LCCC focus period and workshop on Dynamics and Control in Networks

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Workshop: Dynamics and Control in Networks

LUND CENTER FOR CONTROL OF COMPLEX ENGINEERING SYSTEMS

LUND UNIVERSITY

LCCC focus period and workshop on Dynamics and Control in Networks

15-17 Oct 2014

Old Bishop's Palace at Biskopsgatan 1 in Lund

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1. Introduction and organization

The LCCC focus period on Dynamics and Control in Networks took place in Lund in October 2014. It spanned over five weeks with a 3-day workshop in the middle (October 15-17). The highly cross-disciplinary research theme was chosen by the LCCC board to support the strategic vision of the Centre.

The event was organized by the LCCC faculty members Giacomo Como and Anders Rantzer, and local practical arrangements were handled by Eva Westin. A scientific committee was formed, composed by Tryphon Georgiou (University of Minnesota), Steven Low (CalTech), Asuman Ozdaglar (Massahusetts Institute of Technology), Rodolphe Sepulchre (University of Cambridge), Sandro Zampieri (Università di Padova), and the organizers. The scientific committee selected other fifteen world-leading researchers in the field and invited them to give a seminar at the workshop, extend their stay during the focus period compatibly with their commitments, and nominate outstanding young researchers for an extended stay. Seventeen such researchers were selected by the scientific committee and invited to join the Center for periods of 3 to 5 weeks and to give a seminar during the focus period.

In addition to their seminar, all the participants where invited to submit open problems that could motivate and inspire the research community. The initiative was meant as a first step of a wider project to collect open research problems in Control Systems in conjunction with the 2015-16 thematic year in Control Theory and its Applications organized at the Institute for Mathematics and its Applications (IMA). The submitted problems underwent a peer review process and some selected ones were presented during the workshop. The workshop was concluded by a panel discussion moderated by Anders Rantzer.

1.1 SCIENTIFIC THEME

Networks, ranging from infrastructure (such as transportation, communication, and distribution networks) to social and economical networks, play an increasingly central role in many aspects of our lives. Many of the key interactions in these networks are inherently dynamic and the network connectivity and feedback pathways affect robustness and functionality. Such concepts are at the core of a new and rapidly evolving frontier of Dynamical Systems and Control.

In this new technological and scientific realm, the modeling and representation of systems and the role of feedback need to be reevaluated. Traditional thinking, which is limited to a small number of feedback loops, is no longer applicable, e.g., because of its lack of scalability. Feedback control and stability of network dynamics require new approaches. Decentralized control and distributed optimization, as well as layered architectures and plug-and-play paradigms have become of central interest.

The October 2014 LCCC Focus Period and Workshop brought together leading experts and outstanding young researchers with different scientific backgrounds in disciplines such as Control, Power Systems, Operation Research, Communication Networks, and Statistics, to share their results, compare their approaches and attempt to define the goals and boundaries of this exciting new research field.

2. Workshop

2.1 WORKSHOP SEMINARS

Controllability for Complex Networks: Metrics and Limitations
Sandro Zampieri (Università di Padova)

Differentially positive systems
Rodolphe Sepulchre, University of Cambridge, UK

A survey of classical and recent results in RLC circuit synthesis
Malcolm Smith, University of Cambridge, UK

Synchronization and evolvability in nonlinear networks
Jean-Jacques Slotine, Massachusetts Institute of Technology, USA

Community Detection: Fundamental Limits and Optimal Sampling Algorithms
Alexandre Proutiere, Kungliga Tekniska Högskolan, Sweden

Community detection in stochastic block models via spectral methods
Laurent Massoulié, Microsoft Research-INRIA Joint Centre

Complex Energy Systems
Michael Chertkov, Los Alamos National Laboratory, USA

Semidefinite relaxation of optimal power flow
Steven H. Low, CalTech, USA

Distributed Alternating Direction Method of Multipliers for Multi-agent Optimization
Asuman Ozdaglar, Massachusetts Institute of Technology, USA

Learning graphical models: hardness and tractability
Devavrat Shah, Massachusetts Institute of Technology, USA

Load Balancing Using Limited State Information
R. Srikant, University of Illinois at Urbana Champain, USA

On the eigenvalues of large structured matrices and the scalable stability of networks
Glenn Vinnicombe, University of Cambridge, UK

Toward standards for dynamics in future electric energy systems
Marija Ilic, Carnegie Mellon University, USA

Plug-and-Play Control and Optimization in Microgrids
Florian Dörfler, ETH Zurich, Switzerland

Lyapunov Approach to Consensus Problems
Angelia Nedic, University of Illinois at Urbana Champain, USA

Majority Consensus by local polling
Moez Draief, Imperial College, UK

Robust Networked Stabilization: When Gap Meets Two-Port
Li Qiu, Hong Kong University of Science and Technology

Schroedinger bridges: steering of stochastic systems classical and quantum
Tryphon Georgiou, University of Minnesota, USA

Randomized averaging algorithms, when can errors and unreliabilities just be ignored?
Julien Hendrickx, Université Catholique de Louvain, Belgium

Systemic Risk in Finance
Munther A. Dahleh, Massachusetts Institute of Technology, USA

On Distributed Control of Dynamical Flow Networks
Giacomo Como, Lund University, Sweden

Control of convex-monotone systems
Anders Rantzer, Lund University, Sweden

2.2 FOCUS PERIOD SEMINARS

Robust stability of positive systems: a convex characterization
Marcello Colombino (ETH Zurich)

A new approach to sequential rate-distortion problems and applications
Takashi Tanaka (Massachusetts Institute of Technology)

Stability and power sharing in microgrids
Johannes Schiffer (TU Berlin)
Differentially positive systems
Fulvio Forni (University of Liege)

Submodularity and controllability in complex dynamical networks
Tyler Summers (ETH Zurich)

A scalable approach to the design of large networks
Richard Pates (University of Cambridge)

Bounded disturbance amplification for mass chains with passive interconnections
Kaoru Yamamoto (University of Cambridge)

Gossip coverage control for robotic networks: dynamical systems on the space of partitions
Ruggero Carli (Università di Padova)

A communication/control co-design paradigm for networked control systems
Wei Chen (Hong Kong University of Science and Technology)

Resilience of natural gas networks during conflicts, crises and disruptions
Rui Carvalho (University of Cambridge)

New approaches to identification and control of nonlinear systems
Ian Manchester, University of Sydney

On the push-sum algorithm with unreliable communication
Balázs Gerencsér (Université Catholique de Louvain)

Resilience and cascading failures in large-scale networks
Wilbert Rossi (Politecnico di Torino)

Universal convexification via risk aversion
Dvijotham Krishnamurthy (CalTech)

Distributed and event-based state estimation
Sebastian Trimpe, (Max Planck Institute for Intelligent Systems)

Stability of linear consensus processes
Ji Liu (University of Illinois at Urbana Champain)

Control of evolutionary games on networks
James R. Riehl (University of Groningen)

2.3 SELECTED OPEN PROBLEMS

Two Problems on Minimality in RLC Circuit Synthesis
Timothy H. Hughes1, Jason Z. Jiang, and Malcolm C. Smith

The Classical n-Port Resistive Synthesis Problem
Michael Z. Q. Chen

Synchronization of oscillators: Feasibility and Non-local analysis
Julien Hendrickx Florian Dorfler

Modeling for plug and play control in strongly coupled nonlinear networks
Marija D. Ilic and Xia Miao

Stability of Passivity Based Control for Power Systems and Power Electronics
Kevin Bacovchin and Marija D. Ilic

On the asymptotic behavior of differentially positive systems
Fulvio Forni and Rodolphe Sepulchre

3. Panel Disscussion

The workshop was concluded by a panel discussion moderated by Anders Rantzer. The panel was composed by Munther Dahleh (Massachusetts Institute of Technology) Marija Ilic (Carnegie Mellon University), and Glenn Vinnicombe (Cambridge University). The panelists were asked to address the following points:

1. We have had lectures on control, energy, community detection and financial risks. What are the important common issues and the important differences? Do we have enough of a common language?
2. How do we stay relevant and where do we set the boundaries of our discipline?
3. What teaching curriculum do we need to address the common issues?

Concerning point 1, the consensus was that the topics could be classified in two relatively homogeneous categories. On the one hand, in large-scale physical systems such as power, transportation, and financial networks, there are measurable flows among the different units and there is an infrastructure posing constraints on such flows. Issues of efficiency and robustness/ resilience are ubiquitous in this area. On the other hand, there are problems of community detection, consensus, and opinion dynamics in social networks where the central issue is to estimate from data the actual network structure and decision mechanisms in place.

The centrality of ideas such as the design of local decision makers achieving a global objective and layering, e.g., exploiting time-scale separations that are inherent in many complex systems was also brought up.

The discussion on Point 2 highlighted the tension between the need for the community not to loose its rigorous methodology while being close to the problem. The importance and challenges of having good contact with practitioners were also brought up.

Point 3 stimulated the debate on how to go back to fundamentals and get rid of what is no longer so central. The need for engineers with systems knowledge was highlighted along with the fact that computer science and communication should be fundamental parts of control engineering curricula.

PROGRAM LCCC October 2014
Workshop on Dynamics and Control in Networks

Wednesday, October 15

08:30 Registration at the Old Bishop's Palace at Biskopsgatan 1 in Lund

08:45 Opening remarks
Giacomo Como and Anders Rantzer, Lund University, Sweden

09:00 Controllability for Complex Networks: Metrics and Limitations
Sandro Zampieri, Università di Padova, Italy
Differentially positive systems
Rodolphe Sepulchre, University of Cambridge, UK

10:20 Coffee

10:40 A survey of classical and recent results in RLC circuit synthesis
Malcolm Smith, University of Cambridge, UK
Synchronization and evolvability in nonlinear networks
Jean-Jacques Slotine, Massachusetts Institute of Technology, USA

12:00 Lunch

14:00 Community Detection: Fundamental Limits and Optimal Sampling Algorithms
Alexandre Proutiere, Kungliga Tekniska Högskolan, Sweden
Community detection in stochastic block models via spectral methods
Laurent Massoulié, Microsoft Research-INRIA Joint Centre

15:20 Coffee

15:40 Complex Energy Systems
Michael Chertkov, Los Alamos National Laboratory, USA
Semidefinite relaxation of optimal power flow
Steven H. Low, CalTech, USA

Thursday, October 16

09:00 Distributed Alternating Direction Method of Multipliers for Multi-agent Optimization
Asuman Ozdaglar, Massachusetts Institute of Technology, USA
Learning graphical models: hardness and tractability
Devavrat Shah, Massachusetts Institute of Technology, USA

10:20 Coffee

10:40 Load Balancing Using Limited State Information
R. Srikant, University of Illinois at Urbana Champain, USA
On the eigenvalues of large structured matrices and the scalable stability of networks
Glenn Vinnicombe, University of Cambridge, UK

12:00 Lunch

14:00 Toward standards for dynamics in future electric energy systems: The basis for plug-and-play industry paradigm
Marija Ilic, Carnegie Mellon University, USA
Plug-and-Play Control and Optimization in Microgrids
Florian Dörfler, ETH Zurich, Switzerland

15:20 Coffee

15:40 Lyapunov Approach to Consensus Problems
Angelia Nedic, University of Illinois at Urbana Champain, USA
Majority Consensus by local polling
Moez Draief, Imperial College, UK

18:15 Buses leave from Bangatan (green arrow on map)

19:00 Symposium Dinner at Långa bryggan, Bjärreds saltsjöbad

Friday, October 17

09:00 Robust Networked Stabilization: When Gap Meets Two-Port
Li Qiu, Hong Kong University of Science and Technology
Schroedinger bridges: steering of stochastic systems classical and quantum
Tryphon Georgiou, University of Minnesota, USA

10:20 Coffee

10:40 Randomized averaging algorithms, when can errors and unreliabilities just be ignored?
Julien Hendrickx, Université Catholique de Louvain, Belgium
Systemic Risk in Finance
Munther A. Dahleh, Massachusetts Institute of Technology, USA

12:00 Lunch

14:00 On Distributed Control of Dynamical Flow Networks
Giacomo Como, Lund University, Sweden
Control of convex-monotone systems
Anders Rantzer, Lund University, Sweden

15:20 Coffee

15:40 Panel Discussion
Anders Rantzer (moderator)

17:00 Final remarks – end of the workshop

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Workshop on Dynamics and Control in Networks

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Appendix C

PRESENTATIONS

CONTROLLABILITY FOR COMPLEX NETWORKS: METRICS AND LIMITATIONS

Sandro Zampieri, Università di Padova, Italy

This presentation will consider the problem of controlling complex networks, i.e., the joint problem of selecting a set of control nodes and of designing a control input to steer a network to a target state. For this problem, we propose a metric to quantify the difficulty of the control problem as a function of the required control energy and we derive bounds based on the system dynamics (network topology and weights) to characterize the tradeoff between the control energy and the number of control nodes. Our findings show several control limitations and properties. For instance, for symmetric networks, if the number of control nodes is constant, then the control energy increases exponentially with the number of network nodes. On the other hand, if the number of control nodes is a fixed fraction of the network nodes, then certain networks can be controlled with constant energy independently of the network dimension.

Asymmetric networks highlights richer scenarios. Indeed, it can be shown that, while asymmetric but isotropic networks are still difficult to control, sufficiently anisotropic networks are instead easy to control.

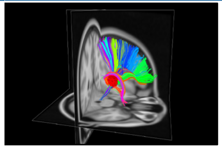
Controllability metrics, limitations and algorithms for complex networks

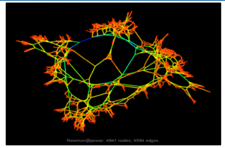
Sandro Zampieri
Universita' di Padova

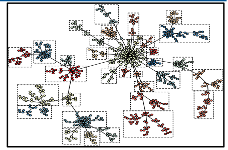
In collaboration with
Fabio Pasqualetti - University of California at Riverside
Francesco Bullo - University of California at Santa Barbara

1

Growing Number of Applications


Brain network


Power network


Social network

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✓ A. Olshevsky, "Minimal Controllability Problems," Arxiv preprint, 2014.

2

Problem formulation

Consider a linear system

$$x(t+1) = Ax(t) + Bu(t)$$

where A is sparse $n \times n$ matrix (the interactions between the states is described by a graph) and

$$B = [e_{i_1} \ \cdots \ e_{i_m}]$$

where

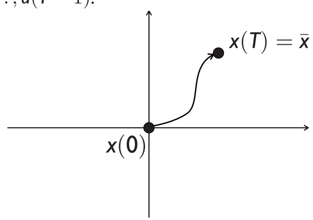
$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Leftarrow i$$

3

Problem formulation

$$x(t+1) = Ax(t) + Bu(t)$$

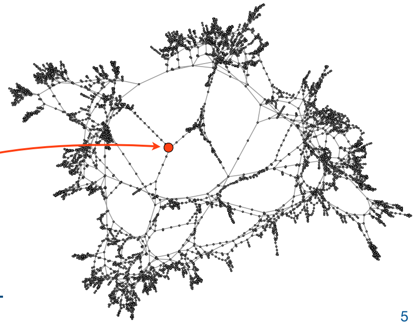
CONTROLLABILTY is the possibility of streering the state from the initial state $x(0) = 0$ to an arbitrary final state $x(T) = \bar{x}$ by applying a suitable input sequence $u(0), u(1), \ldots, u(T-1)$.



4

Need of a controllability metric

Assume that the graph is strongly connected and that it has all the self-loops. Then to have that the resulting system is controllable (generically in the non-zero entries of A) it is enough that only one state is controlled.



5

Need of a controllability metric

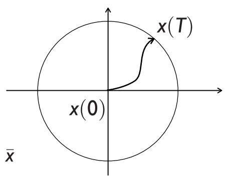
QUESTION: How controllable is the resulting system?

6

A controllability metric

One metric for describing the controllability degree is given by the controllability Gramian

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$



Interpretation: For driving

$$x(0) = 0 \quad \longrightarrow \quad x(T) = \bar{x}$$

where $||\bar{x}||_2 = 1$, we need an input $u(0), u(1), \ldots, u(T-1)$ with L^2 energy $\bar{x}^T W_T^{-1} \bar{x}$.

The energy to drive the state from zero to a norm one state (in the worst case) is given by

$$Energy = \frac{1}{\lambda_{min}(W_T)}$$

7

A controllability metric

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$ Controllability Gramian

$$Energy = \frac{1}{\lambda_{min}(W_T)}$$

Small $\lambda_{min}(W_T)$ ⟷ Small controllability degree

Large $\lambda_{min}(W_T)$ ⟷ Large controllability degree

8

A controllability metric

Alternative controllability metrics

$$W_T := \sum_{t=0}^{T-1} A^t B B^T (A^T)^t$$

$$\lambda_{min}(W_T)$$

$$\frac{1}{n}\mathrm{tr}(W_T^{-1})$$

$$\frac{1}{n}\ln\det(W_T)$$

$$\frac{1}{n}\mathrm{tr}(W_T)$$

9

Few Nodes Cannot Control Symmetric Complex Networks

THEOREM

Assume the matrix A symmetric. Fix any constant $0 < \mu < 1$ and let

$$n(\mu) := |\{\lambda \in \lambda(A) \; : \; |\lambda|^2 \le \mu\}|$$

Then

$$\lambda_{min}(W_T) \le \frac{1}{\mu(1-\mu)}\mu^{n(\mu)/m}$$

10

Few Nodes Cannot Control Symmetric Complex Networks

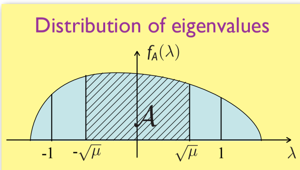
Let

$$F_A(\bar\lambda) := \frac{|\{\lambda \in \lambda(A) \; : \; \lambda \le \bar\lambda\}|}{n}$$

and

$$f_A(\bar\lambda) := \frac{dF(\bar\lambda)}{d\bar\lambda}$$

Then



$$n(\mu) := |\{\lambda \in \lambda(A) \; : \; |\lambda|^2 \le \mu\}| = n \int_{-\sqrt{\mu}}^{\sqrt{\mu}} f_A(\lambda)d\lambda = \mathcal{A}\, n$$

Hence the number $n(\mu)$ typically grows linearly in the network cardinality n.

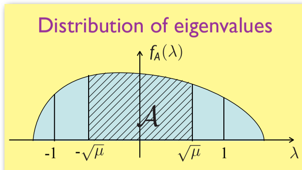
$$\lambda_{min}(W_T) \le \frac{1}{\mu(1-\mu)}\mu^{\mathcal{A}\frac{n}{m}} \qquad \boxed{\mu < 1}$$

11

Few Nodes Cannot Control Symmetric Complex Networks

$$\boxed{\mu < 1}$$

$$\lambda_{min}(W_T) \le \frac{1}{\mu(1-\mu)}\mu^{\mathcal{A}\frac{n}{m}}$$



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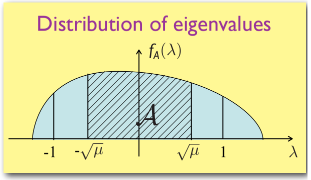
Few Nodes Cannot Control Symmetric Complex Networks

$\mu < 1$

$$\lambda_{min}(W_T) \leq \frac{1}{\mu(1-\mu)} \mu^{\mathcal{A}\frac{n}{m}}$$



✓ For fixed number **m** of control nodes, the controllability degree **decreases exponentially** in the network cardinality **n**.

✓ To have a fixed controllability degree, number **m** of control nodes must grow **linearly** in the network cardinality **n**.

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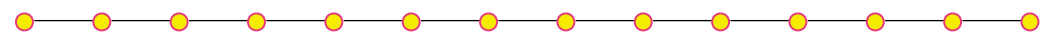
Example: symmetric line graph



$$A = \begin{bmatrix} a & b & 0 & \cdots & \cdots & 0 & 0 \\ b & a & b & \cdots & \cdots & 0 & 0 \\ 0 & b & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a & b \\ 0 & 0 & 0 & \cdots & \cdots & b & a \end{bmatrix}$$

$$\lambda(A) = \left\{ a + 2b\cos\left(\frac{\pi}{n+1}k\right) : k = 1, 2, \ldots, n \right\}$$

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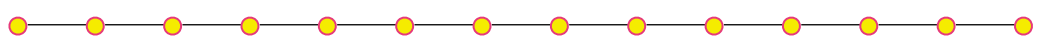
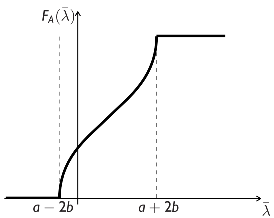
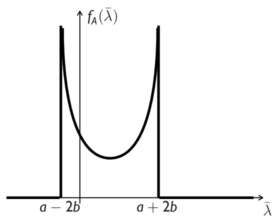
Example: symmetric line graph



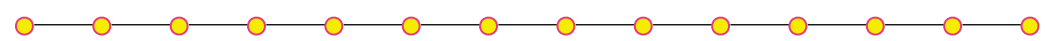
$$F_A(\bar{\lambda}) = 1 - \frac{1}{\pi}\arccos\left(\frac{\bar{\lambda}-a}{2b}\right) \qquad f_A(\bar{\lambda}) = \frac{1/\pi}{\sqrt{1-\left(\frac{\bar{\lambda}-a}{2b}\right)^2}}$$

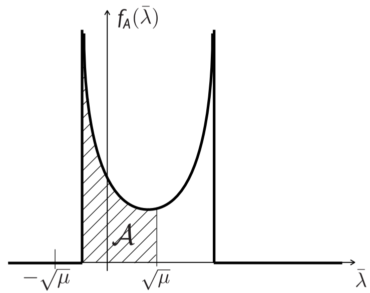




15

Example: symmetric line graph





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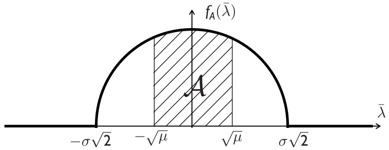
Example: Symmetric random matrix

Assume A is a symmetric matrix with a_{ij} iid random variables $\mathbb{E}[a_{ij}] = 0$ and $\mathbb{E}[a_{ij}^2] = \sigma^2/\sqrt{n}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Then (Wigner's semi-circle law)

$$f_A(\bar{\lambda}) \longrightarrow W(\bar{\lambda})$$



Asymmetric Complex Networks

For asymmetric networks the situation is more complex

- For "**isotropic**" networks (networks with no preferential directions) it seems that the situation is the same as for symmetric networks, namely they are difficult to control.

- For "**anisotropic**" networks (networks with a preferential direction) it seems that few nodes can indeed control large scale networks.

Asymmetric Complex Networks

THEOREM

Assume the matrix A diagonalizable and let V an eigenvector matrix. Fix any constant $0 < \mu < 1$ and let

$$n(\mu) := |\{\lambda \in \lambda(A) \ : \ |\lambda|^2 \leq \mu\}|$$

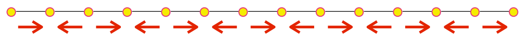
Then

$$\lambda_{min}(W_T) \leq \mathrm{cond}(V)^2 \frac{1}{\mu(1-\mu)} \mu^{n(\mu)/m}$$

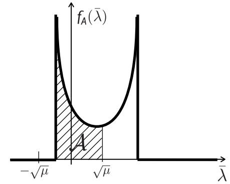
where $\mathrm{cond}(V) = \sigma_{\max}(V)/\sigma_{\min}(V)$ is the condition number of V.

Consequence: If the condition number stays bounded in the network dimension, than the network remains difficult to control.

Example: Asymmetric isotropic line graph



$$A := \frac{1}{3} \begin{bmatrix} 1 & 1/2 & 0 & 0 & \cdots \\ 2 & 1 & 2 & 0 & \cdots \\ 0 & 1/2 & 1 & 1/2 & \cdots \\ 0 & 0 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

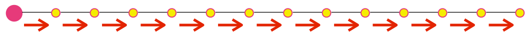


It can be found an eigenvector matrix V such that $\mathrm{cond}(V) = 2$.

$$\lambda(A) = \left\{ \frac{1}{3} + \frac{2}{3} \cos \frac{k\pi}{n+1}, \qquad k = 1, \ldots, n \right\}$$

This network is difficult to control as a symmetric network.

Example: Asymmetric anisotropic line graph



$$A = \begin{bmatrix} a & b & 0 & \cdots & \cdots & 0 & 0 \\ c & a & b & \cdots & \cdots & 0 & 0 \\ 0 & c & a & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & a & b \\ 0 & 0 & 0 & \cdots & \cdots & c & a \end{bmatrix}$$

$$\lambda(A) = \left\{ a + 2\sqrt{bc}\cos\left(\frac{\pi}{n+1}k\right) : k = 1, 2, \ldots, n \right\}$$

Given a, if c is sufficiently larger than b, than this network can be controlled with finite energy by the node on the extreme regardless the network dimension.

Exploiting spatial instability

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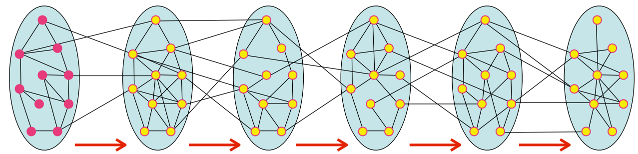
Extension to more general graphs: controllable graphs



$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & \cdots & 0 & 0 \\ C_1 & A_2 & B_2 & \cdots & \cdots & 0 & 0 \\ 0 & C_2 & A_3 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & A_{\ell-1} & B_{\ell-1} \\ 0 & 0 & 0 & \cdots & \cdots & C_{\ell-1} & A_\ell \end{bmatrix}$$

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Extension to more general graphs

$$A = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & \cdots & 0 & 0 \\ C_1 & A_2 & B_2 & \cdots & \cdots & 0 & 0 \\ 0 & C_2 & A_3 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & A_{\ell-1} & B_{\ell-1} \\ 0 & 0 & 0 & \cdots & \cdots & C_{\ell-1} & A_\ell \end{bmatrix}$$

Given the matrices A_i, if the matrices C_i are sufficiently larger than the matrices B_i, than this network can be controlled with finite energy by the nodes on the extreme subgraph regardless the network dimension.

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Extension to more general graphs: uncontrollable graphs

Assume A is a stochastic matrix (diffusion dynamics, consensus, ...). This means that

$$A\mathbb{1} = \mathbb{1}$$

where $\mathbb{1}$ is the vector with entries equal to one. Let w be a vector such that

$$w^T A = w^T \qquad w^T \mathbb{1} = 1$$

which means that w is the invariant measure of the Markov chain associated with A. It is known that the entries of w represent the nodes "centrality" in the network (the bigger the more important).

Result: If the entries of w are all $\leq \frac{cost}{n}$ (all nodes have similar centrality) then the associated network is difficult to control.

In the symmetric case the entries of w are $1/n$.

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Controllers positioning

Decoupled control strategy: "Divide et Impera"

Network partitioning Partition $\mathcal{V} = \{1, \ldots, n\}$ into N disjoint sets $\mathcal{V}_1, \ldots, \mathcal{V}_N$. After relabeling of states and inputs, the matrices read as

$$A = \begin{bmatrix} A_1 & \cdots & A_{1N} \\ \vdots & \vdots & \vdots \\ A_{N1} & \cdots & A_N \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_N \end{bmatrix},$$

The networks dynamics can be written as the interconnection of N subsystems of the form

$$x_i(t+1) = \overbrace{A_i x_i(t)}^{\text{local dynamics}} + \overbrace{\sum_{j \in \mathcal{N}_i} A_{ij} x_j(t)}^{\text{interconnections}} + \overbrace{B_i u_i(t)}^{\text{local controls}},$$

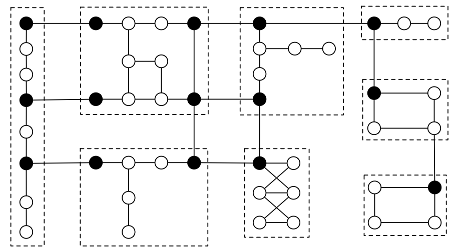
where $i \in \{1, \ldots, N\}$ and $\mathcal{N}_i := \{j \,:\, A_{ij} \neq 0\}$.

Controllers positioning

Decoupled control strategy

1. Partition the network into disjoint connected parts.

2. Select boundary nodes as control nodes.



Controllers positioning

Decoupled control strategy

3. Apply the inputs

$$u_i(t) := v_i(t) - \sum_{j \in \mathcal{N}_i} B_i^T A_{ij} x_j(t)$$

This control law yields N decoupled subsystems

$$x_i(t+1) = A_i x_i(t) + B_i v_i(t)$$

4. Choose v_i which minimizes the energy to steer the subsystem to the desired substate.

Controllers positioning

Local controllability

$$\Lambda := \begin{bmatrix} \lambda_{\min}^{-1}(W_{1,T}) & 0_2 & \cdots & 0 \\ 0 & \lambda_{\min}^{-1}(W_{2,T}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{\min}^{-1}(W_{N,T}) \end{bmatrix}.$$

Coupling strength

$$\Delta := \begin{bmatrix} 1 & \|A_{12}\|_2 & \cdots & \|A_{1N}\|_2 \\ \|A_{21}\|_2 & 1 & \cdots & \|A_{2N}\|_2 \\ \vdots & \vdots & \ddots & \vdots \\ \|A_{N1}\|_2 & \|A_{N2}\|_2 & \cdots & 1 \end{bmatrix}.$$

Controllers positioning

Theorem If we choose a decoupled control law
then we obtain

$$\lambda_{\min}(W_T) \geq \frac{(1-\bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2}$$

where

$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1,\dots,N\}\} < 1$$

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Controllers positioning

Theorem If we choose a decoupled control law
then we obtain

$$\lambda_{\min}(W_T) \geq \frac{(1-\bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2} = \frac{\text{local convergence speed}}{\text{(local controllability)(coupling strength)}}$$

where

$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1,\dots,N\}\} < 1$$

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Controllers positioning

Theorem If we choose a decoupled control law
then we obtain

$$\lambda_{\min}(W_T) \geq \frac{(1-\bar{\lambda}_{\max})^2}{\|\Lambda\|_\infty \|\Delta\|_\infty^2} = \frac{\text{local convergence speed}}{\text{(local controllability)(coupling strength)}}$$

where

$$\bar{\lambda}_{\max} = \max\{\lambda_{\max}(A_i) : i \in \{1,\dots,N\}\} < 1$$

For high controllability degree:

1. Partition so that $\|\Delta\|_\infty$ are small (weakly coupled subsystems)
2. Select local control nodes so that $\|\Lambda\|_\infty$ is small (large local controllability)

With decoupled control, global controllability becomes a local property

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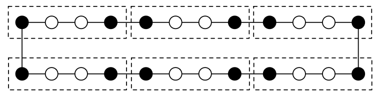
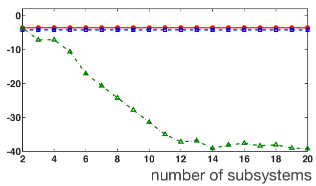
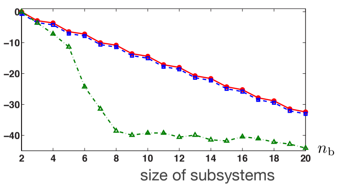
Examples

Circulant graph







$\lambda_{\min}$ with the decoupled control strategy
$\lambda_{\min}$ theoretical lower bound
$\lambda_{\min}$ with random positioning

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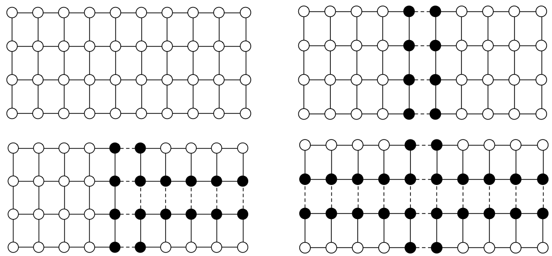
Network partition and selection of the control nodes

Selection of the control nodes

Until the desired number of control nodes have been selected:

1. Bisect the least controllable subsystem via Fiedler partitioning.
2. Include boundary nodes in the control set.



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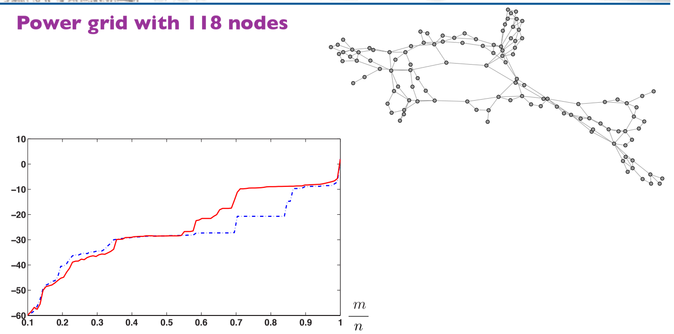
Examples

Power grid with 118 nodes



$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

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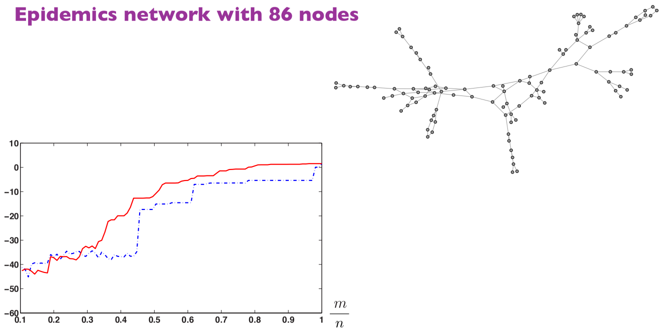
Examples

Epidemics network with 86 nodes



$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

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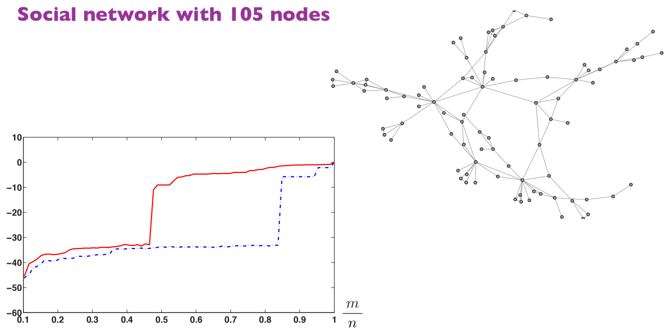
Examples

Social network with 105 nodes



$\lambda_{\min}(W_T)$ with the decoupled control strategy

$\lambda_{\min}(W_T)$ with random positioning

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Conclusions

- Similar results for **observability**
- For symmetric (isotropic) networks we need to control **a fixed fraction** of nodes
- For anisotropic networks it is enough to control **a fixed number** of nodes
- **Random** positioning works pretty well
- **Phase transition** can be noticed (critical fraction of controlled nodes)
- There are a lot of open problems:
  - Understanding isotropic and anisotropic networks
  - Controllability of random and of structured graphs
  - Performance of random positioning
  - Phase transition
  - Different metrics

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Conclusions

Controllability → **Graph theory**

Controllability degree → **Spectral graph theory**

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Thank you

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DIFFERENTIALLY POSITIVE SYSTEMS

Rodolphe Sepulchre, University of Cambridge, UK

The talk will introduce differentially positive systems, as systems whose linearization along an arbitrary trajectory is positive. A generalization of Perron Frobenius theory is developed in this differential framework to show that the property induces a (conal) order that strongly constrains the asymptotic behavior of solutions. The results illustrate that behaviors constrained by local order properties extend beyond the well-studied class of linear positive systems and monotone systems, which both require a constant cone field and a linear state space. Joint work with Fulvio Forni.

Preprint on arxiv: http://arxiv.org/abs/1405.6298.

Differential positivity

Rodolphe Sepulchre -- University of Cambridge, UK

Lund, LCCC workshop, October 2014

The main reference

Differentially positive systems

F. Forni, R. Sepulchre

Abstract

The paper introduces and studies differentially positive systems, that is, systems whose linearization along an arbitrary trajectory is positive. A generalization of Perron Frobenius theory is developed in this differential framework to show that the property induces a (conal) order that strongly constrains the asymptotic behavior of solutions. The results illustrate that behaviors constrained by local order properties extend much beyond the well-studied class of linear positive systems and monotone systems, which both require a constant cone field and a linear state space.

Y] 24 May 2014

This talk: why did we study that property ?

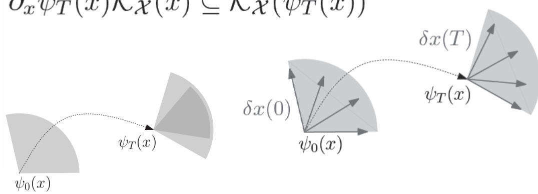
2

Differential positivity

A linear map is positive if it leaves a cone invariant.

A dynamical system is differentially positive if its linearization along an arbitrary trajectory is positive.

$$\partial_x \psi_T(x) \mathcal{K}_{\mathcal{X}}(x) \subseteq \mathcal{K}_{\mathcal{X}}(\psi_T(x))$$

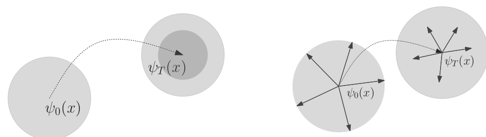


3

Compared to contraction, a cone replaces the ball...

A linear map is (Lyapunov) stable if it leaves a ball invariant.

A dynamical system is differentially stable (non expanding) if its linearization along an arbitrary trajectory is Lyapunov stable.



In systems and control: Lohmiller & Slotine (1998), and many others since then...
Terminology: contraction, convergence, incremental stability, ...

Textbook Perron-Frobenius theory

Perron–Frobenius theorem

From Wikipedia, the free encyclopedia

In linear algebra, the **Perron–Frobenius theorem**, proved by Oskar Perron (1907) and Georg Frobenius (1912), asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components, and also asserts a similar statement for certain classes of nonnegative matrices. This theorem has important applications to probability theory (ergodicity of Markov chains); to the theory of dynamical systems (subshifts of finite type); to economics (Okishio's theorem, Leontief's input-output model);[1] to demography (Leslie population age distribution model),[2] to Internet search engines[3] and even ranking of football teams.[4]

An obstacle to think of positivity as a *geometric* property?

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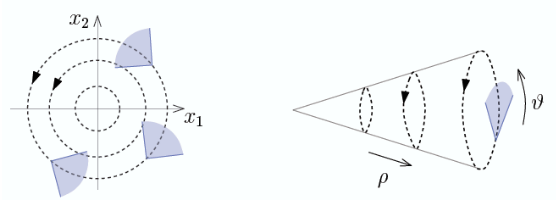
As a geometric concept, positivity is *not* antagonist to oscillations



$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases} \equiv \begin{cases} \dot{\rho} = 0 \\ \dot{\vartheta} = 1 \end{cases}$$

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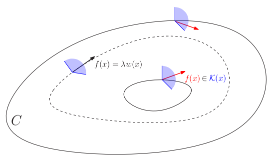
Strict differential positivity and nonlinear oscillations

Corollary 2: Under the assumptions of Theorem 3, consider an open, forward invariant region $\mathcal{C} \subseteq \mathcal{X}$ that does not contain any fixed point. If the vector field $f(x) \in \text{int}\mathcal{K}_{\mathcal{X}}(x)$ for any $x \in \mathcal{C}$, then there exists a unique attractive periodic orbit contained in $\mathcal{C}$.



- Theorem 3 is a *differential* version of Perron-Frobenius theory.
- The corollary is akin to Poincare Bendixon theorem for planar systems.
- Strict differential positivity, similarly to the topology of the plane, enforces a one-dimensional asymptotic behavior.

7

Plan for this talk

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

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Recycling a few old slides...

Consensus theory and Hilbert metric

R. Sepulchre
University of Liege, Belgium

LCCC workshop
January 2010

Classical linear consensus theory

Linear consensus algorithms are linear time-varying systems

$$x(t+1) = A(t)x(t), \quad x(t) \in \mathbb{R}^n$$

where for each t, A(t) is row stochastic, i.e.

A is nonnegative: $a_{ij} \geq 0$

each row sums to one: $A(t)1 = 1$

Uniform convergence to $\alpha 1$ ("consensus: $x_i = \dots = x_j$
is proven under uniform connectivity / irreducibility
(Tsitsiklis, Jadbabaie et al., Moreau, ...)

Convergence analysis and Lyapunov functions

Tsitsiklis (1986) observed that

$$V(x) = \max_{1 \leq i \leq n} x_i - \min_{1 \leq i \leq n} x_i$$

is non increasing along the flow.

Uniform convergence is established by showing the strict decay of over a finite horizon.

It is known that no common quadratic Lyapunov exists in general.
(See *Olshevsky & Tsitsiklis 08* for a discussion)

Birkhoff Theorem

Let K a closed solid cone in X a Banach space, with partial ordering $\preceq$.

A is *positive* if A maps $\mathring{K}$ to $\mathring{K}$

A is *monotone* if $x \preceq y \Rightarrow Ax \preceq Ay$

Theorem (G. Birkhoff, 1957):

Positive linear monotone mappings contract the Hilbert metric in $\mathring{K}$.

The contraction coefficient is $\tanh \frac{1}{4} \Delta(A)$

Note: Perron-Frobenius follows from contraction mapping theorem

Conclusions

Conic geometries are adapted to consensus theory …
Quadratic Lyapunov functions are'nt …

Tsitsiklis Lyapunov function is a measure of contraction of the Hilbert metric.

Birkhoff theorem (positive monotone operators contract the Hilbert metric) applies to more general cones, e.g. the SDP cone.

Opens the way to a consensus theory in noncommutative spaces, with a number of possible applications.

How to bridge the gap between contraction measures and the i/o approach to consensus ?

Plan for this talk

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

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Contraction analysis of linear consensus

Consider the displacements dynamics from (49) given by $\dot{\delta x} = A(t)\delta x$, and the horizontal Finsler-Lyapunov function

$$V(x, \delta x) := \max_i \delta x_i - \min_i \delta x_i, \qquad (50)$$

that coincides with the classical consensus function adopted in [30], [53] lifted to the tangent space. See [43] for its

A differential Lyapunov framework for contraction analysis, F. Forni and RS, TAC 2014.

Consensus theory connects to contraction analysis by interpreting

- Hilbert metric as a *Finsler-Lyapunov function* to study contraction
- the consensus (=Perron-Frobenius) direction as a symmetry to be factored out in the contraction analysis
- Projective contraction + row-stochasticity as horizontal contraction

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 48, NO. 10, OCTOBER 2003

Monotone Control Systems

David Angeli and Eduardo D. Sontag, *Fellow, IEEE*

I. Introduction

ONE OF THE most important classes of dynamical systems in theoretical biology is that of *monotone systems*. Among the classical references in this area are the textbook by Smith [27] and the papers [14] and [15] by Hirsh and [26] by Smale. Monotone systems are those for which trajectories preserve a partial ordering on states. They include the subclass of *cooper-*

Definition II.1: A controlled dynamical system $\phi : \mathbb{R}_{\geq 0} \times X \times \mathcal{U}_\infty \to X$ is *monotone* if the following implication holds for all $t \geq 0$:

$$u_1 \succeq u_2,\ x_1 \succeq x_2 \quad \Rightarrow \quad \phi(t, x_1, u_1) \succeq \phi(t, x_2, u_2).$$

Reading the paper to the end...

Remark VIII.3: Looking at cooperativity as a notion of "incremental positivity" one can provide an alternative proof of the infinitesimal condition for cooperativity, based on the positivity of the variational equation. Indeed, assume that each system (35) is a positive linear time-varying system, along trajectories of (1). Pick arbitrary initial conditions $\xi_1 \succeq \xi_2 \in X$ and inputs $u_1 \geq u_2$. Let $\Phi(h) := \phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2))$. We have (see, e.g., [28, Th. 1]) that $\phi(t,\xi_1,u_1) - \phi(t,\xi_2,u_2) = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(h)dh = \int_0^1 z_h(t, \xi_1 - \xi_2, u_1 - u_2)dh$, where z_h denotes the solution of (35) when $(\partial f/\partial u)(x,u)$ and $(\partial f/\partial u)(x,u)$ are evaluated along $\phi(t, \xi_2 + h(\xi_1 - \xi_2), u_2 + h(u_1 - u_2))$. Therefore, by positivity, and monotonicity of the integral, we have $\phi(t,\xi_1,u_1) - \phi(t,\xi_2,u_2) \succeq 0$, as claimed. □

We remark that monotonicity with respect to other orthants corresponds to generalized positivity properties for linearizations, as should be clear by Corollary III.3.

APPENDIX A

The importance of

Monotone Dynamical Systems

M.W. Hirsch* Hal Smith †

We will see that the long-term behavior of monotone systems is severely limited. Typical conclusions, valid under mild restrictions, include the following:

- If all forward trajectories are bounded, the forward trajectory of almost every initial state converges to an equilibrium
- There are no attracting periodic orbits other than equilibria, because every attractor contains a stable equilibrium.
- In $\mathbb{R}^3$, every compact limit set that contains no equilibrium is a periodic orbit that bounds an invariant disk containing an equilibrium.
- In $\mathbb{R}^2$, each component of any solution is eventually increasing or decreasing.

An analyst viewpoint on Perron-Frobenius theory

and nonlinear operators on Banach spaces. The usefulness of operators that are positive in some sense stems from the theorem of Perron [154] and Frobenius [48], now almost a century old, asserting that for a linear operator on $\mathbb{R}^n$ represented by a matrix with positive entries, the spectral radius is a simple eigenvalue having a positive eigenvector, and all other eigenvalues have smaller absolute value and only nonpositive eigenvectors. In 1912 Jentsch [84] proved the existence of a positive eigenfunction with a positive eigenvalue for a homogeneous Fredholm integral equation with a continuous positive kernel.

In 1935 the topologists Alexandroff and Hopf [2] reproved the Perron-Frobenius theorem by applying Brouwer's fixed-point theorem to the action of a positive $n \times n$ matrix on the space of lines through the origin in $\mathbb{R}^n_+$. This was perhaps the first explicit use of the dynamics of operators on a cone to solve an eigenvalue problem. In 1940 Rutman [169] continued in this vein by reproving Jentsch's theorem by means of Schauder's fixed-point theorem, also obtaining an infinite-dimensional analog of Perron-Frobenius, known today as the Krein-Rutman theorem [103, 213]. In 1957 G. Birkhoff [20] initiated the dynamical use of Hilbert's projective metric for such questions.

The dynamics of cone-preserving operators continues to play an important role in functional analysis; for a survey, see Nussbaum [145, 146]. One outgrowth of this work

(*from* Hirsch and Smith, 2004) 19

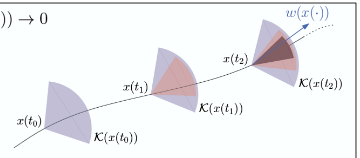
A differential geometric viewpoint on PF theory

VI. DIFFERENTIAL PERRON-FROBENIUS THEORY

A. Contraction of the Hilbert metric

Bushell [10] (after Birkhoff [7]) used the Hilbert metric on cones to show that the strict positivity of a mapping guarantees contraction among the rays of the cone, opening the way to many contraction-based results in the literature of positive operators [10], [30], [39], [8], [26].

$$d_{x(t)}(\delta x(t), \delta y(t)) \to 0$$



B. The Perron-Frobenius vector field

The Perron-Frobenius vector of a strictly positive linear map is a fixed point of the projective space. Its existence is a consequence of the contraction of the Hilbert metric, [10]. To exploit the

The main result: the PF vector field determines the asymptotic behavior

(Theorem 3)

Suppose that the trajectories of Σ are bounded. Then, for every $\xi \in \mathcal{X}$, the ω-limit set $\omega(\xi)$ satisfies one of the following two properties:

(i) The vector field $f(x)$ is aligned with the Perron-Frobenius vector field $w(x)$ for each $x \in \omega(\xi)$, and $\omega(\xi)$ is either a fixed point or a limit cycle or a set of fixed points and connecting arcs;

(ii) The vector field $f(x)$ is nowhere aligned with the Perron-Frobenius vector field $w(x)$ for each $x \in \omega(\xi)$, and either $\liminf_{t\to\infty} |\partial_x \psi(t,0,x) w(x)|_{\psi(t,0,x)} = \infty$ or $\lim_{t\to\infty} f(\psi_t(x)) = 0$.

Plan for this talk

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

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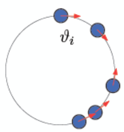
Consensus on nonlinear spaces ☆

R. Sepulchre

Dept. of Electrical and Computer Engineering, Institut Montefiore, B28, Université de Liège, 4000 Liège Sart-Tilman, Belgium

A B S T R A C T

Consensus problems have attracted significant attention in the control community over the last decade. They act as a rich source of new mathematical problems pertaining to the growing field of cooperative and distributed control. This paper is an introduction to consensus problems whose underlying state-space is not a linear space, but instead a highly symmetric nonlinear space such as the circle and other relevant generalizations. A geometric approach is shown to highlight the connection between several fundamental models of consensus, synchronization, and coordination, to raise significant global convergence issues not present in linear models, and to be relevant for a number of engineering applications, including the design of planar or spatial coordinated motions.



Differential positivity and consensus on nonlinear spaces

Current work:

Positivity is the local contraction property of the consensus rule

“move towards the average of your neighbors”

Inferring the cone field from the space geometry.

(Kuramoto model, phase synchronization, ...)

24

Plan for this talk

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

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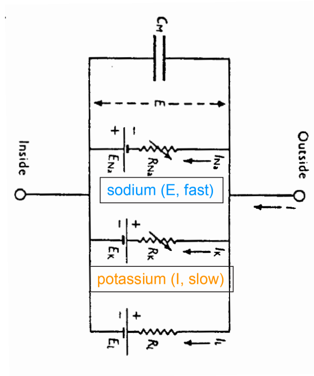
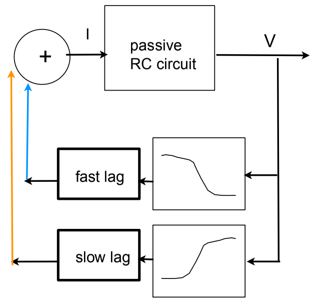
Hodgkin-Huxley electrical circuit is a mixed feedback interconnection of monotone systems





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Differential positivity and interconnection of monotone systems

Current work:

Inferring the cone field from monotonicity of the blocks + the interconnection structure

(Negative feedback is not cone-preserving)

(Biological oscillators, bursters, ...)

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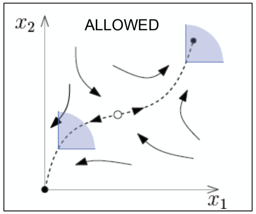
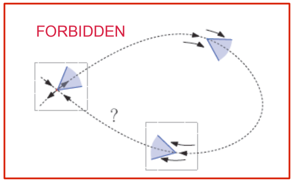
Conclusion: differential positivity = smooth patching of local orders

- Motivation (1): positivity and networks
- Motivation (2): positivity and monotonicity
- Motivation (3): positivity on nonlinear spaces
- Motivation (4): positivity and interconnections

How weak is differential positivity ?





Local ordering is a weak property.
Smooth global patching is a demanding property.

A SURVEY OF CLASSICAL AND RECENT RESULTS IN RLC CIRCUIT SYNTHESIS

Malcolm Smith, University of Cambridge, UK

The talk will recall some of the main results of classical circuit synthesis: Foster's Reactance theorem, the constructions of Brune and Darlington, reactance extraction, and the Bott-Duffin procedure. An introduction will also be provided to the enumerative method for RLC synthesis and the recent results of Hughes and Smith, Jiang and Smith..

A survey of classical and recent results in RLC circuit synthesis

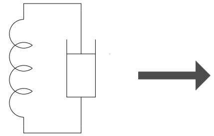
Malcolm C. Smith

Cambridge University Engineering Department

Workshop on “Dynamics and Control in Networks”
Lund University
15-17 October 2014

What is the Most General “Passive” Suspension?

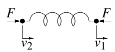
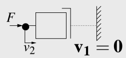
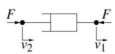
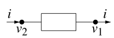
Conceptual step: replace the spring and damper with a “black-box”.



Passive Black-box Mechanism

Can we characterise the properties of the “most general” such mechanism?

Force-current analogy

Mechanical	Electrical
 spring	 inductor
 mass	 capacitor
 damper	 resistor

Mass element only has one terminal—a fundamental restriction for synthesis.

Question

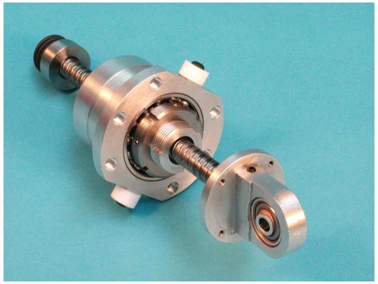
Is it possible to construct a physical device such that the relative acceleration between its endpoints is proportional to the applied force?

$$F = b(\ddot{x}_2 - \ddot{x}_1)$$

Yes! A new word “inerter” was invented to describe such a device.

M.C. Smith, 2002, Synthesis of Mechanical Networks: The Inerter, *IEEE Trans. on Automat. Contr.*, **47**, 1648–1662.

Ballscrew inerter made at Cambridge University Engineering Department (2003) - flywheel removed



Mass $\approx$ 1 kg, Inertance (adjustable) = 60–180 kg

Mechanical Network Synthesis

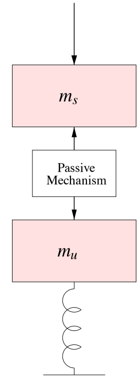
Theorem

It is possible to build a passive mechanism **of small mass** whose impedance (velocity/force) is any rational postive-real function.

Proof

Bott-Duffin, force-current analogy + ideal inerter: $F = b(\ddot{x}_1 - \ddot{x}_2)$, where physical embodiments must satisfy:

- Inertance b (kg) is independent of mass;
- Inertance is independent of travel.



Synthesis methods

LC only:

- Foster (1924)

RC and LC:

- Cauer et al.

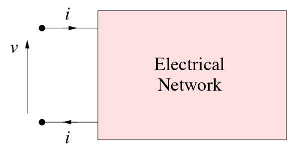
RLC + transformers:

- Brune (1931)
- Darlington (1939)
- Youla and Tissi (1966)

RLC only:

- Bott and Duffin (1949)

Driving-point impedance and admittance

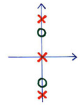


Admittance $= Y(s) = \hat{i}(s)/\hat{v}(s)$. Impedance $= Z(s) = Y(s)^{-1}$.

Foster's Reactance Theorem (1924)

The most general driving-point impedance of a network containing capacitors, inductors, transformers, mutual inductance is:

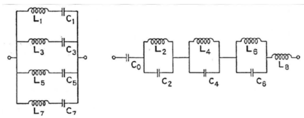
$$Z(s) = \left[k \frac{(s^2+\omega_1^2)(s^2+\omega_3^2)\dots(s^2+\omega_{2n\pm1}^2)}{s(s^2+\omega_2^2)(s^2+\omega_4^2)\dots(s^2+\omega_{2n}^2)} \right]^{\pm 1}$$



where $k \geq 0$ and $0 \leq \omega_1 \leq \omega_2 \dots$.

Proof analogous to a problem in mechanics solved by E.J. Routh (Advanced Rigid Dynamics, 1905).

Foster canonical forms

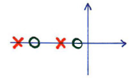


R.M. Foster, "A Reactance Theorem", Bell System Technical Journal, vol. 3, pp. 259–267, 1924

Foster and Cauer

The most general driving-point impedance of an RL network is:

$$Z(s) = k\frac{(s+\sigma_1)(s+\sigma_3)\dots}{(s+\sigma_2)(s+\sigma_4)\dots}$$

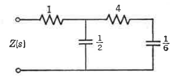


where $k \geq 0$ and $0 \leq \sigma_1 \leq \sigma_2 \dots$, |relative degree| ≤ 1.

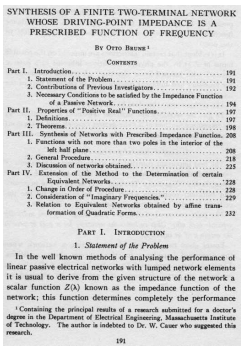
Follows from Foster's reactance theorem using Cauer's square root transformation: $s = p^2$.

Cauer's first form:

$$Z(s) = 1 + \cfrac{1}{\cfrac{s}{2} + \cfrac{1}{4 + \cfrac{1}{s/6}}}$$



Otto Brune


SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY

By Otto Brune [1]

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1. Statement of the Problem 191
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Part I. Introduction

1. *Statement of the Problem*

In the well known methods of analysing the performance of linear passive electrical networks with lumped network elements it is usual to derive from the given structure of the network a scalar function $Z(\lambda)$ known as the impedance function of the network; this function determines completely the performance

[1] Containing the principal results of a research submitted for a doctor's degree in the Department of Electrical Engineering, Massachusetts Institute of Technology. The author is indebted to Dr. W. Cauer who suggested this research.

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Ground-breaking paper (1931).

(1) Introduced the notion of a *positive-real function.*

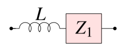
(2) Showed that the impedance of a passive network must be postive-real.

(3) Showed that any positive-real function could be realised as the impedance or admittance of a network comprising resistors, capacitors, inductors *and transformers.*

Foster preamble for a positive-real $Z(s)$

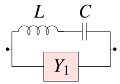
Removal of poles on $j\mathbb{R} \cup \{\infty\}$

$$Z = sL + Z_1, \quad (Z_1 \text{ proper})$$



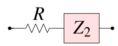
Removal of zeros on $j\mathbb{R} \cup \{\infty\}$

$$Z = \left(\frac{As}{s^2+\omega^2} + Y_1\right)^{-1}$$



Subtract minimum real part

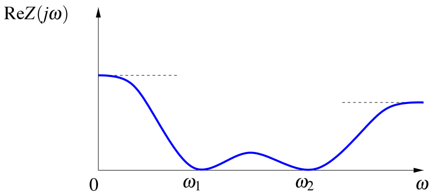
$$Z = R + Z_2$$



Not necessarily a unique process

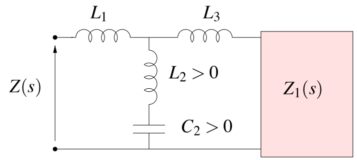
Minimum functions

A **minimum function** $Z(s)$ is a positive-real function with no poles or zeros on $j\mathbb{R} \cup \{\infty\}$ and with the real part of $Z(j\omega)$ equal to 0 at one or more frequencies.



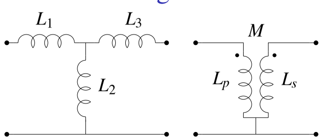
The Brune Cycle

Let $Z(s)$ be a minimum function with $Z(j\omega_1) = jX_1$ $(\omega_1 > 0)$.
It can be shown that the following decomposition is possible with Z_1 positive-real of lower degree than Z.



Problem: $\text{sign}(L_1 L_3) < 0$.

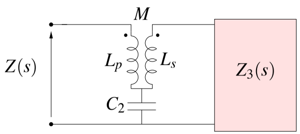
To Remove Negative Inductor:



$L_p = L_1 + L_2$
$L_s = L_2 + L_3$
$M = L_2$

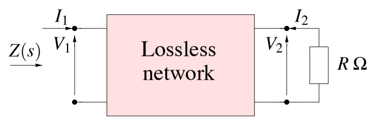
It turns out that: $L_p, L_s > 0$ and $\frac{M^2}{L_p L_s} = 1$ (unity coupling coefficient).

Realisation for completed Brune cycle:



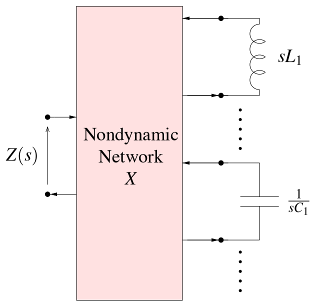
Darlington synthesis

Darlington showed that *any* positive-real $Z(s)$ could be realised by a lossless two-port (containing inductors, capacitors and transformers) terminated in a single resistor.



Darlington, S., "Synthesis of reactance 4-poles which produce prescribed insertion loss characteristics," J. Math. Phys., Vol. 18, 257–353, Sep. 1939.

Synthesis via reactance extraction



Let $L_1 = \cdots = C_1 = \cdots = 1$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the hybrid matrix of X, i.e.

$$\begin{pmatrix} v_1 \\ i_2 \end{pmatrix} = M \begin{pmatrix} i_1 \\ v_2 \end{pmatrix},$$

then

$$Z(s) = M_{11} - M_{12}(sI + M_{22})^{-1} M_{21}.$$

D.C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction, Part I", *IEEE International Convention Record*, 183–205, 1966.

Bott-Duffin Synthesis


Letters to the Editor

Impedance Synthesis without Use of Transformers

An Improvement in the Shadow-Cast Replica Technique

516 JOURNAL OF APPLIED PHYSICS

R. Bott and R.J. Duffin showed that transformers were unnecessary in the synthesis of positive-real functions. (1949)

Bott-Duffin Construction

If $Z(s)$ is positive-real then

$$R(s) = \frac{kZ(s) - sZ(k)}{kZ(k) - sZ(s)}$$

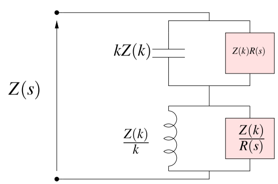
is positive-real for any $k > 0$ (Richard's transformation).

If $Z(s)$ is a minimum function with $Z(j\omega_1) = jX_1$ ($\omega_1 > 0$). (Assume $X_1 > 0$.) Then we can find a k s.t. $R(s)$ has a zero at $s = j\omega_1$.

Bott-Duffin Construction (cont.)

