Preface

1 Background

Ideal fluid flow can be described by the Euler equations. These were formulated by Euler in 1757 and are still the main equations used to model flow in an inviscid fluid. For irrotational, inviscid, incompressible flow in a domain $D = \{(x, y, z) : -h < y < \eta(x, z, t)\}$, where η is a free surface, the Euler equations are given by

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{Dt}} = -\nabla p + \mathbf{F}, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))$ is the velocity, $p(x, y, z, t)$ is the pressure, $\mathbf{F}(x, y, z, t)$ is the force acting on the fluid and

$$\frac{\mathrm{D}}{\mathrm{Dt}} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla,$$

is the material derivative. When the only forcing acting on the fluid is gravity we let $\mathbf{F} \equiv (0, 0, -g)$, where g is the acceleration due to gravity. In addition $\mathbf{u}$ and p must satisfy the boundary conditions

$$v = \eta_t + u\eta_x + w\eta_z \qquad \text{on } y = \eta, \tag{1.3}$$

$$v = 0 \qquad \text{on } y = -h, \tag{1.4}$$

$$p = p_{\mathrm{a}} - T\left(\left[\frac{\eta_x}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_x + \left[\frac{\eta_z}{\sqrt{1 + \eta_x^2 + \eta_z^2}}\right]_z\right) \quad \text{on } y = \eta, \tag{1.5}$$

where p_{a} is the atmospheric pressure. Equations (1.3), (1.4) say that the boundary of D is impermeable, while equation (1.5) says that the jump in pressure across the surface η is proportional to the mean curvature of the surface. We will be considering the case when the fluid flow is irrotational, that is when $\nabla \times \mathbf{u} = 0$. Then there exists a velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. Equations (1.1)–(1.5) then reduce to

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 \qquad -h < y < \eta, \tag{1.6}$$

$$\phi_y = 0 \qquad \text{on } y = -h, \tag{1.7}$$

$$\phi_y = \eta_t + \eta_x\phi_x + \eta_z\phi_z \qquad \text{on } y = \eta, \tag{1.8}$$

and

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2 + \phi_z^2) + g\eta - T\left(\left[\frac{\eta_x}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_x + \left[\frac{\eta_z}{\sqrt{1+\eta_x^2+\eta_z^2}}\right]_z\right) = B, \quad \text{on } y = \eta, \tag{1.9}$$

where B is the Bernoulli constant. The above problem is an example of a partial differential equation with nonlinear boundary conditions prescribed on a free surface and it is well known that such problems are mathematically challenging. Despite these difficulties there have been much progress over the last 250 years, but there are still fundamental questions regarding these equations that remain unanswered.

Steady waves are solutions of (1.6)–(1.9) of the special form $\phi(x, y, z, t) = \phi(x - c_1t, y, z - c_2t)$, $\eta(x, z, t) = \eta(x - c_1t, z - c_2t)$, that is they are uniformly translating in the horizontal direction (c_1, c_2). A steady wave with a localized profile is called a solitary wave. One of the earliest documented observations of such a wave is by John Scott Russel in 1834 [16] where he observed a localized wave propagating in a channel outside Edinburgh. Many leading researchers of the time tried to explain the observations made by Scott Russel, but it wasn't until 1895 that a satisfactory answer was given by Korteweg and de Vries [14]. In that paper they considered wave motion independent of z and derived an approximate equation for the wave profile:

$$\eta_t + \eta\eta_x + \eta_{xxx} = 0. \tag{1.10}$$

Equation (1.10), which is now known as the KdV equation, possesses exact solitary wave solutions of the form

$$\eta(x) = 3c \cdot \operatorname{sech}^2\left(\frac{\sqrt{c}x}{2}\right), \tag{1.11}$$

where c is the wave speed in the x direction.

The KdV equation is an example of a model equation. Such equations can be derived from the full water wave problem by considering a specific physical setting. The KdV equation for example is valid in a certain shallow water regime. Model equations have the benefit of being much simpler to analyze than the full water wave problem but the downside is that the solutions are only approximate.

Before going into the details of each paper we will first explain the general method employed in [Paper I, Paper II, Paper III].

