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On the existence of complex-valued harmonic morphisms

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Abstract

This thesis consists of 4 papers, their content is described below:

Paper I.

We present a new method for manufacturing complex-valued harmonic morphisms from a wide class of Riemannian Lie groups. This yields new solutions from an important family of homogeneous Hadamard manifolds. We also give a new method for constructing left-invariant foliations on a large class of Lie groups producing harmonic morphisms.

Paper II.

We study left-invariant complex-valued harmonic morphisms from Riemannian Lie groups. We show that in each dimension greater than 3 there exist Riemannian Lie groups that do not have any such solutions.

Paper III.

We construct harmonic morphisms on the compact simple Lie group G_2 using eigenfamilies. The construction of eigenfamilies uses a representation theory scheme and the seven-dimensional cross product.

Paper IV.

We study the curvature of a manifold on which there can be defined a complex-valued submersive harmonic morphism with either, totally geodesic fibers or that is holomorphic with respect to a complex structure which is compatible with the second fundamental form.

We also give a necessary curvature condition for the existence of complex-valued harmonic morphisms with totally geodesic fibers on Einstein manifolds.

Populärvetenskaplig sammanfattning

En harmonisk morfi är en avbildning som bevarar harmoniska funktioner. De studerades först av Jacobi år 1848, han studerade dem i syfte att hitta harmoniska avbildningar från rummet till planet.

Man kan visa att harmoniska morfier är precis de avbildingar som bevarar Brownska rörelser. En Brownsk rörelse är en typ av slumpvandring där riktingen i varje steg är likafördelad och steglängden är normalfördelad.

Fuglede och Ishihara visade att en harmonisk morfi måste vara både en harmonisk avbildning och en konform avbildning, detta är för många vilkor för att garantera existens.

Om vi går från två dimensioner till två dimensioner så är harmoniska morfier precis de vinkelbevarande avbildningarna. Mellan två mångfalder av samma dimension som är högre än två är det precis de avståndsbevarande avbildningarna.

Avhandlingen behandlar existens och icke-existens resultat för harmoniska morfier. Artikel 1 och 3 handlar om existens medans artikel 2 och 4 handlar om icke-existens.

I artikel 1 går vi igenom en metod för att producera harmoniska morfier på lösbara Lie grupper. Metoden fungerar utmärkt på nästan alla homogena Hadamardrum.

Detta för oss till artikel 2 som visar att det inte finns vänster-invarianta harmoniska morfier på några av de homogena Hadamardrum där metoden från artikel 1 ej fungerade.

I artikel 3 visar vi att det går att konstruera egenfamiljer på G_2. Vi använder dessa egenfamiljer för att konstruera harmoniska morfier på öppna delmängder av G_2.

I den sista artikeln visar vi att harmoniska morfier med totalt geodetiska fiber måste uppfylla ett starkt geometriskt vilkor. På Einstein-mångfalder får vi ett nödvänligt vilkor som är enkelt att använda för att motbevisa existens.

Acknowledgements

First of all, I would like to express my gratitude to my adviser Sigmundur Gudmundsson for his guidance during the preparation of this thesis. I would also like to thank Martin Svensson for several helpful conversations and John ℂ. Wood for arranging a research visit to Leeds. Third, I thank my colleagues for providing a stimulating working environment.

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Preface

Preface

This Doctoral dissertation is based on the following scientific publications by the author:

Paper I. S. Gudmundsson, J. Nordström, *Harmonic morphisms from homogeneous Hadamard manifolds*, Ann. Global Anal. Geom. **39** (2011), 215-230.
DOI: 10.1007/s10455-010-9229-x

Paper II. J. Nordström, *Riemannian Lie groups with no left-invariant complex-valued harmonic morphisms*, Ann. Global Anal. Geom. **45** (2014), 1-10.
DOI: 10.1007/s10455-013-9383-z

Paper III. J. Nordström, *Harmonic morphisms and eigenfamilies on the exceptional Lie group G_2*, preprint, Lund University 2013.

Paper IV. J. Nordström, *Curvature conditions for complex-valued harmonic morphisms*, preprint, Lund University 2014.

1 Summary

In sections 2, 3 and 4 we introduce necessary background material for this work. The sections 5, 6, 7 and 8 are aimed at describing our results and putting them into context.

