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Scalable Positivity Preserving Model Reduction Using Linear Energy Functions

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Abstract— In this paper, we explore positivity preserving model reduction. The reduction is performed by truncating the states of the original system without balancing in the classical sense. This may result in conservatism, however, this way the physical meaning of the individual states is preserved. The reduced order models can be obtained using simple matrix operations or using distributed optimization methods. Therefore, the developed algorithms can be applied to sparse large-scale systems.

Index Terms— model reduction; positive systems; nonnegative matrices

I. Introduction

Model order reduction is one of the classical problems in systems theory with a vast number of methods reported in the literature (cf. [1], [2]). The classical model reduction techniques are balanced truncation ([3]), optimal Hankel approximation ([4]), methods based on Krylov subspaces ([2], [5]). However, all these methods do not, in general, preserve some properties of the systems. Positivity is one of these properties. Positive systems are linear time-invariant systems, which will always have a nonnegative state vector x given a nonnegative initial condition and nonnegative input signals. Positive systems are met in many applications, such as networked systems, graphs, systems biology etc. In some applications the systems are extremely large, which is a major bottleneck for analysis. Clearly, model reduction could facilitate analysis or simulation of such systems. However, positivity preserving model reduction is a relatively new and not extensively studied topic. In a general setting, the authors are aware of only a few published references, these are [6], [7]. These methods require solving large linear matrix inequalities and therefore, considerable computational effort to obtain an approximation. We should also mention [8] and [9], which deal with a similar problem setting.

The positive systems possess a number of extraordinary properties. For example, Lyapunov functions can be linear with respect to the state-space vector x, while in general they are quadratic (cf. [10]). The computation of the $\mathbb{H}_\infty$ norm requires simple matrix operations, in general, it requires solution of matrix Riccati equations. The goal of this paper is to extend this list to model reduction. Although, the algorithms obtained in this paper are far from optimal in the $\mathbb{H}_\infty$ norm, they are scalable and require simple matrix manipulations. Therefore, the presented algorithms can be

used for large-scale systems, while [6], [7] employ linear matrix inequalities and can be applied to systems of relatively low-order.

The paper is organized as follows. In Section II theoretical background of model reduction is sketched and a theoretical base for the positivity preserving reduction procedure is outlined. In Section III the positivity preserving reduction methods are presented and in Section IV these are illustrated on numerical examples. All the propositions, which are mostly known facts about nonnegative matrices, are formulated and proved in Appendix.

Notation

Throughout the paper we use standard notation for spaces of sequences l_p, as well as standard $\mathbb{H}_\infty$ ($\| \cdot \|_{\mathbb{H}_\infty}$) and Hankel ($\| \cdot \|_H$) norms (cf. [1]). Let $\mathbb{R}_+^{n \times m}$ stand for the positive orthant of $\mathbb{R}^{n \times m}$. Let also A' denote the transpose of a matrix A, and $A \geq B$ stand for an entry-wise inequality for matrices of the same size A and B. We say that a matrix is Schur stable, or simply Schur, if all its eigenvalues have absolute values smaller than one. A matrix A is called nonnegative if $A \geq 0$ and $A \neq 0$, and it is called positive if $A > 0$. Given a partition of matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

let the complements to A_{22} be

$$A_1^c = A_{11} + A_{12}(I - A_{22})^{-1}A_{21}$$
$$B_1^c = B_1 + A_{12}(I - A_{22})^{-1}B_2$$
$$C_1^c = C_1 + C_2(I - A_{22})^{-1}A_{21}$$
$$D_1^c = D + C_2(I - A_{22})^{-1}B_2$$

Similarly, let A_2^c, B_2^c, C_2^c, D_2^c be the complements to A_{11}. Note that inverses of $I - A_{11}$ and $I - A_{22}$ exist if A is a nonnegative Schur matrix. In fact, in this case A_{11} and A_{22} are also nonnegative Schur ([10]).

