, we assume that $m = 1$. (Throughout the rest of the proof, special note should be taken to check that the arguments hold in the particular case when $r = M$.)

By induction, assume that $F_{r+1} \in \mathcal{U}(\mathfrak{n}_-)$. The assumptions on the set T and Lemma 2.23 (ii) then implies that we also have that $F_2 \in \mathcal{U}(\mathfrak{n}_-)$ and that

$$\begin{aligned} F_2 &= \big(\mathcal{L}(f_{i_2}^{s_2}) A_{\gamma_2}^{(i_2)} \cdots \mathcal{L}(f_{i_r}^{s_r}) A_{\gamma_r}^{(i_r)}\big)(F_{r+1}) \\ &= \Big(\prod_{\ell=2}^{r} \Big[\sum_{j=0}^{s_\ell} \binom{\gamma_\ell + j - 1}{j} \mathcal{L}(f_{i_\ell}^{s_\ell - j})(\operatorname{ad} f_{i_\ell})^j\Big]\Big)(F_{r+1}). \end{aligned}$$

Since $s_r = 1$, we obtain that

$$\mathcal{L}(f_{i_r}^{s_r}) A_{\gamma_r}^{(i_r)} = \mathcal{L}(f_{i_r}) A_{\gamma_r}^{(i_r)} = \mathcal{R}(f_{i_r}) A_{\gamma_r+1}^{(i_r)} = \mathcal{R}(f_{i_r}) (A_{-\gamma_r-1}^{(i_r)})^{-1},$$

and, in view of (2.16), this implies that

$$F_r = \mathcal{L}(f_{i_r}^{s_r}) A_{\gamma_r}^{(i_r)}(F_{r+1}) = F_{r+1} f_{i_r} + (-\gamma_r - 1)(-\operatorname{ad} f_{i_r})(F_{r+1}). \quad (2.50)$$

Let $\Theta_j^{(\ell)}$ be the operator given by

$$\Theta_j^{(\ell)} = \binom{\gamma_\ell + j - 1}{j} \mathcal{L}(f_{i_\ell}^{s_\ell - j})(\operatorname{ad} f_{i_\ell})^j,$$

and, for $k = 2, \ldots, r-1$, let J_k denote the set

$$J_k = \big\{ (j_k, \ldots, j_{r-1}) : j_\ell \in \{0, \ldots, s_\ell\} \big\}.$$

Then we can express F_k as

$$F_k = \Big(\prod_{\ell=k}^{r-1} \Big[\sum_{j=0}^{s_\ell} \Theta_j^{(\ell)}\Big]\Big)(F_r) = \Big(\sum_{(j_k, \ldots, j_{r-1}) \in J_k} \Big[\prod_{\ell=k}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big]\Big)(F_r). \quad (2.51)$$

We will write F_k, for $k = 2, \ldots, r-1$, as a sum of terms in a certain way. This will be done by successively expanding products of the form

$$\Big[\prod_{\ell=k}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big](F_r), \quad \text{where } (j_k, \ldots, j_{r-1}) \in J_k, \quad (2.52)$$

according to a specific algorithm. We will now describe how to do this. Let X be the set

$$X = \{\, [f_{i_k}, [f_{i_k}, f_{i_r}]] : k \in \{2, \dots, r-1\} \,\},$$

and let L be the ideal of $\mathcal{U}(\mathfrak{n}_-)$ generated by X, i.e.

$$L = \langle X \rangle \subset \mathcal{U}(\mathfrak{n}_-).$$

The terms that are to make up F_k will be determined modulo L. They will be of four different types, called (A), (B), (C), and (D), respectively. Terms of type (A) will belong to the set $\mathcal{U}(\mathfrak{n}_-)f_{i_r}$. Additionally, the terms of type (B) and (C) will be further subdivided into the types (B_ℓ) and (C_ℓ), for $\ell \in \{k, \dots, r-1\}$, respectively.

Assume now that we have expressed F_{k+1}, modulo L, as a sum of terms of the above-mentioned types. To obtain the terms of F_k we have to let operators of the form $\Theta^{(k)}_{j_k}$, where $0 \le j_k \le s_k$, act on the terms of F_{k+1}, and decide how to express the result as a sum of terms and of which types these terms will be. This is done as follows.

We write the result of the action of $\Theta^{(k)}_{j_k}$ on a term of type (B_ℓ), (C_ℓ) or (D) as a single term, and define it to again be a term of the same respective type. To determine how $\Theta^{(k)}_{j_k}$ transforms a term of type (A), we observe that, by induction, we have for every integer $j \ge 1$ that

$$\begin{aligned}(\operatorname{ad} f_{i_k})^j(yf_{i_r} + L) = (\operatorname{ad} f_{i_k})^j(y)f_{i_r} &- j(\operatorname{ad}[f_{i_k}, f_{i_r}])(\operatorname{ad} f_{i_k})^{j-1}(y)\\ &+ j[f_{i_k}, f_{i_r}](\operatorname{ad} f_{i_k})^{j-1}(y) + L,\end{aligned}$$

for any $y \in \mathcal{U}(\mathfrak{n}_-)$. Thus, when $j_k \ge 1$ we let the result, modulo L, of the action of $\Theta^{(k)}_{j_k}$ on a term of type (A) of the form yf_{i_r} be the three terms

$$\binom{\gamma_k + j_k - 1}{j_k} f_{i_k}^{s_k - j_k}(\operatorname{ad} f_{i_k})^{j_k}(y)f_{i_r},$$

$$-j_k\binom{\gamma_k + j_k - 1}{j_k} f_{i_k}^{s_k - j_k}(\operatorname{ad}[f_{i_k}, f_{i_r}])(\operatorname{ad} f_{i_k})^{j_k-1}(y)$$

and

$$j_k\binom{\gamma_k + j_k - 1}{j_k} f_{i_k}^{s_k - j_k}[f_{i_k}, f_{i_r}](\operatorname{ad} f_{i_k})^{j_k-1}(y).$$

We define these terms to be, in order, of type (A), (B_k) and (C_k), respectively. Similarly, when $j_k = 0$ we also let a term of type (A) produce one term each of the types (A), (B_k) and (C_k) when $\Theta^{(k)}_{j_k}$ acts on it, but we let the terms of type (B_k) and (C_k) be equal to 0 in this case.

Now we have described how to go from the representation of F_{k+1} as a sum of specific terms modulo L to the corresponding representation for F_k. In view of (2.50) we define F_r to consist of one term of type (A) given by

$$F_{r+1} f_{i_r},$$

and one term of type (D) of the form

$$(-\gamma_r - 1)(-\operatorname{ad} f_{i_r})(F_{r+1}).$$

The discussion above then gives a recursive definition of F_k, for $k = r-1, \ldots, 2$, as a sum of terms modulo L of types (A), (B_ℓ), (C_ℓ) and (D), where $\ell \in \{k, \ldots, r-1\}$.

A term of F_k resulting from the expansion of (2.52) will be called a $(j_k, \ldots, j_{r-1})$-term. By convention, we define a $(j_k, \ldots, j_{r-1})$-term to be equal to 0 if $j_\ell > s_\ell$ for some $\ell \in \{k, \ldots, r-1\}$. From the description above, we see that for every $\xi \in J_k$, the expansion of (2.52) gives rise to one ξ-term each of the types (A) and (D), and a total of $r - k$ ξ-terms of the types (B) and (C), respectively. Furthermore, we see that the type (A) $(j_k, \ldots, j_{r-1})$-term is given by

$$\Big(\Big[\prod_{\ell=k}^{r-1} \Theta^{(\ell)}_{j_\ell}\Big](F_{r+1})\Big) f_{i_r}, \tag{2.53}$$

while the type (D) $(j_k, \ldots, j_{r-1})$-term is of the form

$$\Big[\prod_{\ell=k}^{r-1} \Theta^{(\ell)}_{j_\ell}\Big]\big((-\gamma_r - 1)(-\operatorname{ad} f_{i_r})(F_{r+1})\big). \tag{2.54}$$

We now organize the terms of F_k into collections of terms. Given $\xi = (j_k, \ldots, j_{r-1}) \in J_k$, we define the ξ-collection of terms to consist of all the ξ-terms of types (A), (B) and (D), together with each type (C_ℓ) $(j_k, \ldots, j_\ell + 1, \ldots, j_{r-1})$-term, for $\ell = k, \ldots, r-1$. (Recall the convention mentioned above.) In particular, the terms of F_2 are organized into collections. Thus we can write F_2 as

$$F_2 = \sum_{\xi \in J_2} \Delta_\xi + z, \tag{2.55}$$

where $z \in L$ and Δ_ξ is the sum of the terms in the ξ-collection.

Finally, in order to obtain F_1 we let $\mathcal{L}(f_{i_1}^{s_1}) A_{\gamma_1}^{(i_1)}$ act on F_2. We write this operator as $\mathcal{R}(f_{i_1}^{s_1})(A_{-\gamma_1-s_1}^{(i_1)})^{-1}$, and expand it as a series of "right operators" in accordance with (2.16). From Lemma 2.23 (i), we know that F_2 is homogeneous of degree

$$-(s_2\alpha_{i_2} + \ldots + s_M\alpha_{i_M}).$$

Since $(1, r) \in T$ and $s_r = 1$, we have that

$$s_1 = \sum_{k=2}^{M} (-a_{i_k,i_1}) s_k = -2 + \sum_{\substack{k=2\\k\neq r}}^{M} (-a_{i_k,i_1}) s_k.$$

In view of the assumption that no three entries of I_1 are equal, it follows that $(\operatorname{ad} f_{i_1})^{s_1+3}(F_2) = 0$. Consequently, we obtain that the action of $\mathcal{R}(f_{i_1}^{s_1})(A_{-\gamma_1-s_1}^{(i_1)})^{-1}$ on F_2 is given by

$$\Phi = \sum_{j=0}^{s_1+2} \binom{-\gamma_1 - s_1 + j - 1}{j} \mathcal{R}(f_{i_1}^{s_1-j})(-\operatorname{ad} f_{i_1})^j. \tag{2.56}$$

For ease of reference, we introduce the notation

$$\Phi_j = \binom{-\gamma_1 - s_1 + j - 1}{j} \mathcal{R}(f_{i_1}^{s_1-j})(-\operatorname{ad} f_{i_1})^j, \tag{2.57}$$

for $j = 0, \ldots, s_1 + 2$, so that for any $y \in \mathcal{U}(\mathfrak{n}_-)$ we get that

$$\Phi(y) = \sum_{j=0}^{s_1+2} \Phi_j(y). \tag{2.58}$$

We will now show that the result of the action of Φ on F_2 is equal to zero modulo $\mathcal{U}(\mathfrak{n}_-)$. Clearly, we have that

$$\Phi(y) \equiv \Phi_{s_1+1}(y) + \Phi_{s_1+2}(y) \pmod{\mathcal{U}(\mathfrak{n}_-)}, \tag{2.59}$$

for any $y \in \mathcal{U}(\mathfrak{n}_-)$.

First we consider the part of F_2 belonging to L. Let F_2 be represented by the expression in (2.55). By Lemma 2.25 (i), we see that $\Phi_{s_1+2}(z) = 0$. Likewise, Lemma 2.25 (ii) implies that $\Phi_{s_1+1}(z)$ is of the form $z' f_{i_1}^{-1}$, where z' is contained in the ideal generated by all

$$\big[(\operatorname{ad} f_{i_1})^{-a_{i_\ell,i_1}}(f_{i_\ell}), (\operatorname{ad} f_{i_1})^{-a_{i_\ell,i_1}}(f_{i_\ell})\big] = 0,$$

for $\ell = 2, \dots, r-1$, which in turn shows that $\Phi_{s_1+1}(z) = 0$.

Next, we examine what we get modulo $\mathcal{U}(\mathfrak{n}_-)$, when Φ acts on a collection of terms from F_2 of the sort described above. To this end, let $\xi = (j_2, \dots, j_{r-1}) \in J_2$, and let u, v_ℓ, w_ℓ and x be the terms of type (A), (B_ℓ), (C_ℓ) and (D), respectively, belonging to the ξ-collection. Also, let Δ_ξ be the sum of all the terms in this collection. Since $u \in \mathcal{U}(\mathfrak{n}_-) f_{i_r}$, it is immediately clear that

$$\Phi_{s_1+1}(u) \in \mathcal{U}(\mathfrak{n}_-),$$

while Lemma (2.25) (i) implies that

$$\Phi_{s_1+2}(v_\ell) = \Phi_{s_1+2}(w_\ell) = \Phi_{s_1+2}(x) = 0.$$

Thus, modulo $\mathcal{U}(\mathfrak{n}_-)$, we have that

$$\Phi(u) \equiv \Phi_{s_1+2}(u),$$

whereas

$$\Phi(v_\ell) \equiv \Phi_{s_1+1}(v_\ell), \quad \Phi(w_\ell) \equiv \Phi_{s_1+1}(w_\ell) \quad \text{and} \quad \Phi(x) \equiv \Phi_{s_1+1}(x).$$

Since u is given by the expression in (2.53) (with $k = 2$), we get from Lemma 2.25 (ii) that

$$\begin{aligned}\Phi_{s_1+2}(u) = {}& \frac{(s_1+2)!}{\prod_{\substack{k=2\\k\neq r}}^{M} \big((-a_{i_k,i_1})!\big)^{s_k}} \binom{-\gamma_1+1}{s_1+2} \cdot \\ & \cdot \Big(\Big[\prod_{\ell=2}^{r-1} \widehat{\Theta}_{j_\ell}^{(\ell)}\Big](\widehat{F}_{r+1})\Big) f_{i_r}^{-1}, \end{aligned} \tag{2.60}$$

where $\widehat{\Theta}^{(\ell)}_{j_\ell}$ is the operator given by

$$\binom{\gamma_\ell + j_\ell - 1}{j_\ell}\mathcal{L}\Big(\big((-\operatorname{ad} f_{i_1})^{-a_{i_\ell,i_1}}(f_{i_\ell})\big)^{s_\ell - j_\ell}\Big)\cdot$$
$$\cdot\Big(\operatorname{ad}\big((-\operatorname{ad} f_{i_1})^{-a_{i_\ell,i_1}}(f_{i_\ell})\big)\Big)^{j_\ell}$$

and

$$\widehat{F}_{r+1} = (-\operatorname{ad} f_{i_1})^{s_{r+1}}(F_{r+1}).$$

Let $\hat{u}$ be given by

$$\hat{u} = \Big(\Big[\prod_{\ell=2}^{r-1}\widehat{\Theta}^{(\ell)}_{j_\ell}\Big](\widehat{F}_{r+1})\Big)f_{i_r}^{-1}. \tag{2.61}$$

Considering the similarity in the structure of v_ℓ and u, we get from another application of Lemma 2.25 (ii) that

$$\Phi_{s_1+1}(v_\ell) = \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\big((-a_{i_k,i_1})!\big)^{s_k}}(-a_{i_\ell,i_1})\binom{-\gamma_1}{s_1+1}(-j_\ell)\hat{u}. \tag{2.62}$$

Furthermore, if we assume that $j_\ell < s_\ell$, which means that the above-mentioned convention does not apply to w_ℓ, analogous reasoning also yields that

$$\begin{aligned}\Phi_{s_1+1}(w_\ell) &= \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\big((-a_{i_k,i_1})!\big)^{s_k}}(-a_{i_\ell,i_1})\binom{-\gamma_1}{s_1+1}(j_\ell+1)\Big(\frac{\gamma_\ell+j_\ell}{j_\ell+1}\Big)\hat{u}\\ &= \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\big((-a_{i_k,i_1})!\big)^{s_k}}(-a_{i_\ell,i_1})\binom{-\gamma_1}{s_1+1}(\gamma_\ell+j_\ell)\hat{u}.\end{aligned} \tag{2.63}$$

To obtain the above expression for $\Phi_{s_1+1}(w_\ell)$ we have used the fact that

$$\prod_{k=2}^{r-1}\binom{\gamma_k + j_k - 1 + \delta_{k,\ell}}{j_k + \delta_{k,\ell}} = \Big(\frac{\gamma_\ell + j_\ell}{j_\ell+1}\Big)\prod_{k=2}^{r-1}\binom{\gamma_k + j_k - 1}{j_k}.$$

(Notice that w_ℓ is a $(j_2, \dots, j_\ell + 1, \dots, j_{r-1})$-term.)

Finally, as x is given by (2.54) (with $k = 2$), we get from Lemma 2.25 (ii) that

$$\Phi_{s_1+1}(x) = \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\big((-a_{i_k,i_1})!\big)^{s_k}}\Big(\sum_{\ell=r+1}^{M}(-a_{i_\ell,i_1})s_\ell\Big)\binom{-\gamma_1}{s_1+1}(-\gamma_r - 1)\hat{u}.$$

Since $i_1 = i_r$ we have that

$$\sum_{\ell=r+1}^{M}(-a_{i_\ell,i_1})s_\ell = \sum_{\ell=r+1}^{M}(-a_{i_\ell,i_r})s_\ell = s_r = 1,$$

and hence that

$$\Phi_{s_1+1}(x) = \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}\binom{-\gamma_1}{s_1+1}(-\gamma_r-1)\hat{u}. \tag{2.64}$$

Thus, under the assumption that $j_\ell < s_\ell$ for $\ell = 2, \ldots, r-1$, we get from (2.60), (2.62), (2.63) and (2.64) that the action of Φ on Δ_ξ is equal to $c\hat{u}$ modulo $\mathcal{U}(\mathfrak{n}_-)$, where

$$\begin{aligned}
c = {} & \frac{(s_1+2)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}\binom{-\gamma_1+1}{s_1+2} \\
& + \sum_{\ell=2}^{r-1}\frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}(-a_{i_\ell,i_1})\binom{-\gamma_1}{s_1+1}(-j_\ell) \\
& + \sum_{\ell=2}^{r-1}\frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}(-a_{i_\ell,i_1})\binom{-\gamma_1}{s_1+1}(\gamma_\ell+j_\ell) \\
& + \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}\binom{-\gamma_1}{s_1+1}(-\gamma_r-1) \\
= {} & \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}\binom{-\gamma_1}{s_1+1}\cdot \\
& \cdot\left((s_1+2)\left(\frac{-\gamma_1+1}{s_1+2}\right) + \sum_{\ell=2}^{r-1}(-a_{i_\ell,i_1})(-j_\ell+\gamma_\ell+j_\ell) - \gamma_r - 1\right) \\
= {} & \frac{(s_1+1)!}{\prod_{\substack{k=2\\k\neq r}}^{M}\left((-a_{i_k,i_1})!\right)^{s_k}}\binom{-\gamma_1}{s_1+1}\left(-\gamma_1 + \sum_{\ell=2}^{r-1}(-a_{i_\ell,i_1})\gamma_\ell - \gamma_r\right).
\end{aligned} \tag{2.65}$$

Since condition (**C**) is satisfied and $i_1 = i_r$, we have that

$$\gamma_1 = (-a_{i_2,i_1})\gamma_2 + \ldots + (-a_{i_{r-1},i_1})\gamma_{r-1} - \gamma_r.$$

Hence we see that c in (2.65) is equal to 0, which means that this collection of terms does not produce anything nonzero modulo $\mathcal{U}(\mathfrak{n}_-)$ in F_1.

