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Characterising Stability Implying Properties that are Preserved under Feedback

Richard Pates

Abstract— **The passivity approach to the design of large networks is based on preserving the passivity property under different types of interconnection: parallel and negative feedback. Since the passivity property implies stability, this allows large and topologically complex networks to be constructed on the basis of simple local rules without the need for global stability analysis. In this paper we characterise two different stability implying properties that are preserved under negative feedback. The first generalises the passivity approach to electrical network design to other classes of minimum phase impedance functions. The second allows for networks with, for example, Laplacian structures in the feedback loop, to be designed using only local models.**

I. Introduction

The passivity theorem is a central result in control theory. It plays a key role in a number of different areas, including input-output theory, the study of nonlinear systems, and, as is the topic of this paper, network control systems. The importance of the passivity theorem for networks stems from the property that the parallel or negative feedback interconnection of passive networks is also passive. This allows large and topologically complex networks to be constructed in a plug and play manner on the basis of simple rules:

1) A dynamical property: the networks must be passive.
2) An interconnection rule: parallel and negative feedback interconnections are allowed.

The key implication of the above is that to ensure passivity of a large network, one must only check the passivity of the 'base networks' from which it is constructed. Provided these are simple, the associated robustness and synthesis questions are often tractable even when they are combined to form a very large and complex networks, where analysis and synthesis problems are typically intractable. Furthermore the large network can be changed *without* the need for system wide analysis or redesign.

A natural question is then: are there other interconnection rules that preserve a different stability implying property? Such a question is important because there are situations and applications where the above rules are known to be unnecessarily conservative. A significant example is congestion control of the Internet. The design task here could be conducted using the passivity approach, however less restrictive but equally scalable conditions can be obtained by considering products of subsystem dynamics [1]. This is possible because in this application there is additional structure in the network models that can be exploited. It is our objective to develop a notation and theoretical tools to allow such features to be captured.

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In order to generalise 1) and 2), consider the following equations, which capture the notion of a property preserving interconnection:

$$G_i \in \mathcal{P},\ i \in \{1,2,3\}, \tag{1}$$

$$G_3 \in \mathcal{I}(G_1, G_2). \tag{2}$$

The objects G_i correspond to the dynamical models of different networks, assumed to belong to some modelling class $\mathcal{N}$. $\mathcal{P} \subseteq \mathcal{N}$ is the set of all dynamical models with a given property, and $\mathcal{I}(G_1, G_2) \subseteq \mathcal{N}$ the set of models obtained from 'allowable' interconnections of G_1, G_2. Equation (1) is saying that the networks have the property $\mathcal{P}$, and eq. (2) is specifying the rules for constructing a network G_3 from G_1 and G_2. The objective is then to characterise pairs $(\mathcal{P}, \mathcal{I}(\cdot,\cdot))$ that satisfy eqs. (1) and (2). This may be equivalently stated as

$$\begin{aligned} \text{Find }\ & (\mathcal{P}, \mathcal{I}(\cdot,\cdot)) \\ \text{Such that }\ & \mathcal{I}(\cdot,\cdot) : \mathcal{P} \times \mathcal{P} \to \mathcal{Q} \subseteq \mathcal{P}, \\ & \mathcal{P} \subseteq \mathcal{N}. \end{aligned} \tag{3}$$

If such a pair can be found, they can be used exactly as in the passivity design method to recursively build large networks, since they imply that any network constructed according to eq. (2) out of networks in $\mathcal{P}$ will remain in $\mathcal{P}$. This opens the possibility of finding more restricted notions of interconnection (that may still be suitable for a given application) which preserve a *less* restrictive dynamical property, and vice versa.

