A Symmetry Approach for Balanced Truncation of Positive Linear Systems

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A Symmetry Approach for Balanced Truncation of Positive Linear Systems

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Abstract

We consider model order reduction of positive linear systems and show how a symmetry characterization can be used in order to preserve positivity in balanced truncation. The reduced model has the additional feature of being symmetric.

1. INTRODUCTION

Mathematical modeling of biological, chemical and physical systems often leads to complex high-dimensional models, which are hard to analyze and simulate. Approximating high-order models by ones of reduced order is the central goal of model order reduction in control and has received considerable attention e.g. in [7], [14], [19], [20].
Here we consider linear time-invariant systems

$$G: \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

with state vector $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^p$, for small m, p and large dimension n. Our goal is to approximate (1) by a system of the same structure with the same m and p, but with smaller order $r < n$. For this purpose different methods have been developed, the most popular of which are based on linear subspace projection, such as balanced truncation [14] or Krylov subspace methods [1], [7].
In practice, one often deals with so called *(internally) positive* systems ([4]) whose output and state variables are nonnegative, whenever the input and initial states are confined to be nonnegative. Such systems occur e.g. within the discretization of partial differential equations [18], or transport models or compartmental systems [12]. It is desirable that the reduced system also is positive. Unfortunately, positive systems are defined on cones instead of linear subspaces [2], [4], [15] and therefore methods based on linear subspace projection typically do not preserve positivity. As a consequence, new methods have been developed in [5], [10], [17], however with rather conservative results regarding the H_∞-error and the computational effort.

In this paper we present several new results related to positivity preserving model order reduction. On the one hand, we show that balanced truncation to order 1 always gives a positive approximation. On the other hand, for single-input single-output (SISO) systems, we derive a symmetry condition which allows the computation of a positive realization. Since any balanced realization of a SISO-system can be shown to be sign-symmetric with respect to the entries in A, B and C (cf. [14] and [6]), we can describe a procedure to compute a positive reduced order model of a SISO-system, by just comparing signs in the sign-symmetric realization. In the worst case, this procedure only allows for the scalar approximation mentioned above, but in practical examples, it yields positive approximations also of higher order with an acceptable error bound. These approximations have the additional property of being symmetric, which is desirable for instance, in case of linear networks with reaction-diffusion structure [8].

2. Preliminaries

Throughout this paper we use the following notation for real matrices and vectors $X = (x_{ij})$. We say that X is *positive*, $X \gg 0$, if all its entries are positive ($x_{ij} > 0$ for all i, j). It is called *nonnegative*, $X \geqq 0$, if all entries are nonnegative ($x_{ij} \geq 0$ for all i, j). By $|X| = (|x_{ij}|) \geqq 0$

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we denote the entrywise absolute value of X.

A square matrix X is *reducible*, if there exists a permutation matrix $P = [P_1, P_2]$ so that $P_2^T X P_1 = 0$. Otherwise, it is *irreducible* (compare [3]). By $\sigma(X)$ we denote the spectrum of X. If X is square and symmetric, then we write $X > 0$, or $X \geq 0$ if X is positive definite, or nonnegative definite, i.e. $\sigma(X) \subset [0, \infty[$.

We also use these notations to describe the relation between two arbitrary elements, e.g. $A \geq B$ is defined by $A - B \geq 0$. A real vector valued function $u(t) \in \mathbb{R}^n$ is called *nonnegative* if and only if $u(t) \geqq 0$ for all t.

Theorem 1 (Perron-Frobenius [12], [13])**.** *If $A \geqq 0$ is irreducible, then there exist a real $\lambda_0 > 0$ and a vector $x_0 \gg 0$ such that*

i. $Ax_0 = \lambda_0 x_0$.

ii. $\lambda_0 \geq |\lambda|$, $\forall \lambda \in \sigma(A)$.

iii. The algebraic multiplicity of λ_0 is one.