We now notice that if $j_\ell = s_\ell$ and $a_{i_\ell,i_1} = 0$, then the terms v_ℓ and w_ℓ above are both equal to 0. This follows from the fact that v_ℓ contains a factor of the form $[f_{i_\ell}, f_{i_r}] = 0$, while $w_\ell = 0$ by convention. From this observation we see that the calculation in (2.65) also holds if the collection satisfies the condition that $j_k = s_k \Rightarrow a_{i_k,i_1} = 0$, for $k = 2, \ldots, r-1$.

It remains to consider the situation in which the $(j_2, \ldots, j_{r-1})$-collection is such that $j_k = s_k$ and $a_{i_k,i_1} < 0$ for some k. Assume that this is the case for $k = p$. We then obtain that $u = 0$. To see this, notice that u is of the form

$$u = \Big(\Big[\prod_{\ell=2}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big](F_{r+1})\Big) f_{i_r}. \tag{2.66}$$

Since $s_r = 1$, we have that

$$\Big[\prod_{\ell=p+1}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big](F_{r+1})$$

is homogeneous of degree

$$-\sum_{\substack{\ell=p+1 \\ \ell\neq r}}^{M} s_\ell \alpha_{i_\ell}.$$

Furthermore, because $a_{i_p,i_1} < 0$ implies that $a_{i_r,i_p} < 0$, we see that

$$s_p = \sum_{\ell=p+1}^{M} (-a_{i_\ell,i_p}) s_\ell > \sum_{\substack{\ell=p+1 \\ \ell\neq r}}^{M} (-a_{i_\ell,i_p}) s_\ell.$$

It follows that

$$\begin{aligned}\Big[\prod_{\ell=p}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big](F_{r+1}) f_{i_r} &= \binom{\gamma_p + s_p - 1}{s_p} (\operatorname{ad} f_{i_p})^{s_p} \Big(\Big[\prod_{\ell=p+1}^{r-1} \Theta_{j_\ell}^{(\ell)}\Big](F_{r+1})\Big) f_{i_r} \\ &= 0,\end{aligned}$$

whence also $u = 0$. We then obtain from Lemma 2.25 (ii) that $\hat{u} = 0$ (cf. (2.61) and (2.66)). From above, we know that $\Phi_{s_1+2}(u)$, $\Phi_{s_1+1}(v_\ell)$ and $\Phi_{s_1+1}(x)$ are all multiples of $\hat{u}$, and hence equal to 0. If $j_\ell < s_\ell$ this is also true for $\Phi_{s_1+1}(w_\ell)$, whereas if $j_\ell = s_\ell$ then $w_\ell = 0$ by convention. Consequently, $\Phi(\Delta_\xi) \equiv 0$ modulo $\mathcal{U}(\mathfrak{n}_-)$ in this case as well.

We have now seen that none of the collections of terms in F_2 make any contribution to F_1 modulo $\mathcal{U}(\mathfrak{n}_-)$, and that this is also true of the part of F_2 belonging to the ideal L. Thus, we conclude that $F_1 = F(I_1, \gamma) \in \mathcal{U}(\mathfrak{n}_-)$. $\square$

Chapter 3

Vertex operator algebras

The principal aim of this chapter is to study the irreducible modules in the category $\mathcal{O}$ for simple vertex operator algebras of affine type at admissible levels. Our main instrument is the theory afforded by Zhu's algebra. We will start, however, by presenting the required preliminary definitions and results concerning vertex operator algebras and their modules, as well as the concept of admissible weights.

3.1 Definition of vertex operator algebras and their modules

In order to define what a vertex operator algebra is, we need to first establish some notation and conventions about formal series in one and several variables. Thus, our aim is only to briefly discuss certain aspects of formal series that are relevant for this thesis. A thorough exposition of this topic can be found in for instance [LL04].

To this end, let A be an associative algebra over $\mathbb{C}$, and let $A[[z, z^{-1}]]$ be the vector space of formal Laurent series over A, i.e.

$$A[[z, z^{-1}]] = \Big\{ \sum_{n\in\mathbb{Z}} a_n z^n : a_n \in A \Big\}. \tag{3.1}$$

A sequence of elements of $A[[z, z^{-1}]]$ can not in general be multiplied together. To determine if the product of the sequence of series

$$a^{(i)}(z) = \sum_{n\in\mathbb{Z}} a_n^{(i)} z^n, \quad \text{for } i = 1, \ldots, m,$$

is defined, one considers the tentative expression

$$a^{(1)}(z) \cdot \ldots \cdot a^{(m)}(z) = \sum_{n\in\mathbb{Z}} \Big(\sum_{\substack{k_1,\ldots,k_m\in\mathbb{Z} \\ k_1+\ldots+k_m=n}} a_{k_1}^{(1)} \ldots a_{k_m}^{(m)} \Big) z^n. \tag{3.2}$$

If the coefficient of z^n on the right-hand side of (3.2) amounts to a well-defined element in A for every $n \in \mathbb{Z}$, then the product exists and is given by (3.2). (It is important to consider products of sequences of elements of $A[[z, z^{-1}]]$, since a “subproduct” of a well-defined product does not necessarily exist.)

Examples of multiplicative substructures of $A[[z, z^{-1}]]$ include the ring $A((z))$ of lower truncated Laurent series over A (i.e. those elements of (3.1) with $a_n = 0$ for all sufficiently large negative n), the ring $A[[z]]$ of formal power series over A, the ring $A[z, z^{-1}]$ of Laurent polynomials over A, and the ring $A[z]$ of polynomials over A.

Now, let B be a (left or right) A-module, and consider also the formal series in the vector space $B[[z, z^{-1}]]$. The product of a sequence of elements of $A[[z, z^{-1}]]$ with an element of $B[[z, z^{-1}]]$ (to the left or right as appropriate) is interpreted in the obvious way as a generalization of formula (3.2), and, as above, it may or may not yield a well-defined element of $B[[z, z^{-1}]]$.

The definitions just introduced readily carries over to formal expressions in more than one variable. An important aspect of series in several variables is what is called the expansion convention, which we will now describe. Let $\mathbb{K}$ be a field, and recall that the field of fractions of the power series ring $\mathbb{K}[[z]]$ is naturally identified with the field of lower truncated Laurent series $\mathbb{K}((z))$. This reasoning may be applied recursively to the field of fractions of the power series ring $\mathbb{K}[[z_1, \dots, z_n]]$ in the following way. Let $\mathbb{K}_0 = \mathbb{K}$, and define $\mathbb{K}_i$ inductively, for $i = 1, \dots, n$, by

$$\mathbb{K}_i = \mathbb{K}_{i-1}((z_i)).$$

This provides an isomorphism of the field of fractions of $\mathbb{K}[[z_1, \dots, z_n]]$ with $\mathbb{K}_n$, and thereby an embedding into the vector space $\mathbb{K}[[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]]$. The ordering of the formal variables $z_1, \dots, z_n$ is of course essential, and different orderings yield different embeddings. The expansion convention stipulates that, whenever the ordering of the variables is relevant, this should be clear from the way expressions are written. In the present context, the expansion convention will only be used for binomial expressions in two variables, whence it is essentially characterized by the two equations

$$(z_1 + z_2)^n = \sum_{k \in \mathbb{N}} \binom{n}{k} z_1^{n-k} z_2^k \quad \text{and} \quad (z_2 + z_1)^n = \sum_{k \in \mathbb{N}} \binom{n}{k} z_2^{n-k} z_1^k,$$

for $n \in \mathbb{Z}$.

A particular formal series playing an important part in vertex operator algebra is

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^n \in \mathbb{C}[[z]].$$

Particularly relevant are expressions of the form

$$\delta\Big(\frac{z_1 + z_2}{z_0}\Big). \tag{3.3}$$

The expansion convention makes it clear how (3.3) is to be interpreted as a formal series.

We now turn to the context which is relevant for vertex operator algebras. Let V be a complex vector space. The theory of formal series that we have briefly sketched will be applied to the algebra $\mathrm{End}(V)$, as well as to V regarded as a left $\mathrm{End}(V)$-module. Naturally, both V and $\mathrm{End}(V)$ are also considered as $\mathbb{C}$-bimodules.

In order to give meaning to certain infinite sums in $\mathrm{End}(V)$, the following convention is employed. Let $(f_j)_{j \in J} \subseteq \mathrm{End}(V)$ be a set of endomorphisms. If, for each $v \in V$, the element $f_j(v)$ is nonzero for only finitely many $j \in J$, then the sum of the family $(f_j)_{j \in J}$ is defined in the obvious way. In the theory of vertex operator algebras, this convention is frequently used to establish the existence of products in $(\mathrm{End}(V))[[z, z^{-1}]]$.

We have now laid the groundwork necessary to state the definition of a vertex operator algebra.

Definition 3.1. A vertex operator algebra consists of a quadruple

$$(V, Y, \mathbf{1}, \omega), \tag{3.4}$$

where V is a vector space,

$$Y \colon V \longrightarrow (\mathrm{End}(V))[[z, z^{-1}]] \tag{3.5}$$

is a linear map called the vertex operator map, and where $\mathbf{1}$ and ω are distinguished vectors in V referred to as the vacuum vector and the conformal vector, respectively. The image of $u \in V$ under Y is denoted by

$$Y(u, z) = \sum_{n \in \mathbb{Z}} u_n z^{-n-1}, \tag{3.6}$$

and $Y(u, z)$ is called the vertex operator associated with u. The constituents of a vertex operator algebra are required to satisfy the following conditions.

The vector space V is $\mathbb{Z}$-graded,

$$V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}, \tag{3.7}$$

with

$$\dim V_{(n)} < \infty \quad \text{for all } n \in \mathbb{Z}, \tag{3.8}$$

and

$$V_{(n)} = \{0\} \quad \text{for all sufficiently large negative } n \in \mathbb{Z}. \tag{3.9}$$

The gradation of V is said to be given by weights, and the weight of a homogeneous element $v \in V$ is denoted by $\mathrm{wt}\, v$.

For all u, $v \in V$ the truncation condition holds, which means that

$$u_n v = 0 \quad \text{for all sufficiently large } n \in \mathbb{Z}, \tag{3.10}$$

or, equivalently,

$$Y(u, z)v \in V((z)). \tag{3.11}$$

The vacuum vector $\mathbf{1}$ satisfies the vacuum property given by

$$Y(\mathbf{1}, z) = \mathrm{id}_V\,, \tag{3.12}$$

and, furthermore, for every $u \in V$

$$Y(u, z)\mathbf{1} \in V[[z]] \quad \text{and} \quad u_{-1}\mathbf{1} = u, \tag{3.13}$$

which is referred to as the creation property. The conformal vector ω is homogeneous of weight 2, and, with its vertex operator expressed as

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}, \tag{3.14}$$

the components $L(n)$, $n \in \mathbb{Z}$, satisfy the Virasoro algebra relations given by

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}c_V, \quad \text{for } m, n \in \mathbb{Z},$$

where $c_V \in \mathbb{C}$ is called the central charge of V. In addition, the operator $L(0)$ acts semisimply on V, and its eigenvalues correspond to the grading of V, i.e.

$$L(0)v = nv, \quad \text{for } n \in \mathbb{Z} \text{ and } v \in V_{(n)}. \tag{3.15}$$

Moreover, the operator $L(-1)$ satisfies the $L(-1)$-derivative property given by

$$Y(L(-1)u, z) = \frac{d}{dz}Y(u, z) \quad \text{for } u \in V. \tag{3.16}$$

Finally, the vertex operator algebra satisfies the Jacobi identity, which states that

$$\begin{aligned} z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)Y(v, z_2)Y(u, z_1) \\ = z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)Y(Y(u, z_0)v, z_2), \end{aligned} \tag{3.17}$$

for all u, $v \in V$.

Remark 3.2. If the assumptions about the grading of the vector space and the existence and properties of the conformal vector are removed from the above definition of a vertex operator algebra, the resulting algebraic structure is instead called just a vertex algebra. It can be shown that this definition of a vertex algebra is equivalent to the original axioms of a vertex algebra, as they were formulated by R. E. Borcherds in [Bor86] (see e.g. [LL04, pp. 90–91]).

We now fix a vertex operator algebra $(V, Y, \mathbf{1}, \omega)$, which will simply be referred to as V. Next, we will define some different notions of modules for V.

Definition 3.3. A weak V-module consists of a pair

$$(W, Y_W),$$

where W is a vector space and

$$Y_W : V \longrightarrow (\mathrm{End}(W))[[z, z^{-1}]]$$

is a linear map. The image of $u \in V$ under Y_W is denoted by

$$Y_W(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}, \tag{3.18}$$

and the following conditions are satisfied:

(i) the truncation condition:

$$Y_W(u,z)w \in W((z)) \quad \text{for all } u \in V \text{ and } w \in W; \tag{3.19}$$

(ii) the vacuum property:

$$Y_W(\mathbf{1},z) = \mathrm{id}_W\,; \tag{3.20}$$

(iii) the Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\Big(\frac{z_1-z_2}{z_0}\Big)Y_W(u,z_1)Y_W(v,z_2) - z_0^{-1}\delta\Big(\frac{z_2-z_1}{-z_0}\Big)Y_W(v,z_2)Y_W(u,z_1) \\ &\quad = z_2^{-1}\delta\Big(\frac{z_1-z_0}{z_2}\Big)Y_W(Y(u,z_0)v,z_2), \end{aligned} \tag{3.21}$$

for all u, $v \in V$.

Next we define the structure which in the literature is commonly called a vertex operator algebra module, without any further denomination. Since we will mainly focus on weak modules in the subsequent sections, and also to avoid linguistic misunderstandings, we will refer to these modules as ordinary modules.

Definition 3.4. A weak V-module (W, Y_W) is called an ordinary V-module if L_0 acts semisimply on W with finite-dimensional eigenspaces such that for every $h \in \mathbb{C}$ the subset of the spectrum of L_0 contained in $h + \mathbb{Z}$ is bounded from below. In other words, (W, Y_W) is an ordinary V-module if

$$W = \bigoplus_{h\in\mathbb{C}} W_{(h)}, \tag{3.22}$$

where

$$W_{(h)} = \{\, w \in W : L_0 w = hw \,\}, \tag{3.23}$$

and for each $h \in \mathbb{C}$

$$\dim W_{(h)} < \infty \tag{3.24}$$

and

$$W_{(h+n)} = \{0\}, \tag{3.25}$$

for all sufficiently large negative $n \in \mathbb{Z}$.

The modules which are classified by Zhu's algebra (cf. Section 3.3) are the $\mathbb{N}$-gradable weak V-modules, which are defined as follows.

Definition 3.5. An $\mathbb{N}$-gradable weak V-module is a weak V-module (W, Y_W) on which there exists an $\mathbb{N}$-grading,

$$W = \bigoplus_{n\in\mathbb{N}} W_{(n)}, \tag{3.26}$$

such that for all $k \in \mathbb{Z}$ and $u \in V_{(k)}$

$$u_m W_{(n)} \subseteq W_{(n+k-m-1)} \quad \text{for all } m, n \in \mathbb{Z}, \tag{3.27}$$

where by stipulation $W_{(n)} = \{0\}$ for $n \in -\mathbb{Z}^+$. If $W \neq \{0\}$, then the subspace $W_{(t)}$, where

$$t = \min \left\{ n \in \mathbb{N} : W_{(n)} \neq \{0\} \right\}, \tag{3.28}$$

is called the top level of W with respect to the grading in (3.26).

In the general discussion, it will be convenient to have a fixed gradation for the modules just introduced, and therefore we adopt the following principle.

Convention 3.6. If W is an $\mathbb{N}$-gradable weak V-module for which no particular $\mathbb{N}$-grading has been specified, we will assume that W is endowed with a grading expressed by (3.26) such that (3.27) is satisfied, and with the top level given by $W_{(0)}$.

Remark 3.7. As the following reasoning shows, every ordinary V-module $W = \bigoplus_{h\in\mathbb{C}} W_{(h)}$ is an $\mathbb{N}$-gradable weak V-module. First, as a consequence of the grading of W by L_0-eigenvalues, we have that equation (3.27), generalized to allow that $n \in \mathbb{C}$, holds for W. Secondly, let

$$\mathcal{K} = \{ h \in \mathbb{C} : \dim W_{(h)} \neq 0 \text{ and } \dim W_{(h-1)} = 0 \}, \tag{3.29}$$

and for $n \in \mathbb{N}$ let

$$W(n) = \bigoplus_{h\in\mathcal{K}} W_{(h+n)}. \tag{3.30}$$

Then the decomposition $W = \bigoplus_{n\in\mathbb{N}} W(n)$ clearly shows that W is an $\mathbb{N}$-gradable weak V-module.

Definition 3.8. A vertex operator algebra V is called rational if every $\mathbb{N}$-gradable weak V-module decomposes as a direct sum of irreducible $\mathbb{N}$-gradable weak V-modules.

3.2 Vertex operator algebras associated to affine Lie algebras

In this section we will give the basic definitions for vertex operator algebras associated to affine Lie algebras, and mention some results with regard to their modules. We employ the notation of Section 1.2, and start with a simple finite-dimensional Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module U. Let $\ell \in \mathbb{C}$, and extend the action of $\mathfrak{g}$ on U to $(\mathbb{C}[t] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}c$, by letting $t\mathbb{C}[t] \otimes_{\mathbb{C}} \mathfrak{g}$ act trivially and letting c act by ℓ. Then form the induced $\hat{\mathfrak{g}}$-module $M(\ell, U)$ defined by

$$M(\ell, U) = \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{(\mathbb{C}[t]\otimes_{\mathbb{C}}\mathfrak{g})\oplus\mathbb{C}c} U.$$

As was mentioned in Section 1.2, any restricted $\hat{\mathfrak{g}}$-module of level ℓ, where $\ell \neq -g$, can be regarded as a $\tilde{\mathfrak{g}}$-module in a way which is natural in the context of vertex operator algebras. We will describe this in a moment, and assume it temporarily.

Now let U be a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$. The resulting $\hat{\mathfrak{g}}$-module $M(\ell, U)$ is then the restriction to $\hat{\mathfrak{g}}$ of a highest weight $\tilde{\mathfrak{g}}$-module, whose highest weight is determined modulo $\mathbb{C}\delta$ and can be expressed by $\lambda + \ell\Lambda_0$. If we let U be the trivial $\mathfrak{g}$-module $L(0) = \mathbb{C}$, then we denote the $\hat{\mathfrak{g}}$-module $M(\ell, \mathbb{C})$ by $N(\ell\Lambda_0)$. This module is obtained from the Verma module $M(\ell\Lambda_0)$ as the quotient given by

$$N(\ell\Lambda_0) = M(\ell\Lambda_0) \Big/ \sum_{i=1}^{n} \mathcal{U}(\tilde{\mathfrak{g}}) f_i(0) v_{\ell\Lambda_0}.$$

Next, we will show how $N(\ell\Lambda_0)$ can be endowed with the structure of a vertex operator algebra.