1.1 The method of spatial dynamics

In a paper from 1982 [11] Kirchgässner studied a class of semilinear elliptic boundary value problems in an infinite strip $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 \,:\, -1/2 < y < 1/2\}$, of the form

$$\begin{aligned} &u_{xx} + u_{yy} + \delta a(y)u - f(\delta, y, u) = 0, \\ &u(x, \pm\tfrac{1}{2}) = 0, \end{aligned}$$

where a, f are sufficiently smooth functions and $\delta \in \mathbb{R}$ is a parameter. Kirchgässner reformulated the problem as a dynamical system where the unbounded spatial coordinate x plays the role of time:

$$w_x = \mathcal{A}(\delta)w + F(\delta, w), \tag{1.12}$$

with

$$w = \begin{pmatrix} u \\ u_x \end{pmatrix}, \ \mathcal{A}(\delta) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial^2}{\partial y^2} - \delta a & 0 \end{pmatrix},$$
$$F(\delta, w) = \begin{pmatrix} 0 \\ f(\delta, \cdot, u) \end{pmatrix}.$$

In particular, there exists a critical value δ_0 such that when δ is varied through δ_0 the following change in the spectrum of $\mathcal{A}(\delta)$ occurs. For $\delta < \delta_0$, $\mathcal{A}(\delta)$ has two real eigenvalues that collide at the origin when $\delta = \delta_0$ to form the algebraically double eigenvalue 0. When $\delta > \delta_0$ the eigenvalues again separate, but now onto the imaginary axis. Moreover, the rest of the spectrum of $\mathcal{A}(\delta)$ is bounded away from the imaginary axis. This change in the spectrum is called a 0^2 resonance. The next step in Kirchgässner's analysis was to perform a center-manifold reduction. Consider a dynamical system of the form

$$w_x = Lw + F(\delta, w), \tag{1.13}$$

with w belonging to some Hilbert space E. Suppose that the imaginary part of the spectrum of L is finite and that the rest of the spectrum is bounded away from the imaginary axis. Under some additional technical hypothesis (see for example [Paper I, Theorem 1] for a full statement of the center-manifold theorem) it is possible to express all "small" solutions of (1.13) in the form $w_1 + h(\delta, w_1)$, where $w_1 \in PE$, $h(\delta, w_1) \in (I - P)E$, with P being the projection of E onto the eigenspace of L corresponding to the imaginary part of the spectrum, and where w_1 satisfies the reduced equation

$$w_{1x} = Lw_1 + PF(\delta, w_1). \tag{1.14}$$

Note here that due to the finiteness assumption on the imaginary part of the spectrum of L, equation (1.14) is finite dimensional. In Kirchgässner's case the imaginary part of the spectrum of $A(\delta_0)$ consists of the algebraically double eigenvalue 0, so the corresponding reduced equation is two-dimensional. Under certain assumptions on the nonlinear part f, Kirchgässner was able to prove that the reduced equation possesses homoclinic solutions, that is, solutions whose orbits connect an equilibrium to itself. Going back to the original variables, this correspond to solutions u satisfying $\lim_{x\to\pm\infty} u(x, y) = 0$, that is solitary solutions.

Some years later Kirchgässner applied these ideas to the full two-dimensional water wave problem [12]. He obtained a dynamical system of the form (1.13), and found that the spectrum of the linear operator L depends on two dimensionless parameters α, β. In fact he found that λ is an eigenvalue of L is and only if

$$\frac{\lambda}{\tan(\lambda)} = \alpha - \beta\lambda^2. \tag{1.15}$$

If we in (1.15) let $\lambda = \mathrm{i}s$ we obtain the usual dispersion relation for water waves. Kirchgässner found several critical curves in the (β, α) plane where the number of imaginary eigenvalues of L changes. He then continued to investigate a specific region of the parameter plane associated with a 0^2 bifurcation. For parameters in that region, the corresponding truncated reduced equation one gets after performing a center-manifold reduction is given by the steady KdV equation. This yields solitary wave solutions of the full Euler equations, where the wave profiles are up to remainder terms of the form (1.11). Kirchgässner and several other researchers continued to investigate what occurs in other regions of the (β, α) plane, see for example [2, 3, 4, 10]. The method of spatial dynamics has also been extended to more general setups, for instance to three dimensions in [8] and to the rotational case in [9].

