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OPTIMAL CONTROL OF A MULTIMACHINE POWER SYSTEM MODEL

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ABSTRACT.

Linear quadratic control theory is applied to the problem of improving the dynamic behaviour of a power system. A reduced model of the Scandinavian network is considered. The model has three generators, fifteen states and seven inputs. The linear model is simulated using the optimal feedback.

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1. INTRODUCTION

The major part of Swedish electrical power demand is produced in hydroelectric power stations, located in Northern Sweden. The distance between production centres in the north and load centres in the south is 700 - 1000 km and the transmitted power is 5000 - 6000 MW. In the seventies several nuclear power units will be put into commercial operation. The new plants will be located in the central and southern parts of the country and will have a maximum unit size of around 800 MW. The tripping of such a large power plant will subject the power system to a severe disturbance. It is necessary to dimension the system so that the majority of faults do not result in serious power failures.

By providing the generators with stabilizing signals it is possible to improve the dynamic behaviour of the system [1], [2]. Today the introduction of stabilizing signals is based on thorough physical insight and long experience. In fact it is a problem of designing local feedback loops in a large nonlinear multivariable system. If it is possible to linearize the system we can benefit from linear control theory. The application of linear quadratic control theory to power systems has been treated in [3], [4], [5] and [6]. Of course, a feedback law based on a linear model is optimal only for small perturbations around an operating point. On the other hand, if the nonlinear model behaves well it is immaterial that we have found it by using a linear model.

In this report we apply linear quadratic control theory to the problem of improving the dynamic behaviour of a power system. It is assumed that the power system can be described by a set of first order differential equations

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{1.1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1.1b}$$

where x is the n-dimensional state vector, u is the m-dimensional input vector and y is the k-dimensional output vector. A, B, C and D are matrices of dimension n×n, n×m, k×n, and k×m respectively. To apply linear quadratic control theory we assume that the performance of the system can be described by

$$J = \int_0^\infty \{x^T(s)Q_1x(s) + u^T(s)Q_2u(s)\}ds \tag{1.2}$$

where Q_1 is a symmetric nonnegative definite n×n matrix, and Q_2 is a symmetric positive m×m matrix. The task of the control is then to minimize the loss function J. If the system is controllable, the optimal control is a linear time invariant feedback

$$u(t) = - Lx(t) \tag{1.3}$$

where

$$L = Q_2^{-1}B^TS \tag{1.4}$$

and S is a symmetric nonnegative definite solution of the stationary Riccati equation

$$A^TS + SA - SBQ_2^{-1}B^TS + Q_1 = 0 \tag{1.5}$$

If an observability criteria is imposed on the pair [H,A], where $Q_1 = H^TH$ and rank H = rank Q_1, the unique solution S of (1.1) is positive definite, and the optimal system

$$\frac{dx(t)}{dt} = (A-BL)x(t) \tag{1.6}$$

is asymptotically stable [7], [8] and [9].

In Section 2 we summarize the modelling of a multimachine power system and in Section 3 we describe a computer program for the modelling process. The numerical example is presented in Section 4. The order of the model is reduced by eliminating the flux linkage of d- and q-axis armature winding from the state vector.

The eigenvalues of the original model and the eigenvalues of the reduced model are given in Section 5. As the shifts in common eigenvalues are small, the reduced model is used in the sequel. In Section 6 we describe how the matrix Riccati equation (1.5) was solved numerically. Finally, in Section 7, we discuss the choice of loss function and present the results obtained by simulation.

2. A STATE SPACE MODEL OF A MULTIMACHINE POWER SYSTEM.

To apply linear quadratic control theory we have to form the coefficient matrices in (1.1). The nonlinear equations describing a multimachine power system may be derived from basic physical laws. After computation of steady state values it is then necessary to linearize the equations to obtain a linear model valid for small perturbations around the steady state space form is proposed in [10] and only a short summary will be given here.

2.1. The Synchronous Machine and the Exciter.

In deriving the basic equations for the synchronous machine it is assumed that:

A2.1) The stator windings are sinusoidally distributed around the air-gap as far as the mutual effects between them and the rotor are concerned.

A2.2) The stator winding self- and mutual-inductances vary sinusoidally as the stator revolves, and are of the form $a+b\cdot\cos 2\gamma$ and $c+d\cdot\cos(2\gamma-2\pi/3)$ respectively, where a, b, c, and d are constants and γ the rotor angle.

A2.3) Saturation and hysteresis are negligible.