The vacuum vector $\mathbf{1}$ of $N(\ell\Lambda_0)$ is given by the highest weight vector $v_{\ell\Lambda_0}$. For $a \in \mathfrak{g}$, the vertex operator of $a(-1)\mathbf{1}$ is defined by

$$Y(a(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}. \tag{3.31}$$

As a result of the Jacobi identity (cf. (3.17)), the vertex operator map $Y(\cdot, z)$ must satisfy the so-called iterate formula which states that

$$\begin{aligned} Y(Y(a, z_0)b, z_2) = &\operatorname{Res}_{z_1} \Big(z_0^{-1} \delta\Big(\frac{z_1 - z_2}{z_0}\Big) Y(a, z_1) Y(b, z_2) \\ &- z_0^{-1} \delta\Big(\frac{z_2 - z_1}{-z_0}\Big) Y(b, z_2) Y(a, z_1) \Big), \end{aligned} \tag{3.32}$$

for all a, $b \in N(\ell\Lambda_0)$. On the other hand, since $\mathbf{1}$ generates $N(\ell\Lambda_0)$ as a $\hat{\mathfrak{g}}$-module, it follows from repeated application of the iterate formula, that there exists at most one way to define the vertex operator map such that (3.31) and the vacuum property (cf. (3.12)) both hold. It turns out that this process does indeed give rise to a well-defined map $Y(\cdot, z)$.

Next, we define the conformal vector of $N(\ell\Lambda_0)$. Let the vectors $v^{(i)}$, for $i = 1, \dots, \dim \mathfrak{g}$, constitute an orthonormal basis of $\mathfrak{g}$ with respect to the nondegenerate symmetric invariant form $(\cdot, \cdot)$. (Recall that this form was fixed in Section 1.2 by the condition that $(\theta, \theta) = 2$.) Assume that the level ℓ is different from $-g$, where g is the dual Coxeter number of $\mathfrak{g}$. The conformal vector ω of $N(\ell\Lambda_0)$ is then defined via the so-called Segal–Sugawara construction by

$$\omega = \frac{1}{2(\ell + g)} \sum_{i=1}^{\dim \mathfrak{g}} v^{(i)}(-1)v^{(i)}(-1)\mathbf{1}. \tag{3.33}$$

The components $L(m)$ of the vertex operator $Y(\omega, z)$ then satisfy the commutation relations

$$[L(m), a(n)] = -na(m+n) \quad \text{for all } m, n \in \mathbb{Z} \text{ and } a \in \mathfrak{g}. \tag{3.34}$$

We can now state the following important existence theorem for vertex operator algebras associated to affine Lie algebras (cf. e.g. [LL04, Theorem 6.2.18] or [FZ92, Theorem 2.4.1]).

Theorem 3.9. *Let $\ell \in \mathbb{C}$ and assume that $\ell \neq -g$. Then the quadruple $(N(\ell\Lambda_0), Y, \mathbf{1}, \omega)$, where $\mathbf{1} = v_{\ell\Lambda_0}$ and ω is given by* (3.33)*, has a unique structure as a vertex operator algebra such that the vertex operator map Y satisfies* (3.31)*. The $\mathbb{Z}$-grading of $N(\ell\Lambda_0)$ follows from the relations*

$$[L(0), a(k)] = -ka(k), \quad \textit{for } a \in \mathfrak{g} \textit{ and } k \in \mathbb{Z},$$

and the central charge of $N(\ell\Lambda_0)$ is given by $(\dim \mathfrak{g})\ell/(\ell + g)$.

From now on we will regard $N(\ell\Lambda_0)$ as the vertex operator algebra defined by Theorem 3.9, and we will assume that $\ell \neq -g$ whenever $N(\ell\Lambda_0)$ is under consideration. It is clear that $\hat{\mathfrak{g}}$-submodules of $N(\ell\Lambda_0)$ coincide with vertex operator algebra ideals. Thus, we also view any nontrivial quotient of $N(\ell\Lambda_0)$ as a vertex operator algebra via the induced structure.

We now look at modules for the vertex operator algebra $N(\ell\Lambda_0)$. Assume that W is a restricted $\hat{\mathfrak{g}}$-module of level ℓ. We may for example consider $W = M(\ell, U)$ for any $\mathfrak{g}$-module U. To turn W into a weak module (W, Y_W) for $N(\ell\Lambda_0)$, we proceed as above and let

$$Y_W(a(-1)\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}, \quad \text{for } a \in \mathfrak{g}. \tag{3.35}$$

As in the case of the vertex operator map for $N(\ell\Lambda_0)$, there is a unique way to extend Y_W to the whole of $N(\ell\Lambda_0)$, and we have the following result (cf. e.g. [LL04, Theorem 6.2.12]).

Theorem 3.10. *Let W be a restricted $\hat{\mathfrak{g}}$-module of level ℓ, and assume that $\ell \neq -g$. Then the pair (W, Y_W) has a unique structure as a weak module for $N(\ell\Lambda_0)$ such that Y_W satisfies* (3.35).

Henceforth, we will consider restricted $\hat{\mathfrak{g}}$-modules of level ℓ as weak vertex operator algebra modules for $N(\ell\Lambda_0)$ in accordance with Theorem 3.10. There is also a converse to this theorem. If we assume that (W, Y_W) is a weak $N(\ell\Lambda_0)$-module and that (3.35) holds, then W is a restricted $\hat{\mathfrak{g}}$-module of level ℓ, with the action of $\hat{\mathfrak{g}}$ expressed by (3.35) (cf. [LL04, Proposition 6.2.6]). In this way we will also regard weak $N(\ell\Lambda_0)$-modules as restricted $\hat{\mathfrak{g}}$-modules, without further comment. Notice that for modules of this kind there is no need to distinguish between $\hat{\mathfrak{g}}$-submodules and weak $N(\ell\Lambda_0)$-submodules.

Let W be a restricted $\hat{\mathfrak{g}}$-module of level ℓ, and let the action of the conformal vector ω be described by $Y_W(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. Then the relations in (3.34) still hold with respect to the module action on W. This shows in particular that W may be extended to a $\tilde{\mathfrak{g}}$-module by letting $\bar{d}$ act as $-L(0)$. From now on, we will regard any restricted $\hat{\mathfrak{g}}$-module as a $\tilde{\mathfrak{g}}$-module in this way. In particular, we notice that the highest weight of the $\tilde{\mathfrak{g}}$-module $N(\ell\Lambda_0)$ (which was previously only determined modulo $\mathbb{C}\delta$) is indeed equal to $\ell\Lambda_0$.

Let Ω denote the Casimir operator of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$. If we regard U as the subspace $1 \otimes U$ of $M(\ell, U)$, then the action of $L(0)$ on U is given by

$$L(0)|_U = \frac{1}{2(\ell + g)}\Omega. \tag{3.36}$$

If Ω acts on U as a scalar, we see that $M(\ell, U)$ obtains a $\mathbb{Q}$-grading by $L(0)$-eigenvalues, and a slight shift of this grading clearly provides $M(\ell, U)$ with an $\mathbb{N}$-gradation. Since this grading is decided by the action of $L(0)$, it follows that (3.27) holds. Consequently, we have the following corollary to Theorem 3.10.

Corollary 3.11. *Let U be a $\mathfrak{g}$-module and assume that the Casimir operator of $\mathfrak{g}$ with respect to $(\cdot,\cdot)$ acts on U by a scalar. Then $M(\ell, U)$ is an $\mathbb{N}$-gradable weak module for $N(\ell\Lambda_0)$.*

Unless otherwise stated, we will assume that the $\mathbb{N}$-grading of the module $M(\ell, U)$ in Corollary 3.11 is determined according to the paragraph preceding the corollary, in such a way that $M(\ell, U)_{(0)} = U$. In particular, this means that U is the top level of $M(\ell, U)$.

Now let U be a highest weight module with highest weight $\lambda \in \mathfrak{h}^*$. Recall from the theory of semisimple Lie algebras that the action of the Casimir operator Ω on U is given by $(\lambda, \lambda + 2\rho)$. Thus, in view of (3.36), we see that the highest weight of the $\tilde{\mathfrak{g}}$-module $M(\ell, U)$ is given by

$$\lambda + \ell\Lambda_0 + \frac{(\lambda, \lambda + 2\rho)}{2(\ell + g)}\delta. \tag{3.37}$$

Recall also that the irreducible $\mathfrak{g}$-module $L(\lambda)$ is finite dimensional if and only if λ is dominant integral, i.e. $\lambda(\alpha_i^\vee) \in \mathbb{N}$, for $i = 1, \ldots, n$. We have the following classification of irreducible ordinary $N(\ell\Lambda_0)$-modules (cf. e.g. [LL04, Theorem 6.2.23]).

Theorem 3.12. *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and let $\Lambda \in \tilde{\mathfrak{h}}^*$ be the weight given by* (3.37)*. Then $L(\Lambda)$ is an irreducible ordinary $N(\ell\Lambda_0)$-module. Furthermore, every irreducible ordinary module for $N(\ell\Lambda_0)$ can be obtained in this way.*

In our study of the module structure of the irreducible quotient vertex operator algebra $L(\ell\Lambda_0)$ in the subsequent sections, we will have occasion to use the following result which occurs (in a more specialized form) as Lemma 26 in [Per07].

Lemma 3.13. *Let V be a quotient of $N(\ell\Lambda_0)$ with the induced vertex operator algebra structure, and assume that V has only finitely many irreducible ordinary modules in the category $\mathcal{O}$. Then every ordinary module of V belongs to the category $\mathcal{O}$.*

Parallel to Definition 3.8 we also introduce the following terminology.

Definition 3.14. A vertex operator algebra V associated to an affine Lie algebra is called rational in the category $\mathcal{O}$ if every V-module in the category $\mathcal{O}$ decomposes as a direct sum of irreducible V-modules in the category $\mathcal{O}$.

3.3 Zhu's algebra

In the doctoral thesis of Y. Zhu [Zhu90], an associative algebra $A(V)$ is introduced for any vertex operator algebra V. The algebra $A(V)$, called Zhu's algebra, has the property that there is a one-to-one correspondence between irreducible $A(V)$-modules and the irreducible $\mathbb{N}$-gradable weak V-modules. In the present thesis we will apply theorems involving Zhu's algebra that are contained in [Zhu90] and [FZ92].

To define Zhu's algebra, we proceed as follows. Let $*$ be the binary operation on V defined on homogeneous elements $u, v \in V$ by

$$u * v = \mathrm{Res}_z \Big(\frac{(z+1)^{\mathrm{wt}\, u}}{z} Y(u,z)v \Big), \tag{3.38}$$

and on the whole of V by bilinear extension of (3.38). Define $O(V)$ to be the subspace of V spanned by all elements of the form

$$\mathrm{Res}_z \Big(\frac{(z+1)^{\mathrm{wt}\, u}}{z^2} Y(u,z)v \Big), \tag{3.39}$$

for homogeneous $u, v \in V$. Let $A(V)$ be the quotient space

$$A(V) = V/O(V). \tag{3.40}$$

Under the quotient map $V \to A(V)$, we let the image of an element $u \in V$ be denoted by $[u]$, and the image of a subspace $V' \subseteq V$ be denoted by $[V']$. Theorem 2.1.1 in [Zhu90] contains the following result.

Theorem 3.15. *The binary operation $*$ induces a multiplication on $A(V)$, under which $A(V)$ becomes an associative unital algebra with multiplicative identity* $[\mathbf{1}]$.

Furthermore, we have the following proposition concerning quotient vertex operator algebras.

Proposition 3.16 ([FZ92], Proposition 1.4.2)**.** *Let I be an ideal of the vertex operator algebra V, such that $\mathbf{1} \notin I$ and $\omega \notin I$. Then $[I]$ is a two-sided ideal of $A(V)$ and $A(V/I)$ is isomorphic to $A(V)/[I]$.*

A specific calculation in Zhu's algebra is contained in the following lemma (cf. Lemma 2.1.7 in [Zhu90]).

Lemma 3.17. *Let u, $v \in V$ and assume that u is homogeneous of weight 1. Then*

$$[u] * [v] - [v] * [u] = [u_0 v].$$

Let $u \in V$ be a homogeneous element and let W be an $\mathbb{N}$-gradable weak V-module. Then the component u_n of $Y_W(u, z)$ is a homogeneous operator on W of degree $\operatorname{wt} u - n - 1$. In particular, $u_{\operatorname{wt} u - 1}$ is homogeneous of degree 0. We let $o(u)$ denote the restriction of $u_{\operatorname{wt} u-1}$ to the top level $W_{(0)}$, i.e.

$$o(u) = u_{\operatorname{wt} u - 1}|_{W_{(0)}}. \tag{3.41}$$

Extending linearly to the whole of V, we obtain a linear function

$$o\colon V \longrightarrow \operatorname{End}(W_{(0)}). \tag{3.42}$$

One of the main results of Y. Zhu's thesis is given by the following theorem (cf. Theorem 2.1.2 and 2.2.2 in [Zhu90]).

Theorem 3.18. *The linear function o factors through $O(V)$, and induces a representation of the algebra $A(V)$ on $W_{(0)}$. This gives rise to a bijective correspondence between irreducible $\mathbb{N}$-gradable weak V-modules and irreducible $A(V)$-modules.*

Now we concentrate on the case of vertex operator algebras associated to affine Lie algebras. We have the following description of Zhu's algebra for the vertex operator algebra $N(\ell\Lambda_0)$.

Theorem 3.19 ([FZ92], Theorem 3.1.1)**.** *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and assume that $\ell \in \mathbb{C}$ is not equal to the negative of the dual Coxeter number of $\mathfrak{g}$. Then the associative algebra $A(N(\ell\Lambda_0))$ is isomorphic to $\mathcal{U}(\mathfrak{g})$ and an explicit isomorphism is given by*

$$\begin{gathered}\Psi\colon A(N(\ell\Lambda_0)) \longrightarrow \mathcal{U}(\mathfrak{g}),\\ [a_1(-i_1-1)\dots a_n(-i_n-1)v_{\ell\Lambda_0}] \overset{\Psi}{\longmapsto} (-1)^{i_1+\dots+i_n} a_n \dots a_1,\end{gathered}$$

where $a_j \in \mathfrak{g}$ and $i_j \in \mathbb{N}$ for $j = 1, \dots, n$.

Retaining the assumptions of the previous theorem, let $\iota\colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\hat{\mathfrak{g}})$ be the extension to $\mathcal{U}(\mathfrak{g})$ of the natural identification of $\mathfrak{g}$ in $\hat{\mathfrak{g}}$ given by $a \mapsto a(0)$, for $a \in \mathfrak{g}$. The following proposition makes explicit the action of Zhu's algebra in terms of the isomorphism in Theorem 3.19.

Proposition 3.20. *Let W be an $\mathbb{N}$-gradable weak $N(\ell\Lambda_0)$-module and let*

$$\pi_0\colon \iota\big(\mathcal{U}(\mathfrak{g})\big) \longrightarrow \mathrm{End}(W_{(0)})$$

be the representation induced on $W_{(0)}$ by regarding W as a $\hat{\mathfrak{g}}$-module in the natural way. Then

$$o \circ \Psi^{-1} = \pi_0 \circ \iota.$$

Proof. Since $o([a(-1)v_{\ell\Lambda_0}]) = \pi_0(a(0))$, we find that

$$(o \circ \Psi^{-1})(a) = (\pi_0 \circ \iota)(a),$$

for $a \in \mathfrak{g}$. Since $\mathfrak{g}$ generates $\mathcal{U}(\mathfrak{g})$, and since o, Ψ^{-1}, π_0 and ι are all algebra homomorphisms, the proposition follows. □

Retaining the assumptions and notation from the previous theorem, we have the following result which is along the lines of Theorem 3.1.2 in [FZ92].

Theorem 3.21. *Let J be the ideal of the vertex operator algebra $N(\ell\Lambda_0)$ generated by the set $\{\, v_i : i \in I \,\}$, and assume that $v_{\ell\Lambda_0} \notin J$ and $\omega \notin J$. Then $\Psi([J])$ is equal to the two-sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by the elements $\Psi([v_i])$, for $i \in I$.*

Proof. Considering the fact that J is the $\hat{g}$-submodule of $N(\ell\Lambda_0)$ generated by $\{\, v_i : i \in I \,\}$, this theorem follows immediately from Proposition 3.16 and the definition of Ψ. □

3.4 Admissible weights

We will now define the class of weights for affine Lie algebras called admissible weights. In order to motivate this definition, and to show in what ways the results depend on it, we will start with a general discussion. For the original accounts of this theory cf. [DGK82], [KW88] and [KW89]. An exposition of the matter is also given in Sections 6.7 and 6.8 in [MP95]. We also refer to the just mentioned sources for any notions not defined in the present text.

To begin with, assume that $\mathfrak{g}$ is a symmetrizable Kac–Moody algebra. In order to describe the structure of a Verma module $M(\lambda)$, it is important to know for which weights $\mu \in \mathfrak{h}^*$ that $L(\mu)$ occurs as a subquotient of $M(\lambda)$. Motivated by the Šapovalov–Kac–Kazhdan determinant formula (see Theorem 3.39), we introduce the relation $\succcurlyeq\!\!-$ on $\mathfrak{h}^*$, which is defined by the condition that $\lambda \succcurlyeq\!\!- \mu$ if and only if $\mu = \lambda$ or

$$\mu = \lambda - n\alpha, \quad \text{for some } n \in \mathbb{Z}^+ \text{ and } \alpha \in \Delta_+, \tag{3.43}$$

such that

$$2(\lambda + \rho, \alpha) = n(\alpha, \alpha). \tag{3.44}$$

Let $\succ\!\!-$ denote the transitive closure of the relation $\succcurlyeq\!\!-$. Then it can be proved as a corollary to the determinant formula, that $L(\mu)$ is a subquotient of $M(\lambda)$ if and only if $\lambda \succ\!\!- \mu$.

For any subset $H \subseteq \mathfrak{h}^*$, let $\sim_H$ be the transitive and symmetric closure in H of the relation $\succcurlyeq\!\!-$. In other words, $\lambda \sim_H \mu$ means that there exists a finite sequence $\lambda = \delta_0, \delta_1, \ldots, \delta_r = \mu$ of weights in H such that, for $k = 1, \ldots, r$, either $\delta_{k-1} \succcurlyeq\!\!- \delta_k$ or $\delta_k \succcurlyeq\!\!- \delta_{k-1}$.