If $A \geqq 0$ is reducible, then there exists a real $\lambda_0 \geq 0$ and a vector $x_0 \geq 0$ such that

i. $Ax_0 = \lambda_0 x_0$.

ii. $\lambda_0 \geq |\lambda|$, $\forall \lambda \in \sigma(A)$.

Moreover, there exists a permutation matrix π, such that

$$\pi^T A \pi = \begin{pmatrix} B_1 & * & * \\ & \ddots & * \\ & & B_k \end{pmatrix},$$

where each B_i is irreducible or equal to zero.
In particular, if A is diagonalizable and λ_0 has multiplicity m_0, then A has m_0 linearly independent nonnegative eigenvectors.

Now, let us define positive systems (cf. [4]).

Definition 1 (External Positivity)**.** *A linear system (A,B,C,D) as in (1) is called externally positive if and only if its output, corresponding to a zero initial state, is nonnegative for every nonnegative input.*

Definition 2 (Internal Positivity)**.** *A linear system (1) is called (internally) positive if and only if its state and output are nonnegative for every nonnegative input and every nonnegative initial state.*

To characterize a continuous positive system, one needs the notion of a Metzler matrix (or Z-matrix) [3]. A matrix $A \in \mathbb{R}^{n \times n}$ is Metzler if there exists an $\alpha \in \mathbb{R}$ such that $A + \alpha I_n \geqq 0$, where I_n is the $n \times n$ identity matrix [12]. If A is Metzler then $e^{At} \geqq 0$ for all $t \geq 0$.

Theorem 2. *A continuous linear system (1) is positive if and only if A is Metzler and $B, C, D \geqq 0$. [4]*

3. Balanced Truncation to order 1

In the folowing we consider asymptotically stable positive systems (A,B,C,D) as in (1). We assume the reader to be familiar with the concept of *standard balanced truncation* (see e.g. [1, 17]). In general, balanced truncation does not return a positive system – unless the system is reduced to the order $r = 1$.

Theorem 3 (Positive Order-1 Balanced Truncation)**.** *If (A_1, B_1, C_1, D_1) is the reduced system of order 1, obtained by standard balanced truncation of (A,B,C,D), then it has a positive, asymptotically stable realization $(A_1, |B_1|, |C_1|, D_1)$ of order 1.*

Proof. Let P and Q be the Gramians of (A,B,C,D), implicitly given by

$$AP + PA^T = -BB^T, \quad A^T Q + QA = -C^T C, \qquad (2)$$

or in their explicit form by

$$P = \int_0^\infty e^{At} B B^T e^{A^T t} dt, \quad Q = \int_0^\infty e^{A^T t} C^T C e^{At} dt. \quad (3)$$

Obviously, P and Q are nonnegative and thus PQ, too. Balancing the system via a state-space transformation $x = T\xi$ yields $T^{-1} P Q T = \operatorname{diag}\left(\Sigma^2, 0\right)$, where $\Sigma = \operatorname{diag}\left(\sigma_1 I_{k_1}, \ldots, \sigma_N I_{k_N}\right)$, containing the Hankel singular values $\sigma_1 > \cdots > \sigma_N$, with corresponding multiplicities $k_1, \ldots, k_N$ (see [22])
Hence, the columns of T are eigenvectors of PQ and by Theorem 1 there exists a nonnegative right-eigenvector v_1 to the largest eigenvalue σ_1, i.e. $PQv_1 = \sigma_1 v_1$ with $T = \left(v_1, \ldots, v_n\right)$. Analogously, there is a nonnegative left-eigenvector w_1 with $T^{-1} = \left(w_1, \ldots, w_n\right)^T$.
If $k_1 = 1$, the asymptotic stability of the reduced system of order 1 leads to $A_1 = w_1^T A v_1 < 0$, $B_1 = w_1^T B \geqq 0$, $C_1 = C v_1 \geqq 0$, $D_1 = D \geqq 0$.
If $k_1 > 1$, it could happen that $A_1 = 0$. But since the reduced system of order k_1 (belonging to all σ_1) is asymptotically stable, there must exist at least one asymptotically stable first order approximation. By Theorem 1 we conclude the reducibility of PQ and the positivity of each first order approximation. In both cases Theorem 2 concludes the proof. Balanced truncation can also be performed by using $-v_1$ and $-w_1$. In this case we substitute B_1 and C_1 by their elementwise absolute values. □