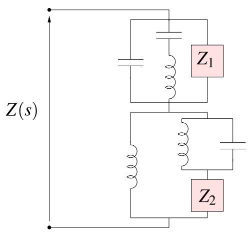
We now write:

$$\begin{aligned} Z(s) &= \frac{kZ(k)R(s) + Z(k)s}{k + sR(s)} = \frac{kZ(k)R(s)}{k + sR(s)} + \frac{Z(k)s}{k + sR(s)} \\ &= \frac{1}{\frac{1}{Z(k)R(s)} + \frac{s}{kZ(k)}} + \frac{1}{\frac{k}{Z(k)s} + \frac{R(s)}{Z(k)}}. \end{aligned}$$



Bott-Duffin Construction (cont.)

We can write: $\frac{1}{Z(k)R(s)} = \text{const} \times \frac{s}{s^2+\omega_1^2} + \frac{1}{R_1(s)}$ etc.



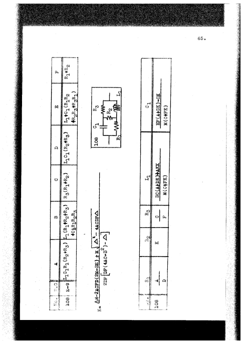
$\delta(Z_1(s)) = \delta(Z_2(s)) = \delta(Z(s)) - 2$ where $\delta =$ (McMillan) degree.

Enumerative approach—Ladenheim's master's thesis (1948)

Ladenheim considered all networks with at most five elements and at most two reactive elements, and reduced the whole set to *108* networks (1948).

Questions not answered:

- What is the totality of biquadratics which may be realised?
- How many different networks are needed?



New approach - the concept of a regularity

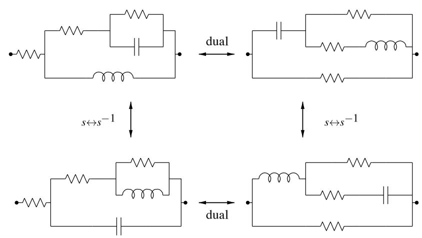
A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\mathrm{Re}(Z(j\omega))$ or $\mathrm{Re}(Z^{-1}(j\omega))$ occurs at $\omega = 0$ or $\omega = \infty$.

Theorem

106 out 108 Ladenheim networks are regular.
6 series-parallel networks are a "generating set" for these 106 regular networks.
2 remaining bridge networks do not realise all the remaining biquadratic positive-real functions.

J.Z. Jiang and M.C. Smith, 2011, Regular Positive-Real Functions and Five-Element Network Synthesis for Electrical and Mechanical Networks, *IEEE Trans on Automat. Contr.*, **56**, pp. 1275–1290.

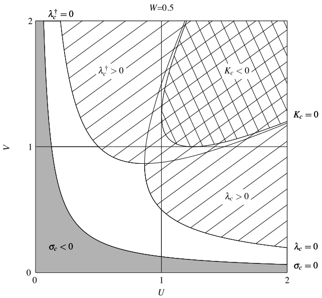
Network quartets



Canonical Form for Biquadratics

$$\frac{s^2+2U\sqrt{W}s+W}{s^2+(2V/\sqrt{W})s+1/W}$$

$$(U,V,W>0)$$



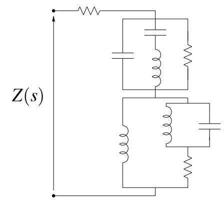
Positive-real boundary: $\sigma_c = 0$

Regular region: $\lambda_c \geq 0$ or $\lambda_c^\dagger \geq 0$,

Bott-Duffin construction again

$$Z(s) = \frac{As^2+Bs+C}{Ds^2+Es+F}$$

General form of Bott-Duffin realisation for a biquadratic:



3 capacitors, 3 inductors and 3 resistors!!

Recent result

T.H. Hughes and M.C. Smith, *On the minimality and uniqueness of the Bott-Duffin realisation procedure*, IEEE AC-Transactions, vol. 59, 1858–1873, July 2014.

Shows that 6 reactive elements are necessary for series-parallel realisation of a biquadratic minimum function.

Sketch of proof

Assume $Z(s)$ = p.r. minimum function $= Z_1(s) + Z_2(s)$ (series). Then

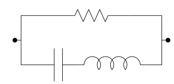
$$\text{Re}(Z(j\omega_0)) = 0 \quad \Rightarrow \quad \text{Re}(Z_1(j\omega_0)) = 0, \qquad (1)$$

$$Z_1 \text{ has no poles on } j\mathbb{R} \cup \{\infty\}. \qquad (2)$$

$$(1) + (2) \quad \Rightarrow \quad \#(Z_1) \geq 2$$

where # = no. of reactive elements in a s.p. realisation.

$$(1) + (2) \text{ and } \#(Z_1) = 2 \quad \Rightarrow \quad Z_1(s) = \frac{s^2+\omega_0^2}{As^2+B\omega_0 s+A\omega_0^2} \text{ (true zero)}$$

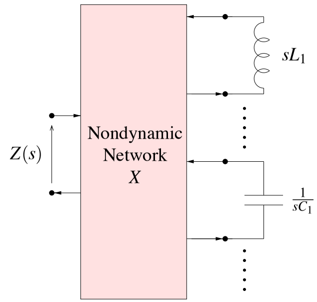


Hence, $\#(Z) = \#(Z_1) + \#(Z_2) \geq 5$.

Rest of talk based on the recent paper:

T. H. Hughes and M. C. Smith, Algebraic criteria for circuit realisations, Mathematical System Theory—Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday, Knut Hüper and Jochen Trumpf (eds.), CreateSpace, 2013, pp. 211–228.

Synthesis via reactance extraction



Let $L_1 = \cdots = C_1 = \cdots = 1$. If

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is the hybrid matrix of X, i.e.

$$\begin{pmatrix} v \\ \underline{v}_1 \\ \underline{i}_2 \end{pmatrix} = M \begin{pmatrix} i \\ \underline{i}_1 \\ \underline{v}_2 \end{pmatrix}$$

then

$$Z(s) = M_{11} - M_{12}(sI + M_{22})^{-1} M_{21}$$

D.C. Youla and P. Tissi, "N-Port Synthesis via Reactance Extraction, Part I", *IEEE International Convention Record*, 183–205, 1966.

Hankel matrix

Assume $Z(s)$ is proper and is realised with p inductors and q capacitors. Suppose $n = \deg(Z(s)) = p + q$ (minimally reactive). Let

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \ldots$$

and define the finite Hankel matrices

$$\mathscr{H}_k = \begin{bmatrix} h_0 & h_1 & \ldots & h_{k-1} \\ h_1 & h_2 & \ldots & h_k \\ \vdots & \vdots & \ddots & \vdots \\ h_{k-1} & h_k & \ldots & h_{2k-2} \end{bmatrix}.$$

Then

$$\mathscr{H}_n = V_o \left(-\Lambda^{-1} \Sigma \right) V_o^T$$

where

$$\begin{aligned} \Lambda &= \operatorname{diag}\{L_1, \ldots, L_p, C_1, \ldots, C_q\} \\ \Sigma &= (I_p \dotplus -I_q) \\ V_0 & \quad \text{non-singular} \end{aligned}$$

Signature of the Hankel matrix

For the (proper) impedance

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \ldots$$

where $n = \deg(Z(s))$. Then

$$\begin{aligned} p &= \#\,\text{inductors} = \#\,\text{neg. eigs. of } \mathscr{H}_n \\ q &= \#\,\text{capacitors} = \#\,\text{pos. eigs. of } \mathscr{H}_n \end{aligned}$$

Algebraic condition:

$$q = \mathbf{P}(1, |\mathscr{H}_1|, \ldots, |\mathscr{H}_n|),$$
$$p = \mathbf{V}(1, |\mathscr{H}_1|, \ldots, |\mathscr{H}_n|)$$

where $\mathbf{P}(\cdot, \ldots, \cdot)$ is the number of permanences of sign and $\mathbf{V}(\cdot, \ldots, \cdot)$ is the number of variations of sign in the sequence.

Cauchy index

For a proper impedance $Z(s)\ldots$

Corollary 1. $q-p=\sigma(\mathscr{H}_n)$ where σ denotes the signature.

Corollary 2. $q-p=I_{-\infty}^{+\infty}Z(s)$ where $I_{-\infty}^{+\infty}Z(s)$ is the difference between the number of jumps of $Z(s)$ from $-\infty$ to $+\infty$ and the number of jumps from $+\infty$ to $-\infty$ as s is increased in $\mathbb{R}$ from $-\infty$ to $+\infty$ (Cauchy index).

Sylvester matrix

Write

$$Z(s)=\frac{a(s)}{b(s)}=\frac{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0}{b_n s^n+b_{n-1}s^{n-1}+\ldots+b_0}.$$

Define the Sylvester matrices

$$\mathscr{S}_{2k}=\begin{bmatrix} b_n & b_{n-1} & \ldots & b_{n-k+1} & b_{n-k} & \ldots & b_{n-2k+1}\\ a_n & a_{n-1} & \ldots & a_{n-k+1} & a_{n-k} & \ldots & a_{n-2k+1}\\ 0 & b_n & \ldots & b_{n-k+2} & b_{n-k+1} & \ldots & b_{n-2k+2}\\ 0 & a_n & \ldots & a_{n-k+2} & a_{n-k+1} & \ldots & a_{n-2k+2}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & b_n & b_{n-1} & \ldots & b_{n-k}\\ 0 & 0 & \ldots & a_n & a_{n-1} & \ldots & a_{n-k} \end{bmatrix}.$$

Then $|\mathscr{S}_{2k}|=b_n^{2k}|\mathscr{H}_k|$ (Gantmacher).
Also, $|\mathscr{S}_{2n}|=$ resultant of $a(s)$ and $b(s)$.

Biquadratic functions

$$Z(s)=\frac{a_2s^2+a_1s+a_0}{b_2s^2+b_1s+b_0}.$$

$$|\mathscr{S}_2|=b_2a_1-b_1a_2,\quad |\mathscr{S}_4|=(b_2a_1-b_1a_2)(b_1a_0-b_0a_1)-(b_2a_0-b_0a_2)^2.$$

$$q=\mathbf{P}(1,|\mathscr{S}_2|,|\mathscr{S}_4|),$$
$$p=\mathbf{V}(1,|\mathscr{S}_2|,|\mathscr{S}_4|).$$

Corollary. Whether the reactive elements are of the same or different kind is determined by the sign of the resultant $|\mathscr{S}_4|$.

Stated (without proof) by Foster (1962), as noted by Kalman (2010).

Thank you!

SYNCHRONIZATION AND EVOLVABILITY IN NONLINEAR NETWORKS

Jean-Jacques Slotine, Massachusetts Institute of Technology, USA

Computation, measurement, and synchronization are key issues in complex networks. Vast nonlinear networks are encountered in biology, for instance, and in neuroscience, where for most tasks the human brain grossly outperforms engineered algorithms using computational elements 7 orders of magnitude slower than their artificial counterparts. We show that nonlinear dynamic systems tools, and in particular contraction analysis, yield simple but highly non-intuitive insights about such issues, and that they also suggest systematic mechanisms to build progressively more refined networks through stable accumulation of functional building blocks and motifs.

COMMUNITY DETECTION: FUNDAMENTAL LIMITS AND OPTIMAL SAMPLING ALGORITHMS

Alexandre Proutiere, Kungliga Tekniska Högskolan, Sweden

We consider networks consisting of a finite number of non-overlapping communities. To extract these communities, the interaction between pairs of nodes may be sampled from a large available data set, which allows a given node pair to be sampled several times. When a node pair is sampled, the observed outcome is a binary random variable, equal to 1 if nodes interact and to 0 otherwise. The outcome is more likely to be positive if nodes belong to the same communities. For a given budget of node pair samples or observations, we wish to jointly design a sampling strategy (the sequence of sampled node pairs) and a clustering algorithm that recover the hidden communities with the highest possible accuracy. We consider both non-adaptive and adaptive sampling strategies, and for both classes of strategies, we derive fundamental performance limits satisfied by any sampling and clustering algorithm. In particular, we provide necessary conditions for the existence of algorithms recovering the communities accurately as the network size grows large. We also devise simple algorithms that accurately reconstruct the communities when this is at all possible, hence proving that the proposed necessary conditions for accurate community detection are also sufficient. The classical problem of community detection in the stochastic block model can be seen as a particular instance of the problems consider here. But our framework covers more general scenarios where the sequence of sampled node pairs can be designed in an adaptive manner. We provide new results for the stochastic block model, and extends the analysis to the case of adaptive sampling. This is a joint work with Se-Young Yun.

Community Detection via Random and Adaptive Sampling

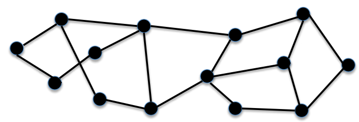
Se-Young Yun (MSR-INRIA)
Marc Lelarge (INRIA-ENS)
Alexandre Proutiere (KTH)

Community detection in networks

Objective: Extract *K* communities in a network of *n* nodes from random *observations*. ***K* << *n***. Here finite *K*, *n* very large

Observations

1. A graph of interactions / similarities



2. General sampling framework

Community detection in networks

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Observations

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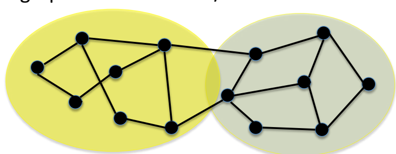


2. General sampling framework

Applications

- Social networks: recommendation systems, targeted advertisement
- Biology: the role of proteins
- Distributed computing: balanced partitions
- Communication networks: caching, pro-active resource allocation (user mobility)
- …

Related work

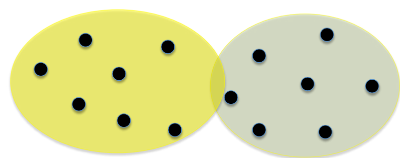
Arora, Rao, Newman, Coja-Oghlan, Jerrum, Chen, Frieze, McSherry, Dyer, Sorkin, Kannan, Vempala, Vetta, Fortunato, Decelle, Krzakala, Karp, Condon, Reichart, Sanghavi, Nadakuditi, Girvan, Mosel, Sly, Rohe, Chatterjee, Yu, Massoulie, Lelarge, Vazirani, Karger, Feld, Fischer, Kleinberg, Gibson, Raghavan, Hopcroft, Khan, Kulis, Santo, Wellman, Hogan, Berg, White, Boorman, Kelley, Xie, Kumar, Mathieu, Schudy, Alon, Krivelevich, Sudakov, Xu, Achlioptas, Kahale, Feige, Zdeborova, Carson, Giesen, Mitshe, Shamir, Tsur, Hassibi, Oymak, Ames, Parrilo, Holland, Laskley, Pothen, Simon, Liou, Girvan, Chauhan, Leone, Ball, Karrer, …

Outline

1. Community detection in the Stochastic Block Model
2. General sampling framework
   a. Fundamental limits
   b. Optimal Algorithms
3. Streaming, Memory Limited Algorithms

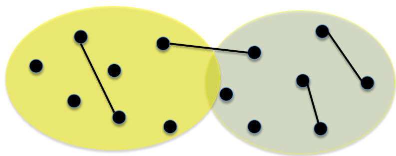
1. Community Detection in the Stochastic Block Model

Stochastic Block Model (SBM)



- The graph is built by considering each pair of nodes once
  - If in the same community: put an edge with probability p
  - Else: put an edge with probability $q < p$

Stochastic Block Model (SBM)



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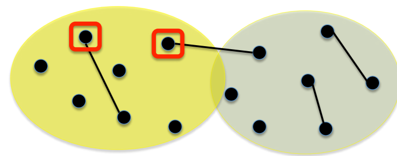

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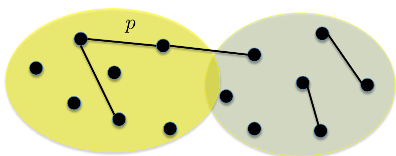

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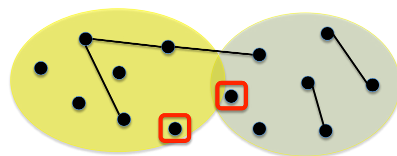

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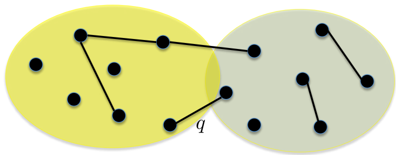
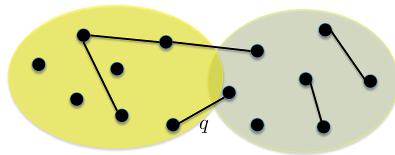
Stochastic Block Model (SBM)



- The graph is built by considering each pair of nodes once
  - If in the same community: put an edge with probability *p*
  - Else: put an edge with probability *q* < *p*

Stochastic Block Model (SBM)



- Network size: *n* nodes, *n* tends to ∞
- Sparse interaction: $p, q = o(1)$
  - Very sparse $p, q \sim 1/n$ (**Massoulie**'s talk)
  - Sparse $p, q \sim f(n)/n,\ f(n) = \omega(1)$
- Dense interaction: $p, q = O(1)$

Performance metrics

Proportion of misclassified nodes under π: $\varepsilon^\pi(n)$

1. **Asymptotic detection**: an algorithm *detects* the clusters if it does better than the algorithm that randomly assigns nodes to clusters

2. **Accurate asymptotic detection**: an algorithm π is asymptotically accurate if $\lim_{n\to\infty} \mathbb{E}[\varepsilon^\pi(n)] = 0$

Asymptotic Detection in the SBM

- Two communities of equal sizes, sparse case $p = \frac{a}{n}, \quad q = \frac{b}{n}$

Theorem (Mossel-Neeman-Sly 2012)
If $a - b < \sqrt{2(a+b)}$, then asymptotic detection is impossible.

Conjectured by Decelle-Krzakala-Moore-Zdeborova 2012

Theorem (Massoulie 2013)
If $a - b > \sqrt{2(a+b)}$, then there exists an algorithm leading to clusters that are positively correlated with the true clusters.

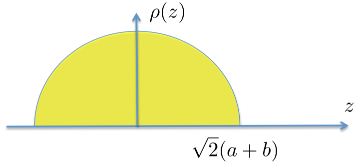
Non-rigorous Spectral Analysis

- Average adjacency matrix

$$\mathbb{E}[A] = \frac{1}{2}(a+b)11^T + \frac{1}{2}(a-b)uu^T$$

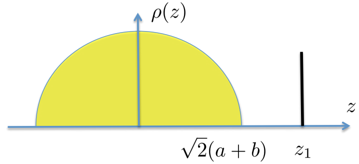
$$1 = \frac{1}{\sqrt{n}}(1,\ldots,1)^T, \quad u = \frac{1}{\sqrt{n}}(1,\ldots,1,-1,\ldots,-1)^T$$

- Noisy observation: $A = \mathbb{E}[A] + X$
- Spectral density of noise matrix X (Wigner semicircle law)



Non-rigorous Spectral Analysis

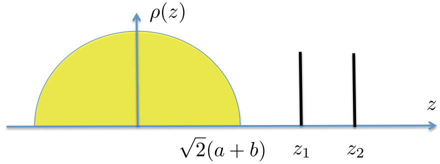
- Spectral density of the modularity matrix: $\frac{1}{2}(a-b)uu^T + X$



$$z_1 = \frac{1}{2}(a-b) + \frac{a+b}{a-b}$$

Non-rigorous Spectral Analysis

- Spectral density of the observed matrix:



- Communities are detectable if $z_1 > \sqrt{2}(a+b)$
- Method: find z_1 and the corresponding eigenvector u

Examples of algorithms

- Maximum Likelihood (NP hard problem)
  - Exact solution: Belief Propagation
  - Compressed sensing: relaxation

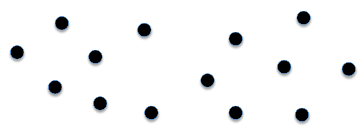
- Spectral method
  - Provide a rank-K approximation of the adjacency matrix
  (+ Trimming + Post-processing)

Open problems

- Very sparse graphs: condition for asymptotic detection with more than two communities
- General graphs: condition on *p, q, n* for asymptotically accurate detection? (**this talk**)
- General graphs: what is the optimal scaling of $\varepsilon^{\pi}(n)$?

2. General Sampling Framework

Sampling Framework



- Large data set available: many *samples* for the interaction of each pair of nodes
- Sample for a given node pair: Bernoulli with mean *p* if nodes are in the same cluster, with mean *q* otherwise
- Sample budget: *T*

Sampling Strategies

- **Non-adaptive Random Strategies**
  - The pair of nodes sampled in round *t* does not depend on past observations, and is chosen uniformly at random
  - S1: sampling with replacement
  - S2: sampling without replacement
- **Adaptive Strategies**
  - The pair of nodes sampled in round *t* depends on past observations
- Classical SBM: random sampling without replacement, and $T = n(n-1)/2$

Objectives

- Performance metric: proportion of misclassified nodes $\varepsilon(n,T)$
  Asymptotically accurate detection: $\lim_{n\to\infty} \mathbb{E}[\varepsilon(n,T)] = 0$
- Non-adaptive sampling:
  - Necessary conditions on *n, T, p, q* for the existence of asymptotically accurate algorithms
  - Asymptotically accurate clustering algorithms
- Adaptive sampling:
  - Necessary conditions *on n, T, p, q* for the existence of asymptotically accurate joint sampling and clustering algorithms
  - Asymptotically accurate sampling and clustering algorithms

Fundamental limits

- Non-adaptive sampling -- necessary conditions for asymptotically accurate detection:

$$\frac{T}{n} = \omega(1), \quad \frac{T}{n}\min(KL(q,p), KL(p,q)) = \omega(1),$$

- Dense interaction: $p, q = \Theta(1)$

$$T(p-q)^2/n = \omega(1)$$

- Sparse interaction: $p, q = o(1)$

$$T(p-q)^2/(pn) = \omega(1)$$

Fundamental limits

- Non-adaptive sampling:

$$\kappa_1(n,T) = T\frac{2(n-2)}{n(n-1)}\min\{KL(q,p), KL(p,q)\}$$
$$+2\sqrt{\frac{4T(n-2)}{n(n-1)}\left[\min\{q,1-p\}\left(\log\frac{p(1-q)}{q(1-p)}\right)^2 + \left(\log(\min\{\frac{p}{q},\frac{1-q}{1-p}\})\right)^2\right]}$$

Theorem Under random sampling strategy S1 or S2, for any clustering algorithm π, we have:

$$\mathbb{E}[\varepsilon^\pi(n,T)] \geq \frac{1}{8}\exp(-\kappa_1(n,T)),$$

Fundamental limits

- Adaptive sampling:

Theorem For asymptotically accurate detection, we need:

$$\min\{p, 1-q\}\frac{T}{n} = \Omega(1) \quad \text{and} \quad \frac{T}{n}\max(KL(q,p), KL(p,q)) = \omega(1).$$

- Example: $p = \frac{\log n}{n} \quad q = \frac{\sqrt{\log n}}{n}$
  - Non-adaptive sampling: $\frac{T}{n} = \omega(\frac{n}{\log(n)})$
  - Adaptive sampling: $\frac{T}{n} = \Omega(\frac{n}{\log(n)})$

Algorithms for non-adaptive sampling

- Spectral algorithms (extension of Coja-Oghlan's algorithm)

1. From random samples, build an observation matrix
2. Trimming (remove nodes with too many interactions)
3. Spectral decomposition (find the largest eigenvalues and corresponding eigenvectors)
4. Greedy improvement (for each node compare the number of interactions with the various clusters)

Performance

Theorem Assume that:

$$\frac{(p-q)^2}{p}\frac{\alpha T}{n} = \omega(1), \quad \frac{(p-q)^2}{p}\frac{\alpha T}{n} \geq \log(p\frac{T}{n}).$$

Then with high probability:

$$\varepsilon^{SP}(n,T) \leq 8\exp\left(-\frac{(p-q)^2}{20p}\frac{\alpha T}{n}\right).$$

- The algorithm is asymptotically accurate under the necessary conditions for accurate detection in the case of random sampling
- The necessary conditions for accurate detection are tight!

Algorithms for adaptive sampling

- Spatial coupling idea: find reference kernels and build the clusters from these kernels

1. Kernels: select n/log(n) nodes and use T/5 samples to classify these nodes (using the previous spectral algorithm)
2. Select one of remaining nodes. Sample T/3n pairs between the selected nodes to each kernel. Classify the node.
3. Repeat 2. until no remaining node or budget

Performance

Theorem Assume that:

$$\frac{(p-q)^2}{p+q}\frac{T}{n} = \Omega(1), \quad \frac{T}{n}\max(KL(q,p), KL(p,q)) = \omega(1).$$

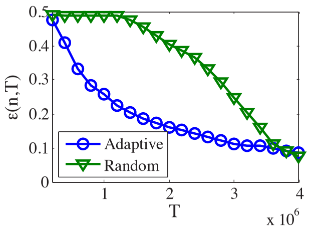
Then with high probability:

$$\varepsilon^{ASP}(n,T) \leq \exp\left(-\frac{T}{6n}(KL(q,p) + KL(p,q))\right).$$

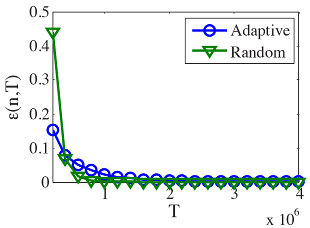
- The algorithm is asymptotically accurate under the necessary conditions for accurate detection in the case of adaptive sampling
- The necessary conditions for accurate detection are tight!

Non-adaptive vs. adaptive sampling

- n = 4000



p = 0.01, q=0.005

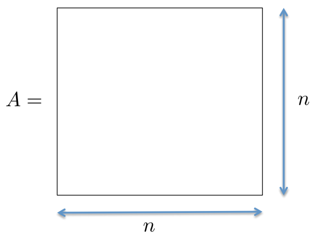


p = 0.1, q=0.05

3. Streaming, Memory-limited Algorithms

Memory and Streaming issues

- Storing and manipulating the adjacency matrix in RAM could be impossible (matlab cannot handle 5000 node networks!)
- Data about a node may arrive sequentially (one column at a time – e.g. recommender systems)



How to deal with $n = 10^7$?

SBM with $f(n) = \omega(1)$

$$p = a\frac{f(n)}{n}$$

$$q = b\frac{f(n)}{n}$$

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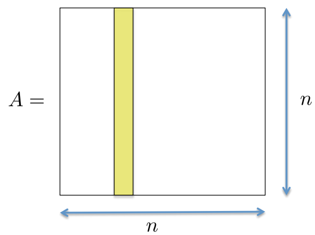
- Storing and manipulating the adjacency matrix in RAM could be impossible (matlab cannot handle 5000 node networks!)
- Data about a node may arrive sequentially (one column at a time – e.g. recommender systems)

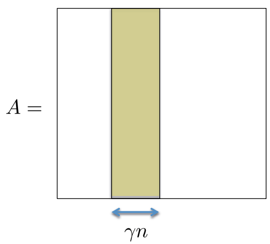


Offline algorithm: return the clusters after all the columns has been observed.

Online algorithm: classify the node immediately after its column is observed.

Classification with Partial Information

- Observe a proportion γ of columns
- Conditions on γ, f(n) to classify the corresponding nodes or all nodes asymptotically accurately?



Classification with Partial Information

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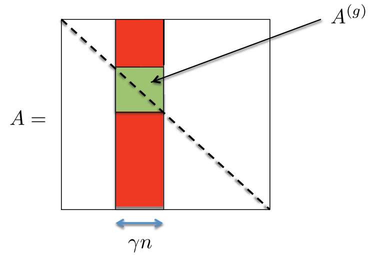


Fundamental limits

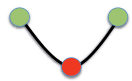
Theorem Assume that $\sqrt{\gamma} f(n) = o(1)$. Then asymptotic detection is impossible.

Theorem (i) If there is an algorithm classifying green nodes asymptotically accurately, then $\sqrt{\gamma} f(n) = \omega(1)$.
(ii) If there is an algorithm classifying all nodes asymptotically accurately, then $\gamma f(n) = \omega(1)$.

Remark: if one uses information about green nodes only, then it is possible to classify these nodes only if $\gamma f(n) = \omega(1)$. We have to use side information provided by red nodes.

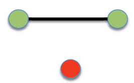
Algorithm for green nodes

- Indirect edges through red nodes



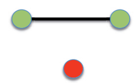
Algorithm for green nodes

- Indirect edges through red nodes



Algorithm for green nodes

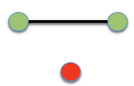
- Indirect edges through red nodes



Do it only for red nodes connected to exactly 2 green nodes (**avoid statistical dependence**!)

Algorithm for green nodes

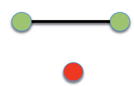
- Indirect edges through red nodes



- Result: a new adjacency matrix A'
- Algorithm: spectral method for $A^{(g)}$ and A' + keep the most informative matrix (with the highest normalized *K*-th eigenvalue)

Algorithm for green nodes

- Indirect edges through red nodes



- Result: a new adjacency matrix A'
- Algorithm: spectral method for $A^{(g)}$ and A' + keep the most informative matrix (with the highest normalized *K*-th eigenvalue)

Theorem Assume that $\sqrt{\gamma}f(n) = \omega(1)$. The above algorithm classifies the green nodes asymptotically accurately.

Algorithm for red nodes

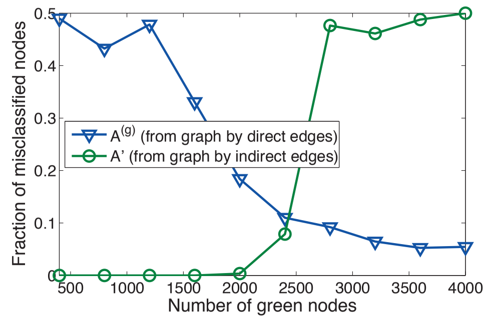
- Use the clusters of green nodes as kernels for the red nodes as in the adaptive sampling algorithm

Theorem Assume that $\gamma f(n) = \omega(1)$. The above algorithm classifies the red nodes asymptotically accurately.

The algorithms are optimal (see fundamental limits) – efficient use of side information provided by red nodes.

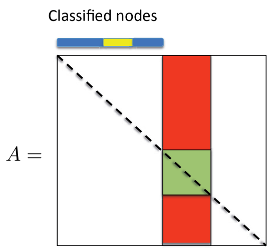
Example

n = 1000,000 ; p = 0.005, q = 0.001



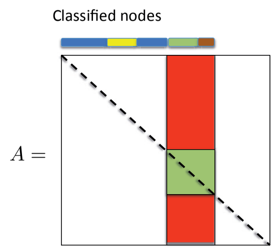
Offline Memory-efficient Algorithm

- Sequentially treat blocks of columns
  - Place the block of B columns in the memory



Offline Memory-efficient Algorithm

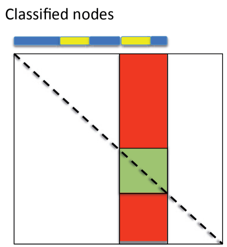
- Sequentially treat blocks of columns
  - Place the block of B columns in the memory
  - Classify the corresponding nodes using previous algorithm



Offline Memory-efficient Algorithm

- Sequentially treat blocks of columns
  - Place the block of B columns in the memory
  - Classify the corresponding nodes using previous algorithm
  - Merge the obtained clusters with the previously observed clusters, and erase the block

$A =$



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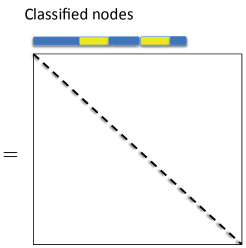


Offline Memory-efficient Algorithm

- Memory-Performance trade-off

Theorem With block size $B = \frac{h(n)n}{\min(f(n), n^{1/3}) \log(n)}$, and memory $\Theta(nh(n) + n)$.
Assume that $h(n) = \omega\left(\frac{\log(n)}{\min(f(n), n^{1/3})}\right)$, then after observing

$T = \omega\left(\frac{n}{\min(f(n), n^{1/3})}\right)$ columns, we have: w.h.p.

$$\varepsilon^{\pi}(n, T) = O\left(\exp(-T\frac{\min(f(n), n^{1/3})}{n})\right)$$

Offline Memory-efficient Algorithm

- Memory-Performance trade-off
- Example: $f(n) = \log(n)^2$
  - linear memory $h(n) = 1$
  - After observing $T = n/\log(n)$, the proportion of misclassified nodes decays faster than $1/n$

Online Memory-efficient Algorithm

- Sequentially treat blocks of columns
  - Place the block of B columns in the memory
  - Classify the corresponding nodes using previous algorithm
  - Merge the obtained clusters with the clusters of the first block, and erase the block
  - Return the classification for the block, and erase it

Theorem With block size $B = \frac{h(n)n}{\min(f(n), n^{1/3})\log(n)}$, and memory $\Theta(nh(n))$.
Assume that $h(n) = \omega\left(\frac{\log(n)}{\min(f(n), n^{1/3})}\right)$, then the above algorithm is asymptotically accurate.

Sublinear memory!

Conclusions

- A generic sampling framework extending the SBM
  - Necessary conditions for asymptotically accurate detection
  - Asymptotically optimal joint sampling and clustering algorithms
  - Our results hold in any regime! (Sparse or dense)
- Memory limited, streaming algorithm
  - Everything can be done with linear memory or even sublinear memory
  - Memory-performance trade-off
- BTW, similar results for more general models than SBM
- Cloud algorithms with communication constraints?

Thanks!

Papers

Yun-Proutiere, COLT 2014

Yun-Lelarge-Proutiere, NIPS 2014

COMMUNITY DETECTION IN STOCHASTIC BLOCK MODELS VIA SPECTRAL METHODS

Laurent Massoulié, Microsoft Research-INRIA Joint Centre

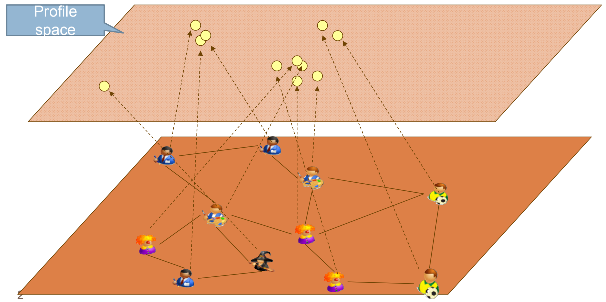
Community detection consists in identification of groups of similar items within a population. In the context of online social networks, it is a useful primitive for recommending either contacts or news items to users. We will consider a particular generative probabilistic model for the observations, namely the so-called stochastic block model, and generalizations thereof. We will describe spectral transformations and associated clustering schemes for partitioning objects into distinct groups. Exploiting results on the spectrum of random graphs, we will establish consistency of these approaches under suitable assumptions, namely presence of a sufficiently strong signal in the observed data. We will also discuss open questions on phase transitions for cluster detectability in such models when the signal becomes weak. In particular we will introduce a novel spectral method which provably allows detection of communities down to a critical threshold, thereby settling an open conjecture of Decelle, Krzakala, Moore and Zdeborová

COMMUNITY DETECTION IN STOCHASTIC BLOCK MODELS VIA SPECTRAL METHODS

Laurent Massoulié (MSR-Inria Joint Centre, Inria)
based on joint work with:
Dan Tomozei (EPFL), Marc Lelarge (Inria), Jiaming Xu (UIUC)

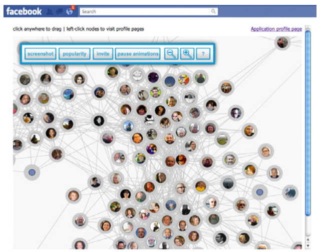
Community Detection

→ Identification of groups of similar objects within overall population
→ Closely related objectives: clustering and embedding



Application 1: contact recommendation in online social networks

Supporting data: e.g. OSN's friendship graph



→recommend members of user's implicit community

Application 2: content recommendation to users of Netflix-like system

Supporting data: user-content ratings matrix

User / Movie	f_1	f_2	…	f_m
u_1	?	**		***
u_2	***	?		?
…				
u_n	*****	**		**

Use content communities to support recommendation
"users who liked this also liked…"

Outline

- The Stochastic Block Model
  - With labels
  - With general types
- Performance of Spectral Methods
  - "rich signal" case
- The weak signal case: sparse observations
  - Phase transition on detectability
  - A modified spectral method

Outline

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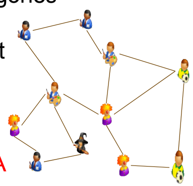
The Stochastic Block Model [Holland-Laskey-Leinhardt'83]

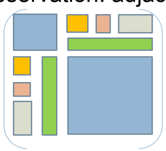
- n "nodes" partitioned into K categories
- Category σ: $\alpha_\sigma n$ nodes
- Edge between nodes u,v present

with probability $b_{\sigma(u)\sigma(v)}\, s/n$

s:"signal strength"



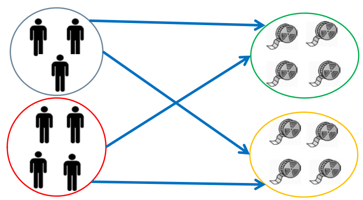
→ Observation: adjacency matrix A

A =  + Noise matrix

The Labeled Stochastic Block Model

- Edges (u-v) labeled by $L_{uv} \in L$ (finite set)
- Drawn from distribution $\mu_{\sigma(u)\sigma(v)}$

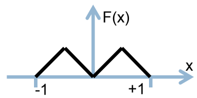
- Netflix case: labels 1-5 stars



The SBM with general types [Aldous'81; Lovász'12]

- User type σ(u) i.i.d. $\sim P$ in general set (e.g. uniform on [0,1])
- Edge (u-v) present w.p. $b_{\sigma(u)\sigma(v)}\ s/n$ for "kernel" b

  e.g. $b_{x,y} = F(x-y)$



- Edges (u-v) labeled by $L_{uv} \in L$ (finite set)
- Drawn from distribution $\mu_{\sigma(u)\sigma(v)}$

- Technical assumptions: compact type set and continuity of symmetric functions b and μ

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Spectral Clustering

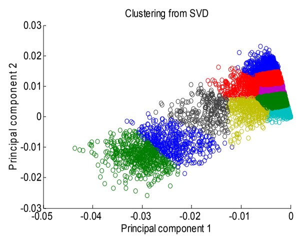
- From Matrix A extract R normalized eigenvectors x_i corresponding to R largest eigenvalues $|\lambda_1| \geq \cdots \geq |\lambda_R|$

- Form R-dimensional node representatives

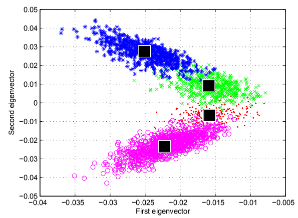
  $$y_u = \sqrt{n}\left(x_i(u)\right)_{i=1\ldots R}$$

- Group nodes u according to proximity of spectral representatives y_u

Illustration for R=2



Netflix dataset



SBM with K=4

Result for “logarithmic” signal strength s

Assume $s=\Omega(\log(n))$ and clusters are distinguishable, i.e.

$$\forall \sigma \neq \sigma' \ \exists \tau \text{ such that } b_{\sigma\tau} \neq b_{\sigma'\tau}$$

→Then spectrum of A consists of

- R eigenvalues λ_i of order $\Omega(s)$ ($R \leq K$) and
- n-R eigenvalues λ_i of order $O(\sqrt{s})$

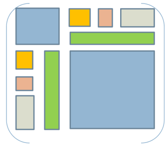
Node representatives y_u based on top R eigenvectors x_i :

Cluster according to underlying “blocks” except for negligible fraction of nodes

Proof arguments

Control spectral radius of noise matrix

+ perturbation of matrix eigen-elements

A =  + random “noise” matrix

Block matrix
non-zero eigenvalues: $\Theta(s)$

spectral separation properties “à la Ramanujan”

s-regular graph Ramanujan if

$$\lambda := \max(|\lambda_2|, |\lambda_n|) \leq 2\sqrt{s-1}$$

[Lubotzky-Phillips-Sarnak’88]

[Friedman’08]: random s-regular graph verifies whp

$$\lambda = 2\sqrt{s-1} + o(1)$$

[Feige-Ofek’05]: for Erdős-Rényi graph $G(n, s/n)$ and $s = \Omega(\log n)$, then whp $\lambda = O(\sqrt{s})$

Also: $\rho(A - \bar{A}) = O(\sqrt{s})$

spectral separation properties “à la Ramanujan”

Corollary: in SBM with $s = \Omega(\log n)$, whp $\rho(A - \bar{A}) = O(\sqrt{s})$ → A’s leading eigen-elements close to those of $\bar{A}$

For $s = \Theta(1)$, $\rho(A - \bar{A}) \sim C\sqrt{\frac{\log n}{\log \log n}}$

→spectral separation is lost

Result for "logarithmic" signal strength s – Labeled SBM

Random projection method: transform categorical labels into numerical data

For each label l generate W(l) i.i.d. uniform on [0,1]

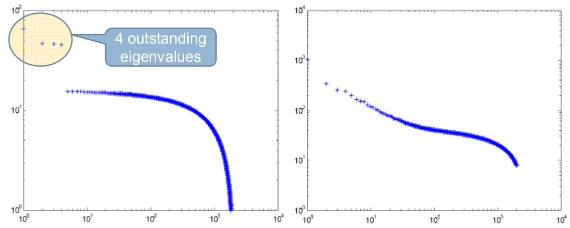
Perform Spectral clustering on matrix $\{A_{ij}W(L_{ij})\}$

→ Under modified distinguishability condition
$\forall \sigma \neq \sigma', \exists \tau, \ell$ such that $b_{\sigma\tau}\nu_{\sigma\tau}(\ell) \neq b_{\sigma'\tau}\nu_{\sigma'\tau}(\ell)$

Same result holds as in unlabeled scenario

Discrepancy between SBM with small K and Netflix

Eigenvalue distributions

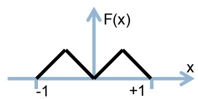


→motivates consideration of SBM with general types

SBM with general types

- User types σ(u) i.i.d. $\sim P$ from general set (e.g. uniform on [0,1])
- Edge (u-v) present w.p. $b_{\sigma(u)\sigma(v)}\ s/n$ for "kernel" b

e.g. $b_{x,y} = F(x-y)$



- Edges (u-v) labeled by $L_{uv} \in L$ (finite set)
- Drawn from distribution $\mu_{\sigma(u)\sigma(v)}$

→Form matrix $\{A_{ij}W(L_{ij})\}$ from random projections $W(l)$ of labels

SBM with general types: Spectral properties for logarithmic s

Define kernel $K(x,y) := \sum_l W(l) b_{xy}\mu_{xy}(l)$ and integral operator $Tf(x) := \int K(x,y)f(y)P(dy)$

→spectrum of $s^{-1}\{A_{ij}W(L_{ij})\} \approx$ spectrum of T

- Eigenvalue convergence: $s^{-1}\lambda_i^{(n)} \to \lambda_i$
- Eigenvector convergence: $x_i(u) \to \varphi_i\ (\sigma_u)$

Associated eigen-function

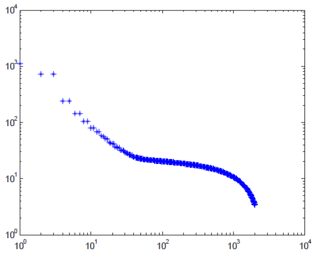
Type of node u

SBM with general types: Spectral properties for logarithmic s

→Flexible model

-power-law spectra (convolution operator + Fourier analysis)

-better matches to Netflix data



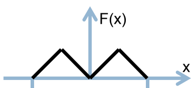
SBM with general types: estimation for logarithmic s

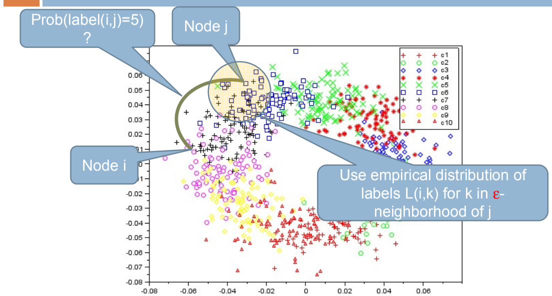
- For fixed R form R-dimensional node representatives

$$y_u = \sqrt{n}\left\{\frac{\lambda_k}{\lambda_1} x_k(u)\right\}_{k=1\ldots R}$$

→Embeds nodes according to pseudo-distance d_R that "captures geometry" of hidden node types $\sigma(u)$ with embedding accuracy controlled by "residual energy" $\varepsilon_R := \sum_{k>R} {\lambda_k}^2$ of operator's spectrum

Illustration with [0,1] types





Embedding allows consistent estimation of label distributions

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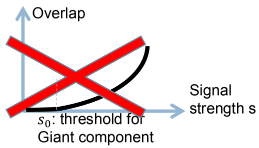
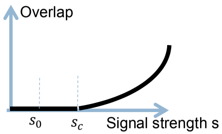
Weak signal strength: s = Θ(1)

- Correct classification of all but negligible fraction of nodes impossible (isolated nodes…)

→ Assess performance of clustering $\hat{\sigma}$ by *overlap* metric:

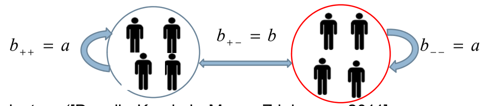
$$\text{ov}(\hat{\sigma}) = \frac{1}{n}\sum_{u=1}^{n} 1\{\sigma_u = \hat{\sigma}_u\} - \max_k(\alpha_k)$$





Weak signal strength : s=1

Symmetric two-communities scenario: $\alpha_+ = \alpha_- = \frac{1}{2}$



Conjecture ([Decelle-Krzakala-Moore-Zdeborova 2011]:

- For $\tau := \frac{(a-b)^2}{2(a+b)} < 1$, overlap tends to zero for any $\hat{\sigma}$

→Proven by [Mossel-Neeman-Sly 2012]

- For $\tau > 1$, positive overlap can be achieved

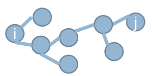
(by Belief Propagation [DKMZ 2011]; by "spectral redemption" [KMMNSZ-Zhang 2013])

No method proven to achieve positive overlap until Nov'13

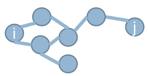
Detection by modified spectral method

Form matrix $B^{(l)}$: $B^{(l)}{}_{ij}$ = nb of self-avoiding paths of length l

Ex: for l=4



$B^{(l)}{}_{ij} = 1$



$B^{(l)}{}_{ij} = 2$

Typical case: for tree-shaped l-neighborhood of i, $B^{(l)}{}_{ij} = 1_{\{d(i,j)=l\}}$

Main result: spectral structure of $B^{(l)}$ for $\tau > 1$ & path length $l \sim c\log(n)$,

Let $\alpha = \frac{a+b}{2}$, $\beta = \frac{a-b}{2}$ (hence $\tau = \frac{\beta^2}{\alpha}$) eigenvalues of

a/2	b/2
b/2	a/2

- Top eigenvalue ~ $\tilde{\Theta}(\alpha^l)$, top eigenvector y: $|\langle y, B^{(l)}e\rangle| \sim |y|\,|B^{(l)}e|$
- 2nd eigenvalue ≥ $\tilde{\Omega}(\beta^l)$, 2nd eigenvector z: $|\langle z, B^{(l)}\sigma\rangle| \sim |z|\,|B^{(l)}\sigma|$
- 3rd eigenvalue = $O(n^{\varepsilon}\sqrt{\alpha^l})$ for all $\varepsilon > 0$

Spectral separation "à la Ramanujan"

- 2nd eigenvector z of $B^{(l)}$ positively correlated with spin vector σ

→Hence positive overlap obtained by estimate $\hat{\sigma}(u) = \begin{cases} +1 \text{ if } z_u\sqrt{n} > T \\ -1 \text{ if } z_u\sqrt{n} \le T \end{cases}$

For suitable threshold T

Proof elements 1) matrix expansion

- Expected adjacency matrix

$$\bar{A} = \frac{a}{n}\left[\frac{1}{2}(ee' + \sigma\sigma') - I\right] + \frac{b}{2n}(ee' - \sigma\sigma')$$

- Centered simple path adjacency matrix

$$\Delta_{ij}^{(\ell)} := \sum_{i_0^\ell \in P_{ij}} \prod_{t=1}^{\ell} (A - \bar{A})_{i_{t-1} i_t}$$

→Expansion: $B^{(\ell)} = \Delta^{(\ell)} + \sum_{m=1}^{\ell} (\Delta^{(\ell-m)} \bar{A} B^{(m-1)}) - \sum_{m=1}^{\ell} \Gamma^{\ell,m}$

"small" terms

"Smallness" of matrix coefficients

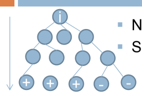
- Trace method: $\rho(M)^{2k} \leq \text{Trace}(M^{2k})$

+ combinatorics (à la [Füredi-Komlós'81])
Here: count contributions of concatenations of simple paths

→Bounds on spectral radii: whp, for all $\varepsilon > 0$

$$\rho(\Delta^{(\ell)}) \leq n^{\epsilon} \alpha^{\ell/2},$$
$$\rho(\Gamma^{\ell,m}) \leq n^{\epsilon - 1} \alpha^{(\ell+m)/2}, \; m = 1, \ldots, \ell.$$

Proof elements 2) Quasi-deterministic growth of node neighborhoods



- Nb of distance t neighbors: $S_t(i)$
- Sum of spins of distance t neighbors: $D_t(i)$

→then whp: $S_t(i) = \alpha^{t-l} S_l(i) + \tilde{O}(\alpha^{t/2})$

$D_t(i) = \beta^{t-l} D_l(i) + \tilde{O}(\alpha^{t/2})$

Proof: Chernoff bounds on binomial random variables

Corollary: For $m \leq l$, whp

Close to vectors $\{S_{m-1}(i)\}$, $\{D_{m-1}(i)\}$

$$\sup_{|x|=1, x'B^{(\ell)}e = x'B^{(\ell)}\sigma = 0} |e'B^{(m-1)}x| = \tilde{O}(\sqrt{n}\alpha^{(m-1)/2})$$
$$\sup_{|x|=1, x'B^{(\ell)}e = x'B^{(\ell)}\sigma = 0} |\sigma'B^{(m-1)}x| = \tilde{O}(\sqrt{n}\alpha^{(m-1)/2})$$

Weak Ramanujan property

- Previous results combined give

$$\sup_{|x|=1, x'B^{(\ell)}e = x'B^{(\ell)}\sigma = 0} |B^{(\ell)}x| \leq n^{\epsilon} \alpha^{\ell/2}.$$

Use spectral radius bounds

$$B^{(\ell)} = \Delta^{(\ell)} + \sum_{m=1}^{\ell} (\Delta^{(\ell-m)} \bar{A} B^{(m-1)}) - \sum_{m=1}^{\ell} \Gamma^{\ell,m}$$

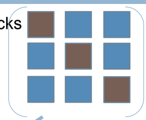
Express in terms of e, σ: $\bar{A} = \frac{a}{n}\left[\frac{1}{2}(ee' + \sigma\sigma') - I\right] + \frac{b}{2n}(ee' - \sigma\sigma')$

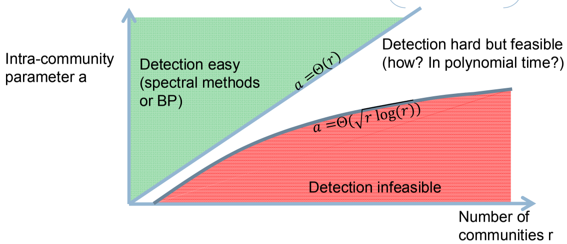
Use bounds from quasi-deterministic growth: $\sup_{|x|=1, x'B^{(\ell)}e = x'B^{(\ell)}\sigma = 0} |e'B^{(m-1)}x|$, $\sup_{|x|=1, x'B^{(\ell)}e = x'B^{(\ell)}\sigma = 0} |\sigma'B^{(m-1)}x|$

Remaining mysteries about SBM’s (1)

Conjectured “phase diagram” for more than 2 blocks

(assuming fixed inter-community parameter b)

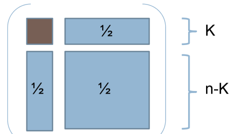




Remaining mysteries about SBM’s (2)

Clique detection problem: add a size-K clique to random graph with edge-probability ½

i.e. a 2-block SBM with unbalanced block sizes:



→for $K = \Omega(\sqrt{n})$ clique easily detectable (e.g. inspection of node degrees)

→are there polynomial-time algorithms for smaller yet large K?

(e.g. $K = \Theta(\sqrt[3]{n})$)

A notoriously hard problem (“planted clique detection” recently proposed as a new benchmark of algorithmic hardness)

Conclusions and Outlook

- “Vanilla” spectral methods efficient for strong (logarithmic) signal strength
- Alternatives needed at low signal strength
  - Belief propagation conjectured optimal
  - Spectral approach on path-expanded matrix proven optimal down to “easy/hard” transition
- Computationally efficient methods for “hard” cases?
  - Detection in SBM = rich playground for analysis of computational complexity with methods of statistical physics
- Does SBM model correctly real-life data?
- Speed of convergence, better-than-random label projections, choice of embedding dimension…

References

- D. Tomozei, L.M., distributed user profiling via spectral methods, ACM Sigmetrics’10
- M. Lelarge, L.M., J. Xu, Reconstruction in the labelled stochastic block model, ITW’13
- J. Xu, L.M., M. Lelarge, Edge label inference in generalized SBM: from spectral theory to impossibility results, COLT’14
- L.M., Community detection thresholds and the weak Ramanujan property, ACM STOC’14

SBM with general types: estimation for logarithmic s

Define Distance $d^2(x,y) = \int [K(x,z) - K(y,z)]^2 P(dz)$

- captures model structure
- Verifies $0 \leq d_R \leq d$
- And $\iint \left[d^2(x,y) - {d_R}^2(x,y)\right] P(dx)P(dy) = \varepsilon_R$

Consistency result for logarithmic s

Inference of label distribution based on

- R-dimensional embedding
- Empirical measures on ε-neighborhoods

For fraction of $1 - \sqrt{\varepsilon_R}$ node pairs, estimation error verifies

$$\lim_{\varepsilon \to 0} \left(\lim_{\varepsilon_R \to 0} \text{Error} \right) = 0$$

COMPLEX ENERGY SYSTEMS

Michael Chertkov, Los Alamos National Laboratory, USA

Today's energy systems, such as electric power grids and gas grids, already demonstrate complex nonlinear dynamics where, e.g., collective effects in one exert uncertainty and irregularities on other. These collective dynamics are not well understood and are expected to become more complex tomorrow as the grids are pushed to reliability limits, interdependencies grow, and appliances become more intelligent and autonomous. Tomorrow's will have to integrate the intermittent power from wind and solar farms whose fluctuating outputs create far more complex stress on power grid operations, often dependent, e.g. in providing fast regulation control, on the gas supply. Conversely, one anticipates significant effect of the wind-following gas fired turbines on reliability of the gas grid. Guarding against the worst of those perturbations will require taking protective measures based on ideas from optimization, control, statistics and physics.

In this talk we introduce a few of the physical, optimization and control principles and phenomena in today's energy grids and those that are expected to play a major role in tomorrow's grids.

We illustrate the new science of the energy grids on three examples: (a) discussing an efficient and highly scalable Chance Constrained Optimal Power Flow algorithm providing risk-aware control of the power transmission system under uncertainty associated with fluctuating renewables (wind farms); (b) describing effect of the intermittent power generation on reliability and compression control of the gas grid operations; and (c) briefly discussing examples of interdependencies, reliability troubles and solutions in the low level (distribution) grids.

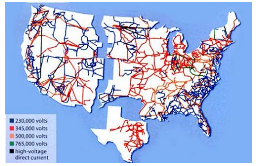
Complex Energy Systems

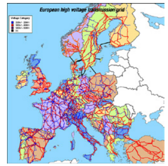
Misha Chertkov
LANL/DOE:OE + LANL/DTRA & NMC/NSF:ECCS

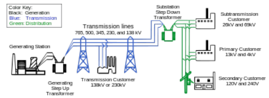
Los Alamos National Laboratory & New Mexico Consortium

Lund, Oct 15, 2014

Power Grids

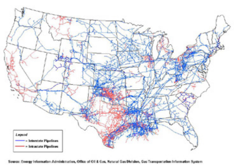


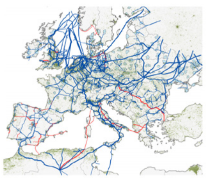




- Power Grids = the greatest engineering achievement of 20th century [IEEE]
- Require smart revolution in 21th century

Gas Grids





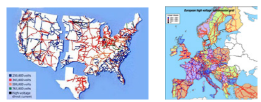
- "smart grids" are not limited to power, should also include other energy grids, e.g. gas grids
- gas networks are younger and less mature
- gas use is expected to grow

energy grids = electric+gas+heat

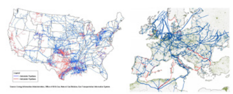
- Energy Hubs (local)
- Energy Interconnections (long distance)
- Vision of Future Energy Networks (green field)

Theme(s) of the talk

- Power and Gas Grids = Basic Energy Grids
- **Fluctuations** & perturbances test the grids
- Calling for new measures and understanding of **reliability**
- Need to deal with emerging **interdependencies**



power grids



gas grids

Complex Energy Systems

Outline

Power System Reliability: from Instanton to Chance Constrained OPF

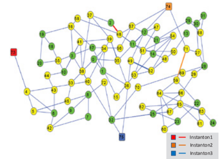
Gas System Reliability: from OGF to controlling pressure fluctuations

Fault Induced Delayed Voltage Recovery: a power distribution trouble

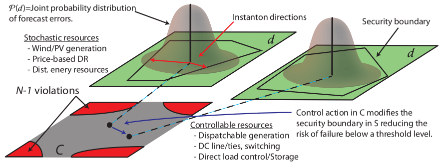
Complex Energy Systems
└Power System Reliability: from Instanton to Chance Constrained OPF
└Instantons = Probabilistic Measure of Uncertainty

Reliability Measure of Power System Under Uncertainty

- Stochastic/uncontrollable participants (e.g. renewables) fluctuate
- Just the standard "N-1"-security gives no guarantees under uncertainty

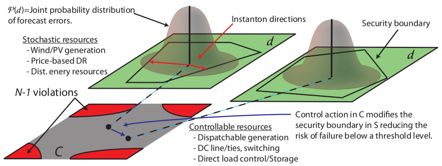


Instantons in Power Systems: MC, F. Pan, M. Stepanov (2010); MC, FP, MS, R. Baldick (2011); S.S. Baghsorkhi, I. Hiskens (2012)



Complex Energy Systems
└Power System Reliability: from Instanton to Chance Constrained OPF
└Instantons = Probabilistic Measure of Uncertainty

Reliability Measure of Power System Under Uncertainty



- **Step one (distance to failure)**: compute the instantons to find the total probability of stochastic failure = $\mathcal{P}_{\text{fail}}$
- **Step two**: if $\mathcal{P}_{\text{fail}}$ > threshold re-dispatch controllable resources so that $\mathcal{P}_{\text{fail}}$ < threshold at minimum cost

Step two — can be built in real-time operations

- e.g. E. Karangelos, P. Panciatici, L. Wehenkel, *Whither probabilistic security management for real-time operation of power systems?*, IREP 2013

Complex Energy Systems
└Power System Reliability: from Instanton to Chance Constrained OPF
└ Chance Constrained Optimum Power Flows

Chance Constrained Re-dispatch

- Or ... instead of **Steps one and two** one can follow another path ⇒
- **Incorporate probabilistic security directly into optimization**

Chance Constrained Re-dispatch

- Or ... instead of **Steps one and two** one can follow another path ⇒
- **Incorporate probabilistic security directly into optimization**

D. Bienstock, MC, S. Harnett (Columbia/LANL) SIAM Review, Aug 2014
R. Bent, DB, MC http://arxiv.org/abs/1306.2972

- **CC-OPF** = make sure that generation is re-dispatched at minimum cost such that $\forall failures: \mathcal{P}_{\text{failure}} <$ threshold

Related, independent work

- E. Sjodin, D. F. Gayme and U. Topcu, *Risk-Mitigated Optimal Power Flow for Wind Powered Grids*, ACC 2012.
- L. Roald, F. Oldewurtel, T. Krause and G. Andersson, *Analytical Reformulation of Security Constrained Optimal Power Flow with Probabilistic Constraints*, Proceedings of the Grenoble PowerTech, Grenoble, France, June 2013.
- M. Vrakopoulou, K. Margellos, J. Lygeros and G. Andersson, *A Probabilistic Framework for Reserve Scheduling and N-1 Security Assessment of Systems with High Wind Power Penetration*, to appear IEEE Transactions on Power Systems.

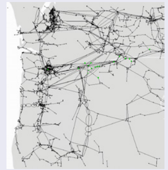
Standard re-dispatch = Optimum Power Flow

Constrained (thermal + generation limits) OPF

$$\min_{p,\theta} \underbrace{c(p)}_{\text{cost of generation}} \left| \begin{array}{ll} B\theta = p - d & \text{[Power flows]} \\ \beta_{ij}(\theta_i - \theta_j) \le u_{ij},\ \forall (i,j) & \text{[Thermal limits]} \\ P_g^{min} \le p_g \le P_g^{max},\ \forall g & \text{[Generation constraints]} \end{array} \right.$$

- $p =$ vector of generations $\in \mathcal{R}^n$, $d =$ vector of loads $\in \mathcal{R}^n$
- $B \in \mathcal{R}^{n\times n}$ — — bus susceptance matrix

- also called tertiary control; done by SO every 5-30 min
- DC-approximation [AC-generalizable]
- may also account for "standard" security (list of contingencies)

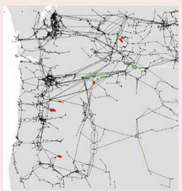


How does OPF handle (renewable) fluctuations?