Now, assume that (3.43) and (3.44) holds for a real root α. Then $(\alpha, \alpha) > 0$ and (3.44) can equivalently be expressed as

$$\langle \lambda + \rho, \alpha^\vee \rangle = n.$$

It is thus natural to consider the set of coroots given by

$$\Delta_\lambda^\vee = \{\, \alpha^\vee \in \Delta^\vee : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \,\}.$$

This set constitutes a subroot system of $\Delta^\vee$ (cf. Chapter 5 in [MP95]). Its Weyl group $\mathscr{W}_\lambda$ is given by

$$\mathscr{W}_\lambda = \langle \{\, r_{\alpha^\vee} : \alpha^\vee \in \Delta_\lambda^\vee \,\} \rangle,$$

and we let $\Pi_\lambda^\vee$ denote the base with respect to which the positive roots of $\Delta_\lambda^\vee$ are given by $\Delta_\lambda^\vee \cap Q_+^\vee$. If $\mathfrak{g}$ is finite dimensional and hence only has real roots, it is straightforward to show that the equivalence classes of $\sim_{\mathfrak{h}^*}$ are characterized by the condition that

$$\mu \sim_{\mathfrak{h}^*} \lambda \quad \iff \quad \mu = w^\rho(\lambda), \quad \text{for some } w \in \mathscr{W}_\lambda. \tag{3.45}$$

To obtain a similar description when $\mathfrak{g}$ is of infinite dimension, we restrict the set of roots under consideration to accommodate for the existence of imaginary roots.

To make the geometry of $\mathfrak{h}^*$ transparent, we temporarily assume that the base field of our discussion is $\mathbb{R}$. Let C be the subset of $\mathfrak{h}^*$ consisting of all weights λ satisfying the following two conditions:

(i) $\langle \lambda, \alpha_i^\vee \rangle \geq 0$, for $i = 1, \ldots, n$;

(ii) $(\lambda, \alpha) \neq 0$, if $\alpha \in \Delta_+$ and $(\alpha, \alpha) = 0$.

Geometrically, C is the fundamental chamber with respect to Π excepting some lower dimensional cones consisting of boundary points. Let K be the orbit of C under the Weyl group, i.e.

$$K = \bigcup_{w \in \mathscr{W}} w(C).$$

It can then be shown that if $\lambda \in -\rho + K$ and $\lambda \succcurlyeq \mu$, then $\mu \in -\rho + K$. Furthermore, for weights λ and μ in $-\rho + K$, we obtain an analogue to the condition in (3.45) given by

$$\mu \sim_{(-\rho+K)} \lambda \quad \Longleftrightarrow \quad \mu = w^\rho(\lambda), \quad \text{for some } w \in \mathscr{W}_\lambda.$$

Considering again $\mathbb{C}$ as the base field, we adopt the following convention to obtain an ordering of the set of complex numbers.

Convention 3.22. We let $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ be ordered lexicographically with respect to the usual order relation on $\mathbb{R}$. Considering $\mathbb{R}$ as the subset $\mathbb{R} \times \{0\}$ of $\mathbb{C}$, this relation extends the ordering of $\mathbb{R}$, and we retain the standard notation for the extended relation.

Applying this convention to condition (i) in the definition of C, the just mentioned results still hold.

The Weyl–Kac character formula for irreducible modules of dominant integral highest weight can be generalized to include certain weights in $-\rho+K$ which can be described as dominant integral with respect to the base $\Pi_\lambda^\vee$ of the subroot system $\Delta_\lambda^\vee$.

Theorem 3.23 ([KW88], Theorem 1)**.** *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, and let $\lambda \in -\rho + K$ be such that $\langle \lambda + \rho, \alpha^\vee \rangle > 0$ for $\alpha^\vee \in \Pi_\lambda^\vee$ (cf.*

Convention 3.22). *Then*

$$\operatorname{ch} L(\lambda) = \sum_{w \in \mathscr{W}_\lambda} \det(w) \operatorname{ch} M\big(w^\rho(\lambda)\big).$$

We will later make use of the following related result.

Theorem 3.24 ([KW88], Corollary 1)**.** *Assume that the conditions of Theorem 3.23 hold. Then*

$$L(\lambda) = M(\lambda) \Big/ \sum_{\alpha^\vee \in \Pi^\vee_\lambda} \mathcal{U}(\mathfrak{g}) v^\alpha,$$

where v^α is a singular vector in $M(\lambda)$ of weight $r^\rho_\alpha(\lambda)$.

Given a subset of weights $H \subseteq \mathfrak{h}^*$, we say that a $\mathfrak{g}$-module M in the category $\mathcal{O}$ is of type H if for every irreducible subquotient $L(\mu)$ of M, the weight μ belongs to H. We let $\mathcal{O}_H$ denote the full subcategory of $\mathcal{O}$, whose objects are the modules of type H. Furthermore, we say that H is a characteristic set if $M(\lambda)$ is of type H for every $\lambda \in H$. In the discussion above, we have seen that $-\rho + K$ is a characteristic set. We have the following general theorem regarding sets of this kind (cf. Theorem 2.12.4 in [MP95]).

Theorem 3.25. *Let $\mathfrak{g}$ be a Kac–Moody algebra, let $H \subseteq \mathfrak{h}^*$ be a characteristic set, and let $\bar{H}$ denote the set of equivalence classes of H under $\sim_H$. If M is a $\mathfrak{g}$-module in the category $\mathcal{O}_H$, then M decomposes as a direct sum*

$$M = \bigoplus_{\bar{\lambda} \in \bar{H}} M_{\bar{\lambda}},$$

where $M_{\bar{\lambda}}$ is the unique maximal submodule of M that belongs to the category $\mathcal{O}_{\bar{\lambda}}$.

Later we will make use of the following result on complete reducibility, which is a slightly more general version of Theorem 4.1 in [KW89].

Theorem 3.26. *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra, and let M be a $\mathfrak{g}$-module in the category $\mathcal{O}_{-\rho+K}$. Assume that for every irreducible subquotient $L(\mu)$ of M, the weight μ satisfies that*

$$\langle \mu + \rho, \alpha^\vee \rangle \notin -\mathbb{Z}^+, \quad \textit{for any } \alpha^\vee \in \Delta^\vee_+. \tag{3.46}$$

Then M is completely reducible.

Proof. Since $-\rho + K$ is a characteristic set, Theorem 3.25 applies, and we use the notation of this theorem with H substituted by $-\rho + K$. Let $\lambda \in \mathfrak{h}^*$ and consider the submodule $M_{\bar{\lambda}}$ of M. Since $\bar{\lambda} = \{\, w^\rho(\lambda) : w \in \mathscr{W}_\lambda \,\}$, we may assume, without loss of generality, that λ is chosen so that

$$\langle \lambda + \rho, \alpha^\vee \rangle \geq 0, \quad \text{for all } \alpha^\vee \in \Pi_\lambda^\vee.$$

Let $L(\mu)$ be a subquotient of $M_{\bar{\lambda}}$. Then $\mu \sim_{(-\rho+K)} \lambda$, and hence $\mu + \rho = w'(\lambda + \rho)$ for some $w' \in \mathscr{W}_\lambda$. Furthermore, from (3.46), we also have that

$$\langle \mu + \rho, \alpha^\vee \rangle \geq 0, \quad \text{for all } \alpha^\vee \in \Pi_\lambda^\vee.$$

It follows by elementary results about root systems (here applied to the root system of $\mathfrak{h}^*$ dual to $\Delta_\lambda^\vee$ with Weyl group $\mathscr{W}_\lambda$), that w' fixes $\lambda + \rho$, and hence $\mu = \lambda$.

It is now clear that $M_{\bar{\lambda}}$ is generated by the homogeneous subspace of weight λ, and that every vector of this weight space is singular. Moreover, we see that every highest weight submodule of $M_{\bar{\lambda}}$ must be isomorphic to $L(\lambda)$. Consequently, $M_{\bar{\lambda}}$ is a sum, and hence a direct sum, of submodules all of which are isomorphic to $L(\lambda)$. Thus M is completely reducible. $\square$

We now define the class of admissible weights. From now on we will work with affine Lie algebras, and in particular those of the untwisted kind. Thus, we will employ the notation introduced in Section 1.2 for untwisted affine Lie algebras. For statements concerning affine Lie algebras in general, we will, without further comment, use the same notation for the corresponding notions for twisted affine Lie algebras.

Definition 3.27. Let $\tilde{\mathfrak{g}}$ be an affine Kac–Moody algebra. A weight $\lambda \in \tilde{\mathfrak{h}}^*$ is said to be admissible if the following holds:

- ⋄ $\langle \lambda + \rho, c \rangle > 0$ (cf. Convention 3.22);
- ⋄ $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}^+$, for all $\alpha^\vee \in \Pi_\lambda^\vee$;
- ⋄ the $\mathbb{Q}$-span of $\Delta_\lambda^\vee$ is equal to the $\mathbb{Q}$-span of $\Delta^\vee$.

If λ is an admissible weight and $\langle \lambda, c \rangle = \ell$, then ℓ is called an admissible level.

It is easily verified that, in the context of affine Lie algebras, the requirement that λ belongs to $-\rho + K$ is equivalent to the first condition in Definition 3.27.

The second condition of this definition is the same as the additional assumption needed in order to prove Theorems 3.23 and 3.24. Notice that the second and third condition of Definition 3.27 imply that an admissible weight λ takes rational values on the coroot lattice $Q^\vee$. In particular, an admissible level must be a rational number. As we will soon see, it turns out that the "denominator" of the admissible level $\langle \lambda, c \rangle$ (i.e. the positive integer u satisfying that $u\langle \lambda, c \rangle \in \mathbb{Z}$ and that u and $u\langle \lambda, c \rangle$ are relatively prime) is an important characteristic of the admissible weight λ.

In [KW89], all admissible weights for every affine Kac–Moody algebra are classified. This is accomplished by first describing the admissible simple sets, which are defined as follows.

Definition 3.28. A finite subset $\mathcal{S}$ of $\Delta_+^\vee$ is called an admissible simple set if it occurs as the base $\Pi_\lambda^\vee$ of the subroot system $\Delta_\lambda^\vee$ for some admissible weight λ.

If $\mathcal{S}$ is an admissible simple set, then the following clearly holds:

(i) if $\alpha, \beta \in \mathcal{S}$, then $\alpha - \beta \notin \Delta_+^\vee$;

(ii) the $\mathbb{Q}$-span of $\mathcal{S}$ is equal to the $\mathbb{Q}$-span of $\Delta^\vee$.

The classification in [KW89] begins with a characterization of all the finite subsets $\mathcal{S}$ of $\Delta_+^\vee$ satisfying (i) and (ii) above.

We will now briefly explain the structure of the admissible simple sets when $\tilde{\mathfrak{g}}$ is an untwisted affine Lie algebra. Given a simple coroot $\alpha^\vee$, express $\Pi^\vee \setminus \{\alpha^\vee\}$ as

$$\Pi^\vee \setminus \{\alpha^\vee\} = \bigcup_{k=1}^{r} \Pi_k^\vee,$$

where $\Pi_k^\vee$, for $k = 1, \ldots, r$, are the indecomposable components of $\Pi^\vee \setminus \{\alpha^\vee\}$ with respect to the corresponding Coxeter–Dynkin diagram. Then $\mathbb{Z}\Pi_k^\vee \cap \Delta^\vee$ is a finite root system. Let θ_k be the highest root of this root system and define $\eta_k \in Q^\vee$ by

$$\eta_k = c - \theta_k,$$

for $k = 1, \ldots, r$. Let $\mathcal{S}_{\alpha^\vee}$ be given by

$$\mathcal{S}_{\alpha^\vee} = \{\eta_1, \ldots, \eta_r\} \cup \big(\Pi^\vee \setminus \{\alpha^\vee\}\big).$$

Denote by $\bar{\mathscr{W}}$ the Weyl group of $\mathfrak{g}$, and consider $\bar{\mathscr{W}}$ as a subgroup of $\mathscr{W}$ in the natural way. Then we have the following description of admissible simple sets.

Proposition 3.29 ([KW89], Proposition 1.1)**.** *Let $\tilde{\mathfrak{g}}$ be an untwisted affine Lie algebra, and let $\mathcal{S}$ be an admissible simple set. Then there exists $\alpha^\vee \in \Pi^\vee$ such that*

$$\mathcal{S} \equiv \bar{w}(\mathcal{S}_{\alpha^\vee}) \pmod{\mathbb{Z}c},$$

for some $\bar{w} \in \bar{\mathscr{W}}$.

Theorem 2.3 in [KW89] gives a more explicit description of the admissible simple sets of an untwisted affine Lie algebra. This theorem also specifies the corresponding admissible weights. Since the formulation of this theorem is rather intricate, we will not recount it in its full generality. Below we restate the theorem for the special case when $\tilde{\mathfrak{g}}$ is a Lie algebra of type $A_n^{(1)}$ ($n \geq 2$). (Recall that θ denotes the highest weight of the associated simple Lie algebra $\mathfrak{g}$, and that g is the dual Coxeter number of $\tilde{\mathfrak{g}}$.)

Theorem 3.30. *Let $\tilde{\mathfrak{g}}$ be an untwisted affine Lie algebra of type $A_n^{(1)}$. For an integer $t \in J = \{0, 1, \dots, n\}$, let $J_t = J \setminus \{t\}$. Given a triple $(t, \vec{k}, u)$, where $t \in J$, $\vec{k} = (k_i)_{i \in J_t} \in \mathbb{N}^{J_t}$ and $u \in \mathbb{Z}^+$, define the set $\mathcal{S}_{(t,\vec{k},u)}$ by*

$$\mathcal{S}_{(t,\vec{k},u)} = \{\, k_i c + \alpha_i^\vee : i \in J_t \,\} \cup \{uc - \theta\}.$$

Then every admissible simple set $\mathcal{S}$ is of the form

$$\mathcal{S} = \bar{w}(\mathcal{S}_{(t,\vec{k},u)}),$$

for some $\bar{w} \in \bar{\mathscr{W}}$.

For $m \in \mathbb{Q}$, let $\mathcal{P}^m_{(t,\vec{k},u,\bar{w})}$ be the set of all admissible weights λ of level m such that $\Pi_\lambda^\vee = \bar{w}(\mathcal{S}_{(t,\vec{k},u)})$. Then $\mathcal{P}^m_{(t,\vec{k},u,\bar{w})}$ is nonempty if and only if the following holds:

- *u is the "denominator" of m, i.e. um is an integer which is relatively prime to u;*
- *$\sum_{i \in J_t} k_i \leq u - 1$;*
- *$u(m + g) - g \geq 0$;*
- *$\bar{w}(\mathcal{S}_{(t,\vec{k},u)}) \subset \Delta_+^\vee$.*

If the above conditions are satisfied, then $\mu \in \mathcal{P}^m_{(t,\vec{k},u,\bar{w})}$ *if and only if*

$$\mu = \bar{w}^\rho\Big(m\Lambda_t + \sum_{i\in J_t}(n_i - k_i(m+g))(\Lambda_i - \Lambda_t)\Big)$$

for some tuple $(n_i)_{i\in J_t} \in \mathbb{N}^{J_t}$ *such that*

$$\sum_{i\in J_t} n_i \leq u(m+g) - g.$$

In the next proposition we specialize the contents of Theorem 3.30 to the case of the affine Lie algebra $\mathfrak{sl}(3,\mathbb{C})^\sim$ and to admissible weights with "denominator" 2. The proof of this proposition consists of elementary computations, and is omitted.

Proposition 3.31. *The admissible levels for* $\mathfrak{sl}(3,\mathbb{C})^\sim$ *of the form* $\frac{1}{2}+\mathbb{Z}$ *are given by the subset* $-\frac{3}{2}+\mathbb{N}$. *The associated admissible simple sets are given by* $\mathcal{S}_\ell$, *for* $\ell = 0, 1, 2, 3$, *where, for* $\ell = 0, 1, 2$ *and* $\{\ell, m, n\} = \{0,1,2\}$

$$\mathcal{S}_\ell = \{2\alpha_\ell^\vee + \alpha_m^\vee + \alpha_n^\vee, \alpha_m^\vee, \alpha_n^\vee\},$$

and

$$\mathcal{S}_3 = \{\alpha_0^\vee + \alpha_1^\vee, \alpha_0^\vee + \alpha_2^\vee, \alpha_1^\vee + \alpha_2^\vee\}.$$

The corresponding admissible weights at level $k \in -\frac{3}{2}+\mathbb{N}$ *are given by the sets* $\mathcal{P}^k_\ell$, *for* $\ell = 0, 1, 2, 3$, *where, for* $\ell = 0, 1, 2$ *and* $\{\ell, m, n\} = \{0,1,2\}$

$$\mathcal{P}^k_\ell = \big\{\,(k-s-t)\Lambda_\ell + s\Lambda_m + t\Lambda_n : s,\, t \in \mathbb{N},\, 0 \leq s+t \leq 2k+3\,\big\},$$

and

$$\begin{aligned}\mathcal{P}^k_3 = \big\{\,&(-k-2+s+t)\Lambda_0 + (k+1-s)\Lambda_1 + (k+1-t)\Lambda_2 : \\ &s,\, t\in\mathbb{N},\, 0 \leq s+t \leq 2k+3\,\big\}.\end{aligned}$$

3.5 Admissible levels

In this section and the next, we will be concerned with vertex operator algebras associated to affine Lie algebras, and with the corresponding modules. More specifically, we will study the simple vertex operator algebra $L(m\Lambda_0)$, where m is an admissible level. The main objective is to obtain information about the irreducible modules in the category $\mathcal{O}$ for $L(m\Lambda_0)$. As mentioned in the introduction, it was conjectured by D. Adamović and A. Milas in [AM95] that for admissible levels m and all affine Lie algebras, the $L(m\Lambda_0)$-modules in the category $\mathcal{O}$ are completely reducible. This conjecture also implicitly assumes the expectation that the highest weights of the irreducible $L(m\Lambda_0)$-modules in the category $\mathcal{O}$ are precisely those belonging to the finite set of admissible weights of level m.

We will examine in detail the series of simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^{\sim}$, whose admissible levels have denominator 2. In particular, we will consider the vertex operator algebra with the minimal admissible level $-\frac{3}{2}$. As far as possible, the results will be presented in a general context, and then they will be applied to the vertex operator algebras of the just mentioned series by means of concrete calculations. We start by setting up the general framework.

Let $\tilde{\mathfrak{g}}$ be an untwisted affine Lie algebra, and consider admissible weights with denominator u. From Theorem 3.30 (if $\tilde{\mathfrak{g}}$ is of type $A_n^{(1)}$) or, in general, from Theorem 2.3 in [KW89], it follows that there exists a minimal admissible level m_0 with this denominator such that

$$m_0 \geq \frac{g}{u} - g$$

(where g is the dual Coxeter number of $\tilde{\mathfrak{g}}$). Moreover, notice that if, in Theorem 3.30, we put $t = 0$, $\vec{k} = (0, \ldots, 0)$, and $\bar{w} = \mathrm{id}_{\mathfrak{h}^*}$ (or if we make the parallel assignments in Theorem 2.3 in [KW89]), then we obtain the admissible simple set $\mathcal{S}_0$ given by

$$\mathcal{S}_0 = \bar{w}(\mathcal{S}_{(t,\vec{k},u)}) = \mathcal{S}_{(0,0,u)} = \{\, \alpha_i^\vee : i = 1, \ldots, n\,\} \cup \{uc - \theta\}.$$

Let $m \in \mathbb{Q}$ be an admissible level with denominator u, and notice that the weight $m\Lambda_0$ is an admissible weight corresponding to $\mathcal{S}_0$.

