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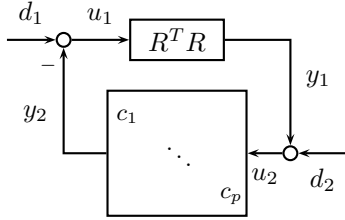
# A Loopshaping approach to Controller Design in Networks of Linear Systems

Richard Pates

**Abstract**—A method for designing a set of controllers to robustly stabilise a network of linear systems is presented. The method allows the design of each controller to be posed as a loopshaping problem. Critically each loopshaping problem requires only local knowledge of the overall system model to formulate, and may be solved separately. Furthermore the approach is inherently scalable, as any local changes to the network model can be accommodated through the design of the corresponding local controllers, leaving all the others untouched.

## I. INTRODUCTION

This paper is about the design of decentralised controllers in networks. More specifically, we consider the design of a set of transfer functions  $\{c_1, \dots, c_p\}$  in the following feedback interconnection:



In the above,  $R$  is a  $q \times p$  matrix of transfer functions. Typically  $R$  will be very large and sparse, and the design task is to select the transfer functions  $c_k$  such that this interconnection is robustly stable.

The robust design of local controllers is fundamental to the successful operation of many networks. For example, controllers located at each generator in an electrical power system play a crucial rôle in maintaining the electrical frequency and voltage. Design methods for the above interconnection are important for two main reasons. Firstly, this problem structure appears in numerous applications. For example, in the models of internet congestion control, consensus, and vehicle platoons in [1], [2], [3], choosing each  $c_k$  corresponds to designing the dynamics of a source, a consensus feedback law, and the dynamics of a vehicle respectively. Furthermore since the matrix  $R^T R$  includes the Laplacian matrices as a special case, the models of numerous physical systems, including electrical networks, also fit into this framework. Secondly, the use of standard methods for design in this setting is complicated by the model size, the fact that they might be slowly changing over time, or that no complete picture of the model is known at all. For example

one cannot consider a dynamical model of the entire internet when designing a protocol, because the exact structure of the internet is unknown, and is continually changing.

For these reasons, a number of tools have been developed for verifying stability of this interconnection on the basis of local network structure (or coarse and globally verifiable structural features of the matrix  $R^T R$ ). These help address many of the issues caused by model size, since the notions of locality are typically independent of model size. Some results that additionally provide insight into the design of the  $c_k$ 's include those based on passivity and dissipativity (e.g. [4], [5]), integral quadratic constraints (IQCs) (e.g. [6], [7]), and S-hulls (e.g. [8], [9]) (some of these results require additional restrictions on the matrix  $R$ , e.g. passivity).

In this paper we present a method, based on the stability result from [10], that decomposes the design of the controllers into a set of decoupled problems. Each of these can be solved by loopshaping one of the following transfer functions:

$$f_k(j\omega) = c_k(j\omega) \sum_{i=1}^q n_i R_{ik}^2(j\omega), \quad k = 1, \dots, p,$$

where  $n_i \in \mathbb{Z}$  equals the number of nonzero entries in the  $i$ th row of  $R$ . Critically this approach allows each  $c_k$  to be designed independently, based only on partial information about  $R$ . Furthermore, as the entry  $R_{ik}$  appears uniquely in the  $k$ th design condition, any changes in the network model can be accommodated by the design of the corresponding controller, without affecting that of the others. In addition to guaranteeing stability, the method also provides performance and robustness guarantees that can be understood in terms of bounds on the singular values of the sensitivity and complementary sensitivity functions of the interconnection. This allows some of the classical intuition from loopshaping scalar transfer functions to be applied in the network setting.

## II. NOTATION

For  $z \in \mathbb{C}$ , let  $|z|, \angle z$  denote its magnitude and argument.  $\mathcal{H}_\infty$  is the space of transfer functions of stable, linear, time-invariant, continuous time systems,  $\mathcal{RH}_\infty$  its rational subspace,  $\mathcal{C}_0$  the class of functions continuous on  $j\mathbb{R} \cup \{\infty\}$  and  $\mathcal{A}_0 := \mathcal{H}_\infty \cap \mathcal{C}_0$ . Finally  $[P, K]$  represents the standard feedback interconnection (negative feedback convention)

$$u_1 = -K u_2 + d_1$$

$$u_2 = P u_1 + d_2$$

where  $P, K$  are transfer functions, and  $d_1, d_2$  the external disturbances to the system.