In general, Theorem 3 does not transfer to *singular perturbation balanced truncation* (cf. [17]). Further, Theorem 3 gives a necessary condition on the positivity, independent of its realization. By numerical experiments, we can observe, that this is a strong condition. Many of the non-positive systems fail at this point.

4. Positive Realization Problem

From the proof of Theorem 3 we can deduce, that even in case of an approximation to order 1, balanced truncation does not necessarily return a positive realization. However, it is straightforward to see, that every first order externally positive system has a positive realization of the same dimension. The same is true for second order SISO-systems (see [15]). But higher-order externally positive systems do not necessarily admit an internally positive realization of the same dimension – even if they possess only real poles (see [15] again). Knowing, that balanced truncation always results in a minimal system, the positive realization problem and its connection to balanced realizations becomes the major obstacle beside the actual positivity preservation.
We call a linear system *quasi-symmetric* if $A = A^T$ and $C = kB^T$ for some $k > 0$. If $k = 1$ the system is said to be *symmetric* (see [11]).

Theorem 4 (Positivity of Symmetric Systems)**.** *Every quasi-symmetric SISO system possesses a symmetric positive minimal realization, which can be computed by Arnoldi's (or Lanczos') algorithm.*

Proof. Let (A,B,C,D) be a quasi-symmetric system with Gramians P and Q. Then from (3) it follows that $Q = k^2P$. Diagonalization of kP gives $kP = T^T\Sigma T$ and $PQ = k^2P^2 = T^T\Sigma^2T = \tilde{T}^{-1}PQ\tilde{T}$, with $\tilde{T} = \frac{1}{\sqrt{k}}T$. Obviously, $\tilde{T}$ is a balancing transformation matrix and the balanced system is given by $(T^{-1}A\tilde{T}, \tilde{T}^{-1}B, C\tilde{T}) = (T^TAT, \sqrt{k}(BT)^T, \sqrt{k}BT)$. Thus, it is always possible to find a symmetric minimal realization of a quasi-symmetric system. Arnoldi's algorithm [1], [21] yields a unitary transformation matrix $V = \left(\frac{B}{\|B\|_2}, *\right)$, such that $V^TV = I$ and V^TAV is upper Hessenberg with positive elements on its lower diagonal. If $A = A^T$ and $C = B^T$, this means that V^TAV is Metzler, $CV = \left(\|B\|_2, 0, \ldots, 0\right) \geqq 0$ and $V^TB = (CV)^T \geq 0$.
Positivity now follows from Theorem 2. □

5. Symmetric Balanced Truncation

If balanced truncation of a SISO system results in a symmetric reduced model, then (by Theorem 4) we are able to compute its positive realization. To this end we recall the following important result of balanced SISO-systems (see also [6],[14]).

Theorem 5. *Let $G(s)$ be the transfer function of an arbitrary SISO-system. Then there exists a balanced realization (A,B,C,D) of $G(s)$, such that (A,B,C,D) is sign symmetric, i.e. $|A| = |A^T|$ and $|B| = |C^T|$.*