- Automatic frequency control: primary [seconds] + secondary [AGC, 1-2 minute]
- Generator output varies up or down **proportionally** to **aggregate** change

Experiment: Bonneville Power Administration data, Northwest US

- data on wind fluctuations at planned farms
- with standard OPF, 7 lines exceed limit ≥ 8% of the time



Want to improve the standard OPF

- standard automatic control [affine, possibly changing rates]
- aware of security (limits)
- not too conservative
- computationally practicable

OPF vs Chance Constrained-OPF

Standard OPF (Dispatch for the mean forecast, not aware of fluctuations)

$$\min_p \underbrace{c(p)}_{\text{cost of generation}} \quad \Bigg| \quad \begin{array}{c}\text{Power Flow Eqs.}\\ \text{Generation limits}\\ \text{Power Flow Thermal Limits}\end{array}$$

Chance Constrained OPF (fluctuations aware dispatch)

- $\min_{\bar p,\alpha} \mathbb{E}\left[c(\bar p,\alpha)\right]$ | Power Flow Eqs. [for mean forecast]; Chance Constraints for Generation; Chance Constraints for line Flows
- Chance Constraints for Line Flows: $\forall (i,j) \in \mathcal{E}: \quad \text{Prob}(|f_{ij}| > f_{ij}^{max}) < \varepsilon_{ij}$.
  Interpretation: overload is allowed for ϵ-fraction of "time".
- $\bar p$ - generation re-dispatch for beginning of the period; α - proportional rates for the period

Chance Constrained OPF (fluctuations aware dispatch)

- $\min_{\bar p,\alpha} \mathbb{E}\left[c(\bar p,\alpha)\right]$ | Power Flow Eqs. [for mean forecast]; Generation satisfies Chance Constraints; Line Power Flows satisfy Chance Constraints
- Chance Constraints for Line Flows: $\forall (i,j) \in \mathcal{E}: \quad \text{Prob}(|f_{ij}| > f_{ij}^{max}) < \varepsilon_{ij}$

▸ CC-OPF detailed formulation

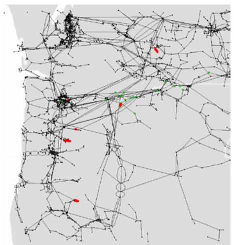
More Technical Details [it is NOT sampling/MCMC]

- Assuming **site-independent**, Gaussian fluctuations enables explicit evaluation [formula] of chance constraints for given $\bar p, \alpha$
- The resulting (after averaging) problem is a **convex (conic) optimization** ▸ [details]
- Constraint violations are few/sparse. **Cutting Plane** method greatly speeds up the optimization ▸ [details]

Back to motivating example

BPA case

- standard OPF: cost 235603, 7 lines unsafe $\geq$ 8% of the time
- CC-OPF: cost 237297, every line safe $\geq$ 98% of the time
- run time = 9.5 seconds (one cutting plane!)



Back to motivating example

BPA case

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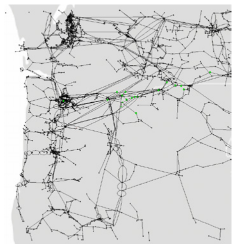


▸ Experiments with CC-OPF

- **CC-OPF succeeds where standard OPF fails**
- Cost of Reliability [CC-OPF saving over standard OPF]
- CC-OPF is not a naive fix. [Changes are nonlocal]
- ...

Performance of the Method

Performance of cutting-plane method on a number of large cases.

Case	Buses	Generators	Lines	Time (s)	Iterations	Barrier iterations
BPA	2209	176	2866	**5.51**	2	256
Polish1	2383	327	2896	**13.64**	13	535
Polish2	2746	388	3514	**30.16**	25	1431
Polish3	3120	349	3693	**25.45**	23	508

Rapid convergence on realistic networks

Typical convergence behavior of cutting-plane algorithm on a large instance.

Iteration	Max rel. error	Objective
1	1.2e-1	7.0933e6
4	1.3e-3	7.0934e6
7	1.9e-3	7.0934e6
10	1.0e-4	7.0964e6
12	8.9e-7	7.0965e6

details

Enhancements of CC-OPF

Out of Sample Tests – can handle either of the two cases

- True distribution is non-Gaussian, but our Gaussian distribution is close
- Parameters of the Gaussian distributions, μ_i, σ_i^2 are mis-estimated

details

Robust (ambigious) CC-OPF

- CC-OPF which is robust with respect to parameters of the Gaussian distribution from a range
- Allows convex tractable reformulation

“Non-linear” OPF & sync-CC OPF [R. Bent, D. Bienstock, MC 2013]

- Convex AC-OPF (losseless, constant voltage - based on an exercise from Boyd, Vandenberghe [book] details)
- Synchronization constrained CC-OPF (based on MC work with F. Dorfler & F. Bullo [PNAS, 2013] details)
- Voltage constraints — work in progress

Summary (CC-OPF) + Extensions:

- DC PF + affine control + independent fluctuations ⇒ conic (tractable) optimization
- Specialized cutting-plane algorithm proves effective
- Commercial solvers do not
- Algorithm efficient even in cases with thousands of buses/lines
- Algorithm can be made robust with respect to data errors
- Allows to account for synchronization constraints

Path Forward (work in progress)

- AC generalizations, convexifications (e.g. FDPF, Energy Function based appr.)
- Ramp Constraints
- Multi-stage CC-OPF
- combined CC-OPF & Unit Commitment

- Consider CC-OPF, or other type of dispatch, responding to uncertainty/wind
- Gas turbines [fast to ramp up/down, relatively clean] are

  - **producers** of electricity: follow wind
  - **consumers** of gas: inducing/transfering fluctuations/stress to the gas network

- Study interdependencies ...
- Start from analysis of gas system under uncertainty (e.g. caused by the wind-induced correlations)

Complex Energy Systems
└ Gas System Reliability: from OGF to controlling pressure fluctuations

2 min crash course on the hydro (gas) dynamics

- single pipe; not tilted (gravity is ignored); constant temperature
- ideal gas, $p \sim \rho$ – pressure and density are in a linear relation
- all fast transients are ignored – gas flow velocity is significantly slower than the speed of sound, $u \ll c_s$
- turbulence is modeled through turbulent friction; mass flow, $\phi = u\rho$, are averaged across the pipe's crossection

$$\left.\begin{array}{c}\underbrace{\partial_t\rho + \partial_x(u\rho) = 0}_{\text{conservation of mass}} \\ \underbrace{\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = -\frac{\rho u|u|}{2D} f}_{\text{conservation of energy}}\end{array}\right\} \approx\Rightarrow \begin{array}{c}\underbrace{c_s^{-2}\partial_t p + \partial_x\phi = 0}_{\text{conservation of mass}} \\ \underbrace{\partial_x p^2 + \frac{\beta}{D}\phi|\phi| = 0}_{\text{conservation of energy}}\end{array}$$

Approximations ... allowing to resolve flows analytically (lamp description)

Stationary, balanced regime [standard]

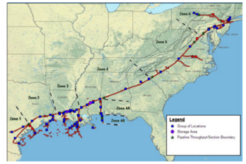
$\phi = \text{const}, \quad p_{in}^2 - (p(x))^2 = x\beta\phi|\phi|/D$

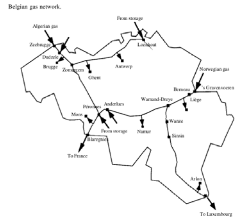
Unbalanced, linearized line-pack [non-standard]

$\phi = \phi_{st}(x) + \delta\phi(t,x), \quad p = p_{st}(x) + \delta p(t,x)$

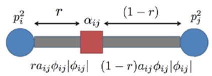
Complex Energy Systems
└ Gas System Reliability: from OGF to controlling pressure fluctuations

Gas Flows. Steady (balanced) Case.





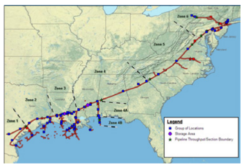
without compressors, $\alpha_{ij} = 1$

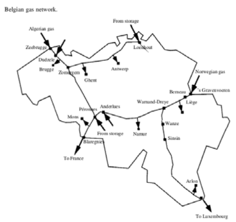


- Gas Flow Equations: $(\sum_i q_i = 0, \quad a_{ij} = L_{ij}\beta_{ij}/D_{ij})$
  $\forall (i,j): \quad p_i^2 - p_j^2 = a_{ij}\phi_{ij}^2$
  $\forall i: \quad q_i = \sum_{j:(i,j)\in\mathcal{E}} \phi_{ij} - \sum_{j:(j,i)\in\mathcal{E}} \phi_{ji}$

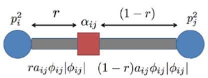
Complex Energy Systems
└ Gas System Reliability: from OGF to controlling pressure fluctuations

Gas Flows. Steady (balanced) Case.





with compressors, $\alpha_{ij} \lesssim 1$



- Gas Flow Equations: $(\sum_i q_i = 0, \quad a_{ij} = L_{ij}\beta_{ij}/D_{ij})$
  $\forall (i,j): \quad \alpha_{ij}^2 = \frac{p_j^2 + (1-r)a_{ij}\phi_{ij}^2}{p_i^2 - ra_{ij}\phi_{ij}|\phi_{ij}|}$
  $\forall i: \quad q_i = \sum_{j:(i,j)\in\mathcal{E}} \phi_{ij} - \sum_{j:(j,i)\in\mathcal{E}} \phi_{ji}$

Complex Energy Systems
└ Gas System Reliability: from OGF to controlling pressure fluctuations

Optimal Gas Flow

Minimizing the cost of compression ($\sim$ work applied externally to compress)

$$\min_{\alpha,p} \sum_{(i,j)} \frac{c_{ij}\phi_{ij}}{\eta_{ij}}\left(\alpha_{ij}^m - 1\right)^+ \Bigg| \begin{array}{l}\forall (i,j): \quad \alpha_{ij}^2 = \frac{p_j^2 + (1-r)a_{ij}\phi_{ij}^2}{p_i^2 - ra_{ij}\phi_{ij}^2} \\ \forall i: \quad 0 \le \underline{p}_i \le p_i \le \overline{p}_i \\ \forall (i,j): \alpha_{ij} \le \overline{\alpha}_{ij}\end{array}$$

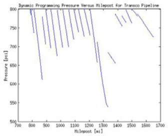
- $0 < m = (\gamma-1)/\gamma < 1$, γ- gas heat capacity ratio (thermodynamics)

- The problem is **convex on trees** (many existing gas transmission systems are trees) $\Leftarrow$ through GeometricProgramming (log-function transformation)

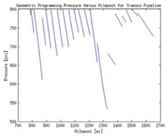
- **S. Misra, M. W. Fisher, S. Backhaus, R. Bent, MC, F. Pan**, *Optimal compression in natural gas networks: a geometric programming approach*, IEEE TCNS 2014

OGF experiments (Transco pipeline)

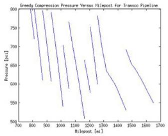
Dynamic Programming of (Wong, Larson '68)



Geometric Programming (ours)



Greedy Compression (current practice)



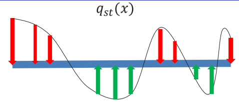
GP is advantageous over DP

- Exact = no-need to discretize.
- Faster. Allows distributed (ADMM) implementation.

- Convexity is lost in the loopy case. However, an efficient heuristics is available. [work in progress]
- This is only one of many possible OGF formulations. Another (Norwegian/European) example – maximize throughput.
- Major handicap of the formulation (ok for scheduling but) = did not account for the **line pack** (dynamics/storage in lines for hours) ⇒

Dynamic Gas Flows (with Line Pack) – Formulation

$q_{st}(x)$



Steady (balanced) continuous profile of gas injection/consumption

- $q(t,x) = q_{st}(x) + \xi(t,x), \quad \xi(t,x) \ll q_{st}(x)$
  - $q_{st}(x)$ is the forecasted consumption/injection of gas
  - $\xi(t,x)$ actual fluctuating/random profile of consumption/injection, e.g. **fluctuations due to gas power plants following wind turbines**

One dimensional (1+1) model – distributed injection/consumption and compression

- mass balance: $c_s^{-2}\partial_t p + \partial_x \phi = -q(t,x)$
- energy balance: $\partial_x p + \frac{\beta}{2d}\frac{\phi|\phi|}{p} = \gamma(x) p$
- $\gamma(x)$ – distributed compression – assumed known

- generalized to an arbitrary graph
- **S. Backhaus, MC, and V. Lebedev, PNAS submitted**

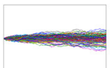
- Describe **spatio-temporal fluctuations of actual pressure** (unbalanced/line pack) on the top of the steady/optimized/inhomogeneous forecast

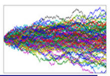
Dynamic Gas Flows (with Line Pack) – Solution

Analytic expressions for the line pack – assuming $\delta p(t) \ll p_{st}, \quad |\delta\phi| \ll |\phi_{st}|$

- analytical (!!) solution:
  - $\delta p(t,x) \approx -Z(x)\frac{c_s^2}{LY}\int_0^t dt' \int_0^L dx\, \xi(t',x)$
  - $Z(x) = \exp\left(\int_0^x dy \frac{\beta\phi_{st}(y)|\phi_{st}(y)|}{d\, p_{st}^2}\right), \quad Y = \int_0^L dx\, Z/L$

- $\delta p(t,x)$ is random zero mean Gaussian
- line pack jitters = grows "diffusively" with time
- the growth rate of the pressure fluctuations, $\sim Z(x)$, is non-uniform, depends (only) on the stationary solution
- $\mathbb{E}\left[\delta p(t,x)^2\right] \to \frac{c_s^4 \tau t}{L^2}\left(\frac{Z(x)}{Y}\right)^2 \int\int_0^L dx_1\, dx_2 \mathbb{E}\left[\xi(t,x_1)\xi(t,x_2)\right]$





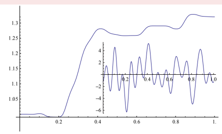
... all of the above is true ...

- When either correlation time or correlation scale of $\xi(t,x)$ is sufficiently short, i.e. $\tau \ll T$ (say minutes vs hours) or $l \ll L$ (say 10km vs 1000km)
- and $\xi(t,x)$ is zero mean and statistically stationary

Spatially Inhomogeneous **Line Pack Jitter**

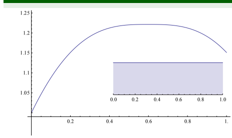
$$\mathbb{E}\left[\delta p(t,x)^2\right] \to \frac{c_s^4 \tau t}{L^2}\left(\frac{Z(x)}{Y}\right)^2 \int\int_0^L dx_1\, dx_2 \mathbb{E}\left[\xi(t,x_1)\xi(t,x_2)\right]$$

Local maxima at the points of flow reversals

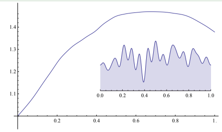


- $\sim Z^2(x)$ [controlling the "frozen" part of the pressure covariance] is shown
- $q_{st}(x)$ is shown in inset – distributed injection/consumption, $q_{st}(0) = q_{st}(L) = 0$
- $\gamma(x)$ is chosen to get p_{st} = const

Injection on two sides of the pipe enhancement and shift of the maximum



uniform consumption



inhomogeneous consumption

Re-cap of the OGF & pressure reliability studies

- Geometric programming offers an efficient way of solving the steady/balanced OGF over tree structures
- Dynamic and stochastic GF (with line pack) is solved perturbatively. Shows diffusive, spatially inhomogeneous **line pack jitter** – extremal at the points of the flow reversal.

Path Forward

- Steady OGF over graphs with loops
- Other OGF formulations, e.g. max-throughput
- Generalize stochastic-line-pack-GF to discrete models with loops
- Extend it to reliability-aware stochastic and non-stationary optimizations

- Fluctuations in power sources (renewable and interdependencies) lead to more frequent (then in the past) interruptions
- In particular voltage faults
- Cleared faults do not cause major direct effect on balanced transmission

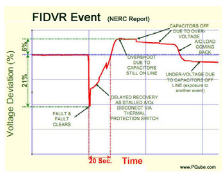
However, ...

- the faults (in spite of being cleared fast) may be of high risks for power distribution

Recordered Distribution/Transmission Voltage Events

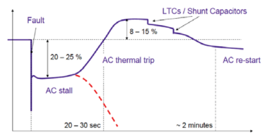
- TVA. Blistering Sat of Aug 22, 1987. Cascading Voltage Collapse in West Tennissee. Fault at 115KV switch. Cleared in 1s. Continued into 161KV and 500KV lines for 10-15s. Resulted in the loss of 700MW in Memphis. Motor loads stalled and drawn large amount of reactive power even after the fault was cleared.
- 1988 event in Florida reported in "Air Conditioner Respond to Transmission Fault" by J. W. Shaffer in 1997 ... "In the last ten years there have been at least eight events in which normally cleared (in 2-3 cycles) multi-phase events in Southern Florida have caused a significant drop in customer load (200-825MW)."
- 1990 Egypt ... 1999 metro area Atlanta, Arizona, Southern California ... NERC Planning Committee White Paper on "Fault Induced Delayed Voltage Recovery" by Transmission Issues Subcommittee

- delays (between cause and the result)
- nonlinearity of loads plays a significant role
- many inductive motors simultaneously affected
- initiated (fault) at the transmission-to-distribution interface, maturates within distribution, cascades into transmission



Typical FIDVR Following a 230-kV Transmission Fault in Southern California

Modeling extended FIDVR C.Duclut, S.Backhaus & MC (PRE '12)



courtesy of D. Kostyrev and B. Lesieutre

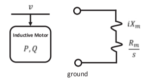
- Observed in feeders with many induction motors (air-conditioning)
- Uncontrolled depressed voltage can spread causing a larger outage
- Hypothesis (Hiskens, Lesieutre, Chassin, ...): the events are caused by many air conditioners stalled
- Modeling the event is a challenge

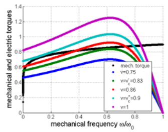
Our Contribution - Modeling of FIDVR over extended feeder

- Observation (simulations – consistent with measurements): soliton-like propagation of "stalled" phase/front
- Coarse-grained (reduced) PDE modeling of the "extended" FIDVR

Modeling Individual Motor — Popovic, Hiskens, Hill '98

minimal model of the motor





$$P = \frac{sR_m v^2}{R_m^2 + s^2 X_m^2}$$
$$Q = \frac{s^2 X_m v^2}{R_m^2 + s^2 X_m^2}$$
$$M\frac{d}{dt}\omega = \frac{P}{\omega_0} - T_0(\omega/\omega_0)^\alpha \text{ (dynamics)}$$
$$s = 1 - \omega/\omega_0$$

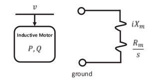
s is the slip against the base frequency)
v is the voltage at the motor
P, Q are real and reactive power consumed by the motor
T_0, α torque constant and scaling coefficient
R_m, X_m resistance and inductance of the motor

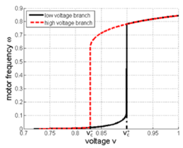
Explanation for "lumped" FIDVR

- Hysteresis: The motor is trapped in the stalled (low-voltage) state!
- First order phase transition. Bifurcation (stability). Spinodal points.

Modeling Individual Motor — Popovic, Hiskens, Hill '98

minimal model of the motor





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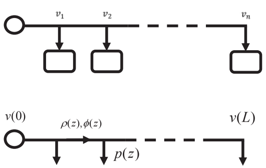
Feeder with Many (distributed) Inductive Motors



- Spatially-continuous version of Dist.Flow [Baran, Wu (1989)]

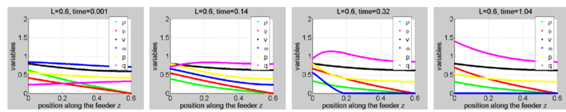
$$\partial_z \rho = -p - r\frac{\rho^2 + \phi^2}{v^2}$$
$$\partial_z \phi = -q - x\frac{\rho^2 + \phi^2}{v^2}$$
$$v\partial_z v = -(r\rho + x\phi)$$
$$p = \frac{sr_m v^2}{r_m^2 + s^2 x_m^2}$$
$$q = \frac{s^2 x_m v^2}{r_m^2 + s^2 x_m^2}$$
$$\mu\frac{d}{dt}\omega = \frac{p}{\omega_0} - \tau_0\left(\frac{\omega}{\omega_0}\right)^\alpha$$
$$v(0) = 1,\ \rho(L) = \phi(L) = 0$$

Reduced model of the "extended" feeder

Easy to analyze dynamics: PDE.

Dynamics/Transitions in an Extended Feeder (I)

Example of a Large Fault → feeder is stalled (Movie Large Fault)

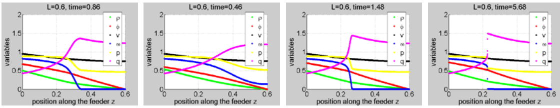


(a) (b) (c) (d)

- (a) Pre fault
- (b) Immediately past fault
- (c) Later in the process
- (d) The feeder is fully stalled

Dynamics/Transitions in an Extended Feeder (II)

Example of a Small Fault → feeder is partially stalled (Movie Small Fault)



(a) (b) (c) (d)

- (a) Immediately past fault
- (b) Later in the process
- (c) Front advances
- (d) Stabilized, part. stalled

▸ dynamics of restoration

- The 1+1 (space+time) continuous model of distribution
- Integrating multiple bi-stable individual motors into power flow
- Emergence of multiple spatially-extended states/transitions

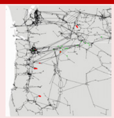
Conclusions Drawn from Experiments/Numerics concern

- Hysteresis
- Self-Similar Transients

... to be done ...

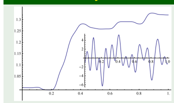
- Inhomogeneity (disorder), stochasticity (noise): what is the probability that the feeder with a given level of disorder will recover?
- Effects of other devices, e.g. distributed generation and control (PV) ...
- Possible cascade – from feeder to feeder (within substation) ... to transmission
- What is the least control effort needed to avoid a FIDVR event/cascade following a given type of fault? ⇐ Voltage control through
  - tap changers, e.g. incorporating in the framework of "primary voltage control of active distributed networks" by Christakou, Tomozei, Le Boudec, Paolone (2014)
  - inverters [distributed power electronics]

Chance Constrained OPF



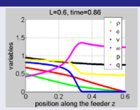
- New reliability measure for uncertainty – the instantons
- Efficient and Scalable Chance-Constrained OPF

Gas Reliability with Line Pack



- Geometric/Signomial Programming for Optimum Gas Flow
- Line-Pack Jitter of Pressure Fluctuations

Distributed Fault Induced Delayed Voltage Recovery



- Soliton-like front describing FIDVR
- PDE approach ⇒ coarse-grained estimation

Take Home High-level message illustrated today on a few examples

New Science of Complex Power/Energy System Engineering

- Old approach focused on individual devices, deterministic – remained valid ... but
- New complexities (renewables, fluctuations, interdependencies) need to be controlled through ...
- Better understanding of the multi-scale, probabilistic science (applied physics/math, operation research, other IT disciplines) ...
- To enable better practical control and optimization of tomorrow grids

Complex Energy Systems

Take Home High-level message illustrated today on a few examples

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Thank You!

Complex Energy Systems
└Appendix: CC-OPF
 └DC approximation

DC-approximation

(0) The amplitude of the complex potentials are all fixed to the same number (unity, after trivial re-scaling): $\forall a: \quad u_a = 1$.

(1) $\forall \{a,b\}: \quad |\varphi_a - \varphi_b| \ll 1$ - phase variation between any two neighbors on the graph is small

(2) $\forall \{a,b\}: \quad r_{ab} \ll x_{ab}$ - resistive (real) part of the impedance is much smaller than its reactive (imaginary) part. Typical values for the r/x is in the $1/27 \div 1/2$ range.

(3) $\forall a: \quad p_a \gg q_a$ - the consumed and generated powers are mainly real, i.e. reactive components of the power are much smaller than their real counterparts

It leads to

- Linear relation between powers and phases (at the nodes): $\hat{B}\varphi = \boldsymbol{p}$
- Losses are ignored: $\sum_a p_a = 0$
- B - graph Laplacian constructed of line susceptances

◂ DC-OPF

Complex Energy Systems
└Appendix: CC-OPF
 └CC-OPF – details

Frequency Control (quasi-static proxy)

For each generator i, two parameters:

- $\overline{p}_i$ = mean output
- α_i = response parameter

Real-time output of generator i:

$$p_i = \overline{p}_i - \alpha_i \sum_j \Delta\omega_j$$

where $\Delta\omega_j$ = change in output of renewable j (from mean).

$$\sum_i \alpha_i = 1$$

$\sim$ primary + secondary control

◂ Chance Constrained OPF

Complex Energy Systems
└Appendix: CC-OPF
 └CC-OPF – details

Computing line flows

wind power at bus i: $\mu_i + \boldsymbol{w}_i$

DC approximation

- $B\boldsymbol{\theta} = \overline{p} - d$ $+(\mu + \boldsymbol{w} - \alpha \sum_{i\in G} \boldsymbol{w}_i)$
- $\boldsymbol{\theta} = B^{+}(\bar{p} - d + \mu) + B^{+}(I - \alpha e^{T})\boldsymbol{w}$
- flow is a linear combination of bus power injections:

$$\boldsymbol{f_{ij}} = \beta_{ij}(\boldsymbol{\theta}_i - \boldsymbol{\theta}_j)$$

◂ Chance Constrained OPF

Computing line flows

$$\boldsymbol{f_{ij}} = \beta_{ij}\left((B_i^+ - B_j^+)^T(\bar{p} - d + \mu) + (A_i - A_j)^T \boldsymbol{w}\right),$$

$$A = B^+(I - \alpha e^T)$$

Given distribution of wind can calculate moments of line flows:

- $E\boldsymbol{f_{ij}} = \beta_{ij}(B_i^+ - B_j^+)^T(\bar{p} - d + \mu)$
- $var(\boldsymbol{f_{ij}}) := s_{ij}^2 \geq \beta_{ij}^2 \sum_k (A_{ik} - A_{jk})^2 \sigma_k^2$ (assuming independence)
- and higher moments if necessary

◂ Chance Constrained OPF

Chance constraints to deterministic constraints

- chance constraint: $P(\boldsymbol{f_{ij}} > f_{ij}^{max}) < \epsilon_{ij}$ **and** $P(\boldsymbol{f_{ij}} < -f_{ij}^{max}) < \epsilon_{ij}$
- from moments of $\boldsymbol{f_{ij}}$, can get conservative approximations using e.g. Chebyshev's inequality
- for Gaussian wind, can do better, since $\boldsymbol{f_{ij}}$ is Gaussian :

$$|E\boldsymbol{f_{ij}}| + var(\boldsymbol{f_{ij}})\phi^{-1}(1 - \epsilon_{ij}) \leq f_{ij}^{max}$$

◂ Chance Constrained OPF

Formulation [convex!]:

Choose mean generator outputs and control to minimize expected cost, with the probability of line overloads kept small.

$$\min_{\overline{p},\alpha} \mathbb{E}[c(\overline{p})]$$

$$\text{s.t.} \sum_{i \in G} \alpha_i = 1, \ \alpha \geq 0$$

$$B\delta = \alpha, \delta_n = 0$$

$$\sum_{i \in G} \overline{p}_i + \sum_{i \in W} \mu_i = \sum_{i \in D} d_i$$

$$\overline{f}_{ij} = \beta_{ij}(\overline{\theta}_i - \overline{\theta}_j),$$

$$B\overline{\theta} = \overline{p} + \mu - d, \ \overline{\theta}_n = 0$$

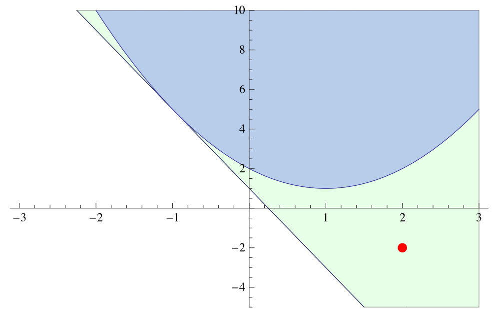
$$s_{ij}^2 \geq \beta_{ij}^2 \sum_{k \in W} \sigma_k^2 (B_{ik}^+ - B_{jk}^+ - \delta_i + \delta_j)^2$$

$$|\overline{f}_{ij}| + s_{ij}\phi^{-1}(1 - \epsilon_{ij}) \leq f_{ij}^{max}$$

◂ Chance Constrained OPF

Cutting-Plane Method

New Solutions still violates conic constraint

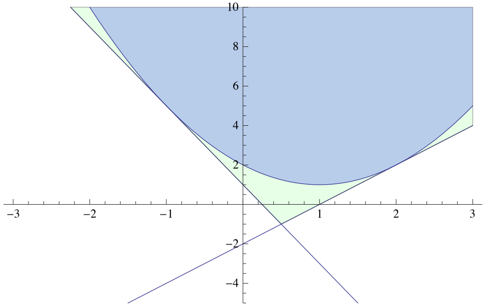


◂ Chance Constrained OPF

Complex Energy Systems
└ Appendix: CC-OPF
└ Cutting-Plane Method

Cutting-Plane Method

Separate again

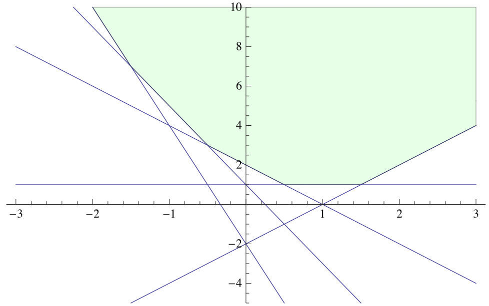


Chance Constrained OPF

Complex Energy Systems
└ Appendix: CC-OPF
└ Cutting-Plane Method

Cutting-Plane Method

We might end up with many linear constraints

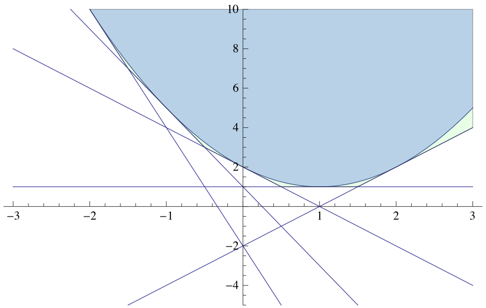


Chance Constrained OPF

Complex Energy Systems
└ Appendix: CC-OPF
└ Cutting-Plane Method

Cutting-Plane Method

… which approximate the conic constraint



Chance Constrained OPF

Complex Energy Systems
└ Appendix: CC-OPF
└ Cutting-Plane Method

Polish 2003-2004 winter peak case

- 2746 buses, 3514 branches, 8 wind sources
- 5% penetration and $\sigma = .3\mu$ each source

CPLEX: the optimization problem has

- 36625 variables
- 38507 constraints, 6242 conic constraints
- 128538 nonzeros, 87 dense columns

CC-OPF Performance

CPLEX:

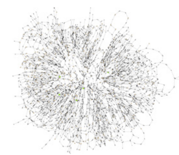
- total time on 16 threads = 3393 seconds
- "optimization status 6"
- solution is wildly infeasible

Gurobi:

- time: 31.1 seconds
- "Numerical trouble encountered"

◂ CC-OPF Performance

Polish 2003-2004 case
CPLEX: "opt status 6"
Gurobi: "numerical trouble"



Example run of cutting-plane algorithm:

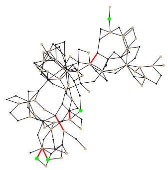
Iteration	Max rel. error	Objective
1	1.2e-1	7.0933e6
4	1.3e-3	7.0934e6
7	1.9e-3	7.0934e6
10	1.0e-4	7.0964e6
12	8.9e-7	7.0965e6

Total running time: 32.9 seconds

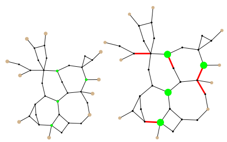
◂ CC-OPF Performance

Experiments with CC-OPF (I)

- **CC-OPF succeeds where standard OPF fails**
- 118-bus case with four wind farms. Standard OPF–Lines in red exceed their limit 8% or more of the time. CC-OPF–finds solution with significantly smaller risk of overload.



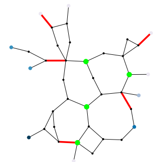
- **Cost of Reliability** [CC-OPF saving over standard OPF]
- 39-bus case under standard solution. Even with a 10% buffer on the line flow limits (for the average wind), five lines exceed their limit over 5% of the time with 30% penetration (right). The penetration must be decreased to 5% before the lines are relieved, but at great cost (left). The CC-OPF model is feasible for 30% penetration at a cost of 264,000. The standard solution at 5% penetration costs 1,275,020 almost 5 times as much.



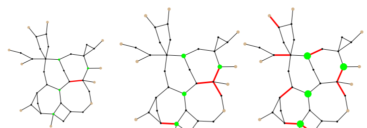
◂ BPA Experiment

Experiments with CC-OPF (II)

- **CC-OPF is not a naive fix.** (Changes are nonlocal.)
- 39-bus case. Darker shades of blue indicating generators with greater change from CC-OPF to standard OPF.



- **What is the penetration that can be tolerated** (without upgrading)?
- 39-bus case. Three levels of penetration. Standard OPF is infeasible for three level of penetrations. CC-OPF is infeasible only with the penetration level > 30 + % .



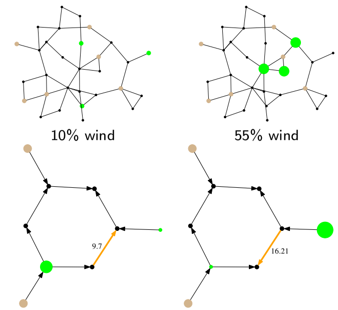
.1% wind 8% wind 30% wind

◂ BPA Experiment

Experiments with CC-OPF (III)

- **Which sites to place wind-farms?**
- 30 bus case with three wind farms. Placement on the right is preferable.
- CC-OPF finds the nodes where the entire network is less susceptible to fluctuations.

- CC-OPF valid configurations may **show significant (allowed!) variability**, e.g. flow reversal.
- 9-bus case, 25% average penetration - two significantly different flows.

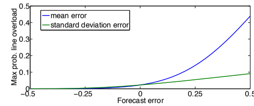


◂ BPA Experiment

Out of Sample Tests

Distribution	Max. prob. violation
Normal	0.0227
Laplace	0.0297
logistic	0.0132
Weibull, $k = 1.2$	0.0457
Weibull, $k = 2$	0.0355
Weibull, $k = 4$	0.0216
t location-scale, $\nu = 2.5$	0.0165
Cauchy	0.0276

Maximum probability of overload for out-of-sample tests. These are a result of Monte Carlo testing with 10,000 samples on the BPA case, solved under the Gaussian assumption and desired maximum chance of overload at 2.27%.



BPA case solved with average penetration at 8% and standard deviations set to 30% of mean. The maximum probability of line overload desired is 2.27%, which is achieved with 0 forecast error on the graph. Actual wind power means are then scaled according to the x-axis and maximum probability of line overload is recalculated (blue). The same is then done for standard deviations (green).

◂ Enhancements of CC-OPF

Constant voltage, lossless, security constrained OPF:

$$\min_{p,\theta} \; c(p)$$

s.t.

$$\sum_{j:ij\in\mathcal{L}} \beta_{ij}\sin(\theta_i-\theta_j) \;=\; p_i - d_i \qquad \forall i\in\mathcal{B}$$

$$|\beta_{ij}\sin(\theta_i-\theta_j)| \;\le\; u_{ij} \quad \text{for each line } ij$$

$$P_g^{min} \;\le\; p_g \;\le\; P_g^{max} \quad \text{for each generator } g$$

Can one convexify this formulation of OPF?

▸ CC-OPF enhancements

PF through Optimization [lossless, constant voltage]

Boyd & Vandenberghe (add. ex. for convex opt. – 2012):

Suppose you solve the **convex optimization problem**:

$$\min_{\rho\text{ - line flows}} \underbrace{\sum_{ij\in\mathcal{L}} \beta_{ij}\Psi(\rho_{ij})}_{\text{reactive losses in lines}}, \quad \Psi(\rho)=\int_{-1}^{\rho}\arcsin(y)\,dy$$

$$\text{s.t.} \quad \underbrace{\sum_{j:ij\in\mathcal{L}} \beta_{ij}\rho_{ij} - \sum_{j:ji\in\mathcal{L}} \beta_{ij}\rho_{ji} \;=\; p_i - d_i}_{\text{network flow conservation}} \qquad \forall i\in\mathcal{B} \;\; (*)$$

$$|\rho_{ij}| \;<\; 1 \quad \text{for each line } ij$$

Then: If θ_i is the optimal dual for (*), $\rho_{ij} \;=\; \sin(\theta_i-\theta_j)$.

The opt. is dual to the Energy Function opt.

How can we incorporate this methodology into OPF-type problems?

▸ CC-OPF enhancements

Theorem: "Exact" AC-OPF [BBC 2013]

Suppose you solve the **convex optimization problem**:

$$\min_{p,\rho,\delta\geq 0} \quad c(p) + D\sum_{ij\in\mathcal{L}} \beta_{ij}\Psi(\rho_{ij}) - K\sum_{ij\in\mathcal{L}} \beta_{ij}\log(\delta_{ij})$$

s.t.

$$\sum_{j:ij\in\mathcal{L}} \beta_{ij}\rho_{ij} - \sum_{j:ji\in\mathcal{L}} \beta_{ij}\rho_{ji} = p_i - d_i \qquad \forall i\in\mathcal{B} \;\; (**)$$

$$|\rho_{ij}| + \min\{1, u_{ij}/\beta_{ij}\}\delta_{ij} < \min\{1, u_{ij}/\beta_{ij}\} \quad \text{for each line } ij$$

$$P_g^{min} \leq p_g \leq P_g^{max} \quad \text{for each generator } g$$

For **appropriate** positive constants ***D*** (small) and ***K*** (large). Then if a feasible solution is found

- The optimal ρ_{ij} are approximate optimal flows [with line flow limits obeyed]
- $\rho_{ij} \approx \sin(\theta_i - \theta_j) \qquad \theta =$ optimal duals to (**)

CC-OPF enhancements

AC-OPF [loseless, constant voltage] formulation

$$\min_{p,\vartheta} \; c(p)$$

s.t.

$$\sum_{j:ij\in\mathcal{L}} \beta_{ij}\sin(\theta_i - \theta_j) = p_i - d_i \qquad \forall i\in\mathcal{B}$$

$$|\sin(\theta_i - \theta_j)| < u_{ij}/\beta_{ij} \quad \text{for each line } ij$$

CC-OPF enhancements

Based on Dörfler, Chertkov, Bullo 2013: an approximation

$$\min_{p,\vartheta} \quad c(p)$$

s.t.

$$\sum_{j:ij\in\mathcal{L}} \beta_{ij}(\vartheta_i - \vartheta_j) = p_i - d_i \qquad \forall i\in\mathcal{B}$$

$$|\vartheta_i - \vartheta_j| < \min\{1, u_{ij}/\beta_{ij}\} \quad \text{for each line } ij$$

[criterium for existence of solution, assumes strong damping]

Sync in Pics

- The ϑ are auxiliary variables only.
- Exact on trees, very accurate for almost all realistic cases tested
- In experiments, $\beta_{ij}(\vartheta_i - \vartheta_j)$ provides a close approximation to the lossless (active) AC power flow on each line ij
- (But does not provide phase angles)

Incorporation into chance-constrained problem:

- On any line ij, we replace $\sin(\theta_i - \theta_j)$ with the quantity $\vartheta_i - \vartheta_j$
- So 'sync' constraint $|\sin(\theta_i - \theta_j)| \leq \gamma_{ij}$ becomes $|\vartheta_i - \vartheta_j| \leq \gamma_{ij}$
- But in **either** case the constraint is stochastic

Results in a (conic) convex optimization

- **Chance-constrained version:** $P(|\vartheta_i - \vartheta_j| > \gamma_{ij}) < \epsilon_{ij}$

All (thermal, gen., sync) Chance Constraints accounted

- Results in the convex (conic) optimization
- Similar to DC CC-OPF – extra sync Chance-constraints added

CC-OPF enhancements

Chance-constrained, thermal and sync-aware (approximate) OPF:
Choose mean generator outputs and control to minimize expected cost, with the probability of line overloads and phase angle excursions kept small. (abridged)

$$\min_{\overline{p},\alpha} \mathbb{E}[c(\overline{p})]$$

$$\text{s.t.} \sum_{i\in G} \alpha_i = 1,\ \alpha \geq 0$$

$$B\delta = \alpha$$

$$\sum_{ij\in\mathcal{L}} \beta_{ij}(\bar{\vartheta}_i - \bar{\vartheta}_j) = \overline{p_i} + \mu_i - d_i$$

$$P(\beta_{ij}|\boldsymbol{\vartheta_i} - \boldsymbol{\vartheta_j}| > u_{ij}) \leq \epsilon_1 \quad \text{for each line } ij$$

$$P(|\boldsymbol{\vartheta_i} - \boldsymbol{\vartheta_j}| > \gamma_{ij}) \leq \epsilon_2 \quad \text{for each line } ij$$

$$P(\boldsymbol{p_g} < P_g^{min} \text{ or } P_g^{max} < \boldsymbol{p_g}) \leq \epsilon_3 \quad \text{for each generator } g$$

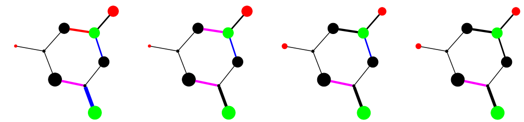
$$\epsilon_2 \ll \epsilon_3 \ll \epsilon_1$$

Again: a conic optimization problem

▸ CC-OPF enhancements

Thermal and Sync Aware CC-OPF: Experiments (I)

Competition of sync and thermal risks guides iterations



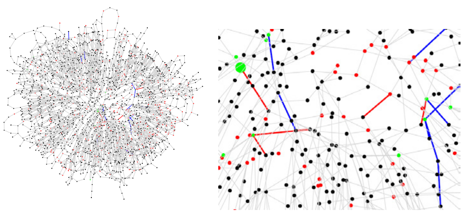
- The case of $\bar{p}_{ij}/\beta_{ij} \leq 1$ but $\varepsilon = 10^{-4} \ll \epsilon_{ij} = 10^{-2}$.
- 1st, 8th, 11th and 13th (final) iteration steps shown.
- Nodes: Loads = black, wind farms = green, regular generators = red
- Lines: sync+therm = red, only sync = magenta, only therm = blue, no viol. = black
- Scaling – with actual values or means

▸ CC-OPF enhancements

Thermal and Sync Aware CC-OPF: Experiments (II)

Pattern(s) of Sync Warnings

- Qualitative value in studying the warning patterns

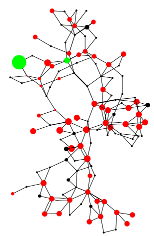


Polish case completed in 11 iterations. Sync overload dominated. Red lines = sync. overloaded with probability $\in [10^{-4}; 10^{-2}]$. Blue lines = weaker overload. Scaling according to cons/prod and mean flows within the optimal solution.

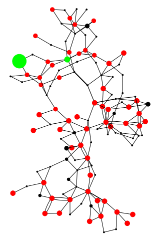
▸ CC-OPF enhancements

Thermal and Sync Aware CC-OPF: Experiments (III)

Sensitivity of the optimal solution to risk awareness



9 cutting plane iterations, both sync and thermal conditions violated [less uniform]

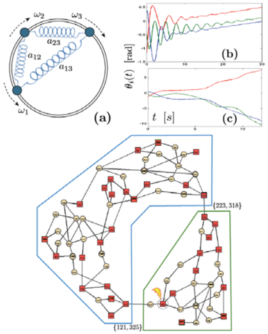


21 cutting plane iterations, only sync conditions violated

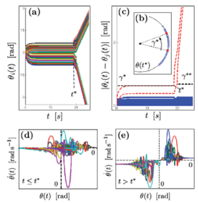
- two slightly different config. of loads
- results distinctly different [cost and distr. of gen.]
- red – regulated generation
- green – renewables [mean]

▸ CC-OPF enhancements

Sync in Pics



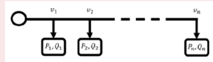
- from F. Dörfler, M. Chertkov, and F. Bullo, PNAS 2013



◂ Sync-constrained OPF

Dist(ributed) Flow Representation [Baran, Wu '89]

graph-linear Element $k = 1, \cdots, N$ of the distribution feeder



$k = 0, \cdots, N, \quad v_0 = 1$

$P_{n+1} = Q_{n+1} = 0$

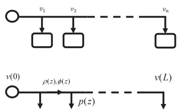
$$P_{k+1} - P_k = p_k - r_k \frac{P_k^2 + Q_k^2}{v_k^2}$$

$$Q_{k+1} - Q_k = q_k - x_k \frac{P_k^2 + Q_k^2}{v_k^2}$$

$$v_{k+1}^2 - v_k^2 = -2(r_k P_k + x_k Q_k) - (r_k^2 + x_k^2) \frac{P_k^2 + Q_k^2}{v_k^2}$$

- nonlinear AC over a line
- generalizable to a tree
- P_k, Q_k real and reactive powers flowing through the segment k
- p_k, q_k, v_k powers injected/consumed and voltage at the bus k

Continuum (one dimensional) static power flows



ODE with mixed boundary conditions
$v(0) = 1, \ \rho(L) = \phi(L) = 0$

From Algebraic Eqs. on a (linear) Graph to Power Flow ODEs

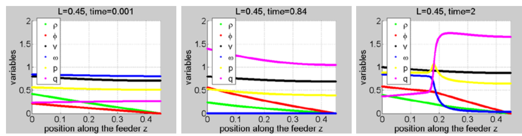
$$0 = \underbrace{p + \beta \partial_r \left(v^2 \partial_r \theta\right) + g v \left(\partial_r^2 v - v\left(\partial_r \theta\right)^2\right)}_{\text{balance of real power}}, \quad 0 = \underbrace{q + \beta v \left(\partial_r^2 v - v\left(\partial_r \theta\right)^2\right) - g \partial_r \left(v^2 \partial_r \theta\right)}_{\text{balance of reactive power}}$$

$$\rho = \underbrace{-\beta v^2 \partial_r \theta - g v \partial_r v}_{\text{real power density flowing through the segment}}, \quad \phi = \underbrace{-\beta v \partial_r v + g v^2 \partial_r \theta}_{\text{reactive power density flowing though the segment}}$$

$$0 = \underbrace{p}_{\text{real consumption}} - \overbrace{\partial_r \rho}^{\text{real transport}} - \underbrace{r \frac{\rho^2 + \phi^2}{v^2}}_{\text{real dissipation}}, \quad 0 = \underbrace{q}_{\text{reactive consumption}} - \overbrace{\partial_r \phi}^{\text{reactive transport}} - \underbrace{x \frac{\rho^2 + \phi^2}{v^2}}_{\text{reactive dissipation}}$$

Dynamics/Transitions in Distributed Feeder (III)

Example of a Short Fault (↓, ↑ to full recovery) (Movie Recovery)



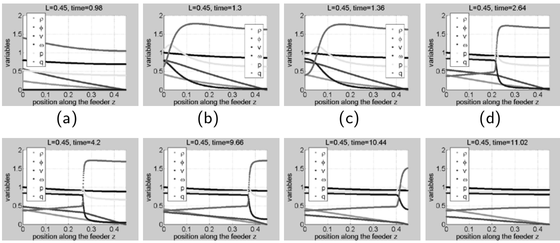
(a) (b) (c)

- (a) Pre fault
- (b) Past voltage drop at the header. Leads to a fully stalled phase.
- (c) Fault is cleared. Front of recovery is advancing towards the tail.

▸ Fault Dynamics

Dynamics/Transitions in Distributed Feeder (IV)

From Stalled to Normal (Movie Recovery)



(a) (b) (c) (d)
(e) (f) (g) (h)

Of interest: “Soliton”-like shape; voltage profile is (almost) frozen

▸ Fault Dynamics

What can one do at the distribution level to mitigate FIDVR?

- Monitor/learn/model distributed motor parameters
- Control voltage at the head of the line (rise it when needed)
- Distributed reactive control

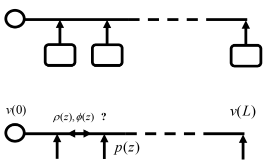
Why should System Operator worry about FIDVR?

- Simple restoration of the transmission network may not drive the circuits back to a running state.
- A transmission fault ⇒ correlated dynamical response in multiple distribution feeders ⇒ individual circuits stalled. Specific to each circuit, there is an energy barrier to the transition back to a running state.
- Once a spatially-correlated stalled state exists, the state of the transmission grid has now fundamentally changed.

What can the system operator do about FIDVR and related?

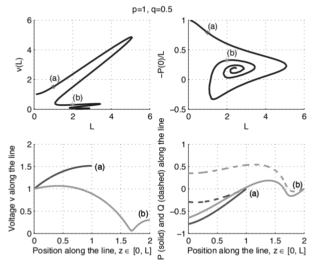
- Consider FIDVR as yet another (and much less analyzed !!) transient stability issue/contingency
- Attempt to predict (monitoring short voltage faults within the transmission) ... and pull it back to normal without relying (or with minimal reliance) on the distribution level protection and response

Feeder with Distributed Generation D. Wang, K. Turitsyn, MC (2012)



- Smart Grid Scenario: Significant Penetration of Photo-Voltaic Systems
- Many Consumers feed back to the system
- (Normally) voltage raises down the feeder and feeder exports, $\rho(0), \eta(0) < 0$
- And what if a fault occurs?

Effect of Distributed Generation

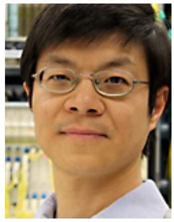


The effect is DISTRIBUTED!
not seen in the two-node model

- PV systems inject both p and q
- New regulations will require ride-through-low-voltage capability
- If the distributed generation is too large, multiple low-voltage states will appear
- Prediction of a potential trouble: after a fault the system may be trapped in the low-voltage state (similar to FIDVR)
- The only (normal) way to get out of the trouble is to disconnect the PVs

SEMIDEFINITE RELAXATION OF OPTIMAL POWER FLOW

Steven H. Low, CalTech, USA



The optimal power flow (OPF) problem seeks to optimize a certain objective, such as power loss, generation cost or user utility, subject to Kirchhoff's laws, power balance as well as capacity, stability and contingency constraints on the voltages and power flows. It is a fundamental problem that underlies many power system operations and planning. It is nonconvex and many algorithms have been proposed to solve it approximately. A new approach via semidefinite relaxation of OPF has been developed in the last few years.

In this tutorial, we present a bus injection model and a branch flow model, formulate OPF within each model, and prove their equivalence. The complexity of OPF formulated here lies in the quadratic nature of power flows, i.e., the nonconvex quadratic constraints on the feasible set of OPF. We characterize these feasible sets and design convex supersets that lead to three different convex relaxations based on semidefinite programming (SDP), chordal extension, and second-order cone programming (SOCP). When a convex relaxation is exact, an optimal solution of the original nonconvex OPF can be recovered from every optimal solution of the relaxation. We summarize three types of sufficient conditions that guarantee the exactness of these relaxations. Finally, we extend the convex relaxations to multiphase unbalanced radial networks that are common in distribution systems.

Convex Relaxation of OPF

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Lund, Sweden

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SCE

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Optimal power flow (OPF)

OPF is solved routinely to determine

- How much power to generate where
- Parameter setting, e.g. taps, VARs
- Market operation & pricing

Non-convex and hard to solve

- Huge literature since 1962
- Common practice: DC power flow (LP)
- Also: Newton-Ralphson, interior point, …

Outline

Optimal power flow (OPF)

- bus injection model, branch flow model

3 convex relaxations

- SDP, chordal, second-order cone (SOCP)
- Relation among them

Sufficient conditions for exact relaxation

- Radial: 3 main conditions
- Mesh: phase shifters

Summary: OPF (bus injection model)

$$\min \quad \operatorname{tr} CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

nonconvex QCQP

Summary: OPF (branch flow model)

$$\min \quad f(x)$$

$$\text{over} \quad x := (S, I, V, s)$$

$$\text{s.t.} \quad \underline{s}_j \le s_j \le \bar{s}_j \qquad \underline{v}_j \le v_j \le \bar{v}_j$$

branch flow model

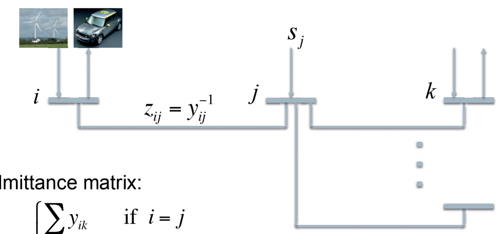
$$\sum_{i \to j}\left(S_{ij} - z_{ij}|I_{ij}|^2\right) - \sum_{j \to k} S_{jk} = s_j$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

nonconvex

details

Bus injection model



admittance matrix:

$$Y_{ij} := \begin{cases} \sum_{k \sim i} y_{ik} & \text{if } i = j \\ -y_{ij} & \text{if } i \sim j \\ 0 & \text{else} \end{cases}$$

graph model G: undirected

Y specifies topology of G and impedance z on lines

Recap

Bus injection model

$$s_j = \mathrm{tr}\left(Y_j V V^*\right)$$

Branch flow model

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2\right) + s_j$$

$(V, s) \in \mathbf{C}^{2(n+1)}$ $\qquad$ $(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$

solution set $\quad \mathbf{V} \qquad \tilde{\mathbf{X}}$

Equivalence

Theorem: $\mathbf{V} \equiv \tilde{\mathbf{X}}$

- BIM and BFM are equivalent in this sense
- Any result in one model is in principle provable in the other,
- … but some results are easier to formulate or prove in one than the other
- BFM seems to be much more numerically stable (radial networks)

$(V, s) \in \mathbf{C}^{2(n+1)}$ $\qquad$ $(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$

solution set $\quad \mathbf{V} \longleftrightarrow \tilde{\mathbf{X}}$

OPF: bus injection model

min $\quad V^* C V$ $\qquad$ gen cost, power loss

over $\quad (V, s)$

subject to $\quad \underline{s}_j \le s_j \le \overline{s}_j \qquad \underline{V}_j \le |V_j| \le \overline{V}_j$

OPF: bus injection model

min $\quad V^* C V$ $\qquad$ gen cost, power loss

over $\quad (V, s)$

subject to $\quad \underline{s}_j \le s_j \le \overline{s}_j \qquad \underline{V}_j \le |V_j| \le \overline{V}_j$

$\quad s_j = \mathrm{tr}\left(Y_j^H V V^H\right)$ $\qquad$ power flow equation

Recap

Bus injection model

$$s_j = \mathrm{tr}\left(Y_j V V^*\right)$$

$$(V,s) \in \mathbf{C}^{2(n+1)}$$

solution set: $\mathbf{V}$

Branch flow model

$$V_i - V_j = z_{ij} I_{ij}$$

$$S_{ij} = V_i I_{ij}^*$$

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij}\left|I_{ij}\right|^2\right) + s_j$$

$$(S,I,V,s) \in \mathbf{C}^{2(m+n+1)}$$

solution set: $\tilde{\mathbf{X}}$

Equivalence

Theorem: $\mathbf{V} \equiv \tilde{\mathbf{X}}$

- BIM and BFM are equivalent in this sense
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- … but some results are easier to formulate or prove in one than the other
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$(V,s) \in \mathbf{C}^{2(n+1)}$ — solution set $\mathbf{V}$ ⟷ $(S,I,V,s) \in \mathbf{C}^{2(m+n+1)}$ — $\tilde{\mathbf{X}}$

OPF: bus injection model

min	$V^* C V$	gen cost, power loss
over	(V,s)	
subject to	$\underline{s}_j \le s_j \le \overline{s}_j$	$\underline{V}_j \le \lvert V_j \rvert \le \overline{V}_j$

OPF: bus injection model

min	$V^* C V$	gen cost, power loss
over	(V,s)	
subject to	$\underline{s}_j \le s_j \le \overline{s}_j$	$\underline{V}_j \le \lvert V_j \rvert \le \overline{V}_j$
	$s_j = \mathrm{tr}\left(Y_j^H V V^H\right)$	power flow equation

OPF: bus injection model

$$\begin{aligned} &\min && \operatorname{tr} CVV^* \\ &\text{subject to} && \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \overline{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \overline{v}_j \end{aligned}$$

quadratically constrained QP (QCQP)
nonconvex, NP-hard

OPF: branch flow model

$$\begin{aligned} &\min && f(x) \\ &\text{over} && x := (S, I, V, s) \\ &\text{s. t.} && \end{aligned}$$

OPF: branch flow model

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branch flow model

$$\sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) - \sum_{j \to k} S_{jk} = s_j$$

$$V_j = V_i - z_{ij} I_{ij} \qquad S_{ij} = V_i I_{ij}^*$$

nonconvexity

Other features

Security constraint OPF
- Solve for operating points after each single contingency (N-1 security)
- N sets of variables and constraints, one for each contingency

Unit commitment
- Discrete variables

Stochastic OPF
- Chance constraints Pr(bad event) < ε

Other constraints
- Line flow, line loss, stability limit, …

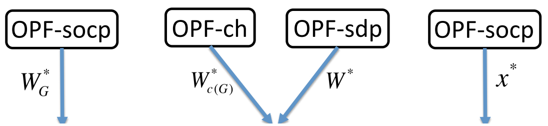
… OPF in practice is a lot harder

Outline

Optimal power flow (OPF)
- bus injection model, branch flow model

3 convex relaxations
- SDP, chordal, second-order cone (SOCP)
- Relation among them

Sufficient conditions for exact relaxation
- Radial: 3 main conditions
- Mesh: phase shifters



What are semidefinite relaxations of OPF?

How to check & recover global optimal ?

details

Literature

Convex relaxation of OPF

relaxation	model	first proposed	first analyzed
SOCP	BIM	Jabr 2006 TPS	
SDP	BIM	Bai et al 2008 EPES	Lavaei, Low 2012 TPS
Chordal	BIM	Bai, Wei 2011 EPES Jabr 2012 TPS	Molzahn et al 2013 TPS Bose et al 2014 TAC

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

Literature

Convex relaxation of OPF

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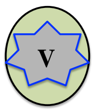
SOCP	BFM	Farivar et al 2011 SGC Farivar, Low 2013 TPS	Farivar et al 2011 SGC Farivar, Low 2013 TPS

Low. Convex relaxation of OPF (I, II), IEEE Trans Control of Network Systems, 2014

Basic idea



$$\min \quad \operatorname{tr} CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

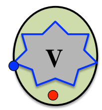
$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{V}}$$

All complexity due to nonconvexity of $\mathbf{V}$

Relaxations:

- design convex supersets of $\mathbf{V}$
- minimize cost over convex supersets

Basic idea



$$\min \quad \operatorname{tr} CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \operatorname{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathbf{V}}$$

All complexity due to nonconvexity of $\mathbf{V}$

Relaxations:

- design convex supersets of $\mathbf{V}$
- minimize cost over convex supersets

Exact relaxation: optimal solution of relaxation happens to lie in $\mathbf{V}$ (when?)

Basic idea

$$\min \quad \mathrm{tr}\, CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

(constraint set: **V**)

Approach

1. Three equivalent characterizations of **V**
2. Each suggests a lift and relaxation

- What is the relation among different relaxations ?
- When will a relaxation be exact ?

Feasible sets

$$\min \quad \mathrm{tr}\, CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

quadratic in V
linear in W

Equivalent problem:

$$\min \quad \mathrm{tr}\, CW$$

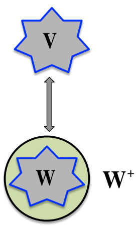
$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j W\right) \le \bar{s}_j \qquad \underline{v}_i \le W_{ii} \le \bar{v}_i$$

$$W \ge 0, \;\; \mathrm{rank}\, W = 1$$

convex in W
except this constraint

Equivalent feasible sets

V:= $\{V$: satisfies quadratic constraints $\}$



instead of n variables
solve for n^2 vars !