Let $\beta_0 \in \Delta_+$ be the root corresponding to the coroot $uc - \theta$, i.e. $\beta_0^\vee = uc - \theta$. From Theorem 3.24, we obtain that the maximal proper submodule of $M(m\Lambda_0)$ is generated by singular vectors $v^{(i)}$, for $i = 0, 1, \ldots, n$, where, for $i = 1, \ldots, n$, the weight of $v^{(i)}$ is $r^\rho_{\alpha_i}(m\Lambda_0) = m\Lambda_0 - \alpha_i$, and $v^{(0)}$ is of weight $r^\rho_{\beta_0}(m\Lambda_0)$. These singular vectors can all be determined in accordance with the theory in [MFF86]. The vectors $v^{(i)}$, for $i = 1, \ldots, n$, are of course already well-known, and we fix them by letting $v^{(i)} = F(r_{\alpha_i}, m\Lambda_0)v_{m\Lambda_0} = f_i v_{m\Lambda_0}$. The quotient given by

$$N(m\Lambda_0) = M(m\Lambda_0) / \sum_{i=1}^{n} \mathcal{U}(\tilde{\mathfrak{g}})v^{(i)},$$

is then the generalized Verma module giving rise to the vertex operator algebra $N(m\Lambda_0)$. Thus, we see that the maximal proper submodule of $N(m\Lambda_0)$ is generated by the single singular vector $v^{(0)} = F(r_{\beta_0}, m\Lambda_0)v_{m\Lambda_0}$, and hence that $L(m\Lambda_0) = N(m\Lambda_0)/\mathcal{U}(\tilde{\mathfrak{g}})v^{(0)}$.

We now set about to examine the representation theory for $L(m\Lambda_0)$. More precisely, we are interested in finding the irreducible $\mathbb{N}$-gradable weak modules for $L(m\Lambda_0)$ which belong to the category $\mathcal{O}$ as $\hat{g}$-modules. Our main instrument in this study is Zhu's algebra, and in particular we will make use of the conclusions of Theorems 3.18 and 3.19.

Consider the image $[v^{(0)}]$ of $v^{(0)}$ in Zhu's algebra $A\big(N(m\Lambda_0)\big)$, and recall the definition of the isomorphism $\Psi\colon A\big(N(m\Lambda_0)\big) \to \mathcal{U}(\mathfrak{g})$ in Theorem 3.19. Since $v^{(0)}$ is a singular vector in $N(m\Lambda_0)$, we have in particular that

$$e_i v^{(0)} = 0, \quad \text{for } i = 1, \ldots, n.$$

In view of Lemma 3.17 this implies that

$$[e_i(-1)v_{m\Lambda_0}] * [v^{(0)}] - [v^{(0)}] * [e_i(-1)v_{m\Lambda_0}] = [e_i v^{(0)}] = 0,$$

for $i = 1, \ldots, n$. Applying Ψ we get that

$$[e_i, \Psi([v^{(0)}])] = e_i\Psi([v^{(0)}]) - \Psi([v^{(0)}])e_i = 0. \tag{3.47}$$

Consider $\mathcal{U}(\mathfrak{g})$ as a $\mathfrak{g}$-module under the adjoint action of $\mathfrak{g}$, and let J denote the submodule generated by $\Psi([v^{(0)}])$. Let η be the weight of $\Psi([v^{(0)}])$. Then (3.47) implies that J is a highest weight module with highest weight η. We make the following observation concerning the weight η.

Proposition 3.32. *The weight η of $\Psi([v^{(0)}])$ lies in Q_+.*

Proof. Consider $N(m\Lambda_0)$ as a $\mathfrak{g}$-module (by means of the usual inclusion of $\mathfrak{g}$ in $\tilde{\mathfrak{g}}$). Let $N' = \mathcal{U}(\mathfrak{g})v^{(0)}$ be the highest weight submodule generated by the singular vector $v^{(0)}$ of weight η. With respect to the vertex operator algebra grading of $N(m\Lambda_0)$, it is clear that N' is contained in a homogeneous subspace of $N(m\Lambda_0)$. Thus, N' is finite dimensional. According to the theory of semisimple Lie algebras, η is a dominant integral weight. The proposition now follows from the fact that a dominant integral weight has positive rational coordinates in the basis $\alpha_1, \ldots, \alpha_n$. This is established, for instance, in Exercises 8 and 9, p. 72 in [Hum72]. $\square$

Let R be the two-sided ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\Psi([v^{(0)}])$, i.e.

$$R = \mathcal{U}(\mathfrak{g})\Psi([v^{(0)}])\mathcal{U}(\mathfrak{g}).$$

By Theorem 3.21 we know that Zhu's algebra $A\big(N(m\Lambda_0)\big)$ is isomorphic to $\mathcal{U}(\mathfrak{g})/R$. Let W be an irreducible $\mathbb{N}$-gradable weak module for $L(m\Lambda_0)$ which belongs to the category $\mathcal{O}$ as a $\hat{g}$-module. By Theorems 3.18 and 3.19, and Proposition 3.20, it follows that this is true if and only if the top level $W_{(0)}$ of W is an irreducible highest weight module for Zhu's algebra $A(L(m\Lambda_0)) \cong \mathcal{U}(\mathfrak{g})/R$. Thus, we now proceed to find the modules of the form $L(\lambda)$ for $\mathcal{U}(\mathfrak{g})/R$.

Let π be the projection

$$\pi : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h})$$

induced by the decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus \big(\mathfrak{n}_-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}_+\big).$$

It is clear that π is a $\mathcal{U}(\mathfrak{h})$-bimodule homomorphism, and hence that $\pi(R)$ is an ideal of the polynomial algebra $\mathcal{U}(\mathfrak{h})$. For $p \in \mathcal{U}(\mathfrak{h})$ and $\lambda \in \mathfrak{h}^*$, we let $p(\lambda)$ denote the polynomial evaluation of p at λ. Moreover, for any subset $T \subseteq \mathcal{U}(\mathfrak{h})$, we let $\mathcal{V}(T) \subseteq \mathfrak{h}^*$ be the algebraic variety defined by T. The next theorem characterizes modules for $\mathcal{U}(\mathfrak{g})/R$ in terms of $\pi(R)$.

Theorem 3.33. *The irreducible highest weight module $L(\lambda)$ is a module for $\mathcal{U}(\mathfrak{g})/R$ if and only if*

$$\lambda \in \mathcal{V}\big(\pi(R)\big).$$

Proof. Clearly, the module action of $\mathcal{U}(\mathfrak{g})$ on $M(\lambda)$ induces an action of $\mathcal{U}(\mathfrak{g})/R$ on $L(\lambda)$, precisely if the submodule $RM(\lambda)$ is contained in the maximal proper submodule of $M(\lambda)$, or equivalently if

$$RM(\lambda) \neq M(\lambda).$$

This, in turn, is equivalent to the condition that

$$v_\lambda \notin RM(\lambda). \tag{3.48}$$

Since $RM(\lambda) = Rv_\lambda$, it follows that (3.48) is true if and only if

$$v_\lambda \notin \pi(R)v_\lambda,$$

which proves the theorem. □

The next proposition relates the ideal $\pi(R)$ to the module J, and thereby gives us a description of $\pi(R)$ which is more suitable for our purposes.

Proposition 3.34. *The ideal $\pi(R)$ is equal to $\mathcal{U}(\mathfrak{h})\pi(J)$.*

Proof. Notice first that $R = \mathcal{U}(\mathfrak{g})J$. Writing $\mathcal{U}(\mathfrak{g})$ as $\mathcal{U}(\mathfrak{n}_-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{n}_+)$ we then find that

$$\begin{aligned}\pi(R) &= \pi\big(\mathcal{U}(\mathfrak{g})J\big) = \pi\big(\mathcal{U}(\mathfrak{n}_-)\mathcal{U}(\mathfrak{h})\mathcal{U}(\mathfrak{n}_+)J\big) = \pi\big(\mathcal{U}(\mathfrak{n}_-)\mathcal{U}(\mathfrak{h})J\big)\\ &= \pi\big(\mathcal{U}(\mathfrak{h})J\big) = \mathcal{U}(\mathfrak{h})\pi(J),\end{aligned}$$

which proves the proposition. □

Recall that η denotes the weight of $\Psi([v^{(0)}])$. We fix a basis $u_1, \ldots, u_\ell$ for the homogeneous subspace $\mathcal{U}(\mathfrak{n}_-)_{-\eta}$. Then we have the following corollary to the previous proposition.

Corollary 3.35. *The ideal $\pi(R)$ is generated by the polynomials*

$$\pi\big(\operatorname{ad}(u_i)\big(\Psi([v^{(0)}])\big)\big), \quad \textit{for } i = 1, \ldots, \ell.$$

Proof. Since $\Psi([v^{(0)}])$ generates the highest weight module J, we get from Proposition 3.34 that

$$\pi(R) = \mathcal{U}(\mathfrak{h})\pi(J) = \mathcal{U}(\mathfrak{h})\pi\big(\operatorname{ad}\big(\mathcal{U}(\mathfrak{n}_-)_{-\eta}\big)\big(\Psi([v^{(0)}])\big)\big),$$

which proves the corollary. □

We now apply the theory just developed to the series of simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^{\sim}$, whose levels are admissible with denominator 2. In the process we will employ notation developed in Section 2.4 concerning the representation of singular vectors. By Proposition 3.31, we see that the vertex operator algebras under consideration are of the form $L(m\Lambda_0)$, with $m \in -\frac{3}{2} + \mathbb{N}$. Moreover, this proposition (or the discussion at the beginning of this section) shows that the admissible weight $m\Lambda_0$ is associated to the admissible simple set

$$\mathcal{S}_0 = \{\alpha_1^\vee, \alpha_2^\vee, 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee\}.$$

The root β_0 is thus given by

$$\beta_0 = 2\alpha_0 + \alpha_1 + \alpha_2,$$

and can be expressed as

$$\beta_0 = r_{\alpha_0}(r_{\alpha_2}(\alpha_1)) = r_{\alpha_0}(\alpha_1 + \alpha_2) = 2\alpha_0 + \alpha_1 + \alpha_2. \tag{3.49}$$

For $m \in -\frac{3}{2}+\mathbb{N}$, let $v_m^{(0)}$ be the generator of the maximal submodule of $N(m\Lambda_0)$ given by

$$v_m^{(0)} = F(r_{\beta_0}, m\Lambda_0)v_{m\Lambda_0}. \tag{3.50}$$

From (3.49), we see that $F(r_{\beta_0}, m\Lambda_0)$ may be defined by letting $F(r_{\beta_0}, m\Lambda_0) = F(I_1, m\Lambda_0)$, where

$$I_1 = (0, 2, 1).$$

Furthermore, equation (3.49) then shows that

$$s_2 = 1 \quad \text{and} \quad s_1 = 2.$$

The matrix $C_A^{I_1}$ (where $A = A_2^{(1)}$) is given by

$$C_A^{I_1} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix},$$

and hence the numbers γ_1, γ_2, γ_3 are determined by the system of equations

$$\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} \big(m\Lambda_0 + \rho\big)(\alpha_0^\vee) \\ \big(m\Lambda_0 + \rho\big)(\alpha_2^\vee) \\ \big(m\Lambda_0 + \rho\big)(\alpha_1^\vee) \end{pmatrix} = \begin{pmatrix} m+1 \\ 1 \\ 1 \end{pmatrix},$$

which gives that

$$\gamma_1 = -m-1, \quad \gamma_2 = -m-2, \quad \text{and} \quad \gamma_3 = -2m-4.$$

Consequently, we get that

$$\begin{aligned} F(r_{\beta_0}, m\Lambda_0) = F(I_1, m\Lambda_0) &= f_0^{3m+7} f_2^{m+2} f_1^{2m+4} f_2^{m+2} f_0^{m+1} \\ &= \big(\mathcal{L}(f_0^{4m+8})A^{(0)}_{-m-1}\mathcal{L}(f_2^{2m+4})A^{(2)}_{-m-2}\big)(f_1^{2m+4}). \end{aligned} \tag{3.51}$$

We now consider the case when $m = -\frac{3}{2}$. Then we have

$$\begin{aligned}
F\big(r_{\beta_0}, -\frac{3}{2}\Lambda_0\big) &= F\big(I_1, -\frac{3}{2}\Lambda_0\big) = \Big(\mathcal{L}(f_0^2)A_{\frac{1}{2}}^{(0)}\mathcal{L}(f_2)A_{-\frac{1}{2}}^{(2)}\Big)(f_1) \\
&= \Big(\mathcal{L}(x_\theta(-1)^2)A_{\frac{1}{2}}^{(0)}\mathcal{L}(x_{-\alpha_2}(0))A_{-\frac{1}{2}}^{(2)}\Big)\big(x_{-\alpha_1}(0)\big) \\
&= \Big(\mathcal{L}(x_\theta(-1)^2)A_{\frac{1}{2}}^{(0)}\Big)\big(x_{-\alpha_2}(0)x_{-\alpha_1}(0) - \frac{1}{2}x_{-\theta}(0)\big) \\
&= x_\theta(-1)^2x_{-\alpha_2}(0)x_{-\alpha_1}(0) - \frac{1}{2}x_\theta(-1)^2x_{-\theta}(0) \\
&\quad + \frac{1}{2}x_\theta(-1)x_{\alpha_1}(-1)x_{-\alpha_1}(0) - \frac{1}{2}x_\theta(-1)x_{-\alpha_2}(0)x_{\alpha_2}(-1) \\
&\quad - \frac{1}{4}x_\theta(-1)h_\theta(-1) - \frac{3}{4}x_{\alpha_1}(-1)x_{\alpha_2}(-1) + \frac{3}{8}x_\theta(-2).
\end{aligned}$$

Since $F(r_{\beta_0}, -\frac{3}{2}\Lambda_0)$ will act on $v_{-(3/2)\Lambda_0} \in N(-\frac{3}{2}\Lambda_0)$, we express it as

$$\begin{aligned}
F\big(r_{\beta_0}, -\frac{3}{2}\Lambda_0\big) = x_\theta(-1)^2x_{-\alpha_2}(0)x_{-\alpha_1}(0) - \frac{1}{2}x_\theta(-1)^2x_{-\theta}(0) \\
+ \frac{1}{2}x_\theta(-1)x_{\alpha_1}(-1)x_{-\alpha_1}(0) + \frac{1}{2}x_\theta(-1)h_{\alpha_2}(-1) \\
- \frac{1}{2}x_\theta(-1)x_{\alpha_2}(-1)x_{-\alpha_2}(0) - \frac{1}{4}x_\theta(-1)h_\theta(-1) \\
- \frac{3}{4}x_{\alpha_1}(-1)x_{\alpha_2}(-1) + \frac{3}{8}x_\theta(-2).
\end{aligned}$$

Hence in $N(-\frac{3}{2}\Lambda_0)$ we have

$$\begin{aligned}
v_{-3/2}^{(0)} = \Big(\frac{1}{2}x_\theta(-1)h_{\alpha_2}(-1) - \frac{1}{4}x_\theta(-1)h_\theta(-1) \\
- \frac{3}{4}x_{\alpha_1}(-1)x_{\alpha_2}(-1) + \frac{3}{8}x_\theta(-2)\Big)v_{-(3/2)\Lambda_0}.
\end{aligned}$$

Clearing fractions, we consider instead the singular vector $8v_{-3/2}^{(0)}$. The image of $[8v_{-3/2}^{(0)}] \in A\big(N(-\frac{3}{2}\Lambda_0)\big)$ under Ψ is given by

$$[8v_{-3/2}^{(0)}] \overset{\Psi}{\longmapsto} 4h_{\alpha_2}x_\theta - 2h_\theta x_\theta - 6x_{\alpha_2}x_{\alpha_1} - 3x_\theta. \tag{3.52}$$

From (3.52), we see that the weight of $\Psi([8v_{-3/2}^{(0)}])$ is equal to $\alpha_1 + \alpha_2$, and hence Corollary 3.35 implies that $\pi(R)$ is the ideal of $\mathcal{U}(\mathfrak{h})$ generated by the two polynomials p_1 and p_2 given by

$$p_1 = \pi\big(\operatorname{ad}(x_{-\theta})\Psi([8v^{(0)}_{-3/2}])\big)$$

and

$$p_2 = \pi\big(\operatorname{ad}(x_{-\alpha_1})\operatorname{ad}(x_{-\alpha_2})\Psi([8v^{(0)}_{-3/2}])\big).$$

Computing p_1 and p_2, we get from (3.52) that

$$\begin{aligned}
p_1 &= \pi\big(\operatorname{ad}(x_{-\theta})(4h_{\alpha_2}x_\theta - 2h_\theta x_\theta - 6x_{\alpha_2}x_{\alpha_1} - 3x_\theta)\big)\\
&= \pi(-4h_{\alpha_2}h_\theta + 2h_\theta^2 - 6x_{\alpha_2}x_{-\alpha_2} + 3h_\theta)\\
&= -4h_{\alpha_2}h_\theta + 2h_\theta^2 - 6h_{\alpha_2} + 3h_\theta\\
&= -4h_{\alpha_2}(h_{\alpha_1} + h_{\alpha_2}) + 2(h_{\alpha_1} + h_{\alpha_2})^2 - 6h_{\alpha_2} + 3(h_{\alpha_1} + h_{\alpha_2})\\
&= 2h_{\alpha_1}^2 + 3h_{\alpha_1} - 2h_{\alpha_2}^2 - 3h_{\alpha_2}\\
&= (h_{\alpha_1} - h_{\alpha_2})(2h_{\alpha_1} + 2h_{\alpha_2} + 3),
\end{aligned} \tag{3.53}$$

and that

$$\begin{aligned}
p_2 &= \pi\big(\operatorname{ad}(x_{-\alpha_1})\operatorname{ad}(x_{-\alpha_2})(4h_{\alpha_2}x_\theta - 2h_\theta x_\theta - 6x_{\alpha_2}x_{\alpha_1} - 3x_\theta)\big)\\
&= \pi\big(\operatorname{ad}(x_{-\alpha_1})(-4h_{\alpha_2}x_{\alpha_1} + 2h_\theta x_{\alpha_1} + 6h_{\alpha_2}x_{\alpha_1} + 3x_{\alpha_1})\big)\\
&= 4h_{\alpha_2}h_{\alpha_1} - 2h_\theta h_{\alpha_1} - 6h_{\alpha_2}h_{\alpha_1} - 3h_{\alpha_1}\\
&= 4h_{\alpha_2}h_{\alpha_1} - 2(h_{\alpha_1} + h_{\alpha_2})h_{\alpha_1} - 6h_{\alpha_2}h_{\alpha_1} - 3h_{\alpha_1}\\
&= -4h_{\alpha_1}h_{\alpha_2} - 2h_{\alpha_1}^2 - 3h_{\alpha_1} = -h_{\alpha_1}(2h_{\alpha_1} + 4h_{\alpha_2} + 3).
\end{aligned} \tag{3.54}$$

We now readily obtain the variety $\mathcal{V}\big(\pi(R)\big)$ for the admissible level $-\frac{3}{2}$. In terms of the coordinates of the basis of $\mathfrak{h}^*$ dual to the basis $(h_{\alpha_1}, h_{\alpha_2})$ of $\mathfrak{h}$, we have that

$$\mathcal{V}\big(\pi(R)\big) = \mathcal{V}\big(\{p_1, p_2\}\big) = \Big\{(0,0), (-\frac{3}{2}, 0), (0, -\frac{3}{2}), (-\frac{1}{2}, -\frac{1}{2})\Big\}. \tag{3.55}$$

In view of Theorems 3.18, 3.19, 3.33 and Proposition 3.20 we know that the irreducible $\mathbb{N}$-gradable weak modules from the category $\mathcal{O}$ for the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ are precisely those whose top level are irreducible highest weight $\mathfrak{g}$-modules with the highest weight belonging to the finite set in (3.55). If $L(\lambda)$ is an $L(-\frac{3}{2}\Lambda_0)$-module, then λ is necessarily of level $-\frac{3}{2}$ (i.e. $\lambda(c) = -\frac{3}{2}$).