Proof. Let (A,B,C,D) have simple Hankel singular values $\{\sigma_1, \ldots, \sigma_n\}$. By definition of a balanced system, its Lyapunov equations can be written as

$$A\Sigma + \Sigma A^T = -BB^T \;\Leftrightarrow\; a_{ij}\sigma_j + \sigma_i a_{ji} = -b_ib_j,$$
$$A^T\Sigma + \Sigma A = -C^TC \;\Leftrightarrow\; a_{ij}\sigma_i + \sigma_j a_{ji} = -c_ic_j, \quad (4)$$

for $i, j = 1, \ldots, n$ and $\Sigma := \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. In particular it holds for $i = j$:

$$2a_{ii}\sigma_i = -b_i^2 = -c_i^2 \;\Rightarrow\; b_i = \pm c_i. \quad (5)$$

If $i \neq j$ we can deduce from (4) and (5)

$$\begin{pmatrix} \sigma_j & \sigma_i \\ \sigma_i & \sigma_j \end{pmatrix} \begin{pmatrix} a_{ij} \\ a_{ji} \end{pmatrix} = \begin{pmatrix} b_ib_j \\ c_ic_j \end{pmatrix} = \begin{pmatrix} b_ib_j \\ \pm b_ib_j \end{pmatrix}.$$

Solving for $\begin{pmatrix} a_{ij} & a_{ji} \end{pmatrix}^T$ yields

$$\begin{pmatrix} a_{ij} \\ a_{ji} \end{pmatrix} = \frac{b_ib_j}{\sigma_j^2 - \sigma_i^2} \begin{pmatrix} \sigma_j \mp \sigma_i \\ \pm(\sigma_j \mp \sigma_i) \end{pmatrix}$$

and hence $a_{ij} = \pm a_{ji}$.
In case of multiple Hankel singular values we can assume w.l.o.g. $\Sigma = \mathrm{diag}(\sigma_1 I_{k_1}, \sigma_2, \ldots, \sigma_n)$ for $k_1 > 1$. By partitioning $A = \begin{pmatrix} A_1 & * \\ * & * \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ * \end{pmatrix}$ correspondingly to $\sigma_1 I_{k_1}$, we can write $\sigma_1(A_1 + A_1^T) = B_1B_1^T$. Diagonalizing $B_1B_1^T = U^T\mathrm{diag}\left(\lambda, 0\right)U$ with $\lambda > 0$ gives

$$\sigma_1(UA_1U^T + UA_1^TU^T) = UB_1B_1^TU^T = \mathrm{diag}\left(\lambda, 0\right)$$

and it follows for $\tilde{A} := UA_1U^T$, that $\tilde{a}_{ij} = -\tilde{a}_{ji}$. By $T := \mathrm{diag}(U, I)$ we define a balanced sign symmetric realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) := (TAT^T, TB, CT^T, D)$. □

Note that $b_ib_j = -c_ic_j$ if and only if $a_{ij} = -a_{ji}$. Hence, balanced truncation returns an k-th order symmetric approximation as long as $c_i = b_i$ for all $i = 1, \ldots, k$. From Theorem 3 we know, that $k \geq 1$.
Theorems 3–5 provide the basis of the following *Symmetric Balanced Truncation Algorithm* (SBT).

Algorithm 1 (Symmetric Balanced Truncation Algorithm)**.** *Let G be a given linear system as in (1), then:*

i. Compute a balanced realization (A_b, B_b, C_b, D_b).

ii. Compare the entries of B_b and C_b in order to identify the smallest k, where $c_k \neq b_k$.

iii. Perform the truncation of (A_b, B_b, C_b) to obtain a reduced symmetric system G_r of the order $r < k$.

iv. Obtain a positive realization of G_r with the help of Lanczos Algorithm.

Due to the symmetry constraint the reduced models possess only real eigenvalues. Thus, we can expect to approximate a system well, only if its dominating poles are real. Such systems often occur in the context of sparse large-scale systems, i.e. $n \gg 1000$. For such high dimensions balanced truncation may not be applicable and therefore a pre-approximation is required. In [7] it is shown empirically, that the Iterative Rational Krylov Algorithm (IRKA) gives comparable good results as balanced truncation. The same can be said about the size of the symmetric part after balancing a reduced model, which is obtained by IRKA. This makes IRKA an advisable pre-approximator for our method.