In section 2 we define harmonic morphisms, show their connection to harmonic maps and discuss some of their properties. Section 3 deals with an equivalent formulation on harmonic morphism, that of foliations, which is sometimes more convenient. Finally, section 4 deals with some existence results.

In section 5 we show that there is a large amount of solvable groups on which we can find harmonic morphisms [**Paper I**]. Section 6 concerns examples of solvable groups on which there are no left-invariant harmonic morphisms [**Paper II**]. In section 7 we present our results on the existence of eigenfamilies on G_2 [**Paper III**]. The last section 8 is about harmonic morphisms with totally geodesic fibres and holomorphic harmonic morphisms. These maps are particularly rigid and allow us to get some necessary conditions [**Paper IV**].

2 Harmonic morphisms

The first to show an interest in harmonic morphisms was Jacobi. In [29] he studied maps from $\mathbb{R}^3$ to $\mathbb{C}$ that pull back holomorphic functions to harmonic functions. The maps he wanted are exactly the harmonic morphisms. For a more detailed history of harmonic morphisms see [7].

Definition 2.1. Let $\phi : (M, g) \to (N, h)$ be a C^2-map. We say that ϕ is a **harmonic morphism** if for every subset V of N such that $U = \phi^{-1}(V)$ is non-empty and for every harmonic function $f : V \to \mathbb{R}$ the function $f \circ \phi : U \to \mathbb{R}$ is harmonic.

Fuglede [11] and Ishihara [28] characterized harmonic morphisms in terms of harmonic maps and weak horizontal conformality. We have

Theorem 2.2. *A C^2-map $\phi : (M, g) \to (N, h)$ is a harmonic morphism if and only if it is a harmonic map and horizontally weakly conformal.*

Given a map $\phi : (M, g) \to (N, h)$ we define the vertical space by $\mathcal{V}_p = \ker(\mathrm{d}\phi_p)$ and the horizontal space $\mathcal{H}_p$ as its orthogonal complement.

Definition 2.3. A C^1-map $\phi : (M, g) \to (N, h)$ is said to be **horizontally weakly conformal** if for each $p \in M$

(i) $\mathrm{d}\phi_p = 0$, or

(ii) $\mathrm{d}\phi_p|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\phi(p)}N$ is conformal.

The function $\lambda : M \to \mathbb{R}_0^+$ defined by

$$h_{\phi(p)}(\mathrm{d}\phi_p(X), \mathrm{d}\phi_p(Y)) = \lambda(p)^2 g_p(X_p, Y)$$

is called the **dilation** of ϕ. A point $p \in M$ is said to be a **critical point** if $\lambda(p) = 0$, otherwise it is said to be a **regular point**.

Definition 2.4. We say that a map $\phi : (M, g) \to (N, h)$ is harmonic if it is a critical point of the energy functional

$$E(\phi) = \frac{1}{2} \int_K |\mathrm{d}\phi|^2 \mathrm{d}\nu$$

for any compact subset K of M.

The tension field of a C^2-map $\phi : (M, g) \to (N, h)$ is given by

$$\tau(\phi) = \mathrm{trace}(\nabla \mathrm{d}\phi).$$

A map $\phi : (M, g) \to (N, h)$ is harmonic if and only if it satisfies the Euler-Lagrange equation for the energy functional i.e. $\tau(\phi) = 0$.

Introduce local coordinates $x = (x^i) : U \to \mathbb{R}^m$ and $y = (y^\alpha) : V \to \mathbb{R}^n$ on M and N, respectively. If we set $\phi^\gamma = y^\gamma \circ \phi : U \to \mathbb{R}$ then the n equations for ϕ to be a harmonic map are given by

$$\tau(\phi)^\gamma = \sum_{ij} g^{ij} \left(\frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^i} - \sum_m \Gamma^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + \sum_{\alpha\beta} (\Gamma^\gamma_{\alpha\beta} \circ \phi) \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right) = 0,$$

where Γ^k_{ij} and $\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of M and N, respectively.