It follows that the just mentioned correspondence is bijective, and we easily obtain from (3.55) that the $L(-\frac{3}{2}\Lambda_0)$-modules that we are looking for are of the form $L(\lambda)$ where

$$\lambda \in \Big\{ -\frac{3}{2}\Lambda_0, -\frac{3}{2}\Lambda_1, -\frac{3}{2}\Lambda_2, -\frac{1}{2}\rho \Big\}. \tag{3.56}$$

Comparing with Proposition 3.31, we see that the set in (3.56) coincides with the set of admissible weights of level $-\frac{3}{2}$ for $\mathfrak{sl}(3,\mathbb{C})^{\sim}$. Thus, we have the following theorem.

Theorem 3.36. *The irreducible $\mathbb{N}$-gradable weak modules from the category $\mathcal{O}$ for the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ are given by the modules $L(\lambda)$, where λ belongs to the set of admissible weights of level $-\frac{3}{2}$, which are listed in* (3.56).

Combining Theorem 3.36 and Theorem 3.26, we immediately obtain our next result, which shows that the conjecture of Adamović–Milas is true in the case of the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$.

Theorem 3.37. *Let M be a module for $L(-\frac{3}{2}\Lambda_0)$, which belongs to the category $\mathcal{O}$ as a $\hat{\mathfrak{g}}$-module. Then M is completely reducible.*

Considering ordinary modules for $L(-\frac{3}{2}\Lambda_0)$, we can draw the following conclusion.

Theorem 3.38. *The only irreducible ordinary module for $L(-\frac{3}{2}\Lambda_0)$ is its adjoint module, and every ordinary $L(-\frac{3}{2}\Lambda_0)$-module is a direct sum of a finite number of copies of this module.*

Proof. From Theorem 3.36 we obtain that the only irreducible $\mathbb{N}$-gradable weak $L(-\frac{3}{2}\Lambda_0)$-module belonging to the category $\mathcal{O}$ with a finite dimensional top level is given by $L(-\frac{3}{2}\Lambda_0)$ itself. Thus, the theorem follows immediately from Theorems 3.12 and 3.37 together with Lemma 3.13. □

As pointed out in the introduction of this thesis, the contents of Theorems 3.36, 3.37 and 3.38 already appear in the article [Per08] by O. Perše. These conclusions are therefore only to be considered as a confirmation of his results.

3.6 An application of the Sapovalov form

In this section we show how the Šapovalov form naturally occurs in connection with the ideal $\pi(R)$ introduced in the previous section, and how its determinant can be used to obtain information about the algebraic variety $\mathcal{V}\big(\pi(R)\big)$. We start by defining the Šapovalov form.

Let $\mathfrak{g}$ be a Kac–Moody algebra. The Šapovalov form $\mathcal{B}$ is the symmetric bilinear map

$$\mathcal{B}\colon \mathcal{U}(\mathfrak{g}) \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{h}) \tag{3.57}$$

defined by

$$\mathcal{B}(x, y) = \pi(\sigma(x)y), \quad \text{for } x, y \in \mathcal{U}(\mathfrak{g}), \tag{3.58}$$

where σ is the anti-involution of $\mathfrak{g}$ given in (1.1) extended to an anti-involution of $\mathcal{U}(\mathfrak{g})$. Notice that $\mathcal{B}$ is also bilinear when we consider $\mathcal{U}(\mathfrak{g})$ as a right $\mathcal{U}(\mathfrak{h})$-module, and hence, in this sense, $\mathcal{B}$ is a bilinear form over $\mathcal{U}(\mathfrak{h})$. Thus, if $D \subseteq \mathcal{U}(\mathfrak{g})$ is a free right $\mathcal{U}(\mathfrak{h})$-submodule of finite rank, then the determinant of $\mathcal{B}|_{D\times D}$ is defined up to a nonzero complex number, which depends on the choice of a basis for D. In this case, we let $\det(\mathcal{B}|_{D\times D})$ denote this determinant with respect to some basis—it will be irrelevant which basis we choose. It is clear that the root-lattice grading of $\mathcal{U}(\mathfrak{g})$ provides an orthogonal decomposition with respect to $\mathcal{B}$. We let $\mathcal{B}^{(\mu)}$ denote the restriction of $\mathcal{B}$ to $\mathcal{U}(\mathfrak{n}_- \oplus \mathfrak{h})_{-\mu} \times \mathcal{U}(\mathfrak{n}_- \oplus \mathfrak{h})_{-\mu}$, which means that the determinant $\det(\mathcal{B}^{(\mu)})$ is defined according to the just mentioned convention.

For every Verma module $M(\lambda)$, the Šapovalov form induces a symmetric bilinear form

$$\mathcal{B}_\lambda\colon M(\lambda) \times M(\lambda) \longrightarrow \mathbb{C}, \tag{3.59}$$

which is given by

$$\mathcal{B}_\lambda(xv_\lambda, yv_\lambda) = \big(\mathcal{B}(x,y)\big)(\lambda), \quad \text{for } x, y \in \mathcal{U}(\mathfrak{g}). \tag{3.60}$$

It is easy to show that the radical of $\mathcal{B}_\lambda$, $\operatorname{rad}(\mathcal{B}_\lambda)$, is equal to the maximal submodule of $M(\lambda)$. We let $\mathcal{B}_\lambda^{(\mu)}$ be the restriction of $\mathcal{B}_\lambda$ to $M(\lambda)_{\lambda-\mu} \times M(\lambda)_{\lambda-\mu}$.

An explicit formula for the determinant $\det(\mathcal{B}^{(\mu)})$ for symmetrizable Kac–Moody algebras was given by V. G. Kac and D. A. Kazhdan in [KK79]. This formula generalizes the formula proved by N. N. Šapovalov for finite-dimensional semisimple Lie algebras in [Šap72], and shows that $\det(\mathcal{B}^{(\mu)})$ factorizes into linear factors in $\mathcal{U}(\mathfrak{h})$.

In order to express this result, it will be convenient to introduce a set which repeats the positive roots according to their multiplicity. Thus, we let Δ^+_{mult} be given by

$$\Delta^+_{\text{mult}} = \{\, (\alpha, \ell) : \alpha \in \Delta_+, \ell = 1, \ldots, \dim \mathfrak{g}_\alpha \,\}.$$

Let $K \colon Q \to \mathbb{N}$ denote the Kostant partition function. For $\beta \in Q$, $K(\beta)$ is defined as the number of sequences $(m_{(\alpha,\ell)})$ of non-negative integers indexed by Δ^+_{mult} such that

$$\beta = \sum_{(\alpha,\ell)\in\Delta^+_{\text{mult}}} m_{(\alpha,\ell)}\alpha.$$

In particular K is identically equal to zero on the complement of Q_+. The next theorem gives the Šapovalov–Kac–Kazhdan determinant formula.

Theorem 3.39 ([KK79], Theorem 1)**.** *Let $\mathfrak{g}$ be a symmetrizable Kac–Moody algebra. Then for every $\beta \in Q_+ \cup \{0\}$, the determinant of $\mathcal{B}^{(\beta)}$ is, up to a nonzero constant, given by*

$$\det(\mathcal{B}^{(\beta)}) = \prod_{(\alpha,\ell)\in\Delta^+_{mult}} \prod_{n=1}^{\infty} \Big(\nu^{-1}(\alpha) + \rho(\nu^{-1}(\alpha)) - n\frac{(\alpha,\alpha)}{2}\Big)^{K(\beta-n\alpha)}.$$

We will now show how the Šapovalov form appears in relation to the study of modules for simple vertex operator algebras associated to affine Lie algebras. To this end, we adopt the framework introduced in the previous section. Recall that $\Psi([v^{(0)}])$ is homogeneous of weight $\eta \in Q_+$, and that we have fixed a basis $u_1, \ldots, u_\ell$ ($\ell = K(\eta)$) for $\mathcal{U}(\mathfrak{n}_-)_{-\eta}$. We now write $\Psi([v^{(0)}])$ as

$$\Psi([v^{(0)}]) = q_1\sigma(u_1) + \ldots + q_\ell\sigma(u_\ell) + r, \tag{3.61}$$

where $q_1, \ldots, q_\ell \in \mathcal{U}(\mathfrak{h})$ and $r \in \mathfrak{n}_-\mathcal{U}(\mathfrak{g})$ are uniquely determined. The following theorem explains the structure of the polynomial ideal $\pi(R)$.

Theorem 3.40. *Assume that $\Psi([v^{(0)}])$ is expressed by equation* (3.61)*. Let M be the matrix of $\mathcal{B}^{(\eta)}$ with respect to the ordered basis $(u_1, \ldots, u_\ell)$, and let v be the column vector $(q_1, \ldots, q_\ell)^\intercal$. Then the ideal $\pi(R)$ is generated by the entries of the column vector Mv.*

Proof. Let τ be the anti-involution of $\mathcal{U}(\mathfrak{g})$ determined by the condition that

$$x \overset{\tau}{\longmapsto} -x, \quad \text{for } x \in \mathfrak{g} \subset \mathcal{U}(\mathfrak{g}). \tag{3.62}$$

Corollary 3.35 applied to the basis $\tau(u_1), \ldots, \tau(u_\ell)$ of $\mathcal{U}(\mathfrak{n}_-)_{-\eta}$ then shows that $\pi(R)$ is generated by the polynomials

$$\pi\big((\operatorname{ad}\tau(u_i))\big(\Psi([v^{(0)}])\big)\big), \tag{3.63}$$

for $i = 1, \ldots, \ell$. Note that

$$\pi((\operatorname{ad}\tau(x))(y_1)y_2) = \pi(y_1 x y_2), \quad \text{for } x \in \mathfrak{n}_- \text{ and } y_1, y_2 \in \mathcal{U}(\mathfrak{g}),$$

and hence, by induction and linearity, that

$$\pi((\operatorname{ad}\tau(u))(y)) = \pi(yu), \quad \text{for } u \in \mathcal{U}(\mathfrak{n}_-) \text{ and } y \in \mathcal{U}(\mathfrak{g}).$$

Note also that the set $\mathfrak{n}_-\mathcal{U}(\mathfrak{g})$ is invariant under the adjoint action of $\mathfrak{n}_-$. By means of these observations, and using the expression for $\Psi([v^{(0)}])$ in (3.61), we obtain that the polynomial in (3.63) is equal to

$$\begin{aligned} &\sum_{j=1}^{\ell} \pi\big((\operatorname{ad}\tau(u_i))\big(q_j\sigma(u_j)\big)\big) + \pi((\operatorname{ad}\tau(u_i))(r)) \\ &\qquad = \sum_{j=1}^{\ell} \pi(q_j\sigma(u_j)u_i) = \sum_{j=1}^{\ell} q_j\pi(\sigma(u_j)u_i) = \sum_{j=1}^{\ell} q_j\mathcal{B}^{(\mu)}(u_i, u_j), \end{aligned}$$

which proves the theorem. □

Theorem 3.41. *Assume that* $\Psi([v^{(0)}])$ *is expressed by equation* (3.61). *Then*

$$\mathcal{V}\big(\pi(R)\big) \subseteq \mathcal{V}\big(\{\det(\mathcal{B}^{(\eta)})\}\big) \cup \mathcal{V}\big(\{q_1, \ldots, q_\ell\}\big).$$

Furthermore,

$$\mathcal{V}\big(\{q_1, \ldots, q_\ell\}\big) \subseteq \mathcal{V}\big(\pi(R)\big).$$

Proof. We adopt the notation of Theorem 3.40. By this theorem, it is clear that if $\lambda \in \mathcal{V}\big(\pi(R)\big)$, then either $q_i(\lambda) = 0$, for $i = 1, \ldots, \ell$, or the matrix M gives rise to a singular matrix when its entries are evaluated at λ. The latter is the same as saying that $\det(\mathcal{B}^{(\eta)})(\lambda) = 0$. If $q_i(\lambda) = 0$, for $i = 1, \ldots, \ell$, then it is again clear by Theorem 3.40 that $\lambda \in \mathcal{V}\big(\pi(R)\big)$. □

Remark 3.42. Notice that it is clear from the definition of the polynomials $q_1, \ldots, q_\ell$ in (3.61) that the variety $\mathcal{V}(\{q_1, \ldots, q_\ell\})$ is independent of how the basis $u_1, \ldots, u_\ell$ is chosen.

The next proposition shows that the ideal $\pi(R)$ is not equal to the trivial ideal $\{0\}$, or, in other words, that every irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ ($\lambda \in \hat{\mathfrak{h}}^*$) is not a weak $L(m\Lambda_0)$-module. According to Theorem 3.41, this is the same as saying that the polynomials $q_1, \ldots, q_\ell$ in (3.61) are not all equal to 0.

Proposition 3.43. *The ideal* $\pi(R)$ *is nontrivial, or, equivalently*

$$\mathcal{V}(\pi(R)) \neq \mathfrak{h}^*.$$

Proof. Let $\hat{\mathfrak{h}}^*_m$ denote the hyperplane in $\hat{\mathfrak{h}}^*$ consisting of weights of level m, i.e.

$$\hat{\mathfrak{h}}^*_m = \{\, \lambda \in \hat{\mathfrak{h}}^* : \lambda(c) = m \,\}.$$

For any $\lambda \in \hat{\mathfrak{h}}^*$, identify $M(\lambda)$ with $\mathcal{U}(\hat{\mathfrak{n}}_-)$ by means of the linear isomorphism given by $uv_\lambda \mapsto u$ for $u \in \mathcal{U}(\hat{\mathfrak{n}}_-)$. With this identification, we let ϕ be the mapping defined by

$$\begin{aligned} \phi \colon \hat{\mathfrak{h}}^*_m &\longrightarrow \mathcal{U}(\hat{\mathfrak{n}}_-), \\ \lambda &\overset{\phi}{\longmapsto} (v^{(0)})_{-1} v_\lambda, \end{aligned}$$

where $(v^{(0)})_{-1} = \operatorname{Res}_z Y_{M(\lambda)}(v^{(0)}, z)$. Regard $\hat{\mathfrak{h}}^*$ and $\mathcal{U}(\hat{\mathfrak{n}}_-)$ as normed linear spaces with respect to some norms. The topologies induced on $\hat{\mathfrak{h}}^*_m$ and the finite-dimensional vector space $\phi(\hat{\mathfrak{h}}^*)$ are independent of the choice of norms. It is clear that ϕ becomes a continuous map in relation to these topologies.

From the determinant formula in Theorem 3.39 we see that the Verma module $M(\lambda)$ is irreducible for all $\lambda \in \hat{\mathfrak{h}}^*$, except those lying in a countable union of hyperplanes. Furthermore, this union of hyperplanes does not contain the hyperplane $\hat{\mathfrak{h}}^*_m$. To prove this, assume that the linear polynomial $c - m \in \mathcal{U}(\hat{\mathfrak{h}})$ is a factor in the polynomial $\det(\mathcal{B}^{(\beta)})$ in Theorem 3.39, for some $\beta \in Q_+$. Since $(c, c) = 0$, we see from the factorization of the determinant $\det(\mathcal{B}^{(\beta)})$ that this implies that $c - m = c + \rho(c)$. But this is impossible because $-\rho(c)$ is the negative of the dual Coxeter number of $\mathfrak{g}$ and is hence different from m. Thus, there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of weights of level m such that

$$\lambda_n \longrightarrow m\Lambda_0, \quad \text{as } n \longrightarrow \infty,$$

and such that $M(\lambda_n)$ is irreducible for all $n \in \mathbb{N}$.

Now assume that $\pi(R)$ is equal to the trivial ideal $\{0\}$. By Theorems 3.18 and 3.33 it follows that any irreducible highest weight $\mathfrak{g}$-module can serve as top level in an irreducible $L(m\Lambda_0)$-module in the category $\mathcal{O}$. In particular, this implies that $M(\lambda_n)$ is an irreducible $L(m\Lambda_0)$-module for every $n \in \mathbb{N}$. Hence we obtain that

$$Y_{M(\lambda_n)}(v^{(0)}, z) = 0, \quad \text{for } n \in \mathbb{N},$$

whence $\phi(\lambda_n) = 0$ for all $n \in \mathbb{N}$. However, $\phi(m\Lambda_0) \neq 0$ since, by the creation property of vertex operator algebras, the image of $(v^{(0)})_{-1}v_{m\Lambda_0}$ in $N(m\Lambda_0)$ is the nonzero vector $v^{(0)}$. Hence we have arrived at a contradiction to the continuity of ϕ. This shows that $\pi(R) \neq \{0\}$. □

Remark 3.44. Notice that Proposition 3.43 implies that the weight η of $\Psi([v^{(0)}])$ must belong to the subset Q_+ of the root lattice, which was also proved in Proposition 3.32.