II. Energy Functions, Hankel Norms and Model Reduction

In this paper we focus on discrete-time positive systems, which means that the state-space matrices A, B and C have only nonnegative entries. However, the same techniques can be successfully applied to continuous time models (in this

case, all entries of B and C, and only off diagonal entries of A are nonnegative). Let G be a scalar-valued positive system:

$$G = \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad\quad y(k) = Cx(k) \end{cases} \quad (1)$$

where $A \in \mathbb{R}_+^{n \times n}$, $B \in \mathbb{R}_+^{n \times 1}$, and $C \in \mathbb{R}_+^{1 \times n}$. Let also A be Schur stable. In classical model reduction algorithms the, so called, energy functions are employed:

$$L_o(x_0, l_2) = \|y\|_{l_2 \, [0,+\infty)}$$
$$L_c(x_0, l_2) = \min_u \|u\|_{l_2 \, (-\infty,0]}$$

where $x(0) = x_0$, $x(-\infty) = 0$, $u(k) = 0$ for all $k \geq 0$. The function L_o is called the observability energy function and L_c the controllability one, and they can be readily computed

$$L_o(x_0, l_2) = \langle x_0, Q x_0 \rangle^{1/2} \quad L_c(x_0, l_2) = \langle x_0, P^{-1} x_0 \rangle^{1/2}$$
$$A'QA - Q + C'C = 0 \quad\quad APA' - P + BB' = 0$$

These functions are also used to define the norm of the Hankel operator induced by l_2 signals (or simply the Hankel norm, cf. [11]):

$$\|G\|_H = \max_{x_0, \, u} \frac{\|y\|_{l_2 \, [0,+\infty)}}{\|u\|_{l_2 \, (-\infty,0]}} = \max_{x_0} \frac{L_o(x_0, l_2)}{L_c(x_0, l_2)}$$

Now, for model reduction purposes one can use directly the matrices P and Q (see, [3]), which yields a simple, but powerful heuristic. Or one can compute an optimal approximation in the Hankel norm ([4]).

The functions L_o^2 and L_c^2 can be used as Lyapunov functions for the dynamical system (1). It is known that for positive systems as (1) there are linear Lyapunov functions in the form $v'x$ for some nonnegative vector v (cf. [10]). This lead the authors to an assertion of the existence of L_o and L_c with a similar property. Unfortunately this assertion is only partially true. The observability energy function with such a property is easy to obtain, simply by changing the norm of the output signal:

$$L_o(x_0, l_1) = \|y\|_{l_1 \, [0,+\infty)}$$

where $x(0) = x_0$ and $u(k) = 0$ for all nonnegative k. Since the model is confined to a positive orthant, x_0 and CA^k are nonnegative, which entails that:

$$L_o(x_0, l_1) = \|y\|_{l_1 \, [0,+\infty)} = \sum_{k=0}^{\infty} CA^k x_0$$

Using the well-known matrix series formula $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$ (see, Proposition 1 in Appendix for a simple proof), the expression above can be computed as:

$$L_o(x_0, l_1) = q' x_0 \quad \text{where} \quad q' = C(I-A)^{-1}$$

The vector q can be computed by inverting the matrix $I - A$ or using distributed programming methods as in [12]. Note, if A is Schur stable then the matrix $(I-A)^{-1}$ is nonnegative ([10]). This means that for stable G, the vector q is nonnegative, as well. It is possible to introduce a "dual"

to q vector, i.e., $p = (I-A)^{-1}B$. However, the controllability functions $L_c(x_0, l_1)$ or $L_c(x_0, l_\infty)$ do not explicitly depend on p, which certainly complicates any reduction procedure. However, by introducing a notion similar to the Hankel norm, but induced by l_1 and l_∞ signal norms, an easy approximation procedure can be obtained. Introduce:

$$\|G\|_{H,1,\infty} = \max_{x_0 \in \mathbb{R}_+^n, \, u} \frac{\|y\|_{l_1 \, [0,+\infty)}}{\|u\|_{l_\infty \, (-\infty,0]}}$$

where $y = Gu$, $x(0) = x_0$, $x(-\infty) = 0$, $u(k) = 0 \; \forall k \geq 0$.