2 Main results

2.1 Paper I

This paper is concerned with proving the existence of several types of internal solitary waves, using the spatial dynamics approach. More specifically we consider waves propagating on the interface between two fluids of different density, with the lighter fluid on top of the heavier one. Furthermore we assume that the fluids are inviscid, incompressible, irrotational, that the interfacial tension is positive and that the fluids are bounded above by a rigid lid and below by a flat bottom. Surface waves can be seen as a special case of internal waves, where the upper fluid has density zero (air). When the governing equations are non-dimensionalized four parameters α, β, ρ and h emerge. The parameters α and β play the same role as in the surface wave case, while ρ and h are the ratios of the densities and the surface depths respectively. We apply the method of spatial dynamics and find a dynamical system of the form (1.13). Here we find that λ is an eigenvalue of L if and only if

$$\lambda \left(\frac{\rho}{\tan(\lambda)} + \frac{1}{\tan(h\lambda)} \right) = \alpha - \beta\lambda^2,$$

and we find several critical curves in the (β, α) plane where the number of imaginary eigenvalues of L changes. These curves are qualitatively the same as the the ones Kirchgässner found in [12]; the difference is that they now depend on the additional parameters ρ and h. We consider three regions of the (β, α)-plane to investigate further, which we within this subsection refer to as regions I, II and III (see [Paper I, Figure ??]). For each parameter region we perform a center-manifold reduction, which yields three different reduced equations. When disregarding higher order terms, the reduced equation in region I is given by the Kawahara equation:

$$\eta_{xxxx} - 2(1+\delta)\eta_{xx} + \eta - d(\rho - \frac{1}{h^2})\eta^2 = 0, \tag{2.1}$$

where d is a positive constant and $\delta < 0$ is a parameter. This equation possesses homoclinic solutions that correspond to multi-throughed internal solitary waves of depression when $\rho - 1/h^2 < 0$ and of elevation when $\rho - 1/h^2 > 0$. In the critical case when $\rho \approx 1/h^2$ one has to include a cubic term in equation (2.1). We prove in [Paper I] that in this case we still have solutions that correspond to solitary internal waves of either depression or of elevation. Region I was studied in the surface wave setting by Buffoni, Groves and Toland in [3], where they found solitary waves of depression and we can recover their results by choosing $\rho = 0$.

For parameters in region II the leading order part of the reduced equation is given by the nonlinear Schrödinger equation:

$$\eta_{xx} + \operatorname{sgn}(\delta) c_1 \eta + 2c_3 \eta |\eta|^2 = 0.$$

Here c_1 is found to be negative, while c_3 can be either positive or negative. When $c_3 > 0$ the equation is called focusing and there exists a family of bright solitary wave solutions. When $c_3 < 0$ the equation is said to be defocusing and in this case we find dark solitary wave solutions. In comparison, the corresponding parameter region for surface waves was studied in [10] and there the coefficient of the cubic term is always positive. The fact that c_3 can take negative values in the internal wave case is due to the dependence of c_3 on the extra parameters ρ and h.

Finally, region III corresponds to the region Kirchgässner studied in [12]. We also find that the leading order part of the reduced equation is given by the the KdV-equation, however the equation we find possesses solitary wave solutions of elevation when $\rho > 1/h^2$ and of depression when $\rho < 1/h^2$. When $\rho - 1/h^2$ is small we include higher order terms in the reduced equation and show that it possesses both solitary waves of elevation and of depression. Kirrmann [13] also studied this parameter region and we recover his results.

2.2 Paper II

Here we consider the following setup. A ferrofluid surrounds a current-carrying wire. The current in the wire gives rise to a magnetic field which stabilizes the setup. One may then consider waves on the surface of the ferrofluid cylinder, that propagate under the influence of the magnetic force and surface tension. As in [Paper I] we prove the existence of solitary waves by using the spatial dynamics approach. We obtain a dynamical system of the form (1.13) and find that the spectrum of L depends on two parameters γ and β. More precisely we find that