The $\binom{n+1}{2} - 1$ equations for ϕ to be horizontally conformal are given by

$$\sum_{ij} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} = \lambda^2 (h^{\alpha\beta} \circ \phi).$$

Thus the number of equations for a map to be a harmonic morphism increases quadratically with the dimension of the codomain.

3 Foliations

A foliation is a decomposition of a manifold into submanifolds in such a way that locally the submanifolds are the fibres of a submersion.

Definition 3.1. Let M be a manifold. A **foliation** $\mathcal{F}$ on M is a partition $\{L_\alpha\}$ into connected subsets, called **leaves**, such that, for each $p \in M$ there is a local submersion $\phi : W \to N$ where $L_\alpha \cap W$ are the fibres of ϕ.

A related concept to foliations are **distributions** which are smooth subbundles of the tangent bundle. We say that a distribution $\mathcal{V}$ is **integrable** if $U, V \in \Gamma(\mathcal{V})$ implies $[U, V] \in \Gamma(\mathcal{V})$. The tangent spaces of the leaves of a foliation will generate an integrable distribution. Conversely, if a distribution is integrable, then the Frobenius theorem tells us that it is locally the tangent spaces of the leaves of a foliation.

The **second fundamental form** $B^{\mathcal{V}}$ of $\mathcal{V}$ is defined by

$$B^{\mathcal{V}}(U,V) = \frac{1}{2}\mathcal{H}(\nabla_U V + \nabla_V U)$$

for $U, V \in \mathcal{V}$ and the **mean curvature vector field** by

$$\mu^{\mathcal{V}} = \text{trace}(B^{\mathcal{V}}).$$

Definition 3.2. A distribution $\mathcal{V}$ is said to be

(i) totally geodesic if $B^{\mathcal{V}} = 0$,

(ii) minimal if $\mu^{\mathcal{V}} = 0$ and

(iii) conformal if $(\mathcal{L}_V g)(X,Y) = \nu(V)g(X,Y)$ for some $\nu : \Gamma(\mathcal{V}) \to C^\infty(M)$.

A foliation is said to, have totally geodesic leaves, have minimal leaves, or to be conformal, corresponding to its induced distribution.

An additional property of harmonic morphisms was given by Baird and Eells in [3], they showed that the tension field of a horizontally weakly conformal map ψ is given by

$$\tau(\psi) = (n-2)\mathcal{V}(\text{grad}(\ln\lambda)) + (m-n)\mu^{\mathcal{V}}.$$

From this the obtain the following result.

Theorem 3.3 ([3])**.** *Let $\phi : (M,g) \to (N,h)$ be a horizontally weakly conformal map. If $\dim(N) = 2$, then ϕ is a harmonic morphism if and only if it has minimal fibres at regular points. If $\dim(N) \geq 3$, then any two of the following imply the third:*

(i) ϕ is harmonic.

(ii) ϕ is horizontally homothetic, i.e., $\text{grad}(\lambda) \in \mathcal{V}$ at regular points.

(iii) ϕ has minimal fibres at regular points.

Furthermore Fuglede showed in [12] that a horizontally homothetic harmonic morphism has no critical points. Observe that there exist harmonic morphisms that do not have minimal fibres, see Example 5.4 of [3].

Using foliations we can obtain local existence of harmonic morphisms without involving the codomains.

Theorem 3.4 ([37])**.** *A foliation on a Riemannian manifold of codimension* 2 *produces harmonic morphisms if and only if it is conformal and has minimal leaves.*

There is a similar result about foliations of codimension not equal to 2, but since we are interested in complex-valued harmonic morphisms we will not discuss it here.

4 Existence of harmonic morphisms

The fact that harmonic morphisms are harmonic maps means that they are subject to a maximum principle and thus we can not have any non-constant morphism from a compact manifold to a non-compact one. This is the simplest of the topological obstructions to harmonic morphisms. Since we are mostly interested in the geometric consequences we will not go into this any further.

In [11] Fuglede gave a few examples of harmonic morphisms, such as, Riemannian projections and holomorphic maps from Kähler manifolds to surfaces. Baird showed in Section 8 of [1] that for $m \geq n$ there exist harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$ given by polynomials and that some of these restrict to harmonic morphisms from $S^{m-1} \to S^{n-1}$

The equations for a harmonic morphism form an over-determined system, thus we do not expect harmonic morphisms to exist on all Riemannian manifolds. There is no general existence theory, but in 3 and 4 dimensions there are some classification results, see [5], [38] and [2]. Thus finding examples of Riemannian manifolds on which we can define a harmonic morphism is of interest.