After d-q-transformation the flux linkages in field, d-axis armature and q-axis armature windings are given by

$$\psi_f = L_f i_f - L_{af} i_d \qquad (2.1)$$

$$\psi_d = L_{af} i_f - L_d i_d \qquad (2.2)$$

$$\psi_q = - L_q i_q \tag{2.3}$$

The equations for induced voltages are

$$v_f = \frac{d\psi_f}{dt} + r_f i_f \tag{2.4}$$

$$v_d = \frac{d\psi_d}{dt} - r_a i_a - \omega\psi_q \tag{2.5}$$

$$v_q = \frac{d\psi_q}{dt} - r_a i_a + \omega\psi_d \tag{2.6}$$

The air gap torque is given by

$$M_e = \psi_d i_q - \psi_q i_d \tag{2.7}$$

It is assumed that the exciter can be described by a first order linear differential equation

$$\frac{dv_f}{dt} = (-v_f+v_e)/T_e \tag{2.8}$$

2.2. The Transmission Network.

The transmission network is assumed to be completely described by the complex nodal admittance matrix $\tilde{Y}$. The nodal admittance equation can be written

$$\tilde{I} = \tilde{Y}\tilde{V} \tag{2.9}$$

which can be partitioned for generator and load nodes

$$\begin{vmatrix} \tilde{I}_G \\ \tilde{I}_L \end{vmatrix} = \begin{vmatrix} \tilde{Y}_{GG} & \tilde{Y}_{GL} \\ \tilde{Y}_{LG} & \tilde{Y}_{LL} \end{vmatrix} \begin{vmatrix} \tilde{V}_G \\ \tilde{V}_L \end{vmatrix} \tag{2.10}$$

It is assumed that all loads can be described by fixed admittances to ground. The loads are included in the admittance matrix and the load node voltages are eliminated giving

$$\tilde{I}_G = (\tilde{Y}_{GG} + \tilde{y}_{LG} - \tilde{Y}_{GL}(\tilde{Y}_{LL}+\tilde{y}_{LL})^{-1}\tilde{Y}_{LG})\tilde{V}_G \tag{2.11}$$

which after separation into real and imaginary parts can be written

$$\begin{vmatrix} I_D \\ I_Q \end{vmatrix} = \begin{vmatrix} G_N & -B_N \\ B_N & G_N \end{vmatrix} \begin{vmatrix} V_D \\ V_Q \end{vmatrix} \tag{2.12}$$

After inversion of the admittance matrix (2.12) can be written

$$\begin{vmatrix} V_D \\ V_Q \end{vmatrix} = \begin{vmatrix} R_N & -X_N \\ X_N & R_N \end{vmatrix} \begin{vmatrix} I_D \\ I_Q \end{vmatrix} \tag{2.13}$$

The equations for each generator are expressed with reference to pairs of axes (d,q) which rotate in synchronism with the rotor of the generators. On the other hand, the equations for the network refer to axes (D,Q) rotating at constant speed (ω_0).

Under transient conditions the angles θ_i, defined in Fig. 2.1, will vary as the machine speeds vary. The angles θ_i are state variables and i_d and i_q are linear combina-

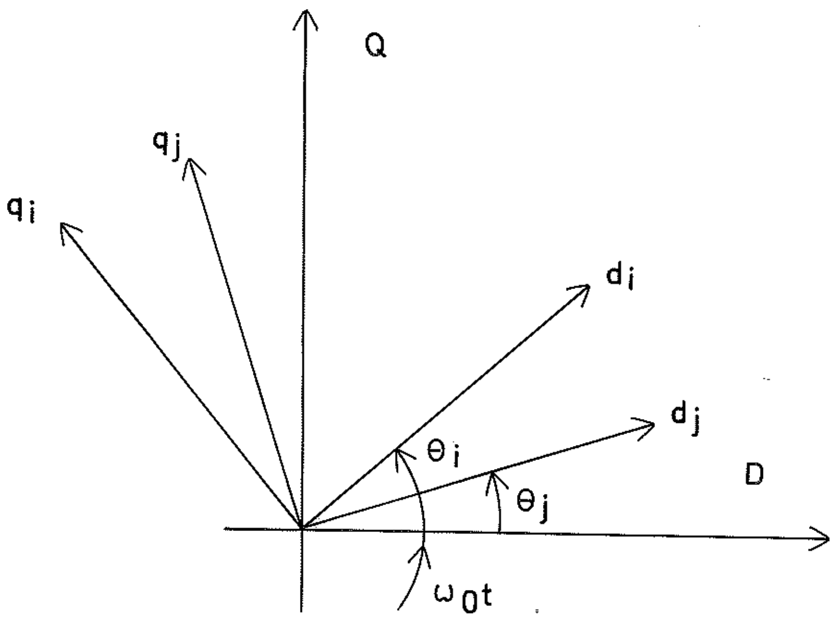

Fig. 2.1 - Angular relationships between network and synchronous machine reference axes.