To clarify the above we will sketch out a such a characterisation for the passivity case. In the linear setting, the networks may be taken as square matrices of transfer functions, the dynamical property strict positive realness (SPR)

$$\mathcal{P} = \left\{ Z \in \mathbf{RH}_\infty^{n \times n} : Z \text{ is SPR} \right\},$$

and the interconnection constraint parallel or negative feedback

$$\mathcal{I}(G_1, G_2) = \left\{ G_1 + G_2, \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} I & G_2 \\ -G_1 & I \end{bmatrix}^{-1} \right\}.$$

In this paper we will show how preserving pointwise properties written in terms of the numerical range can be used to prove this classical passivity result (we will actually give a slight variant). We will additionally use this approach to show two other ways in which a stability implying property can be preserved under feedback. The first new characterisation gives an alternative notation for describing series-parallel interconnections in the electrical sense. This characterisation is better suited for this role than $\mathcal{I}(G_1, G_2)$ described above (which is more general), and can be used to show that less restrictive dynamical properties can be preserved. This generalises the classical passivity based approach for the design of electrical networks to classes of impedance functions with minimum phase, but not necessarily passive dynamics. The second characterisation shows how stability implying properties of the return ratio of a feedback interconnection can be preserved. We show how this result can be used to deduce stability of networks with, for example, Laplacian structures in the feedback loop, on the basis of only local dynamics.

II. Preserved properties of the Numerical Range

In this section we will argue for the suitability of the numerical range for characterising pairs $(\mathcal{P}, \mathcal{I}(\cdot,\cdot))$ that satisfy eq. (3), when $\mathcal{N} = \mathbb{C}^{n\times n}$. These can be applied frequency by frequency to give preserved properties of interconnections of transfer functions.

The numerical range of a matrix $A \in \mathbb{C}^{n\times n}$ is defined to be the set

$$\mathrm{W}(A) := \{x^* A x : x \in \mathbb{C}^n, x^* x = 1\}.$$

$\mathrm{W}(A)$ is always a compact convex set of the complex plane. For a general introduction to the many appealing properties of this object, see [2, ch.1]. We will now give some results that preserve the location of the numerical range in the complex plane.

A. Preserving the right half plane

The most basic type of region that can be preserved is the open right half plane $\mathbb{C}_+$. These results are the constant matrix analogues of classical passivity preserving interconnection results.

Lemma 1: If $\mathrm{W}(A_i) \subset \mathbb{C}_+,\ i \in \{1, 2\}$, then:

(i) $\mathrm{W}(A_1 + A_2) \subset \mathbb{C}_+$;

(ii) $\mathrm{W}\left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} I & A_2 \\ -A_1 & I \end{bmatrix}^{-1}\right) \subset \mathbb{C}_+$.

Proof: To see (i), note that any point $p \in \mathrm{W}(A_1 + A_2)$ can be written as

$$p = x^* A_1 x + x^* A_2 x.$$

This implies that $p \in \{\mathrm{W}(A_1) + \mathrm{W}(A_2)\}$, which is contained in $\mathbb{C}_+$. For (ii), observe that $0 \notin \mathrm{W}(A_i)$, which implies that A_i^{-1} exist. Hence

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} I & A_2 \\ -A_1 & I \end{bmatrix}^{-1} = \left(\begin{bmatrix} I & A_2 \\ -A_1 & I \end{bmatrix}\begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{bmatrix}\right)^{-1} = \begin{bmatrix} A_1^{-1} & I \\ -I & A_2^{-1} \end{bmatrix}^{-1} =: M^{-1}.$$

Next note that $\mathrm{W}(A) \subset \mathbb{C}_+$ if and only if $\mathrm{W}(A + A^*) \subset \mathbb{C}_+$. Since

$$M + M^* = \begin{bmatrix} A_1^{-1} + A_1^{-*} & 0 \\ 0 & A_2^{-1} + A_2^{-*}, \end{bmatrix},$$

it follows that

$$\mathrm{W}(M + M^*) \subset \mathbf{Co}\left(\mathrm{W}\left(A_1^{-1} + A_1^{-*}\right), \mathrm{W}\left(A_2^{-1} + A_2^{-*}\right)\right),$$

which implies that $\mathrm{W}(M) \subset \mathbb{C}_+$. The result follows because $\mathrm{W}(M) \subset \mathbb{C}_+$ implies that $\mathrm{W}\left(M^{-1}\right) \subset \mathbb{C}_+$ (we will prove a stronger version of this in the next section). ∎

B. Preserving angular sectors

Angular sectors can also be preserved under operations resembling series-parallel interconnections. An angular sector is defined to be

$$\Lambda(\theta_1, \theta_2) := \{z : \arg(z) \in (\theta_1, \theta_2)\}.$$

In the above $\arg(z)$ denotes the argument of a complex number. This quantity is undefined for $z = 0$, so 0 is not in the above sets.