λ is an eigenvalue of L if and only if

$$\lambda J_0(\lambda) = (\gamma - \beta\lambda^2) J_1(\lambda), \tag{2.2}$$

where J_0, J_1 are Bessel functions. We see that even though the physical setup of [Paper II] is quite different from the water wave problem, equation (2.2) is still of the same form as (1.15). In particular we find that the same type of bifurcations can occur. Again there are three interesting regions in the (β, γ) plane for which we are able to obtain homoclinic solutions and the corresponding reduced equations are the Kawahara, nonlinear Schrödinger and KdV equation, respectively. Recall that there were critical cases in [Paper I] for which the coefficients of the quadratic terms in the Kawahara and KdV equations were equal to 0. Such critical cases occur for the ferrofluid setup as well, and in fact there are additional cases to consider. For example, the Kawahara equation is given by

$$\eta_{xxxx} - 2(1+\delta)\eta_{xx} + \eta - 3c_1\eta^2 = 0, \tag{2.3}$$

where $c_1 \propto (3m_1'(1) - 8)$. The function m_1 is called the magnetization of the ferrofluid and it describes how the ferrofluid is affected by the magnetic field. When $m_1'(1)$ is close to $8/3$ we need to include cubic terms in (2.3). However, we find that the coefficient of the cubic terms is proportional to $1264 - 75m_1''(1)$. Hence, for $m_1''(1)$ close to $1264/75$ we would need to include a quintic term in (2.3). This pattern seems to repeat itself with coefficients of higher order terms depending on higher order derivatives of m_1. However, we still have existence of solitary waves for these critical cases, see [Paper II, Theorem 5.2]. A similar situation occurs for the KdV equation, while the NLS equation always seems to be of focusing type. So in particular we do not find any dark solitary waves propagating on the surface of the ferrofluid jet.

2.3 Paper III

In [Paper III] we turn our attention to three-dimensional steady internal waves. More specifically, we consider waves that are uniformly translating in the direction X, have a bounded profile in some direction x and are periodic in some other direction z. In the paper we prove the existence of such waves, using again the method of spatial dynamics. The idea of applying the method of spatial dynamics to the three-dimensional water wave problem is due to Groves and Mielke [8]. In that paper they consider the situation described above, but for surface waves, with

x aligned in the direction X and z being orthogonal to X. This was continued by Groves and Haragus [7], where they consider the general case of arbitrary directions x and z. The dispersion relation for the three-dimensional setting is a lot more complicated than in the two-dimensional one. Indeed, as was pointed out in [7] all known bifurcations in Hamiltonian systems theory can occur by varying the different parameters involved. In [Paper III] we find the same type of dispersion relation for internal waves, with the two additional parameters ρ and h. Just like in [Paper I], these extra parameters do not yield any new bifurcation scenarios, however they may affect the solution set associated with a specific bifurcation.

We prove the existence of two types of waves: doubly periodic waves and waves that are periodic in z with a bright or dark solitary wave profile in x-direction. Doubly periodic waves are waves that are periodic in both x and z. Such waves were found in [7] using the Lyapunov-center theorem. However, those waves are best described as oblique line waves, that is they only depend upon one horizontal coordinate. In [Paper III] we use a different approach involving a variational Lyapunov-Schmidt reduction, which yields truly doubly periodic waves. For the other case the situation is similar to what occurred for parameter values in region II in [Paper I]. Just like in that case we perform a center-manifold reduction and find that the truncated reduced equation is a focusing or defocusing NLS equation. This means that the corresponding waves can have either a bright or dark solitary wave profile in the x direction, while being periodic in z. In comparison, the corresponding surface waves found in [7] always have a bright solitary wave profile in the direction x.

2.4 Paper IV

The Green-Nagdhi system of equations is a model that can be used to describe long waves, that is waves with wavelengths that are large in comparison with the water depth. In this paper we study a class of model equations that was suggested by one of the coauthors and his collaborators in [5] as an improvement of the Green-Naghdi system of equations for interfacial flow. The equations are of the form

$$\begin{cases} \partial_t\zeta \;+\; \partial_x w \;=\; 0, \\[2ex] \partial_t\left(\dfrac{h_1+\gamma h_2}{h_1h_2}w + \mathcal{Q}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w\right) \;+\; (\gamma+\delta)\partial_x\zeta \\ \qquad\qquad\qquad\qquad +\dfrac{1}{2}\partial_x\left(\dfrac{h_1^2-\gamma h_2^2}{(h_1h_2)^2}w^2\right) = \partial_x\big(\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w]\big) \end{cases} \tag{2.4}$$