tions of state variables (ψ_f, ψ_d and ψ_q) but v_d and v_q are needed for the computation of $d\psi_d/dt$ and $d\psi_q/dt$. Therefore it is necessary to express v_{di} and v_{qi} in θ_i, i_{di} and i_{qi}. The transformation relating rotor-based voltages to network-based voltages is given in [11] and can be obtained from Fig. 2.1.

$$v_{di} = \cos\theta_i v_{Di} + \sin\theta_i v_{Qi} \tag{2.14}$$

$$v_{qi} = -\sin\theta_i v_{Di} + \cos\theta_i v_{Qi} \tag{2.15}$$

In compact form (2.14) and (2.15) can be written

$$\begin{vmatrix} V_d \\ V_q \end{vmatrix} = \begin{vmatrix} C & S \\ -S & C \end{vmatrix} \begin{vmatrix} V_D \\ V_Q \end{vmatrix} \tag{2.16}$$

We require the power ($I^T V$) to be invariant under the change of bases and obtain [11]

$$\begin{vmatrix} I_D \\ I_Q \end{vmatrix} = \begin{vmatrix} C & -S \\ S & C \end{vmatrix} \begin{vmatrix} I_d \\ I_q \end{vmatrix} \tag{2.17}$$

Substitution of (2.16) and (2.17) into (2.13) now yields

$$\begin{vmatrix} V_d \\ V_q \end{vmatrix} = \begin{vmatrix} C & S \\ -S & C \end{vmatrix} \begin{vmatrix} R_N & -X_N \\ X_N & R_N \end{vmatrix} \begin{vmatrix} C & -S \\ S & C \end{vmatrix} \cdot \begin{vmatrix} I_d \\ I_q \end{vmatrix} \tag{2.18}$$

which can be written

$$\begin{vmatrix} V_d \\ V_q \end{vmatrix} = \begin{vmatrix} R_m(\theta) & -X_m(\theta) \\ X_m(\theta) & R_m(\theta) \end{vmatrix} \begin{vmatrix} I_d \\ I_q \end{vmatrix} \tag{2.19}$$

2.3. The Prime Movers.

The state variable associated with hydro turbines is the velocity of water in the penstock. In [10] the following equations were derived

$$\frac{dz_p}{dt} = \left(1 - z_p^3/u_t^3\right)/T_w \tag{2.20}$$

$$P_m = P_{max} z_p^3/u_t^2 \tag{2.21}$$

where z_p is the velocity of water, u_t is the gate opening, T_w is a time constant, P_m is the mechanical output power from the turbine and P_{max} is the maximum mechanical output power from the turbine.

Åström and Eklund [12] have derived a reasonable accurate model of a boiler and steam turbine. The state variable z_p is drum pressure. Assuming that the boiler is equipped with a feedwater regulator, which provides the boiler with feedwater proportional to the steam flow ($\sim u_1 \cdot \sqrt{z_p}$). Then the model is given by

$$\frac{dz_p}{dt} = \left[-\left((1+\alpha)u_t z_p^{5/8} - \alpha\right) + (1+\beta)u_f - \beta u_t z_p^{1/2}\right]/T_b \qquad (2.22)$$

$$P_m = P_{max}\left((1+\alpha)u_t z_p^{5/8} - \alpha\right) \qquad (2.23)$$

where u_t is the steam valve setting, u_f is the fuel flow, α, β and T_b parameters.

As before P_m is the mechanical output power from the turbine and P_{max} is the maximum mechanical output power from the turbine.

By differentiation of the expression

$$\theta_i(t) = \theta_{i0} + \int_0^t (\omega_i - \omega_0)ds$$

for the rotor angles we obtain

$$\frac{d\theta_i}{dt} = \omega_i - \omega_0 \qquad (2.24)$$

The fundamental torque balance for the rotor and the turbine gives

$$\frac{d\omega_i}{dt} = (P_{mi}/\omega_i - M_{ei} - M_{di})/J_i \qquad (2.25)$$

2.4. Construction of the System Matrices.

The states are: rotor angle, rotor angular velocity, flux linkages of field, d- and q-axis armature windings, excitation voltage, velocity of water (hydro plants) and steam pressure (steam plants).
The inputs are: excitation input, gate opening (hydro plants), steam valve setting (steam plants) and fuel flow (steam plants).
The outputs are: rotor angular velocity, terminal voltage, excitation voltage and steam pressure (steam plants).