A program to find Riemannian manifolds allowing harmonic morphisms was started by Gudmundsson. His idea was to look at Riemannian manifolds with lots of "symmetries". Starting with Riemannian manifolds of constant curvature, which are locally isometric to one of S^n, $\mathbb{R}^n$ or $\mathbb{R}H^n$, see [14]. Then existence of non-holomorphic harmonic morphisms on complex projective space $\mathbb{C}P^n$ in [15]. Since $\mathbb{C}P^n$ is Kähler each component of the coordinate charts are harmonic morphisms. This shows that the equations are not too rigid. Examples of harmonic morphisms from real hyperbolic space with non-totally geodesic fibres, were found in [17] in the odd-dimensional case. This lead to the following conjecture.

Conjecture 4.1 (Gudmundsson 1995)**.** *Let* (M, g) *be an irreducible Riemannian symmetric space of dimension* $m \geq 2$*. For each point* $p \in M$ *there exists a non-constant* *complex-valued harmonic morphism $\phi : U \to \mathbb{C}$ defined on an open neighbourhood U of p. If M is of non-compact type then the domain U can be chosen to be the whole of M.*

Other symmetric spaces of non-compact type are the complex and quaternionic hyperbolic spaces where existence was proven in [18]. This together with the fact that several non-compact symmetric spaces have natural holomorphic harmonic morphisms, give some credibility to the conjecture.

5 Existence on solvable Lie groups

Much progress has been made on the Gudmundsson conjecture and currently harmonic morphisms have been found to exist on all symmetric spaces except for $G_2/SO(4)$ and its non-compact dual, see [23].

The duality principle of Theorem 7.1 in [21] states that a harmonic morphism exists on a symmetric space of compact type if and only if there exists one on its non-compact dual. Thus we study existence on the symmetric spaces of non-compact type.

It is know that all symmetric spaces of non-compact type are solvable Lie groups, thus we are lead to finding harmonic morphisms on solvable Lie groups.

In [23] the authors give two methods for producing harmonic morphisms on solvable Lie groups. The first method is given in Theorems 5.2 and 11.3 and is suited for symmetric spaces of rank at least 3. The second method, that of Theorem 12.1 from [23], is suited for symmetric spaces with restricted root spaces of dimension at least 2.

The next result is a generalized version of Theorem 12.1 in [23].

Theorem 5.1 ([**Paper I**], Theorem 3.1)**.** *Let $G = N \rtimes A$ be a semi-direct product of the simply-connected Lie groups A and N. Let G be equipped with a left-invariant metric g and $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k} \oplus \mathfrak{m}$ be an orthogonal decomposition of the Lie algebra $\mathfrak{g}$ of G such that $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{k} \oplus \mathfrak{m}$ are the Lie algebras of A and N, respectively. Let K be a closed simply connected subgroup of N with Lie algebra $\mathfrak{k}$ such that*

(i) $[\mathfrak{a}, \mathfrak{k}] \subseteq \mathfrak{k}$,

(ii) $[\mathfrak{a}, \mathfrak{m}] \subseteq \mathfrak{m}$,

(iii) $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{k}$,

(iv) $\mathrm{trace}(\mathrm{ad}_Z) = 0$ *for all* $Z \in \mathfrak{m}$,

(v) *there exists an* $\lambda \in \mathfrak{a}^*$ *such that for each* $H \in \mathfrak{a}$ *and* $Z \in \mathfrak{m}$

$$(\mathrm{ad}_H + \mathrm{ad}_H^t)(Z) = 2\lambda(H)Z.$$

Then there exists a harmonic morphism on $\Phi : G \to \mathbb{R}^m$, *where* $m = \dim(\mathfrak{m})$.

This result works particularly well on homogeneous Hadamard manifolds, since they have to be solvable Lie groups and have natural candidates for $\mathfrak{m}$, the generalized root spaces of the almost normal operators ad_A, $A \in \mathfrak{a}$. This is not enough for existence, but almost, as we see in the following result which follows easily from the above theorem.