Lemma 2: If $\mathrm{W}(A_i) \subset \Lambda(\theta_1, \theta_2),\ i \in \{1, 2\}$, and $\theta_2 - \theta_1 \le \pi$, then

(i) $\mathrm{W}(A_1 + A_2) \subset \Lambda(\theta_1, \theta_2)$;

(ii) $\mathrm{W}\left(\left(A_1^{-1} + A_2^{-1}\right)^{-1}\right) \subset \Lambda(\theta_1, \theta_2)$.

Proof: (i) is true for exactly the same reason as property (i) in Lemma 1. To see (ii), observe that for any $x \neq 0$, there exists a $y \neq 0$, such that $x = Ay$, and that

$$x^* A_i^{-1} x = y^* A_i^* y.$$

This implies that if $\mathrm{W}(A_i) \subset \Lambda(\theta_1, \theta_2)$, then $\mathrm{W}\left(A_i^{-1}\right) \subset \Lambda(\theta_1, \theta_2)^*$ (this gives the result on $\mathbb{C}_+$, because $\mathbb{C}_+^* = \mathbb{C}_+$). Therefore by (i),

$$\mathrm{W}\left(A_1^{-1} + A_2^{-1}\right) \subset \Lambda(\theta_1, \theta_2)^*,$$

which in turn implies (ii). ∎

C. Preserving Convex Sets

Arbitrary convex sets can also be preserved, albeit under a more abstract looking notion of interconnection.

Lemma 3: If $\mathrm{W}(A_i) \subset \mathcal{C},\ i \in \{1, 2\}$, where $\mathcal{C} \subset \mathbb{C}$ is a convex set, then:

$$\left\{\mathrm{W}\left(X\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}X\right) : X^* X = I\right\} \subset \mathcal{C}.$$

Proof: This result is well known. Let $y = Xx$:

$$x^* X \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} X x = y^* \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} y.$$

Since $X^* X = I$, $y^* y = 1$. Hence the above is contained in $\mathrm{Co}(\mathrm{W}(A_1), \mathrm{W}(A_2)) \subset \mathcal{C}$. ∎

III. Stability Implying Properties of the Numerical Range

In this section we will show how to infer stability from pointwise numerical range conditions. The following shows that if the numerical range of a real rational (not necessarily proper) matrix of transfer functions $G(s) \in \mathbf{R}^{n\times n}$ can be constrained to lie within a convex region along a closed contour Γ (using for example, any of the results in the previous section), one can deduce

$$\mathrm{wno}_\Gamma \det G(s).$$

In the above wno_Γ denotes the winding number evaluated on the contour Γ. When Γ is the Nyquist D-contour, this is typically all that is required to deduce stability using Nyquist type arguments.

Theorem 1: Let $\mathcal{R}(s) \subset \mathbb{C} \setminus \{0\}$ be convex $\forall s \in \Gamma$. Assume there exists a $g \in \mathbf{R}$ such that $g(s) \in \mathcal{R}(s), \forall s \in \Gamma$. If $G \in \mathbf{R}^{n\times n}$ satisfies

$$\mathrm{W}(G(s)) \subset \mathcal{R}(s),\ \forall s \in \Gamma,$$

then

$$\text{wno}_\Gamma \det G(s) = n(\text{wno}_\Gamma\ g(s)).$$

Proof: The following (and much of this paper) leans heavily on [3][chapter 1].Define

$$X(\lambda) = \begin{bmatrix} \sqrt{(1-\lambda^2)}I_n \\ \sqrt{\lambda}I_n \end{bmatrix},\ M(s) = \begin{bmatrix} g(s)I_n & 0 \\ 0 & G(s) \end{bmatrix}.$$

Observe that for any $\lambda[0,1]$, $X(\lambda)^* X(\lambda) = I_n$. Since $\mathcal{R}(s)$ is convex, by Lemma 3

$$\mathrm{W}\left(X(\lambda)^* M(s) X(\lambda)\right) \subset \mathcal{R}(s), \forall s \in \Gamma, \forall \lambda[0,1].$$