We now turn back to the class of simple vertex operator algebras associated to $\mathfrak{sl}(3,\mathbb{C})^\sim$ with admissible level belonging to the set $-\frac{3}{2} + \mathbb{N}$. Let η_m denote the weight of $\Psi([v_m^{(0)}])$. According to (3.51) this means that

$$\eta_m = (2m+4)\alpha_1 + (2m+4)\alpha_2.$$

As we have seen in (3.52), for level $-\frac{3}{2}$ we have that $\Psi([8v_{-3/2}^{(0)}])$ is given by

$$\begin{aligned}\Psi([8v_{-3/2}^{(0)}]) &= 4h_{\alpha_2}x_\theta - 2h_\theta x_\theta - 6x_{\alpha_2}x_{\alpha_1} - 3x_\theta \\ &= -6x_{\alpha_2}x_{\alpha_1} + (-2h_{\alpha_1} + 2h_{\alpha_2} - 3)x_\theta.\end{aligned}$$

Regarding this as an instance of the expression in (3.61), we get in this case that

$$\mathcal{V}\big(\{q_1, \dots, q_\ell\}\big) = \mathcal{V}\big(\{-6, -2h_{\alpha_1} + 2h_{\alpha_2} - 3\}\big) = \varnothing,$$

and hence, by Theorem 3.41, that

$$\mathcal{V}\big(\pi(R)\big) \subseteq \mathcal{V}\big(\{\det(\mathcal{B}^{(\eta_{-3/2})})\}\big).$$

The next proposition shows that the corresponding statement is true for all $m \in -\frac{3}{2} + \mathbb{N}$. Unlike the previous results in this section, the proof of this proposition depends on the explicit structure of $\Psi([v_m^{(0)}])$ obtained from (3.51).

Proposition 3.45. *Let η_m denote the weight of $\Psi([v_m^{(0)}])$. Then*

$$\mathcal{V}\big(\pi(R)\big) \subseteq \mathcal{V}\big(\{\det(\mathcal{B}^{(\eta_m)})\}\big).$$

Proof. Let

$$\{0\} \subset \hat{U}_0 \subset \hat{U}_1 \subset \ldots \subset \hat{U}_i \subset \ldots \subset \mathcal{U}(\hat{\mathfrak{g}})$$

denote the standard filtration of $\mathcal{U}(\hat{\mathfrak{g}})$, and let $(U_i)_{i\in\mathbb{N}}$ be the corresponding filtration of $\mathcal{U}(\mathfrak{g})$. In accordance with the notation in the proof of Theorem 2.19, we let

$$\begin{aligned} F_3 &= x_{-\alpha_1}(0)^{2m+4}, \\ F_2 &= \big(\mathcal{L}(x_{-\alpha_2}(0)^{2m+4})A^{(2)}_{-m-2}\big)(F_3), \end{aligned}$$

and

$$F_1 = \big(\mathcal{L}(x_\theta(-1)^{4m+8})A^{(0)}_{-m-1}\big)(F_2).$$

It is then clear that for some constant $c_1 \in \mathbb{Q} \setminus \{0\}$, we have that

$$F_2 \equiv c_1 x_{-\alpha_2}(0)^{2m+4} x_{-\alpha_1}(0)^{2m+4} \pmod{\hat{U}_{4m+7}}. \tag{3.64}$$

Likewise, it is clear that

$$F_1 \equiv c_2 \operatorname{ad}\big(x_\theta(-1)\big)^{4m+8}(F_2) \pmod{x_\theta(-1)\mathcal{U}(\hat{\mathfrak{g}})}, \tag{3.65}$$

for some $c_2 \in \mathbb{Q} \setminus \{0\}$. Combining (3.64) and (3.65) we obtain that there exists a constant $c_3 \in \mathbb{Q} \setminus \{0\}$ such that

$$F_1 \equiv c_3 x_{\alpha_1}(-1)^{2m+4} x_{\alpha_2}(-1)^{2m+4} \pmod{\hat{U}_{4m+7} + x_\theta(-1)\mathcal{U}(\hat{\mathfrak{g}})}. \tag{3.66}$$

Let Υ be the mapping given by

$$\begin{aligned} \Upsilon \colon \mathcal{U}(\hat{\mathfrak{g}}) &\longrightarrow \mathcal{U}(\mathfrak{g}), \\ u &\overset{\Upsilon}{\longmapsto} \Psi([u v_{m\Lambda_0}]), \end{aligned}$$

which means that $\Upsilon(F_1) = \Psi([v_m^{(0)}])$. From the definition of Ψ in Theorem 3.19 we see that $\Upsilon(\hat{U}_i) = U_i$. (Note, however, that a monomial in $\mathcal{U}(\hat{\mathfrak{g}})$ of length n with respect to the standard filtration, is in general mapped by Υ to a sum of monomials in $\mathcal{U}(\mathfrak{g})$, whose lengths are less than or equal to n.) Thus, we get from (3.66) that

$$\Psi([v_m^{(0)}]) \equiv c_3 x_{\alpha_2}^{2m+4} x_{\alpha_1}^{2m+4} \pmod{U_{4m+7} + \mathcal{U}(\mathfrak{g})x_\theta}. \tag{3.67}$$

Consider the basis elements for $\mathfrak{n}_-$ introduced in Section 1.2, and let $\prec$ be the ordering of them given by

$$x_{-\theta} \prec x_{-\alpha_1} \prec x_{-\alpha_2}. \tag{3.68}$$

Let $\{u_1, \dots, u_\ell\}$ be the basis for $\mathcal{U}(\mathfrak{n}_-)_{-\eta_m}$, obtained as a subset of the Poincaré–Birkhoff–Witt basis for $\mathcal{U}(\mathfrak{n}_-)$ corresponding to the order relation $\prec$. Let u_1 be the basis element given by

$$u_1 = x_{-\alpha_1}^{2m+4} x_{-\alpha_2}^{2m+4}.$$

Express $\Psi([v_m^{(0)}])$ in terms of $u_1, \dots, u_\ell$ as in equation (3.61) and compare the result with (3.67). We then find that the polynomial $q_1 \in \mathcal{U}(\mathfrak{h})$ is equal to the nonzero constant c_3, and consequently

$$\mathcal{V}\big(\{q_1, \dots, q_\ell\}\big) = \varnothing.$$

In view of Theorem 3.41, this proves the proposition. $\square$

We apply Theorem 3.39 to calculate the Šapovalov determinant for the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. For $\mu = k_1\alpha_1 + k_2\alpha_2$, with k_1, $k_2 \in \mathbb{N}$, we obtain

$$\begin{aligned}\det(\mathcal{B}^{(\mu)}) = &\Big(\prod_{j=1}^{k_1}(h_{\alpha_1} + 1 - j)^{\min\{k_1-j,k_2\}+1}\Big)\\ &\cdot\Big(\prod_{j=1}^{k_2}(h_{\alpha_2} + 1 - j)^{\min\{k_1,k_2-j\}+1}\Big)\\ &\cdot\Big(\prod_{j=1}^{\min\{k_1,k_2\}}(h_{\alpha_1} + h_{\alpha_2} + 2 - j)^{\min\{k_1-j,k_2-j\}+1}\Big).\end{aligned}$$

Specializing to $\eta_m = (2m+4)\alpha_1 + (2m+4)\alpha_2$ ($m \in -\frac{3}{2} + \mathbb{N}$), which is the weight of $\Psi([v_m^{(0)}])$, we get that

$$\det(\mathcal{B}^{(\eta_m)}) = \prod_{j=1}^{2m+4} \big((h_{\alpha_1}+1-j)(h_{\alpha_2}+1-j)(h_{\alpha_1}+h_{\alpha_2}+2-j)\big)^{2m+4-j+1}.$$

When comparing this polynomial with the explicit description of admissible weights for $\mathfrak{sl}(3,\mathbb{C})^\sim$ given in Proposition 3.31, we see that the restriction to $\mathfrak{h}$ of the admissible weights of level m are zeros of $\det(\mathcal{B}^{(\eta_m)})$, i.e.

$$\big(\det(\mathcal{B}^{(\eta_m)})\big)(\lambda) = 0, \quad \text{for } \lambda \in \mathcal{P}^{(m)}. \tag{3.69}$$

This is of course in agreement with the conjecture of Adamović–Milas mentioned earlier. In fact, the assertion in (3.69) is not true if we replace $\det(\mathcal{B}^{(\eta_m)})$ in this equation with any divisor of $\det(\mathcal{B}^{(\eta_m)})$ with fewer distinct irreducible factors.

In the particular case when $m = -\frac{3}{2}$, we have that

$$\det(\mathcal{B}^{(\alpha_1+\alpha_2)}) = h_{\alpha_1} h_{\alpha_2}(h_{\alpha_1} + h_{\alpha_2} + 1).$$

Comparing with the generators p_1 and p_2 of the ideal $\pi(R)$ obtained in the previous section (cf. (3.53) and (3.54)), we find that

$$\det(\mathcal{B}^{(\alpha_1+\alpha_2)}) = -\frac{1}{6} h_{\alpha_1} p_1 - \frac{1}{6}(h_{\alpha_1} + h_{\alpha_2}) p_2,$$

and hence that $\det(\mathcal{B}^{(\eta_{-3/2})}) \in \pi(R)$. In general, for $m > -\frac{3}{2}$, the only conclusion along these lines that we can draw from Proposition 3.45 is that $\det(\mathcal{B}^{(\eta_m)})$ belongs to the radical of $\pi(R)$.

3.7 Intertwining operators

In this section our objective is to examine the intertwining operators of the irreducible $\mathbb{N}$-gradable weak $L(-\frac{3}{2}\Lambda_0)$-modules, determined in Theorem 3.36. Our approach is to use the theory of bimodules for Zhu's algebra developed by I. Frenkel and Y. Zhu in [FZ92]. In their article they apply this theory to the case of vertex operator algebras of affine type at positive integer levels, but in the present case we have to make some modifications in order to consider admissible levels.

In the study of intertwining operators we have to consider yet more general formal series than the ones previously introduced. For any vector space A, we let $A\{z\}$ denote the space

$$A\{z\} = \Big\{ \sum_{n\in\mathbb{Q}} a_n z^n : a_n \in A \Big\}.$$

Intertwining operators of modules of a vertex operator algebra are then defined as follows (cf. e.g. [FHL93]).

Definition 3.46. Let V be a vertex operator algebra and let (W^i, Y_i) be weak V-modules for $i = 1, 2, 3$. An intertwining operator of type $\binom{W^3}{W^1\ W^2}$ is a linear map

$$\mathcal{Y}\colon W^1 \longrightarrow \big(\operatorname{Hom}(W^2, W^3)\big)\{z\},$$
$$w \longmapsto \sum_{n\in\mathbb{Q}} w_n z^{-n-1},$$

which satisfies the following conditions:

(i) the truncation condition:

$$\text{for } w^{(i)} \in W^i,\, i = 1, 2, \quad w^{(1)}_n w^{(2)} = 0, \quad \text{for sufficiently large } n \in \mathbb{Q};$$

(ii) the $L(-1)$-derivative property:

$$\mathcal{Y}(L(-1)w, z) = \frac{d}{dz}\mathcal{Y}(w, z), \quad \text{for all } w \in W^1;$$

(iii) the Jacobi identity:

$$\begin{aligned} &z_0^{-1}\delta\Big(\frac{z_1 - z_2}{z_0}\Big)Y_3(v, z_1)\mathcal{Y}(w^{(1)}, z_2)w^{(2)} \\ &\quad - z_0^{-1}\delta\Big(\frac{z_2 - z_1}{-z_0}\Big)\mathcal{Y}(w^{(1)}, z_2)Y_2(v, z_1)w^{(2)} \\ &= z_2^{-1}\delta\Big(\frac{z_1 - z_0}{z_2}\Big)\mathcal{Y}(Y_1(v, z_0)w^{(1)}, z_2)w^{(2)}, \end{aligned}$$

for all $v \in V$, $w^{(1)} \in W^1$ and $w^{(2)} \in W^2$.

The vector space of all intertwining operators of type $\binom{W^3}{W^1\ W^2}$ is denoted by $\mathcal{I}\binom{W^3}{W^1\ W^2}$, and is called the fusion rule of corresponding type.

We notice that if (W, Y_W) is a weak V-module, then Y_W is clearly an intertwining operator of type $\binom{W}{V\ W}$. Moreover, there exist certain skew-symmetry constructions, which give rise to linear isomorphisms between the spaces $\mathcal{I}\binom{W^3}{W^1\ W^2}$ and $\mathcal{I}\binom{W^3}{W^2\ W^1}$. The result of the following proposition is obtained as a direct generalization of the arguments in Proposition 7.1 in [HL95] to the case of weak modules.

Proposition 3.47. *Let V be a vertex operator algebra, and let W^i be weak V-modules for $i = 1, 2, 3$. Then*

$$\dim \mathcal{I}\binom{W^3}{W^1\, W^2} = \dim \mathcal{I}\binom{W^3}{W^2\, W^1}.$$

Next, we will show how to create a bimodule for Zhu's algebra from a weak vertex operator algebra module. The definitions and basic results are in many ways analogous to corresponding statements about Zhu's algebra. Indeed, for the adjoint module of a vertex operator algebra, the following construction is just a reformulation of the definition of Zhu's algebra.

Let V be a vertex operator algebra and let W be a weak V-module. Introduce left and right actions of V on W (both denoted by $*$) by letting

$$u * w = \mathrm{Res}_z \left(Y(u, z) \frac{(z+1)^{\mathrm{wt}\, u}}{z} w \right), \tag{3.70}$$

and

$$w * u = \mathrm{Res}_z \left(Y(u, z) \frac{(z+1)^{\mathrm{wt}\, u - 1}}{z} w \right), \tag{3.71}$$

for $w \in W$ and homogeneous $u \in V$, and extend to all of V by linearity. Also, define $O(W)$ to be the subspace of W spanned by all elements of the form

$$\mathrm{Res}_z \left(Y(u, z) \frac{(z+1)^{\mathrm{wt}\, u}}{z^2} w \right),$$

for $w \in W$ and homogeneous $u \in V$. Let $A(W)$ be the quotient space given by

$$A(W) = W/O(W).$$

As in the case of Zhu's algebra, we use the notations $[w]$ and $[W']$ to denote the images of an element $w \in W$ and a subspace $W' \subseteq W$, respectively, under the quotient map $W \to A(W)$.

The space $A(W)$ is an $A(V)$-bimodule in accordance with the following result.

Theorem 3.48 ([FZ92], Theorem 1.5.1)**.** *Let V be a vertex operator algebra and let W be a weak V-module. Then the binary operations denoted by $*$ in* (3.70) *and* (3.71) *induce on $A(W) = W/O(W)$ the structure of a bimodule for $A(V) = V/O(V)$.*

The next result explains the effect on the bimodule $A(W)$ when we transfer to a quotient of W (cf. Proposition 3.16).

Proposition 3.49 ([FZ92], Proposition 1.5.4)**.** *Retain the assumptions of the previous theorem, and let W^1 be a submodule of W. Then $[W^1]$ is a submodule of the $A(V)$-bimodule $A(W)$, and the quotient $A(W)/[W^1]$ is isomorphic to the bimodule $A(W/W^1)$ associated to the quotient V-module W/W^1. Moreover, if I is an ideal of V, $\mathbf{1}, \omega \notin I$ and $IW \subseteq W^1$ (i.e. $u_n w \in W^1$, for all $u \in I$, $w \in W$ and $n \in \mathbb{Z}$), then $[I]A(W) \subseteq [W^1]$ and the induced $A(V/I)$-bimodule $A(W)/[W^1]$ is isomorphic to the $A(V/I)$-bimodule associated to the V/I module W/W^1.*

We now consider fusion rules again. For $i = 1, 2, 3$, let W^i be $\mathbb{N}$-gradable weak V-modules, and let $\mathcal{Y} \in \mathcal{I}\binom{W^3}{W^1\ W^2}$ be an intertwining operator. Furthermore, assume that there exists $r_i \in \mathbb{C}$ such that $L(0)$ acts on $W^i_{(n)}$ by the scalar $r_i + n$, for $i = 1, 2, 3$ and $n \in \mathbb{N}$. Then, if we express $\mathcal{Y}(w, z)$ as

$$\mathcal{Y}(w, z) = \sum_{n \in \mathbb{Z}} w_n z^{-n-1-r_1-r_2+r_3},$$

it follows that if w is homogeneous, we have that

$$w_n W^2_{(m)} \subseteq W^3_{(m+\mathrm{wt}\, w-n-1)}.$$

In particular, we see that $w_{\mathrm{wt}\, w-1}$ maps $W^2_{(0)}$ into $W^3_{(0)}$. As in the case of Zhu's algebra, we thus introduce a linear function

$$o\colon W^1 \longrightarrow \mathrm{Hom}(W^2_{(0)}, W^3_{(0)}),$$

by letting

$$o(w) = w_{\mathrm{wt}\, w-1}|_{W^2_{(0)}},$$

for homogeneous $w \in W^1$, and extending linearly to the whole of W^1.

The next theorem, which is parallel to the first statement in Theorem 3.18, shows how intertwining operators are connected to the previously introduced bimodules via the map o (cf. Lemma 1.5.2 in [FZ92]).

Theorem 3.50. *The linear map o factors through $O(W^1)$ and induces an $A(V)$-bimodule homomorphism from $A(W^1)$ to $\mathrm{Hom}(W^2_{(0)}, W^3_{(0)})$, where the $A(V)$-bimodule structure of $\mathrm{Hom}(W^2_{(0)}, W^3_{(0)})$ is determined by Theorem 3.18.*

For rings R and S, use the notations ${}_RM_S$ and ${}_RN_S$ to specify that M and N are (R,S)-bimodules, and let $\mathrm{Hom}_R^S(M,N)$ denote the space of (R,S)-bimodule homomorphisms from M to N. We then have the following result, which is a generalization of e.g. Proposition 2.2, Ch. 9, in [CE56]. The proof consists of simple verifications and is omitted.

Proposition 3.51. *Let K be a commutative ring, let R, S and T be associative K-algebras, and let ${}_RM_S$, ${}_SN_T$ and ${}_RP_T$ be bimodules as indicated. Then, as K-linear spaces,*

$$\mathrm{Hom}_R^T(M\otimes_S N, P)\cong \mathrm{Hom}_S^T\big(N, \mathrm{Hom}_R(M,P)\big) \tag{3.72}$$

and

$$\mathrm{Hom}_R^T(M\otimes_S N, P)\cong \mathrm{Hom}_R^S\big(M, \mathrm{Hom}^T(N,P)\big). \tag{3.73}$$

The conclusion of Theorem 3.50 is that there exists a linear map ϕ determined by

$$\begin{aligned}\phi\colon \mathcal{I}\binom{W^3}{W^1\;W^2} &\longrightarrow \mathrm{Hom}_{A(V)}^{A(V)}\big(A(W^1), \mathrm{Hom}(W^2_{(0)}, W^3_{(0)})\big),\\ \mathcal{Y}(\cdot,z) &\overset{\phi}{\longmapsto} o.\end{aligned}$$

We will now rewrite the codomain of ϕ with the help of Proposition 3.51. Considering (3.73) with $R=S=A(V)$, $T=\mathbb{C}$, $M=A(W^1)$, $N=W^2_{(0)}$ and $P=W^3_{(0)}$, we obtain a linear isomorphism

$$\begin{aligned}\Omega\colon\ &\mathrm{Hom}_{A(V)}^{A(V)}\big(A(W^1), \mathrm{Hom}(W^2_{(0)}, W^3_{(0)})\big)\longrightarrow\\ &\qquad\mathrm{Hom}_{A(V)}\big(A(W^1)\otimes_{A(V)} W^2_{(0)}, W^3_{(0)}\big).\end{aligned}$$

We have thus arrived at a linear map $\pi=\Omega\circ\phi$, where

$$\pi\colon \mathcal{I}\binom{W^3}{W^1\;W^2}\longrightarrow \mathrm{Hom}_{A(V)}\big(A(W^1)\otimes_{A(V)} W^2_{(0)}, W^3_{(0)}\big), \tag{3.74}$$

and is given by

$$\big(\pi(\mathcal{Y})\big)(w^{(1)}\otimes w^{(2)})=\big(o(w^{(1)})\big)(w^{(2)}),$$

for $\mathcal{Y}\in\mathcal{I}\binom{W^3}{W^1\;W^2}$, $w^{(1)}\in A(W^1)$ and $w^{(2)}\in W^2_{(0)}$.