Theorem 5.2 ([**Paper I**], Theorem 10.3)**.** *Let the solvable Riemannian Lie group* S *be a homogeneous Hadamard manifold with Lie algebra* $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$. *Furthermore assume that there exists a generalized root space* $\mathfrak{n}_{\alpha,\beta}$ *of* $\mathfrak{n}$ *of dimension* $m \geq 2$ *such that*

(i) $\mathrm{ad}_H|_{\mathfrak{n}_{\alpha,\beta}}$ *is normal for all* $H \in \mathfrak{a}$,

(ii) $[\mathfrak{n}, \mathfrak{n}]$ *is contained in the orthogonal complement* $\mathfrak{n}_{\alpha,\beta}^\perp$ *of* $\mathfrak{n}_{\alpha,\beta}$ *in* $\mathfrak{n}$,

(iii) $\mathrm{ad}_H(\mathfrak{n}_{\alpha,\beta}^\perp) \subseteq \mathfrak{n}_{\alpha,\beta}^\perp$ *for all* $H \in \mathfrak{a}$.

Then there exists a harmonic morphism $\Phi : S \to \mathbb{R}^m$.

The harmonic morphisms produced will foliate S by left-translated Lie subgroups of S, thus we are lead to the concept of a left-invariant foliation.

Definition 5.3. A foliation $\mathcal{F} = \{\ell_g\}_{g \in G}$ of a Lie group G is said to be left-invariant if $\ell_e = K$ is a closed subgroup and $\ell_g = L_g(\ell_e)$.

In the case of the non-compact dual of $G_2/SO(4)$, which is a symmetric space of rank 2, all the eigenspaces of ad_A for $A \in \mathfrak{a}$ have dimension 1. Thus this method will not produce harmonic morphisms.

6 Non-existence on solvable Lie groups

In [**Paper II**] we investigate the existence of left-invariant foliations. We answer the following question: Are there solvable Lie groups of dimension larger than 3 that do not have left-invariant conformal foliations with minimal fibres?

For a Riemannian manifold (M, g) and a point $p \in M$ we define $\mathrm{Iso}_p(M)$ to be the set of isometries $F : M \to M$ such that $F(p) = p$.

Proposition 6.1 ([**Paper II**], Theorem 3.1)**.** *Let $\mathfrak{s}$ be a Euclidean Lie algebra with orthonormal basis*

$$\{X_0 = A, X_1, \dots, X_n\}$$

and brackets $[A, X_i] = \lambda_i X_i$. Let S be the simply connected Lie group with Lie algebra $\mathfrak{s}$ and the induced left-invariant Riemannian metric. If $0 < \lambda_1 < \dots < \lambda_n$ then $\mathrm{Iso}_e(S)$ is finite.

To give a non-existence result we show that if there is a left-invariant harmonic morphism on this space then there must exist a fix-point isometry.

Theorem 6.2 ([**Paper II**], Theorem 5.1)**.** *Let $\mathfrak{s}$ be a solvable Euclidean Lie algebra with $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ Abelian and strictly negative sectional curvature. If there exist an orthogonal decomposition $\mathfrak{s} = \mathfrak{h} \oplus \mathfrak{v}$, with $\dim(\mathfrak{h}) \geq 2$, such that*

(i) $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{v}$,

(ii) $\mathrm{trace}_{\mathfrak{v}}(\mathrm{ad}_H) = 0$ *for all $H \in \mathfrak{h}$,*

(iii) there exists $\lambda \in \mathfrak{v}^*$ such that

$$\langle \mathrm{ad}_V H_1, H_2 \rangle + \langle H_1, \mathrm{ad}_V H_2 \rangle = 2\lambda(V) \langle H_1, H_2 \rangle$$

for all $V \in \mathfrak{v}$ and all $H_1, H_2 \in \mathfrak{h}$, and

(iv) $\mathrm{ad}_A : \mathfrak{n} \to \mathfrak{n}$ *is normal for all $A \in \mathfrak{a} = \mathfrak{n}^\perp$,*

then there exists a non-zero skew-symmetric derivation T of $\mathfrak{s}$.