Lemma 1: Let G be a positive model with the state x confined to $\mathbb{R}_+^n$, then

$$\|G\|_{H,1,\infty} = C(I-A)^{-2}B = q'p$$

Proof. First, show that $\|G\|_{H,1,\infty} \leq C(I-A)^{-2}B$.

Since $x_0 = \sum_{m=0}^{\infty} A^m Bu(-m)$, then

$$\|y\|_{l_1} = \sum_{k=0}^{\infty} CA^k x_0 = C(I-A)^{-1} \sum_{m=0}^{\infty} A^m Bu(-m)$$

Now using the Hölder's inequality we get:

$$\sum_{m=0}^{\infty} C(I-A)^{-1}A^m Bu(-m) \leq \|C(I-A)^{-1}A^m B\|_{l_1} \|u\|_{l_\infty}$$

It can be verified that $\|C(I-A)^{-1}A^m B\|_{l_1} = C(I-A)^{-2}B$ and finally:

$$\|G\|_{H,1,\infty} = \max_{x_0, \, u} \frac{\|y\|_{l_1 \, [0,+\infty)}}{\|u\|_{l_\infty \, (-\infty,0]}} \leq$$
$$\frac{C(I-A)^{-2}B\|u\|_{l_\infty \, (-\infty,0]}}{\|u\|_{l_\infty \, (-\infty,0]}} = C(I-A)^{-2}B$$

To prove the converse, let $p = (I-A)^{-1}B$. If we choose $u(-k)$ equal to one for all k, then the state x_0 is equal to $\sum_{k=0}^{\infty} A^k B$ and therefore $x_0 = p$. This gives us that

$$\|y\|_{l_1 \, [0,+\infty)} = \sum_{k=0}^{\infty} CA^k(I-A)^{-1}B = C(I-A)^{-2}B$$

$$\|u\|_{l_\infty \, (-\infty,0]} = 1$$

and $\|G\|_{H,\infty,1} \geq C(I-A)^{-2}B$. ∎

Remark 1: The norm $\|\cdot\|_{H,1,\infty}$ does not change by adding the constraint $u(-k) \geq 0$, since the maximizing sequence $u(-k)$ is nonnegative as shown above.

Remark 2: In the multivariable case (for the matrix-valued G with m_2 outputs and m_1 inputs)

$$\|G\|_{H,1,\infty} = \bar{C}(I-A)^{-2}\bar{B}$$

where $\bar{C} = \sum_{i=1}^{m_2} C_i$, $\bar{B} = \sum_{i=1}^{m_1} B_i$ and B_i denotes the individual columns of B, C_i denotes the individual rows of C.

Remark 3: It can be verified using similar techniques that:
1) $\|G\|_{H,1,1} = C(I-A)^{-1}B = q'(I-A)p$
2) $\|G\|_{H,\infty,\infty} = C(I-A)^{-1}B = q'(I-A)p$
3) $\|G\|_{H,\infty,1} = \max_{k \geq 0} |CA^k B|$

Note that $\|G\|_{H,1,1} = \|G\|_{H,\infty,\infty} = C(I - A)^{-1}B$. Recall that for scalar-valued G with $D = 0$ the $\mathbb{H}_\infty$ norm is also equal to $C(I - A)^{-1}B$. Moreover, all induced norms are equal (cf. [12]). Which gives us an unexpected result that for $r = 1$ and $r = \infty$:

$$\|G\|_{H,r,r} = \max_{x_0 \in \mathbb{R}_+^n, u} \left\{ \left. \frac{\|y\|_{l_r\,[0,+\infty)}}{\|u\|_{l_r\,(-\infty,0]}} \right| \begin{array}{l} x(0) = x_0, \\ x(-\infty) = 0 \end{array} \right\} =$$

$$\max_u \frac{\|y\|_{l_r\,[-\infty,+\infty)}}{\|u\|_{l_r\,(-\infty,+\infty]}} = \|G\|_{\mathrm{r-ind}}$$

Note, the relation $\|G\|_{H,2,2} \neq \|G\|_{2-\mathrm{ind}}$ does not generally hold, where $\|G\|_{H,2,2}$ is the Hankel norm and $\|G\|_{2-\mathrm{ind}}$ is the $\mathbb{H}_\infty$ norm. This fact becomes clear in light of results in [13]. Based on [13] it can be shown that for scalar-valued positive systems the sum of the Hankel singular values of G is equal to $\|G\|_{2-\mathrm{ind}}/2$. At the same time, $\|G\|_{H,2,2}$ is equal to the maximal Hankel singular value of G.