Define $f_\lambda(s) = \det\left(X(\lambda)^* M(s) X(\lambda)\right)$. Since $0 \notin \mathcal{R}(s)$, the above implies that $f_\lambda \neq 0, \forall s \in \Gamma, \forall\lambda[0,1]$. Therefore since $f_\lambda(s)$ is continuous in both λ and s, $\text{wno}_\Gamma\ f_\lambda(s)$ is constant for all $\lambda \in [0,1]$. The result then follows because $\text{wno}_\Gamma\ f_0(s) = \text{wno}_\Gamma \det(g(s)I_n) = n(\text{wno}_\Gamma\ g(s))$, and $\text{wno}_\Gamma\ f_1(s) = \text{wno}_\Gamma \det G(s)$. ■

IV. Scalable Network Protocols

In this section we will combine Theorem 1 with the preserved properties in Section II to define some properties and interconnection rules $(\mathcal{P}, \mathcal{I}(\cdot,\cdot))$ that satisfy eq. (3). A key feature of the found pairs is that they can be used to deduce and preserve stability under feedback. Throughout the modelling class $\mathcal{N}$ will be $\mathbf{R}^{n\times n}$, and a model $G_i \in \mathcal{N}$ is said to be stable if in addition it is in $\mathbf{H}_\infty$. We will have frequent need to take inverses within the model class, and so explicitly point out that a matrix $A \in \mathbf{R}^{n\times n}$ has an inverse in $\mathbf{R}^{n\times n}$ if and only if the matrix $A(s)$ is invertible for some value of $s \in \mathbb{C}$. Finally, from here on Γ will refer to 'the usual Nyquist D-contour', though this will be given formal clarification when it is used below.

A. Passivity

Existing stability criteria based on passivity ideas allow for different classes $\mathcal{P}$, depending on the 'type' of passivity being considered (strictly positive real, strictly output passive, etc.), and the definition of stability used. Rather than considering standard definitions, we instead focus on finding the largest class of transfer functions that can be shown to define a pair satisfying Equation (3) using the results in the previous sections. This class is most easily written in terms of the following class of transfer functions

$$\mathcal{P}^n_{\text{pass}} = \left\{Z \in \mathbf{R}^{n\times n} : \mathrm{W}(Z(s)) \subset \mathbb{C}_+ \forall s \in \Gamma\right\}.$$

In the above when we say that the numerical range condition holds on Γ, we mean is that it holds for any sufficiently large D-contour, since this is what is required to apply Nyquist type arguments. This is clearly an abuse of notation, and precisely we mean that

$$\mathrm{W}(G(s)) \subset \mathbb{C}_+, \forall s = j\omega,$$

and that there exists an R_0 such that for all $r > R_0$ and $\theta \in [-\pi/2, \pi/2]$,

$$\mathrm{W}\left(G\left(re^{j\theta}\right)\right) \subset \mathbb{C}_+.$$

This means that the numerical range condition is satisfied on all Nyquist contours with sufficiently large semicircular arcs.

The similarities with passivity type properties is more apparent when the above is rewritten in terms of matrix inequities, and $\mathcal{P}^n_{\text{pass}}$ could be equivalently defined as those transfer functions for which

$$G(j\omega) + G(j\omega)^* > 0, \forall\omega \in \mathbb{R},$$

and in addition $G(s) + G(s)^*$ is positive definite along all sufficiently large semicircular arcs in the right half plane. When restricted to $\mathbf{H}_\infty$ the above is closely related to, but not equal to, the class of strictly positive real transfer functions. As a simple example

$$g(s) = \frac{1}{s+1}$$

is both in $\mathcal{P}_{\text{pass}}$ and is SPR, but

$$g(s) = \frac{s+1}{s^2+s+1} = \left(s + \frac{1}{s+1}\right)^{-1}$$

is in $\mathcal{P}_{\text{pass}}$, but is not SPR.

The following result is reminiscent of standard passive interconnection results, and states that the parallel or negative feedback interconnection of stable transfer functions in $\mathcal{P}_{\text{pass}}$, remains stable and in $\mathcal{P}_{\text{pass}}$. Here the term passivity should be interpreted very loosely, and we claim no physical interpretation of $\mathcal{P}_{\text{pass}}$. We use the term passivity simply because the transfer function condition here is very similar to that obtained with standard passivity results.