Corollary 6.3 ([**Paper II**], Corollary 5.2)**.** *Let S be a 2-step solvable Riemannian Lie group with strictly negative sectional curvature. Let $\mathfrak{s}$ be the Lie algebra of S, set $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ and $\mathfrak{a} = \mathfrak{n}^\perp$. If $\mathrm{ad}_A : \mathfrak{n} \to \mathfrak{n}$ is normal and there exists a left-invariant conformal foliation with minimal fibres on S, then the isotropy group at e is of dimension at least one.*

Combining the results of Proposition 6.1 and Corollary 6.3, we get the following.

Theorem 6.4 ([**Paper II**], Theorem 1.1)**.** *Let $\mathfrak{s}$ be a Euclidean Lie algebra with orthonormal basis*

$$\{A, X_1, \dots, X_n\}$$

and brackets $[A, X_i] = \lambda_i X_i$. If $0 < \lambda_1 < \dots < \lambda_n$ then there exist no complex-valued left-invariant harmonic morphisms on the simply connected Lie group S with Lie algebra $\mathfrak{s}$ and the induced left-invariant Riemannian metric.

We do not know if harmonic morphisms on a Lie group have to be left-invariant, except in some cases: The rank one symmetric spaces of non-compact type, where there exist non left-invariant harmonic morphisms. Opposite, on 3-dimensional Riemannian Lie groups with non-constant curvature all harmonic morphisms are left-invariant, as was shown in Proposition 13.4 in [23].

7 Existence on G_2

In this section we discuss existence of harmonic morphisms on the compact Lie groups, the eigenfamily method for finding harmonic morphisms, and the existence of eigenfamilies on G_2 proven in [**Paper III**].

The compact simple Lie groups are divided into the classical and the exceptional ones. The classical are $SO(n)$, $SU(n)$ and $Sp(n)$. While the exceptional are G_2, F_4, E_6, E_7 and E_8. Any of these Lie groups when equipped with a bi-invariant metric is a symmetric space of compact type.

The homogeneous projection $G \to G/K$ is a Riemannian submersion with totally geodesic fibres, hence a harmonic morphism. More generally Bérnard-Bergery and Bourguignon proved in [8] that for mild conditions on the Lie subgroups $K \subseteq H$ the projection $G/H \to G/K$ is a Riemannian submersion with totally geodesic fibres. Thus in the case that we have existence of complex-valued harmonic morphisms from G/K we also get existence on G/H. This was used in [13] to get existence of harmonic morphism on the classical compact Lie groups.

A more general construction, taking a compact Lie group to a flag-manifold was introduced in [36]. In [21] the authors use this to show the existence of harmonic morphisms on all the compact simple Lie groups by mapping to a flag manifold

$$G \to G/(G \cap P) = G^{\mathbb{C}}/P$$

where P is a parabolic subgroup of $G^{\mathbb{C}}$. Since the latter is a cosymplectic Hermitian manifold, each component of a coordinate chart is a complex-valued harmonic morphism.

In [19], Gudmundsson and Sakovich introduced a different method to find harmonic morphisms on compact Lie groups, that of eigenfamilies.

Introduce the operators Δ and κ by

$$\Delta(\phi) = \operatorname{div}(\operatorname{grad}(\phi)) \quad \text{and} \quad \kappa(\phi, \psi) = \langle \nabla\phi, \nabla\psi \rangle,$$

for $\phi, \psi : (M, g) \to \mathbb{C}$.

Definition 7.1. Let (M, g) be a Riemannian manifold and $\lambda, \mu \in \mathbb{C}$. A set $\mathcal{E}$ of functions $\phi : (M, g) \to \mathbb{C}$ such that

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu\phi\psi$$

for all $\phi, \psi \in \mathcal{E}$ is said to be an **eigenfamily**.

Once we have an eigenfamily it follows from the following that there exist harmonic morphisms.