The vectors p and q are not classical Gramians, however, one of their properties can be very useful:

Lemma 2: If an entry of the vector p (the vector q) is equal to zero, then the corresponding state is not controllable (not observable).

Proof. Without loss of generality, we prove only the controllability part. By Proposition 3 (see, Appendix), for an arbitrary partitioning of the vector $p' = \begin{pmatrix} p_1' & p_2' \end{pmatrix}$ we have $p_2 = (I - A_2^c)^{-1}B_2^c$ and $p_1 = (I - A_1^c)^{-1}B_1^c$. Now let p_2 be a zero vector and p_1 be a positive vector, which leads to:

$$(I - A_2^c)^{-1}B_2^c = 0$$

and thus

$$0 = B_2^c = B_2 + A_{21}(I - A_{11})^{-1}B_1$$

The sum of nonnegative matrices can be zero, only if the summands (i.e., B_2 and $A_{21}(I - A_{11})^{-1}B_1$) are zero matrices. Now, we have

$$(I - A_{22}^c)^{-1}A_{21}(I - A_{11})^{-1}B_1 = 0$$

Due to Proposition 2 and the fact that $B_1 = B_1^c$ (due to $B_2 = 0$)

$$(I - A_{22})^{-1}A_{21}(I - A_{11}^c)^{-1}B_1^c = 0$$

Since $(I - A_{11}^c)^{-1}B_1^c$ is equal to the positive vector p_1, $(I - A_{22})^{-1}A_{21}$ is a zero matrix, which also proves that A_{21} is a zero matrix. Finally, due to A_{21} and B_2 being zero matrices, the system G is not controllable. ∎

Having a zero entry in p is a sufficient condition for uncontrollability. In fact, it is fairly easy to construct uncontrollable systems with a positive vector p. But, a more interesting question is how p related to positive controllability, which is not addressed in this paper. To conclude the section, we remark that vectors p and q are appearing in various norms and have interesting properties. However, it is not entirely clear why and how p and q measure the importance of states in the $\mathbb{H}_\infty$ norm, but there certainly is a connection.

III. MODEL REDUCTION METHODS

In relation to $p = (I - A)^{-1}B$ and $q' = C(I - A)^{-1}$, introduce the vector σ as $\sigma_i^2 = q_i p_i$ for all entries i. This vector determines the weight of each state in the input-output relationship. The first step in the reduction procedures presented in this section is to determine an invertible matrix T, such that the new state-space matrices are TAT^{-1}, TB, CT^{-1}. Moreover, $T(I - A)^{-1}B = (C(I - A)^{-1}T^{-1})' = \sigma$ and $\sigma_i \geq \sigma_j$ if $i \geq j$. Such T is the product of a permutation and the diagonal matrix with entries $\sqrt{q_i/p_i}$ on the diagonal, which also implies that T and its inverse are nonnegative matrices. If p and q have zero entries, these states can be truncated, since they correspond to uncontrollable or unobservable states (based on Lemma 2). Based on σ partition the state-space as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} C' = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}$$

where x_2 corresponds to small entries of σ.

A. Model Reduction by Truncation of States

The simplest version of model reduction in this setting is the state truncation, which is summarized in Algorithm 1. Basically, we throw away the states corresponding to small σ_i, which determine the weight of every state in input-output relationship. The $\mathbb{H}_\infty$ error of approximation in this case can be readily computed.