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### III. PRELIMINARY RESULTS

The results presented in this section form the basis of the loopshaping design procedure. They can be used to analyse the interconnection

$$[R^T R, C], \quad (1)$$

where  $R^T R C \in \mathcal{A}_0^{p \times p}$ ,  $R(j\omega) \in \mathbb{C}^{q \times p}$  and  $C = \text{diag}(c_k \in \mathcal{RH}_\infty : k \in \{1, \dots, p\})$ . In particular, they allow stability of this interconnection to be tested in a distributed manner, and give bounds on the singular values of the following sensitivity and complementary sensitivity functions:

$$\begin{aligned} S &= (I + R C R^T)^{-1}, \\ T &= R C R^T (I + R C R^T)^{-1}. \end{aligned} \quad (2)$$

All the results are proved in the appendix.

#### A. Distributed Stability Test

*Definition 1:* For  $z \in \mathbb{C}$  and  $\theta \in (-\pi, \pi)$ , define  $g(z, \theta)$  as

$$g(z, \theta) = \frac{|z|}{\sqrt{|z|^2 + 4|z| \cos \theta \cos(\theta + \angle z) + 4 \cos^2 \theta}},$$

where the positive square root is taken.

The following theorem gives a condition for testing stability of the interconnection in eq. (1). The condition takes the form of a set tests, each dependent on the dynamics of a column of the matrix  $R$ , and a function  $\theta : j\mathbb{R} \mapsto (-\pi, \pi)$ .

*Theorem 1:* Let  $R^T R C \in \mathcal{A}_0^{q \times q}$  where  $R(j\omega) \in \mathbb{C}^{q \times p}$  and  $C = \text{diag}(c_k : k \in \{1, \dots, p\})$ . Define

$$\begin{aligned} f_k &= c_k \sum_{i=1}^q n_i R_{ik}^2, \\ \varepsilon_k &= \frac{|\sum_{i=1}^q n_i R_{ik}^2|}{\sum_{i=1}^q |n_i R_{ik}^2|}, \end{aligned}$$

where  $n_i \in \mathbb{Z}$  is equal to the number of nonzero entries in the  $i$ th row of  $R$ . If there exists a  $\theta : j\mathbb{R} \mapsto (-\pi, \pi)$  such that  $\forall k \in \{1, \dots, p\}$ :

$$\varepsilon_k(j\omega) > g(f_k(j\omega), \theta(j\omega)), \quad \forall \omega \geq 0, \quad (3)$$

then  $[R^T R, C]$  is stable.

#### B. Singular Value Inequalities

*Definition 2 (Ellipse):* For  $f \in \mathbb{C}$  and  $\varepsilon \in [0, 1]$ , define  $\mathbb{E}(f, \varepsilon)$  as

$$\mathbb{E}(f, \varepsilon) = \left\{ z \in \mathbb{C} : |z| + |z - f| < \frac{|f|}{\varepsilon} \right\}.$$

The two propositions in this subsection give upper bounds on the singular values of the functions  $S$  and  $T$  in eq. (2). The first can be used to determine frequency ranges in which  $\bar{\sigma}(T(j\omega))$  is small ( $\ll 1$ ), and  $\bar{\sigma}(S(j\omega))$  is not too large (e.g.  $\lesssim 2$ ) ( $\bar{\sigma}(\cdot)$  denotes the largest singular value). These provide information about the robustness of the interconnection. The second can be used to determine frequency ranges where the harmonic mean of the singular values of  $S(j\omega)$  is

small. This provides information about the performance and disturbance rejection of the interconnection.

*Proposition 1:* Let  $R, C, f_k, \varepsilon_k$  be as in Theorem 1, and define  $d_k(j\omega)$  to be the shortest distance from the  $-1$  point to  $\mathbb{E}(f_k(j\omega), \varepsilon_k(j\omega))$ . Assume that

$$L(j\omega) = f_1(j\omega) = f_2(j\omega) = \dots = f_p(j\omega).$$

Then

$$\begin{aligned} \bar{\sigma}(S(j\omega)) &\leq \max_{k \in \{1, \dots, p\}} \frac{1}{d_k(j\omega)}, \\ \bar{\sigma}(T(j\omega)) &\leq \max_{k \in \{1, \dots, p\}} \frac{|L(j\omega)|}{\varepsilon_k(j\omega) d_k(j\omega)}. \end{aligned}$$

Loosely speaking, Proposition 1 shows that at frequencies where  $f_k(j\omega) \approx L(j\omega)$  and  $\varepsilon_k(j\omega)$  is not too small:

- 1) If  $L(j\omega)$  is not close to the  $-1$  point, then  $\bar{\sigma}(S(j\omega))$  will not be too large.
- 2) If  $|L(j\omega)|$  is small, then  $\bar{\sigma}(T(j\omega))$  will be small.