Theorem 7.2 ([19], Theorem 2.5)**.** *Let (M, g) be a semi-Riemannian manifold and $\mathcal{E} = \{\phi_1, \ldots, \phi_n\}$ be a finite eigenfamily of complex-valued functions on M. If $P, Q : \mathbb{C}^n \to \mathbb{C}$ are linearly independent homogeneous polynomials of the same positive degree then the quotient*

$$\frac{P(\phi_1, \ldots, \phi_n)}{Q(\phi_1, \ldots, \phi_n)}$$

is a harmonic morphism on the open and dense set

$$\{p \in M \,|\, Q(\phi_1, \ldots, \phi_n)(p) \neq 0\}.$$

Gudmundsson and Sakovich proved the existence of eigenfamilies on the classical compact Lie groups, $SO(n)$, $SU(n)$ and $Sp(n)$. In particular they showed: If $p \in \mathbb{C}^n$ be an isotropic vector, i.e. $\langle p, p \rangle = 0$, then

$$\mathcal{E}_p = \{SO(n) \ni g \mapsto \langle a, g \cdot p \rangle | a \in \mathbb{C}^n\}$$

is an eigenfamily on $SO(n)$. If $q \in \mathbb{C}^n$ and $G = SU(n)$ or $G = Sp(n)$, then

$$\mathcal{E}_q = \{G \ni g \mapsto \langle a, g \cdot q \rangle | a \in \mathbb{C}^n\}$$

is an eigenfamily on G.

In [26] the authors show that existence of the eigenfamilies above follow easily from the irreducibility of the exterior product for $SO(n)$ and symmetric product for $SU(n)$ and $Sp(n)$. They also found eigenfamilies that do not come from the standard representation, the families given above.

Let $\rho : G_2 \to \mathrm{Aut}(V)$ denote the standard real representation of G_2. The exterior square representation $\tilde{\rho} : G_2 \to \mathrm{Aut}(\bigwedge^2 V)$ has two irreducible orthogonal subspaces $\bigwedge^2 V = \mathfrak{g}_2 \oplus W$, see page 541 in [9]. As $\dim(\mathfrak{g}_2) = 14$ we have $\dim(W) = 7$.

Since neither the exterior nor the symmetric product of the standard representation are irreducible we can not proceed as in [26].

In [**Paper III**] we show that the projection onto $\mathfrak{g}_2$ is in fact given by the operator κ at the point e. We then use the seven dimensional cross product to understand the projection onto W. In essence we show that $|P_W(a \wedge b)|^2 = \frac{1}{3}|a \times b|^2$.

Theorem 7.3 ([**Paper III**], Theorem 1.1)**.** *Let $\rho : G_2 \to \mathrm{Aut}(\mathbb{R}^7)$ be the standard representation of the exceptional compact Lie group G_2. Extend $\rho(g)$ to act on complex vectors by $\rho(g)(u + iv) = \rho(g)u + i\rho(g)v$. For any $a, b \in \mathbb{C}^7$ define*

$$\phi_{ab}(g) = \langle a, \rho(g)b \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the complex-bilinear extension of the standard scalar product on $\mathbb{R}^7$. Suppose that $p \in \mathbb{C}^7$ satisfies $\langle p, p \rangle = 0$. Then

$$\mathcal{E}_p = \{\phi_{ap} \,|\, a \in \mathbb{C}^7\}$$

is an eigenfamily on G_2 i.e. there exist non-positive $\lambda, \mu \in \mathbb{R}$ such that

$$\Delta(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu\phi\psi$$

for all $\phi, \psi \in \mathcal{E}_p$.

The harmonic morphisms produced by these eigenfamilies live on the complex quadratic $Q_5 = G_2/U(2)_-$.

8 Necessary curvature conditions for harmonic morphisms

Baird and Wood show in the paper [6] that for a harmonic morphism from a 3-dimensional Riemannian manifold to a surface the Ricci curvature satisfies

$$\mathrm{Ric}(X + iY, X + iY) = 0, \text{ for } X, Y \in \mathcal{H}.$$

In three dimensions this condition is in fact equivalent to $K(X_\theta, U)$ not depending on θ for $U \in \mathcal{V}$. From this condition they can find necessary conditions for the existence of harmonic morphisms. They apply this to the unimodular solvable Lie group Sol_3 and show that this does not have any harmonic morphism, not even locally.

In three dimensions the minimal fibres are curves thus they are geodesics. To generalize to higher dimensions we assume that the fibres are totally geodesic.