The applicability to large-scale systems and the general independence of a specific state-space representation can be considered the main advantages of SBT. This method often is preferable to those presented in [5], [10], [17], for the following reasons.
The methods in [5] and [10] have the common goal of satisfying the Bounded Real Lemma [22] for the error-system, i.e. between the original and the reduce model. Both are using an iterative linearization approach and consequently do not have a convergence guarantee. The linear matrix inequalities (LMIs), which need to be solved, are usually very expensive to solve (cf. [16]).
The method in [17] is based on LMIs, consisting only of $2n$ variables. In the following we refer to this method as Generalized Balanced Truncation (GBT). It generalizes the idea of balanced truncation by using diagonal solutions $\tilde{P} \geq 0$ and $\tilde{Q} \geq 0$ of the LMIs

$$A\tilde{P} + \tilde{P}A^T \leq -BB^T, \quad A^T\tilde{Q} + \tilde{Q}A \leq -C^TC\,. \qquad (6)$$

Such solutions exist, since A is Metzler (see [3]). Balanced truncation based on the generalized Gramians $\tilde{P}$ and $\tilde{Q}$ preserves the error formula [20], but the bound is more conservative, as the following proposition shows.

Proposition 1. *Let (A,B,C,D) be a minimal system, $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of PQ given by (2), and $\tilde{\lambda}_1 \geq \cdots \geq \tilde{\lambda}_n$ be the eigenvalues of $\tilde{P}\tilde{Q}$ as defined in (2). Then $\lambda_i \leq \tilde{\lambda}_i$ for all $i = 1,\ldots,n$.*

Proof. By subtracting equations (2) from the inequalities (6) it follows by the stability of the system [22], that $\tilde{P} - P \geq 0$, or equivalently that $\tilde{P} \geq P > 0$. In the same way we receive $\tilde{Q} \geq Q > 0$. It holds, that

$$\sigma(PQ) = \sigma(P^{-\frac{1}{2}}(PQ)P^{\frac{1}{2}}) = \sigma(RQR)$$

where $R = P^{\frac{1}{2}}$. Analogously, $\sigma(\tilde{P}\tilde{Q}) = \sigma(\tilde{R}\tilde{Q}\tilde{R})$ with $\tilde{R} \geq R > 0$. Since

$$\begin{aligned}\tilde{R}\tilde{Q}\tilde{R} - RQR &= \tilde{R}\tilde{Q}\tilde{R} - \tilde{R}Q\tilde{R} + \tilde{R}Q\tilde{R} - RQR = \\ &= R(\tilde{Q}-Q)R + Q^{-\frac{1}{2}}\left((Q^{\frac{1}{2}}\tilde{R}Q^{\frac{1}{2}})^2 - (Q^{\frac{1}{2}}RQ^{\frac{1}{2}})^2\right)Q^{-\frac{1}{2}},\end{aligned}$$

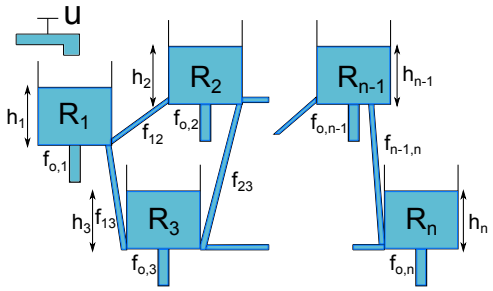

Figure 1. System of n water reservoirs.

it follows by $\tilde{Q} \geq Q$, as well as $Q^{\frac{1}{2}}\tilde{R}Q^{\frac{1}{2}} \geq Q^{\frac{1}{2}}RQ^{\frac{1}{2}}$, that $\tilde{R}\tilde{Q}\tilde{R} \geq RQR$. The inequalities for the eigenvalues now follow from the Courant-Fischer theorem [9]. □

From a geometric point of view this is clear, since balancing with respect to the generalized Gramians does not project the system onto the controllable and observable subspace. In particular, standard balanced truncation with diagonal Gramians is essentially a permutation of the states followed by truncation.
In contrast, SBT inherits the good H_∞-error behavior of balanced truncation. For that reason even a small symmetric part often yields good results. In section 6 we compare SBT and GBT numerically. Since GBT can also be used for singular perturbation balanced truncation, we always present the error of the better one.