Since  $0 \in \mathbb{E}(L(j\omega), \varepsilon_k(j\omega))$  for any  $L(j\omega)$  and  $\varepsilon_k(j\omega)$ , the bound on  $\bar{\sigma}(S(j\omega))$  in Proposition 1 can never be less than 1. This limitation is reasonable, since if  $R C R^T$  is rank deficient, then  $\bar{\sigma}(S(j\omega)) = 1$ . However, as shown in Proposition 2 below, even in this case the harmonic mean of the singular values of  $S(j\omega)$  can be small, indicating that a wide range of disturbances can be effectively attenuated by using high gain controllers.

*Proposition 2:* Let  $R, C$  be as in Theorem 1, and define

$$S_i = \frac{1}{1 + \sum_{k=1}^p c_k R_{ik}^2}.$$

Then

$$\left( \frac{1}{q} \sum_{i=1}^q \frac{1}{\sigma_i(S(j\omega))} \right)^{-1} \leq \left( \frac{1}{q} \sum_{i=1}^q \frac{1}{|S_i(j\omega)|} \right)^{-1},$$

where  $\sigma_i(\cdot)$  denotes the  $i$ th singular value.

Proposition 2 shows that harmonic mean of the singular values of  $S(j\omega)$  is bounded by the harmonic mean of the functions  $S_i(j\omega)$ . These correspond to the sensitivity functions of the following interconnections:

$$[(R C R^T)_{ii}, 1],$$

where  $(R C R^T)_{ii}$  denotes the  $i$ th diagonal entry of  $R C R^T$ .

### IV. DISTRIBUTED LOOPSHAPING DESIGN PROCEDURE

In this section we present a two step procedure that can be used to design the controllers in eq. (1). This procedure decomposes the design task into a set of decoupled loopshaping problems, one for each  $c_k$ . Each entry of  $R$  appears uniquely in a single problem. This approach therefore completely decouples the model structure from the perspective of design. In addition to alleviating issues caused by model size or lack of global knowledge, this makes the design method highly robust to changes in  $R$ . Any single change can only impact its design problem, and can hence be accommodated for by local measures of robustness (or if necessary by retuning the corresponding local controller).

*Procedure 1 (Distributed loopshaping):*

*Step 1:* Select a loopshape  $L(j\omega)$ . Compute

$$\theta^*(j\omega) = \arctan \left( \frac{-\text{Im}\{L(j\omega)\}}{1 + \text{Re}\{L(j\omega)\} + |1 + L(j\omega)|} \right).$$

*Step 2:* For each  $k \in \{1, \dots, p\}$ : Compute

$$f_k = c_k \sum_{i=1}^q n_i R_{ik}^2$$

$$\varepsilon_k = \frac{\left| \sum_{i=1}^q n_i R_{ik}^2 \right|}{\sum_{i=1}^q |n_i R_{ik}^2|}.$$

Then tune  $c_k$  such that  $f_k(j\omega) \approx L(j\omega)$ , and

$$\varepsilon_k(j\omega) > g(f_k(j\omega), \theta^*(j\omega)), \quad \forall \omega \geq 0. \quad (4)$$

◇

The basic idea is to first ignore the multivariate nature of the model, and try to meet the design criteria with a scalar transfer function  $L(j\omega)$  (see e.g. [11, chapters 7–8]). Then provided the controllers  $c_k$  can be tuned such that  $f_k(j\omega) \approx L(j\omega)$ , as a result of Propositions 1 and 2, much of the intuition from the scalar case can be carried into the network setting. For example, if  $|L(j\omega)|$  is small, then as a result of Proposition 1  $\bar{\sigma}(T(j\omega))$  will also be small, ensuring good robustness to multiplicative uncertainty.

Satisfying eq. (4) guarantees that the controllers are stabilising by Theorem 1, and the function  $\theta^*$  is optimal for  $f_k = L$  (as discussed in the appendix, it makes  $g(L, \theta)$  as small as possible). This requirement should also be considered when choosing  $L$ . It is recommended that in addition to the standard considerations,  $L$  should be chosen such that  $g(L(j\omega), \theta^*(j\omega))$  is small in frequency ranges where  $\varepsilon_k(j\omega)$  is expected to be small. This can always be achieved by making  $|L(j\omega)|$  small in these frequency ranges, essentially treating  $\varepsilon_k(j\omega)$  as a type of uncertainty. We will give an example of this in section V.

To aid with the tuning of  $c_k$ , we give a second procedure that allows part of the feasible region of eq. (4) to be plotted on a conventional Bode plot. For a given  $t > 0$ , this procedure produces the shaded region corresponding to the largest angular sector of radius  $t|L(j\omega)|$  such that eq. (4) is satisfied. Hence if  $f_k(j\omega)$  lies within such a shaded region, eq. (4) is guaranteed to be satisfied.

*Procedure 2 (Feasible regions):*

*Step 1:* Select a  $t > 0$ .

*Step 2:* Shade the region

$$M(j\omega) = \{x | |L(j\omega)| : 0 \leq x < t\}$$

on the magnitude plot.