Lemma 3: Assume G is positive and asymptotically stable. Let G_{tr} be a reduced order system obtained from Algorithm 1, then G_{tr} is positive and asymptotically stable. Moreover,

$$\|G - G_{tr}\|_{\mathbb{H}_\infty} = G(1) - G_{tr}(1) = q_2'(I - A_2^c)p_2$$

Proof. It is straightforward to show that G_{tr} is positive and asymptotically stable, therefore, we continue with the proof of the error expression. The transfer function $G - G_{tr}$ can be written using impulse response as

$$G(z) = \sum_{i=0}^{\infty} (CA^k B - C_1(A_{11})^k B_1)z^{-k}$$

Due to Proposition 4 (see, Appendix) we have

$$(A^k)_{11} \geq (A_{11})^k$$

and therefore

$$CA^k B \geq C_1(A_{11})^k B_1$$

which means that $G - G_{tr}$ has a positive impulse response. Using the same proof as in [12] it can be shown, that

$$\|G - G_{tr}\|_{\mathbb{H}_\infty} = \sum_{i=0}^{\infty} (CA^k B - C_1 A_{11}^k B_1)$$

and $\|G - G_{tr}\|_{\mathbb{H}_\infty} = G(1) - G_{tr}(1)$. Due to Proposition 5 (see, Appendix)

$$C(I - A)^{-1}B = C_2^c(I - A_2^c)^{-1}B_2^c + C_1(I - A_{11})^{-1}B_1$$

which entails that

$$G(1) - G_{tr}(1) = C_2^c(I - A_2^c)^{-1}B_2^c$$

According to Proposition 3 (see, Appendix) $q_2' = C_2^c(I - A_2^c)^{-1}$ and $p_2 = (I - A_2^c)^{-1}B_2^c$, and finally:

$$C_2^c(I - A_2^c)^{-1}B_2^c = q_2'(I - A_2^c)p_2$$

∎

Algorithm 1 Model Reduction Based on Truncation

Perform a state-space transformation such that $\sigma_i = p_i = q_i$ and $\sigma_i \geq \sigma_j$ if $i \geq j$

Based on σ partition the state-space as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C' = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}$$

where x_2 corresponds to small entries of σ.
Compute the reduced order model as:

$$G_{tr} = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array} \right]$$

B. Model Reduction Based on Singular Perturbation

Consider the case of truncation based on singular perturbation as in [14]. Since for positive models the largest value $|G(e^{\jmath\omega})|$ occurs at $\omega = 0$, it seems more important to match the DC gains of the full and the reduced order models. Truncation of the positive systems based on singular perturbation is concluded in Algorithm 2, which possesses the following properties:

Lemma 4: Assume G is positive and asymptotically stable. Let G_{sp} be obtained using Algorithm 2, then:

- The system G_{sp} is positive and asymptotically stable
- The $\mathbb{H}_\infty$ norms of the models match , i.e., $\|G\|_{\mathbb{H}_\infty} = \|G_{sp}\|_{\mathbb{H}_\infty}$
- The norm $\|G_{sp}\|_{H,1,\infty}$ is equal to $q_1'p_1 = \|G\|_{H,1,\infty} - q_2'p_2$

Proof. For the proof of stability the reader is referred to [14]. Therein it is shown that $G(1) = G_{sp}(1)$, i.e., the DC gains match.

Let us show positivity of the reduced order model. Since A_{22} is nonnegative and asymptotically stable, the matrix $(I - A_{22})^{-1}$ is also nonnegative (cf. [10]). This implies that matrices A_1^c, B_1^c and C_1^c are nonnegative and G_{sp} is a positive system. This completes the proof of the second bullet, since $\|G_{sp}\|_{\mathbb{H}_\infty} = G_{sp}(1)$ and $G_{sp}(1)$ is equal to $\|G\|_{\mathbb{H}_\infty} = G(1)$.