where $h_1 = 1-\zeta$, $h_2 = \delta^{-1}+\zeta$, $w = h_1h_2(\overline{u}_2 - \gamma\overline{u}_1)/(h_1+\gamma h_2)$, ζ is the deformation of the interface, δ is the ratio of the water depths, γ is the ratio of the densities, $\mathcal{Q}^{\mathsf{F}}_{\gamma,\delta}[\zeta]w = \mathcal{Q}^{\mathsf{F}}_2[h_2](h_2^{-1}w) + \gamma\mathcal{Q}^{\mathsf{F}}_1[h_1](h_1^{-1}w)$ and $\mathcal{R}^{\mathsf{F}}_{\gamma,\delta}[\zeta,w] = \mathcal{R}^{\mathsf{F}}_2[h_2,h_2^{-1}w] - \gamma\mathcal{R}^{\mathsf{F}}_1[h_1,h_1^{-1}w]$, with

$$\mathcal{Q}^{\mathsf{F}}_i[h_i]\overline{u}_i = -\frac{1}{3}h_i^{-1}\partial_x\mathsf{F}_i\big\{h_i^3\partial_x\mathsf{F}_i\{\overline{u}_i\}\big\},$$

$$\mathcal{R}^{\mathsf{F}}_i[h_i,\overline{u}_i] = \overline{u}_ih_i^{-1}\partial_x\mathsf{F}_i\big\{h_i^3\partial_x\mathsf{F}_i\{\overline{u}_i\}\big\} + \frac{1}{2}\big(h_i\partial_x\mathsf{F}_i\{\overline{u}_i\}\big)^2.$$

The operators F_i, $i=1,2$ are Fourier multipliers satisfying certain admissibility conditions. By choosing $\mathsf{F}_i \equiv 1$ we obtain the original Green-Naghdi system of equations for interfacial flow, and by choosing

$$\mathsf{F}_1(k) = \sqrt{\frac{3}{|k|\tanh(|k|)} - \frac{3}{|k|^2}},\ \mathsf{F}_2(k) = \sqrt{\frac{3}{\delta^{-1}|k|\tanh(\delta^{-1}|k|)} - \frac{3}{\delta^{-2}|k|^2}},$$

the system (2.4) has the same frequency dispersion as the Euler equations.

The main result of [Paper IV] is that the system (2.4) possesses solitary wave solutions. In order to prove this we use a variational formulation of the problem, together with a penalization argument and concentration-compactness. This strategy was for example used in [1, 6].

The first step is to identify solitary wave solutions as solutions of the constrained minimization problem $\arg\min_{\zeta\in V_{q,R}}\mathcal{E}(\zeta)$, with

$$V_{q,R} = \{\zeta \in H^\nu(\mathbb{R}) \,:\, \|\zeta\|_{H^\nu(\mathbb{R})} < R,\ (\gamma+\delta)\,\|\zeta\|^2_{L^2(\mathbb{R})} = q\},$$

where $q>0$, $R>0$, $\nu>1/2$, and

$$\mathcal{E}(\zeta) = \int_{-\infty}^{\infty} \frac{\zeta^2}{1-\zeta} + \frac{1}{3}(1-\zeta)^3(\partial_x\mathsf{F}_1\{\frac{\zeta}{1-\zeta}\})^2 + \frac{\zeta^2}{\delta^{-1}+\zeta} + \frac{1}{3}(1+\zeta)^2(\partial_x\mathsf{F}_2\{\frac{\zeta}{\delta^{-1}+\zeta}\})^2\,\mathrm{d}x.$$

The functional $\mathcal{E}$ is not coercive and since we are working on an unbounded domain we do not have the Rellich–Kondrachov theorem at our disposal. This prevents us from using direct methods to obtain a minimizer. Instead we first consider the problem of minimizing the penalized functional $\mathcal{E}_{P,\varrho}(\zeta) = \varrho(\|\zeta\|^2_{H^\rho_P}) + \mathcal{E}_P(\zeta)$, over the set

$$V_{P,q,R} = \{\zeta \in H^\nu_P \,:\, (\gamma + \delta)\,\|\zeta\|^2_{L^2_P} = q \text{ and } \|\zeta\|_{H^\nu_P} < 2R\},$$