6. Examples

We discuss some properties of SBT and compare it with the method in [17].

6.1. Water Reservoirs

We start with the same water reservoir example as in [17]. As schematically shown in Fig. 1, we consider a system of n connected water reservoirs. All reservoirs $R_1,\ldots,R_n$ are assumed to be located on the same level. Base area and fill level of reservoir R_i are denoted by a_i and h_i, respectively. Further, R_i and R_j are connected by a pipe of diameter $d_{ij} = d_{ji} \geq 0$, resulting in a flow f_{ij} from R_i to R_j, where f_{ij} is assumed to be linear dependent on the pressure difference at both ends. The external inflow to reservoir R_1 serves as the single input of the system. The output is the sum of all outflows $f_{o,i}$ of R_i through a pipe with diameter $d_{o,i}$. According to Pascal's law the system flows are described by

$$\begin{aligned} f_{ij}(t) &= d_{ij}^2 \cdot k \cdot (h_i(t) - h_j(t))\,, \\ f_{o,i}(t) &= d_{o,i}^2 \cdot k \cdot (h_i(t) - h_j(t))\,, \end{aligned}$$

where k is a constant representing gravity as well as viscosity and density of the medium. Thus, the fill level h_i of R_i is subject to the differential equation

$$\dot{h}_i = \frac{k}{a_i}\left(-d_{o,i}^2 h_i(t) + \sum_{j=1}^{n} d_{ij}^2 (h_j(t) - h_i(t))\right) + \frac{1}{a_i}\delta_{1i}u(t)$$

where $\delta_{1i} = 1$ if $i = 1$ and zero otherwise. Writing these equations as a linear state-space system results in a SISO-system (A,B,C,D) given by $B = \left(\frac{1}{a_1}, 0, \ldots, 0\right)^T$, $C = k\begin{pmatrix} d_{o,1}^2 & \cdots & d_{o,n}^2 \end{pmatrix}$ and a symmetric A with entries

$$a_{ij} := \frac{k}{a_i}\begin{cases} -d_{o,i}^2 - \sum_{m=1}^{n} d_{im}^2, & i = j \\ d_{ij}^2 & i \neq j, \end{cases} \quad \text{with } d_{ii} := 0.$$

In [17] the system is supposed to consist of two substructures of five reservoirs each. In both substructures each reservoir is connected to every other by a pipe of diameter 1, i.e $d_{ij} = 1$ for $i \neq j$ and $i, j = 1, \ldots, 5$ and $i, j = 6, \ldots, 10$, respectively. The connection of the substructures is established by a pipe of diameter $d_{1,10} = d_{10,1} = 0.2$, between reservoir 1 and 10. For simplicity, $a_i = 1$ and $k = 1$. One can show that the transfer function is just $G(s) = \frac{1}{s+1}$.
Applying SBT to this system yields an exact realization of G. In contrast, since GBT does not return a minimal realization, we get $\tilde{G}(s) = \frac{3.039}{s+3.039}$, with a relative H_∞-error of 0.5014.
Now we modify the system to get a minimal example with unsymmetric A. First, set $d_{o,i} = 0.01 \cdot i$ to get minimality. Further assume, that the first substructure admits a flow from R_1 to R_j, but not vice versa, i.e. $d_{j1} = 0$ for $j = 2, \ldots, \frac{n}{2}$. For 50 water tanks per substructure, SBT gives a symmetric model of order 2

$$A_2 = \begin{pmatrix} -0.1305 & 0.0914 \\ 0.0914 & -0.2676 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.0457 \\ 0 \end{pmatrix},$$
$$C_2 = \begin{pmatrix} 0.0457 & 0 \end{pmatrix}, \quad D_2 = 0,$$

with error 0.0032. About the same error is achieved by GBT only for reduction order 91. We conclude that SBT performs fairly well even for systems with non-symmetric A-matrix.