Proposition 1: If G_1, G_2 satisfy:

(i) $G_i \in \mathbf{RH}^{n_i\times n_i}_\infty$,
(ii) $G_i \in \mathcal{P}^{n_i}_{\text{pass}}$;

then

$$G_3 \in \left\{(G_1+G_2),\ \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}\begin{bmatrix} I & G_2 \\ -G_1 & I \end{bmatrix}^{-1}\right\},$$

also satisfies (i) and (ii).

Proof: Satisfaction of (ii) follows immediately from Lemma 1 for both interconnections. It is also immediate that (i) holds for $G_1 + G_2$ since $\mathbf{RH}_\infty$ is closed under addition. To show that (i) holds for the second interconnection, first recall that for any nonsingular $H \in \mathbf{R}^{n\times n}$, by the principle of the argument

$$\text{wno}_\Gamma \det H(s) = \zeta(H) - \eta(H). \tag{4}$$

In the above $\zeta(H), \eta(H)$ are the number of zeros and poles of $H(s)$ in the open right half plane (since the transfer functions we consider only have finitely many poles and zeros, there always exists a D-contour with sufficiently large, but finite, radius such that this holds). Since G_i satisfies (i) and (ii), the choice $g(s) = 1$ in Theorem 1 shows that $\mathrm{wno}_\Gamma \det G_i(s) = 0$. By eq. (4), this means that

$$\zeta(G_i) = 0.$$

Define

$$\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} I & G_2 \\ -G_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} G_1^{-1} & I \\ -I & G_2^{-1} \end{bmatrix}^{-1} =: M^{-1}.$$

Clearly $\eta(M) \leq \zeta(G_1)+\zeta(G_2) = 0$, and hence $\eta(M) = 0$. Since $M = G_3^{-1}$, this means that $\zeta(G_3) = 0$. Running the same argument as before with eq. (4) and Theorem 1 this time implies that $\eta(G_3) = 0$. The result then follows since G_3 is clearly proper, and is therefore in $\mathbf{RH}_\infty$. ■

B. Angular Sector Conditions

Lemma 2 suggests that an angular sector based property can be preserved for the series parallel interconnection. Under an additional restriction to the functions θ_1, θ_2 this property can additionally be made stability implying, as the following shows.

Proposition 2: Let $\theta_1(s), \theta_2(s)$ satisfy $\theta_2(s) - \theta_1(s) \leq \pi, \forall s \in \Gamma$, and assume that there exists a $g \in \mathbf{R}$ such that $g(s) \in \Lambda(\theta_1(s), \theta_2(s)), \forall s \in \Gamma$ and $\mathrm{wno}_\Gamma\ g = 0$. If G_1, G_2 satisfy:

(i) $G_i \in \mathbf{RH}_\infty^{n_i \times n_i}$,

(ii) $\mathrm{W}(G(s)) \subset \Lambda(\theta_1(s), \theta_2(s)), \forall s \in \Gamma$;

then

$$G_3 \in \left\{ (G_1 + G_2),\ \left(G_1^{-1} + G_2^{-1}\right)^{-1} \right\}$$

satisfies (i) and (ii).

Proof: As before (ii) is guaranteed by Lemma 2, and satisfaction of (i) for $G_1 + G_2$ is again immediate. (i) is guaranteed for the second interconnection by using the same argument as in the proof of Proposition 1, but with $M = G_1^{-1} + G_2^{-1}$. ■

The requirements on the angular sector is simply that pointwise in s, the arclength is less than or equal to π (as in Lemma 2), and that the sector is making no net encirclements of the origin (this is due to the Nyquist type requirement). These requirements force the transfer functions to be minimum phase by Theorem 1. However they allow for a more general class of transfer functions, because $\mathbb{C}_+$ is a special instance of the angular sector requirement corresponding to

$$\theta_2(s) = -\theta_1(s) = \frac{\pi}{2}.$$

Under different choices of $\theta_1(s), \theta_2(s)$, the dynamical property in Proposition 2 can cover minimum phase transfer functions that are not passive under both our unconventional, as well as conventional, definitions. This is particularly interesting because the interconnection rules in Proposition 2 also capture series and parallel interconnection in the electrical sense, because they correspond to the series and parallel interconnection rule for resistors. This extends the applicability of the passivity approach for electrical networks to allow for the interconnection of impedance functions described by broader families of minimum phase transfer functions.