*Step 3:* Compute

$$\Delta = \frac{t^2 |L|^2 \left( \frac{1}{\varepsilon_k^2} - 1 \right) - 4 \cos^2 \theta^*}{4t |L| \cos \theta^*},$$

and shade the region

$$\Phi(j\omega) = \{\theta^*(j\omega) + x : |x| < \arccos(\text{sat}(\Delta(j\omega)))\}$$

on the phase plot. In the above  $\text{sat}(x)$  returns  $x$  if  $x \in (-1, 1)$ , and  $\text{sign}(x)$  otherwise. ◇

## V. EXAMPLE

In this section Procedure 1 is applied to design controllers to guarantee that a specific instance eq. (1) is robustly stable to unmodelled dynamics, while achieving good disturbance rejection at low frequencies. Only  $c_1$  and  $c_2$  will be explicitly designed; any others could be tuned in a similar manner with knowledge of the corresponding column of  $R$ .

*Model Description:* Suppose that

$$R_{ik}(s) = \begin{cases} \frac{1}{\sqrt{n_i}} e^{-D_{ik}s} & \text{if } D_{ik} > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where  $D \in \mathbb{R}^{q \times p}$ ,  $D_{ik} \geq 0$ , and each column of  $D$  is expected to have 3 nonzero entries, each drawn from an exponential distribution with mean 1.  $R$  is only considered an accurate model for  $\omega < 10$  rad/s. Finally, the nonzero entries of the first two columns of  $D$  are  $(0.1, 0.1, 0.5)$ , and  $(0.1, 0.1, 0.5, 3, 3)$  respectively. This model could be loosely interpreted as a simplified model for congestion control in the internet (valid for low frequencies), where  $D_{ik}$  is the delay time between the  $i$ th link and  $k$ th source, and the task is to design the source law.

*Step 1:* For simplicity, Bode's ideal loop transfer function

$$L(s) = \left( \frac{s}{\omega_c} \right)^{-\frac{3}{2}},$$

was chosen with  $\omega_c = 1$ . Since  $|L(j\omega)|$  is small for  $\omega > 10$ , Proposition 1 guarantees that  $\bar{\sigma}(T(j\omega))$  is small, ensuring good robustness to multiplicative uncertainty in this frequency range. Furthermore, since  $|L(j\omega)|$  is large for low frequencies (necessitating the use of high gain controllers), Proposition 2 guarantees that a proportion of the singular values of  $S(j\omega)$  will be small. Since  $L(j\omega)$  is never closer than  $1/\sqrt{2}$  from the  $-1$  point, Proposition 1 additionally guarantees that the peak in  $\bar{\sigma}(S(j\omega))$  should remain  $\lesssim \sqrt{2}$ . Together these ensure reasonable disturbance rejection at low frequencies.

To get a feel for the suitability of this loop transfer function with respect to eq. (4),  $10^5$  samples of columns of  $R$  were generated (3 nonzero entries, each exponentially distributed), and used to approximate the distribution of  $\varepsilon_k(j\omega)$ . The interquartile range of this distribution across frequency, along with  $g(L(j\omega), \theta^*(j\omega))$  is plotted in fig. 1. Observe that in the frequency range  $\omega \approx [0.1, 1]$  rad/s,  $g(L(j\omega), \theta^*(j\omega))$  lies within the interquartile range. This indicates that the constraint imposed by eq. (4) is likely to be violated if  $f_k(j\omega) \approx L(j\omega)$ , and hence this loop transfer function is not suitable. The problem is essentially that  $\omega_c = 1$  is too large given the expected values of the delay times  $D_{ik}$ . Consequently the crossover frequency was adjusted to  $\omega_c = 0.5$ , and  $\theta^*(j\omega)$  recalculated. This  $L(j\omega)$  and  $\theta^*(j\omega)$  were deemed acceptable as  $g(L(j\omega), \theta^*(j\omega))$  lies comfortably below the interquartile range.