W:= $\{W$: satisfies linear constraints $\} \cap \{W \ge 0$ ~~rank-1~~$\}$
idea: $W = VV^*$

Feasible set

only $n+2m$ vars !

linear in (W_{jj}, W_{jk}) ⟵ W_{jj} W_{jk}

$$\sum_{k:k\sim j} y_{jk}^* \left(|V_j|^2 - V_j V_k^*\right) : \text{ **only** } |V_j|^2 \text{ and } V_j V_k^*$$

corresponding to edges (j,k) in G!

$$\min \quad \mathrm{tr}\, CVV^*$$

$$\text{subject to} \quad \underline{s}_j \le \mathrm{tr}\left(Y_j VV^*\right) \le \bar{s}_j \qquad \underline{v}_j \le |V_j|^2 \le \bar{v}_j$$

(constraint set: **V**)

Feasible set

only $n+2m$ vars !

linear in (W_{jj}, W_{jk}) ← W_{jj} W_{jk}

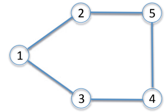
$$\sum_{k:k\sim j} y_{jk}^* \left(|V_j|^2 - V_j V_k^* \right) : \textbf{only } |V_j|^2 \text{ and } V_j V_k^*$$

partial matrix W_G defined on G

$$W_G := \{ [W_G]_{jj}, [W_G]_{jk}, [W_G]_{kj} \mid j, jk \in G \}$$

Kirchhoff's laws depend directly only on W_G

Example



G

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} \\ W_{51} & W_{52} & W_{53} & W_{54} & W_{55} \end{bmatrix}$$

SDP solves for $W \in \mathbf{C}^{n^2}$

n^2 variables

$$W_G = \begin{bmatrix} W_{11} & W_{12} & W_{13} & & \\ W_{21} & W_{22} & & & W_{25} \\ W_{31} & & W_{33} & W_{34} & \\ & & W_{43} & W_{44} & W_{45} \\ & W_{52} & & W_{54} & W_{55} \end{bmatrix}$$

Want to solve for W_G

$n+2m$ variables

Feasible sets

OPF $\mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j V V^*\right) \le \bar{s}_j, \ \ \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$

SDP

$$\mathbf{W} := \left\{ W \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j W\right) \le \bar{s}_j, \ \underline{v}_j \le W_{jj} \le \bar{v}_j \right\} \cap \left\{ W \ge 0, \text{rank-}1 \right\}$$

depend only on W_G

depend on all entries of W

Feasible sets

OPF $\mathbf{V} := \left\{ V \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j V V^*\right) \le \bar{s}_j, \ \ \underline{v}_j \le |V_j|^2 \le \bar{v}_j \right\}$

SDP

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first idea:

$$\mathbf{W}_G := \left\{ W_G \,\middle|\, \underline{s}_j \le \operatorname{tr}\left(Y_j W_G\right) \le \bar{s}_j, \ \underline{v}_j \le [W_G]_{jj} \le \bar{v}_j \right\} \cap \left\{ W_G \ge 0, \text{rank-}1 \right\}$$

W_G is equivalent to V when G is chordal
Not equivalent otherwise

Equivalent feasible sets

$$\mathbf{W}_G := \left\{ \begin{array}{l} W_{jj}, W_{jk} : (j,k) \text{ in } G \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{array} \right\} \cap \left\{ \begin{array}{l} W(j,k) \geq 0 \text{ rank-1}, \\ \text{cycle cond on } \angle W_{jk} \end{array} \right\}$$

idea: $W_G = \left(VV^* \text{ only on } G \right)$

$$\mathbf{W}_{c(G)} := \left\{ \begin{array}{l} W_{jj}, W_{jk} : (j,k) \text{ in } c(G) \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{array} \right\} \cap \left\{ W_{c(G)} \geq 0 \text{ rank-1} \right\}$$

idea: $W_{c(G)} = \left(VV^* \text{ on } c(G) \right)$

matrix completion [Grone et al 1984]

$$\mathbf{W} := \left\{ W : \text{satisfies } \underline{\text{linear}} \text{ constraints} \right\} \cap \left\{ W \geq 0 \text{ rank-1} \right\}$$

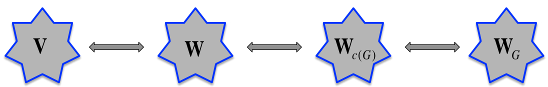
idea: $W = VV^*$

Cycle condition

local $\quad W_G(j,k) \succeq 0, \ \text{rank } W_G(j,k) = 1, \quad (j,k) \in E$

global $\quad \sum_{(j,k) \in c} \angle [W_G]_{jk} = 0 \mod 2\pi$ ← cycle cond

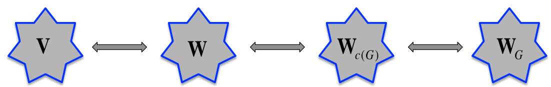
Equivalent feasible sets



Theorem: $\mathbf{V} \equiv \mathbf{W} \equiv \mathbf{W}_{c(G)} \equiv \mathbf{W}_G$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Equivalent feasible sets



Theorem: $\mathbf{V} \equiv \mathbf{W} \equiv \mathbf{W}_{c(G)} \equiv \mathbf{W}_G$

Given $W_G \in \mathbf{W}_G$ or $W_{c(G)} \in \mathbf{W}_{c(G)}$ there is unique completion $W \in \mathbf{W}$ and unique $V \in \mathbf{V}$

Can minimize cost over **any** of these sets, but …

Relaxations

$$\mathbf{W}_G := \left\{ \begin{matrix} W_{jj}, W_{jk}: (j,k) \text{ in } G \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{matrix} \right\} \cap \left\{ \begin{matrix} W(j,k) \geq 0 \ \text{~~rank-1~~}, \\ \text{~~cycle cond on } \angle W_{jk}\text{~~} \end{matrix} \right\}$$

idea: $W_G = \left(VV^* \text{ only on } G\right)$

$$\mathbf{W}_{c(G)} := \left\{ \begin{matrix} W_{jj}, W_{jk}: (j,k) \text{ in } c(G) \\ \text{satisfy } \underline{\text{linear}} \text{ constraints} \end{matrix} \right\} \cap \left\{ W_{c(G)} \geq 0 \ \text{~~rank-1~~} \right\}$$

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matrix completion [Grone et al 1984]

$$\mathbf{W} := \left\{ W: \text{satisfies } \underline{\text{linear}} \text{ constraints} \right\} \cap \left\{ W \geq 0 \ \text{~~rank-1~~} \right\}$$

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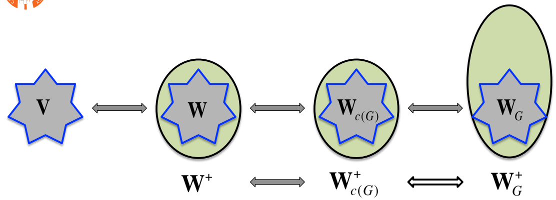
Relaxations

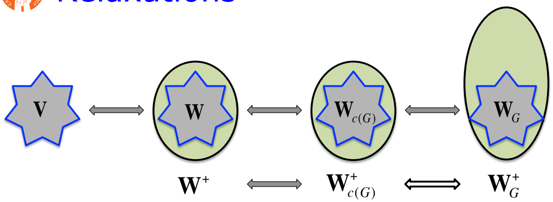


Theorem

- Radial G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \cong \mathbf{W}^+_G$
- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

Bose, Low, Chandy Allerton 2012
Bose, Low, Teeraratkul, Hassibi TAC2014

Relaxations



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- Mesh G : $\mathbf{V} \subseteq \mathbf{W}^+ \cong \mathbf{W}^+_{c(G)} \subseteq \mathbf{W}^+_G$

For radial networks: always solve SOCP !

Convex relaxations

OPF

$$\min_V \ C(V) \quad \text{subject to } V \in \mathbf{V}$$

OPF-sdp:

$$\min_W \ C(W_G) \quad \text{subject to} \quad W \in \mathbb{W}^+$$

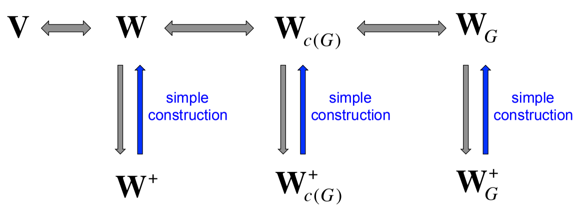
OPF-ch:

$$\min_{W_{c(G)}} \ C(W_G) \quad \text{subject to} \quad W_{c(G)} \in \mathbb{W}^+_{c(G)}$$

OPF-socp:

$$\min_{W_G} \ C(W_G) \quad \text{subject to} \quad W_G \in \mathbb{W}^+_G$$

Recap: convex relaxations

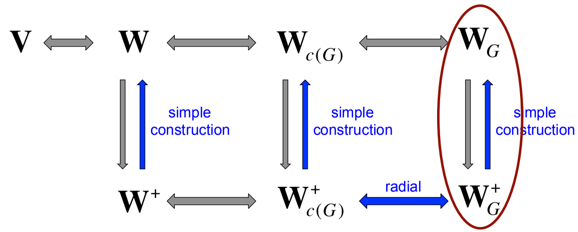


SDP relaxation
- tightest superset
- max # variables
- slowest

Chordal relaxation
- equivalent superset
- much faster for sparse networks

SOCP relaxation
- coarsest superset
- min # variables
- fastest

Recap: convex relaxations



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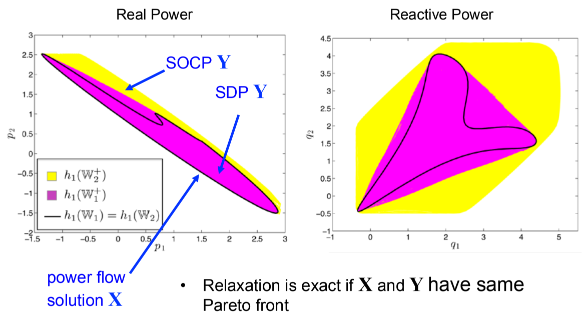
For radial network: always solve SOCP !

Examples



- Relaxation is exact if $\mathbf{X}$ and $\mathbf{Y}$ have same Pareto front
- SOCP is faster but coarser than SDP

Bose, Low, Teeraratkul, Hassibi TAC 2014

Without PS: SDP vs SOCP

Test case	Objective values ($/hr)		Running times (sec)		
	SDP	SOCP	SDP		SOCP
9 bus	5297.4	5297.4	0.2		0.2
14 bus	8081.7	8075.3	0.2		0.2
30 bus	574.5	573.6	0.4		0.3
39 bus	41889.1	41881.5	0.7		0.3
57 bus	41738.3	41712.0	1.3		0.3
118 bus	129668.6	129372.4	6.9		0.6
300 bus	720031.0	719006.5	109.4		1.8
2383 bus	1840270	1789500.0	-		155.3

SOCP inexact

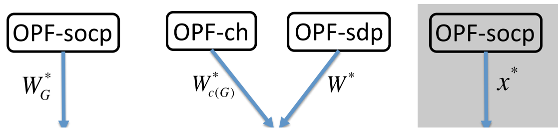
SDP not scalable

Examples

Test case	Objective values ($/hr)		Running times (sec)		
	SDP/ch	SOCP	SDP	chordal	SOCP
9 bus	5297.4	5297.4	0.2	0.2	0.2
14 bus	8081.7	8075.3	0.2	0.2	0.2
30 bus	574.5	573.6	0.4	0.3	0.3
39 bus	41889.1	41881.5	0.7	0.3	0.3
57 bus	41738.3	41712.0	1.3	0.5	0.3
118 bus	129668.6	129372.4	6.9	0.7	0.6
300 bus	720031.0	719006.5	109.4	2.9	1.8
2383 bus	1840270	1789500.0	-	1005.6	155.3

SOCP inexact

SDP not scalable



What are semidefinite relaxations of OPF?

How to check & recover global optimal ?

Branch flow model

Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^* = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$

SOCP relaxation

$$\sum_{j \to k} P_{jk} = \sum_{i \to j} \left(P_{ij} - r_{ij} |I_{ij}|^2 \right) + p_j$$

$$\sum_{j \to k} Q_{jk} = \sum_{i \to j} \left(Q_{ij} - x_{ij} |I_{ij}|^2 \right) + q_j$$

Branch flow model

$\ell_{ij} := |I_{ij}|^2$

$v_i := |V_i|^2$

Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^* = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$

SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2 \, \mathrm{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} = |S_{ij}|^2$$

$$(S, \ell, v, s) \in \mathbf{R}^{3(m+n+1)}$$

Branch flow model

$\ell_{ij} := |I_{ij}|^2$
$v_i := |V_i|^2$

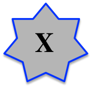
Branch flow model

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} |I_{ij}|^2 \right) + s_j$$

$$V_i - V_j = z_{ij} I_{ij}$$

$$V_i I_{ij}^* = S_{ij}$$

$$(S, I, V, s) \in \mathbf{C}^{2(m+n+1)}$$



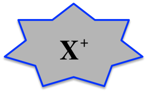
SOCP relaxation

$$\sum_{j \to k} S_{jk} = \sum_{i \to j} \left(S_{ij} - z_{ij} \ell_{ij} \right) + s_j$$

$$v_i - v_j = 2 \operatorname{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$v_i \ell_{ij} \ge |S_{ij}|^2$$

$$(S, \ell, v, s) \in \mathbf{R}^{3(m+n+1)}$$



Branch flow model

power flow solutions: $x := (S, \ell, v, s)$ satisfy

$$\sum_{j \to k} S_{jk} = S_{ij} - z_{ij} \ell_{ij} + s_j$$

$$v_i - v_j = 2 \operatorname{Re}\left(z_{ij}^* S_{ij} \right) - |z_{ij}|^2 \ell_{ij}$$

$$\ell_{ij} v_i = |S_{ij}|^2$$

Advantages
- Recursive structure (radial networks)
- Variables represent physical quantities
- More numerically stable

$\ell_{ij} := |I_{ij}|^2$
$v_i := |V_i|^2$

Baran and Wu 1989
for radial networks

Branch flow model

$$\mathbf{X}^+ := \left\{ \begin{matrix} x : \text{satisfies linear} \\ \text{constraints} \end{matrix} \right\} \cap \left\{ \ell_{jk} v_j \ge |S|^2 \right\} \quad \text{SOC}$$

$$C := \left\{ \begin{matrix} \ell_{jk} v_j = |S|^2 \\ \text{cycle cond on } x \end{matrix} \right\}$$

Theorem $\mathbf{X} \equiv \mathbf{X}^+ \cap C$

Cycle condition

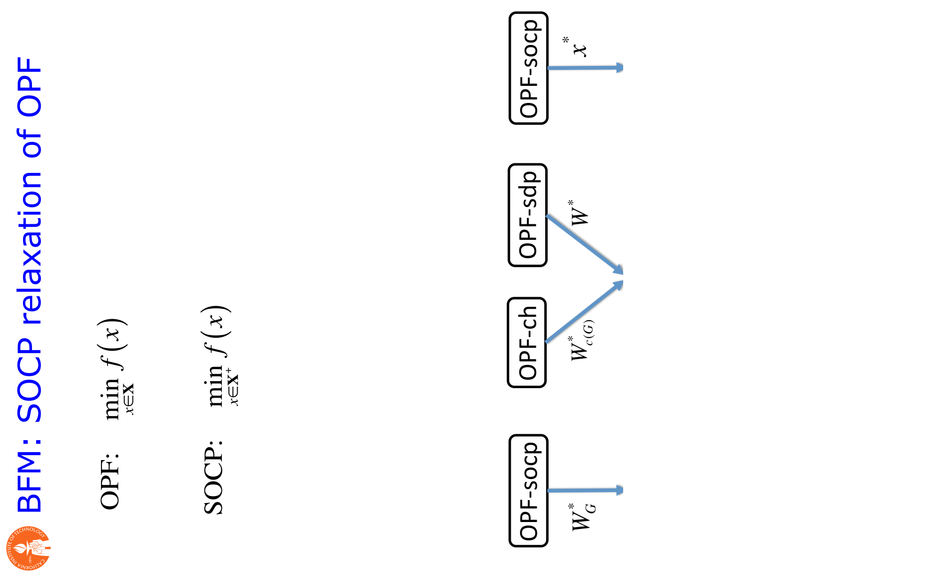
A relaxed solution x satisfies the cycle condition if

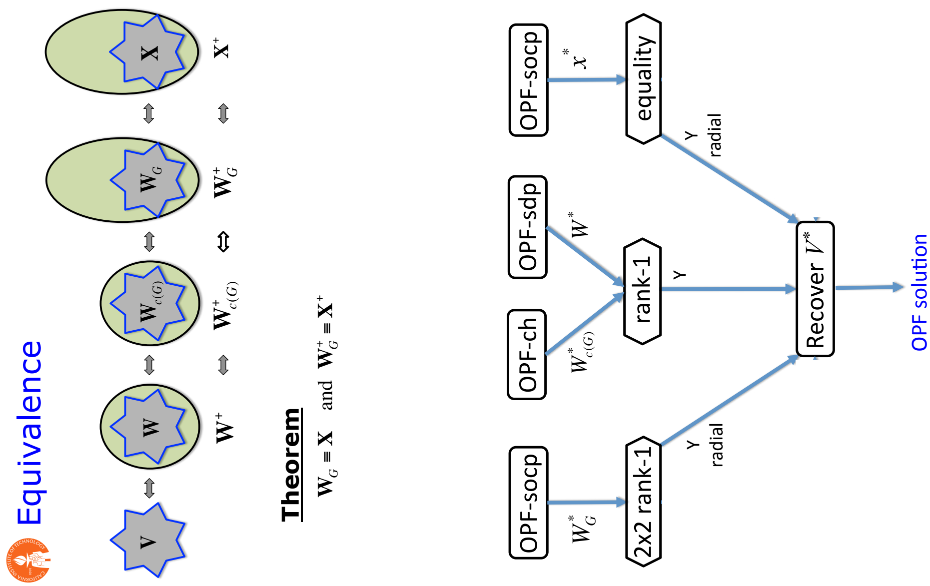
$$\exists \theta \quad \text{s.t.} \quad B\theta = \beta(x) \quad \mod 2\pi$$

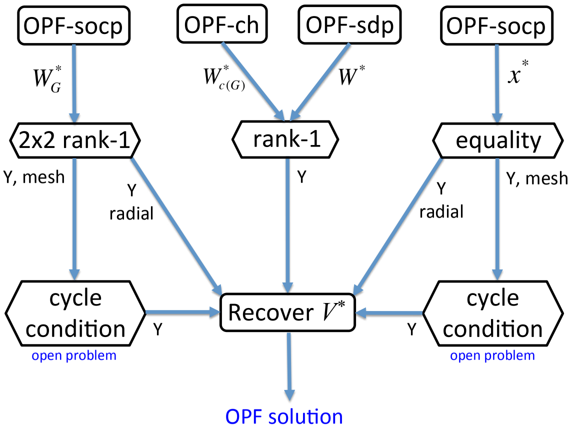
incidence matrix; depends on topology (B)

$x := (S, \ell, v, s)$

$\beta_{jk}(x) := \angle \left(v_j - z_{jk}^H S_{jk} \right)$


BFM: SOCP relaxation of OPF
OPF: $\min_{x \in \mathbf{X}} f(x)$
SOCP: $\min_{x \in \mathbf{X}^+} f(x)$
OPF-socp
W_G^*
OPF-ch
$W_{c(G)}^*$
OPF-sdp
W^*
OPF-socp
x^*


Equivalence
V
W
$\mathbf{W}_{c(G)}$
$\mathbf{W}_G$
X
$\mathbf{W}^+$
$\mathbf{W}_{c(G)}^+$
$\mathbf{W}_G^+$
$\mathbf{X}^+$
Theorem
$\mathbf{W}_G \equiv \mathbf{X}$ and $\mathbf{W}_G^+ \equiv \mathbf{X}^+$
OPF-socp
W_G^*
2x2 rank-1
Y
radial
OPF-ch
$W_{c(G)}^*$
OPF-sdp
W^*
rank-1
Y
OPF-socp
x^*
equality
Y
radial
Recover V^*
OPF solution



Outline

Optimal power flow (OPF)

- bus injection model, branch flow model

3 convex relaxations

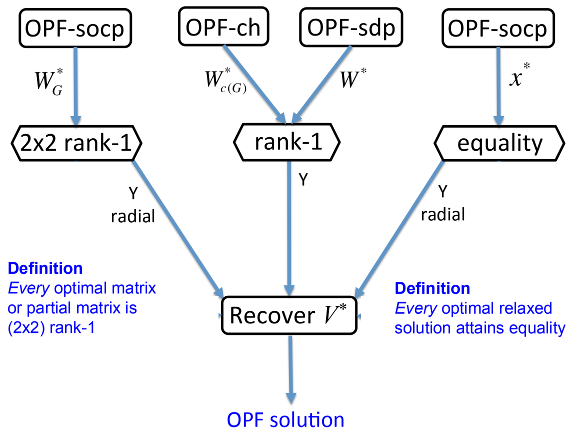
- SDP, chordal, second-order cone (SOCP)
- Relation among them

Sufficient conditions for exact relaxation

- Radial: 2/3 main conditions
- Mesh: phase shifters

Exact relaxation

A relaxation is exact if an optimal solution of the original OPF can be recovered from *every* optimal solution of the relaxation



Summary of sufficient conds

type	condition	model	reference	remark
A	power injections	BIM, BFM	[25], [26], [27], [28], [29] [30], [16], [17]	
B	voltage magnitudes	BFM	[31], [32], [33], [34]	allows general injection region
C	voltage angles	BIM	[35], [36]	makes use of branch power flows

TABLE I: Sufficient conditions for radial (tree) networks.

network	condition	reference	remark
with phase shifters	type A, B, C	[17, Part II], [37]	equivalent to radial networks
direct current	type A	[17, Part I], [19], [38]	assumes nonnegative voltages
	type B	[39], [40]	assumes nonnegative voltages

TABLE II: Sufficient conditions for mesh networks

1. QCQP over tree

QCQP (C, C_k)

$$\begin{aligned} \min \quad & x^* C x \\ \text{over} \quad & x \in \mathbf{C}^n \\ \text{s.t.} \quad & x^* C_k x \le b_k \qquad k \in K \end{aligned}$$

graph of QCQP

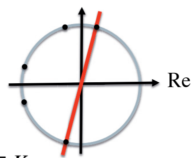
$G(C, C_k)$ has edge (i, j) $\Leftrightarrow$

$C_{ij} \neq 0$ or $[C_k]_{ij} \neq 0$ for some k

QCQP over tree

$G(C, C_k)$ is a tree

1. Linear separability



QCQP (C, C_k)

$$\begin{aligned} \min \quad & x^* C x \\ \text{over} \quad & x \in \mathbf{C}^n \\ \text{s.t.} \quad & x^* C_k x \le b_k \qquad k \in K \end{aligned}$$

Key condition

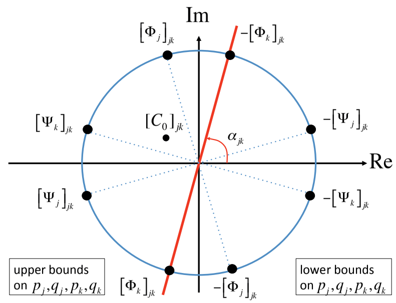
$i \sim j$: $\left(C_{ij}, [C_k]_{ij}, \forall k\right)$ lie on half-plane through 0

Theorem

SOCP relaxation is exact for QCQP over tree

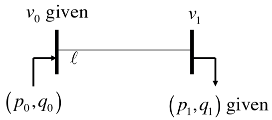
Bose et al 2012
Sojoudi, Lavaei 2013

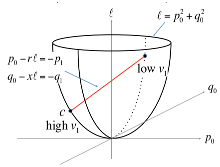
Implication on OPF



Not both lower & upper bounds on real & reactive powers at both ends of a line can be finite

2. Voltage upper bounds

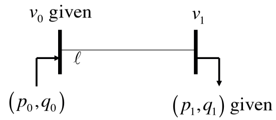


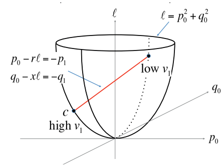


when there is no voltage constraint

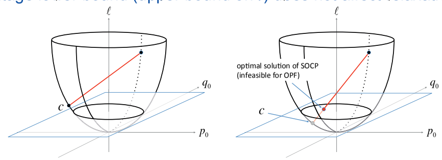
- feasible set : 2 intersection pts
- relaxation: line segment
- exact relaxation: c is optimal

2. Voltage upper bounds





voltage lower bound (upper bound on l) does not affect relaxation



(a) Voltage constraint not binding (b) Voltage constraint binding

2. Voltage upper bounds

OPF: $\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \Sigma$

SOCP: $\min_{x \in \mathbf{X}^+} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \Sigma$

Key condition:

- $L(s) \le \overline{v}$ — voltages if network were lossless
- Jacobian condition — if upward current were reduced then all subsequent powers dec

$\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$ for all $1 \le t \le t' < k$

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014

2. Voltage upper bounds

OPF: $\min_{x \in \mathbf{X}} f(x) \quad \text{s.t.} \quad \underline{v} \le v \le \overline{v}, \quad s \in \Sigma$

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Key condition:

- $L(s) \le \overline{v}$
- Jacobian condition

$\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$ for all $1 \le t \le t' < k$

satisfied with large margin in IEEE circuits and SCE circuits

Theorem

SOCP relaxation is exact for radial networks

Gan, Li, Topcu, Low TAC2014

MULTI-AGENT OPTIMIZATION

Asuman Ozdaglar, Massachusetts Institute of Technology, USA

We consider a network of agents solving a global optimization problem, where the objective function is the sum of privately known local convex objective functions. Recent literature presented subgradient based methods for distributed solution of this problem with O(1/ \sqrt{k}) rate of convergence (where k is the iteration number). In this talk, we present distributed Alternating Direction Method of Multipliers (ADMM) based methods for solving this problem over undirected and directed networks. We present convergence rate estimates that show that these methods converge at rate $O(1/k)$ and highlight the dependence of performance on network structure.

Distributed Alternating Direction Method of Multipliers for Multi-agent Optimization

Asu Ozdaglar

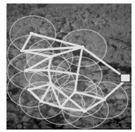
Laboratory for Information and Decision Systems
Operations Research Center
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology

Lund Workshop on Dynamics and Control in Networks
October, 2014

Motivation

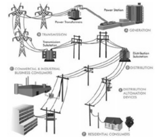
- Many networks are large-scale and comprise of agents with local information and heterogeneous preferences.
- This motivated much interest in developing distributed schemes for control and optimization of multi-agent networked systems.

Routing and congestion control in wireline and wireless networks



Parameter estimation in sensor networks

Multi-agent cooperative control and coordination



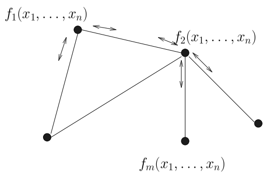
Smart grid systems

Distributed Multi-agent Optimization

- Many of these problems can be represented within the general formulation:
- A set of agents (nodes) $\{1, \ldots, N\}$ connected through a network (graph).
- The goal is to cooperatively solve

$$\min_x \sum_{i=1}^{N} f_i(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n,$$

$f_i(x) : \mathbb{R}^n \to \mathbb{R}$ is a convex (possibly nonsmooth) function, known only to agent i.



- Since such systems often lack a centralized processing unit, algorithms for this problem should involve each agent performing computations locally and communicating only with neighbors.

Machine Learning Example

- A network of agents $i = 1, \ldots, N$.
- Each agent i has access to local feature vectors A_i and output b_i.
- System objective: train weight vector x to

$$\min_x \sum_{i=1}^{N-1} L(A_i' x - b_i) + p(x),$$

for some loss function L (on the prediction error) and penalty function p (on the complexity of the model).

  - Example: Least-Absolute Shrinkage and Selection Operator (LASSO):

$$\min_x \sum_{i=1}^{N-1} ||A_i' x - b_i||_2^2 + \lambda ||x||_1 .$$

  - Other examples from ML estimation, low rank matrix completion, image recovery [Schizas, Ribeiro, Giannakis 08], [Recht, Fazel, Parrilo 10], [Steidl, Teuber 10].

Introduction

Literature: Parallel and Distributed Optimization

- Lagrangian relaxation and dual optimization methods:
  - Dual gradient ascent, coordinate ascent methods.
- Parallel computation and optimization:
  - [Tsitsiklis, Bertsekas, Athans 86], [Bertsekas and Tsitsiklis 89].
- Consensus and cooperative control:
  - [Jadbabaie, Lin, Morse 03], [Olfati-Saber, Murray 04], [Boyd et al. 05], [Olshevsky, Tsitsiklis 07], [Fagnani, Zampieri 09].
- Multi-agent optimization
  - Distributed primal subgradient methods [Nedic, Ozdaglar 07].
  - Distributed dual averaging methods [Duchi, Agarwal Wainwright 12].
  - These methods converge at the rate $O(1/\sqrt{k})$, where k is the number of iterations.

5

Introduction

Literature: Alternating Direction Method of Multipliers

- A large literature on alternating direction method of multipliers (ADMM) due to fast computational performance and distributed-memory, parallel implementations:
  - [Glowinski, Marrocco 75], [Fortin, Glowinski 83], [Gabay 83], [Eckstein, Bertsekas 92].
  - Recent tutorials: [Boyd et al. 10], [Eckstein 12].
  - Relation to proximal point algorithm: [Rockafellar 76, 76], [Luque 84].
  - Decentralized estimation and compressive sensing applications: [Schizas, Ribeiro, Giannakis 08], [Mota, Xavier, Aguiar, Puschel 11].

6

Introduction

This Talk

- We present distributed ADMM-type algorithms for multi-agent optimization.
  - Distributed ADMM over undirected networks [Shtern, Wei, and Ozdaglar 14].
  - Distributed ADMM over directed networks [Makhdoumi and Ozdaglar 14].
- In both cases, we show that these algorithms converge at the faster rate $O(1/k)$.

7

Introduction

Standard ADMM

- Standard ADMM solves a separable problem, where decision variable decomposes into two (linearly coupled) variables:

$$\min_{x,y} \quad f(x) + g(y) \qquad \text{s.t.} \quad Ax + By = c.$$

- Consider an Augmented Lagrangian function:

$$L_\beta(x, y, p) = f(x) + g(y) - p'(Ax + By - c) + \frac{\beta}{2} ||Ax + By - c||_2^2 .$$

- ADMM: approximate version of classical Augmented Lagrangian method.
  - Primal variables: approximately minimize augmented Lagrangian through a single-pass coordinate descent (in a Gauss-Seidel manner).
  - Dual variable: updated through gradient ascent.

8

Standard ADMM

More specifically, updates are as follows:

$$\begin{aligned} x^{k+1} &= \text{argmin}_x \quad L_\beta(x, y^k, p^k), \\ y^{k+1} &= \text{argmin}_y \quad L_\beta(x^{k+1}, y, p^k), \\ p^{k+1} &= p^k - \beta(Ax^{k+1} - By^{k+1} - c). \end{aligned}$$

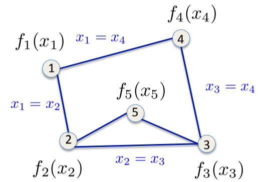
- Each minimization involves (quadratic perturbations of) functions f and g separately.
  - In some applications, these minimizations are easy (quadratic minimization, l_1 minimization, which arises in Huber fitting, basis pursuit, LASSO, total variation denoising).
- Best known convergence rate: $O(1/k)$ [He, Yuan 11]. [1]

[1] Under stronger assumptions (strong convexity, Lipschitz gradient), ADMM converges linearly [Goldfarb et. al 10], [Deng, Yin 12], [Hong, Luo 12].

ADMM for Multi-agent Optimization Problem

- Multi-agent optimization can be reformulated in the ADMM framework:
- Consider a set of agents $V = \{1, \dots, N\}$ in an undirected connected graph $G = \{V, E\}$.
- $\mathcal{N}(i)$: agent i's neighborhood.
- We introduce a local copy x_i in $\mathbb{R}^n$ for each of the agents and impose $x_i = x_j$ for all $(i, j) \in E$.

$$\begin{aligned} \min_x \quad & \sum_{i=1}^N f_i(x_i) \\ s.t. \quad & x_i = x_j, \quad \text{for} \quad (i,j) \in E, \end{aligned}$$

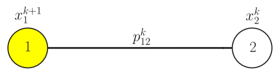


Special Case Study: 2-agent Optimization Problem

- Multi-agent optimization problem with two agents:

$$\begin{aligned} \min_{x_1, x_2} \quad & f_1(x_1) + f_2(x_2) \\ \text{s.t.} \quad & x_1 = x_2. \end{aligned}$$

- ADMM applied to this problem yields:



- $x_1^{k+1} = \text{argmin}_{x_1} \quad f_1(x_1) + f_2(x_2^k) - (p_{12}^k)'(x_1 - x_2^k) + \frac{\beta}{2} ||x_1 - x_2^k||_2^2$

Special Case Study: 2-agent Optimization Problem

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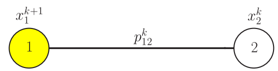

- $x_1^{k+1} = \text{argmin}_{x_1} \quad f_1(x_1) - (p_{12}^k)'x_1 + \frac{\beta}{2} ||x_1 - x_2^k||_2^2$

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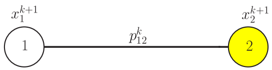

- $x_2^{k+1} = \text{argmin}_{x_2} \quad f_1(x_1^{k+1}) + f_2(x_2) - (p_{12}^k)'(x_1^{k+1} - x_2) + \frac{\beta}{2} \|x_1^{k+1} - x_2\|_2^2$

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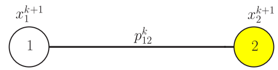

- $x_2^{k+1} = \text{argmin}_{x_2} \quad f_2(x_2) + (p_{12}^k)'x_2 + \frac{\beta}{2} \|x_1^{k+1} - x_2\|_2^2$

Special Case Study: 2-agent Optimization Problem

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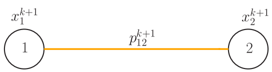
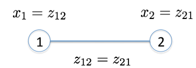


- $p_{12}^{k+1} = p_{12}^k - \beta(x_1^{k+1} - x_2^{k+1})$.

Multi-agent Optimization Problem: Reformulation

- Requires a globally known order on the agents [Wei, Ozdaglar 12].
- Reformulate to remove ordering: technique from [Bertsekas, Tsitsiklis 89].
- Rewrite each constraint $x_i - x_j = 0$ for edge $e = (i,j)$ as

$$x_i = z_{ij}, \quad x_j = z_{ji},$$
$$z_{ij} = z_{ji}.$$



- The reformulated problem can be written compactly as

$$\begin{aligned} \min_{x \in \mathbb{R}^{Nn}, z \in Z} \quad & F(x) = \sum_{i=1}^{N} f_i(x_i) \\ s.t. \quad & Dx + z = 0. \end{aligned} \tag{1}$$

where $Z = \{z \in \mathbb{R}^{2Mn} \mid z_{ij} = z_{ji}, \text{ for } (i,j) \text{ in } E\}$.

- Assumption: The optimal solution set of this problem is nonempty.

ADMM for Multi-agent Optimization

a The primal variable x is updated as

$$x^{k+1} \in \operatorname*{argmin}_{x} F(x) - (p^k)'Dx + \frac{\beta}{2}\left|\left|Dx + z^k\right|\right|^2.$$

b The primal variable z is updated as

$$z^{k+1} \in \operatorname*{argmin}_{z \in Z} -(p^k)'Hz + \frac{\beta}{2}\left|\left|Dx^{k+1} + z\right|\right|^2.$$

c The dual variable p is updated as

$$p^{k+1} = p^k - \beta(Dx^{k+1} + z^{k+1}).$$

- The minimizers in the update equations exist (since matrix $D'D$ has full rank).
- Update in z is a quadratic program with linear constraint: has a closed form solution and can be computed using communication between neighboring nodes.

Distributed Implementation

- Each agent i maintains x_i^k and p_{ij}^k, z_{ij}^k, for $j \in \mathcal{N}(i)$.
- At iteration k,

  a Agent i updates the primal variable x_i^k as

$$x_i^{k+1} \in \operatorname*{argmin}_{x_i} f_i(x_i) - \sum_{j \in \mathcal{N}(i)} (p_{ij}^k)'x_i + \frac{\beta}{2} \sum_{j \in \mathcal{N}(i)} \left|\left|x_i - z_{ij}^k\right|\right|^2.$$

  The value x_i^{k+1} is sent to all neighbors.

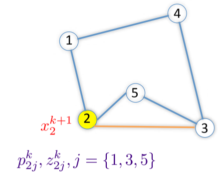
  b Agent i computes

$$z_{ij}^{k+1} = z_{ji}^{k+1} = \frac{1}{2}(x_i^{k+1} + x_j^{k+1}).$$

  c Agent i computes

$$p_{ij}^{k+1} = p_{ij}^k - \frac{\beta}{2}(x_i^{k+1} - x_j^{k+1}),$$

  for all neighbors in j in $\mathcal{N}(i)$.



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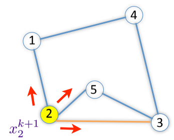

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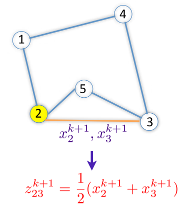

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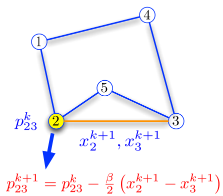


Rate of Convergence

- Let $\{x^k, z^k, p^k\}$ be the sequence generated by the synchronous ADMM algorithm. We define the ergodic averages:

$$\bar{x}^k = \frac{1}{k}\sum_{l=0}^{k-1} x^k, \qquad \bar{z}^k = \frac{1}{k}\sum_{l=0}^{k-1} z^k.$$

  We also define the ergodic time average of the residual: $\bar{r}^k = D\bar{x}^k + \bar{z}^k$.

Theorem

Let (x^*, z^*, p^*) be a primal-dual optimal solution for problem (1). The following hold at each iteration k:

$$|F(\bar{x}^k) - F(x^*)| \leq \frac{1}{2\beta k}\left(\beta^2\left\|z^0 - z^*\right\|^2 + \max\left\{\left\|p^0\right\|^2, \left\|p^0 - 2p^*\right\|^2\right\}\right),$$

$$\|\bar{r}(k)\| \leq \frac{1}{2\beta k}\left(\beta^2\left\|z^0 - z^*\right\|^2 + \left(\left\|p^0 - p^*\right\| + 1\right)^2\right).$$

Proof Idea

- Using optimality conditions and dual update, we obtain for all $p \in \mathbb{R}^{2Mn}$,

$$F(x^{k+1}) - F(x^*) - p' r^{k+1} \leq \frac{1}{2\beta}\left(\left\|p^k - p\right\|^2 - \left\|p^{k+1} - p\right\|^2\right) + \frac{\beta}{2}\left(\left\|z^k - z^*\right\|^2 - \left\|z^{k+1} - z^*\right\|^2\right).$$

- First bound follows from adding this over a window and using convexity to generate function values at ergodic averages.
- Second bound follows from picking $p = p^* - \frac{\bar{r}^k}{\|\bar{r}^k\|}$.
- One can use $\left\|p^k - p^*\right\|^2 + \beta^2\left\|z^k - z^*\right\|^2$ as a Lyapunov function to show that the sequence $\{x^k, z^k, p^k\}$ converges to a primal-dual optimal solution of problem (1).

Network Effect

- The performance depends on the network topology through $\|p^*\|$.

Theorem

Let (x^*, z^*, p^*) be an optimal primal-dual solution for problem (1). Then,

$$\|p^*\|^2 \leq \frac{2C^2}{\rho_2(L(G))},$$

where C is a bound on the norms of all vectors in $\partial F(x^*)$ and $\rho_2(L(G))$ is the second smallest positive eigenvalue of the Laplacian matrix $L(G)$ of the underlying graph.[2]

- The performance depends on the algebraic connectivity of the graph: the more connected it is, the larger $\rho_2(L(G))$ is, the better the performance.

[2] $L(G)$ is a matrix with elements $[L(G)]_{ij} = \begin{cases} \text{degree}(i) & i = j, \\ -1 & (i,j) \in E. \end{cases}$

Dependence of Convergence Rate on Network

Theorem

Let (x^*, z^*) be a primal optimal solution for problem (1). Then with initialization $x^0 = 0$, $z^0 = 0$ and $p^0 = 0$, we have

$$|F(\bar{x}^k) - F(x^*)| \leq \frac{M\beta \,||x^*||^2}{kN} + \frac{4C^2}{k\beta\rho_2(L(G))},$$

$$||\bar{r}^k|| \leq \frac{1}{2k\beta}\left(\frac{C\sqrt{2}}{\sqrt{\rho_2(L(G))}} + 1\right)^2 + \frac{M\beta\,||x^*||^2}{kN},$$

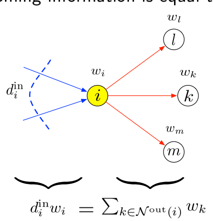
where M is the number of edges.

Distributed ADMM over Directed Networks

- So far, we have assumed communication among nodes bidirectional.
- This does not hold when nodes have different local interference patterns and different power transmission levels.
- Question: Can we design a distributed ADMM-like algorithm over directed networks with fast convergence? [Makhdoumi, Ozdaglar 14]
  - Every agent updates primal and dual variables based on information received from incoming links.
- Key idea: Normalize the information entering a node with the *right weight* so that in the limit the network becomes balanced (i.e., amount of information entering and leaving a node is equalized).
- Two approaches:
  - Push-sum [Kempe, Dobra, Gehrke 03], [Nedich, Olshevsky 13,14].
    - Leads to nonlinear updates and a non convergent algorithm for ADMM.
  - Graph balancing: [Gharesifard, Cortes 09], [Priolo et al. 13], [Rikos et al. 14].

Weight Balanced Graph

- Let $G = (V, E)$ be a strongly connected directed network.
- $\mathcal{N}^{\text{in}}(i)$: in-neighbors of node i; $\mathcal{N}^{\text{out}}(i)$: out-neighbors of node i.
- Each node has a weight w_i representing how much he scales his incoming information.
- The node weights $(w_1, \ldots, w_N)$ make the network balanced if for each node i, total weight of i's incoming information is equal to the total weight of its outgoing information.



Distributed Balancing

- Laplacian of the directed graph L: $L_{ii} = d_i^{\text{in}}$ and $L_{ij} = -1$, $j \in \mathcal{N}^{\text{in}}(i)$.
- The balancing weight vector $\mathbf{w} = (w_1, \ldots, w_N)'$ is the left eigenvector of L corresponding to eigenvalue 0, i.e., $\mathbf{w}'L = 0$; or for $i = 1, \ldots, N$,

$$(w_1, \ldots, w_i, \ldots, w_N)\begin{bmatrix} & -1 & \\ & \vdots & \\ \cdots & d_i^{\text{in}} & \cdots \\ & \vdots & \\ & \underbrace{}_{i\text{th column}} & \end{bmatrix} \Rightarrow d_i^{\text{in}} w_i = \sum_{k \in \mathcal{N}^{\text{out}}(i)} w_k.$$

- We will use a slight variation of [Priolo et al. 13] to design a distributed balancing algorithm.
- For each node i, let $0 < \delta_i < \frac{1}{d_i^{\text{in}}}$.
- Let $C = I - \Delta L$, where $\Delta = \text{diag}(\delta_1, \ldots, \delta_N)$.

Distributed Balancing Algorithm

Each agent i maintains weight vector $\tilde{\mathbf{w}}_i^k \in \mathbb{R}^N$. At iteration k:

- The weight matrix $\tilde{W}^k = [\tilde{\mathbf{w}'}_1^k; \dots; \tilde{\mathbf{w}'}_N^k]$ is updated as
$$\tilde{W}^{k+1} = C\tilde{W}^k.$$
- This corresponds to each agent i updating his weight vector $\tilde{\mathbf{w}}_i^k$ using local information from his in-neighbors:
$$\tilde{\mathbf{w}'}_i^{k+1} = (1 - \delta_i d_i^{in})\tilde{\mathbf{w}'}_i^k + \sum_{j \in N^{\text{in}}(i)} \delta_i \tilde{\mathbf{w}'}_j^k.$$
- Agent i sends $\tilde{\mathbf{w}}_i^{k+1}$ to all his out-neighbors.

Convergence of Balancing Algorithm

Theorem

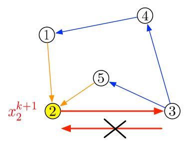
Starting from $\tilde{W}^0 = I$, we have
$$\lim_{k\to\infty} \tilde{W}^k = \lim_{k\to\infty} C^k = \mathbf{1}\mathbf{v}' = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \\ v_1 & v_2 & \cdots & v_N \\ \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & \cdots & v_N \end{bmatrix},$$
where $\mathbf{v}'$ is the left eigenvector of C corresponding to eigenvalue 1. Convergence is exponentially fast with exponent given by the second largest eigenvalue of L.

Proof: Follows from Perron-Frobenius Theorem (C is a primitive matrix since G is strongly connected).

- Hence the weight vector of agent i converges to $\tilde{\mathbf{w}}_i^\infty = [v_1, \dots, v_N]'$.
- The balancing weight for agent i is $w_i^\infty = \delta_i \tilde{\mathbf{w}}_i^\infty(i) = \delta_i v_i$.
$$v_i = (1 - \delta_i d_i^{\text{in}})v_i + \sum_{j \in N^{\text{out}}(i)} \delta_j v_j \Rightarrow d_i^{\text{in}} \underbrace{\delta_i v_i}_{w_i^\infty} = \sum_{j \in N^{\text{out}}(i)} \underbrace{\delta_j v_j}_{w_j^\infty}$$

Implementation of ADMM over Directed Graphs

- Consider the distributed ADMM iteration over undirected networks.
- At iteration k, agent i updates the primal variable x_i^k as
$$x_i^{k+1} \in \operatorname*{argmin}_{x_i} f_i(x_i) - \sum_{j \in \mathcal{N}^{\text{in}}(i)} (p_{ij}^k)' x_i + \frac{\beta}{2} \sum_{j \in \mathcal{N}^{\text{in}}(i)} \left\| x_i - z_{ij}^k \right\|^2 - \sum_{j \in \mathcal{N}^{\text{out}}(i)} (p_{ij}^k)' x_i + \frac{\beta}{2} \sum_{j \in \mathcal{N}^{\text{out}}(i)} \left\| x_i - z_{ij}^k \right\|^2.$$
- This requires node i to receive information from $j \in \mathcal{N}^{\text{out}}(i)$.



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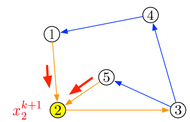
- Fix:
  - Use only incoming information in the update.



Implementation of ADMM over Directed Graphs

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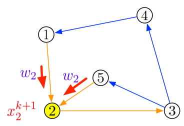
- Fix:
  - Use only incoming information in the update.
  - Scale incoming information by balancing weight w_i.
  - Update w_i^k using distributed balancing in the same time scale.



Distributed ADMM over Directed Networks

- Each agent i maintains x_i^k, $\tilde{\mathbf{w}}_i^k$ (w_i^k) and p_{ij}^k, z_{ij}^k, for $j \in \mathcal{N}^{\text{in}}(i)$.
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    The values x_i^{k+1} and $\tilde{\mathbf{w}}_i^k$ are sent to all out-neighbors.
  - b For all $j \in \mathcal{N}^{\text{in}}(i)$, agent i computes

$$z_{ij}^{k+1} = \frac{1}{2}(x_i^{k+1} + x_j^{k+1}),$$

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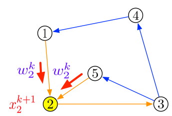
  - d Agent i lets $w_i^{k+1} = \delta_i \tilde{\mathbf{w}}_i^{k+1}(i)$, where

$$\tilde{\mathbf{w}}'^{k+1}_i = (1 - d_i^{\text{in}}\delta_i)\tilde{\mathbf{w}}'^k_i + \sum_{j\in N^{\text{in}}(i)} \delta_i \tilde{\mathbf{w}}'^k_j$$



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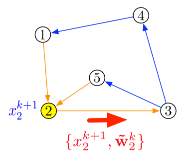

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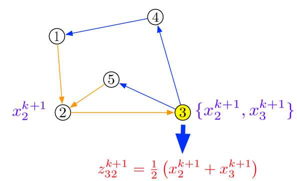

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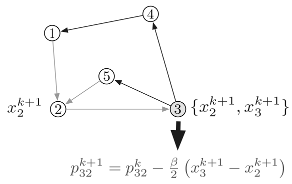

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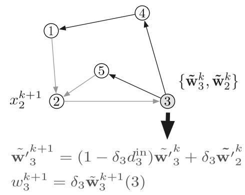


Rate of Convergence

- Let $\{x^k, z^k, p^k, w^k\}$ be the sequence generated by the Directed ADMM algorithm. We define the ergodic average:

$$\bar{x}^k = \frac{1}{k} \sum_{l=0}^{k-1} x^k.$$

- Assume the sequence $\{x^k\}$ is bounded.

Theorem

Let (x^*, z^*, p^*) be a primal-dual optimal solution for problem (1). The following hold at each iteration k:

$$|F(\bar{x}^k) - F(x^*)| \leq \frac{1}{k} D, \qquad \sum_{(i,j) \in E} |\bar{x}_i^k - \bar{x}_j^k| \leq \frac{1}{k} C,$$

where the constants C and D depends on number of nodes N, number of edges M, $||p^*||$, and the second largest eigenvalue of Laplacian.

Conclusions and Future Work

- We presented distributed ADMM-based algorithms for solving multi-agent optimization problems over undirected and directed networks.
- For general convex cost functions, we showed that these methods converge at the faster rate $O(1/k)$ and provided rate estimates that highlighted dependence on network structure.
- Simulation results illustrate the superior performance of ADMM (even for network topologies with slow mixing).
  - This comes at the expense of a more complex local optimization step.
- Extensions and Future Work:
  - Asynchronous distributed ADMM algorithms [Wei, Ozdaglar 13].
  - Broadcast-based ADMM algorithms [Makhdoumi, Ozdaglar 14].
  - Graph balancing for distributed optimization over directed networks.
  - ADMM-type algorithms for time-varying networks.
  - Distributed and incremental second order methods [Gurbuzbalaban, Ozdaglar, Parrilo 14].

LEARNING GRAPHICAL MODELS: HARDNESS AND TRACTABILITY

Devavrat Shah, Massachusetts Institute of Technology, USA

Our understanding of the question of learning graphical model, despite it's fundamental importance, has been far from satisfactory. In this talk, I will discuss recent progresses we have made towards improving this status quo. Specifically, we will discuss the reasons for learning graphical model being hard, both computationally and statistically. We shall also discuss, somewhat surprising, conditions under which it becomes tractable. Time permitting, we shall discuss a variant of the 'standard' graphical model learning problem from literature that might be more relevant to practice.

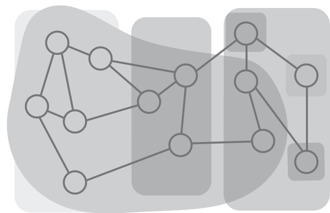
This talk describes joint work with Guy Bresler and David Gamarnik, both at MIT.

Learning graphical models: hardness and tractability

Guy Bresler David Gamarnik Devavrat Shah
MIT

graphical models

$G = (V, E) \quad |V| = p \quad |\partial i| \leq d$



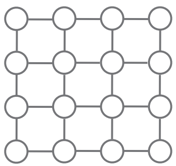
$X_A \perp\!\!\!\perp X_B \mid X_S$

$X_i \perp\!\!\!\perp X_{V \setminus \partial i \cup \{i\}} \mid X_{\partial i}$

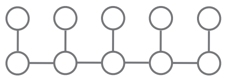
efficient inference

belief propagation can be used to do inference

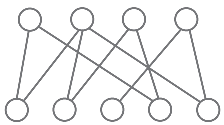
historically people knew what models to use



lattice

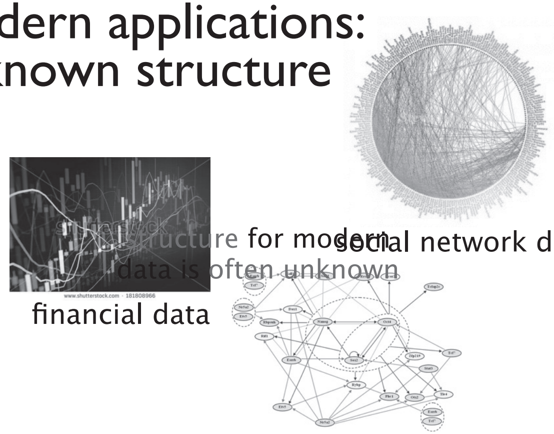


hidden Markov model



LDPC code

modern applications: unknown structure



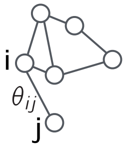
financial data

social network data

gene regulatory network

learning graphical model

$G = (V, E)$
$|V| = p$
$|\partial i| \le d$
$X \in \{0,1\}^p$



graphical model: $\mathrm{P}(X) = \frac{1}{Z} \exp\left(\sum_{\{i,j\} \in E} \theta_{ij} X_i X_j + \sum_{i \in V} \theta_i X_i \right)$

$\alpha \le |\theta_{ij}| \le \beta$

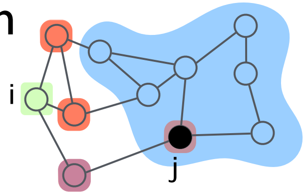
data: $X^{(1)}, X^{(2)}, \ldots, X^{(n)} \quad X \sim \mathrm{P}$ (i.i.d. samples)

task: reconstruct graph and parameters from the data

w.prob. $\to 1$ as $n, p \to \infty$

baseline: exhaustive search algorithm

[Abbeel-Koller-Ng '06]
[Bresler-Mossel-Sly '08]



$X_i \perp\!\!\!\perp X_{V \setminus \partial i \cup \{i\}} \mid X_{\partial i}$

test whether $U \subseteq \partial i$

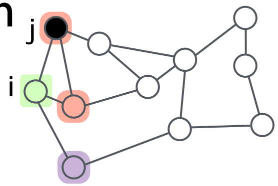
if $U \subsetneq \partial i$ then for some $j \in U$ and $W \supseteq \partial i \setminus U$

$|\mathrm{P}(X_i = +1 | X_U = x_U, X_W = x_W)$
$- \mathrm{P}(X_i = +1 | X_U = \mathrm{flip}_j(x_U), X_W = x_W)| = 0$

"*U* fails test"

baseline: exhaustive search algorithm

[Bresler-Mossel-Sly '08]



$X_i \perp\!\!\!\perp X_{V \setminus \partial i \cup \{i\}} \mid X_{\partial i}$

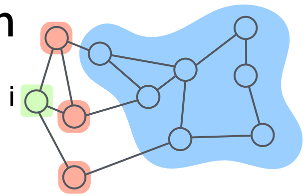
test whether $U \subseteq \partial i$

if $U \subseteq \partial i$ then for all $j \in U$ and $W \supseteq \partial i \setminus U$

$|\mathrm{P}(X_i = +1 | X_U = x_U, X_W = x_W)$
$- \mathrm{P}(X_i = +1 | X_U = \mathrm{flip}_j(x_U), X_W = x_W)| \ge \frac{\tanh 2\alpha}{2e^{2(d-1)\beta}}$

"*U* passes test"

baseline: exhaustive search algorithm



$X_i \perp\!\!\!\perp X_{V \setminus \partial i \cup \{i\}} \mid X_{\partial i}$

Algorithm:

for each node *i*
test all possible neighborhoods *U*
choose largest *U* passing test

Theorem: [Bresler-Mossel-Sly '08]

algorithm recovers with prob. $1 - o(1)$ using $n = O(2^{2d} e^{(4\beta+h)d} \log p)$ samples, w runtime $\widetilde{O}(p^{2d+1})$

our notion of computational efficiency

exhaustive search: $p^{\Theta(d)}$

efficient: $f(d)p^c$

can be exponential

indep. of d!

want to have no restrictions on graph structure

question: for what types of interactions can we learn *efficiently*?

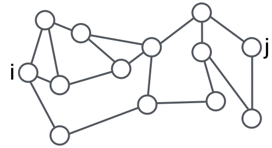
correlation decay

Theorem: [Bresler-Mossel-Sly '08]

if have *correlation decay* and $EX_iX_j \geq \kappa$ for $\{i,j\} \in E$ can learn using $n = O(2^{8d}e^{16\beta d}\log p)$ samples in time $O(np^2)$

$f(d)p^c$: $c = 2$, $f(d) = 2^{8d}e^{16\beta d}$

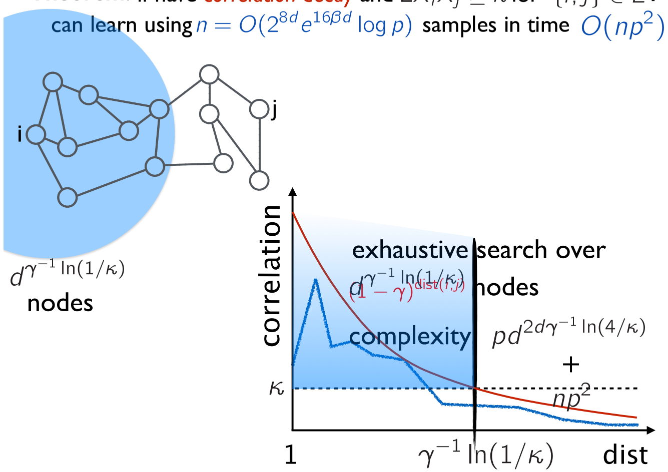
exponential decay of correlations: $EX_iX_j \leq (1-\gamma)^{\mathrm{dist}(i,j)}$



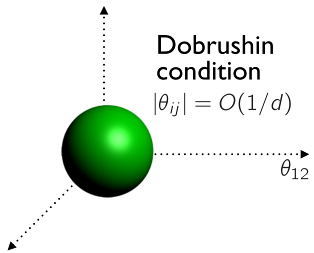
various models satisfy CDP:

[Dobrushin '70, Dobrushin-Shlosman '85, Martinelli '95, Weitz '06, Salas-Sokal '97, Bandyopadhyay-Gmarnik '08, Gamarnik-Goldberg-Weber '13, and many others…]

Theorem: if have *correlation decay* and $EX_iX_j \geq \kappa$ for $\{i,j\} \in E$: can learn using $n = O(2^{8d}e^{16\beta d}\log p)$ samples in time $O(np^2)$



correlation decay



other low-complexity approaches to learning:

[Ravikumar-Lafferty-Wainwright '06]
[Lee-Ganapathi-Koller '06]
[Anandkumar-Tan-Huang-Willsky '12]
[Ray-Sanghavi-Shakkottai '12]
[Wu-Srikant-Ni '13]
[many many others]

[Bento-Montanari '09]

several low-complexity algorithms explicitly or implicitly require correlation decay

all known low-complexity algorithms fail without correlation decay

can we learn efficiently without correlation decay?

repelling models

$$P(X) = \frac{1}{Z} \exp\left(\sum_{\{i,j\} \in E} \theta_{ij} X_i X_j + \sum_{i \in V} \theta_i X_i \right) \qquad X \in \{0,1\}^p$$

$$\alpha \le |\theta_{ij}| \le \beta$$

repelling → $\theta_{ij} \le -\alpha$

$\theta_i \le h$

$\alpha \ge d(h + \ln 2)$

Theorem: [Bresler-Gamarnik-Shah '14a]

can learn these models with prob. $1 - o(1)$ using $n = O(2^{2d} e^{4\beta d} \log p)$ samples, with runtime $O(np^2)$

repelling models

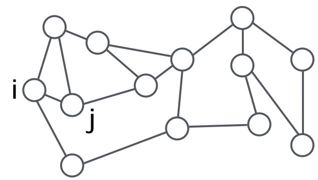
Ex. Independent set model

$$\alpha \to \infty, \text{ i.e. } \theta_{ij} \to -\infty$$

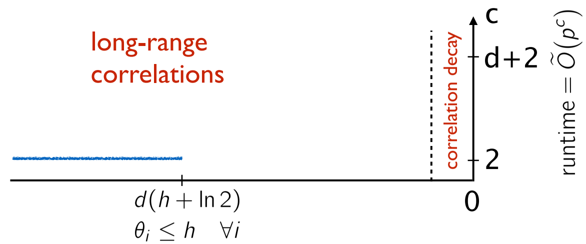
Key observation:

i and j are neighbors: $(X_i, X_j) \neq (1,1)$ $w.p.$ 1

i and j are not neighbors: $(X_i, X_j) = (1,1)$ $w.p. \ge \gamma(d)$



repelling models

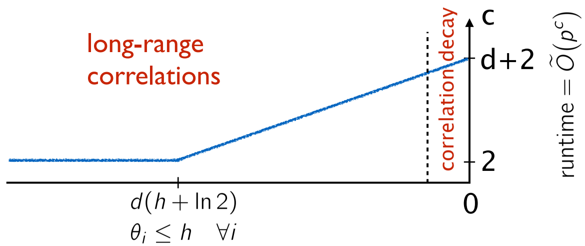


Theorem: [Bresler-Gamarnik-Shah '14a]

can learn these models with prob. $1 - o(1)$ using $n = O(2^{2d} e^{4\beta d} \log p)$ samples, with runtime $O(np^2)$

repelling models

~~$\alpha \geq d(h + \ln 2)$~~
$\alpha \geq (d - \tau)(h + \ln 2)$



Theorem: [Bresler-Gamarnik-Shah '14a]

can learn these models with prob. $1 - o(1)$ using $n = O(2^{2d} e^{4\beta d} \log p)$ samples, with runtime ~~$O(np^2)$~~ $O(np^{2+\tau})$

learning parameters

once you know the graph, learning parameters is easy

Algorithm (node-wise parameters)

1. consider samples with $X_{\partial i} = 0$
2. this allows to estimate

$$P(X_i = 1 | X_{\partial i} = \mathbf{0}) = \frac{e^{h_i}}{1 + e^{h_i}}$$

3. solve for h_i

Theorem:

algorithm recovers with prob. $1 - o(1)$ using $n = O(2^{2d} e^{(4\beta + h)d} \log p)$ samples, with runtime $O(np)$

can we do better in general

exhaustive search: $p^{\Theta(d)}$

Theorem: [Bresler-Gamarnik-Shah '14a]

No algorithm can do better than $p^{\Theta(d)}$ under the computation model of "statistical algorithms" in general.

a general approach to simplifying:

reduce to sufficient statistics

reducing to sufficient statistics

$$P(X) = \frac{1}{Z} \exp\Big(\sum_{\{i,j\} \in E} \theta_{ij} X_i X_j + \sum_{i \in V} \theta_i X_i \Big) \qquad X \in \{0,1\}^p$$

$$(\mu_i)_i = (\mathrm{E}X_i)_i = (\mathrm{P}(X_i = i))_i$$
$$(\mu_{ij})_{ij} = (\mathrm{E}X_i X_j)_{ij} = (\mathrm{P}(X_i = X_j i))_{ij}$$

sufficient statistics

try to estimate $\mu \mapsto \theta$ (feasible in principle!)

physicists try to estimate this map using various "expansions"

[Ricci-Tersenghi '12]
[Sessak-Monasson '08]
[Cocco-Monasson '12]
...many others

reducing to sufficient statistics

$$P(X) = \frac{1}{Z} \exp\Big(\sum_{\{i,j\} \in E} \theta_{ij} X_i X_j + \sum_{i \in V} \theta_i X_i \Big) \qquad X \in \{0,1\}^p$$

$\theta_{ij} \to -\infty$ (special case of repelling model)

$P(X) = \frac{1}{Z} \exp\Big(\sum_{i \in V} \theta_i X_i \Big) \mathbb{1}(X$ is an independent set$)$

$\mu = (\mathrm{E}(X_i))_i = (\mathrm{P}(X_i = 1))_i$ sufficient statistics

Theorem: [Bresler-Gamarnik-Shah '14b] [Montanari '14]

learning parameters of graphical models from sufficient statistics is NP-hard

some remarks on proof

Reduction:

Suppose there exists efficient algorithm for $\mu \mapsto \theta$

Use it as a black-box to solve a known difficult problem

The difficult problem: given θ find corresponding $\mu \equiv \mu(\theta)$

For independent set with $\theta = \mathbf{0}$ this corresponds to

counting # of independent sets in G
a known hard (to approximate) problem

[Dyer-Frieze-Jerrum '02]
[Sly '10]
[Sly-Sun '12]

some remarks on proof

Reduction:

The difficult problem: given θ find corresponding $\mu \equiv \mu(\theta)$

Solve using black-box $\mu \mapsto \theta$

$$\mu(\theta) \in \arg\max_{\nu \in \mathcal{M}} \langle \nu, \theta \rangle + H_{\mathrm{ER}}(\nu)$$

Gradient ascent:

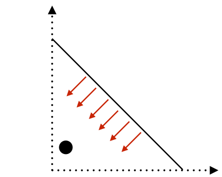
$\mu^t \to \theta^t$

$$\mu^{t+1} = \mu^t + \frac{1}{t}\Big(\theta - \theta^t\Big)$$

Key challenge:

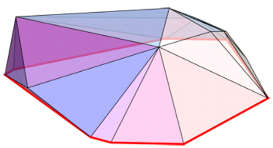
μ^{t+1} needs to be projected on marginal polytope $\mathcal{M}$

some remarks on proof



what if algorithm naturally
avoids boundary

Lemma: [Bresler-Gamarnik-Shah '14b]
For the objective of interest, the polytope boundary has an inherent repulsion property



marginal polytope is
very complicated

once you know the graph,
learning parameters is easy

graph tells you on which
higher order statistics to focus

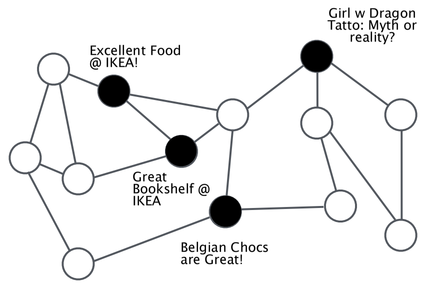
learning from sufficient statistics
is probably not a good idea

so far: i.i.d. data

revisit original goal: learning from data

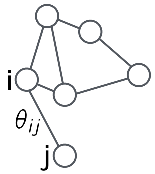
what sort of data?