To understand why we do not see this more general class in Proposition 1, observe that to use the feedback interconnection

$$\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} I & G_2 \\ -G_1 & I \end{bmatrix}^{-1}$$

to describe a parallel interconnection (in the electrical sense), one G_i must correspond to an impedance, and the other an admittance, and the resulting transfer function is actually a mixture of the two. This means that our network models must be able to describe both admittances and impedances (and mixtures) for the interconnection rules to make sense electrically, and hence the properties (i) and (ii) are implicitly required on both these forms also. This is not the case for Proposition 2. We may take our family of network models to correspond to either all impedances, or all addmittances, and the interconnections generated by the interconnection rule will make sense and remain of the same type. This allows for a more flexible characterisation, since it must only apply to one of these forms, and not all.

To see the relevance of an angular sector condition for the interconnection of minimum phase transfer functions, consider the simple example

$$G_1 = \frac{(s+1)^3}{(s/20+1)^3},\ G_2 = k > 0.$$

Both are minimum phase, but there exists no angular sector meeting the conditions of Proposition 2 that contains both these transfer functions. This is because for some frequency, G_1 cuts the negative real axis, and it is impossible for a positive and negative number to both lie within an angular sector of arclenth at most π. Examining the root locus of G_1^{-1} shows that there are values of k for which $\left(G_1^{-1} + G_2^{-1}\right)^{-1}$ on the above is unstable (e.g. $k = 1/20$). This provides a type of converse result for Proposition 2, because it shows that for certain minimum phase transfer functions, violating the angular sector requirement results in instability under feedback.

C. Convex Set Conditions

The convex set conditions can be used to study stability of feedback interconnections by preserving properties of the return ratio. This result can be used to study the stability of networks with dynamics characterised by the standard feedback interconnection:

$$[P_i, C_i] := \begin{bmatrix} P_i \\ I \end{bmatrix} (I + P_i C_i)^{-1} \begin{bmatrix} C_i & I \end{bmatrix}.$$

By this we mean that the network dynamics are specified by (P_i, C_i), but given by the closed loop transfer function $[P_i, C_i]$. This feedback characterisation is useful because descriptions of this form, where the network structure is in (P_i, C_i), are available for a wide range of applications, including consensus problems, vehicle platoons, Internet congestion control, etc., e.g. [4], [5], [6] (note that a frequently considered subclass of the above is that P_i is diagonal, and C_i is a sparse 'interconnection matrix').

Proposition 3: Let $\mathcal{C}(j\omega) \subset \mathbb{C} \setminus \{0\}$ be convex $\forall \omega \in \mathbb{R}$, and assume that there exists a $g, g^{-1} \in \mathbf{RH}_\infty$ such that $g(j\omega) \in \mathcal{C}(j\omega), \forall \omega \in \mathbb{R}$. If G_1, G_2 satisfy:

(i) $(I+G_i), (I+G_i)^{-1} \in \mathbf{RH}_\infty^{n_i \times n_i}$,

(ii) $\mathrm{W}(I+G_i) \subset \mathcal{C}(j\omega), \forall \omega \in \mathbb{R}$;

then

$$G_3 \in \left\{ X^* \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} X : X^*X = I, X \in \mathbb{R}^{(n_1+n_2)\times n_3} \right\}$$

satisfies (i) and (ii).

Proof: To see that (ii) is satisfied as a result of Lemma 3, note that

$$I + G_3 = X^* \begin{bmatrix} I+G_1 & 0 \\ 0 & I+G_2 \end{bmatrix} X.$$

The result then follows by using the same argument as in the proof of Proposition 1 with $M = I + G_3$. Note that we do not need to make the argument about the Nyquist contour since we additionally assume biproperness of the relevant transfer functions, and hence this requirement is satisfied automatically. This is because $\lim_{s\to\infty}(I + G(s))$ is nonzero and independent of the direction in which ∞ is approached. ■

To see that the above can be used to deduce the stability of families of feedback interconnections, note that for a feedback interconnection $[P_i, C_i]$, if P_i, C_i are both stable, then $[P_i, C_i]$ is internally stable if and only if $(I + P_iC_i)^{-1} \in \mathbf{RH}_\infty^{n_i \times n_i}$ (e.g. [7]). Hence if it can be additionally guaranteed that the elements in the feedback loop for our network application are stable (which is often the case), (i) is stability implying.