*Step 2* ( $k = 1$ ): Computing  $f_1, \varepsilon_1$  gives

$$f_1(j\omega) = c_1(j\omega) (2e^{-0.2j\omega} + e^{-j\omega})$$

$$\varepsilon_1(j\omega) = \frac{1}{3} (2e^{-0.2j\omega} + e^{-j\omega}).$$

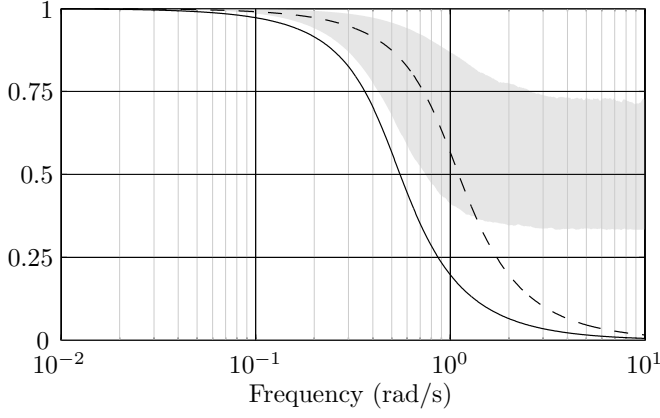


Fig. 1: The dashed and solid curves show  $g(L, \theta^*)$  with  $\omega_c = 1$ , and 0.5 respectively. The shaded region shows the interquartile range of the samples from the distribution of  $\varepsilon_k$ . As the dashed curve does not lie below the shaded region, designing  $f_k \approx L$  will result in  $f_k$ 's that are likely to violate eq. (4). This is not the case for the solid curve, indicating that this is a more suitable loop transfer function.

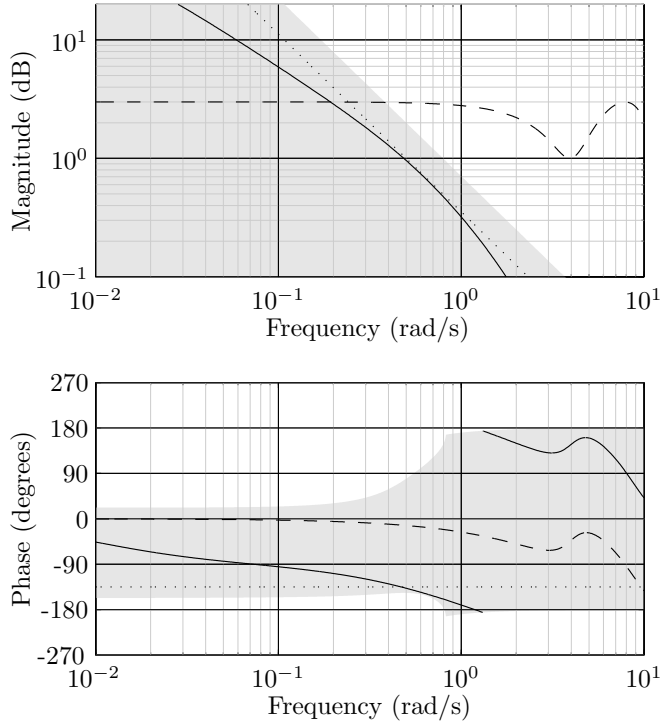


Fig. 2: The dashed curve shows  $f_1$  with  $c_1 = 1$ , the dotted curve  $L$ , and the solid curve the designed  $f_1$ . The shaded region shows the regions obtained by applying Procedure 2 with  $t = 2$ . The designed  $f_1$  closely matches  $L$  at the crossover frequency, and approximates the high and low frequency behaviour. Furthermore, as it lies within the shaded region it is guaranteed to satisfy the constraint imposed by eq. (4).

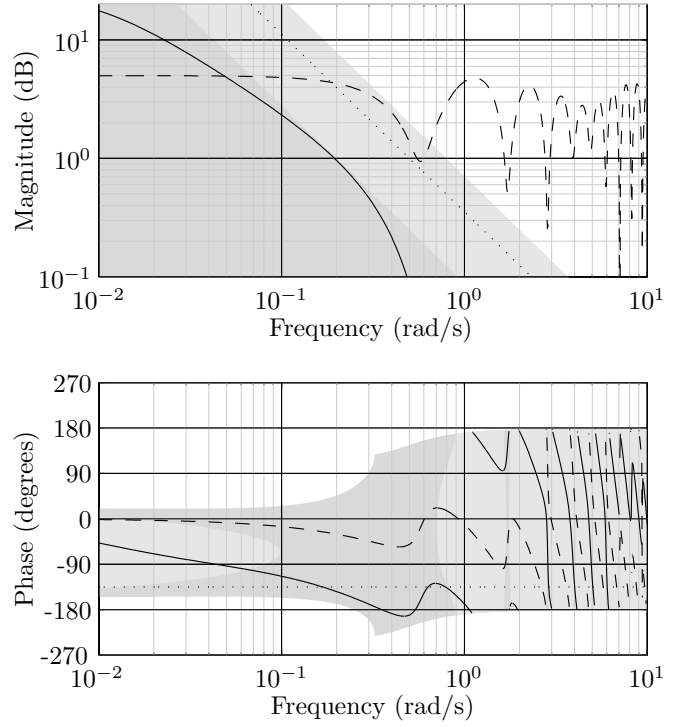


Fig. 3: The dashed curve shows  $f_2$  with  $c_2 = 1$ , the dotted curve  $L$ , and the solid curve the designed  $f_2$ . The lightly and darkly shaded regions show the regions obtained by applying Procedure 2 with  $t = 2$  and 0.25 respectively. Since there is a gap in the lightly shaded region of the phase plot for frequencies between  $[0.1, 0.7]$ , there is no  $c_k$  that will closely match  $L$  and satisfy eq. (4) in this frequency range. As a compromise  $f_2$  was designed to approximate the high and low frequency behaviour of  $L$ , but with a lower crossover frequency. Since this  $f_2$  lies entirely within the darker shaded region, eq. (4) is guaranteed to be satisfied.