Harmonic morphisms with totally geodesic fibres were studied by Mustafa. His interest was in the case of codimension at least 4. He got some global results in the case of non-positive curvature on the domain see Theorem 4.1 in [32]. Later this was extended to codimension at least 2 assuming the harmonic morphism is horizontally homothetic [34].

There is a classification of all complex-valued harmonic morphisms with totally geodesic fibres on spaces of constant curvature due to Baird and Wood in Section 6.8 of [7]. This was later extended to Riemannian manifolds which are conformaly equivalent to a constant curvature metric by Pantilie [33].

Holomorphic harmonic morphisms have been studied extensively. It is a natural class of maps to study, since any (pseudo)-holomorphic map from an almost Hermitian manifold to a surface is conformal. Eells and Sampson showed in [10] that if the domain is Kähler then any holomorphic map is also harmonic. Lichnerowicz showed in [30] that the Kähler condition is too strong, and that it is enough that the domain is cosymplectic, div $J = 0$, for any holomorphic map to a surface to be harmonic. The complete description came with [27] where Gudmundsson and Wood show that the tension field of a holomorphic map is given by

$$\tau(\phi) = -\mathrm{d}\phi(J \operatorname{div} J)$$

thus ϕ is a harmonic morphism if and only if div $J \in \ker \mathrm{d}\phi$.

In [31] Loubeau and Pantilie study twistorial harmonic morphisms, which in a sense are holomorphic maps with respect to an integrable Hermitian structure.

They show that in four dimensions a holomorphic harmonic morphism satisfies the Ricci curvature condition

$$\mathrm{Ric}(X + iY, X + iY) = 0.$$

In the case of harmonic morphisms with minimal but not totally geodesic fibres we have Examples 6.1 and 6.2 from [25]. These two maps are not holomorphic with respect to an integrable Hermitian structure.

We are interested in the case of codimension 2, where the curvature operator has to satisfy a strong symmetry property, rotations in the horizontal plane do not change the sectional curvature.

Theorem 8.1 ([**Paper IV**], Theorem 1.1)**.** *Let (M, g) and (N^2, h) be Riemannian manifolds, let $\phi : (M, g) \to (N^2, h)$ be a submersive harmonic morphism with totally geodesic fibres and $p \in M$. Given $U, V \in \mathcal{V}_p = \ker(\mathrm{d}\phi)$ and any orthonormal basis $\{X, Y\}$ for $\mathcal{H}_p = \mathcal{V}_p^\perp$, set $X_\theta = \cos(\theta)X + \sin(\theta)Y$. Then*

$$\langle R(X_\theta \wedge U), X_\theta \wedge V\rangle$$

is independent of θ.

If we consider an Einstein manifold we have the following proposition which is easy to test.

Proposition 8.2 ([**Paper IV**], Proposition 3.1)**.** *Let (M, g) be an Einstein manifold and (N^2, h) be a Riemannian surface. Let R be the curvature operator of (M, g) at $p \in M$. If there is a submersive harmonic morphism $\phi : (M, g) \to (N^2, h)$ with totally geodesic fibres then R has at least $\dim(M) - 2$ pairs of eigenvalues.*

This gives an example of an Einstein metric which does not have any harmonic morphism with totally geodesic fibres, see Example 4.1 in [**Paper IV**].

We are not able to show that the conditions of Theorem 11 in [**Paper IV**] are valid for holomorphic harmonic morphisms. However, we can show that the Ricci curvature condition holds if we also add that the second fundamental form is compatible with the complex structure. So we show that for a Hermitian metric we can get the Ricci condition even if the fibres are not superminimal.

Theorem 8.3 ([**Paper IV**], Theorem 5.1). *Let $\phi : M^{2m} \to N^2$ be a harmonic morphism between Hermitian manifolds (M^{2m}, g, J) and (N^2, h, J^N). Suppose that J is adapted to ϕ and compatible with the second fundamental form B. Then*

$$\mathrm{Ric}(X,X) = \mathrm{Ric}(Y,Y) \textit{ and } \mathrm{Ric}(X,Y) = 0$$

for $X, Y \in \mathcal{H}$ orthonormal.

Note that this does not imply that X and Y are eigenvectors of the Ricci curvature operator, since we can have $\mathrm{Ric}(X,U) \neq 0$ for some $U \in \mathcal{V}$.

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