Social Behavior: Purchases, Likes, …



dynamics over time

learning models from data



$$P(X) = \frac{1}{Z}\exp\left(\sum_{\{i,j\}\in E}\theta_{ij}X_iX_j + \sum_{i\in V}\theta_iX_i\right)$$

$$X \in \{0,1\}^p \quad \alpha \le |\theta_{ij}| \le \beta$$

data: $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ ~~i.i.d. samples~~ *n* steps of some process

task: reconstruct graph and parameters from the data

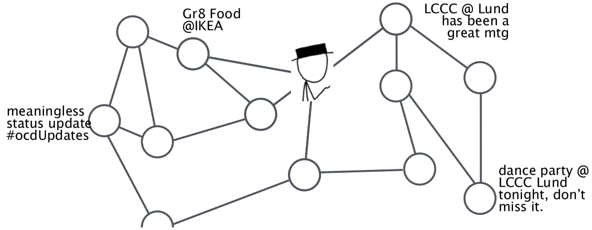
w.prob. $\to 1$ as $n, p \to \infty$

Glauber dynamics

1. each node has a Poisson(1) clock

2. when clock rings, update variable according to

$$P(X_i = 1 | X^t_{\partial i}) = \frac{\exp\left(2\sum_{j\in\partial i}\theta_{ij}X^t_j\right)}{1+\exp\left(2\sum_{j\in\partial i}\theta_{ij}X^t_j\right)}$$



slow mixing

i.i.d. sampling is NP-hard for some models
but Glauber dynamics defined for any graphical model

for models without correlation decay, the Glauber dynamics is known to mix exponentially slowly in *p*

samples will be *far* from i.i.d.

efficient learning from the Glauber dynamics

Theorem: [Bresler-Gamarnik-Shah '14c] with $n = O(e^{4d\beta}\log p)$ samples per node, and runtime $O(np^2)$ can learn *any pairwise model* even *without correlation decay*

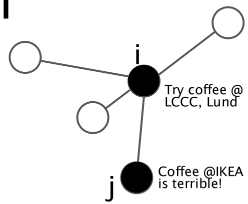
learning theory:

[Aldous-Vazirani '90]
[Bartlett-Fischer-Hoffgen '94]
[Bshouty-Mossel-O'Donnell-Servedio '03]

epidemic models:

[Netrapalli-Sanghavi '12]
[Dahleh-Tsitsiklis-Zoumpoulis '13]

estimating effect of a neighbor



imaginary scenario: node i updates, then node j flips, then node i again

test for existence of an edge:

$$\exp\left(\theta_{ij}\right) = \frac{p^+(1-p^-)}{p^-(1-p^+)}$$

$$p^+ = \mathbb{P}\big(X_i = +1 | X_{\partial i \setminus j} = +\mathbf{1}, X_j = +1\big)$$
$$p^- = \mathbb{P}\big(X_i = +1 | X_{\partial i \setminus j} = +\mathbf{1}, X_j = -1\big)$$

this would require $\Omega(e^{d\beta} p^2)$ samples per node

a more delicate argument is required to get to $O_d(\log p)$

summary

correlation decay is <u>not</u> necessary to learn efficiently

however exhaustive algorithm seems the best in general

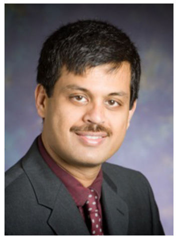
reducing to sufficient statistics is computationally suboptimal

observing dynamics over time can make things easy

insight: often makes sense to learn structure first
and only then estimate parameters

LOAD BALANCING USING LIMITED STATE INFORMATION

R. Srikant, University of Illinois at Urbana Champain, USA



Social network providers, search engine providers, and other big data companies operate massive data centers with thousands of processors handling data processing requests from incoming tasks. A central problem in such data centers is to design good load balancing algorithms using very little state information, where the state of each processor is the queue length of tasks waiting to be processed at that processor. We will present an algorithm which does remarkably well while sampling approximately only one queue per arriving task. This reduces the sampling requirement by half compared to previously known algorithms, while matching the delay performance of the best known algorithms. Joint work with Lei Ying and Xiaohan Kang

Load Balancing Using Limited State Information

R. Srikant*
Joint work with Lei Ying+ and Xiaohan Kang+

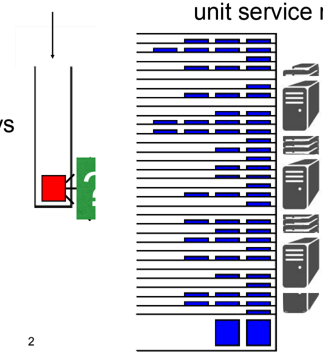
*University of Illinois at Urbana-Champaign
+Arizona State University

1

Load Balancing

- Arriving tasks have to be routed to a server
- Requirement: small delays
- Join-the-shortest-queue
- Expensive feedback overhead

n servers with unit service rate



2

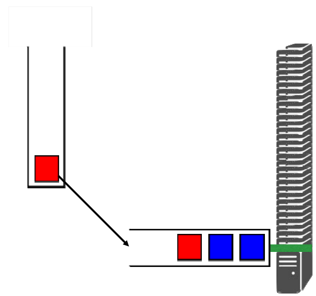
Data Centers are Large

Yahoo! Hadoop cluster
42,000 nodes

3

Random Routing

- No overhead
- Delay ~ $\frac{1}{1-\rho}$
- ρ: Traffic Intensity, i.e., the ratio of the arrival rate of tasks to the maximum rate at which they can be processed by the servers



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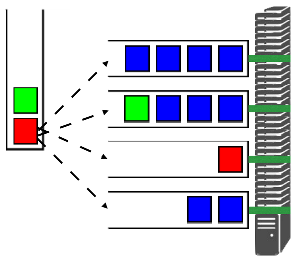
Power-of-Two-Choices

- Mitzenmacher 1996,
- Vvedenskaya, Dobrushin & Karpelevich 1996
- Delay, many-servers limit

$$\sum_{l=1}^{\infty} \rho^{2^l-2}$$

$$\rightarrow \log_2 \frac{1}{1-\rho} \quad (\rho \approx 1)$$



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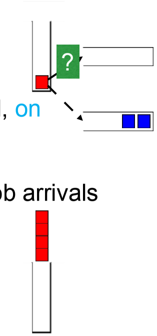
Key Question

- Question: sample one queue per arrival, on average, instead of two?
- Observation (Ousterhout et al, 2013): job arrivals occur in large batches (parallel tasks)



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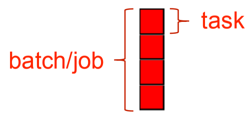
Model and Main Results

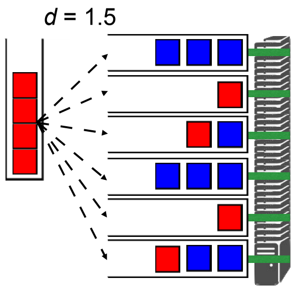
Model

- n servers
- Fixed batch size m $\quad 1 \ll m \ll n$
- Poisson batch arrivals with rate $n\rho/m$
- Note: Job is completed when all tasks in the job are completed
- Exponentially distributed service



Batch-Sampling (BatchSamp)

- Ousterhout et al. 2013
- Probe ratio d
- One job in each of the smallest queues



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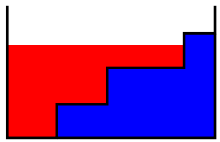
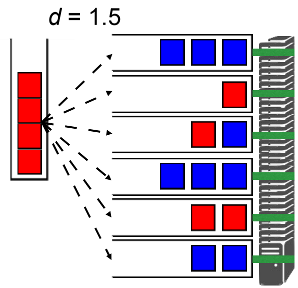
Batch-Filling (BatchFill)

Our algorithm:
Batch Sampling
+
WaterFilling





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BatchSamp vs BatchFill

- Suppose we sample six queues to distribute five tasks: let's say that the queue lengths of the sampled queues are 0, 1, 2, 15, 20, 25
- Under BatchSamp, the resulting queue lengths are 1, 2, 3, 16, 21, 25
- Under BatchFill, they are 2, 3, 3, 15, 20, 25

BatchFill

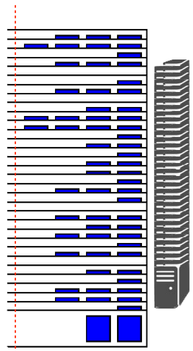
- Many-server heavy-traffic delay

$$\frac{\log \frac{1}{1-\rho}}{\log(1+d)}$$

- Queue length upper-bounded by

$$\left\lceil \frac{\log \frac{1}{1-\rho}}{\log(1+\rho d)} \right\rceil$$

$\rho = 0.9$, $d = 1.1$, upper bound 4



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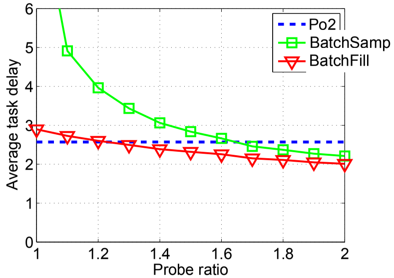
Simulation Results: Average Task Delay



n = 10,000, m = 100, ρ = 0.7

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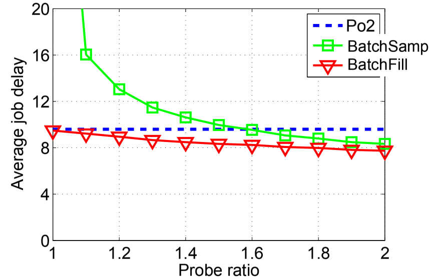
Simulation Results: Average Job Delay



n = 10,000, m = 100, ρ = 0.7

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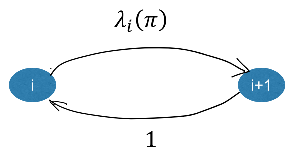
Mean Field Approximation

General Idea

- Focus on a particular queue: Assume all other queues are in steady-state (stationary distribution $\boldsymbol{\pi}$) and independent of each other
- The arrival rate when there are i tasks in the queue is a function of the state of the other queues, and hence a function of $\boldsymbol{\pi}$
- $\boldsymbol{\lambda_i(\pi)\pi_i = \pi_{i+1}}$



Calculating $\lambda_i(\pi)$: Po2

- Task arrival rate $n\rho$
- Prob(a particular queue is sampled) = $2/n$
- Probability being chosen $\pi_i/2+\pi_{i+1}+\pi_{i+2}+\ldots$
- $\lambda_i = \rho(\pi_i + 2\pi_{i+1} + 2\pi_{i+2} + \cdots)$

Stationary Distribution: Po2

- Queue length distribution: $\pi_l = \rho^{2^l-1} - \rho^{2^{l+1}-1}$
- Delay: $\frac{\sum_{l=1}^{\infty} l\pi_l}{\rho} = \sum_{l=1}^{\infty} \rho^{2^l-2}$

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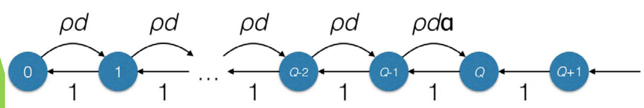
BatchSamp

- No waterfilling
- Reducible to finite states

$0 < a \leq 1$



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Why is the queue finite?

- md out of n queues are sampled
- We see the empirical distribution: $md\pi_0$ empty queues, $md\pi_1$ queues with one task, etc.
  $0, 0, 0, 1, 1, 1, 1, 1, \ldots, j, j, j, j, \ldots$
- The first m of these queues gets one task each
- Under the mean-field approximation, the number of tasks in the m^{th} queue is fixed, as a function of π

π for BatchSamp

- $\pi_0 = 1-\rho$
- $\pi_i = (1-\rho)\rho^i d^i, \quad 1 \le i \le Q-1$
- $\pi_Q = 1-(1-\rho)(\rho^Q d^Q-1)/(\rho d-1)$

Cutoff queue length

$$Q = \left\lceil \frac{\log \frac{d-1}{d(1-\rho)}}{\log(\rho d)} \right\rceil$$

- Delay $\frac{\sum_{i=1}^{\infty} i\pi_i}{\rho} \approx \frac{\log \frac{1}{1-\rho}}{\log d}$ (heavy-traffic)

- When d = 2, asymptotically equivalent to Po2

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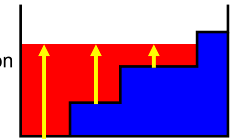
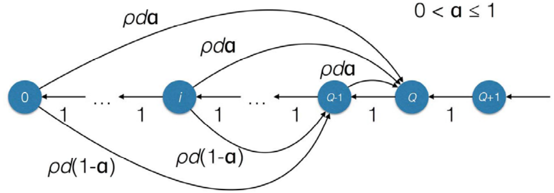
BatchFill

- Constant up-crossing rate to Q or Q-1, determined by the stationary distribution π



$0 < a \le 1$



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π for BatchFill

- $\pi_0 = 1-\rho$
- $\pi_i = (1-\rho)\rho d(1+\rho d)^{i-1}, \quad 1 \le i \le Q-1$
- $\pi_Q = 1-(1-\rho)(1+\rho d)^{Q-1}$

Cutoff queue length

$$Q = \left\lceil \frac{\log \frac{1}{1-\rho}}{\log(1+\rho d)} \right\rceil$$

- Delay $\frac{\sum_{i=1}^{\infty} i\pi_i}{\rho} \approx \frac{\log \frac{1}{1-\rho}}{\log(1+d)}$ (heavy-traffic)

- Better asymptotic delay than Po2 when d > 1!

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Justifying the Mean-Field Approximation

Ordinary Differential Equation (ODE) for BatchFill

- Deriving an ODE: Derivative is given by

$$\lim_{n\to\infty} \lim_{\delta\to 0^+} E\left(\frac{change\ in\ state}{\delta} \middle| current\ state\right)$$

- ODE

$$\frac{dx_i}{dt} = \begin{cases} -(1+\rho d)x_i + x_{i+1} & i \le \bar{X}_x - 2 \\ \rho d(1-\alpha_x)\sum_{j=0}^{i-1} x_j - (1+\rho d\alpha_x)x_i + x_{i+1} & i = \bar{X}_x - 1 \\ \rho d\alpha_x \sum_{j=0}^{i} x_j - x_i + x_{i+1} & i = \bar{X}_x \\ -x_i + x_{i+1} & \text{otherwise} \end{cases}$$

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Global Asymptotic Stability of ODE

- $s_i(t)$ fraction of servers with queue size ≥ i
- Lyapunov function

$$V(s) = \sum_{i=1}^{\infty} |s_i - \hat{s}_i|$$

- $s(t)$ converges to the equilibrium point for any $s(0)$

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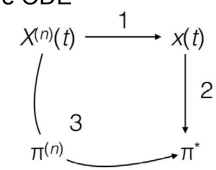
Justifying the Mean-Field Approximation

1. The ODE approximation (in a finite time interval) works well. The deviation from the ODE goes to zero as n goes to infinity
2. Global asymptotic stability of the ODE
3. Interchange of limits



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Conclusions

- One sample can be powerful in randomized load-balancing
- Batch arrivals can be exploited to reduce sampling overhead
- Extensions: (i) Variable batch sizes, (ii) Batch arrivals are not necessary, (iii) General service-time distributions and Processor Sharing

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ON THE EIGENVALUES OF LARGE STRUCTURED MATRICES AND THE SCALABLE STABILITY OF NETWORKS

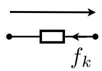
Glenn Vinnicombe, University of Cambridge, UK

A class of scalable stability results for interconnections of linear dynamical systems depend ultimately on tools for locating the eigenvalues of large structured matrices using only local information. This talk will summarise some existing results in this area and introduce some open problems.

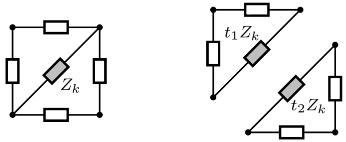
Scalable Design Rules for Physical Systems

Richard Pates *, and Glenn Vinnicombe *,

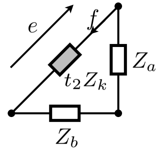
$$e_k = Z_k(s) f_k$$
$$\text{or } e_k = \phi_k(f_k)$$



Component



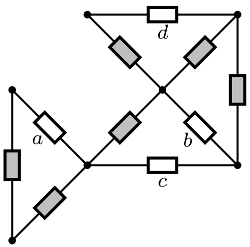
$$\frac{1}{t_1} + \frac{1}{t_2} = 1$$



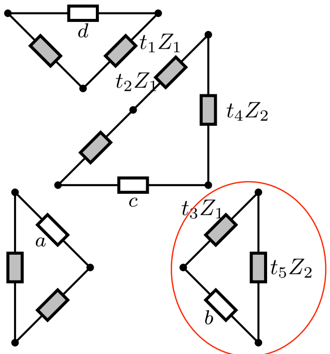
$$e = t_2 Z_k(s) f$$
$$-f = \frac{1}{(Z_a(s) + Z_b(s))} e$$

$$\implies \text{use Nyquist with } L(s) = \frac{t_2 Z_k(s)}{Z_a(s) + Z_b(s)}$$

(we will be assuming that Z_k is open circuit stable and Z_a, Z_b short circuit stable).

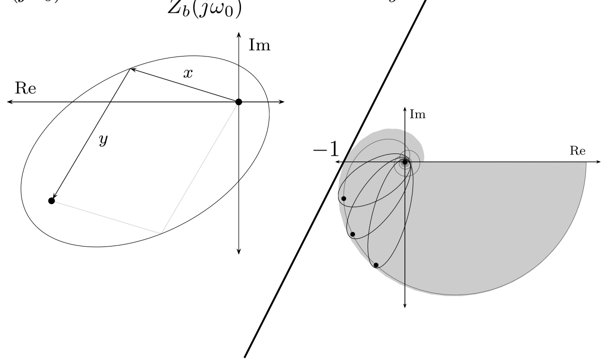


$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} = 1$$
$$\frac{1}{t_4} + \frac{1}{t_5} = 1$$



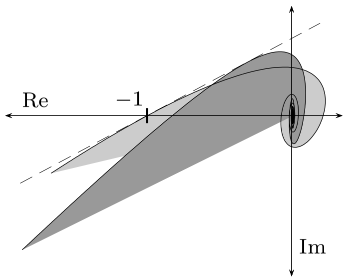
$$L(s) = \frac{t_3 Z_1 + t_5 Z_2}{Z_b}$$

$$L(j\omega_0) = \frac{t_3 Z_1(j\omega_0) + t_5 Z_2(j\omega_0)}{Z_b(j\omega_0)} = x + y$$



Network is stable if this *and all other* curves lies to the right of a globally specified line through -1

Can do Popov too!



So, what's going on?

Return Ratio of form GA

$$G = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & g_n \end{bmatrix}, \quad [A]_{ij} \in \Re$$

If $A = A^T > 0$ then

$$\sigma(GA) \in \mathrm{Co}\{g_i\}\rho(A) \qquad \text{e.g. V (2000)}$$

If $g_1 = g_2 = \cdots = g_n = g$ then

$$\sigma(GA) = g\sigma(A) \qquad \text{e.g. Fax \& Murray (2001)}$$

e.g $A = R\,\mathrm{diag}\{l_i\}R^T$ or $A =$ a Laplacian (consensus problems).

Can these be put together?

$$AGv = \lambda v$$
$$Gv = A^{-1}v$$
$$\implies v^*Gv = \lambda v^*A^{-1}v$$

So $W(G) \cap W(A^{-1}) = \emptyset \implies$ no eigenvalue at λ
(where $W(X) = \left\{\frac{v^*Xv}{v^*v} : v \in \mathbb{C}^n, v \neq 0\right\}$)

also

$$v^*G^*Gv = \lambda^2 v^*A^{-*}A^{-1}v$$

So $DW(G) \cap DW(A^{-1}) = \emptyset \implies$ no eigenvalue at λ ([Jönsson and Kao 2010, Lestas 2012])
(where $DW(X) = \left\{\Re\frac{v^*Xv}{v^*v}, \Im\frac{v^*Xv}{v^*v}, \frac{v^*X^*Xv}{v^*v} : v \in \mathbb{C}^n, v \neq 0\right\}$)

Also

$$DW(G) \cap DW(A^{-1}) = \emptyset \iff DW(G^{-1}) \cap DW(A) = \emptyset$$

What about neighbouring dynamics

Could consider $\sqrt{G}A\sqrt{G}$, but strongest results are in the bipartite case:

e.g. $G = \mathrm{diag}(f_1, f_2, \ldots, h_1, h_2, \ldots)$ $A = [0\ R; R^T\ 0]$, $A_{ij} \in \{-1, 0, 1\}$

$$\sigma\left(\mathrm{diag}(g_i)R^T\,\mathrm{diag}(h_i)R\right) \subset \mathrm{Co}\{m_i h_i S(n_j g_j : R_{ij} \neq 0)\}$$

where $S(X) = \mathrm{Co}(\sqrt{X})^2$ [V (2002), Lestas & V (2006)]

Open question: Local conditions for eigenvalue locations

Richard Pates & Glenn Vinnicombe

$$G = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & & \\ \vdots & & \ddots & \\ 0 & \dots & & g_n \end{bmatrix}, \quad [A]_{ij} \in \{0, -1, 1\}$$

Identify A with the adjacency matrix of a *directed* graph, with $g_i \in \mathbb{C}$ labelling the nodes and R_{ij} labelling the edges.

What can we say about the $\sigma(GA)$ in terms of *local information* about the cycles of A (including those of length 2)?

What we know (bipartite & symmetric)

If

$$G = \text{diag}(f_1, f_2, \dots, h_1, h_2, \dots),\ A = [0\ R; R^T\ 0], A_{ij} \in \{0, 1\}$$

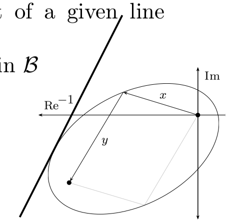
$$\sigma\,(GA)^2 \subset \text{Co}\{f_i E(n_j h_j : R_{ij} \neq 0)\}$$

where $E(x_i)$ is ellipse with foci 0, $\sum x_i$ and major axis $\sum |x_i|$, n_j = in-degree of node h_j.

For $\mathcal{B}$ being the region to the right of a given line through -1,

and question is "does your ellipse lie in $\mathcal{B}$

If all answer "yes", then $\sigma(GA) \in \mathcal{B}$



A better result is

$$\sigma\left(\text{diag}(g_i) R^T \text{diag}(h_i) R\right) \subset \text{Co}\{h_i E(n_j g_j : R_{ij} \neq 0)\}$$

where $E(x_i)$ is ellipse with foci 0, $\sum x_i$ and major axis $\sum |x_i|$. [Pates & V (2012)]

proof :

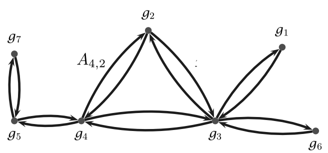
$$\sigma\left(\text{diag}(g_j) R^T \text{diag}(h_i) R\right) = \sigma\left(\text{diag}(\sqrt{g_j}) R^T \text{diag}(h_i) R\, \text{diag}(\sqrt{g_j})\right)$$
$$= \sigma\left(\sum_i \text{diag}(\sqrt{g_j}) R_{i\cdot}^T h_i R_{i\cdot}\, \text{diag}(\sqrt{g_j})\right)$$

etc

Each node knows all cycles it participates in, and for each of those cycles g_i, A_{ij}, n_i along that cycle.

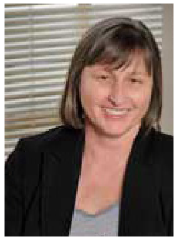
Is it possible, for some region $\mathcal{B}$, to come up with a yes/no question such that if all nodes say "yes", based on their local information, then $\sigma(GA) \in \mathcal{B}$?

What is the smallest $\mathcal{B}$ (and associated question).



TOWARD STANDARDS FOR DYNAMICS IN FUTURE ELECTRIC ENERGY SYSTEMS: THE BASIS FOR PLUG-AND-PLAY INDUSTRY PARADIGM

Marija Ilic, Carnegie Mellon University, USA



In this talk we propose that by taking a step back and understanding the fundamental physics of interconnected electric power systems one can begin to think systematically about the role of both competitive and cooperative control, and their pros and cons for these rapidly evolving systems. To support this, we organize our presentation in several parts: 1) physics and operation of power grids; 2) the role of control; 3) systematic specification of performance objectives (dynamic standards) for plug-and-play operations; 4) use of cyber for implementing standards (sensing, communications and control architectures for supporting implementation of these standards); and, 5) illustration of how several industry problems can be solved when following proposed plug-and-play standards for dynamics (use of power electronics for transient stabilization during faults and sudden equipment failures; use of PMUs for ensuring voltage and frequency quality by an intelligent balancing authority (iBA) – generalization of today's control areas; storage control in microgrids/systems with small inertia).

In the first part we give a somewhat new, dynamic systems view, of physical operation in several qualitatively different power grids (bulk power regulated; bulk power with markets; hybrid mix of emerging grids; micro-girds for developing countries; and micro-grids for developed countries). The physical operation dictates how to define internal states of given (groups of) physical components and the interaction variables between different (groups of) physical components. Based on understanding physical principles, we propose a new state space model of an interconnected grid comprising (groups of) different physical components. This model has a transparent physical interpretation, and, it is, therefore used as the basis for explicit performance specifications in terms of interactions of components. Performance specifications become standards for plug-and-play dynamic interactions of components (behavior) over several time horizons within an otherwise complex dynamical grid, which, when followed, ensure system-wide performance. We emphasize that the proposed approach is a framework for thinking about the necessary specifications in future electric energy systems, which enables both choice of technology at the (groups of) component levels and, minimal interaction specifications to align these with the performance of the over overall system. It is not a specific method. This framework could possibly overcome the roadblock of integrating distributed local grid technologies with the bulk power grid without running into major coordinating complexity. Specific proposed approaches to distributed control for smart grids are interpreted.

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Electric Energy Systems Group — Electrical & Computer ENGINEERING

Toward standards for dynamics in future electric energy systems— The basis for plug-and-play industry paradigm

Marija Ilić milic@ece.cmu.edu
Electric Energy Systems Group (EESG) http://www.eesg.ece.cmu.edu/, Director
Invited talk, LCCC Workshop
Lund, Sweden
October 15-17, 2014

Outline

- Overview of current NERC standards and evolving standards for wind and solar plants
- Issues with current standards
- Our proposal :
  - Plug-and-play (TCP/IP) like protocols/standards
  - Introduction of intelligent Balancing Authority (iBAs)
- Examples of iBAs
- Theoretical foundations for new standards (TCP/IP like)
- Proof-of-concept examples of controller designs which meet such protocols

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NERC standards Transmission Planning Standards

- System simulations and associated assessments are needed periodically to ensure that reliable systems are developed that meet specified performance (http://www.nerc.com)

Category	Contingencies	System Stable and both Thermal and Voltage Limits within Applicable Rating	Loss of Demand
A	No contingency	Yes	No
B	Event resulting in the loss of a single element.	Yes	No
C	Event(s) resulting in the loss of two or more (multiple) elements.	Yes	Planned/ Controlled
D	Extreme event resulting in two or more (multiple) elements removed or Cascading out of service.	Evaluate for risks and Consequences. - May involve substantial loss of customer Demand and generation in a widespread area or areas. - Portions or all of the interconnected systems may or may not achieve a new, stable operating point. -Evaluation of these events may require joint studies with neighboring systems.	

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Evolving standards for Wind and Solar Generation Technologies

- voltage/var control/regulation
- voltage ride-through
- power curtailment and ramping
- primary frequency regulation
- inertial response

http://www.nerc.com/files/2012_IVGTF_Task_1-3.pdf
NERC 2012 Special Assessment: Interconnection Requirements for Variable Generation September 2012

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Need for a new paradigm

- Today's industry approach– the worst case approach, inefficient and does not rely on on-line automation and regulation other than energy feed-forward economic dispatch
- Emphasis on large-scale time-domain system simulations for transient stability, voltage, collapse, power flow feasibility, etc
- Primary control is constant gain tuned assuming no dynamic interactions with the rest of the system
- Existing and emerging system-level unacceptable interactions; no incentives for ``smarts" of modules

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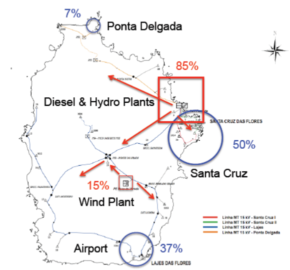
From old to new paradigm—Flores Island Power System, Portugal [11]



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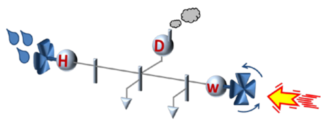
Controllable components—today's operations (very little dynamic control, sensing)



H – Hydro
D – Diesel
W – Wind

*Sketch by Milos Cvetkovic

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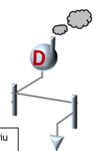
Two Bus Equivalent of the Flores Island Power System



State	Equilibrium
$e_q'[pu]$	0.9797
$\delta[rad]$	0.0173
$\omega[pu]$	1
$v_r[pu]$	0.8527
$e_{fd}[pu]$	0.7482
$v_f[pu]$	0
$P_m[pu]$	0.01
$a[pu]$	0

Generator	Diesel
$x_d[pu]$	8.15
$x_q[pu]$	8.15
$x_d'[pu]$	0.5917
$x_q'[pu]$	0.5917
$T_{q0}'[s]$	2.35
$T_{d0}'[s]$	2.35
$J[s]$	2.26
$D[pu]$	0.005

Transmission line	From Diesel to Load bus
$R[pu]$	0.3071
$L[pu]$	0.1695

Base values $S_b = 10MVA$ $V_b = 15KV$

AVR	Diesel
$K_A[pu]$	400
$T_A[s]$	0.02
$K_E[pu]$	1.3
$T_E[s]$	1
$S_E[pu]$	0.1667
$K_F[pu]$	0.03
$T_F[s]$	1

Governor	Diesel
$k_t[pu]$	40
$T_g[s]$	0.6
$r[pu]$	1/0.03
$T_t[s]$	0.2

Base values $S_b = 10MVA, V_b = 0.4KV$

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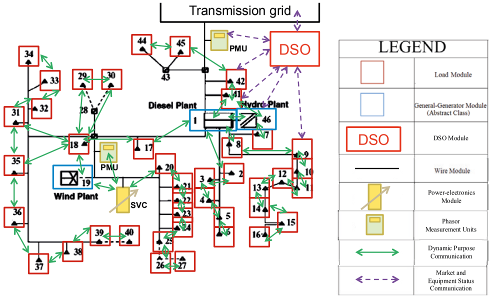
Information exchange in the case of Flores---new (lots of dynamic control and sensing)



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Possible dynamical problems seen by particular dynamic components

		Dynamical problems						
		Small signal instab.	Transient instab.	SSR	SSCI	Freq. instab.	Volt. Instab.	Power flow imbalance
Types of Component	Synchronous generators	?	?	?	?	?	?	?
	Wind generators	?	?	?	?	?	?	?
	Solar plants	?	?	?	?	?	?	?
	FACTS	?	?	?	?	?	?	?
	Storage	?	?	?	?	?	?	?

Table 1.

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Our proposal: TCP/IP like standards

❖ Given **specified disturbances and range of operating conditions within a known system**:
- specified with e.g voltage, power
- similar to LVRT curves for wind turbines
- with specified duration

❖ **All components** (synchronous gens, wind gens) **should guarantee** that they would not create any of the problems in Table 1. (Clear objectives goals for components, assigned responsibility for system reliability)

❖ Two key **questions**: Q1-- Why does it matter?
Q2)--- Can this be technically done?
Not one way to achieve these!

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Not one way to meet the standards -iBAs

❖ **iBAs (intelligent Balancing Authorities)**
- **Single component** or **group of components** which meet the desired objectives: Given **specified disturbances** their components do not cause any of the dynamical problems in Table 1.
- **Dynamic notion** of Control Areas—intelligent Balancing Authorities (iBAs)

❖ iBAs **would utilize advanced control design** methods to meet the protocol; could be either decentralized or wide area control (cooperative control to save on number of controllers and energy used within the iBA)
- Huge potential for **exploiting efficiently new technologies** like **storage** and **FACTS** and at the same time have guaranteed system performance

S.Baros, M.Ilic intelligent Balancing Authorities (iBAs) for Transient Stabilization of Large Power Systems IEEE PES General Meeting 2014

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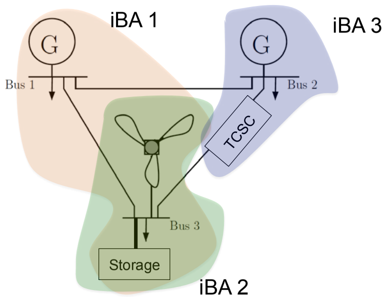
A1: Examples of iBAs—it matters for ensuring both reliable and efficient operations [13]

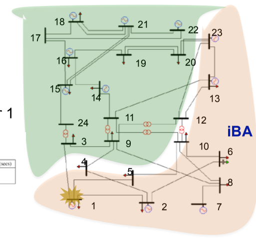


Possible to create iBAs for meeting transient stability distributed standard

Given disturbance
Tripping of generator 1

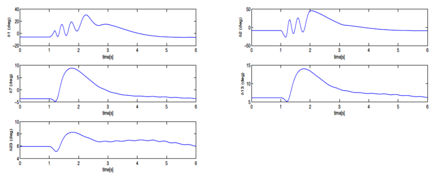
TABLE I
CONTINGENCY CONSIDERED ON THE IEEE RTS 24 BUS SYSTEM

CONTINGENCY (CATEGORY B)	DURATION (secs)
Tripping of generator 1 after 3-phase fault on its terminal bus	0.17



S.Baros, M.Ilic intelligent Balancing Authorities (iBAs) for Transient Stabilization of Large Power Systems IEEE PES General Meeting 2014

Rotor angle response of iBA generators



(b) Transient stabilization of critical generators i=1,2,7,13,23 with iBA-based control in low-load scenario

Q2: Can we have a unifying theoretically sound approach to TCP/IP like standards for smart grids? [12]

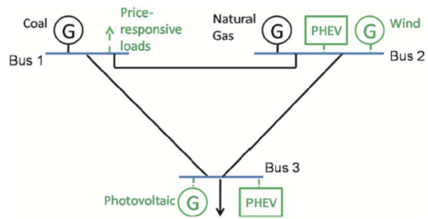


Fig. 5. Small example of the future electric energy system.

Basic functionalities

- Simple transparent TCP/IP like functionalities
- Transparency based on a unifying modular modeling of network system dynamics
- Provable performance-difficult
- Proposal—use interaction variables to specify family of standards sufficient to avoid operating problems
  - Measure of how well modules balance themselves in steady state
  - Measure of rate of exchange of stored energy between a module and the rest of the system over different time horizons

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Unifying modeling and control approach— use of multi-scale interaction variables

- Standards/protocols --- specifications of module interactions for plug-and-play operations; architectures define how are sets of protocols organized
- Cyber design for managing multi-layered interactions
- New physics-based modeling and control as the basis for interaction variables-based protocols
- Illustrations of possible standards-based enhancements (transient stabilization using power electronics switching; storage control in micro-grids)

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New physics-based modeling and control using interaction variables [12]

- Mechanical system representation of electric power grids
- Physics-based state transformation for multi-layered dynamics
- Defining interaction variables over different time horizons (to capture bounds on change in stored energy over T and on the rate of change of stored energy)
- Multi-layered specifications for interaction variable-based standards

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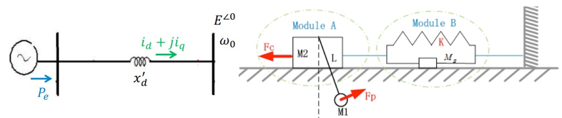
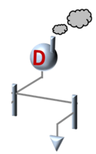
Two-module power system in Flores



- Module A—power plant with its governor and excitation control
- Module B—transmission line controlled by FACTS



Ilic, ECE Smart Grid and Future Electric Energy Systems course, Fall 2014

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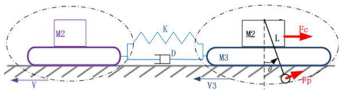
Interconnected power system and its mechanical representation

Induction Machine-

Conventional generator



- Each mechanical sub-system is analogous to a generator within interconnected power system (with Xia Miao)
- Spring and damper between pendulum-mass systems the same as transmission line in interconnected power system;
- Only conveyor components of mechanical systems are connected and placed on the same reference (ground);
- If speed of each pendulum-mass systems are not same, interactions will happen.

Analogy Table

- Component level

Mechanical Quantity	Power System Analogue
Force, F_p	Mechanical Power Input, P_m
Velocity, V	Current, I
Force, F_C	Exciter input, E_{fd}
Force between M_2 & M_3, F_{32}	Back EMF

- System level

Mechanical Quantity	Power System Analogue
Velocity, V_3	Terminal Voltage, V
Interaction Force, F_s	Current, I

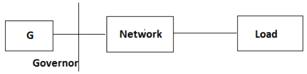
Very intriguing questions

- Can this system synchronize around inverted pendulum position? Is this acceptable? What should be a standard/protocol for plug-and-play?
- Which controller works? Why?
- Answers depend on the actual operating conditions (close to the stable or unstable equilibrium); on control laws applied to Fp and/or Fc; if the equilibrium is not known (stabilization) energy-based controller will work, while constant gain controller will not. It will also depend on the load model used. The control effort and quality of response vary .
- Not all ``smarts" are the same in smart grids!

Three different architectures of interest

- Bulk power system can be represented as an infinite bus (ideal voltage source)
- Flores Island (micro-grid)– diesel power plant has dynamics—hydro and wind negative constant power load
- Flores Island (micro-grid)-diesel power plant dynamics and the load is constant impedance

Governor of AC Generator



- Governor is the main synchronizing controller.
- In mechanical representation, Fp responds to the deviations of speed from synchronous speed.
- Does not synchronize around inverted pendulum position.
- It does synchronize when Fp also responds to x3-xB (phase lock loop).

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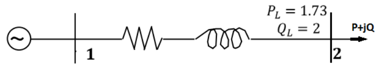
Constant power load model

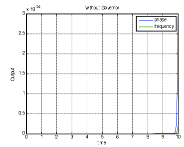


- ***Unstable Equilibrium Set***

$$\begin{cases} V_2^{(2)} = 2.7419 \\ \theta_2^{(2)} = -149.348 \end{cases}$$

- ***System Matrix***

$$A_{open} = \begin{bmatrix} 0 & 377 \\ 1.398 & -0.0221 \end{bmatrix} \qquad \begin{cases} \lambda_1 = 22.9569 \\ \lambda_2 = -22.9569 \end{cases}$$



Unstable equilibrium

- ***Participation Factor***

$$P = \begin{bmatrix} 0.5 & 0.4998 \\ 0.4998 & 0.5 \end{bmatrix}$$

- θ and ω contribute same to the unstable eigenvalue which means governor should take care of θ as well as ω (PI control)

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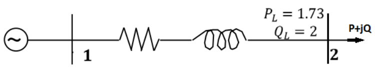
Synchronized system to constant power load



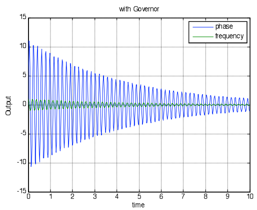
- ***Governor Design***

$$Governor: K_d(\omega - \omega^*) + K_i(\theta - \theta^*)$$

- ***System Matrix***

$$A_{closed} = \begin{bmatrix} 0 & 377 \\ -3.03 & -0.46 \end{bmatrix}$$

$$\begin{cases} \lambda_1 = -0.23 + 33.779i \\ \lambda_2 = -0.23 - 33.779i \end{cases}$$



- ***System response***

Stable

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Second control –fast Fc

- Exciter in power plant
- Fc responds to acceleration of rotating pendulum
- Mechanical analogy—control acceleration of the pendulum pivot
- Internal interactions non-unique—can control with Fp, Fc or a combination (competitive at the module A level)
- Module B could also control stiffness of the spring ***FACTS is controllable inerter** to help stabilize module A (cooperative control)

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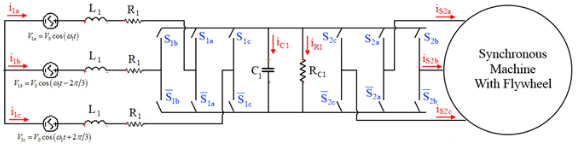
Gets complicated fast—
Variable Speed Drives for Flywheels



- Stator voltages of synchronous machine with flywheel are NOT assumed to be directly controllable
- The only controllable inputs are the switch positions of the power electronics
- Use AC/DC/AC converter to regulate the speed of the flywheel to a different frequency than the grid frequency

Source: K. D. Bachovchin, M. D. Ilić, "Passivity-Based Control Using Three Time-Scale Separations of Variable Speed Drives for Flywheel Energy Storage Systems," EESG Working Paper No. R-WP-6-2014, October 2014.

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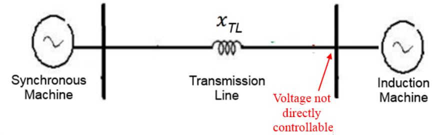
Standalone vs. Interconnected Variable Speed Drives

- For standalone variable speed drives, the voltage of the stator windings is directly controllable
- For interconnected systems, the voltage of the stator windings is not directly controllable and depends on the dynamics of the rest of the system
  - Need power electronics (AC/DC/AC converter) in order to control the stator voltages



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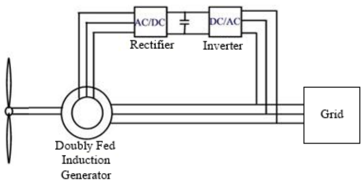
Doubly-fed induction machines (DFIG)—wind power plant

- AC/DC/AC converter interfacing between the stator windings and the rotor windings of the DFIM
- The rotor voltage is controlled through switches in the AC/DC/AC converters



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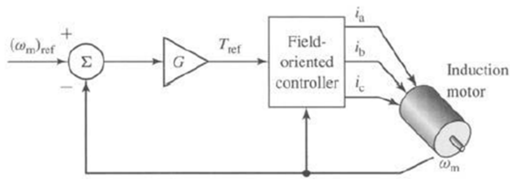
Field Oriented Control for Variable Speed Drives



- Outer (proportional) controller regulates the speed of the induction machine
- Inner (field oriented) controller regulates the torque to the set point given by outer controller

Source: A. Fitzgerald, C. Kingsley Jr., and S. Umans, *Electric Machinery*. Piscataway, NJ: McGraw-Hill, 2003.

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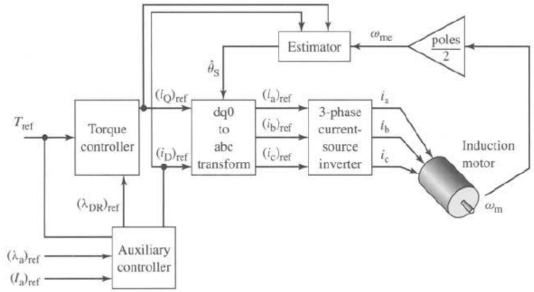
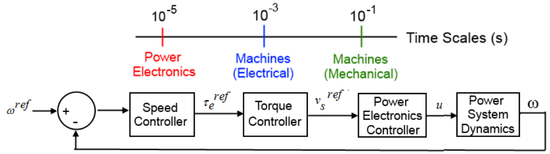
Field Oriented Controller Block Diagram



Source: A. Fitzgerald, C. Kingsley Jr., and S. Umans, *Electric Machinery*. Piscataway, NJ: McGraw-Hill, 2003.

Controller Implementation--detailed

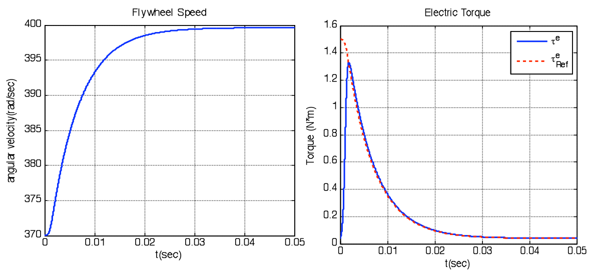
- Can use time-scale separation to simplify the control design



- Speed controller uses dynamic model at the mechanical machine (slowest) time-scale
- Torque controller uses dynamic model at the electrical machine time-scale
- Power electronics controller uses dynamic model at the power electronics (fastest) time-scale

Source: K. D. Bachovchin, M. D. Ilić, "Passivity-Based Control Using Three Time-Scale Separations of Variable Speed Drives for Flywheel Energy Storage Systems," EESG Working Paper No. R-WP-6-2014, October 2014.

Control design—can be done



- Can control speed of the flywheel to set point $\omega_2 = 400\ rad/sec$
- The actual electric torque converges very quickly to the reference electric torque

Source: K. D. Bachovchin, M. D. Ilić, "Passivity-Based Control Using Three Time-Scale Separations of Variable Speed Drives for Flywheel Energy Storage Systems," EESG Working Paper No. R-WP-6-2014, October 2014.

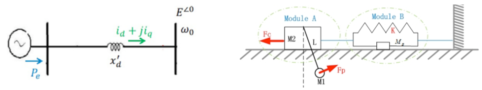
Must simplify!

- Utilities are having hard time adding all these new components and their smarts for simulating system-wide dynamics
- Is there a ``smarter" way to model and define modular functionalities so that the interconnected system meets system-level performance (Table 1)?
- 80% of each solution is modeling (Petar Kokotovic, Chalenges in Control Theory, Santa Clara, circa 1982)

From acceleration to stored energy

- Basis for Furuta stabilization of pendulum around the inverted equilibrium; actuator dynamics neglected, acceleration control Fc. No Fp. Requires lots of effort
- A better approach: Energy approach Astrom, Furuta
- For power systems this would mean: Avoiding blackout by changing the logic of Fc when close to voltage ``collapse"

Dynamics of interconnected system in new state space



- Standard state space of interconnected system

$$\dot{\overline{\overline{X}}}_A = f_A\left(\overline{\overline{X}}_A, Z_A, P_A, u_A\right)$$

$$\dot{Z}_A = f_{ZA}\left(\overline{\overline{X}}_A, Z_A, P_B\right)$$

Dynamics of Interaction variables

$$\dot{P}_A = f_{PA}\left(\overline{\overline{X}}_A, P_A, \dot{P}_B\right)$$

$$\dot{Z}_B = f_{ZB}\left(Z_B, P_A, u_B\right)$$

$$\dot{P}_B = f_{PB}\left(P_B, \dot{P}_A\right)$$

Interaction variable-based two-layer model

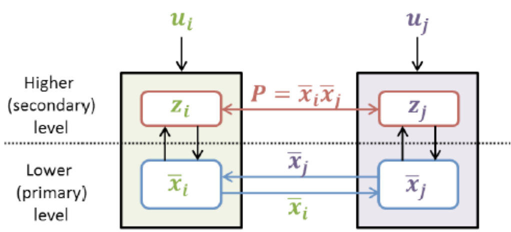
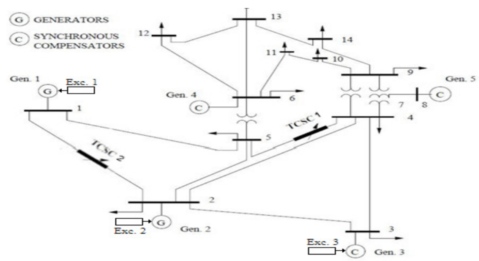


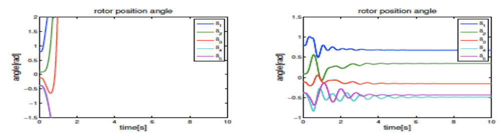
Fig. 1. An example of a system comprised of two modules i and j.

Nonlinear cooperative control of FACTS in the new state space to synchronize generator 2



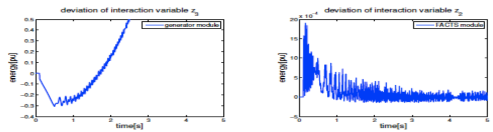
3. IEEE 14 bus system with two TCSC and five excitation controllers.

With Milos Cvetkovic



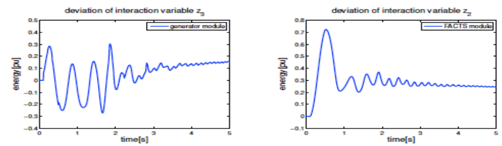
(a) Uncontrolled system (b) With cooperative controllers

Fig. 4. Generator rotor angle response with and without the controller.



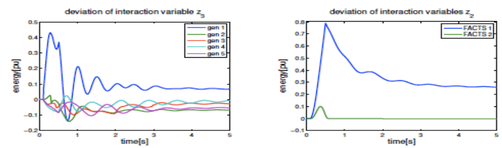
(a) Generator modules (b) FACTS modules

Fig. 5. Interaction variables without any controllers in *Case 1*.



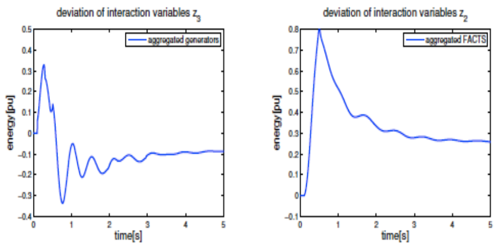
(a) Generator modules (b) FACTS modules

Fig. 6. Interaction variables in the controlled system in *Case 1*.



(a) Generator modules (b) FACTS modules

Fig. 7. Interaction variables of the controlled system in *Case 2*.



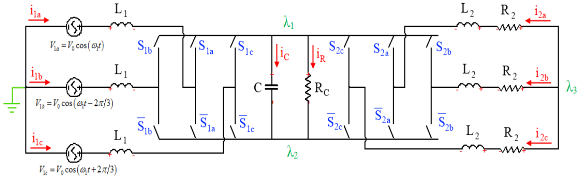
(a) Generator modules (b) FACTS modules

Fig. 8. Aggregated interaction variables in *Case 2*.

Interaction variables in new state space

- System dynamics separable into multi-layer system: internal layer and interaction layer;
- Rate of change of stored energy zero when module disconnected
- Natural evolution of control area specifications (ACE) – linearized, steady-state notion; can cooperate for AGC
- Essential for iBA-level transient stabilization, power flow feasibility, SSCI, SSR standards (non-existent today)
- Using this modeling framework, different control strategy can be used and designed: competitive or cooperative control
- Note on synchronization—in the new state space entirely decentralized controller

Passivity-based AC/DC/AC Converter control



- Choose to directly regulate the direct and quadrature components of the load and source currents
- Desired capacitor charge has dynamics

$$\frac{dq_C{}^D}{dt} = -\frac{\left(q_C{}^D\right)^2 - C^2R_C\left(V_{1d}i_{1d}{}^* + V_{1q}i_{1q}{}^*\right) + C^2R_CR_2\left(i_{2d}{}^{*2} + i_{2q}{}^{*2}\right) + C^2R_CR_1\left(i_{1d}{}^{*2} + i_{1q}{}^{*2} - i_{1d}i_{1d}{}^* - i_{1q}i_{1q}{}^*\right)}{CR_Cq_C{}^D}$$

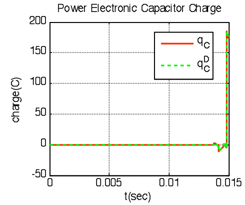
Source: K. D. Bachovchin, M. D. Ilić, "Automated Passivity-Based Control Law Derivation for Electrical Euler-Lagrange Systems and Demonstration on Three-Phase AC/DC/AC Converter," EESG Working Paper No. R-WP-5-2014, August 2014.

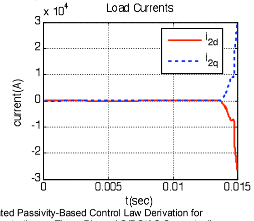
Existence of stable equilibrium

- A stable equilibrium for $q_C{}^D$ only exists when

$$\underbrace{V_{1d}i_{1d}{}^* + V_{1q}i_{1q}{}^*}_{\text{power input by source}} \geq \underbrace{R_2 i_{2d}{}^{*2} + R_2 i_{2q}{}^{*2}}_{\text{power dissipated by load}}$$

- If this condition is violated, then the passivity-based controller will be unstable





Source: K. D. Bachovchin, M. D. Ilić, "Automated Passivity-Based Control Law Derivation for Electrical Euler-Lagrange Systems and Demonstration on Three-Phase AC/DC/AC Converter," EESG Working Paper No. R-WP-5-2014, August 2014.

Interaction variables between complex modules

- Need to understand both module-level dynamics and inter-modular dynamics
- Cooperative control between modules w/o detailed knowledge of internal dynamics is possible according to functional performance quantifiable in terms for interaction variables requirements for several time horizons.
- Competitive control of individual modules without much information exchange with the rest of the system also possible?
- Potential for unstable interactions --sub--synchronous control instabilities (instabilities created by power electronics controllers on wind power plants and FACTS controllers in Texas power grid) managed by protocols avoiding problems in Table 1.

Conclusions

- Our proposal: Interaction variable-based
- Standards/protocols for interactive iBAs can set the basis for plug-and-play in smart grids—bounds on stored energy change and on rate of change of stored energy for T of interest
- Standards need to define transparent protocols for all dynamic components
  - Complexity of smart grids can be managed this way
  - At the same time system performance is guaranteed
- With current NERC standards system performance cannot be mapped into responsibilities of different components

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PLUG-AND-PLAY CONTROL AND OPTIMIZATION IN MICROGRIDS

Florian Dörfler, ETH Zurich, Switzerland

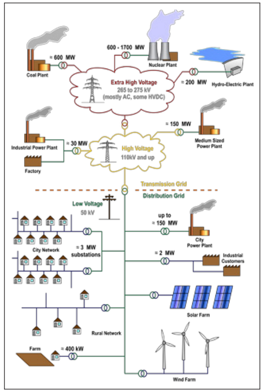
Microgrids are low-voltage electrical distribution networks, heterogeneously composed of distributed generation, storage, load, and managed autonomously from the larger transmission network. Modeled after the hierarchical control architecture of power transmission systems, a layering of primary, secondary, and tertiary control has become the standard operation paradigm for microgrids. Despite this superficial similarity, the control objectives in microgrids across these three layers are varied and ambitious, and they must be achieved while allowing for robust plug-and-play operation and maximal flexibility, without hierarchical decision making and time-scale separations. In this seminar, we explore control strategies for these three layers and illuminate some possibly-unexpected connections and dependencies among them. We build upon a first-principle model and different decentralized primary control strategies such as droop, quadratic droop, and virtual oscillator control. We motivate the need for additional secondary regulation and study centralized, decentralized, and distributed secondary control architectures. We find that averaging-based distributed controllers using communication among the generation units offer the best combination of flexibility and performance. We further leverage these results to study constrained AC economic dispatch in a tertiary control layer. Surprisingly, we show that the minimizers of the economic dispatch optimization problem are in one-to-one correspondence with the set of steady-states reachable by droop control. This equivalence results in simple guidelines to select the droop coefficients, which include the known criteria for power sharing. Finally, we illustrate the performance and robustness of our designs through through hardware experiments.

Plug-and-Play Control and Optimization in Microgrids

Florian Dörfler
ETH Zürich

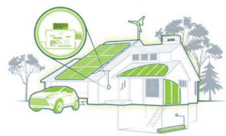
LCCC Dynamics and Control in Networks Workshop

Paradigm shifts in the operation of power networks



Traditional **top to bottom** operation:
- generate/transmit/distribute power
- hierarchical control & operation

Smart & green **power to the people**:
- high renewable penetration
- distributed generation & deregulation
- demand response & load control



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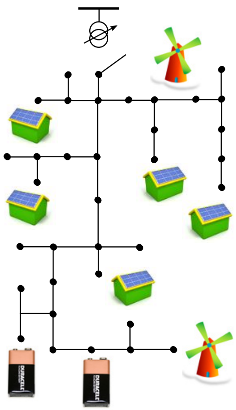
Microgrids

Structure
- low-voltage distribution networks
- grid-connected or islanded
- autonomously managed

Applications
- hospitals, military, campuses, large vehicles, & isolated communities

Benefits
- naturally distributed for renewables
- flexible, efficient, & reliable

Operational challenges
- volatile dynamics & low inertia
- plug'n'play & no central authority



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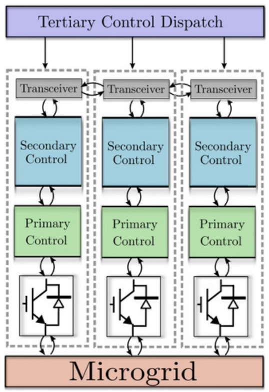
Conventional control architecture from bulk power ntwks



3. **Tertiary control** (offline)
   - Goal: optimize operation
   - Strategy: centralized & forecast

2. **Secondary control** (slower)
   - Goal: maintain operating point
   - Strategy: centralized

1. **Primary control** (fast)
   - Goal: stabilization & load sharing
   - Strategy: decentralized

Microgrids: distributed, model-free, online & without time-scale separation
⇒ **break** vertical & horizontal **hierarchy**

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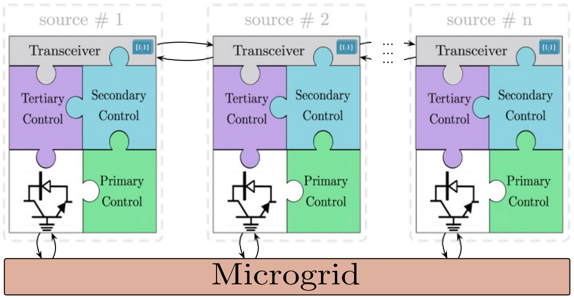
A preview – plug-and-play control and optimization

flat hierarchy, distributed, no time-scale separations, & model-free . . .



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Outline

Introduction

Primary Control

Tertiary Control

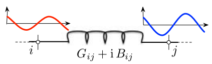
Secondary Control

P-n-P Experiments

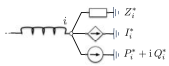
Conclusions

Modeling: a microgrid is a circuit

❶ synchronous (& acyclic) **AC circuit** with harmonic waveforms $E_i e^{i(\theta_i + \omega^* t)}$



❷ **ZIP loads**: constant impedance, current, & power $P_i^* + iQ_i^*$ (today)



❸ **coupling** via Kirchhoff & Ohm — injection $= \sum$ power flows

❺ purely inductive lines $G/B \approx 0$ (can be relaxed to $G/B = const.$)

❻ decoupling: $P_i \approx P_i(\theta)$ & $Q_i \approx Q_i(E)$ (near operating point)

- active power: $P_i = \sum_j B_{ij} E_i E_j \sin(\theta_i - \theta_j) + G_{ij} E_i E_j \cos(\theta_i - \theta_j)$
- reactive power: $Q_i = -\sum_j B_{ij} E_i E_j \cos(\theta_i - \theta_j) + G_{ij} E_i E_j \sin(\theta_i - \theta_j)$

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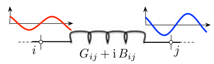
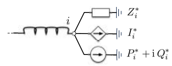
Modeling: a microgrid is a circuit

❶ synchronous (& acyclic) **AC circuit** with harmonic waveforms $E_i e^{i(\theta_i + \omega^* t)}$



❷ **ZIP loads**: constant impedance, current, & power $P_i^* + iQ_i^*$ (today)



❸ **coupling** via Kirchhoff & Ohm — injection $= \sum$ power flows

❺ purely inductive lines $G/B \approx 0$ (can be relaxed to $G/B = const.$)

❻ decoupling: $P_i \approx P_i(\theta)$ & $Q_i \approx Q_i(E)$ (near operating point)

- trigonometric active power flow: $P_i(\theta) = \sum_j B_{ij} \sin(\theta_i - \theta_j)$
- polynomial reactive power flow: $Q_i(E) = -\sum_j B_{ij} E_i E_j$ (not today)

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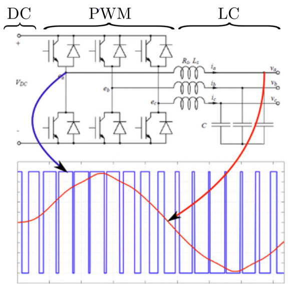
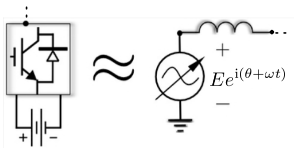
Modeling: sources interfaced with inverters

(all results also apply to synchronous machines & frequency-dependent loads)

Power **inverters** are . . .