With  $c_1 = 1$ , the Bode plot of  $f_1$  is given by the dashed curves in fig. 2. To aid with the tuning of  $c_1$ , Procedure 2 was applied with  $t = 2$ . Provided  $c_1$  can be tuned such that  $f_1(j\omega)$  lies anywhere within the shaded region, then eq. (4) is guaranteed to be satisfied. This is very simple to achieve in this case. For example

$$c_1(s) = \frac{20}{(1 + 100s)(1 + 7s)}$$

meets this requirement while approximately matching  $L(j\omega)$ . This is shown in fig. 2.

Step 2 ( $k = 2$ ):  $f_2$  and  $\varepsilon_2$  were computed. The Bode plot of  $f_2$  with  $c_2 = 1$  is given by the dashed curve in fig. 3. The design of  $c_2$  is slightly more involved, as the delay times in the second column are more numerous, and larger (note the lag introduced around  $\omega_c$ ). It can be quickly verified that for these parameter values

$$\varepsilon_2(j\omega) < g(L(j\omega), \theta^*(j\omega))$$

for frequencies around  $\omega_c$ , and hence that there does not exist a  $c_2$  such that  $f_2(j\omega)$  satisfies eq. (4) and closely

matches  $L(j\omega)$ . This is confirmed by applying Procedure 2 with  $t = 2$ , which shows no feasible region in the phase plot for this frequency range (this in fact shows that we cannot even closely match  $|L(j\omega)|$  while satisfying eq. (4)). However by designing  $c_2$  such that  $f_2$  has a lower crossover frequency, we can at least ensure eq. (4) is satisfied while matching the phase of  $L(j\omega)$  at crossover (phase margin), and approximating its high and low frequency behaviour. This is shown in fig. 3, where the simple design

$$c_2(s) = \frac{5}{(1 + 100s)(1 + 4s)}$$

is chosen.

## VI. CONCLUSIONS

A method for designing a set of controllers to stabilise a feedback interconnection representative of a range of network models has been given. This method poses the design problem as a set of decoupled loopshaping problems. Each loopshaping problem requires only local knowledge of the overall system model to formulate, and may be solved separately. Furthermore as each element of the network model appears in a single design problem, changes in the network can be accommodated by updating the design of the local controllers only, leaving the others untouched.

## VII. ACKNOWLEDGMENTS

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## APPENDIX

The main results of the paper are derived in this section. These derivations are contingent on the following preliminary results and definitions.

**Definition 3 (Numerical Range):** For  $A \in \mathbb{C}^{p \times p}$ , define  $W(A)$  as

$$W(A) = \{x^*Ax : x \in \mathbb{C}^p, x^*x = 1\}.$$

**Definition 4 (Halfplane):** For  $a \in \mathbb{C}$  and  $\theta \in [0, 2\pi)$ , define  $\mathbb{P}(a, \theta)$  as

$$\mathbb{P}(a, \theta) = \{z \in \mathbb{C} : \operatorname{Re}\{(z - a)e^{j\theta}\} > 0\}.$$

**Definition 5:** For  $\varepsilon \in [0, 1]$  and  $\theta \in [0, 2\pi)$ , define  $\mathbb{D}(\varepsilon, \theta)$  as

$$\mathbb{D}(\varepsilon, \theta) = \left\{z \in \mathbb{C} : \left|z - \frac{2\varepsilon^2 \cos \theta}{1 - \varepsilon^2} e^{-j\theta}\right| < \frac{2\varepsilon \cos \theta}{1 - \varepsilon^2}\right\}.$$

**Lemma 1:** Let  $z \in \mathbb{C}$ ,  $\varepsilon \in [0, 1]$ , and  $\theta \in (-\pi, \pi)$ . Then the following are equivalent:

- (i)  $\mathbb{E}(z, \varepsilon) \subset \mathbb{P}(-1, \theta)$
- (ii)  $\varepsilon > g(z, \theta)$ .
- (iii)  $z \in \mathbb{D}(\varepsilon, \theta)$ .

**Proof:** We will show the result for  $\theta = 0$ . The general case follows by making the substitution  $z = \frac{e^{-j\theta}w}{\cos \theta}$ , as this maps  $\mathbb{P}(-1, 0) \mapsto \mathbb{P}(-1, \theta)$ .