According to Proposition 3 (see, Appendix) it is possible to decompose $q'p$ into the sum $q_1'p_1 + q_2'p_2$, where

$$q_1'p_1 = C_1^c(I - A_1^c)^{-2}B_1^c$$

Note that A_1^c, B_1^c, C_1^c is the state-space representation of G_{sp}. Therefore,

$$\|G_{sp}\|_{H,1,\infty} = q_1'p_1 = \|G\|_{H,1,\infty} - q_2'p_2$$

∎

Computing the $\mathbb{H}_\infty$ approximation error for Algorithm 2 is more demanding than for Algorithm 1, since $G - G_{sp}$ does not have a positive impulse response and Riccati equations have to be involved. Moreover, the approximation error $\|G - G_{sp}\|_{\mathbb{H}_\infty}$ does not explicitly depend on σ, which seems as a weaker theoretical result. However, numerical experiments show that Algorithm 2 is competitive in comparison to Algorithm 1 and many examples the former outperforms the latter.

Algorithm 2 Model Reduction Based on Singular Perturbation

Perform a state-space transformation such that $\sigma_i = p_i = q_i$ and $\sigma_i \geq \sigma_j$ if $i \geq j$

Based on σ partition the state-space as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad C' = \begin{pmatrix} C_1' \\ C_2' \end{pmatrix}$$

where x_2 corresponds to small entries of σ.
Compute the reduced order model as:

$$G_{sp} = \left[\begin{array}{c|c} A_1^c & B_1^c \\ \hline C_1^c & D_1^c \end{array} \right]$$

IV. EXAMPLES

The major interest in the numerical examples is comparing the presented algorithms to [7], where a somewhat similar procedure was employed. In [7] the states are truncated based on diagonal matrices P and Q, which are the solutions to linear matrix inequalities. Let [7]-tr be a simple truncation method from [7], and [7]-sp be a reduction based on singular perturbation from [7]. For brevity, we are going to use the following notation for the state-space representations:

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

Example 1: First, we are going to compare the Algorithms 1 and 2 to algorithms from [7] on random examples, which depict clearly the overall trends. Consider the positive systems

$$G_1 = \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & 0 \end{array} \right] \quad \text{and} \quad G_2 = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & 0 \end{array} \right]$$

$$A = \begin{pmatrix} 0.05 & 0.08 & 0.01 & 0.1 & 0.04 & 0.09 \\ 0.02 & 0.09 & 0.02 & 0.03 & 0.01 & 0.05 \\ 0.04 & 0.05 & 0.02 & 0.03 & 0.06 & 0.01 \\ 0.01 & 0.08 & 0.02 & 0.04 & 0.04 & 0.09 \\ 0.04 & 0.07 & 0.02 & 0.03 & 0.04 & 0.03 \\ 0.08 & 0 & 0.03 & 0.08 & 0.01 & 0.06 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 7 & 2 & 10 & 7 & 5 & 9 \end{pmatrix}^T \quad C_2 = \begin{pmatrix} 6 & 0 & 5 & 8 & 7 & 6 \end{pmatrix}$$

Both systems are controllable and observable. Tables I and II depict the result of reduction procedures. It is noticeable that Algorithm 1 performs similarly to [7]-tr, while Algorithm 2 performs similarly to [7]-sp. This is due the structure of truncation, which is quite similar in both approaches. The only

Reduction order k	2	3	4	5
Algorithm 1	5.33	3.37	1.70	0.63
Algorithm 2	4.58	2.55	1.15	0.43
[7]-tr	5.33	3.79	1.70	0.63
[7]-sp	4.58	3.03	1.15	0.43

Reduction order k	2	3	4	5
Algorithm 1	59.00	39.08	19.68	2.77
Algorithm 2	69.53	46.22	15.92	1.92
[7]-tr	59.00	40.69	19.68	2.77
[7]-sp	69.53	37.58	15.92	1.92

Reduction order k	1	2	3	4	5
Algorithm 1	0.77	0.26	0.05	0.02	$1.45 \cdot 10^{-2}$
Algorithm 2	0.44	0.08	0.02	0.01	$0.4 \cdot 10^{-2}$

difference is the way of determining the state's contribution into the input-output relationship. However, in some cases the presented algorithms perform better than their counterparts from [7], and in some cases it is the opposite. Authors at the present moment cannot elaborate on the reasons for such a behavior. It can be related to properties of matrices B and C, however, no clear connection has been established so far.