6.2. Heat Equation

Consider the 2-dimensional heat equation

$$\dot{T} = \triangle T = \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T \tag{7}$$

on the unit square. The Dirichlet boundary conditions on the four edges are interpreted as inputs. Using a finite difference discretization on a uniform grid of step size $h = \frac{1}{N+1}$ sketched in Fig. 2 we get the relations

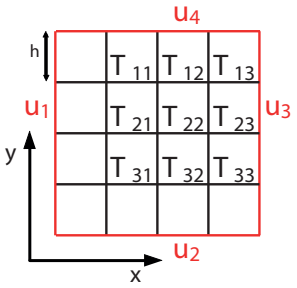

Figure 2. Discretized heat equation on unit square.

$$\triangle T_{ij} \approx -\frac{1}{h^2}(4T_{ij} - T_{i+1,j} - T_{i,j+1} - T_{j-1,j} - T_{i,j-1}),$$

for the temperatures at the inner grid points. Let A denote the $N^2 \times N^2$ Poisson-matrix and $B := [b_{ij}] \in \mathbb{R}^{N^2 \times 4}$, where $b_{ij} = 0$ except for the following cases:

$$\begin{aligned} b_{i1} &:= 1, \quad \text{for } i = 1, 2, \ldots, N \\ b_{i2} &:= 1, \quad \text{for } i = N, 2N, \ldots, N^2 \\ b_{i3} &:= 1, \quad \text{for } i = N(N-1)+1, N(N-1)+2, \ldots, N^2 \\ b_{i4} &:= 1, \quad \text{for } i = 1, N+1, \ldots, N(N-1)+1 \end{aligned}$$

This gives the discretized system

$$\dot{x} = \frac{1}{h^2}Ax + \frac{1}{h^2}Bu \quad \text{with } u \in \mathbb{R}^4 \text{ and } x \in \mathbb{R}^{N^2}. \tag{8}$$

As the output we take the average temperature, i.e.

$$y = \frac{1}{N^2}Cx, \text{ with } C := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{1 \times N^2}.$$

For small h the system will be very large. Starting the comparison between SBT and GBT with a SISO-system, i.e. $u_2 = u_3 = u_4 = 0$ and $N = 10$, yields for SBT a realization of order 15 without any error. In contrast, GBT gives a relative H_∞-error of $3.9087 \cdot 10^{-5}$ by just reducing one state. Moreover, if GBT halves the order it has nearly the same error as balanced truncation to order 1. For $N = 50$, we get a system of order 2500, for which it takes GBT hours to calculate a reduced model, due to the high complexity of conventional LMI-solvers [16]. In case of a large-scale system we apply IRKA to decrease the system to a order lower than 1000, followed by the usual symmetry argument. These computations consume less than half an hour and return a 15-th order model. The Bode diagram of the error system, as shown in Fig. 3, indicates that the reduction error is zero up to machine precision.

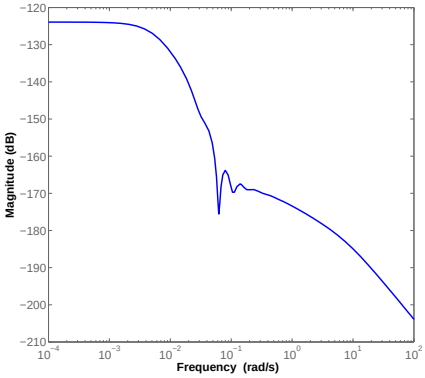

Figure 3. Bode plot: Error system of the heat equation with 2500 states (N = 50).

Applying balanced truncation to the full MISO-system results in a reduced system (A_r, B_r, C_r), with $A_r = A_r^T$ and $C_r = B_r^1 = \cdots = B_r^4$, where $B_r^1, \ldots, B_r^4$ denote the columns of B_r. In case of $N = 10$ SBT returns as in the SISO-case a reduced system of order 15 with zero error. However, the error of reducing just one state by GBT increases to 0.0070.