where $\mathcal{E}_P$ is the same functional as $\mathcal{E}$ but where the integration is over $[-P/2, P/2]$, $\varrho : [0, (2R)^2) \mapsto [0, \infty)$ is a penalization function such that $\varrho(t) = 0$ for $0 \leq t \leq R^2$ and $\varrho(t) \to \infty$, as $t \to (2R)^2$. The penalization function makes $\mathcal{E}_{P,\varrho}$ coercive, and the fact that we are now working in H^ν_P allows us to use the Rellich–Kondrachov theorem. It is then an easy task to show that there exists a minimizer $\zeta_P \in V_{P,q,2R}$ of $\mathcal{E}_{P,\varrho}$. A key result then tells us that such minimizers must satisfy $\|\zeta_P\|_{H^\nu_P} \lesssim q$, so by choosing q sufficiently small we may assume that the minimizers of $\mathcal{E}_{P,\varrho}$ belong to $V_{P,q,R}$, which means that they are in fact minimizers of $\mathcal{E}_P$.

Using the minimizers of the periodic problem we can construct a special minimizing sequence $\{\zeta_n\}$ for the problem on the real line. This sequence has the following properties:

$$(\gamma + \delta)\,\|\zeta_n\|^2_{L^2} = q, \quad \|\zeta_n\|^2_{H^\nu} \leq Mq,$$

and

$$\lim_{n\to\infty} \mathcal{E}(\zeta_n) = I_q := \inf_{\zeta \in V_{q,R}} \mathcal{E}(\zeta) < q(1 - mq^{\frac{2}{3}}),$$

for some constants $M, m > 0$. This special minimizing sequence is used to show that the map $q \mapsto I_q$ is subadditive, and as we shall see this is an important detail.

The final part of the existence proof consists of applying Lions' concentration compactness principle [15].

Theorem 1 (Concentration-compactness). *Any sequence $\{e_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R})$ of non-negative functions such that*

$$\lim_{n\to\infty} \int_{\mathbb{R}} e_n \,\mathrm{d}x = I > 0$$

admits a subsequence, denoted again $\{e_n\}_{n\in\mathbb{N}}$, for which one of the following phenomena occurs.

- *(Vanishing) For each* $r > 0$*, one has*

$$\lim_{n\to\infty}\left(\sup_{x\in\mathbb{R}}\int_{x-r}^{x+r} e_n \,\mathrm{d}x\right) = 0.$$

- *(Dichotomy) There are real sequences* $\{x_n\}_{n\in\mathbb{N}}, \{M_n\}_{n\in\mathbb{N}}, \{N_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *and* $I^* \in (0, I)$ *such that* $M_n, N_n \to \infty$*,* $M_n/N_n \to 0$*, and*

$$\int_{x_n-M_n}^{x_n+M_n} e_n \,\mathrm{d}x \to I^* \quad \textit{and} \quad \int_{x_n-N_n}^{x_n+N_n} e_n \,\mathrm{d}x \to I^*$$

  as $n \to \infty$*.*

- *(Concentration) There exists a sequence* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *with the property that for each* $\epsilon > 0$*, there exists* $r > 0$ *with*

$$\int_{x_n-r}^{x_n+r} e_n \,\mathrm{d}x \geq I - \epsilon$$

  for all $n \in \mathbb{N}$*.*

In our case e_n is the integrand of $\mathcal{E}(\zeta_n)$, where $\{\zeta_n\}$ is a minimizing sequence. The idea is to exclude the cases of vanishing and dichotomy and then use the concentration property to prove the existence of a minimizer of $\mathcal{E}$. The main work lies in excluding the dichotomy case, see [Paper IV, Lemma 40], and this is were the subadditivity of I_q is used.

Apart from the existence result we also describe how the solitary wave solutions obtained behave in the long wave regime. More precisely we show that

$$\sup_{\zeta\in D_{q,R}}\inf_{x_0\in\mathbb{R}}\left\|q^{-\frac{2}{3}}\zeta(q^{-\frac{1}{3}}\cdot) - \xi_{KdV}(\cdot - x_0)\right\|_{H^1(\mathbb{R})} = \mathcal{O}(q^{\frac{1}{6}}),$$

where

$$\xi_{KdV}(x) = \frac{\alpha_0(\gamma+\delta)}{\delta^2-\gamma}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{3\alpha_0(\gamma+\delta)}{\gamma+\delta^{-1}}}x\right),$$

is a solitary wave solution of the KdV equation, and $D_{q,R}$ is the set of minimizers of $\mathcal{E}$ over $V_{q,R}$.

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