The following theorem contains results from Proposition 2.10 and Corollary 2.13 in [Li99], which show that the map π (or, equivalently, ϕ) can give extensive information concerning fusion rules.

Theorem 3.52. *Let V be a vertex operator algebra and let W^i be irreducible $\mathbb{N}$-gradable weak V-modules, for $i = 1, 2, 3$. Then the linear map π in* (3.74) *is injective. Furthermore, if V is rational, then π is a linear isomorphism.*

Remark 3.53. The first statement in the above theorem amounts to a slight reformulation of Proposition 2.10 in [Li99], which is proved by rather straightforward arguments. However, the second statement is a corollary of Theorem 2.11 in [Li99], which is a much more elaborate result.

We now turn our attention to fusion rules for vertex operator algebras associated to affine Lie algebras, and adopt the notation developed in Section 3.2. The arguments used to prove the second statement in Theorem 3.52 can easily be modified to obtain the following variation of that result (cf. Remark 2.14 in [Li99]).

Theorem 3.54. *Assume that the vertex operator algebra $V = L(k\Lambda_0)$ is rational in the category $\mathcal{O}$, and let $W^i = L(\Lambda_i)$ $(\Lambda_i \in \hat{\mathfrak{h}}^*)$ be irreducible $L(k\Lambda_0)$-modules in the category $\mathcal{O}$, for $i = 1, 2, 3$. Then the map π in* (3.74) *is a linear isomorphism.*

For $\lambda \in \mathfrak{h}^*$, we consider $L(\lambda)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})$ as a left tensor product $\mathfrak{g}$-module, and also as a right $\mathfrak{g}$-module with the action induced by its right component $\mathcal{U}(\mathfrak{g})$. In other words, for $u \otimes x \in L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ and $a \in \mathfrak{g}$, we let

$$a(u \otimes x) = au \otimes x + u \otimes ax \quad \text{and} \quad (u \otimes x)a = u \otimes xa.$$

The following theorem is analogous to Theorem 3.19, and explains how the structure of the bimodule $A(M(k, L(\lambda)))$ can be identified with that of $L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$.

Theorem 3.55 ([FZ92], Theorem 3.2.1)**.** *Let $\lambda \in \mathfrak{h}^*$ and regard $L(\lambda)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})$ as an $A(N(k\Lambda_0))$-bimodule by means of the isomorphism $\Psi\colon A(N(k\Lambda_0)) \to \mathcal{U}(\mathfrak{g})$ (cf. Theorem 3.19). Then $A(M(k, L(\lambda)))$ and $L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ are isomorphic as $A(N(k\Lambda_0))$-bimodules, and an isomorphism is given by*

$$\Upsilon\colon A(M(k, L(\lambda))) \longrightarrow L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g}),$$
$$[a_1(-i_1 - 1) \dots a_n(-i_n - 1)v] \stackrel{\Upsilon}{\longmapsto} (-1)^{i_1+\dots+i_n}(v \otimes a_n \dots a_1),$$

where $v \in L(\lambda)$, and $a_j \in \mathfrak{g}$ and $i_j \in \mathbb{N}$, for $j = 1, \dots, n$.

We now restrict our consideration further to the case of vertex operator algebras at admissible level. Let $\lambda + k\Lambda_0$ ($\lambda \in \mathfrak{h}^*$) be an admissible weight, and let P_λ be the set given by

$$P_\lambda = \{\, \Upsilon([v^\alpha]) : \alpha^\vee \in \Pi^\vee_{\lambda+k\Lambda_0} \,\} \subset L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g}),$$

where the vectors v^α are (the images in $M(k, L(\lambda))$ of) the singular vectors generating the maximal submodule of $M(\lambda + k\Lambda_0)$, as described in Theorem 3.24. We define Γ_λ to be the sub-bimodule of $L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ generated by P_λ, i.e.

$$\Gamma_\lambda = \langle P_\lambda \rangle.$$

For later use, we also express the element $\Upsilon([v^\alpha]) \in L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ explicitly as

$$\Upsilon([v^\alpha]) = \sum_{j \in J_\alpha} (u_\alpha^{(1,j)} v_\lambda \otimes u_\alpha^{(2,j)}), \tag{3.75}$$

where $u_\alpha^{(i,j)} \in \mathcal{U}(\mathfrak{g})$, for $i = 1, 2$ and $j \in J_\alpha$.

Applying Proposition 3.49, we immediately obtain the following result, which is a more general form of Theorem 3.2.2 in [FZ92].

Theorem 3.56. *If $\lambda + k\Lambda_0$ ($\lambda \in \mathfrak{h}^*$) is an admissible weight, then the map Υ of Theorem 3.55 induces an $A(N(k\Lambda_0))$-bimodule isomorphism*

$$\Upsilon' \colon A(L(\lambda + k\Lambda_0)) \longrightarrow \big(L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})\big)/\Gamma_\lambda.$$

Furthermore, if $L(\lambda + k\Lambda_0)$ is a weak $L(k\Lambda_0)$-module, then the ideal $\langle [v^{(0)}] \rangle$ of $A(N(k\Lambda_0))$ acts trivially on $A(L(\lambda + k\Lambda_0))$, and the induced bimodule for $A(L(k\Lambda_0)) \cong A(N(k\Lambda_0))/\langle [v^{(0)}] \rangle$ is the bimodule associated to the weak $L(k\Lambda_0)$-module $L(\lambda + k\Lambda_0)$.

Provided that $L(\lambda + k\Lambda_0)$ is an $L(k\Lambda_0)$-module, the above theorem tells us that the $A(L(k\Lambda_0))$-bimodule $A(L(\lambda + k\Lambda_0))$ can be identified with the $\mathcal{U}(\mathfrak{g})/R$-bimodule $\big(L(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})\big)/\Gamma_\lambda$. Now assume that the weights $\lambda_i + k\Lambda_0$ ($\lambda_i \in \mathfrak{h}^*$), for $i = 1, 2, 3$, are admissible, and that $W^i = L(\lambda_i + k\Lambda_0)$ are $L(k\Lambda_0)$-modules. Then Theorem 3.52 implies that the map π in (3.74) takes the form of an injective linear mapping (which we continue to denote by π) given by

$$\begin{aligned} \pi \colon \mathcal{I}\binom{L(\lambda_3 + k\Lambda_0)}{L(\lambda_1 + k\Lambda_0)\ L(\lambda_2 + k\Lambda_0)} &\longrightarrow \\ \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})/R}\big(\big(L(\lambda_1) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})\big)/\Gamma_{\lambda_1} &\otimes_{\mathcal{U}(\mathfrak{g})/R} L(\lambda_2), L(\lambda_3)\big). \end{aligned} \tag{3.76}$$

In order to attain a better understanding of the fusion rule above, we will express the codomain of the injection π somewhat differently. Consider the map τ given by

$$\begin{aligned} \tau\colon L(\lambda_1)\otimes_{\mathbb{C}} L(\lambda_2) &\longrightarrow \big(L(\lambda_1)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})\big)\otimes_{\mathcal{U}(\mathfrak{g})} L(\lambda_2), \\ u_1\otimes u_2 &\overset{\tau}{\longmapsto} (u_1\otimes 1)\otimes u_2, \end{aligned} \tag{3.77}$$

for $u_i\in L(\lambda_i)$, $i=1,2$. Clearly τ is an isomorphism of vector spaces. Now, regard the domain of τ as a tensor product module for $\mathfrak{g}$, and let the codomain be equipped with the $\mathfrak{g}$-module structure obtained from its component $L(\lambda_1)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})$. Then we have that

$$\begin{aligned} &\tau(a(u_1\otimes u_2)) = \tau(au_1\otimes u_2 + u_1\otimes au_2) = \\ &(au_1\otimes 1)\otimes u_2 + (u_1\otimes 1)\otimes au_2 = (au_1\otimes 1)\otimes u_2 + (u_1\otimes a)\otimes u_2 = \\ &(au_1\otimes 1 + u_1\otimes a)\otimes u_2 = a((u_1\otimes 1)\otimes u_2) = a\tau(u_1\otimes u_2), \end{aligned}$$

for $a\in\mathfrak{g}$, and $u_i\in L(\lambda_i)$, $i=1,2$. Thus, τ is a $\mathfrak{g}$-module isomorphism.

We can now give our desired characterization of the fusion rule between three irreducible $L(k\Lambda_0)$-modules with admissible highest weights.

Theorem 3.57. *For $i=1,2,3$, let $\lambda_i+k\Lambda_0$ ($\lambda_i\in\mathfrak{h}^*$) be admissible weights, and assume that $L(\lambda_i+k\Lambda_0)$ are $L(k\Lambda_0)$-modules. For $\alpha^\vee\in\Pi^\vee_{\lambda_1+k\Lambda_0}$, use the notation of* (3.75) *to define the subspace $U_\alpha\subset L(\lambda_1)\otimes_{\mathbb{C}} L(\lambda_2)$ by*

$$U_\alpha = \sum_{j\in J_\alpha}\mathbb{C}u_\alpha^{(1,j)}v_{\lambda_1}\otimes_{\mathbb{C}} u_\alpha^{(2,j)}L(\lambda_2).$$

Then there exists a linear injection

$$\begin{aligned} \tilde{\pi}\colon\ &\mathcal{I}\binom{L(\lambda_3+k\Lambda_0)}{L(\lambda_1+k\Lambda_0)\ L(\lambda_2+k\Lambda_0)} \longrightarrow \\ &\quad \big\{\, f\in\operatorname{Hom}_{\mathfrak{g}}\big(L(\lambda_1)\otimes_{\mathbb{C}} L(\lambda_2), L(\lambda_3)\big) : f|_{U_\alpha}=0, \text{ for } \alpha^\vee\in\Pi^\vee_{\lambda_1+k\Lambda_0}\,\big\}. \end{aligned}$$

Furthermore, if the vertex operator algebra $L(k\Lambda_0)$ is rational in the category $\mathcal{O}$, then $\tilde{\pi}$ is an isomorphism of vector spaces.

Proof. The mapping τ in (3.77) induces a $\mathfrak{g}$-module epimorphism

$$\bar{\tau}\colon L(\lambda_1)\otimes_{\mathbb{C}} L(\lambda_2)\longrightarrow \big(\big(L(\lambda_1)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})\big)/\Gamma_{\lambda_1}\otimes_{\mathcal{U}(\mathfrak{g})/R} L(\lambda_2).$$

It is clear that the image of $\Gamma_{\lambda_1} \otimes_{\mathcal{U}(\mathfrak{g})} L(\lambda_2)$ in $\big(L(\lambda_1) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})\big) \otimes_{\mathcal{U}(\mathfrak{g})} L(\lambda_2)$ is generated, as a $\mathfrak{g}$-submodule, by the subspaces

$$\tilde{U}_\alpha = \mathbb{C}\Upsilon([v^\alpha]) \otimes_{\mathcal{U}(\mathfrak{g})} L(\lambda_2),$$

for $\alpha^\vee \in \Pi^\vee_{\lambda_1+k\Lambda_0}$. Since U_α corresponds to $\tilde{U}_\alpha$ under the isomorphism τ, it follows that the kernel of $\bar{\tau}$ is generated by the subspaces U_α ($\alpha^\vee \in \Pi^\vee_{\lambda_1+k\Lambda_0}$). Thus, $\bar{\tau}$ induces a linear isomorphism

$$\begin{aligned}\psi\colon\ &\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})/R}\left(\big(L(\lambda_1)\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g})\big)/\Gamma_{\lambda_1}\otimes_{\mathcal{U}(\mathfrak{g})/R} L(\lambda_2), L(\lambda_3)\right) \longrightarrow \\ &\left\{ f \in \mathrm{Hom}_{\mathfrak{g}}\left(L(\lambda_1)\otimes_{\mathbb{C}} L(\lambda_2), L(\lambda_3)\right) : f|_{U_\alpha} = 0, \text{ for } \alpha^\vee \in \Pi^\vee_{\lambda_1+k\Lambda_0} \right\}.\end{aligned}$$

Let $\tilde{\pi} = \psi \circ \pi$, where π is the map in (3.76). The asserted properties of $\tilde{\pi}$ in the statement of the theorem then follow immediately from Theorems 3.52 and 3.54. □

We will now focus on the case of the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ related to the affine Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{sl}(3,\mathbb{C})^\sim$ and at admissible level $-\frac{3}{2}$. Theorem 3.37 established that $L(-\frac{3}{2}\Lambda_0)$ is rational in the category $\mathcal{O}$, and its four irreducible $\mathbb{N}$-gradable weak modules from the category $\mathcal{O}$ were determined in Theorem 3.36. With the help of Theorem 3.57 we will be able to compute the dimensions of the fusion rules between these modules.

Theorem 3.58. *Let W^i, for $i = 1, 2, 3$, be irreducible $\mathbb{N}$-gradable weak modules from the category $\mathcal{O}$ for the vertex operator algebra $V = L(-\frac{3}{2}\Lambda_0)$. Then*

$$\dim \mathcal{I}\binom{W^3}{W^1\ W^2} = \begin{cases} 1 & \text{if } W^1 = V \text{ and } W^2 = W^3 \\ 1 & \text{if } W^2 = V \text{ and } W^1 = W^3 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. According to Theorem 3.36, the vertex operator algebra $L(-\frac{3}{2}\Lambda_0)$ is rational in the category $\mathcal{O}$, and its irreducible $\mathbb{N}$-gradable weak modules are those $L(\lambda)$ where $\lambda \in \tilde{\mathfrak{h}}^*$ is admissible of level $-\frac{3}{2}$. The corresponding top levels of these modules are the $\mathfrak{g}$-modules of the form $L(\mu)$, where $\mu \in \mathfrak{h}^*$ belongs to the set of weights in (3.55). In the coordinates of the basis of $\mathfrak{h}^*$ dual to the basis $(h_{\alpha_1}, h_{\alpha_2})$ of $\mathfrak{h}$, we express these weights as

$$\mu_0 = (0,0), \quad \mu_1 = (-\frac{3}{2}, 0), \quad \mu_2 = (0, -\frac{3}{2}) \quad \text{and} \quad \mu_3 = (-\frac{1}{2}, -\frac{1}{2}).$$

The simple roots α_1 and α_2 are, in the same basis, given by

$$\alpha_1 = (2, -1) \quad \text{and} \quad \alpha_2 = (-1, 2).$$

From Theorem 3.57, we know that for any i, j, $k \in \{0, 1, 2, 3\}$, the fusion rule

$$\mathcal{I}\binom{L(\mu_k - \frac{3}{2}\Lambda_0)}{L(\mu_i - \frac{3}{2}\Lambda_0)\ L(\mu_j - \frac{3}{2}\Lambda_0)}$$

is linearly isomorphic to a subspace of $\mathrm{Hom}_{\mathfrak{g}}\left(L(\mu_i) \otimes_{\mathbb{C}} L(\mu_j), L(\mu_k)\right)$. The weight $\mu_i + \mu_j$ is a maximal weight for the $\mathfrak{g}$-module $L(\mu_i) \otimes_{\mathbb{C}} L(\mu_j)$ in the sense that every other weight is of the form $\mu_i + \mu_j - \beta$, for some $\beta \in Q_+ \cup \{0\}$. In order for a nonzero $\mathfrak{g}$-module homomorphism from $L(\mu_i) \otimes_{\mathbb{C}} L(\mu_j)$ to the simple module $L(\mu_k)$ to exist, it is therefore necessary that

$$\mu_i + \mu_j - \mu_k \in Q_+ \cup \{0\}.$$

Considering the occurrences of half-integers in the coordinate representations of μ_1, μ_2 and μ_3 above, we immediately find that

$$\mu_i + \mu_j - \mu_k \notin Q$$

if

- ◇ the indices i, j, k are all different and one is equal to 0 (18 cases);
- ◇ two of the indices i, j, k are equal and the remaining one is not equal to 0 (27 cases);
- ◇ $i = j = k \neq 0$ (3 cases).

With $i = j \neq k = 0$, the expression $\mu_i + \mu_j - \mu_k$ gives rise to 3 cases:

$$\begin{aligned}
\mu_1 + \mu_1 - \mu_0 &= -2\alpha_1 - \alpha_2 \\
\mu_2 + \mu_2 - \mu_0 &= -\alpha_1 - 2\alpha_2 \\
\mu_3 + \mu_3 - \mu_0 &= -\alpha_1 - \alpha_2
\end{aligned}$$

If the indices i, j, k are all different and none is equal to 0, we obtain a further 6 cases:

$$\begin{aligned}
\mu_1 + \mu_2 - \mu_3 = \mu_2 + \mu_1 - \mu_3 &= -\alpha_1 - \alpha_2 \\
\mu_1 + \mu_3 - \mu_2 = \mu_3 + \mu_1 - \mu_2 &= -\alpha_1 \\
\mu_2 + \mu_3 - \mu_1 = \mu_3 + \mu_2 - \mu_1 &= -\alpha_2
\end{aligned}$$

In all the instances considered so far $\mu_i + \mu_j - \mu_k \notin Q_+ \cup \{0\}$, and hence the corresponding fusion rules have dimension 0. The remaining possibilities occur when $i = k$ and $j = 0$, or when $j = k$ and $i = 0$. In these 7 cases we obtain that

$$\mu_i + \mu_j - \mu_k = 0.$$

It follows that the corresponding fusion rules have dimension at most 1. These fusion rules are of the configurations

$$\mathcal{I}\binom{W}{W\ V} \quad \text{and} \quad \mathcal{I}\binom{W}{V\ W},$$

where $V = L(-\frac{3}{2}\Lambda_0)$ and $W = L(\mu_i - \frac{3}{2}\Lambda_0)$, for some $i \in \{0, 1, 2, 3\}$. From Proposition 3.47 we obtain that

$$\dim \mathcal{I}\binom{W}{W\ V} = \dim \mathcal{I}\binom{W}{V\ W} \geq 1,$$

which completes the proof of the theorem. $\square$

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