- **interfaces** between
  - ◇ the AC microgrid and
  - ◇ DC & variable AC sources
- **controllable** (voltage) sources
  [Zhong & Hornik, '12]





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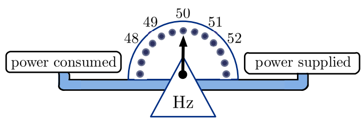
primary control

Decentralized primary control of active power

Inverters are controlled to emulate the physics of synchronous generators.
[Chandorkar et. al. '93]



Intuition: Recall...

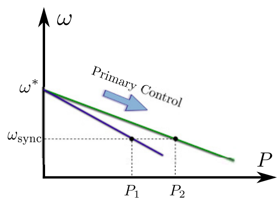
$P_i(\theta) = \sum_{j=1}^{n} B_{ij} \sin(\theta_i - \theta_j)$

$P/\dot{\theta}$ droop control:

$(\omega_i - \omega^*) \propto (P_i^* - P_i(\theta))$

$\Updownarrow$

$D_i \dot{\theta}_i = P_i^* - P_i(\theta)$



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Putting the pieces together...

differential-algebraic closed loop

network physics

load power balance: $P_i^* = \sum_j B_{ij} \sin(\theta_i - \theta_j)$

source injections: $P_i(\theta) = \sum_j B_{ij} \sin(\theta_i - \theta_j)$

$\dot{\theta}_i$ $\quad$ $P_i(\theta)$

droop control

$\dot{\theta}_i = \frac{1}{D_i}(P_i^* - P_i(\theta))$

loads: $0 = P_i^* - \sum_j B_{ij} \sin(\theta_i - \theta_j)$

sources: $D_i \dot{\theta}_i = P_i^* - \sum_j B_{ij} \sin(\theta_i - \theta_j)$

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Closed-loop stability under droop control

Theorem: stability of droop control [J. Simpson-Porco, FD, & F. Bullo, '12]

$\exists$ unique & exp. stable frequency sync $\iff$ active power flow is feasible

Main **proof ideas** and some **further results**:

- synchronization frequency: $\omega_{\text{sync}} = \omega^* + \dfrac{\sum_{\text{inverters}} P_i^* + \sum_{\text{loads}} P_i^*}{\sum_{\text{inverters}} D_i}$
  ($\propto$ power balance)
- steady-state power injections: $\mathcal{P}_i = \begin{cases} P_i^* & \text{for loads} \\ P_i^* - D_i(\omega_{\text{sync}} - \omega^*) & \text{for inverters} \end{cases}$
  (depend on D_i & P_i^*)
- unique steady-state branch flows: $\xi_{ij} = B_{ij} \sin(\theta_i^* - \theta_j^*) \Rightarrow \boxed{B_{ij} \geq \xi_{ij}}$
  ($\mathcal{P}_i \mapsto \xi_{ij}$)

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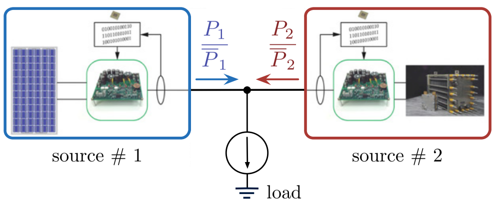
tertiary control

Objective I: decentralized proportional load sharing

1) Inverters have **injection constraints**: $P_i(\theta) \in [0, \overline{P}_i]$
2) Load must be **serviceable**: $0 \leq \left|\sum_{\text{loads}} P_j^*\right| \leq \sum_{\text{inverters}} \overline{P}_j$
3) **Fairness:** load should be shared proportionally: $P_i(\theta)/\overline{P}_i = P_j(\theta)/\overline{P}_j$



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Objective I: decentralized proportional load sharing

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Theorem: fair proportional load sharing [J. Simpson-Porco, FD, & F. Bullo, '12]

Let the droop coefficients be selected **proportionally**:

$$\boxed{D_i/\overline{P}_i = D_j/\overline{P}_j \;\;\&\;\; P_i^*/\overline{P}_i = P_j^*/\overline{P}_j}$$

The the following statements hold:

(i) Proportional load sharing: $P_i(\theta)/\overline{P}_i = P_j(\theta)/\overline{P}_j$

(ii) Constraints met: $0 \leq \left|\sum_{\text{loads}} P_j^*\right| \leq \sum_{\text{inverters}} \overline{P}_j \;\Leftrightarrow\; P_i(\theta) \in [0, \overline{P}_i]$

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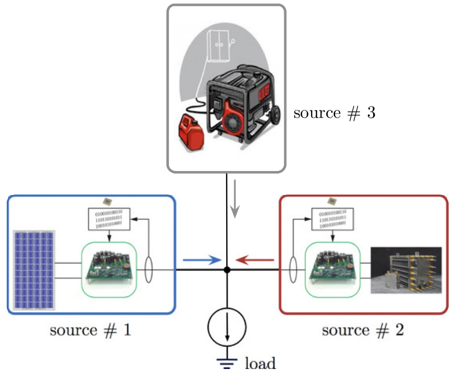
Objective I: fair proportional load sharing

proportional load sharing is not always the right objective



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Objective II: optimal economic dispatch

minimize the total accumulated generation

minimize $_{\theta \in \mathbb{T}^n,\, u \in \mathbb{R}^{n_I}}$ $\quad f(u) = \sum_{\text{inverters}} \alpha_i u_i^2$

subject to

inverter power balance: $P_i^* + u_i = P_i(\theta)$

load power balance: $P_i^* = P_i(\theta)$

branch flow constraints: $|\theta_i - \theta_j| \leq \gamma_{ij} < \pi/2$

inverter injection constraints: $P_i(\theta) \in [0, \overline{P}_i]$

Problem is generally non-convex and feasible only if the load is serviceable

In conventional power system operation, the economic dispatch is

- solved **offline**, in a **centralized** way, & with a **model & load forecast**

In an autonomously managed microgrid, the economic dispatch should be

- solved **online**, in a **decentralized** way, & **without knowing a model**

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Objective II: decentralized dispatch optimization

Insight: droop-controlled microgrid = decentralized primal algorithm

Theorem: optimal droop [FD, J. Simpson-Porco, & F. Bullo, '14]

The following statements are equivalent:

(i) the economic dispatch with cost coefficients α_i is **strictly** feasible with global minimizer (θ^*, u^*).

(ii) $\exists$ droop coefficients D_i such that the microgrid possesses a unique & locally exp. stable sync'd solution θ satisfying $P_i(\theta) \in [0, \overline{P}_i)$.

If (i) & (ii) are true, then $\theta_i \sim \theta_i^*$, $u_i^* = -D_i(\omega_{\text{sync}} - \omega^*)$, & $\boxed{D_i \alpha_i = D_j \alpha_j}$.

- similar results hold for the general **constrained** case
- similar results in transmission ntwks with DC flow [E. Mallada & S. Low, '13] & [N. Li, L. Chen, C. Zhao, & S. Low '13] & [X. Zhang & A. Papachristodoulou, '13] & [M. Andreasson, D. V. Dimarogonas, K. H. Johansson, & H. Sandberg, '13] & ...

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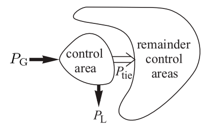
secondary control

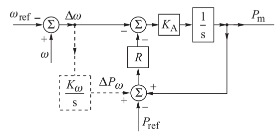
Secondary frequency control in power networks

Problem: steady-state frequency deviation ($\omega_{\text{sync}} \neq \omega^*$)

Solution: integral control [Chandorkar et. al. '93, Lopes et al. '05, Bevrani '09, . . .]

Interconnected Systems	Isolated Systems
• **Centralized** automatic generation control (AGC)	• **Decentralized** PI control





compatible with econ. dispatch [N. Li, L. Chen, C. Zhao, & S. Low '13]

is *globally* stabilizing [C. Zhao, E. Mallada, & FD, '14]

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Secondary frequency control in power networks

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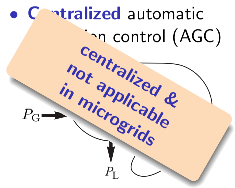

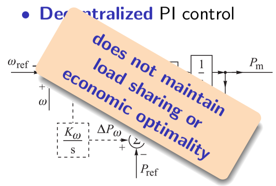

compatible with econ. dispatch

is *globally* stabilizing

Microgrids require **distributed** (**!**) secondary control strategies.

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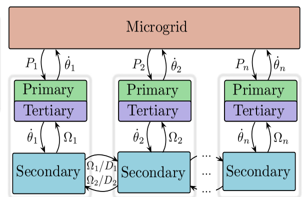
Distributed Averaging PI (DAPI) control

$$D_i\dot{\theta}_i = P_i^* - P_i(\theta) - \Omega_i$$

$$k_i\dot{\Omega}_i = D_i\dot{\theta}_i - \sum_{j \subseteq \text{inverters}} a_{ij} \cdot \left(\frac{\Omega_i}{D_i} - \frac{\Omega_j}{D_j}\right)$$



- no tuning & no time-scale separation: $k_i, D_i > 0$
- distributed & modular: connected comm. $\subseteq$ inverters
- recovers primary op. cond. (load sharing & opt. dispatch)

$\Rightarrow$ plug'n'play implementation

Theorem: stability of DAPI [J. Simpson-Porco, FD, & F. Bullo, '12]

primary droop controller works

$\Longleftrightarrow$

secondary DAPI controller works

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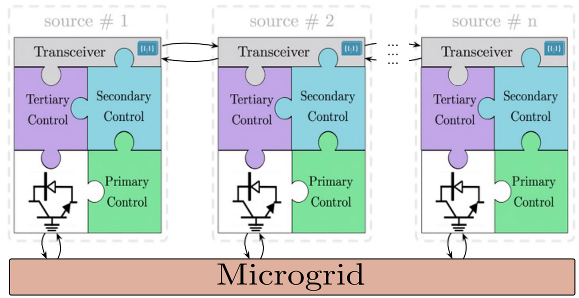
plug-and-play experiments

Plug'n'play architecture

flat hierarchy, distributed, no time-scale separations, & model-free



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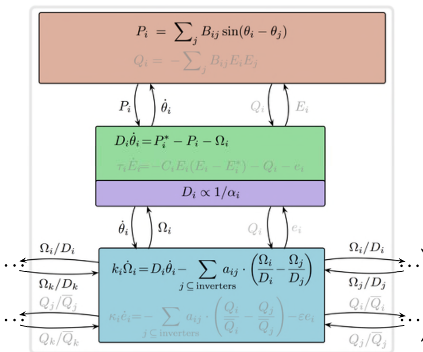
Plug'n'play architecture

recap of detailed signal flow (active power only)



Microgrid: physics & loadflow

Primary control: mimic oscillators & polyn. symmetry

Tertiary control: marginal costs $\propto$ 1 /control gains

Secondary control: diffusive averaging of injection ratios

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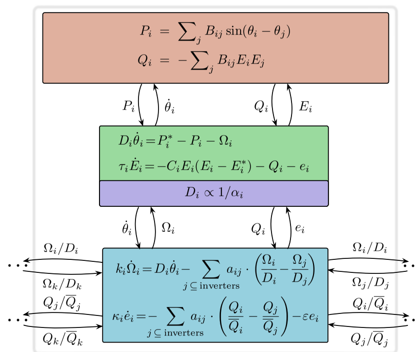
Plug'n'play architecture

similar results in the reactive case



Microgrid: physics & loadflow

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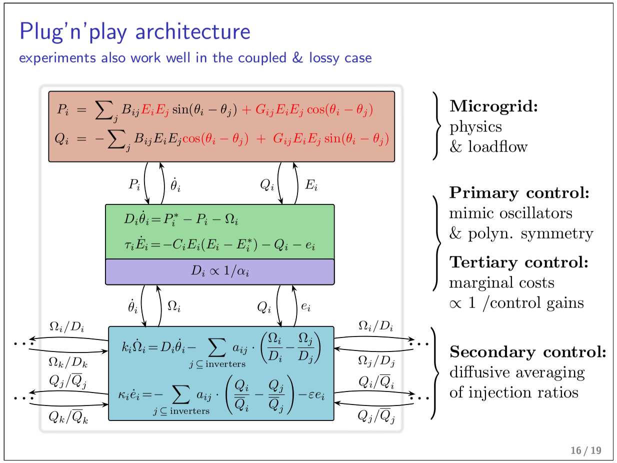
Plug'n'play architecture
experiments also work well in the coupled & lossy case
Microgrid: physics & loadflow
Primary control: mimic oscillators & polyn. symmetry
Tertiary control: marginal costs ∝ 1 /control gains
Secondary control: diffusive averaging of injection ratios
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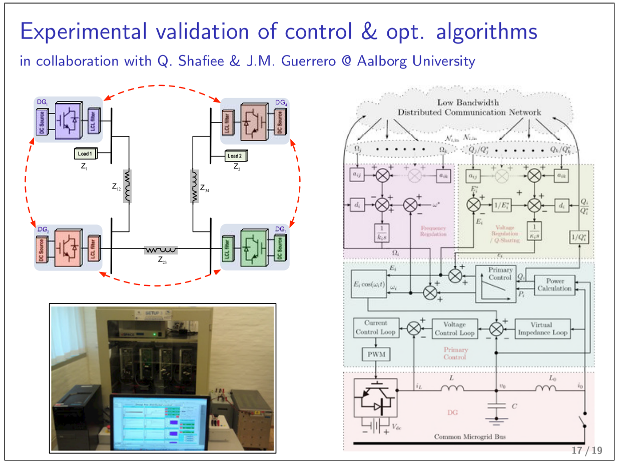
Experimental validation of control & opt. algorithms
in collaboration with Q. Shafiee & J.M. Guerrero @ Aalborg University
Low Bandwidth Distributed Communication Network
Primary Control
Power Calculation
Current Control Loop
Voltage Control Loop
Virtual Impedance Loop
PWM
Common Microgrid Bus
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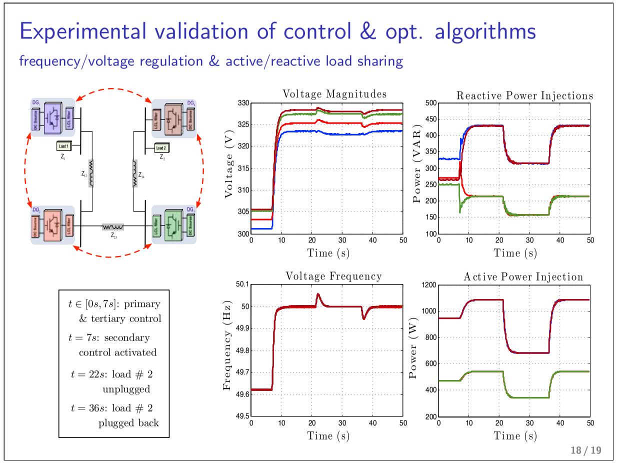
Experimental validation of control & opt. algorithms
frequency/voltage regulation & active/reactive load sharing
Voltage Magnitudes
Reactive Power Injections
Voltage Frequency
Active Power Injection
Voltage (V)
Power (VAR)
Frequency (Hz)
Power (W)
Time (s)
t ∈ [0s, 7s]: primary & tertiary control
t = 7s: secondary control activated
t = 22s: load # 2 unplugged
t = 36s: load # 2 plugged back
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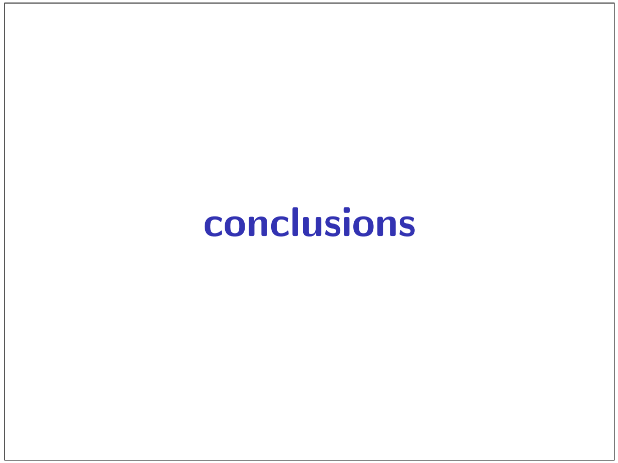
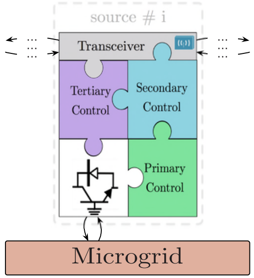

conclusions

Conclusions

Summary
- primary $P/\dot{\theta}$ droop control
- fair proportional load sharing & economic dispatch optimization
- distributed secondary control strategies based on averaging
- experimental validation

Further results
- reactive power control
- virtual oscillator control

Open conjecture
- solve these problems without comm



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Acknowledgements

LYAPUNOV APPROACH TO CONSENSUS PROBLEMS

Angelia Nedic, University of Illinois at Urbana Champain, USA

This talk is focused on the weighted-averaging dynamic for unconstrained and constrained consensus problems. Through the use of a suitably defined adjoint dynamic, quadratic Lyapunov comparison functions are constructed to analyze the behavior of weighted-averaging dynamic. As a result, new convergence rate results are obtained that capture the graph structure in a novel way. In particular, the exponential convergence rate is established for unconstrained consensus with the exponent of the order of O(1/(mlog2m)). Also, the exponential convergence rate is established for constrained consensus, which extends the existing results limited to the use of doubly stochastic weight matrices.

LYAPUNOV APPROACH TO CONSENSUS PROBLEMS

Angelia Nedić
angelia@illinois.edu
Industrial and Enterprise Systems Engineering Department
and Coordinated Science Laboratory
University of Illinois at Urbana-Champaign

joint work with Ji Liu (ECE, UIUC)

Motivation

Say we are given an $m \times m$ row-stochastic matrix A, think of it as a transition matrix of a Markov chain over m states.

Suppose that the chain is ergodic, so that

$$\lim_{t! \;!} A^t = \mathbf{1} \;\; !,$$

where is the stationary distribution of the chain and every state has positive charge, i.e.,

$$_i > 0 \qquad \text{for all } i.$$

Convergence rate for ! $A^t - \mathbf{1}$!! is known to be exponential

$$! A^t - \mathbf{1} \;\; !!^2 ! \;\; c \;\; 1 - \frac{0}{f(m)} \;\; ^t \qquad \text{for all } t ! \;\; 0,$$

where c, $_0$ are some universal constants that do not depend on m.

What is the best known convergence rate (in terms of the dependence on m)?

- For deterministic graphs? $f(m)$! $O(m^2)$
- For random graphs? $f(m)$! $O(m \log_2 m)$

Back to Weighted Averaging Models for Consensus

- The literature is vast: Tsitsiklis 1984, Jadbabie Morse and Lin 2003 and the stream of publications that followed
- Unconstrained consensus problem - can be used as a mechanism for information diffusion in networks to deal with optimization, tracking, estimation, learning in networks
- Understanding of the convergence rate is critical for making use of the consensus as a building block for other algorithms over the network AN and Ozdaglar 2007, 2008, 2009, 2010
- Convergence rate results did not explicitly capture the network structure
- Alternative approach by using a different Lyapunov function to measure the progress of the consensus-dynamics allows us to capture the rate in terms of network structure

Unconstrained Consensus Problem

- Consider a system consisting of m agents (nodes, sensors, robots, etc), represented by a set $[m] = \{1, \ldots, m\}$
- We assume that a sequence $\{G_t\}$ of directed graphs is given externally, where each graph G_t represents communication structure among the agents, where $G_t = ([m], E_t)$

We consider the unconstrained (scalar) consensus problem, formalized as follows.
[Unconstrained Consensus] *Design a distributed algorithm obeying the communication structure given by graph G_t at each time t and ensuring that, for every set of initial values $x_i(0)$ 2 R, i 2 $[m]$, the following limiting behavior emerges:*

$$\lim_{t2 \;2} x_i(t) = c \qquad \text{for all } i \text{ 2 } [m] \text{ and some } c \text{ 2 R}.$$

Literature: VAST

Roughly speaking there are three approaches

- Push-sum or Ratio Consensus Algorithm (discussed yesterday)

- Laplacian-Based Algorithm:

$$x(t+1) = \left(I - \frac{1}{\eta}L(t)\right) x(t)$$

where $x(t) = [x_1(t), \ldots, x_m(t)]'$ and $\eta \geq m$ (see Jadbabaie et al. 2003)

- Weighted-Averaging Algorithm

$$x(t+1) = A(t)x(t)$$

where $A(t)$ is a row-stochastic matrix with sparsity pattern matching the graph G_t structure (Tsitsiklis 1984)

Most of the Laplacian-based algorithms require that each $L(t)$ is also *symmetric* which implicitly require bidirectional communication links.

Weighted-averaging algorithm gets around this limitation.

Weighted-Averaging Algorithm

$$x(t+1) = A(t)x(t),$$

where the weight matrices $A(t)$ are row-stochastic and compliant with the graph G_t structure (to be discussed soon)

The existing analysis of the weighted-averaging is based on studying the behavior of the left-matrix products

$$x(t) = A(t)A(t-1)\cdots A(s+1)A(0)x(0) \qquad \text{for } t \geq 0,$$

In particular, when the matrices $A(t)A(t-1)\cdots A(1)A(0)$ converge to a rank one matrix, the iterates $x(t)$ converge to a consensus

Some conditions on the graphs G_t and the matrices $A(t)$ that convergence are:
(Tsitsiklis 1984, Nedić and Ozdaglar 2009, Nedić, Olshevsky, Ozdaglar, Tsitsiklis 2009)

Assumption Let $\{G_t\}$ be a graph sequence and $\{A(t)\}$ be a sequence of $m \times m$ matrices that satisfy the following conditions:

(a) Graph Compliance Each $A(t)$ is a stochastic matrix that is compliant with the graph G_t, i.e., $A_{ij}(t) > 0$ when $(j,i) \in E_t$, for all t.

(b) Aperiodicity The diagonal entries of each $A(t)$ are positive, $A_{ii}(t) > 0$ for all t and $i \in [m]$.

(c) Uniform Positivity There is a scalar $\alpha > 0$ such that $A_{ij}(t) \geq \alpha$ whenever $A_{ij}(t) > 0$.

(d) Irreducibility Each G_t is strongly connected.

Convergence Rate

Under the preceding Assumption we have

$$\lim_{t\to\infty} A(t)\cdots A(k+1)A(k) = \mathbf{1}\,\phi'(k) \qquad \text{for all } k \geq 0,$$

where each $\phi(k)$ is stochastic vector and $\mathbf{1} = [1,\ldots,1]'$.

Furthermore, the convergence rate is geometric: for all $t \geq k \geq 0$,

$$\|A(t)\cdots A(k+1)A(k) - \mathbf{1}\,\phi'(k)\|^2 \leq Cq^{t-k},$$

where the constants $C > 0$ and $q \in (0,1)$ depend only on m and α.

When the matrices $A(t)$ are doubly stochastic, we have for all $t \geq k \geq 0$,

$$\left\| A(t)\cdots A(k+1)A(k) - \frac{1}{m}\mathbf{1}\mathbf{1}' \right\|^2 \leq \left(1 - \frac{\alpha}{2m^2}\right)^{t-k}.$$

Refs. Tsitsiklis 1984, AN and Ozdaglar 2009; AN, Olshevsky, Ozdaglar, and Tsitsiklis 2009

These and other existing rate results *are not explicitly capturing the structure of the graph* G_t such as the longest shortest path or the maximum node degrees for example.

Convergence Rate: Lyapunov Approach

New rate results are possible by adopting dynamic system point of view and applying Lyapunov approach

This approach allows us to characterize the convergence of the weighted-averaging algorithm with a more explicit dependence on the graph structure than the existing results

In particular, we work with a quadratic Lyapunov comparison function proposed by Touri 2011, and Touri and AN 2014

In this approach, *an absolute probability sequence of matrices* $A(t)$ *play a critical role* in the construction of a Lyapunov comparison function and in establishing its rate of decrease along the iterates of the algorithm.

Definition For a chain of row-stochastic matrices $\{A(t)\}$ we say that the vector sequence $\{\pi(t)\}$ is an absolute probability sequence of the chain if each $\pi(t)$ is a probability vector and

$$\pi'(t) = \pi'(t+1)A(t) \qquad \text{for all } t \geq 0.$$

NOTE The absolute probability sequence is an adjoint dynamic for the weighted-averaging dynamic

Assumptions

An undirected tree $\mathcal{T}$ is weakly spanning tree of a directed graph $G = ([m], E)$, if for every $\{i, j\}$ 2 $\mathcal{T}$ we have either (i, j) 2 E or (j, i) 2 E

Assumption 1

Let $\{G_t\}$ be a graph sequence and $\{A(t)\}$ be a matrix sequence such that:

(a) (*Partial Irreducibility*)
Each graph G_t contains a weakly spanning tree and each $A(t)$ is a stochastic matrix that is compliant with a weakly spanning tree T_t of G_t, i.e., $A_{ij}(t) > 0$ whenever (j, i) 2 T_t for all t 2 0.

(b) (*Aperiodicity*)
The diagonal entries of each $A(t)$ are positive, $A_{ii}(t) > 0$ for all t, and i 2 $[m]$.

(c) (*Partial Uniform Positivity*)
There is a scalar > 0 such that $A_{ii}(t)$ 2 and $A_{ij}(t)$ 2 for all (j, i) 2 T_t and for all t 2 0.

(d) (*Adjoint Dynamics / Absolute Probability Sequence*)
The matrix sequence $\{A(t)\}$ has an absolute probability sequence $\{ (t)\}$ that is uniformly bounded away from zero, i.e., there is 2 (0, 1) such that $_i(t)$ 2 for all i and t.

Basic Result

Theorem Under Assumption 1, for the iterates $\{x(t)\}$ generated by the weighted-averaging algorithm with any initial vector $x(0)$ 2 R^m, we have for any t 2 k 2 0,

$$\sum_{i=1}^{m} {}_i(t)\ \ x_i(t) - (0)^{2}x(0)\ ^2\ 2\ \ 1 - \frac{2}{p^2}\ ^{t-k}\ \sum_{j=1}^{m} {}_j(k)\ \ x_j(k) - (0)^{2}x(0)\ ^2,$$

where

- > 0 is the uniform lower bound on the entries in $A(t)$ (Assumptions 1(c))
- > 0 is the uniform lower bound on the entries of the absolute probability vectors (t) (Assumption 1(d)),
- while
$$p^2 = \max_{s 2 0} p^2(s)$$
where $p^2(s)$ is the longest shortest path in the tree T_s of Assumption 1(a).

Proof Sketch

Proof: Based on the use of Lyapunov Function

$$V(x, \ ) = \sum_{i=1}^{m} {}_i(x_i - \ {}^0 x)^2$$

and the relation

$$V(x(t+1), \ (t+1)) = V(x(t), \ (t)) - D(t),$$

where

$$D(t) = \frac{1}{2}\sum_{i=1}^{m} {}_i(t+1) \sum_{j=1}^{m}\sum_{\ =1}^{m} A_{ij}(t)A_{i\ }(t)\,(x_j(t) - x\ (t))^2.$$

Abound for the decrement is obtained (using Assumption 1)

$$D(t)\ 0\ \frac{2}{p^0(t)} \max_{j,\ 0\,[m]} (x_j(t) - x\ (t))^2 \quad \text{for } t\ 0\ 0,$$

$p^0(t)$ is the maximum number of links in any of the shortest paths in the tree T_t of Assumption 1(a).

We also use

$$^0(t)x(t) = \ ^0(0)x(0)$$

Implications of Theorem 1: Doubly Stochastic

Let Assumption 1 hold, and assume also that the weight matrices $A(t)$, t k 0, are doubly stochastic. Then, we have $(t) = \frac{1}{m}\mathbf{1}$ and the relation of Theorem 1 reduces to:

$$\mathrm{k}x(t) - \bar{x}(0)\mathbf{1}\mathrm{k}^2\ \mathrm{k}\ \ 1 - \frac{2}{mp^{\mathrm{k}}}\ ^{t-k}\ \mathrm{k}x(k) - \bar{x}(0)\mathbf{1}\mathrm{k}^2, \tag{1}$$

with $\bar{x}(0) = \frac{\mathbf{1}^{\mathrm{k}}x(0)}{m}$. Since the maximum path length in a tree is no larger than $m-1$, i.e., $p^{\mathrm{k}}(s)$ k $m-1$, it follows that

$$\mathrm{k}x(t) - \bar{x}(0)\mathbf{1}\mathrm{k}^2\ \mathrm{k}\ \ 1 - \frac{2}{m(m-1)}\ ^{t-k}\ \mathrm{k}x(k) - \bar{x}(0)\mathbf{1}\mathrm{k}^2.$$

Thus, when does not depend on m, the convergence rate has dependency of $O(m^2)$ in terms of the number m of agents, which is the same as the rate result in our earlier work (see slide 4)

Think of $= \frac{1}{\max_{i,t}|N_i(t)|}$.

Given m, what graphs on m nodes will be the best?

Implications of Theorem 1: New Bound

Suppose now that we want to construct the graphs G_t such that Assumption 1 holds and we want to get the most favorable rate dependency on m. In this case, the following result is valid.

Theorem 1 *There is a sequence* $\{G_t\}$ *of regular undirected graphs such that for all* $x(0)$ 2 R^m *and all* t 2 k 2 0,

$$2x(t) - \bar{x}(0)12^2 \; 2 \; q^{t-k} 2x_j(k) - \bar{x}(0)12^2,$$

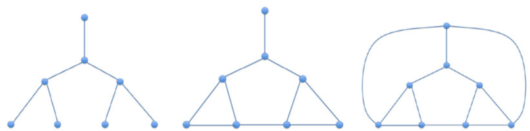
with $\bar{x}(0) = \frac{1^2 x(0)}{m}$ *and*

$$q = 1 - \frac{1}{4^3 m \, 2\log_2 m2}$$

The rate of this order is provided in Diaconis and Stroock 1991 (Proposition 3 and Example 2.3) for a *static ergodic Markov Chain.*

Proof

We will construct an undirected graph sequence $\{G_t\}$ that satisfies Assumption 1.
Let $m = 2^d$ for some integer d 2 1. Let t be arbitrary but fixed time. Select $2^d - 1$ agents and construct an undirected binary tree with these agents as nodes. Next, add one extra agent as a root with a single child (see left plot in the Figure). Thus, each agent i except for the root and the leaf agents has the degree equal to 3.
Consider, now connecting all leaf-nodes with undirected edges (see middle plot in Figure). Now, all leaf-agents have degree equal to 3 except for the far most left and far most right agents, each of which has the degree equal to 2. Connect these two agents to the root node (see right plot in Figure). In this way, the far most left and far most right leaf agents, as well as the root agent have degree 3. In the resulting regular undirected graph, we let $A_{ij}(t) = \frac{1}{4}$ for all j 2 $N_i(t)$ 2 $\{i\}$ and for all i, so that $= \frac{1}{4}$. The shortest path from the root agent to any other agent in the graph is at most $2\frac{d}{2}2$ (going down from the root of the tree to the nodes at the depth $2\frac{d}{2}2$, and going through the leaf nodes to reach those that are the depth larger than d).
Using the same construction, for all times t, we have that $\{A(t)\}$ is a sequence of doubly stochastic matrices, and therefore $(t) = \frac{1}{m}\mathbf{1}$ for all t. Thus, Assumption 1 is



satisfied, and the estimate in (1) reduces to

$$\|x(t) - \bar{x}(0)1\|^2 \; \mathrm{k} \; \left(1 - \frac{1}{4^2 md}\right)^{t-k} \|x(k) - \bar{x}(0)1\|^2.$$

The result follows by noting that $d = \log_2 m$.

Theorem 1 shows that the exponential convergence rate with the ratio of the order $1 - O(\frac{1}{m \log_2 m})$ is achievable for consensus on some tree-like regular undirected graphs. This improves the best known bound with the ratio of the order $1 - O(\frac{1}{m^2})$ for undirected graphs and doubly stochastic matrices (see slide 4).

Implication for Matrix Sequence

We next consider an implication of Theorem 1 for the convergence of matrix products

$$A(t:k) \; , \; A(t) \cdots A(k+1)A(k) \qquad \text{for all } t \; 0 \; k \; 0 \; 0,$$

where $A(t:k)$, $A(k)$ whenever $t = k$.

Theorem 2 *If Assumption 1 holds, then for all* t 0 k 0 0,

$$A(t:k) - \mathbf{1} \; (k)^{0\,2} \; 0 \; \frac{1}{\,} \; \left(1 - \frac{2}{p^0}\right)^{t-k} \; I - \mathbf{1} \; (k)^{0\,2},$$

where I is the identity matrix.

Corollary 3 *Under Assumption 1, the sequence* $\{A(t)\}$ *is ergodic:*

$$\lim_{t0\;0} A(t) \cdots A(k) = \mathbf{1} \; (k)^0$$

for all k 0 0.

Constrained Consensus

Assumption 2 We are given a collection of sets $X_i \setminus$ R^n which closed and convex, and their intersection is nonempty, i.e., X , $\setminus_{i=1}^m X_i \models \setminus$.

The constrained consensus problem is as follows.

[Constrained Consensus] *Assuming that each agent i knows only its set X_i, design a distributed algorithm obeying the communication structure given by graph G_t at each time t and ensuring that, for every set of initial values $\mathbf{x}_i(0) \setminus$ R^n, $i \setminus [m]$, the following limiting behavior emerges:*

$$\lim_{t \setminus \setminus} \mathbf{x}_i(t) = c \qquad \text{for all } i \setminus [m] \text{ and some } c \setminus X.$$

We have considered this problem in AN, Ozdaglar, and Parrilo 2010, proposing the following algorithm

$$\mathbf{w}_i(t+1) = \sum_{j=1}^{m} A_{ij}(t)\mathbf{x}_j(t),$$

$$\mathbf{x}_i(t+1) = \mathsf{P}_{X_i}[\mathbf{w}_i(t+1)], \qquad (2)$$

with a restriction to the use of doubly stochastic matrices

Convergence

The following result proves that the iterates of the algorithm converge to a common point in the set X.

Theorem 4 *Let Assumption 1 and Assumption 2 hold. Then, the sequences $\{\mathbf{x}_i(t)\}$, i 2 $[m]$ of the algorithm (2) are bounded, i.e., there is a scalar >0 such that*

$$2\mathbf{x}_i(t)2\ 2 \qquad \text{for all } i \text{ 2 } [m] \text{ and all } t \text{ 2 } 0,$$

and they converge to a common point x^2 2 X:

$$\lim_{t2\ 2} \mathbf{x}_i(t) = x^2 \qquad \text{for some } x^2 \text{ 2 } X \text{ and for all } i \text{ 2 } [m].$$

Some notes on the determining

Boundedness of iterates: for all y 2 X and all t 2 0,

$$\sum_{i=1}^{m} {}_i(t)2\mathbf{x}_i(t) - y2^2\ 2 \sum_{j=1}^{m} {}_j(0)2\mathbf{x}_j(0) - y2^2$$

yields (under Assumption 1) for any y 2 X, all t and i,

$$2\mathbf{x}_i(t)2\ 2\ 2y2 + \frac{1}{=} \max_j 2\mathbf{x}_j(0) - y2$$

Convergence Rate

We have results for the sets X_i that satisfy a certain regularity condition which relates the distances from a given point to the sets X with the distance from the point to the intersection set $X = \setminus_{i=1}^m X_i$.

In particular, since $X \setminus X_i$ for all i, it follows that

$$\mathsf{dist}(x, X_i) \setminus \mathsf{dist}(x, X) \text{ for all } x \setminus \mathsf{R}^n \text{ and } i \setminus [m]. \qquad (3)$$

In our analysis, we need an upper bound on $\mathsf{dist}(x, X)$ in terms of the distances $\mathsf{dist}(x, X_i)$, $i \setminus [m]$.

We will use the following definition of set regularity.

Let $Z \setminus$ R^n be a nonempty set. We say that a (finite) collection of closed convex sets $\{Y_i, i \setminus \mathcal{I}\}$ is regular (in Euclidian norm) with respect to the set Z, if there is a constant $r \setminus 1$ such that

$$\mathsf{dist}(y, Y) \setminus r \max_{i \setminus \mathcal{I}} \{\mathsf{dist}(y, Y_i)\} \qquad \text{for all } y \setminus Z.$$

We refer to the scalar r as a *regularity constant.* When the preceding relation holds with $Z = \mathsf{R}^n$, we say that the sets $\{Y_i, i \setminus \mathcal{I}\}$ are *uniformly regular.* In view of relation (3) it follows that the regularity constant r must satisfy $r \setminus 1$.

Theorem 5 *Let Assumption 1 and Assumption 2 hold. Assume further that the sets $\{X_i, i$ 2 $[m]\}$ are regular, with a regularity constant r 2 1, with respect to a ball $B(0, ')$ which contains all the iterates $\{\mathbf{x}_i(t)\}$ generated by the algorithm (2). Consider the following Lyapunov comparison function:*

$$'(t, y) \ , \sum_{i=1}^{m} '_i(t)\, 2\mathbf{x}_i(t) - y2^2. \qquad (4)$$

Then, the Lyapunov comparison function $'(t, \mathbf{v}(t))$ decreases at a geometric rate:

$$'(t+1, \mathbf{v}(t+1))\ 2\ \left(1 - \frac{''\,2}{p^2(r+1)^2}\right) '(t, \mathbf{v}(t)) \qquad \text{for all } t \text{ 2 } 0,$$

where

$$\mathbf{u}(t) = \sum_{i=1}^{m} '_i(t)\mathbf{x}_i(t), \quad \mathbf{v}(t) = \mathsf{P}_X[u(t)], \text{ for all } t \text{ 2 } 0. \qquad (5)$$

and the scalars $'$, $'$ 2 $(0, 1)$ and the integer p^2 2 1 are the same as in Theorem 1.

Theorem 6 *Under the assumptions of Theorem 5, for all t 2 0,*

$$\sum_{j=1}^{m} \mathsf{dist}^2(\mathbf{x}_j(t), X)\ 2\ \frac{1}{'} \left(1 - \frac{''\,2}{4p^2(r+1)^2}\right)^{t} '(0, \mathbf{v}(0)),$$

where $\mathbf{v}(0) = \mathsf{P}_X[\mathbf{u}(0)]$ with $\mathbf{u}(0) = \sum_{j=1}^{m} '_j(0)\mathbf{x}_j(0)$

MAJORITY CONSENSUS BY LOCAL POLLING

Moez Draief, Imperial College, UK

We consider a network where each of n anonymou) nodes holds some initial binary opinion. Our goal is to design efficient and fully-distributed algorithms that enable the network to reach a state where all the nodes hold the initial majority opinion. This problem has received a lot of interest from a number of communities such as distributed computing, communication networks, biological and chemical networks and social networks.

In this talk, I will present two distributed procedures for solving the majority consensus. In the first one we allow nodes to hold undecided states, provide a simple dynamics for solving the exact majority consensus problem on general connected graphs and derive bounds on the time to convergence of the algorithm. The second algorithm lets nodes interact with more than one neighbour at a time and show that, for some family of graphs of interest, and provided there is sufficient bias in the population towards the major it opinion, this algorithm solves majority consensus with high probability. We compute its asymptotic convergence time.

This is a joint work with M. Vojnovic and M. Abdullah.

Majority Consensus by Local Polling

Moez Draief

Imperial College London

Lund Workshop on Dynamics and Control in Networks
October 2014

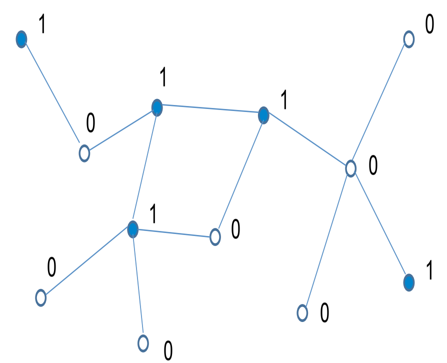
Aggregating through Majority

1785, Marquis de Condorcet's weak law of large numbers

- in a large population of voters, and each one independently votes correctly with probability $\alpha > 1/2$
- as population size grows, probability that the outcome of a majority vote is correct converges to one

Information is efficiently aggregated

Aggregating in a network



Binary majority consensus

Desired outcome and metrics

- Nodes end with opinion held by majority of nodes
- Node can probe neighbours and update opinion accordingly using little (constant) memory
- Probability of error (convergence to incorrect consensus)
- Time to convergence

Applications

- Occurrence of a given event in cooperative decision making
- Voting in distributed systems
- Routine to solve more elaborate distributed decision making instances

Local Majority Dynamics

$G = (V, E)$ simple connected graph on $|V| = n$ vertices

Each vertex either red (**1**) or blue (**0**).
Initial proportion of blues is $\alpha \in (1/2, 1)$

GOAL: Local algorithm for inferring the majority state.

- Does the graph settle into one colour?
- If so, how does the graph structure and the initial distribution affect which colour wins?
- How long does it take?

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Outline

- Distributed consensus [known results]
- Interval consensus [Draief, Vojnovic '12]
- Local polling [Abdullah, Draief '14]

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Continuous-time Interaction Model

- Connected undirected graph $G = (V, E)$, $|V| = n$
- αn nodes hold 0 and $(1-\alpha)n$ nodes hold 1, $\alpha \in (1/2, 1)$
- Nodes i and j interact at rate $q_{ij} = q_{ji}$, $q_{ij} \neq 0$ iff $(i, j) \in E$

Markov chain

- $(X_t)_{t\geq 0}$ continuous-time Markov chain with rate matrix Q, $q_{ii} = -\sum_{i\neq j} q_{ij}$
- $(\pi_i)_{i\in V}$ stationary distribution is uniform on V. Mixing time:

$$|\mathbb{P}_j(X_t = i) - 1/n| = O\left(e^{-\lambda_2(Q)t}\right)$$

where $\lambda_2(Q) = \inf\{\sum_{i,j} q_{ij}(x_i - x_j)^2/2, ||x|| = 1,\ x^T\mathbf{1} = 0\}$

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Performance of voter model

Node i contacts j at rate q_{ij} and i updates to j's state

Theorem [Hassin-Peleg '01]

- The number of nodes in state 1 is a martingale.
- Probability of reaching (wrong) consensus at 1 is $1 - \alpha$.
- Time to convergence of voter model $O(n/(\lambda_2(Q)))$.

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Time to convergence

Complete graph

- Each edge has rate $1/(n-1)$. Number of agents with opinion 1 evolves as Birth-Death process

$$\lambda_{k,k+1} = \lambda_{k,k-1} = \frac{k(n-k)}{n-1} .$$

- Time to convergence $= O(n)$

General graphs

- Conductance $\eta(Q) = \inf_{A \subset V} \frac{\sum_{i \in A, j \in A^c} q_{ij}}{|A||A^c|/n}$
- Markov chain tracking the number of nodes in state 0 evolves at least $\eta(Q)$ times as fast as on the complete graph, since

$$\sum_{i \in A,\, j \in A^c} q_{ij} \geq \eta(Q) \underbrace{\frac{|A||A^c|}{n}}_{\text{complete graph}}$$

- Time to convergence $O(n/\eta(Q))$,

D. Aldous, "Interacting particle systems as stochastic social dynamics" Bernoulli 19(4), 1122-1149, 2013.

Time to convergence

Cheeger's inequality

- Conductance: $\eta(Q) = \inf_{A \subset V} \frac{\sum_{i \in A, j \in A^c} q_{ij}}{|A||A^c|/n}$
- Spectral Gap: $\lambda_2(Q) = \inf\{\sum_{i,j} q_{ij}(x_i - x_j)^2/2, ||x|| = 1,\ x^T\mathbf{1} = 0\}$

$$2\lambda_2(Q) \leq \eta(Q) .$$

- Time to convergence of voter model $O(n/(\lambda_2(Q)))$.

Let S of size k be the subset realising the inf in $\eta(Q)$ and let x such that $x_i = -\sqrt{\frac{n-k}{kn}}$, $i \in S$ and $x_i = \sqrt{\frac{k}{(n-k)n}}$, $i \in S^c$.

Distributed averaging

At each interaction of (i,j) occurring at rate q_{ij}

$$x_i(t) = x_j(t) = \frac{x_i(t-) + x_j(t-)}{2} .$$

Theorem [Boyd et al '06, Aldous '12]

- Algorithm converges to the average value, using $O(Poly(\log(n))$ memory per node
- Time to convergence to up $O(1/n)$ error of the average is

$$\Theta(\log(n)/\lambda_2(Q)) ,$$

Distributed averaging: Proof

Let $R(t) = ||x(t)||^2$. When an i, j interaction takes place $R(t)$ reduces by $(x_i - x_j)^2/2$.

$$\begin{aligned}\mathbb{E}(dR(t) \mid x(t) = x) &= \sum_{i,j} q_{ij} \left(2 \left(\frac{x_i + x_j}{2} \right)^2 - (x_i^2 + x_j^2) \right) \\ &= -\sum_{i,j} q_{ij} \frac{(x_i - x_j)^2}{2} dt \\ \text{(Assume that } \textstyle\sum_i x_i(0) = 0) &\leq -\lambda_2(Q)||x||^2 dt\end{aligned}$$

In particular

$$\mathbb{E}||x(t)||^2 \leq ||x(0)||^2 e^{-\lambda_2(Q)t}$$

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Small memory

Could we use less memory and still guarantee small error?

Theorem: Impossibility

- Connected undirected graph $G = (V, E)$, $|V| = n$,
- αn nodes in 0 and $(1 - \alpha)n$ nodes in 1, $\alpha \in (1/2, 1)$, $2\alpha - 1$ is the *voting margin*.

No 1-bit distributed algorithm can solve the majority consensus problem.

Land, Belew, "No perfect two-state cellular automata for density classification exists", PRL 74, 5148-5150, 1995

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Ternary Consensus

- αn nodes hold 0 and $(1 - \alpha)n$ nodes hold 1,
- Additional state e for undecided nodes, $q_{i,j} = 1/n$, $\forall i, j$

Theorem [PVV '09]

Probability of reaching wrong consensus **1**. For n large,

$$P_{error} = (1 + o(1))2^{-D(\alpha||\frac{1}{2})n}$$

where $D(\alpha||\frac{1}{2})$ is the Kullback-Leibler divergence. T time to convergence, $\mathbb{E}(T) = (1 + o(1)) \log n$.

- Results (seem to) hold for expander but fail for the line.
- Generalises beyond binary consensus [Babaee, Draief '14]

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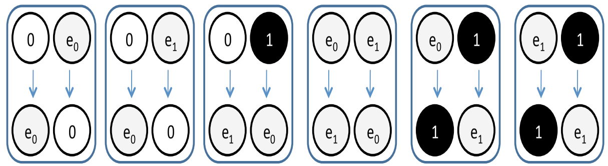
Binary Consensus with two undecided states

Averaging-like updates: States $0 < e_0 < e_1 < 1$.
Rules: Swaps + Annihilation



Kashyap, Basar, Srikant, "Quantized consensus" Automatica, 1192-1203, 2007

Bénézit, Thiran, Vetterli, Interval consensus: From quantized gossip to voting, ICASSP 2009

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Mean-field analysis (Complete graph)

Let $q_{ij} = \frac{1}{n-1}$, $i \neq j$ and $\mathbf{X}(t) = (|S_0(t)|, |S_{e_0}(t)|, |S_{e_1}(t)|, |S_1(t)|)$ is a Markov process with the following transition rates

$$\to \begin{cases} (|S_0(t)|-1, |S_{e_0}(t)|+1, |S_{e_1}(t)|+1, |S_1(t)|-1) & : \frac{|S_0(t)||S_1(t)|}{n-1} \\ (|S_0(t)|, |S_{e_0}(t)|-1, |S_{e_1}(t)|+1, |S_1(t)|) & : \frac{|S_{e_0}(t)||S_1(t)|}{n-1} \\ (|S_0(t)|, |S_{e_0}(t)|+1, |S_{e_1}(t)|-1, |S_1(t)|) & : \frac{|S_0(t)||S_{e_1}(t)|}{n-1}. \end{cases}$$

By Kurtz's theorem, $\mathbf{X}(t)/n$ converges to $(s_0(t), s_{e_0}(t), s_{e_1}(t), s_1(t))$

$$\begin{aligned} s_0'(t) &= -s_1(t)s_0(t) \\ s_1'(t) &= -s_0(t)s_1(t) \\ s_{e_1}'(t) &= s_1(t)(1-s_1(t)) - (s_0(t)+s_1(t))s_{e_1}(t) \end{aligned}$$

with $s_{e_0}(t) = 1 - s_0(t) - s_{e_1}(t) - s_1(t)$, $t \geq 0$.

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Proposition [Draief, Vojnovic '10]

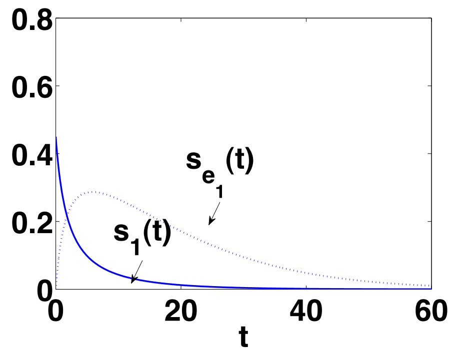
For large t,

$$\begin{aligned} s_{e_1}(t) &\sim (2\alpha-1)\frac{1-\alpha}{\alpha} t e^{-(2\alpha-1)t} \\ s_1(t) &\sim (2\alpha-1)\frac{1-\alpha}{\alpha} e^{-(2\alpha-1)t}. \end{aligned}$$

In particular, $t^1_{n,\alpha}$ and $t^{e_1}_{n,\alpha}$ times nodes in 1 and e_1 to disappear

$$\begin{aligned} t^1_{n,\alpha} &= \frac{1}{2\alpha-1}\log(n) + O(1) \\ t^{e_1}_{n,\alpha} &= \frac{1}{2\alpha-1}[\log(n) + \log(\log(n))] + O(1). \end{aligned}$$

Minority states



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General bound

Theorem [Draief, Vojnovic '12]

Let T be the time until there are only nodes in state 0 and e_0.

$$\mathbb{E}(T) = O(\log n / \delta(Q,\alpha))$$

where $\delta(Q,\alpha) = \min_{S\subset V, |S|=(2\alpha-1)n} \min_{\lambda\in Spec(Q_S)} |\lambda|$

$$Q_S = \left[\begin{array}{c|c} \text{diag}(q_{ii},\ i\in S) & \mathbf{0} \\ (q_{ij})_{i\in S^c,\, j\in S} & (q_{ij})_{i,j\in S^c} \end{array}\right]$$

Master Equation

First phase: $Z_i(t)$ ($A_i(t)$) indicator that i in state 0 (1) at t

$$(Z,A) \to \begin{cases} (Z-e_i, A-e_j) & : \quad q_{i,j}Z_iA_j \\ (Z-e_i+e_j, A) & : \quad q_{i,j}Z_i(1-A_j-Z_j) \\ (Z, A-e_i+e_j) & : \quad q_{i,j}A_i(1-A_j-Z_j) \end{cases}$$

Second phase: $B_i(t)$ indicator that node i is in state e_1 at t

$$(Z,B) \to \begin{cases} (Z-e_i+e_j, B-e_j) & : \quad q_{i,j}Z_iB_j \\ (Z-e_i+e_j, B) & : \quad q_{i,j}Z_i(1-B_j-Z_j) \\ (Z, B-e_i+e_j) & : \quad q_{i,j}B_i(1-B_j-Z_j) \end{cases}$$

(random) Piecewise-linear dynamical system

$$\frac{d}{dt}\mathbb{E}(Y_i(t)) = -\left(\sum_{l\in V} q_{i,l}\right)\mathbb{E}(Y_i(t)) + \sum_{j\in V} q_{i,j}\mathbb{E}\left(Y_j(t)(1-Z_i(t))\right).$$

Dynamics reduces to $Y(t) = (Y_i(t))_{i\in V}$,

$$\frac{d}{dt}\mathbb{E}_k(Y(t)) = Q_{S_k}\mathbb{E}_k(Y(t)),$$

for $t\in[t_k, t_{k+1})$ during which $\{S_0(t) = S_k\}$ and Q_{S_k} is given by

$$Q_S(i,j) = \begin{cases} -\sum_{l\in V} q_{i,l}, & i=j \\ q_{i,j}, & i\notin S, j\neq i \\ 0, & i\in S, j\neq i. \end{cases}$$

Solution

Proposition

$$\mathbb{E}(Y(t)) = \mathbb{E}\left[e^{\lambda(t)}Y(0)\right]$$

where $\lambda(t) = Q_{S_k}(t-t_k) + \sum_{l=0}^{k-1} Q_{S_l}(t_{l+1}-t_l)$.

Lemma

For any finite graph G, there exists $\delta(Q,\alpha) > 0$ such that, for any non-empty subset of vertices S with $|S|\in[(2\alpha-1)n, \alpha n]$, if λ is an eigenvalue of the matrix Q_S, then

$$\lambda \leq -\delta(G,\alpha) < 0.$$

Proof: Spectrum of Q_S

$$Q_S = \left[\begin{array}{c|c} \text{diag}(q_{ii},\ i \in S) & \mathbf{0} \\ \hline (q_{ij})_{i\in S^c,\, j\in S} & (q_{ij})_{i,j\in S^c} \end{array}\right]$$

- First $\left(q_{ii} = -\sum_{l\neq i} q_{i,l}\right)$, $i \in S$ are eigenvalues of Q_S
- The remaining eigenvalues correspond to eigenvectors $\underline{x} = (\underbrace{0,\ldots,0}_{S}, \underbrace{x}_{S^c})^T$. Let $W \subset S^c$, for $i \in W$, $x_i \neq 0$

$$\begin{aligned} -\lambda &= \underline{x}^T Q_S \underline{x} \\ &= \sum_{i\in W}\sum_{j\in S} q_{i,j}x_i^2 + \sum_{i\in W, j\in S^c\setminus W} q_{i,j}x_i^2 + \frac{1}{2}\sum_{i,j\in W} q_{i,j}(x_i - x_j)^2 \end{aligned}$$

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Proof

Note that

$$\mathbb{E}(Y(t)) = \mathbb{E}\left[e^{\lambda(t)}Y(0)\right]$$

where $\lambda(t) = Q_{S_k}(t - t_k) + \sum_{l=0}^{k-1} Q_{S_l}(t_{l+1} - t_l)$
By Jensen's and matrix norm inequalities,

$$||\mathbb{E}(Y(t))||_2 \leq \mathbb{E}\left[||e^{Q_{S_k}(t-t_k)}|| \prod_{l=0}^{k-1} ||e^{Q_{S_l}(t_{l+1}-t_l)}||\, ||Y(0)||_2\right] \leq \sqrt{n}e^{-\delta(G,\alpha)t}$$

Therefore, by Cauchy-Schwartz, we have

$$\mathbb{P}(Y(t) \neq \mathbf{0}) \leq \sum_{i\in V} \mathbb{E}(Y_i(t)) \leq n\, e^{-\delta(G,\alpha)t}$$

We conclude since $\mathbb{E}(T_0) = \int_0^\infty \mathbb{P}(Y(t) \neq \mathbf{0})dt$.

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Complete graph

Corollary

An application of the theorem to complete graph $q_{i,j} = \frac{1}{n-1}$ for all $i \neq j$, yields

$$\mathbb{E}(T) \leq 2\frac{1}{2\alpha - 1}\log(n).$$

Exact asymptotics

A direct analysis of the dynamics of the 1st phase

$$\mathbb{E}(T_1) = \frac{n-1}{|S_0| - |S_1|}\left(H_{|S_1|} + H_{|S_0|-|S_1|} - H_{|S_0|}\right)$$

where $H_k = \sum_{i=1}^k \frac{1}{i}$

Moez Draief — Majority Consensus by Local Polling

Various initial conditions

- $|S_0| - |S_n| = (2\alpha - 1)n$, α a constant larger than $1/2$

$$\mathbb{E}(T_1) = \frac{1}{2\alpha - 1}\log(n) + O(1).$$

- If $|S_0| = |S_1|$

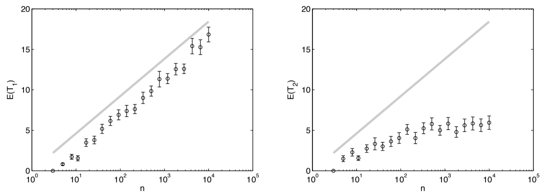
$$\mathbb{E}(T_1) = \frac{\pi^2}{6}n(1 + o(1)).$$

- $\mu_n = (|S_0| - |S_1|)/n$ is strictly positive but small $(o(1))$,

$$\mathbb{E}(T_1) = \frac{1}{\mu_n}\log(n\mu_n) + O(1).$$

Moez Draief — Majority Consensus by Local Polling

Complete Graph: Theory v. Simulation



Star

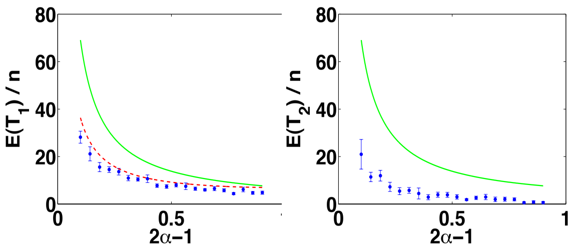
- **Star Network:** $q_{1,i} = q_{i,1} = \frac{1}{n-1}$, $i \neq 1$ and $q_{i,j} = 0$, $i, j \neq 1$. $\mathbb{E}(T_i) \leq \frac{1}{2\alpha-1} n \log(n)$. Using, direct calculation

$$\mathbb{E}(T_1) = \frac{1}{(2\alpha-1)(3-2\alpha)} n \log(n) + O(n)$$

- **ER-graph:** $q_{i,j} = \frac{1}{np_n} X_{i,j}$ $X_{i,j}$ i.i.d. Bernoulli r.v. with mean $c\frac{\log(n)}{n}$, $c > \frac{2}{2\alpha-1}$, for h^{-1} the inverse of $h(x) = x\log(x) + 1 - x$,

$$\mathbb{E}(T_i) \leq \frac{1}{(2\alpha-1)h^{-1}\left(\frac{2}{c(2\alpha-1)}\right)} \log(n) + O(1)$$

- **Path**: $\mathbb{E}(T_i) \leq \frac{16(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1)$
- **Ring**: $\mathbb{E}(T_i) \leq \frac{4(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1)$.



ER-graph

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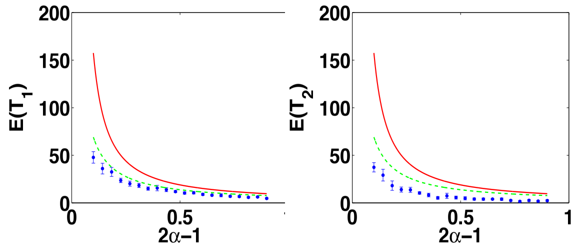

Path and Ring

- **Star Network:** $q_{1,i} = q_{i,1} = \frac{1}{n-1}$, $i \neq 1$ and $q_{i,j} = 0$, $i, j \neq 1$. $\mathbb{E}(T_i) \leq \frac{1}{2\alpha-1} n \log(n)$. Using, direct calculation

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Summary

- Upper bound on the expected convergence time for a number of distributed for solving Majority consensus
- Bounds based on the location of the spectral gap of rate matrix (generalised-cut: quick for expander graphs).
- For binary consensus, expected convergence time critically depends on the voting margin
- Application to particular network topologies: complete graphs, stars, ER graph, paths, cycles.

(Discrete-time k–choice local majority protocol

At $t = 0$, each vertex of G is blue **independently with constant probability** $\alpha \in (1/2, 1)$.

Local Majority

We then run $\mathcal{MP}^k$ on G. Choose k odd ($k \geq 5$ in what follows).

- At each time t, each vertex v polls k neighbours **uar**, and assumes majority colour
- If v doesn't have k neighbours, poll all, or all minus one

What is the probability that there will be a red consensus?

How long does it take to reach consensus?

Graphs of a given degree sequence

Let $V = [n]$

$\mathcal{G}_n(\mathbf{d})$: the set of connected simple graphs with degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, where d_i is the degree of vertex $i \in V$.

Need some restrictions on degree sequence to make it graphical, e.g., $\sum_i d_i$ is even

Nice degree sequences

Let $V_j = \{i \in V : d_i = j\}$, $= \frac{1}{n}\sum_{i=1}^{n} d_i$ be the average degree, $0 < \kappa \leq 1$, $0 < c < 1/8$ constants, and let $\gamma = (\sqrt{\log n}/)^{1/3}$. A degree sequence **d** is **nice** if it satisfies

(i) Average degree $= o(\sqrt{\log n})$.

(ii) Minimum degree $\delta \geq 3$.

(iii) Let $d \geq 5$ be such that $|V_d| = \kappa n + o(n)$. We call d the **effective minimum degree**.

(iv) Number of little vertices $\sum_{j=\delta}^{d-1} |V_j| = O(n^{\frac{1}{11}})$; a vertex i is **little** if $d_i \leq d - 1$.

(v) Maximum degree $\Delta = O(n^{\frac{1}{11}})$.