7. Conclusion

We have presented a positivity preserving model reduction method for SISO-systems based on the sign-symmetry of balanced SISO-systems. It always yields at least some positive approximation since the reduced model of order 1 is guaranteed to be positive. Application of this idea to MIMO systems provides a necessary condition for positivity, which is preferable over a consideration of the impulse response [4].
Furthermore, the reduction method works independently of a positive state-space realization. Hence, large-scale systems can be treated by pre-approximations with methods such as the Iterative Rational Krylov algorithm [7]. Besides, the method preserves and provides symmetry in the A-matrix.

References

[1] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. SIAM, 2005, ch. 10-11.

[2] A. Berman, M. Neumann, and R. J. Stern, *Nonnegative Matrices in Dynamical Systems*. John Wiley & Sons, 1989.

[3] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. Academic Press, 1979.

[4] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*. John Wiley & Sons, 2000.

[5] J. Feng *et al.*, "Internal positivity preserved model reduction," in *International Journal of Control*. Taylor & Francis, 2010, vol. 83, no. 3, pp. 575–585.

[6] K. V. Fernando and H. Nicholson, "On the structure of balanced and other principal representations of siso systems," in *IEEE Transactions On Automatic Control*, 1983, vol. AC-28, no. 2.

[7] S. Gugercin, A. C. Antoulas, and C. Beattie, "$\mathscr{H}_2$ model reduction for large-scale linear dynamical systems," in *SIAM Journal on Matrix Analysis and Applications*, 2008, vol. 3, pp. 609–638.

[8] T. Ishizaki, K. Kashima, and J. Imura, "Extraction of 1-dimensional reaction-diffusion structure in siso linear dynamical networks," in *49th IEEE Conference on Decision and Control*, 2010, pp. 5350–5355.

[9] P. Lancester and M. Tismenetsky, *The Theory Of Matrices Second Edition*. Academic Press Inc., 1985, pp. 286–302.

[10] P. Li *et al.*, "Positivity-preserving $\mathscr{H}_\infty$ model reduction for positive systems," in *Automatica*, 2011, vol. 47, pp. 1504 – 1511.

[11] W. Liu, V. Sreeram, and K. Teo, "Model reduction for state-space symmetric systems," in *Systems & Control Letters*. ELSEVIER, 1998, no. 34, pp. 209–215.

[12] D. G. Luenberger, *Introduction to Dynamic Systems: Theory, Models & Applications*. John Wiley & Sons, 1979.

[13] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra Book and Solutions Manual*. SIAM, 2001, ch. 8, pp. 670 – 678.

[14] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," in *IEEE Transactions On Automatic Control*, 1981, vol. AC-26, no. 1, pp. 17–32.

[15] Y. Ohta, H. Maeda, and S. Kodama, "Reachability, observability, and realizability of continuous-time positive systems," in *SIAM J. Control and Optimization*, 1984, vol. 22, no. 2, pp. 171–180.

[16] D. Peaucelle *et al.*, *User's Guide for SeDuMi Interface 1.04*, 2002, p. 4.

[17] T. Reis and E. Virnik, *Positivity preserving model reduction*. Springer-Verlag, 2009, vol. 389, pp. 131 – 139.

[18] Y. Saad, *Iterative Methods for Sparse Linear Systems*, 3rd ed., 2000.

[19] M. G. Safonov and R. Y. Chiang, "A schur method for balanced-truncation model reduction," in *IEEE Transactions on Automatic Control*, 1989, vol. 34, no. 7, pp. 729–733.

[20] H. Sandberg and A. Rantzer, "Balanced truncation of linear time-varying systems," in *IEEE Transactions on Automatic Control*, 2004, vol. 49, no. 2.

[21] L. N. Trefethen and D. Bau, *Numerical Linear Algebra*. SIAM, 1997, ch. 32-39.

[22] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Prentice Hall, 1999.