We will first rewrite (i) as an inequality. Note that the boundary of  $\mathbb{E}(z, \varepsilon)$  can be written in polar coordinates  $(r, \phi)$  as

$$r(\phi) = \frac{|z|(1 - \varepsilon^2)}{2\varepsilon(1 - \varepsilon \cos(\phi - \angle z))}.$$

Therefore (i) is equivalent to

$$\begin{aligned} & \min_{\phi} r(\phi) \cos \phi > -1 \\ \Leftrightarrow & \max_{\phi} \frac{\varepsilon \cos(\phi - \angle z) - 1}{\cos \phi} > \frac{|z|(1 - \varepsilon^2)}{2\varepsilon} \\ & \max_{\phi} (\varepsilon(\cos(\angle z) + \sin(\angle z) \tan \phi) - \sec \phi) > \frac{|z|(1 - \varepsilon^2)}{2\varepsilon} \end{aligned}$$

It is simple to show that the maximiser in the above is  $\phi^* = \pi - \arcsin(\varepsilon \sin(\angle z))$ , which gives the following inequality

$$\varepsilon \cos(\angle z) + \sqrt{1 - \varepsilon^2 \sin^2(\angle z)} > \frac{|z|(1 - \varepsilon^2)}{2\varepsilon},$$

where the positive square root is taken. Now write (iii) as an inequality:

$$\left|z - \frac{2\varepsilon^2}{1 - \varepsilon^2}\right| < \frac{2\varepsilon}{1 - \varepsilon^2}.$$

Therefore each condition in the theorem can be written as an inequality in  $z, \varepsilon$ . All that is required is to show that these inequalities are equivalent. It can then be readily verified that (i)  $\Leftrightarrow$  (iii) by writing  $z = |z|(\cos(\angle z) + j \sin(\angle z))$ , and expanding the above. Finally (i)  $\Leftrightarrow$  (ii) follows from expanding

$$1 - \varepsilon^2 \sin^2(\angle z) > \left(-\varepsilon \cos(\angle z) + \frac{|z|(1 - \varepsilon^2)}{2\varepsilon}\right)^2,$$

and solving for  $\varepsilon$ . ■

**Lemma 2:** Let  $A \in \mathbb{C}^{p \times p}$ . Then

$$\underline{\sigma}(A) \geq \min\{|z| : z \in W(A)\},$$

where  $\underline{\sigma}(A)$  denotes the smallest singular value of  $A$ .

**Proof:** By the Cauchy-Schwartz inequality, for any  $x \in \mathbb{C}^p$  such that  $\|x\| = 1$ ,  $|x^*Ax| \leq \|Ax\|$ . Therefore

$$\begin{aligned} \underline{\sigma}(A) & \geq \min\{|x^*Ax| : x \in \mathbb{C}^p, x^*x = 1\} \\ & = \min\{|z| : z \in W(A)\}, \end{aligned}$$

as required. ■

### A. Proof of Theorem 1

By the equivalence of (i) and (ii) in Lemma 1, if eq. (3) is satisfied, then  $\forall k \in \{1, \dots, p\}$ :

$$\mathbb{E}(f_k(j\omega), \varepsilon_k(j\omega)) \subset \mathbb{P}(-1, \theta(j\omega)), \quad \forall \omega > 0.$$

This is sufficient for stability of  $[R^T R, C]$  by Proposition 1 in [10].

### B. Proof of Proposition 1

Since  $L = f_k$ , by Lemma 1 in [10]  $W(RCR^T) \subset \mathbb{E}(L, \min_k \varepsilon_k)$ . Therefore

$$\min\{z : z \in W(RCR^T + I)\} \geq \min_{k \in \{1, \dots, p\}} d_k.$$

Therefore by Lemma 2

$$\bar{\sigma}(S) = \frac{1}{\underline{\sigma}(RCR^T + I)} \leq \max_{k \in \{1, \dots, p\}} \frac{1}{d_k}.$$

The bound on  $T$  then follows since  $\bar{\sigma}(T) \leq \bar{\sigma}(RCR^T)\bar{\sigma}(S)$ , and

$$\bar{\sigma}(RCR^T) \leq \bar{\sigma}(RC^{\frac{1}{2}})^2 \leq \max_{i \in \{1, \dots, p\}} \frac{|L|}{\varepsilon_i}.$$

Note that

$$\left( \max_{k \in \{1, \dots, p\}} \frac{1}{d_k} \right) \left( \max_{i \in \{1, \dots, p\}} \frac{|L|}{\varepsilon_i} \right) = \max_{j \in \{1, \dots, p\}} \frac{|L|}{\varepsilon_j d_j},$$

since both the maxima on the left hand side must be achieved by the same index (smaller eccentricity means that the distance to the  $-1$  point is also smaller).