The methods from [7] are much more numerically demanding than the presented algorithms, since [7] require solving linear matrix inequalities. Taking into account the performance in terms of the approximation errors, we can state that our approach has a few advantages over [7].

Example 2: The second example is a continuous-time multi-input-multi-output positive system described in [6]. Even though explicit $\mathbb{H}_\infty$ approximation errors were not computed for the matrix valued case, we can still compare the performance of the algorithms. In this example, the Algorithms 1 and 2 have exactly the same results as algorithms from [7]. In [6] the reduction order was set to 2, which resulted in $\|G - G_r\|_{\mathbb{H}_\infty} / \|G\|_{\mathbb{H}_\infty} = 0.04$ relative error. Recall, that using Algorithms 1 and 2 we can preserve an interpretation of the states x_i, when with [6] such interpretation is lost. Moreover, the iterative semidefinite approach [6] does not seem to be competitive for large systems due to numerical efficiency.

$$G = \left[\begin{array}{cccccc|cc} -1.5 & .6 & 1.0 & 0 & 0 & 0 & 1 & 0 \\ 0.3 & -1.9 & 0.2 & 0 & 0 & 0 & 0 & 1 \\ 0.2 & 0.5 & -2.7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & -3 & 0.6 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & -1.6 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.5 & -1.6 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array} \right]$$

V. CONCLUSION

This paper presents two model reduction algorithms which preserve stability and positivity. The algorithms are scalable and can be applied to sparse large-scale systems, for which inversion of matrix A can be performed efficiently. As an alternative one can employ distributed methods to compute the vectors p and q.

The main aspect of the algorithm is that balancing in the classical sense is not performed, while computing the approximation. This fact can be seen as a drawback or an advantage depending on the view point. The methods are conservative with respect to balanced truncation, in fact, even first order balanced reduced order models sometimes have a better match than all reduced order models obtained by the presented algorithms. However, the presented algorithms preserve the interpretation of the individual states of the systems, which can be a good asset for analysis.

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APPENDIX

Proposition 1: Let A be a Schur matrix with A, B and C being nonnegative matrices, then $p = \sum_{k=0}^{\infty} A^k B = (I - A)^{-1} B$ and $q' = \sum_{k=0}^{\infty} C A^k = C(I - A)^{-1}$.

Proof. We are going to show only that $\sum_{k=0}^{\infty} C A^k = C(I - A)^{-1}$, the second statement can be obtained by transposing the relations.

Assume $\sum_{k=0}^{\infty} C A^k = q'$, then

$$q' A = \sum_{k=1}^{\infty} C A^k = \sum_{k=0}^{\infty} C A^k - C = q' - C$$

Finally, $q' = C(I - A)^{-1}$. ∎

Proposition 2: Let $A \in \mathbb{R}^{n \times n}$ $B \in \mathbb{R}^{k \times n}$, $C \in \mathbb{R}^{n \times k}$ and $D \in \mathbb{R}^{k \times k}$. If A, D, $A - BD^{-1}C$ and $D - CA^{-1}B$ are invertible, then:

$$(A - BD^{-1}C)^{-1} BD^{-1} = A^{-1} B(D - CA^{-1}B)^{-1}$$

Proof. Assume the assertion is true.

$$(A - BD^{-1}C)^{-1} BD^{-1} = A^{-1} B(D - CA^{-1}B)^{-1}$$

multiply from the left with $A - BD^{-1}C$ and from the right with $D - CA^{-1}B$

$$BD^{-1}(D - CA^{-1}B) = (A - BD^{-1}C)A^{-1}B$$
$$B - BD^{-1}CA^{-1}B = B - BD^{-1}CA^{-1}B$$

We arrive to identity valid for any matrices A, B, C, D satisfying the assumptions of the proposition. This concludes the proof. ∎

Proposition 3: Let A be a Schur, nonnegative matrix. Given an arbitrary partitioning of $q = \begin{pmatrix} q_1 & q_2 \end{pmatrix}$

$$q_1' = C_1^c (I - A_1^c)^{-1}$$
$$q_2' = C_2^c (I - A_2^c)^{-1}$$

Proof. The proposition is going to be proved by direct computation. First it is required to compute the inverse of the matrix $I - A$:

$$\begin{pmatrix} I - A_{11} & -A_{12} \\ -A_{21} & I - A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