(vi) Upper tail size $\sum_{j=\gamma}^{\Delta} n_j = O(\Delta)$.

The effective minimum degree

(iii) Let $d \geq 5$ be such that $|V_d| = \kappa n + o(n)$. We call d the **effective minimum degree**.

Need not be a constant, can have $d \to \infty$ as $n \to \infty$

Not necessarily the minimum degree (though it can be)

Can have “little” vertices with smaller degree, as long as not too many of them:

(iv) Number of little vertices $\sum_{j=\delta}^{d-1} |V_j| = O(n^{\frac{1}{11}})$; a vertex i is **little** if $d_i \leq d - 1$.

Examples of nice degree sequences

- Any d-regular graph with $d \geq 5$ and $d = o(\sqrt{\log n})$
- ‘Bi-regular’ graph where half the vertices are degree $d \geq 5$ and half of degree $\Delta = o(\sqrt{\log n})$.
- Truncated power-law

Results: informal statement

Suppose G is typical with effective min degree d. If we run $\mathcal{MP}^k$ then

Upper bound

If $d/k = O(1)$ and α is 'not too close' to 1/2, then **whp**, correct consensus is reached within $(A \log_k d) \log_k \log_k n$ steps

$(A \leq 5$ and $A \to 1$ if $k \to \infty)$

Lower bound

Any algorithm where a vertex keeps its colour if same as all neighbours, will take at least $\log_d \log_d n$ steps to reach correct consensus, **whp**

Bias condition

"α is not too close to 1/2" means

$$\left[\left(1 + \frac{1}{\sqrt{k}}\right) 2\right]^{\frac{2}{k-2}} \alpha(1-\alpha) < 1/4$$

Since $\alpha \neq 1/2 \Rightarrow \alpha(1-\alpha) < 1/4$, so inefficiency is in $\left[\left(1 + \frac{1}{\sqrt{k}}\right) 2\right]^{\frac{2}{k-2}}$

$k = 5$ needs $1 - \alpha < 0.143$

$k = 20$ needs $1 - \alpha < 0.350$

$k = 100$ needs $1 - \alpha < 0.437$

Compare with other works

E. Mossel, J. Neeman, O. Tamuz ('14) Study local majority on d-regular λ-expanders. Show sufficient bias implies certain correct consensus.

better bias condition but only regular graphs, no timing information, full polling only

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+better bias condition, -only infinite regular graph

J. Cruise and A. Ganesh ('10) Study (m,d)-generalisation of local majority on complete graphs with unit rate exponential on each vertex. Give exponential decay error probability and $O(\log n)$ timing

+stronger error probability, -only complete graph

Typical graphs

Typical graphs: For a nice degree sequence **d**, the space $\mathcal{G}_n(\mathbf{d})$ is the set of nice graphs

We do not analyse for the whole space, only for those graphs called **typical**

Informally, G is typical if it is nice and:

- most vertices are locally tree-like
- little vertices and very high-degree vertices, should they exist, are far from each other and small cycles

Let $\mathcal{G}'_n(\mathbf{d}) \subset \mathcal{G}_n(\mathbf{d})$ be the typical graphs, then $|\mathcal{G}'_n(\mathbf{d})|/|\mathcal{G}_n(\mathbf{d})| \to 1$ as $n \to \infty$

Modified Majority

Let $\mathcal{T} = G[v, c \log_k \log_k n]$.
At $t+1$, each $x \in V$ randomly picks a $x(k)$-subset of neighbours $N_x(t+1)$

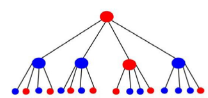
- $x \notin \mathcal{T}$ then x becomes at $t+1$ the majority colour of the vertices in $N_x(t+1)$.

$$X_{t+1}^{\mathcal{MMP}^k(v,s)}(x) = \mathbf{1}_{\left\{\left(\sum_{y \in N_x(t+1)} X_t^{\mathcal{MP}^k}(y)\right) > x(k)/2\right\}}.$$

- non-leaf $x \in \mathcal{T}$ and Par(x) the parent of x in $\mathcal{T}$. At $t+1$, x becomes the majority colour of the vertices in $N_x(t+1)$, with the added assumption that Par(x) was red at time t.

$$X_{t+1}^{\mathcal{MMP}^k(v,s)}(x) = \mathbf{1}_{\left\{\left(\sum_{y \in N_x(t+1)\setminus\{\text{Par}(x)\}} X_t^{\mathcal{MMP}^k(v,s)}(y)\right) > x(k)/2\right\}}.$$

Modified majority protocol



For a vertex v, let $X_v(t)$ be the indicator v is red at time t under $\mathcal{MPP}^k$. Let $k = 2r+1$.

- At time $t = 0$, for each level 2 (i.e., leaf) vertex v, $\mathbb{P}(X_v(0) = 0) = p_0 = 1 - \alpha$
- At time $t = 1$, for each level 1 vertex v $\mathbb{P}(X_v(1) = 0) = p_1 = \mathbb{P}(\text{Bin}(2r, p_0) \geq r)$
- At time $t = 2$, for each level 0 vertex v (i.e., the root) $\mathbb{P}(X_v(2) = 0) = p_2 = \mathbb{P}(\text{Bin}(2r, p_1) \geq r)$

Modified majority protocol

If height of the tree is H, then given p_t, at $t+1$, for v at distance $H-t-1$ from root,

$$\mathbb{P}(X_v(t+1)=0) = p_{t+1} = \mathbb{P}(\text{Bin}(2r, p_t) \geq r)$$

and we get a rapidly decaying sequence $p_0 > p_1 > \ldots > p_t$ with $p_0 = \alpha \gg p_t$ when t large

When $t = \Omega(\log\log n)$, p_t is very small and we conclude by union bound over all n vertices

The root will have the correct colour.

Now we are left to deal with vertices not locally tree-like...

Theorem: Erdös-Renyi graphs

Let $p = \frac{c \log n}{n}$ where $c > 2+\epsilon$ for some constant $\epsilon > 0$, $k \geq 5$ and $\nu = \lfloor \frac{k-1}{2} \rfloor$. Run $\mathcal{MP}^k$ on $G \in \mathcal{G}(n,p)$.
Let $A = \frac{1+\varepsilon}{\log_k(k-1) - \log_k 2}$ where $\varepsilon > 0$ is a small constant. Subject to condition

$$\left[\left(1 + \frac{1}{\sqrt{2\nu}}\right) 2\right]^{\frac{1}{\nu-1}} 4\alpha(1-\alpha) < 1$$

by time $A \log_k \log_k n$, $\mathcal{MP}^k$ will have reached consensus on the initial majority **whp**.

Summary

- Asymptotic correct and efficient consensus using local polling. What happens for other values of k? [Cooper-Elsasser-Radzik'14]
- Analysis for a sparse family of graphs and dense E-R graphs.
- Still lot of ongoing interest...

Related work

- Kanoria, Montanari, Majority dynamics on trees and the dynamic cavity method, Annals Applied Prob., 2011
- Jung, Kim, Vojnovic, Distributed Ranking in Networks with Limited Memory and Communication, IEEE ISIT 2012
- Mossel, Neeman and Tamuz, Majority Dynamics and Aggregation of Information in Social Networks, Autonomous Agents and Multi-Agent Systems, 2014
- Mertzios, Nikoletseas, Raptopoulos, Spirakis, Determining Majority in Networks with Local Interactions and very Small Local Memory, ICALP 2014
- Becchetti, Clementi, Natale, Pasquale, Silvestri, Trevisan, Simple dynamics for plurality consensus. SPAA 2014.

ROBUST NETWORKED STABILIZATION: WHEN GAP MEETS TWO-PORT

Li Qiu, Hong Kong University of Science and Technology

In robust networked stabilization, if the plant and controller uncertainties are described by the gap metric (or ν-gap metric), and the communication network between the plant and the controller is described by a two-port with an uncertain transmission matrix, magic happens.

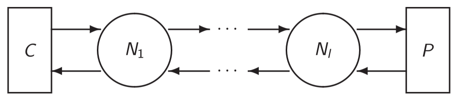
Networked Robust Stabilization: when gap meets two-port

Guoxiang Gu and Li Qiu

Louisiana State University
Hong Kong University of Science and Technology

October 2014

Networked control system (NCS)



- The plant and controller are uncertain and the uncertainty is described by the gap metric.
- The communication network is a cascade of two-port networks, modeling bidirectional transmission with relays.
- We only consider SISO plants and controllers in this talk, for the sake of simplicity.

Gap metric: a review

(Zames; Vidyasagar; Georgiou and Smith; Qiu and Davison; Vinnicombe, from 1980 to ~1995.)

- P is LTI and possibly unstable.
- Graph of P

$$\mathcal{G}_P = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{H}_2 \times \mathcal{H}_2 : y = Pu \right\}.$$

  a subspace of $\mathcal{H}_2 \times \mathcal{H}_2$.
- Gap metric between P_1 and P_2.

$$\delta(P_1, P_2) = \|\Pi_{\mathcal{G}_{P_1}} - \Pi_{\mathcal{G}_{P_2}}\|.$$

- Uncertain systems can be described by gap balls

$$\mathcal{B}(P, r) = \{\tilde{P} : \delta(P, \tilde{P}) \leq r\}.$$

- Gap ball viewed as rotation of the graph:

$$\{\tilde{P} : \mathcal{G}_{\tilde{P}} = (I + \Delta)\mathcal{G}_P, \|\Delta\|_\infty \leq r\} \subset \mathcal{B}(P, r).$$

  The maximal rotating angle is arcsin r.

Gap metric: a review (continued)

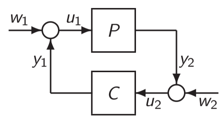


Figure: Feedback System (P, C).

- The gang of four

$$GoF(P, C) = \begin{bmatrix} \frac{1}{1-PC} & \frac{-C}{1-PC} \\ \frac{P}{1-PC} & \frac{-PC}{1-PC} \end{bmatrix} = \begin{bmatrix} 1 \\ P \end{bmatrix} (1 - PC)^{-1} \begin{bmatrix} 1 & -C \end{bmatrix}.$$

- Need to work on the inverse graph of C

$$\mathcal{G}'_C = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{H}_2 \times \mathcal{H}_2 : u = Cy \right\}.$$

- (P, C) is stable if $\mathcal{G}_P$ and $\mathcal{G}'_C$ are complementary.

Gap metric: a review (continued)

- $(\tilde{P}, \tilde{C})$ is stable for all $\tilde{P} \in \mathcal{B}(P, r_P)$ and $\tilde{C} \in \mathcal{B}(C, r_C)$ iff

$$\arcsin r_P + \arcsin r_C < \arcsin \|GoF(P, C)\|_\infty^{-1}.$$

(arcsin theorem)

- Optimal robust control problem:

$$\min_C \|GoF(P, C)\|_\infty$$

an "easy" $\mathcal{H}_\infty$ control problem.

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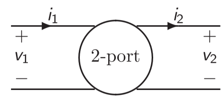
Two-port circuit: a review



- Transmission representation

$$\begin{bmatrix} v_1(s) \\ i_1(s) \end{bmatrix} = A(s) \begin{bmatrix} v_2(s) \\ i_2(s) \end{bmatrix}.$$

- Ideal transmission

$$A_0(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Transmission with distortion

$$A(s) = I + \Delta(s).$$

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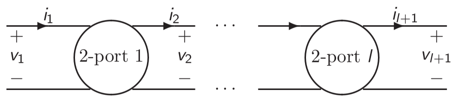
Cascade connection of two-port circuits



- The transmission matrices are multiplied

$$\begin{bmatrix} v_1(s) \\ i_1(s) \end{bmatrix} = A_1(s)A_2(s)...A_l(s) \begin{bmatrix} v_{l+1}(s) \\ i_{l+1}(s) \end{bmatrix}.$$

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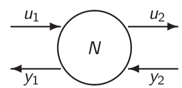
Two-port communication network



- Voltages → down-link signals, currents → up-link signals.
- Transmission matrix

$$\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} = A \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}.$$

- Ideal transmission

$$A_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

- Transmission with distortion

$$A = I + \Delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \Delta_\div & \Delta_- \\ \Delta_+ & \Delta_\times \end{bmatrix}, \quad \|\Delta\|_\infty < r.$$

(Notation invented by Halsey and Glover)

- We allow Δ to be nonlinear.

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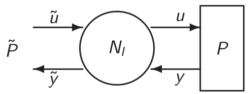
Plant with two-port distortion



- Linear fractional transformation (LFT)

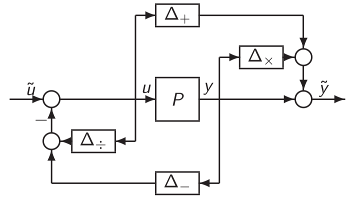
$$\tilde{P} = \frac{(1+\Delta_\times)P + \Delta_+}{1+\Delta_\div + \Delta_- P}.$$

- Graph of the distorted system

$$\mathcal{G}_{\tilde{P}} = (I+\Delta)\mathcal{G}_P.$$

The same type of rotation as in the gap ball.

System with $+, -, \times, \div$ uncertainty.



Two-port transmission model of NCS

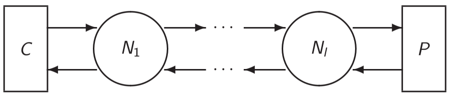


Figure: Networked control system.

- $\tilde{P}, \tilde{C}$ are only known to belong to gap balls

$$\tilde{P} \in \mathcal{B}(P, r_P), \tilde{C} \in \mathcal{B}(C, r_C).$$

- N_i is only known to have transmission matrix

$$A_i = I + \Delta_i, \|\Delta_i\|_\infty \le r_i.$$

Main result

- The NCS is robustly stable iff

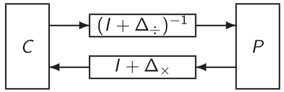
$$\arcsin r_P + \arcsin r_C + \sum_{i=1}^{l} \arcsin r_i < \arcsin \|GoF(P,C)\|_\infty^{-1}.$$

(Networked arcsin theorem)

- Optimal design

$$\min_C \|GoF(P,C)\|_\infty.$$

Hammerstein-Wiener model of NCS



- $\Delta_\times$ and $\Delta_\div$ are nonlinear time-varying systems satisfying

$$\|\Delta_\times\|_\infty \le r, \quad \|\Delta_\div\|_\infty \le r.$$

- Two-port network with diagonal transmission matrix.
- $\Delta_\times$ and $\Delta_\div$ can be used to model logarithmic quantizations.
- The NCS is robustly stable iff

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \Delta_\div & 0 \\ 0 & \Delta_\times \end{bmatrix} GoF$$

is stably invertible for all $\Delta_\times$ and $\Delta_\div$.

μ-synthesis of Hammerstein-Wiener system

- Introducing scaling: the closed-loop system is robustly stable iff

$$\begin{aligned} &\inf_{\gamma\in(0,\infty)} \left\| \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} GoF(P,C) \begin{bmatrix} 1 & 0 \\ 0 & \gamma^{-1} \end{bmatrix} \right\|_\infty \\ = &\inf_{\gamma\in(0,\infty)} \left\| \begin{bmatrix} \frac{1}{1-PC} & \frac{-\gamma^{-1}C}{1-PC} \\ \frac{\gamma P}{1-PC} & \frac{PC}{1-PC} \end{bmatrix} \right\|_\infty \\ = &\inf_{\gamma\in(0,\infty)} \|GoF(\gamma P, \gamma^{-1}C)\|_\infty < \frac{1}{r}. \end{aligned}$$

- Optimal design

$$\inf_{\gamma\in(0,\infty)} \inf_C \|GoF(\gamma P, \gamma^{-1}C)\|_\infty.$$

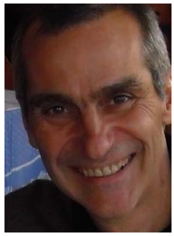
- The function $\gamma \mapsto \inf_C \|GoF(\gamma P, \gamma^{-1}C)\|_\infty$ was mistakenly conjectured to be unimodal.

Conclusions

- Trade-off between the capacities of the down-link and the up-link channels.
- Optimal robust networked control, linking the history.
- H_2 vs H_∞ theory

SCHROEDINGER BRIDGES: STEERING OF STOCHASTIC SYSTEMS CLASSICAL AND QUANTUM

Tryphon Georgiou, University of Minnesota, USA



The classical Schroedinger bridge seeks the most likely probability law for a diffusion process, in path space, that matches marginals at two end points in time; the likelihood is quantified by the relative entropy between the sought law and a prior, and the law dictates a controlled path that abides by the specified marginals. Schreodinger proved that the optimal steering of the density between the two end points is effected by a multiplicative functional transformation of the prior; this transformation represents an automorphism on the space of probability measures and has since been studied by Fortet, Beurling and others. A similar question can be raised for processes evolving in a discrete time and space as well as for processes defined over non-commutative probability spaces. Ultimately, in all of the above Schrodinger's question relates to a corresponding stochastic control problem to steer a stochastic system from an initial to a final distribution. In the talk we will begin with a treatment of the Schroedinger bridge problem for Markov chains, Quantum channels, and then for classical stochastic systems where we will present in the linear-quadratic case (linear dynamics, Gaussian evolution) the solution to the minimum energy steering problem in closed form. We will discuss potential applications in active suppression of noise and in steering swarms of inertial diffusive particles. The presentation will be based on joint work with Michele Pavon and with Yongxin Chen.

Schrödinger Bridges classical and quantum evolution

Tryphon Georgiou
University of Minnesota

Joint work with Michele Pavon and Yongxin Chen

LCCC
Workshop on Dynamic and Control in Networks
Lund, October 2014

- History of Schrodinger bridges
- Bridges for Markov chains
- The Hilbert metric
- Bridges for quantum (TPTP) evolutions
- Bridges for Gauss-Markov process

Schrodinger 1931/32:
The time reversal of the laws of nature

Kolmogoroff:
The reversibility of the statistical laws of nature

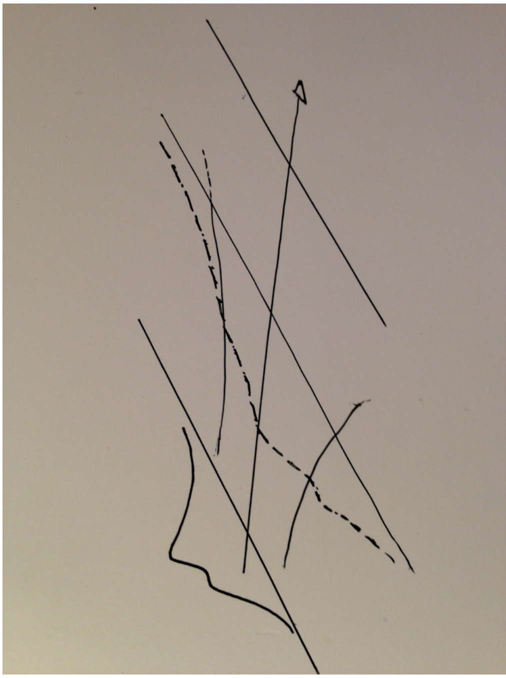
Bernstein 1932
Fortet 1940
Beurling 1960
Jamison 1974/75
Follmer 1988

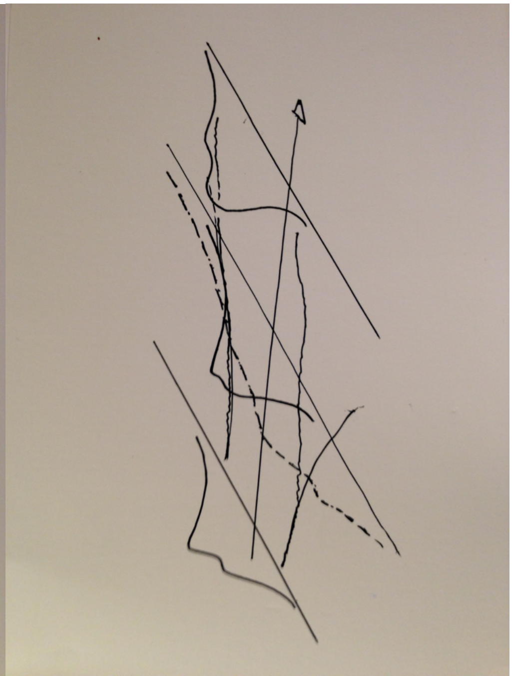
connections to Nelson's stochastic mechanics
Zambrini, Wakolbinger, Dai Pra, Pavon,Ticozzi
and others

Hilbert metric:

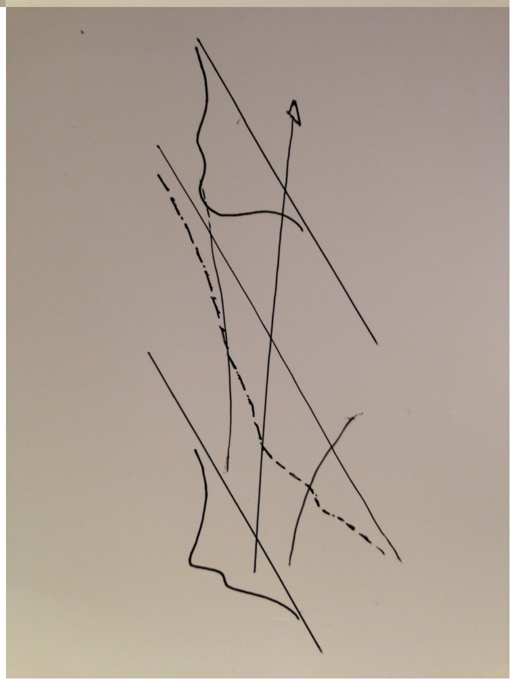
Hilbert 1895
Birkhoff 1957
Bushell 1973

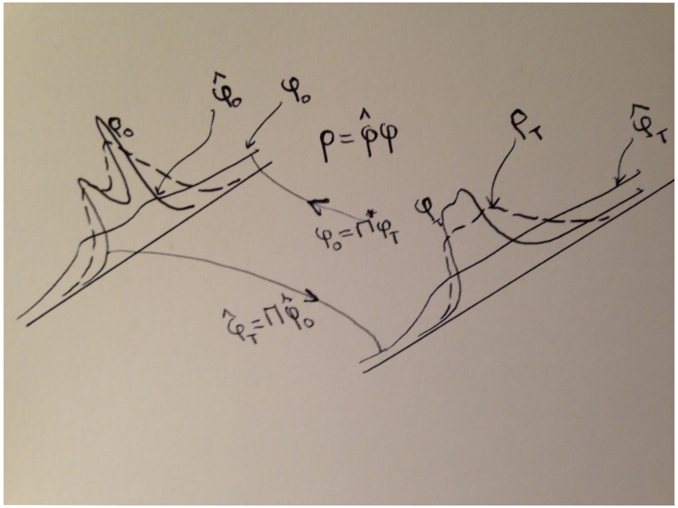
Sepulchre, Sarlette, Rouchon 2010
Reeb, Kastoryano, Wolf 2011





- Schrodinger 1931/1932: suppose a large number of Brownian particles observed at two different times to evolve between two empirical distributions. What is the most likely intermediate distribution at any point in time?





Given initial and final distribution $p_0(x)$, $p_T(x)$ and transition $p(x,y)$

Schrödinger hypothesised that

$$p_T(\cdot) \neq \int p(x,\cdot)p_0(x)dx$$
$$=: \Pi_{0T}(p_0(x))$$

$\Pi_{0t} \; : \; q_0(x) \to q_t(x)$

$\frac{\partial q(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 q(t,x)}{\partial x^2}, \quad q(0,x) = q_0(x).$

Large deviations
Sample paths
Relative entropy
Stochastic Control

Schrödinger system

discretized time, space, N-particles
Stirling's approximation
optimized, lagrange multipliers

the most likely joint density and transition probability

$\boxed{P^\star(x_0,x_T) = \hat\phi(x_0)p(x_0,x_T)\phi(x_T)}$ and $\boxed{p^\star(x_0,x_T) = p(x_0,x_T)\frac{\phi(x_T)}{\phi(x_0)}}$

$$p_0(x_0) = \hat\phi(x_0)\int p(x_0,x_T)\phi(x_T)dx_T = \hat\phi(x_0)\overbrace{\Pi^\dagger_{0,T}\underbrace{\phi(x_T)}_{\phi_T(x_T)}}^{\phi_0(x_0)}$$

$$p_T(x_T) = \phi(x_T)\int p(x_0,x_T)\hat\phi(x_0)dx_0 = \phi(x_T)\overbrace{\Pi_{0,T}\underbrace{\hat\phi(x_0)}_{\hat\phi_0(x_0)}}^{\hat\phi_T(x_T)}$$

and $\boxed{p_t(x_t) = \hat\phi_t(x_t)\phi(x_t)}$ where $\phi_t(x_t) := \Pi^\dagger_{t,T}\phi(x_T)$, $\hat\phi_t(x_t) := \Pi_{0,t}\hat\phi(x_0)$

Schrödinger system

Schrödinger: there exists a solution "except possibly for very nasty p_0, p_T because the question leading to the pair of equations is so reasonable."

Existence/uniqueness
Fortet 1940
Résolution d'un system d'equations de M. Schrd̈inger

Beurling 1960
An automorphism of product measures

Markov chains

$\{1,\dots,N\}$ states, $x=(x_0,x_1,\dots,x_T)$ sample path
Π_t stochastic matrices, $t\in\{1,\dots,T\}$
$P\in$ probability induced by Π's on $\{1,\dots,N\}^{T+1}$

$P(x_0,\dots,x_T)=P(x_0,x_T)P(x_1,\dots,x_{T-1}\mid x_0,x_T)$

Schrödinger question

given p_0, p_T
$p_T\neq\Pi_T\cdots\Pi_1 p_0$

find
$Q(x_0,\dots,x_T)=Q(x_0,x_T)Q(x_1,\dots,x_{T-1}\mid x_0,x_T)$
such that
$\sum_{x_T}Q(x_0,x_T)=p_0(x_0)$
$\sum_{x_0}Q(x_0,x_T)=p_T(x_T)$
and minimizes the relative entropy

$$\sum_{all}Q\log\frac{Q}{P}=\sum_{x_0,x_T}Q(x_0,x_T)\log\frac{Q(x_0,x_T)}{P(x_0,x_T)}+\sum_{all}Q(\cdot\mid x_0,x_T)\log\frac{Q(\cdot\mid x_0,x_T)}{P(\cdot\mid x_0,x_T)}Q(x_0,x_T)$$

Lagrangian

$$L(Q)=\sum_{x_0,x_T}Q(x_0,x_T)\log\frac{Q(x_0,x_T)}{P(x_0,x_T)}+\sum_{x_0}\lambda(x_0)\left(\sum_{x_T}Q(x_0,x_T)-p_T(x_T)\right)+\sum_{x_T}\mu(x_T)\left(\sum_{x_0}Q(x_0,x_T)-p_0(x_0)\right)$$

$\lambda(x_0)\sim\hat\phi_0$
$\mu(x_T)\sim\phi_T$
such that with $\Pi=\Pi_T\cdots\Pi_1$

Schrödinger system

$$\hat\phi_T=\Pi\hat\phi_0$$
$$\phi_0=\Pi^\dagger\phi_T$$
$$p_0=\phi_0\circ\hat\phi_0$$
$$p_T=\phi_T\circ\hat\phi_T$$

if there is a solution

$$\Pi^\star=D_{\phi_T}\Pi D_{\phi_0}^{-1}$$
$$[Q(x_0,x_T)]_{x_0,x_T}=D_{\phi_T}\Pi D_{\hat\phi_0}$$

Hilbert metric

S real Banach space
K closed solid cone in S

$$x\preceq y\Leftrightarrow y-x\in K,$$

$$M(x,y) := \inf\{\lambda\mid x\preceq\lambda y\}$$
$$m(x,y) := \sup\{\lambda\mid \lambda y\preceq x\}.$$

define the Hilbert metric:

$$d_H(x,y):=\log\left(\frac{M(x,y)}{m(x,y)}\right).$$

Examples:
i) positive cone in $\mathbb{R}^n$
ii) positive definite Hermitian matrices

d_H-gain bound of positive maps

Π is a positive map:

$$\Pi \; : \; K\backslash\{0\} \to K\backslash\{0\}.$$

Projective diameter

$$\Delta(\Pi) := \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K\backslash\{0\}\}$$

Contraction ratio, or gain/H-norm

$$\|\Pi\|_H := \inf\{\lambda \mid d_H(\Pi(x), \Pi(y)) \leq \lambda d_H(x, y), \text{ for all } x, y \in K\backslash\{0\}\}.$$

Birkhoff-Bushell theorem

Let Π positive, monotone, homogeneous of degree m, i.e.,

$$x \preceq y \Rightarrow \Pi(x) \preceq \Pi(y),$$

and

$$\Pi(\lambda x) = \lambda^m \Pi(x),$$

then

$$\|\Pi\|_H \leq m.$$

For the special case where Π is also linear, the (possibly stronger) bound

$$\|\Pi\|_H = \tanh(\frac{1}{4}\Delta(\Pi))$$

also holds.

Solution of the Schrödinger system

Lemma

Let $\Pi >_e 0$ (element-wise positive) stochastic matrix
p_0, p_T probability vectors
then $\|\Pi\|_H < 1$.

proof
i) $\Delta(\Pi) = \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K\backslash\{0\}\}$
remains the same if we restrict x, y
to be probability vectors
ii) $d_H(\Pi(x), \Pi(y)) < \infty \; \forall x, y$.
iii) the probability simplex is compact.

Solution of the Schrödinger system

Consider

$$\begin{array}{ccc} & \hat{\varphi}_0 \xrightarrow{\Pi} \hat{\varphi}_T & \\ \hat{\varphi}_0(x_0) = \frac{\mathbf{p}_0(x_0)}{\varphi_0(x_0)} \uparrow & & \downarrow \varphi_T(x_T) = \frac{\mathbf{p}_T(x_T)}{\hat{\varphi}_T(x_T)} \\ & \varphi_0 \xleftarrow{\Pi^\dagger} \varphi_T & \end{array}$$

where

$$\mathcal{D}_T \; : \; \varphi_0 \mapsto \hat{\varphi}_0(x_0) = \frac{\mathbf{p}_0(x_0)}{\varphi_0(x_0)}$$
$$\mathcal{D}_T \; : \; \hat{\varphi}_T \mapsto \varphi_T(x_T) = \frac{\mathbf{p}_T(x_N)}{\hat{\varphi}_T(x_T)}$$

are componentwise division of vectors $\Rightarrow$ d_H-isometries!

The composition

$$\hat{\varphi}_0 \xrightarrow{\Pi} \hat{\varphi}_T \xrightarrow{\mathcal{D}_T} \varphi_T \xrightarrow{\Pi^\dagger} \varphi_0 \xrightarrow{\mathcal{D}_0} (\hat{\varphi}_0)_{\text{next}}$$

is strictly contractive in the Hilbert metric.

Sinkhorn's theorem

If $\Pi >_e 0$,
then $\exists\ a_i, b_j$
such that $[\pi_{ij} a_i b_j]_{i,j}$ doubly stochastic.

Cf. $p_0 = \mathbb{1}$, $p_T = \mathbb{1}$
$\Pi^\star = D_{\phi_T} \Pi D_{\phi_0}^{-1}$ doubly stochastic

i.e., $(\Pi^\star)^\dagger \mathbb{1} = \mathbb{1}$
but also $(\Pi^\star)\mathbb{1} = \mathbb{1}$

Quantum analogues

Density matrices: $\mathfrak{D} = \{\rho \geq 0 \mid \text{trace}(\rho) = 1\}$

TPTP: $\mathcal{E} : \mathfrak{D} \to \mathfrak{D} : \rho \longrightarrow \sigma = \sum_{i=1}^{n_\mathcal{E}} E_i \rho E_i^\dagger$
with

$$\sum_{i=1}^{n_\mathcal{E}} E_i^\dagger E_i = I$$

i.e., $\mathcal{E}^\dagger(I) = I$

$\mathcal{E}$ is positivity improving: if $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$

Reference quantum evolution

TPCP maps $\{\mathcal{E}_t; 0 \leq t \leq T-1\}$
with Kraus representation

$$\mathcal{E}_t : \sigma_t \mapsto \sigma_{t+1} = \sum_i E_{t,i} \sigma_t E_{t,i}^\dagger, \quad t = 0, 1, \ldots, T-1.$$

Consider the composition

$$\mathcal{E}_{0:T} := \mathcal{E}_{T-1} \circ \cdots \circ \mathcal{E}_0.$$

initial and a final ρ_0 and ρ_T
Problem
Find $\mathcal{F}_{0:T} = \mathcal{F}_{T-1} \circ \cdots \circ \mathcal{F}_0$ such that

$$\mathcal{F}_{0:T}(\rho_0) = \rho_T.$$

and $\mathcal{F}$ "close to" $\mathcal{E}$

"rank-1" corrections

$$\mathcal{F}_t(\cdot) = \chi_{t+1} \left(\mathcal{E}_t(\chi_t^{-1}(\cdot)\chi_t^{-\dagger}) \right) \chi_{t+1}^\dagger$$

i.e., $\mathcal{F}_t = \Phi_{t+1} \circ \mathcal{E}_t \circ \Phi_t^{-1}$ where
Φ are rank-1 Kraus maps, $n_\Phi = 1$

Corresponds to the commutative case via: $\chi^\dagger \chi = \phi$

Quantum version of Sinkhorn's thm

Suppose $\mathcal{E}_{0:T}$ is positivity improving
Then, $\exists$ observables ϕ_0, ϕ_T such that,
for any factorization

$$\begin{aligned}\phi_0 &= \chi_0^\dagger \chi_0, \text{ and}\\ \phi_T &= \chi_T^\dagger \chi_T,\end{aligned}$$

the map

$$\mathcal{F}(\cdot) := \chi_T \left(\mathcal{E}_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$

is a *doubly stochastic* Kraus map,
in that $\mathcal{F}(I) = I$ as well as $\mathcal{F}^\dagger(I) = I$.

General case

Given $\mathcal{E}_{0:T}^\dagger$ and ρ_0 and ρ_T
if $\exists$ ϕ_0, ϕ_T, $\hat{\phi}_0$, $\hat{\phi}_T$ solving

$$\begin{aligned}\mathcal{E}_{0:T}^\dagger(\phi_T) &= \phi_0,\\ \mathcal{E}_{0:T}(\hat{\phi}_0) &= \hat{\phi}_T,\\ \rho_0 &= \chi_0 \hat{\phi}_0 \chi_0^\dagger,\\ \rho_T &= \chi_T \hat{\phi}_T \chi_T^\dagger.\end{aligned}$$

Then, for any factorization

$$\begin{aligned}\phi_0 &= \chi_0^\dagger \chi_0, \text{ and}\\ \phi_T &= \chi_T^\dagger \chi_T,\end{aligned}$$

the map

$$\mathcal{F}(\cdot) := \chi_T \left(\mathcal{E}_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$

is a quantum bridge for $(\mathcal{E}_{0:T}^\dagger, \rho_0, \rho_T)$, namely $\mathcal{F}(I) = I$
and $\mathcal{F}^\dagger(\rho_0) = \rho_T$.

Proof

$$\begin{array}{ccc} \hat{\phi}_0 & \xrightarrow{\mathcal{E}_{0,T}} & \hat{\phi}_T \\ \hat{\phi}_0 = \phi_0^{-1} \ \uparrow & & \downarrow \ \phi_T = \hat{\phi}_T^{-1} \\ \phi_0 & \xleftarrow{\mathcal{E}_{0,T}^\dagger} & \phi_T \end{array}$$

The composition map

$$\mathcal{C} \ : \ \left(\hat{\phi}_0\right)_{\text{starting}} \xrightarrow{\mathcal{E}_{0,T}} \hat{\phi}_T \xrightarrow{(\cdot)^{-1}} \phi_T \xrightarrow{\mathcal{E}_{0,T}^\dagger} \phi_0 \xrightarrow{(\cdot)^{-1}} \left(\hat{\phi}_0\right)_{\text{next}}$$

is strictly contractive
the steps are identical

Conjecture

The quantum Schrödinger system has a solution
for arbitrary ρ_0, ρ_T

Snag in the proof:
$\phi \to \hat{\phi}$ and $\hat{\phi} \to \phi$ are not isometries, e.g.,

$$\begin{aligned} D_T \ &: \ \hat{\phi}_T \mapsto \phi_T = \left(\rho_T^{1/2} \left(\rho_T^{-1/2} \hat{\phi}^{-1} \rho_T^{-1/2} \right)^{1/2} \rho_T^{1/2} \right)^2 \\ \hat{D}_0 \ &: \ \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{1/2} \rho (\phi_0)^{1/2} \end{aligned}$$

Extensive numerical evidence that
the composition has a fixed point
Software for numerical experimentation
`http://www.ece.umn.edu/~georgiou/papers/schrodinger_bridge/`

Pinned bridge

$\mathcal{E}_{0:T}$ positivity improving and two pure states

$$\rho_0 = v_0 v_0^\dagger \text{ and } \rho_T = v_T v_T^\dagger$$

(i.e., v_0, v_T are unit norm vectors), define

$$\phi_0 := \mathcal{E}(v_T v_T^\dagger)$$
$$\phi_T := v_T v_T^\dagger,$$

and

$$\mathcal{F}^\dagger(\cdot) := \phi_T^{1/2}\mathcal{E}^\dagger(\phi_0^{-1/2}(\cdot)\phi_0^{-1/2})\phi_T^{1/2}$$

(where, clearly, $\phi_T^{1/2} = \phi_T = v_T v_T^\dagger$). Then, $\mathcal{F}^\dagger$ is TPTP and satisfies the marginal conditions

$$\rho_T = \mathcal{F}^\dagger(\rho_0).$$

Example

$$\mathcal{E}(\cdot) = E_1(\cdot)E_1^\dagger + E_2(\cdot)E_2^\dagger + E_3(\cdot)E_3^\dagger$$

$$E_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{bmatrix},\ E_3 = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{bmatrix}.$$

$$\rho_0 = \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix} \text{ and } \rho_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$\phi_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$
$$\phi_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$
$$\hat{\phi}_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix}$$
$$\hat{\phi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} \sqrt{2/3} & 0 \\ 0 & 0 \end{bmatrix},\ F_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1/3} \end{bmatrix},\ F_3 \begin{bmatrix} 0 & \sqrt{2/3} \\ \sqrt{1/3} & 0 \end{bmatrix}$$

Recap

Hilbert metric $\Rightarrow$ constructive existence proofs for

i) classical Schrödinger systems

ii) quantum Sinkhorn version (uniform marginals)

iii) general case open

Final topic:

Schrödinger bridges for "degenerate" classical linear stochastic systems

$\equiv$ a new type of optimal control problem

Optimal steering of state-densities

min relative entropy $\overset{\text{Girsanov}}{\leftrightarrow}$ minimum energy stochastic control

$dx = bdt + dw$ diffusion

$dx = (b+u)dt + dw$ controlled diffusion

$\min\{E\{\|u\|^2\} \mid p_0, p_T\} \sim$ relative entropy from prior (dai Pra)

our interest:

inertial particles, cooling of oscillators

$dx = vdt$

$dv = (b+u)dt + dw$ controlled degenerate diffusion

Optimal steering of state-densities

$dx(t) = A(t)x(t)dt + B(t)u(t)dt + B(t)dw(t)$

Given initial and terminal (target) Gaussian densities with covariances Σ_0, Σ_T.

Find $u(t)$ with $t \in [0, T]$ that steers the system from the initial to the target state density and minimizes

$$E\{\int_0^T u(t)'u(t)dt\}$$

The optimal control is $u(t) = -B(t)'Q(t)^{-1}x(t)$
The controlled degenerate diffusion is the closest to the uncontrolled diffusion in the relative entropy sense.

$$Q(0) = N(T,0)^{1/2}S_0^{1/2}\left(S_0 + \frac{1}{2}I - \left(S_0^{1/2}S_T S_0^{1/2} + \frac{1}{4}I\right)^{1/2}\right)^{-1} S_0^{1/2}N(T,0)^{1/2}$$

$N(T,0)$ is the controllability Grammian.

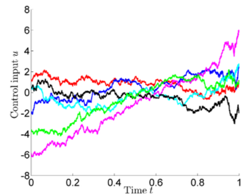
Optimal steering of state-densities

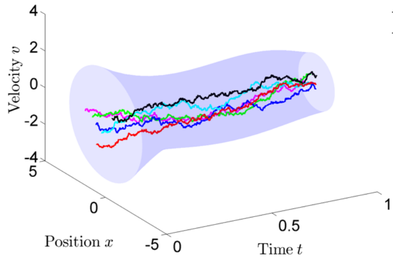
Theorem (Gauss-Markov Schrödinger bridge):
There exists a unique solution to the following (analogue of the Schrödinger system)

$Q(T)$, $P(0)$ values for matrices satisfying
$\Sigma_0^{-1} = Q(0)^{-1} + P(0)^{-1}$
$\Sigma_T^{-1} = Q(T)^{-1} + P(T)^{-1}$
and $Q(0)$, $P(T)$ obtained via
$\dot{Q}(t) = A(t)Q(t) + Q(t)A(t)' + B(t)B(t)'$
$\dot{P}(t) = A(t)P(t) + P(t)A(t)' - B(t)B(t)'$
with $Q(t)$ invertible over $[0, T]$.

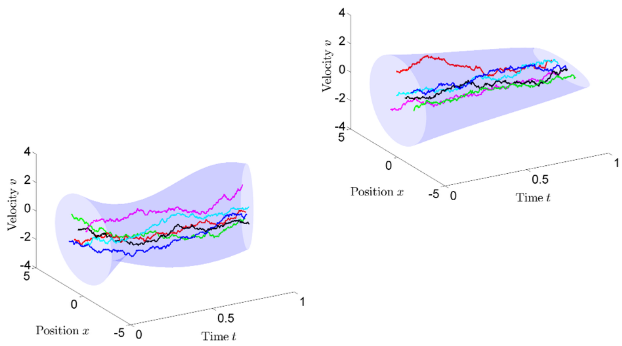
Gauss Markov model for inertial particles

$$\begin{aligned} dx(t) &= v(t)dt \\ dv(t) &= u(t)dt + dw(t) \end{aligned}$$



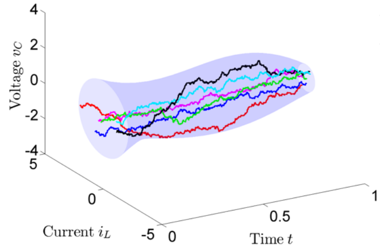


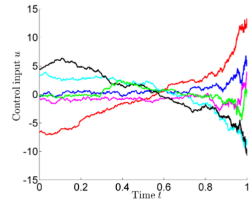
Gauss Markov model for inertial particles

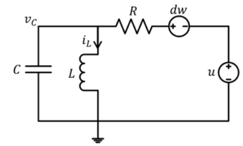


Gauss Markov model for Nyquist-Johnson noise driven oscillator

$$
\begin{aligned}
L di_L(t) &= v_C(t)dt \\
RC dv_C(t) &= -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t)
\end{aligned}
$$







Gauss Markov model for inertial particles: state-cost ~ particles with losses

$$dX(t) = f(X(t),t)dt + \sigma(X(t),t)dw(t)$$

$$
\begin{aligned}
&\inf_{(\tilde\rho,\tilde u)} \int_{\mathbb{R}^N}\int_0^T \left[\frac{1}{2}\|u\|^2 + V(x,t)\right]\tilde\rho(x,t)dtdx, \\
&\frac{\partial\tilde\rho}{\partial t} + \nabla\cdot((f+\sigma u)\tilde\rho) = \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}\tilde\rho)}{\partial x_i\partial x_j}, \\
&\tilde\rho(0,x) = \rho_0(x), \quad \tilde\rho(T,y) = \rho_T(y).
\end{aligned}
$$

$$a_{ij}(x,t) = \sum_k \sigma_{ik}(x,t)\sigma_{kj}(x,t)$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(f(x,t)\rho) + V(x,t)\rho = \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}(x,t)\rho)}{\partial x_i\partial x_j}$$

Schrödinger system

$$
\begin{aligned}
\frac{\partial\varphi}{\partial t} + f(x,t)\cdot\nabla\varphi + \frac{1}{2}\sum_{i,j=1}^N a_{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j} &= V\varphi, \\
\frac{\partial\hat\varphi}{\partial t} + \nabla\cdot(f(x,t)\hat\varphi) - \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}\hat\varphi)}{\partial x_i\partial x_j} &= -V\hat\varphi,
\end{aligned}
$$

$$\varphi(x,0)\hat\varphi(x,0) = \tilde\rho_0(x), \quad \varphi(x,T)\hat\varphi(x,T) = \tilde\rho_T(x)$$

$$
\begin{aligned}
u^*(x,t) &= \sigma'\nabla\log\varphi(x,t), \\
\frac{\partial\tilde\rho}{\partial t} + \nabla\cdot((f + a\nabla\log\varphi)\tilde\rho) &= \frac{1}{2}\sum_{i,j=1}^N \frac{\partial^2(a_{ij}\tilde\rho)}{\partial x_i\partial x_j},
\end{aligned}
$$

Controllability of Fokker-Planck - Linear-Gaussian

$$dx(t) = Ax(t)dt + Bu(t)dt + B_1 dw(t)$$
$$\text{with } x(0) = x_0 \text{ a.s.}$$

Thm: (A,B) controllable is sufficient to steer the system from any initial Gaussian distribution to a final one at t=T.

Thm: A Gaussian state-pdf can be “sustained” with constant state-feedback iff the state covariance satisfies

$$(A - BK)\Sigma + \Sigma(A' - K'B') + B_1 B_1' = 0.$$

equivalently, $\text{rank}\begin{bmatrix} A\Sigma + \Sigma A' + B_1 B_1' & B \\ B & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}$

Schrödinger system

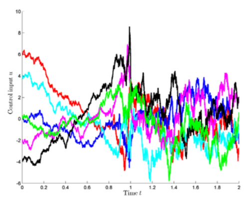
$$\begin{aligned} \dot{\Pi} &= -A'\Pi - \Pi A + \Pi BB'\Pi \\ \dot{\mathrm{H}} &= -A'\mathrm{H} - \mathrm{H}A - \mathrm{H}BB'\mathrm{H} \\ &\quad + (\Pi + \mathrm{H})\,(BB' - B_1 B_1')\,(\Pi + \mathrm{H}) \\ \Sigma_0^{-1} &= \Pi(0) + \mathrm{H}(0) \\ \Sigma_T^{-1} &= \Pi(T) + \mathrm{H}(T). \end{aligned}$$

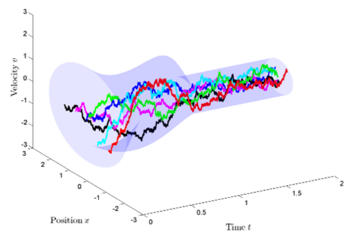
Compare with conditions for:
i) steering the system to a given state - controllability
ii) steering within the positive cone?
iii) maintaining the state at a given value

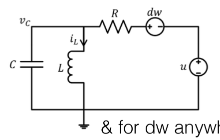
$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$0 = (A - BK)\xi + Bu$$

Fast “cooling” + stationary control







& for dw anywhere

Open problem

Density matrices: e.g.
$\mathfrak{D} = \{\rho \geq 0 \mid \text{ symmetric } \rho \in \mathbb{R}^{n\times n} \text{ with } \mathrm{trace}(\rho) = 1\}$

E_i with $i = 1, \ldots, n_{\mathcal{E}}$ and $\sum_{i=1}^{n_{\mathcal{E}}} E_i^\dagger E_i = I$
(typically $n_{\mathcal{E}} \sim n^2$
for "positivity-improving": $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$)

TPTP: $\mathcal{E} : \mathfrak{D} \to \mathfrak{D} : \rho \longrightarrow \sigma = \sum_{i=1}^{n_{\mathcal{E}}} E_i \rho E_i^\dagger$

Data: ρ_0, ρ_T, $\mathcal{E}$.
Problem: Prove that the iteration:

$$\begin{aligned}
\mathcal{E} &: \hat{\phi}_0 \mapsto \hat{\phi}_T = \mathcal{E}(\hat{\phi}_0) \\
D_T &: \hat{\phi}_T \mapsto \phi_T = \left(\rho_T^{1/2}\left(\rho_T^{-1/2}\hat{\phi}^{-1}\rho_T^{-1/2}\right)^{1/2}\rho_T^{1/2}\right)^2 \\
\mathcal{E}^\dagger &: \phi_T \mapsto \phi_0 = \mathcal{E}^\dagger(\phi_T) \\
\hat{D}_0 &: \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{1/2}\rho(\phi_0)^{1/2}
\end{aligned}$$

has an attractive fixed point.

Software for numerical experimentation
http://www.ece.umn.edu/~georgiou/papers/schrodinger_bridge/

Thank you for your attention

http://arxiv.org/abs/1405.6650
Positive contraction mappings for classical and quantum Schrodinger systems

http://arxiv.org/abs/1407.3421
Stochastic bridges of linear systems

http://arxiv.org/abs/1410.1605
Optimal steering of inertial particles diffusing anisotropically with losses

arxiv.org/abs/1408.2222
Optimal steering of a linear stochastic system to a final probability distribution

arxiv.org/abs/1410.3447
Optimal steering of a linear stochastic system to a final probability distribution, Part II

RANDOMIZED AVERAGING ALGORITHMS, WHEN CAN ERRORS AND UNRELIABILITIES JUST BE IGNORED?

Julien Hendrickx, Université Catholique de Louvain, Belgium

We consider randomized discrete-time consensus systems that preserve the average ”on average”, and provide a new upper bound on the mean square deviation of the final consensus value from the initial average.

We show that a certain asymptotic accuracy can be guaranteed when there are few simultaneous interactions or when the simultaneous interactions are sufficiently uncorrelated. Our results are easily applicable to many classes of systems, and we particularize them to various algorithms having been proposed in the literature, obtaining bounds that match or outperform all previously available ones..

Randomized averaging algorithms:
when can errors & unreliability just be ignored?

Paolo Frasca & ***Julien Hendrickx***

Lund – LCCC workshop Oct 2014

1

(Linear) Averaging

Agents have values $x_i(t)$

Averaging iteration:

$$x_i(t+1) = \sum a_{ij}(t)x_j(t)$$

$$a_{ij}(t) \geq 0 \qquad \sum_j a_{ij}(t) = 1$$

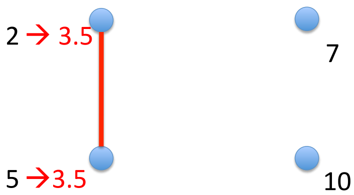
If average preserving (ex: if $a_{ij} = a_{ji}$) and enough interactions

$$x_i \to \bar{x} = \frac{1}{n}\sum x_i(0)$$

Basis for many decentralized algorithms

2

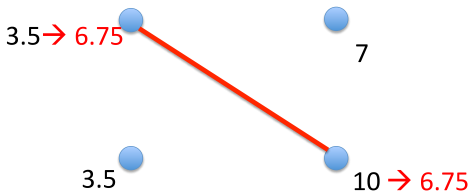
Ex: Gossip



$\bar{x} = 6$

3

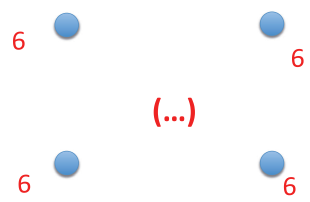
Ex: Gossip



$\bar{x} = 6$

4

Ex: Gossip



$\bar{x} = 6$

Convergence to the average

What if transmissions are asymmetric/uncertain?

5

Communication requirements for averaging

- Simple consensus:

$$x(t+1) = A(t)x(t) \qquad A(t) \text{ stochastic}$$

Average preserved if *A(t)* doubly stochastic. $\mathbf{1}^T A(t) = \mathbf{1}^T$. Needs
- ***symmetric / balanced*** interactions
- therefore ***synchronous*** interactions

- Push sum *[Kempe et al 03]*, based on mass preservation

Directed asynchronous communications OK
But needs ***reliability***: agents must know
- if message received,
- how many agents receive message

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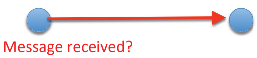
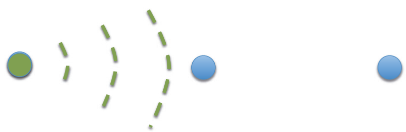
Unpredictability and Unreliability

- Directed & uncertain communication



- Packet loss
- Broadcasts at random times



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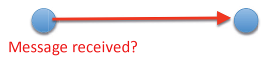
Unpredictability and Unreliability

- Directed & uncertain communication



- Packet loss
- Broadcasts at random times
- Collisions due to interferences

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Unpredictability and Unreliability: 2 approaches

1) Verification and correction mechanisms

idea: *errors may create big trouble*

2) Ignore issues, assume average behavior and hope for the best

Idea: *errors will cancel out*

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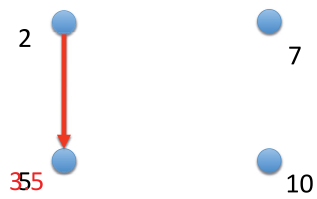
Naïve extension: Asymmetric Gossip

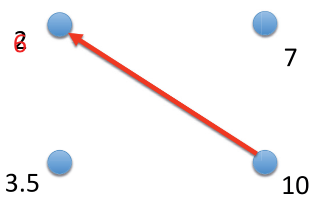


$\bar{x}(0) = 6$

$\bar{x} = 5.625$

10

Naïve extension: Asymmetric Gossip



$\bar{x}(0) = 6$

$\bar{x} = 6.625$

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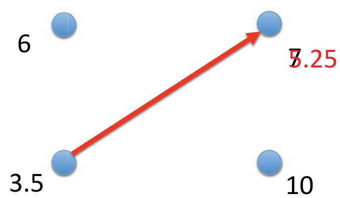
Naïve extension: Asymmetric Gossip



$\bar{x}(0) = 6$

$\bar{x} = 6.1875$

Does not converge to average, but seems close…

Guarantee?

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Outline

- Introduction
- ***Formulation and previous results***
  - ***Formulation***
  - ***Exact approach***
  - ***Convergence-based approach***
- Our approach
- Particularization and Applications
- Conclusions

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Problem formulation

$$x(t+1) = A(t)x(t)$$

where $A(t) \in \Re_+^{n \times n}$

- Averaging matrices ($A(t)\mathbf{1} = \mathbf{1}$)
- ***i.i.d. random variables***

Any distribution, but
events at different times independent

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Problem formulation

$$x(t+1) = A(t)x(t)$$

where $A(t) \in \Re_+^{n \times n}$

- Averaging matrices ($A(t)\mathbf{1} = \mathbf{1}$)
- ***i.i.d. random variables***
- preserve average on expectation ($\mathbf{1}^T EA(t) = \mathbf{1}^T$)

→ $E\bar{x}(t) = \bar{x}(0)$

Convergence to "consensus" w.p. 1 under conditions

$$x(t) \rightarrow \mathbf{1}x_\infty \quad \text{with} \quad Ex_\infty = \bar{x}(0)$$

Goal: bound ***standard asymptotic error*** $E\left(x_\infty - \bar{x}(0)\right)^2$

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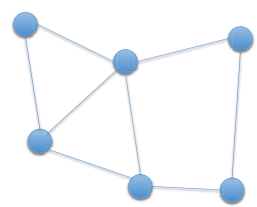
Examples: nominal network



1. Synchronous symmetric updates,

$$x_i(t+1) = x_i(t) + \sum_j a_{ij}(x_j(t) - x_i(t))$$

but some messages are lost

2. Nodes wake up and update at random times
3. Nodes wake up, send messages to neighbor(s) at random time

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Exact approach

Standard asymptotic error computed exactly using x(0) and V:

$$E(A^T V A) = V \qquad \mathbf{1}^T V \mathbf{1} = 1$$

But, very hard to obtain expression for V

→ Theoretical computation only for specific cases (involved!)

Ex: *[Fagnani & Zampieri, SICON 2009]*
Fixed averaging algorithm with packet losses deployed on Cayley graphs of Abelian groups

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Exact approach

Standard asymptotic error computed exactly using x(0) and V:

$$E(A^T V A) = V \qquad \mathbf{1}^T V \mathbf{1} = 1$$

But, very hard to obtain expression for V

→ Theoretical computation only for specific cases (involved!)
→ Numerical verification for given system (hard)

- Numerical issues
- May need enumeration of all A
often many (one for each possible combination of updates)

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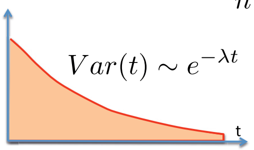
Convergence based approach

$Var(t)$ Individual variance E $\frac{1}{n}\sum (x_i(t) - \bar{x}(t))^2$

$err(t)$ Expected group error E $(\bar{x}(t) - \bar{x}(0))^2$

Idea $\Delta err(t) \leq \frac{K}{n} Var(t)$

$Var(t) \sim e^{-\lambda t}$



$$err(\infty) \leq O\left(\frac{K}{n\lambda}\right) Var(0)$$

λ linked to algebraic connectivity of "network"

Ex: [Fagnani & Frasca, 2011]

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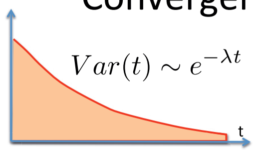
Convergence based approach

$Var(t) \sim e^{-\lambda t}$



$$err(\infty) \leq O\left(\frac{K}{n\lambda}\right) Var(0)$$

λ linked to algebraic connectivity of "network"

Conclusion: keep λ large when n grows → expanders, etc.

But, expanders not easy to build physically (range constraints, etc.)

and λ often not critical in experimental results

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Outline

- Introduction
- Formulation and previous results
- ***Our approach***
- Particularization and Applications
- Conclusion and further works

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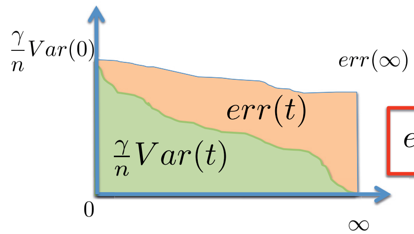
Our approach: conservation

Find γ s.t. $err(t) + \frac{\gamma}{n} Var(t)$ non-increasing



$$err(\infty) \leq \frac{\gamma}{n} Var(0)$$

- If γ independent of n (or o(n)), "***accuracy***" for large n
- Not always the case, exist systems with large errors

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How to pick γ ?