### C. Proof of Proposition 2

For any matrix  $B \in \mathbb{C}^{q \times q}$  with spectrum  $(\lambda_1(B), \dots, \lambda_q(B))$ ,

$$\sum_{i=1}^q \sigma_i(B) \geq \sum_{i=1}^q |\lambda_i(B)| \geq |\text{Tr } B|.$$

Therefore for any invertible matrix  $A \in \mathbb{C}^{q \times q}$

$$\begin{aligned} \left( \frac{1}{q} \sum_{i=1}^q \frac{1}{\sigma_i(A)} \right)^{-1} &= \left( \frac{1}{q} \sum_{i=1}^q \sigma_i(A^{-1}) \right)^{-1} \\ &\leq \left( \frac{1}{q} |\text{Tr } A^{-1}| \right)^{-1}. \end{aligned}$$

Since the singular values of a matrix are not changed by unitary transformations, the above shows that

$$\left( \frac{1}{q} \sum_{i=1}^q \frac{1}{\sigma_i(S)} \right)^{-1} \leq \left( \frac{1}{q} |\text{Tr}(U^*(I + RCR^T))| \right)^{-1}$$

for any  $U \in \mathbb{C}^{q \times q}$  such that  $U^*U = I$ . The result then follows by choosing the unitary matrix  $U$  to be diagonal, with same phase as the corresponding diagonal entries of  $(I + RCR^T)$ , since this means that

$$\text{Tr}(U^*(I + RCR^T)) = \sum_{i=1}^q |1 + (RCR^T)_{ii}| = \sum_{i=1}^q \frac{1}{|S_i|}.$$

### D. Choice of $\theta(j\omega)$ in Theorem 1

The following lemma shows that if  $f_k(j\omega) = L(j\omega)$ , then  $\theta^*(j\omega)$  is optimal (makes  $g(L(j\omega), \theta(j\omega))$  as small as possible). Therefore  $\theta^*(j\omega)$  represents a sensible *a priori* choice, since we are designing  $c_k$  to make  $f_k(j\omega) \approx L(j\omega)$ .

**Lemma 3:** Let  $z \in \mathbb{C}$ . Then

$$\arg \min_{\theta \in (-\pi, \pi)} g(z, \theta) = \arctan \left( \frac{-\text{Im}\{z\}}{1 + \text{Re}\{z\} + |1 + z|} \right).$$

**Proof:** This can be shown by solving  $\frac{\partial g}{\partial \theta} = 0$ . An equivalent condition is

$$\begin{aligned} \frac{\partial}{\partial \theta} (|z| \cos(\theta) \cos(\theta + \angle z) + \cos^2(\theta)) &= 0 \\ |z| \sin(\angle z + 2\theta) + \sin(2\theta) &= 0 \\ \tan 2\theta + \frac{\text{Im}\{z\}}{1 + \text{Re}\{z\}} &= 0 \end{aligned}$$

The result follows from applying the half angle formula for  $\tan$  to the above, and solving for  $\theta$ . ■

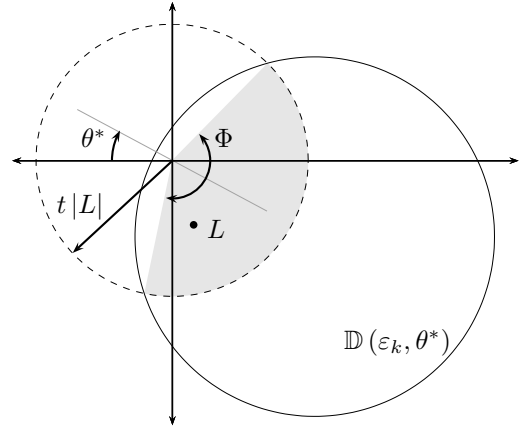


Fig. 4: The point shows  $L(j\omega)$  at a particular frequency, the dashed curve shows a circle of radius  $t|L(j\omega)|$  centred at the origin, and the solid circle,  $\mathbb{D}(\varepsilon_k(j\omega), \theta^*(j\omega))$ . The shaded region shows the largest angular sector of radius  $t|L|$  that lies within both circles. By Lemma 1,  $f_k$  may lie anywhere within  $\mathbb{D}(\varepsilon_k, \theta^*)$  and satisfy eq. (4) at this frequency. However  $\mathbb{D}(\varepsilon_k, \theta^*)$  cannot be easily sketched on a Bode plot. The shaded region can, and it is this region that is being computed in Procedure 2. The function  $\Phi(j\omega)$  itself is straightforward to derive from geometric arguments.

### E. Derivation of Procedure 2

See fig. 4.

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