The entries X_{11} and X_{12} are computed in a straightforward manner using the Schur's complement:

$$X_{11} = (I - A_{11} - A_{12}(I - A_{22})^{-1}A_{21})^{-1} = (I - A_1^c)^{-1}$$
$$X_{21} = (I - A_{22})^{-1} A_{21}(I - A_1^c)^{-1}$$
$$X_{12} = (I - A_1^c)^{-1} A_{12}(I - A_{22})^{-1}$$

Using Proposition 2 rewrite the expression for X_{12} as:

$$X_{12} = (I - A_1^c)^{-1} A_{12}(I - A_{22})^{-1} = $$
$$(I - A_{11})^{-1} A_{21}(I - A_2^c)^{-1}$$

Using the same proposition it is possible to show that:

$$X_{22} = (I - A_2^c)^{-1}$$

All is required to show now is that:

$$q_1' = C_1 X_{11} + C_2 X_{21} = C_1^c (I - A_1^c)^{-1}$$
$$q_2' = C_1 X_{12} + C_2 X_{22} = C_2^c (I - A_2^c)^{-1}$$

which follows immediately given the matrix X. ∎

Proposition 4: For any nonnegative matrix A and any positive integer k the following inequality is true: $(A^k)_{11} \geq (A_{11})^k$

Proof. A similar statement can be found in [9] for the continuous time systems. Let us prove the proposition by induction. For $k = 1$, the assertion is obviously valid, since $A_{11} \geq A_{11}$. Assume $(A^k)_{11} \geq (A_{11})^k$, let us show it for $k + 1$:

$$(A^{k+1})_{11} = (A^k A)_{11} = $$
$$\left(\begin{pmatrix} (A^k)_{11} & (A^k)_{12} \\ (A^k)_{21} & (A^k)_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right)_{11} = $$
$$(A^k)_{11} A_{11} + (A^k)_{12} A_{21} \geq (A_{11})^k A_{11} = (A_{11})^{k+1}$$

The last inequality is valid since matrices $(A^k)_{12}$ and A_{21} are nonnegative and $(A^k)_{11} \geq (A_{11})^k$ by assumption. ∎

Proposition 5: Let A be a Schur, nonnegative matrix, then the following equality holds:

$$C(I - A)^{-1} B = C_1^c (I - A_1^c)^{-1} B_1^c + C_2 (I - A_{22})^{-1} B_2$$

Proof. Using the Schur's complement it is possible to rewrite $(I - A)^{-1}$ in terms of block matrices A_{ij} as:

$$\begin{pmatrix} I - A_{11} & -A_{12} \\ -A_{21} & I - A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ (1 - A_{22})^{-1}A_{21} & I \end{pmatrix} \cdot$$
$$\begin{pmatrix} (I - A_1^c)^{-1} & 0 \\ 0 & (1 - A_{22})^{-1} \end{pmatrix} \begin{pmatrix} I & A_{12}(1 - A_{22})^{-1} \\ 0 & I \end{pmatrix}$$

Note that

$$\begin{pmatrix} C_1^c & C_2 \end{pmatrix} = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ (1 - A_{22})^{-1}A_{21} & I \end{pmatrix}$$
$$\begin{pmatrix} B_1^c \\ B_2 \end{pmatrix} = \begin{pmatrix} I & A_{12}(1 - A_{22})^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Therefore

$$C(I - A)^{-1} B = \begin{pmatrix} C_1^c & C_2 \end{pmatrix} \cdot$$
$$\begin{pmatrix} (I - A_1^c)^{-1} & 0 \\ 0 & (1 - A_{22})^{-1} \end{pmatrix} \begin{pmatrix} B_1^c \\ B_2 \end{pmatrix} = $$
$$C_1^c (I - A_1^c)^{-1} B_1^c + C_2 (I - A_{22})^{-1} B_2$$

∎