$err(t) + \frac{\gamma}{n} Var(t)$ non-increasing

γ valid if $E\left(L^T \mathbf{1}\mathbf{1}^T L\right) \leq \gamma E\left(L + L^T - L^T L\right)$

with $L(t) = I - A(t)$

Appears involved, but

- Generic γ can be found for large classes of systems
- Results very easy to use
- Often leads to strong bounds

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Outline

- Introduction
- Formulation and previous results
- Our approach
- ***Particularization and Applications***
  1. ***Limited updates***
  2. ***Uncorrelated updates***
  3. ***Uncorrelated broadcasts***
  4. ***Limited correlations***
- Conclusions

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1. Limited updates

$A_{\max} \geq \sum_i \sum_{j \neq i} a_{ij}(t)$ Max total simultaneous updates

$a_{ii}^{\min} \leq a_{ii}(t)$ Min. self confidence

Then $\gamma = \dfrac{A_{\max}}{a_{ii}^{\min}}$

$$err(\infty) \leq \frac{1}{n}\left(\frac{A_{\max}}{a_{ii}^{\min}}\right) V(0)$$

"Accuracy" if self confidence + amount of updates << n

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Asymmetric gossip

$$A_{\max} \geq \sum_i \sum_{j \neq i} a_{ij}(t) \rightarrow err(\infty) \leq \frac{1}{n}\left(\frac{A_{\max}}{a_{ii}^{\min}}\right) V(0)$$

Iteration: At each t, ***1 random node*** i chooses* j and updates to

$$x_i(t) + q\,(x_j(t) - x_i(t))$$

$A_{\max} = q$

$a_{ii}^{\min} = 1 - q$ → $$err(\infty) \leq \frac{1}{n}\left(\frac{q}{1-q}\right) V(0)$$

Accuracy for large n. Bound similar to *[Fagnani & Zampieri 08]*

* With symmetric probabilities

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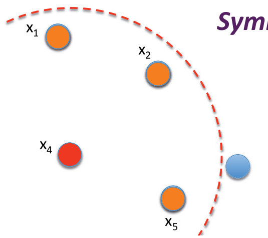
Broadcast gossip

- At each t, a node j wakes up with uniform probability
- j transmits x_j to all within listening range (assumed symmetric)
- update $x_i(t+1) = x_i(t) + q(x_j(t) - x_i(t))$

$q \in (0,1)$



Symmetry → $E(\bar{x}(t)) = \bar{x}(0)$

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Broadcast gossip

- At each t, a node j wakes up with uniform probability
- j transmits x_j to all within listening range (assumed symmetric)
- update $x_i(t+1) = x_i(t) + q(x_j(t) - x_i(t))$

$q \in (0,1)$

__Previous results__

- Involved, rely on spectral quantities related to network defined by listening ranges
- No general guarantee of accuracy for large n

Ex: $err(\infty) \leq O\left(\dfrac{d_{\max}^2}{\lambda_1 n}\right) V(0)$

$d_{\max}$ Largest degree

λ_1 Smallest nonzero eigenvalue laplacian

[Aysal et al. 2009], [Fagnani & Delvenne 2010], [F. Fagnani and P. Frasca. 2011a,b]

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Broadcast gossip: our approach

- At each t, a node j wakes up with uniform probability
- j transmits x_j to all within listening range (assumed symmetric)
- update $x_i(t+1) = x_i(t) + q(x_j(t) - x_i(t))$

$q \in (0,1)$

Limited updates rule:

$$A_{\max} = q d_{\max}, \quad a_{ii}^{\min} = 1 - q \quad \rightarrow \quad err(\infty) \le \frac{d_{\max}}{n}\left(\frac{q}{1-q}\right) V(0)$$

- Accuracy for large n if $d_{\max} < o(n)$
- Independent of graph spectrum

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2. Independent updates

$$err(\infty) \le \frac{\gamma}{n} Var(0)$$

If nodes make independent decisions about updates and weights

a_{ij}, a_{kl} Uncorrelated if $i \neq k$ Then

$$\gamma = \frac{1 - a_{ii}^{\min}}{a_{ii}^{\min}}$$

→ Always accurate when $n \to \infty$ if $a_{ii}^{\min}$ bounded

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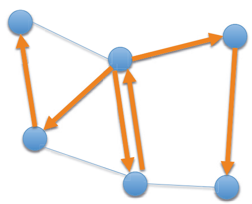
Synchronous Asymmetric Gossip

Iteration: Every node i chooses neighbor j_i and updates to

$$x_i(t+1) = (1-q)x_i(t) + q x_{j_i}(t)$$

Probabilities s.t $E(\bar{x}(t)) = \bar{x}(0)$



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Synchronous Asymmetric Gossip

Iteration: Every node i chooses neighbor j_i and updates to

$$x_i(t+1) = (1-q)x_i(t) + q x_{j_i}(t)$$

Probabilities s.t $E(\bar{x}(t)) = \bar{x}(0)$

Previous results

- Related to eigenvalues of graph Laplacian
- Accuracy for particular cases

[Fagnani & Zampieri, 2008]

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Synchronous Asymmetric Gossip

Iteration: Every node i chooses neighbor j_i and updates to

$$x_i(t+1) = (1-q)x_i(t) + q x_{j_i}(t)$$

Probabilities s.t $E(\bar{x}(t)) = \bar{x}(0)$

Indep updates: $err(\infty) \le \frac{\gamma}{n} Var(0)$ $\gamma = \frac{1-a_{ii}^{\min}}{a_{ii}^{\min}}$

$a_{ii}^{\min} = 1 - q$

$$err(\infty) \le \frac{1}{n}\left(\frac{q}{1-q}\right) V(0)$$

For all graphs, independently of spectral quantities

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3. Independent broadcasts

$$err(\infty) \le \frac{\gamma}{n} Var(0)$$

- 2. Independent decisions about updates and weights

$$\gamma = \frac{1-a_{ii}^{\min}}{a_{ii}^{\min}}$$

- 3. Independent broadcast decisions (i.e. columns in A)

$$\gamma = \frac{a_{col}^{\max}}{a_{ii}^{\min}} \qquad a_{col}^{\max} \ge \sum_{i:i \ne j} a_{ij}(t) \qquad \forall j,t$$

Max importance of a node

Minimal self confidence

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4. General correlation rule

$$err(\infty) \le \frac{\gamma}{n} Var(0)$$

If ***no*** a_{ij} ***correlated to more than 4C*** other coefficients

$$\gamma = \frac{C}{a_{ii}^{\min}}$$

(conservative)

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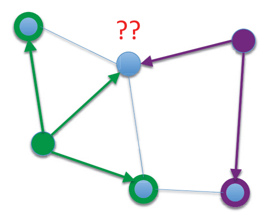
Broadcast with collisions

At every time:

- Some nodes awake and broadcast their values
- Nodes receiving one value update
- Nodes receiving two or more values do not



Weights and probabilities s. t. $E(\bar{x}(t)) = \bar{x}(0)$

Challenges:
- Multiple updates
- Multiple correlations

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Broadcast with collisions

At every time:

- Some nodes awake and broadcast their values
- Nodes receiving one value update
- Nodes receiving two or more values do not

Previous results

- Results on Cayley graphs of Abelian Groups

- Accuracy for certain sparse networks when n grows

[Fagnani & Frasca, 2011]

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Broadcast with collisions

$$\gamma = \frac{C}{a_{ii}^{\min}}$$

If ***no*** a_{ij} ***correlated to more than 4C*** other coefficients

How many correlations?

$a_{12} > 0$ If 1 transmits

Unless 3 transmits

Which also affects a_{34}

Correlations up to distance 3

→ $4C \leq d_{max}^3$

38

Broadcast with collisions

$$\gamma = \frac{C}{a_{ii}^{\min}}$$

If ***no*** a_{ij} ***correlated to more than 4C*** other coefficients

Correlations up to distance 3 $\quad 4C \leq d_{max}^3$

$$err(\infty) \leq \frac{d_{\max}^3}{n} \frac{1}{a_{ii}^{\min}} V(0)$$

Accuracy for large n if $d_{max} \leq o(n^{1/3})$

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Outline

- Introduction
- Formulation and previous results
- Our approach
- Particularization and Applications
- ***Conclusions***

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Summary

- $\frac{\gamma}{n}$ group error + individual var. non-increasing*
- γ often easy to compute
- Bound independent of convergence speed and spectral properties; only local properties
- Prove asymptotical accuracy of many schemes
- Equals or outperforms (almost) all previous ad hoc results

* actually $\frac{\gamma}{n+\gamma}$

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Summary

$$err(\infty) \leq \frac{\text{Largest "correlated event"}}{\text{Min "self confidence"}} \frac{Var(0)}{n}$$

Ignoring problems and hoping for the best is OK if

- Limited correlations *(often OK in multi-agent systems)*
- Sufficient self-confidence for agents

Robustness of large class algorithms with respect to important fluctuations

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Possible developments

- Expected average not entirely preserved
- Correlation between different times
- More detail → Less conservative bounds
- Other algorithms

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Thank you for your attention!

References

Paolo Frasca and J.H, ***On the mean square error of randomized averaging algorithms,*** Automatica 2013 arXiv:1111.4572v1

Paolo Frasca and J.H., ***Large network consensus is robust to packet losses and interferences***, Proceedings of ECC 2013, Zurich, July 2013

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SYSTEMIC RISK IN FINANCE

Munther A. Dahleh, Massachusetts Institute of Technology, USA

We study welfare of competitive equilibria in an economy with banks runs and costly fire sales, formalizing the importance of "animal spirits" for financial stability. These forces are not the product of irrationality, but a consequence of uncertainty of a self-fulfilling nature. Short-term debt issued by banks is safe, and thus functions as private money. Feedback between financial constraints and market prices results in a systemic externality not internalized by banks in the process of money creation: during credit booms aggregate short-term debt is excessive; during recessions it is insufficient. With a multi-stage game formulation we show that there are exactly two equilibria corresponding to a Boom-Bust scenarios.

Work is co-authored with Diego Fejer.

Systemic Risk in Financial Systems

Diego Feijer and Munther Dahleh
LIDS, MIT

Cascades and Systemic Risk

Systemic Risk is a term used to describe fragility in interconnected systems that result in cascades of failures due to either relatively small shocks at the subsystem level or larger and more malicious types of disruptions affecting the whole system.

Air Traffic Congestion: $31.2B

Power Outages: $80B-$150B

Financial Crisis 2008: $500B + …

Major Disruptions: Fukushima, H1N1

2

Market Expectations and Economic Activity

- **Aggregate expectations** play a key role in credit markets and **macroeconomic activity**
- **Optimism** results in **credit booms**; **pessimism** leads to **credit crunches**, and potential recession
    - Most recent financial crisis
- **Fluctuations** in market **sentiments** often seem to happen without apparent reason; they seem to be driven by **animal spirits**

Animal Spirits and Regulation

“History –including recent history– shows that without regulation, animal spirits will drive the economic activity to extremes.”

G. Akerlof and R. Shiller

“Good Government and Animal Spirits.” WSJ, Apr. 24, 2009

Related Literature

- Dominant theory to justify regulation is based on **fire sales** and **pecuniary externalities**:
  - Occasionally binding **collateral constraint**
  - **Over-borrowing** during credit booms

    Caballero & Krishnamurthy'03; Lorenzoni'08; Bianchi'11; **Stein'12**
  - **Silent about credit crunches**
- **In this model:**
  - Collateral constraint + (binding) **credit risk constraint**
  - **Endogenous** state of the economy: **booms** and **crunches**

Preview of the Model

$t=0$	$t=1$	$t=2$
Banks raise funds from creditors to invest in risky asset	Short-term creditors receive noisy signal about banks outlooks	Pay off from investment is realized
They choose a mix of short and long-term debt	They decide whether to rollover credit or not	Insolvent banks fail, and creditors receive zero
	Banks that experience runs can liquidate assets in a fire sale	

Households

- Initial endowment of consumption good
- Consume endowment at date 0, or invest in financial assets (lend to banks) and consume proceeds at date 2
- **Two financial assets:** $(m, 1-m)$
  - Risky "bonds" with gross real return R_B (lend long-term)
  - **Riskless "money"** with gross real return R_M (lend short-term)
- Linear preferences over consumption:

  $$U(\{C_t\}) = C_0 + \beta \mathbb{E}[C_2] + \gamma m R_M \quad \text{with} \quad \beta + \gamma < 1$$
- Derive utility from **"monetary" services** provided by **safe assets**

Banks

- Continuum of identical banks, with total mass one
- **Risky investment opportunity** at date 0, with real return

  $\theta \sim F$ with mean $\bar{\theta} > \frac{1}{\beta}$ and support over $[\theta_{\min}, \theta_{\max}]$
- Banks need 1 unit of consumption good to undertake investment. They raise money (short-term debt)
- and finance issuing bonds (long-term debt): $(m, 1-m)$
- Short-term debt claims need to be **rolled over** at date 1
- To meet redemption demands, banks can sell any fraction of their assets in a **secondary market**.

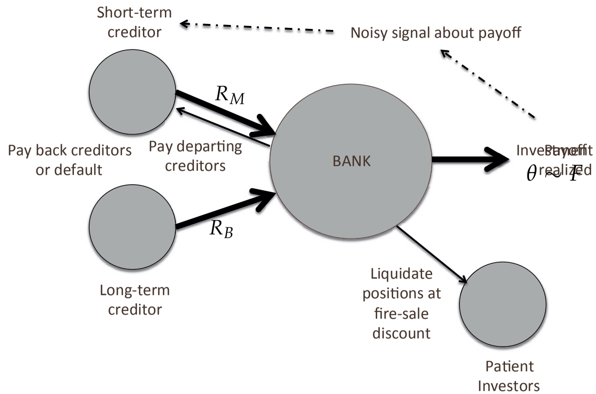
Fire Sales and Patient Investors

- Sales yield $k\theta$ per unit liquidated. **Fire-sale discount** $k \in [0,1]$
- **Collateral constraint:** Short-term debt has to be **riskless**. If a bank issues m units of short-term finance, then

$$mR_M \leq k\theta_{\min}$$

- Patient investors receive **fixed** endowment $I > \frac{1}{\beta + \gamma}$ at **date 1**
- Investment technology: $g(\cdot)$ increasing and strictly concave
- **No Crisis:** they invest all resources in new, late arriving projects that yield total output of $g(I)$
- **Crisis:** they provide liquidity to banks purchasing assets at fire-sale discount, and invest the rest of resources in projects

Recap



Coordination Problem

- A bank **fails** at date 2 when unable to honor its liabilities
- **Assumption (Zero recovery value).** In case of failure, **all creditors** holding debt claims at date 2 receive a payoff of 0
- Suppose the capital structure of a bank is $(m, 1-m)$
- At date 1, θ is realized and its value revealed to banks and patient investors. Each creditor j receives a **noisy private signal**: $\theta_j = \theta + \epsilon_j$ with $\epsilon_j \sim \mathcal{U}[-\epsilon, \epsilon]$ iid
- If a fraction λ of creditors decide to withdraw, the bank needs to liquidate a fraction q: $qk\theta = \lambda m R_M$. The bank fails if:

$$(1-q)\theta < (1-\lambda)mR_M + (1-m)R_B$$

Defines: $\theta_{\text{run}}(\lambda)$

Panic Runs

- The bank fails if

$$\theta < \theta_{\text{run}}(\lambda) \equiv mR_M + (1-m)R_B + \lambda m R_M \left(\frac{1}{k} - 1\right)$$

Lower dominance region $[\theta_{\min}, \theta_{\text{run}}(0))$

Upper dominance region $(\theta_{\text{run}}(1), \theta_{\max}]$

- **Switching strategy.** In the limit as $\epsilon \to 0$, the coordination game has a **unique (symmetric) equilibrium** in which agents run whenever their signal is below the threshold:

$$\theta^* = mR_M + (1-m)R_B + mR_M \left(\frac{1}{k} - 1\right)$$

Insolvency threshold ($mR_M + (1-m)R_B$)

Illiquidity component ($mR_M \left(\frac{1}{k} - 1\right)$)

Self-fulfilling illiquidity fears may drive a solvent bank to failure

Prices

- **Credit Market:** households indifference condition

$$R_M = \frac{1}{\beta+\gamma} < R_B = \frac{1}{\beta(1-F(\theta^*))}$$

Cheaper because of convenience yield

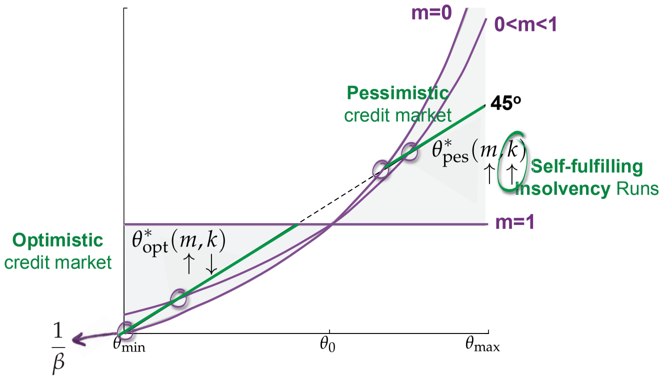
- **Fire Sales:** patient investors indifference condition; investment activities should yield same marginal returns

$$\frac{1}{k} = g'(I - mR_M)$$

Captures how funding decisions at the individual level affect the whole system $\frac{dk}{dm} < 0$

Sentiments

$$\theta^* = \frac{mR_M}{k} + \frac{1-m}{\beta(1-F(\theta^*))}$$ has **two solutions**



Private Money Creation

- **Assumption (Looting).** The owner of a bankrupt bank steals any remaining assets for personal consumption
- Taking k as given, banks choose m to maximize profits:

$$\Pi(m;k) = \left(\bar{\theta} - \frac{1}{\beta}\right) + m\left(\frac{1}{\beta} - R_M\right) - F(\theta^*)mR_M\left(\frac{1}{k} - 1\right)$$

Gains — Fire-sale cost

subject to:
$$\begin{cases} mR_M \leq k\theta_{min} & \textbf{(collateral)} \\ \theta^* = \frac{mR_M}{k} + \frac{1-m}{\beta(1-F(\theta^*))} & \textbf{(credit risk)} \end{cases}$$

- **First term:** expected profits if only long-term debt
- **Second term:** gains from moving to cheaper short-term debt
- **Third term:** cost (risk) that comes along with short-term debt

Competitive Equilibrium

- **Definition.** A CE is a pair (m^*, k^*) such that,
  - m^* maximizes:

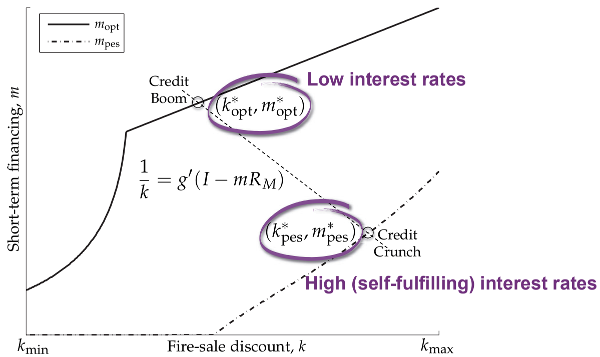
$$\left(\bar{\theta} - \frac{1}{\beta}\right) + m\left(\frac{1}{\beta} - R_M\right) - F(\theta^*_s)mR_M\left(\frac{1}{k} - 1\right)$$

subject to:
$$\begin{cases} mR_M \leq k\theta_{min} & \textbf{(collateral)} \\ s \in \{opt, pes\} & \textbf{(sentiment)} \end{cases}$$

multiplicity

  - and: $\frac{1}{k^*} = g'(I - m^*R_M)$

Credit Booms and Crunches



Fluctuations in market sentiment are driven by **animal spirits**

Welfare

- Proceeds from investments by banks and patient investors are rebated back to households
- Welfare at date 2 is $W(m)$:

$$\left(\bar{\theta} - \frac{1}{\beta}\right) + m\left(\frac{1}{\beta} - R_M\right)$$
$$+ \left\{(1 - F(\theta^*))g(I) + F(\theta^*)\left(g(I - mR_M) + mR_M\right) - \frac{I}{\beta}\right\}$$

- **First term:** net expected return to investment by banks
- **Second term:** monetary services
- **Third term:** net expected return to investment by patient investors

Welfare

- Proceeds from investments by banks and patient investors are rebated back to households
- **Consumption at date 2:**

probability

$$\mathbb{E}[\theta|\theta > \theta^*] + g(I) \qquad 1 - F(\theta^*)$$
$$\mathbb{E}[(1-q)\theta|\theta < \theta^*] + g(I - mR_M) + mR_M + \mathbb{E}[q\theta|\theta < \theta^*] \qquad F(\theta^*)$$

- Welfare at date 2 is $W(m)$:

$$\left(\bar{\theta} - \frac{1}{\beta}\right) + m\left(\frac{1}{\beta} - R_M\right)$$
$$+ \left\{(1 - F(\theta^*))g(I) + F(\theta^*)\left(g(I - mR_M) + mR_M\right) - \frac{I}{\beta}\right\}$$

Planning Problem

- The planner solves:

$$\max_{m \in [0,1]} W(m) \quad \text{subject to:}$$

multiplicity

$$\begin{cases} mR_M \leq k\theta_{min} & \text{(collateral)} \\ \theta^*_s = \theta^*_s(m, k) \quad s \in \{opt, pes\} & \text{(sentiment)} \\ \frac{1}{k} = g'(I - mR_M) & \text{(fire sale discount)} \end{cases}$$

Inefficiency

- **Theorem.** CE, in general, not constrained efficient.
- **Wedge** between private and social solutions:

$$\tau_s^* = -\mu_s^P \frac{\theta_{\min}}{R_M} \frac{dk}{dm} + C(m) \frac{dF(\theta_s^*)}{dm} - mR_M \left(\frac{1}{k} - 1\right) \frac{\partial F(\theta_s^*)}{\partial m}$$

with $C(m) \equiv \Big(g(I) - g(I - mR_M) - mR_M\Big) \geq 0$

  - $\tau_{\text{opt}}^* \geq 0 \iff$ **excessive** money creation
  - $\tau_{\text{pes}}^* \leq 0 \iff$ **insufficient** money creation

- **Efficiency-restoring tax:**

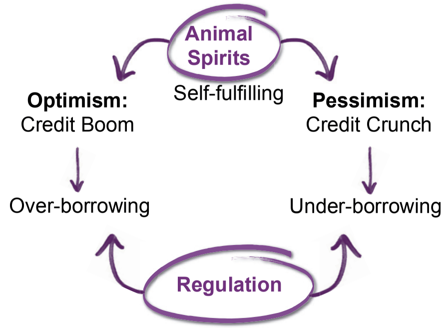
$$\max_m W(m) = \max_m \{\Pi_s(m; k^*) - \tau_s^* m\}$$

Concluding Remarks

- The economy benefits from private money creation, but **banks incentives are often distorted** because they don't internalize cost of runs and fire sales
- State of the economy is endogenous: interaction between insolvency risk and rollover risk. **Multiple equilibria**
- **Unified setting (boom/crunch)** to justify interventions; direction depends on the sentiment in the credit market
- Highlights the need for **dynamic regulatory frameworks**
- Restoring tax/subsidy **doesn't prevent ups/downs, but prevents the economy from swinging to extremes**

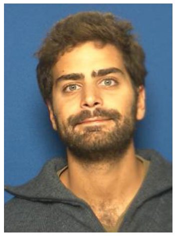
Key Results

Sign of restoring tax depends on the collective **sentiment** in the credit market:



questions

- Where does the money go if the bank is insolvent? Would be an incentive to always fail
- Can we induce a dynamic behavior between equilibria that says theta_opt changes as a function of k



ON DISTRIBUTED CONTROL OF DYNAMICAL FLOW NETWORKS

Giacomo Como, Lund University, Sweden

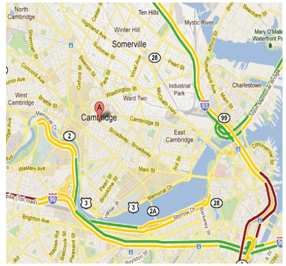
This talk focuses on distributed control of dynamical flow networks. First, we show that optimal throughput and resilience can be achieved by feedback policies that depend only on local information and require no global knowledge of the network. Then, we prove how the optimal selection of a stable equilibrium and the optimal control of the transient can be cast as convex problems which are amoenable to dsitributed solutions. Applications to arterial traffic control are discussed.

Resilient distributed control of dynamical flow networks

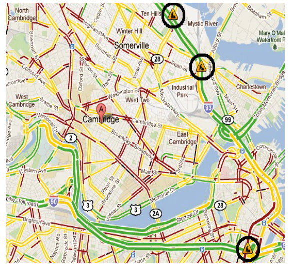
Giacomo Como
Department of Automatic Control
Lund University

LCCC Workshop on Dynamics and Control in Networks
October 17, 2014

Large-scale infrastructure networks



typical Monday at 18:30

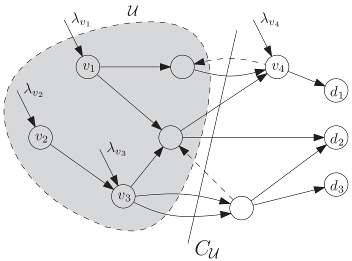


Monday, July 11, at 18:30

- good in business as usual, prone to disruptions
- cascade effects

$\Longrightarrow$ network vulnerability $\gg \sum$ component vulnerabilities

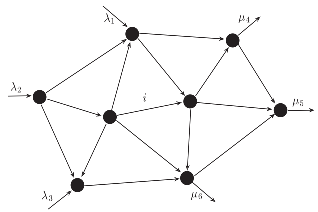
Max-flow min-cut theorem ('56)



$\exists$ feasible equilibrium flow $\iff$ $\min_{\mathcal{U}} \{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\} > 0$

- static, centralized, global information

Optimal network flow

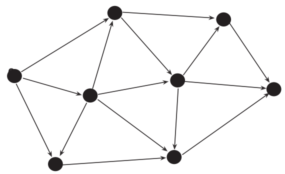


$$\min \sum_{i,j} c_{ij}(f_{ij})$$

$$\lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i = 0$$

- static, convex, rich duality theory

Wardrop equilibrium ('56)



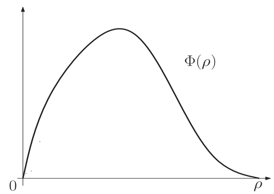
- 'the journey times in all routes actually used are equal and less than those which would be experienced on any unused route'

  $f^W \Leftrightarrow$ Nash equilibrium of congestion game

- user optimum vs social optimum: price of anarchy
- using tolls on links allows one to align f^W with any desired f^*

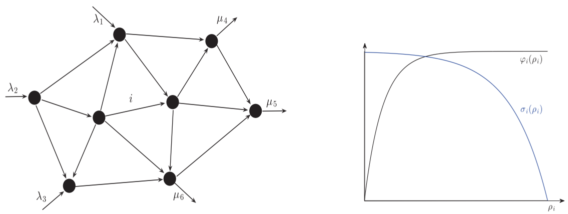
Lighthill-Whitham-Richards traffic model ('55)

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\Phi(\rho) = 0$$



fundamental diagram

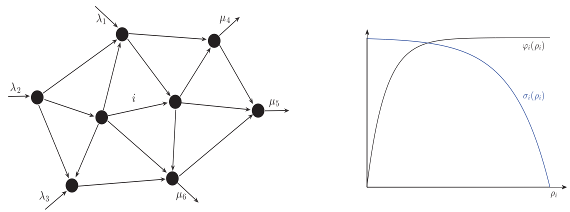
Dynamical flow networks



$$\dot\rho_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \le \varphi_i(\rho_i) \qquad \sum_j f_{ji} \le \sigma_i(\rho_i)$$

Dynamical flow networks



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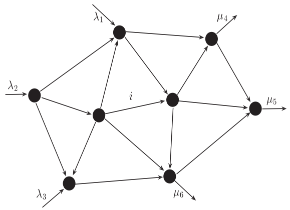
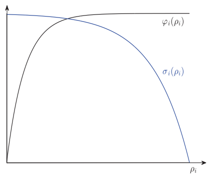
$$\mu_i + \sum_j f_{ij} \le \varphi_i(\rho_i) \qquad \sum_j f_{ji} \le \sigma_i(\rho_i)$$

- Daganzo'92, Gomes&Horowitz'06, Pisarski&CanudasDeWit'12, Varaiya'08, Coogan&Arcak'14, Karafyllis&Papageorgiou'14, ...

Dynamical flow networks



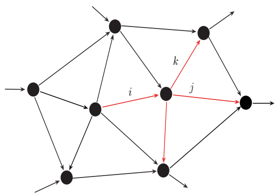


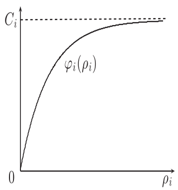
$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i) \qquad \sum_j f_{ji} \leq \sigma_i(\rho_i)$$

- goal: scalable design of f_{ij} with good performance and resilience

Decentralized routing





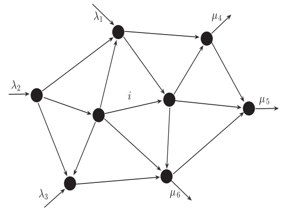
$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

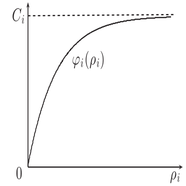
flow f_{ij} depends only on local information ρ^i

$$f_{ij} = \varphi_i(\rho_i) \; R_{ij}(\rho^i)$$

flow from i to j — demand on i — fraction routed to j

Resilient decentralized routing in dynamical flow networks

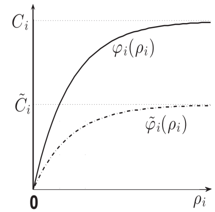




$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

$$\mu_i + \sum_j f_{ij} \leq \varphi_i(\rho_i)$$

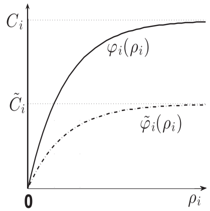
Resilience



- magnitude of perturbation $\delta := \sum_i |\tilde{\lambda}_i - \lambda_i| + \sum_i |\tilde{C}_i - C_i|$
- throughput loss $\gamma := \liminf_{t \to +\infty} \frac{1}{t} \int_0^t \sum_i (\tilde{\lambda}_i(s) - \tilde{\mu}_i(s)) \mathrm{d}s$

$$\dot{\rho}_i = \tilde{\lambda}_i + \sum_j \tilde{f}_{ji} - \sum_j \tilde{f}_{ij} - \tilde{\mu}_i \qquad \tilde{f}_{ij} = \tilde{\varphi}_i(\rho_i) R_{ij}(\rho^i)$$

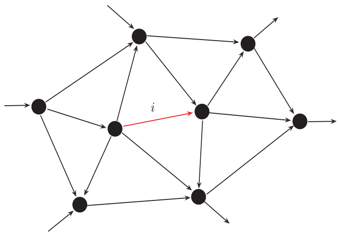
Resilience



margin of resilience $:= \inf\{\delta : \gamma > 0\}$

- $\delta :=$ magnitude of perturbation
- $\gamma :=$ throughput loss

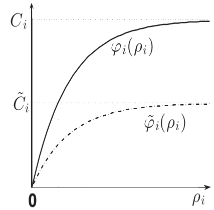
Resilience with fixed routing



- $f_{ij} = R_{ij}\varphi_i(\rho_i)$ with constant R_{ij}
- start from equilibrium $f_i^* = \varphi_i(\rho_i^*)$

$$\nu = \min_i \{\underbrace{C_i - f_i^*}_{\text{link residual capacity}}\}$$

Resilience



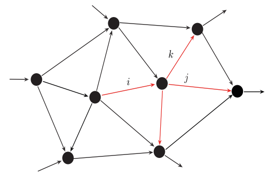
margin of resilience $\nu := \inf\{\delta : \gamma > 0\}$

- necessarily: $\nu \leq \min_{\mathcal{U}}\{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\}$
- Problem: max resilience ν over decentralized routing policies R

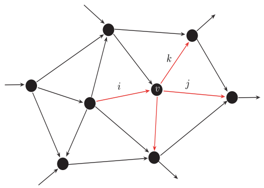
Resilience with locally responsive routing

(a) $R_{ij}(\rho^i)$ depends on local info

$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$



Resilience with locally responsive routing



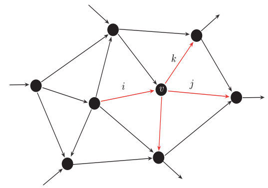
(*a*) $R_{ij}(\rho^i)$ depends on local info

$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$

▶ Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

$$\underset{\text{initial equilibrium } f^*}{(a)} \quad \Longrightarrow \quad \nu \leq \min_v \underbrace{\sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)}_{\text{node residual capacity}}$$

Resilience with locally responsive routing



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$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$

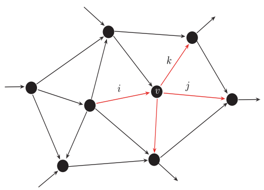
(*b*) $\frac{\partial}{\partial \rho_k} R_{ij} \geq 0 \qquad \forall k \neq j$

$\rho_j \to \infty \Rightarrow \; R_{ij} \to 0$

▶ Example

$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j + \alpha_j)}}{\sum_k e^{-\beta(\rho_k + \alpha_k)}} \qquad \beta > 0$$

Resilience with locally responsive routing



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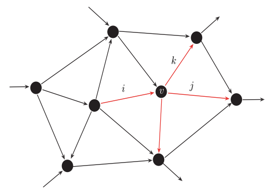
▶ Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

In acyclic networks

$$(a) \text{ and } (b) \quad \Longrightarrow \quad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

f^* initial equilibrium $\qquad$ f^* globally attractive

Resilience with locally responsive routing



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$$(a) \text{ and } (b) \quad \Longrightarrow \quad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

▶ shocks (and information) propagate only downstream

sub-additive effects of perturbations

Resilience with locally responsive routing

(*a*) $R_{ij}(\rho^i)$ depends on local info

$$R_{ij}(\rho^i) \geq 0 \qquad \sum_{j \in \mathcal{E}_i^+} R_{ij}(\rho^i) \equiv 1$$

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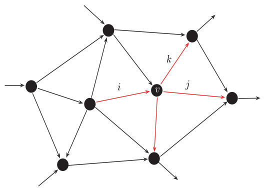
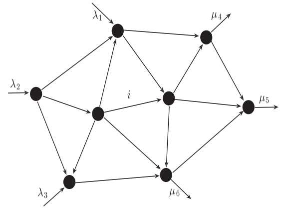


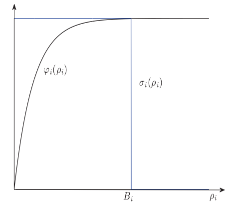
- Theorem [G.C.,K.Savla,D.Acemoglu,M.Dahleh,E.Frazzoli,TAC'13]

$$(a) \text{ and } (b) \quad \Longrightarrow \quad \nu = \min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*)$$

- typically $$\min_v \sum_{j \in \mathcal{E}_v^+} (C_j - f_j^*) \ll \min_{\mathcal{U}} \{ C_{\mathcal{U}} - \lambda_{\mathcal{U}} \}$$

Dynamical flow networks with cascading failures



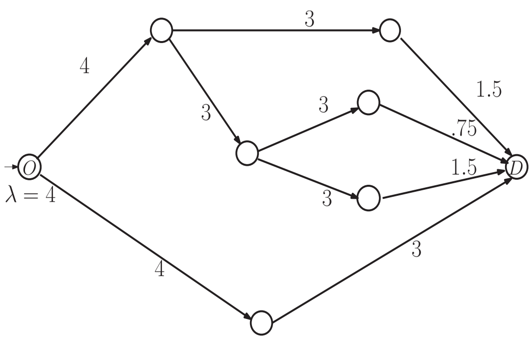


$$\dot{\rho}_i = \lambda_i + \sum_j f_{ji} - \sum_j f_{ij} - \mu_i$$

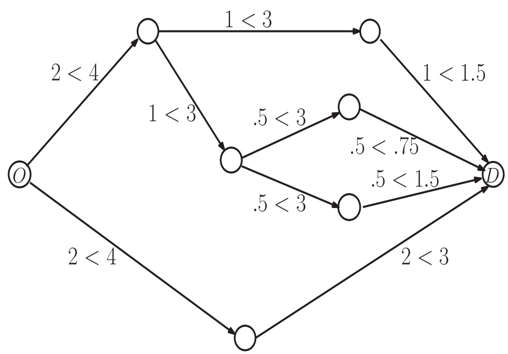
$$f_{ij} = \varphi_i(\rho_i) R_{ij}(\rho^i)$$

- link i goes down irreversibly when $\rho_i(t) = B_i$

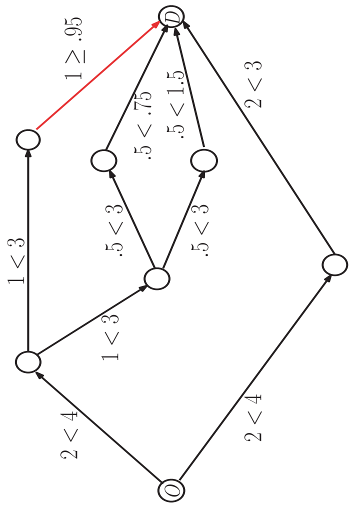
Dynamical flow networks with cascading failures



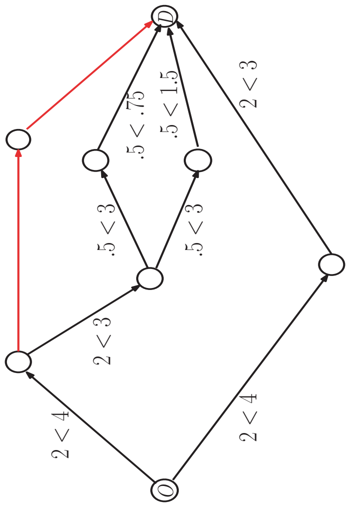
Dynamical flow networks with cascading failures



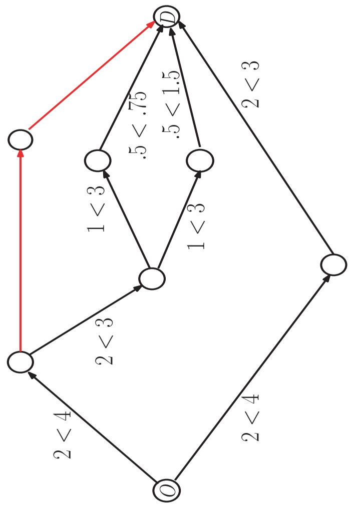
Dynamical flow networks with cascading failures



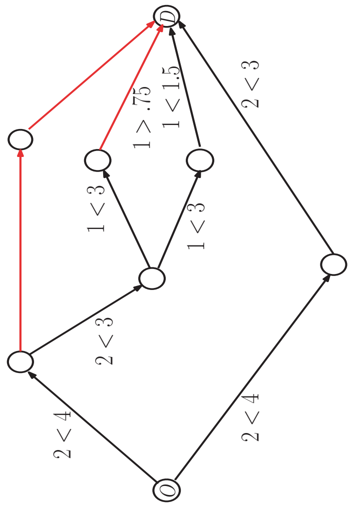
Dynamical flow networks with cascading failures



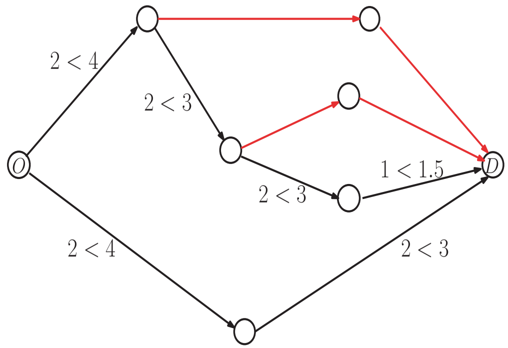
Dynamical flow networks with cascading failures



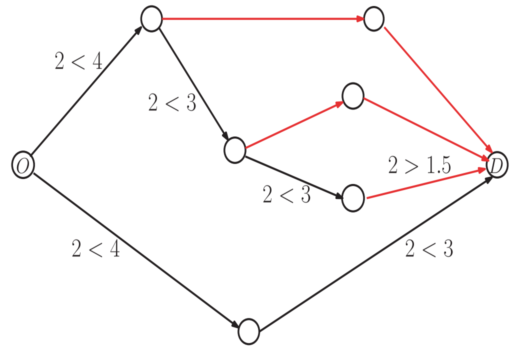
Dynamical flow networks with cascading failures



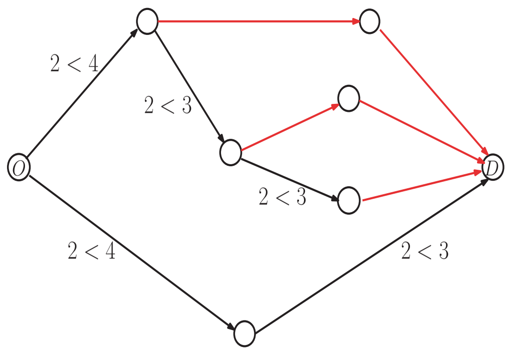
Dynamical flow networks with cascading failures



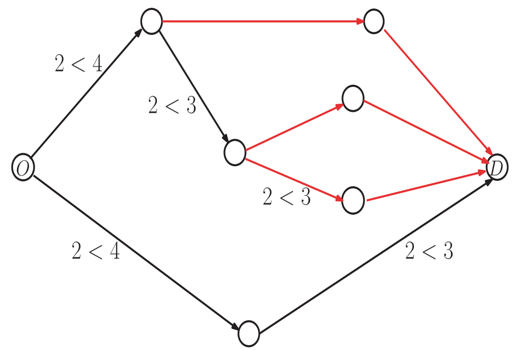
Dynamical flow networks with cascading failures

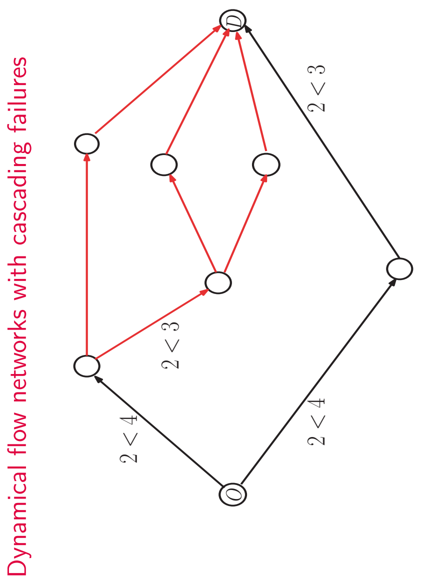


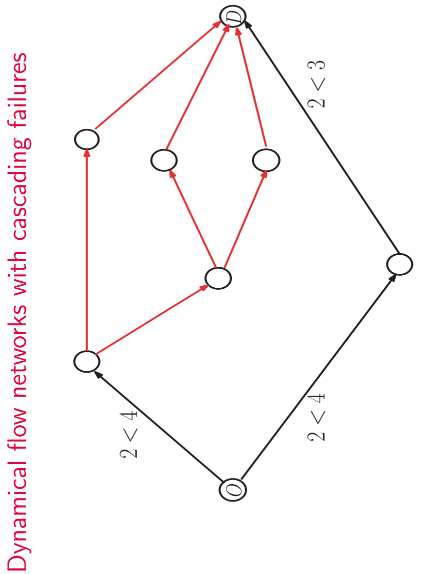
Dynamical flow networks with cascading failures

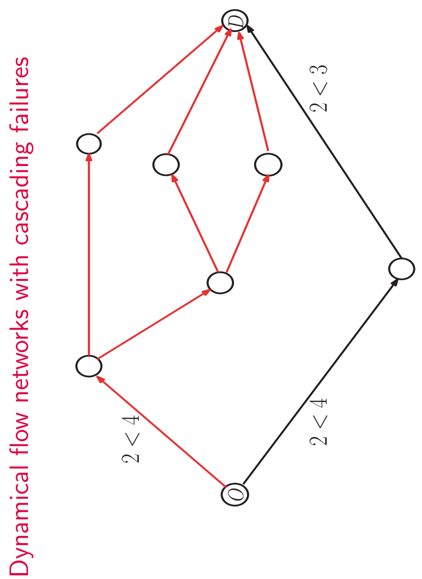


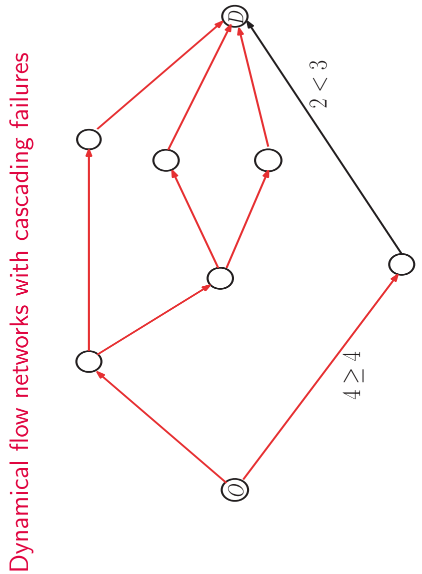
Dynamical flow networks with cascading failures



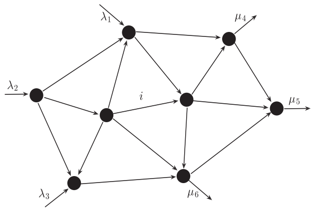

Dynamical flow networks with cascading failures
O
D
2 < 4
2 < 4
2 < 3
2 < 3


Dynamical flow networks with cascading failures
O
D
2 < 4
2 < 4
2 < 3


Dynamical flow networks with cascading failures
O
D
2 < 4
2 < 4
2 < 3


Dynamical flow networks with cascading failures
O
D
4 ≥ 4
2 < 3

Dynamical flow networks with cascading failures

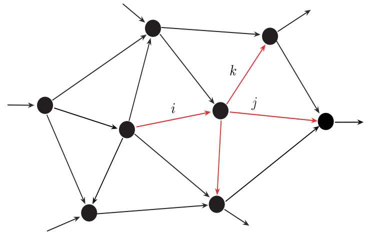


Theorem [K.Savla, G.C., M.Dahleh,'14] In acyclic networks

optimal resilience ν can be computed by dynamic programming

- shocks propagate both up- and downstream
- typically $\nu \ll \min_{\mathcal{U}}\{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\}$

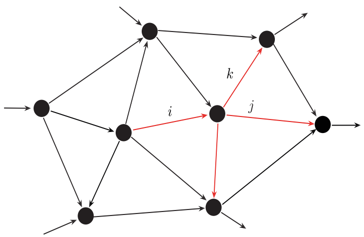
Decentralized routing with flow control



$$f_{ij} = \varphi_i(\rho_i) \quad R_{ij}(\rho^i)$$

flow from i to j — max outflow from i — fraction routed to j

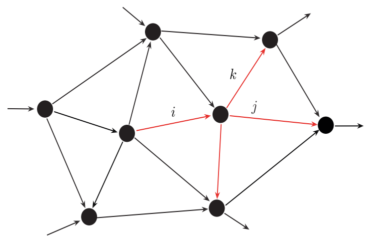
Decentralized routing with flow control



$$f_{ij} = \varphi_i(\rho_i) R_{ij}(\rho^i)$$

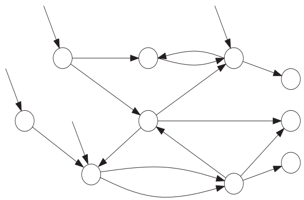
- relax $\sum_{j \in \mathcal{E}_i^+} R_{ij} = 1$ to $\sum_{j \in \mathcal{E}_i^+} R_{ij} \leq 1$, still decentralized

Decentralized monotone routing with flow control



$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j + \alpha_{ij})}}{e^{-\beta(\rho_i + \alpha_{ii})} + \sum_{k \in \mathcal{E}_i^+} e^{-\beta(\rho_k + \alpha_{ik})}}$$

Decentralized monotone routing with flow control

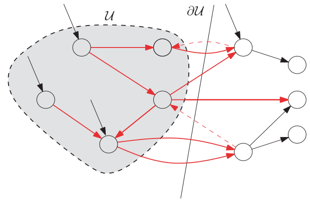


Theorem [G.C.,E.Lovisari,K.Savla,TCONES'14] :

Decentralized monotone routing, both with finite and infinite buffer

- $\min_{\mathcal{U}}\{C_{\mathcal{U}}-\lambda_{\mathcal{U}}\} > 0 \Longrightarrow \quad \exists$ equilibrium ρ^* s.t. $\rho(t) \to \rho^* \quad \forall \rho(0)$

Decentralized monotone routing with flow control

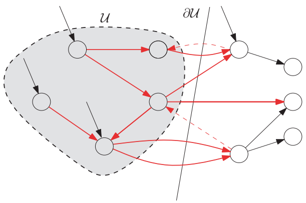


Theorem [G.C.,E.Lovisari,K.Savla,TCONES '14] :

Decentralized monotone routing, infinite buffer

- $\min_{\mathcal{U}}\{C_{\mathcal{U}}-\lambda_{\mathcal{U}}\} < 0 \Longrightarrow$ minimal throughput loss (graceful degradation)

Decentralized monotone routing with flow control

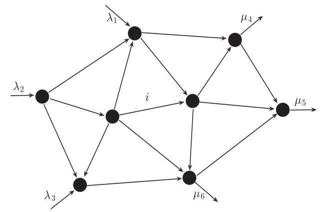


Theorem [G.C.,E.Lovisari,K.Savla,TCONES '14] :

Decentralized monotone routing, finite buffer

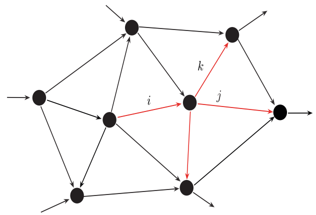
- $\min_{\mathcal{U}}\{C_{\mathcal{U}}-\lambda_{\mathcal{U}}\} < 0 \Longrightarrow$ links left of cut fail simultaneously

Beyond throughput and resilience



- decentralized routing + flow control achieves optimal resilience implicitly propagating information through the network
- for other performance measures (e.g., delay at equilibrium) communication / distributed optimization layer necessary

What equilibrium?



$$R_{ij}(\rho^i) = \frac{e^{-\beta(\rho_j+\alpha_{ij})}}{e^{-\beta(\rho_i+\alpha_{ii})} + \sum_{k\in\mathcal{E}_i^+} e^{-\beta(\rho_k+\alpha_{ik})}}$$

$$\alpha_{ij} = -\rho_j^* - \beta^{-1}\log(f_{ij}^*) \qquad \alpha_{ii} = -\rho_i^* - \beta^{-1}\log(\varphi_i(\rho_i^*) - \textstyle\sum_j f_{ij}^*)$$

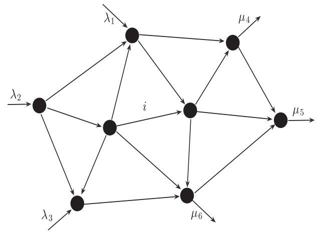
Equilibrium optimization

$$\min_{x,y} \sum_i \Psi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} = \sum_j f_{ij} + \mu_i$$

$$\mu_i + \sum_j f_{ij} \le \varphi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} \le \sigma_i(x_i)$$



[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

If φ_i, σ_i, and $-\psi_i$ are concave (linear):

- ▶ equilibrium optimization is convex (linear) problem

suitable for distributed solutions (ADMM)

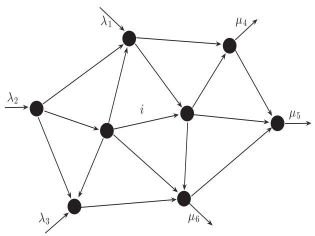
Equilibrium optimization

$$\min_{x,y} \sum_i \Psi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} = \sum_j f_{ij} + \mu_i$$

$$\mu_i + \sum_j f_{ij} \le \varphi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} \le \sigma_i(x_i)$$



[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

If φ_i, σ_i, and $-\psi_i$ are concave (linear):

- ▶ equilibrium optimization is convex (linear) problem

analogous problem when routing is fixed (or constrained)

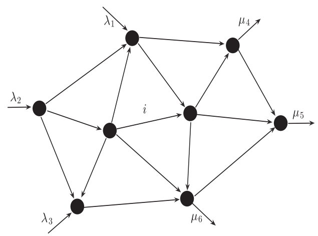
Optimal control

$$\min_{x,y} \sum_i \int_0^T \Psi_i(x_i(t))\mathrm{d}t$$

$$\dot{x}_i = \lambda_i + \sum_j (f_{ji} - f_{ij}) + \mu_i$$

$$\mu_i + \sum_j f_{ij} \le \varphi_i(x_i)$$

$$\lambda_i + \sum_j f_{ji} \le \sigma_i(x_i)$$

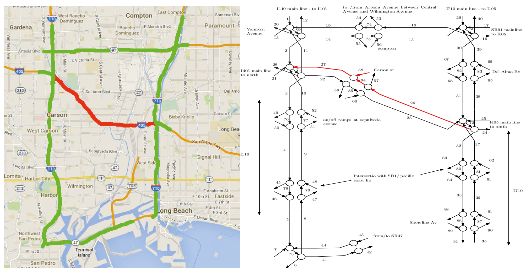


[E.Lovisari,G.C.,A.Rantzer,K.Savla,'14]

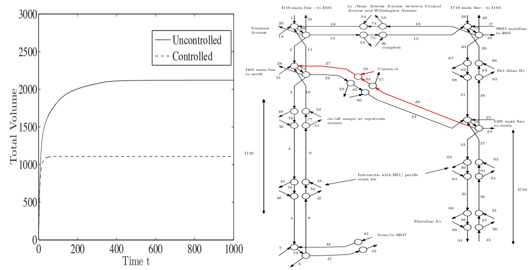
If φ_i, σ_i, and $-\psi_i$ are concave (linear):

- ▶ optimal control is convex (linear) problem

Highway net in Long Beach



Highway net in Long Beach



Conclusion

Dynamical flow networks:

- resilience and cascading failures with decentralized routing
- distributed equilibrium optimization and optimal control
- proofs: exploit monotonicity and l_1-contraction

Dynamical flow networks beyond transportation:

- production networks
- distribution networks

CONTROL OF CONVEX-MONOTONE SYSTEMS

Anders Rantzer, Lund University

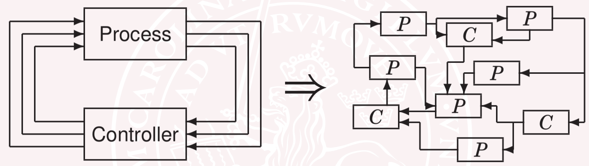
We define the notion of convex-monotone system and prove that for such systems the state trajectory is a convex function of the initial state and the input trajectory. This observation gives a useful class of nonlinear dynamical systems for which optimal trajectories can be performed by convex optimization. Applications to evolutionary dynamics of diseases and voltage stability in power networks are presented. In particular, first order convex optimization methods enable computation of optimal trajectories with a complexity that grows only linearly with the number of network edges.

Scalable Control of Convex-Monotone Systems

Anders Rantzer and Bo Bernhardsson

LCCC Linnaeus Center, Lund University, Sweden

Towards a Scalable Control Theory



Can we find distributed controllers by distributed computation?

Outline

- **Positive and Convex-Monotone Systems**
- Voltage Stability
- HIV and Cancer Treatment

Positive systems

A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

$$\frac{dx}{dt} = Ax + Bu \qquad\qquad y = Cx$$

Equivalently, A, B and C have nonnegative coefficients except for the diagonal of A.

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Traffic Networks.

Positive Systems and Nonnegative Matrices

Classics:

Mathematics: Perron (1907) and Frobenius (1912)

Economics: Leontief (1936)

Books:

Nonnegative matrices: Berman and Plemmons (1979)

Dynamical Systems: Luenberger (1979)

Recent control related work:

Biology inspired theory: Angeli and Sontag (2003)

Synthesis by linear programming: Rami and Tadeo (2007)

Switched systems: Liu (2009), Fornasini and Valcher (2010)

Distributed control: Tanaka and Langbort (2010)

Robust control: Briat (2013)

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Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exits a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iii) There exits a vector $z > 0$ such that $A^T z < 0$.

(iv) There is a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$

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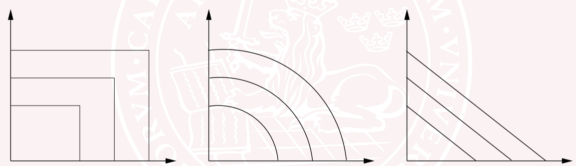
Lyapunov Functions of Positive systems

Solving the three alternative inequalities gives three different Lyapunov functions:

$A\xi < 0$ $\quad$ $A^T P + PA \prec 0$ $\quad$ $A^T z < 0$



$V(x) = \max_k (x_k/\xi_k)$ $\quad$ $V(x) = x^T Px$ $\quad$ $V(x) = z^T x$

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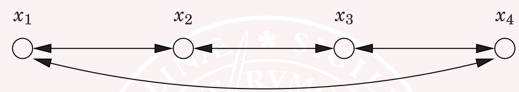
A Scalable Stability Test for Positive Systems



Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

...

Verification is scalable!

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A Distributed Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11}-\ell_1 & a_{12} & 0 & a_{14} \\ a_{21}+\ell_1 & a_{22}-\ell_2 & a_{23} & 0 \\ 0 & a_{32}+\ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0,1]$.

For stabilizing gains ℓ_1, ℓ_2, find $0 < \mu_k < \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Examples: Transportation Networks

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics

Externally Positive Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m\times n}$ is called *externally positive* if if the corresponding impulse response $g(t)$ is nonnegative for all t. The set of all such matrices is denoted $\mathbb{PH}_\infty^{m\times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$. Then

- $\mathbf{GH} \in \mathbb{PH}_\infty^{n\times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{PH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{PH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

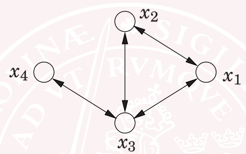
Positively Dominated Systems

$\mathbf{G} \in \mathbb{RH}_\infty^{m\times n}$ is called *positively dominated* if $|\mathbf{G}_{jk}(i\omega)| \leq \mathbf{G}_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{DH}_\infty^{m\times n}$.

Suppose $\mathbf{G}, \mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$. Then

- $\mathbf{GH} \in \mathbb{DH}_\infty^{n\times n}$
- $a\mathbf{G} + b\mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$.
- $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.
- $(I - \mathbf{G})^{-1} \in \mathbb{DH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Example 3: Mass-spring system



$$\ddot{x}_i + d_i\dot{x} + k_i x_i = \sum_j \ell_{ij}(x_j - x_i) + w_i$$

$$\left(s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij}\right) X_i(s) = \sum_j \left(\ell_{ij} X_j(s) + (\bar{\ell}_{ij} - \ell_{ij}) X_i(s)\right) + W_i(s)$$

$$X = (\mathbf{A} + \mathbf{E}L\mathbf{F})X + \mathbf{B}W$$

The transfer matrices $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{A} + \mathbf{E}L\mathbf{F}$ are positively dominated for all $L \in \mathcal{D}$ provided that $d_i \geq k_i + \sum_j \bar{\ell}_{ij}$.

Max-separable Lyapunov Functions

Max-separable: $V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}$

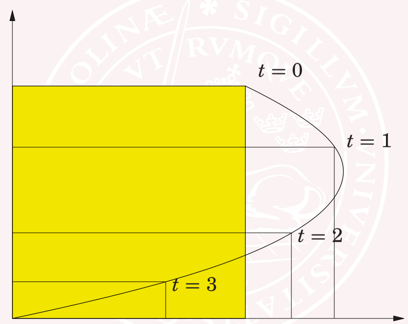
Theorem. Let $\dot{x} = f(x)$ be a monotone system such that the origin globally asymptotically stable and the compact set $X \subset \mathbb{R}^n_+$ is invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \dots, n$, such that $V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}$ satisfies

$$\frac{d}{dt} V(x(t)) = -V(x(t))$$

along all trajectories in X.

[Rantzer, Rüffer, Dirr, CDC-13]

Proof idea



Convex-Monotone Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

is a *monotone system* if its linearization is a positive system. It is a *convex monotone system* if every row of f is also convex.

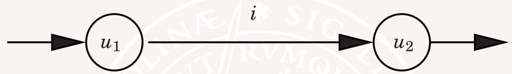
Theorem. [Rantzer/ Bernhardsson (2014)]

For a convex monotone system $\dot{x} = f(x, u)$, each component of the trajectory $\phi_t(a, u)$ is a convex function of (a, u).

Outline

- Positive and Convex -Monotone Systems
- **Voltage Stability**
- HIV and Cancer Treatment

One Transmission Line



The power $p = iu_2$ delivered to the load is upper bounded by

$$p = i(u_1 - Ri) \leq \frac{u_1^2}{4R}.$$

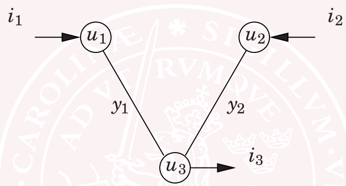
An active load:

$$\frac{di}{dt} = \frac{\widehat{p}}{u_1 - Ri} - i.$$

where $\widehat{p}$ is the power demand.

Voltage collapse occurs if $\widehat{p}$ is too large!

Two Transmission Lines



Node 3 is an active load with

$$\frac{di_3}{dt} = \frac{\widehat{p}(y_1 + y_2)}{y_1u_1 + y_2u_2 - i_3} - i_3$$

For constant generator voltages u_1 and u_2, the load voltage $u_3 = y_1u_1 + y_2u_2 - i_3$ could shrink to zero in finite time, which means voltage collapse.

Arbitrary Networks

Voltages at generators u^G and loads u^L are mapped into external currents i^G and i^L according to

$$\begin{bmatrix} -i^G(t) \\ i^L(t) \end{bmatrix} = \begin{bmatrix} Y^{GG} & Y^{GL} \\ Y^{LG} & Y^{LL} \end{bmatrix} \begin{bmatrix} u^G(t) \\ u^L(t) \end{bmatrix}$$

The load model: $\frac{di_k^L}{dt}(t) = \frac{\widehat{p}_k}{u_k^L(t)} - i_k^L(t)$ gives

$$\frac{di^L}{dt}(t) = \widehat{p}./[(Y^{LL})^{-1}(i^L - Y^{LG}u^G)] - i^L(t)$$

This system is convex-monotone with state i^L and input $-u^G$, so

$$i^G, -u^L, i^L, \frac{di^L}{dt} \text{ and } \frac{di^G}{dt}$$

are all convex functions of u^G

Outline

- ∘ Positive and Convex-Monotone Systems
- ∘ Voltage Stability
- • **HIV and Cancer Treatment**

Combination Therapy is a Control Problem

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i\right) x$$

Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

A describes the mutation dynamics without drugs, while $D^1, \dots, D^m$ are diagonal matrices modeling drug effects.

Determine $u_1, \dots, u_m \geq 0$ with $u_1 + \dots + u_m \leq 1$ such that x decays as fast as possible!

[Jonsson, Rantzer,Murray, ACC 2014]

Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row k, this means

$$A_k \xi - \sum_i u_i D^i_k \xi_k + \gamma \xi_k < 0$$

or equivalently

$$\frac{A_k \xi}{\xi_k} - \sum_i u_i D^i_k + \gamma < 0$$

Maximizing γ is convex optimization in $(\log \xi_i, u_i, \gamma)$!

Using Measurements of Virus Concentrations

Evolutionary dynamics:

$$\dot{x}(t) = \left(A - \sum_i u_i(t) D^i\right) x(t)$$

Can we get faster decay using time-varying $u(t)$ based on measurements of $x(t)$?

Using Measurements of Virus Concentrations

The evolutionary dynamics can be written as a convex monotone system:

$$\frac{d}{dt}\log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D_k^i$$

Hence the decay of $\log x_k$ is a convex function of the input and optimal trajectories can be found even for large systems.

Example

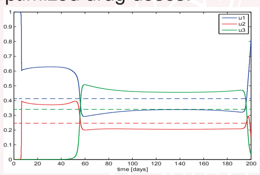
$$A = \begin{bmatrix} -\delta & \mu & \mu & 0 \\ \mu & -\delta & 0 & \mu \\ \mu & 0 & -\delta & \mu \\ 0 & \mu & \mu & -\delta \end{bmatrix}$$

clearance rate $\delta = 0.24\,\text{day}^{-1}$, mutation rate $\mu = 10^{-4}\,\text{day}^{-1}$ and replication rates for viral variants and therapies as follows

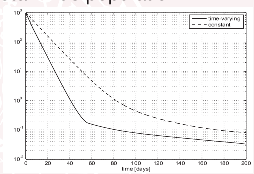
Virus variant	Therapy 1	Therapy 2	Therapy 3
Type 1 (x_1)	$D_1^1 = 0.05$	$D_1^2 = 0.10$	$D_1^3 = 0.30$
Type 2 (x_2)	$D_2^1 = 0.25$	$D_2^2 = 0.05$	$D_2^3 = 0.30$
Type 3 (x_3)	$D_3^1 = 0.10$	$D_3^2 = 0.30$	$D_3^3 = 0.30$
Type 4 (x_4)	$D_4^1 = 0.30$	$D_4^2 = 0.30$	$D_4^3 = 0.15$

Example

Optimized drug doses:



Total virus population:



Summary

- Scalability for Positive and Convex-Monotone Systems
- Voltage Stability
- HIV and Cancer Treatment

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