Unlike in Propositions 1 and 2 the restriction on G_i *does not* require that the elements in the feedback loop (P_i, C_i in this case, G_1, G_2 from before) are minimum phase, allowing them to include right half plane zeros.

To see the utility of Proposition 3, consider networks with dynamics described by the interconnection

$$\left[\begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_p \end{bmatrix}, L \right]. \tag{5}$$

In the above $g_i(s)$ are scalar transfer functions, and L a symmetric Laplacian matrix. Proposition 3 can of course be used to analyse far more complex feedback interconnections, though it is reasuring that when additional structure is available it can be exploited to give simple stability conditions.

In the following discussion, we will show that any interconnection of the form in eq. (5) can be written in the appropriate form, with:

$$P_iC_i = 2g_i(s)\mathbf{1}_{n_i}. \tag{6}$$

In the above n_i is equal to L_{ii}, and $\mathbf{1}_{n_i}$ an $n_i \times n_i$ matrix of ones. It is easy to show that

$$\mathrm{W}(2g_i(s)\mathbf{1}_{n_i} + I) = 1 + \mathbf{Co}(2n_ig_i(s), 0).$$

This means that conducting design using Proposition 3 is equivalent to requiring that k_ig_i lies in convex regions of the complex plane for a range of values of the constant k_i. This can be tackled with a wide array of standard tools, and the resulting analysis will be valid for any possible Laplacian interconnection of the form in eq. (5).

To show this, let us first rewrite eq. (5). Since L is a Laplacian matrix it can be factorised as $L = BB^T$, where B is an oriented incidence matrix. Since the g_i's are assumed stable[1], we can equivalently consider internal stability of

$$\left[B^T \begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_p \end{bmatrix} B, I \right]. \tag{7}$$

The following lemma shows the return ratio for this interconnection can be written in the appropriate form. Hence Proposition 3 shows that we can conduct design as discussed on the subsystems described by eq. (6).

Lemma 4: $A = \text{blkdiag}(a_1, \ldots, a_p)$ and $B \in \mathbb{C}^{p\times m}$. If B is an oriented incidence matrix, then there exists an X such that $X^*X = I$ and

$$B^*AB = X^*\text{blkdiag}\left(2a_1\mathbf{1}_{n_1}, \ldots, 2a_p\mathbf{1}_{n_p}\right)X,$$

where $n_i = (BB^*)_{ii}$.

Proof: Let Y_i equal $\text{diag}(B_{i\bullet})$, except with all the zero rows deleted. Observe that

$$B_{i\bullet} = 1_{n_i}^T Y_i,$$

where 1_{n_i} is the vector of n_i ones. Therefore

$$\begin{aligned} B^*AB &= \sum_{i=1}^p B_{i\bullet}^* a_i B_{i\bullet} \\ &= Y^*\text{blkdiag}\left(a_1\mathbf{1}_{n_1}, \ldots, a_p\mathbf{1}_{n_p}\right)Y, \end{aligned}$$

where

$$Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_p \end{bmatrix}.$$

Observe that $Y^*Y = 2I$. This follows because B has two nonzero entries in each column, and

$$(Y^*Y)_{kk} = \sum_{i=1}^p B_{ik}^* B_{ik} = 2.$$

The result follows by putting $X = \frac{1}{\sqrt{2}}Y$, and rescaling the diagonal matrix. ■

[1] It is often desirable to allow the g_i's to contain an integrator. By indenting the Nyquist contour into the right half plane around this pole, the presented method can be extended with minimal changes to this case. However the resulting criteria will be inconclusive about the number of poles that remain at the origin in the closed loop. A separate argument will be required (based on the properties of B) to draw conclusions about this and any assoicated 'consensus subspace'.

V. Conclusions

A method for conducting network design by preserving stability implying properties under interconnection is presented. The method hinges on showing that properties of the frequency responses of the numerical range are preserved under various notions of interconnectoin. When combined with a Nyquist type argument, this is sufficient for stability. It is shown that classical passivity results arise as a special case of this approach, and that the passivity approach to electrical network design can be extended to other impedance classes with non-passive dynamics. In addition a stability preserving property of the return ratio of feedback interconnections, which has applications to networks with Laplacian structures, is given.

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