The state vector x is partitioned into seven subvectors in the following manner

$$x^T = (x_1^T, x_2^T, x_3^T, x_4^T, x_5^T, x_6^T, x_7^T) \quad (2.26)$$

where

$$x_1^T = (\delta\theta_1, \delta\theta_2, \ldots, \delta\theta_n) \quad (2.27)$$

$$x_2^T = (\delta\omega_1/\omega_0, \delta\omega_2/\omega_0, \ldots, \delta\omega_n/\omega_0) \quad (2.28)$$

$$x_3^T = (\delta\omega_0\psi_{f1}, \delta\omega_0\psi_{f2}, \ldots, \delta\omega_0\psi_{fn}) \quad (2.29)$$

$$x_4^T = (\delta\omega_0\psi_{d1}, \delta\omega_0\psi_{d2}, \ldots, \delta\omega_0\psi_{dn}) \quad (2.30)$$

$$x_5^T = (\delta\omega_0\psi_{q1}, \delta\omega_0\psi_{q2}, \ldots, \delta\omega_0\psi_{qn}) \quad (2.31)$$

$$x_6^T = (\delta e_{f1}, \delta e_{f2}, \ldots, \delta e_{fn}) \quad (2.32)$$

$$x_7^T = (\delta z_{p1}, \delta z_{p2}, \ldots, \delta z_{pn}) \quad (2.33)$$

and $e_{fi} = X_{afi}v_{fi}/r_{fi}$. The input vector u is partitioned into three subvectors in a similar way.

$$u^T = (u_1^T, u_2^T, u_3^T) \tag{2.34}$$

where

$$u_1^T = (\delta u_{e1}, \delta u_{e2}, \ldots, \delta u_{en}) \tag{2.35}$$

$$u_2^T = (\delta u_{t1}, \delta u_{t2}, \ldots, \delta u_{tn}) \tag{2.36}$$

$$u_3^T = (\delta u_{f1}, \delta u_{f2}, \ldots, \delta u_{fn}) \tag{2.37}$$

Finally the output vector y is partitioned into four subvectors

$$y^T = (y_1^T, y_2^T, y_3^T, y_4^T) \tag{2.38}$$

where

$$y_1^T = (\delta\omega_1/\omega_0, \delta\omega_2/\omega_0, \ldots, \delta\omega_n/\omega_0) \tag{2.39}$$

$$y_2^T = (\delta v_{t1}, \delta v_{t2}, \ldots, \delta v_{tn}) \tag{2.40}$$

$$y_3^T = (\delta e_{f1}, \delta e_{f2}, \ldots, \delta e_{fn}) \tag{2.41}$$

$$y_4^T = (\delta z_{p1}, \delta z_{p2}, \ldots, \delta z_{pn}) \tag{2.42}$$

Observe that δz_{pi} is included in the output vector only for steam plants.

The system matrices are partitioned in a similar way and the nonzero submatrices are A_{12}, A_{22}, A_{23}, A_{24}, A_{25}, A_{33}, A_{34}, A_{35}, A_{36}, A_{41}, A_{42}, A_{43}, A_{44}, A_{45}, A_{51}, A_{52}, A_{53}, A_{54}, A_{55}, A_{66}, A_{77}, B_{22}, B_{61}, B_{72}, B_{73}, C_{12}, C_{21}, C_{23}, C_{24}, C_{25}, C_{36} and C_{47}.

The expressions for the submatrices are given in [10].

3. A LINEAR POWER SYSTEM MODELLING PROGRAM.

It is a very tedious work to establish a linearized mathematical model of an industrial process of some complexity. Often it is desirable to develop linearized models for different operating conditions and to investigate the sensitivity of the model to a set of parameters. The modelling of a process involves the following steps:

1) Computation of steady state values
2) Linearization of equations
3) Choice of state variables
4) Forming of the coefficient matrices

The first step requires the solution of a large set of nonlinear simultaneous equations, which can be a tough problem. Fortunately there are loadflow programs available, which resolve most of the computational problems, associated with the computation of steady state values. To find the other steady state values we exploit the structure of the problem. Once the steady state values are obtained we can linearize the equations and form the coefficient matrices.

The computer available is a UNIVAC 1108 with 64 k of core memory. It is possible to model power systems with 30 nodes, 50 lines and 10 generators without using backup storage. In a model with 10 generators there are 50-70 states, 20-30 inputs, and 30-40 outputs. The actual size of the state vector depends on whether ψ_d and ψ_q are included in the state vector. The number of inputs and outputs depend on whether the prime movers are hydro turbines or steam turbines. The flow chart of the main program is shown in Fig. 3.1. The forming of the coefficient matrices is divided into six straightforward steps as illustrated in Fig. 3.2.

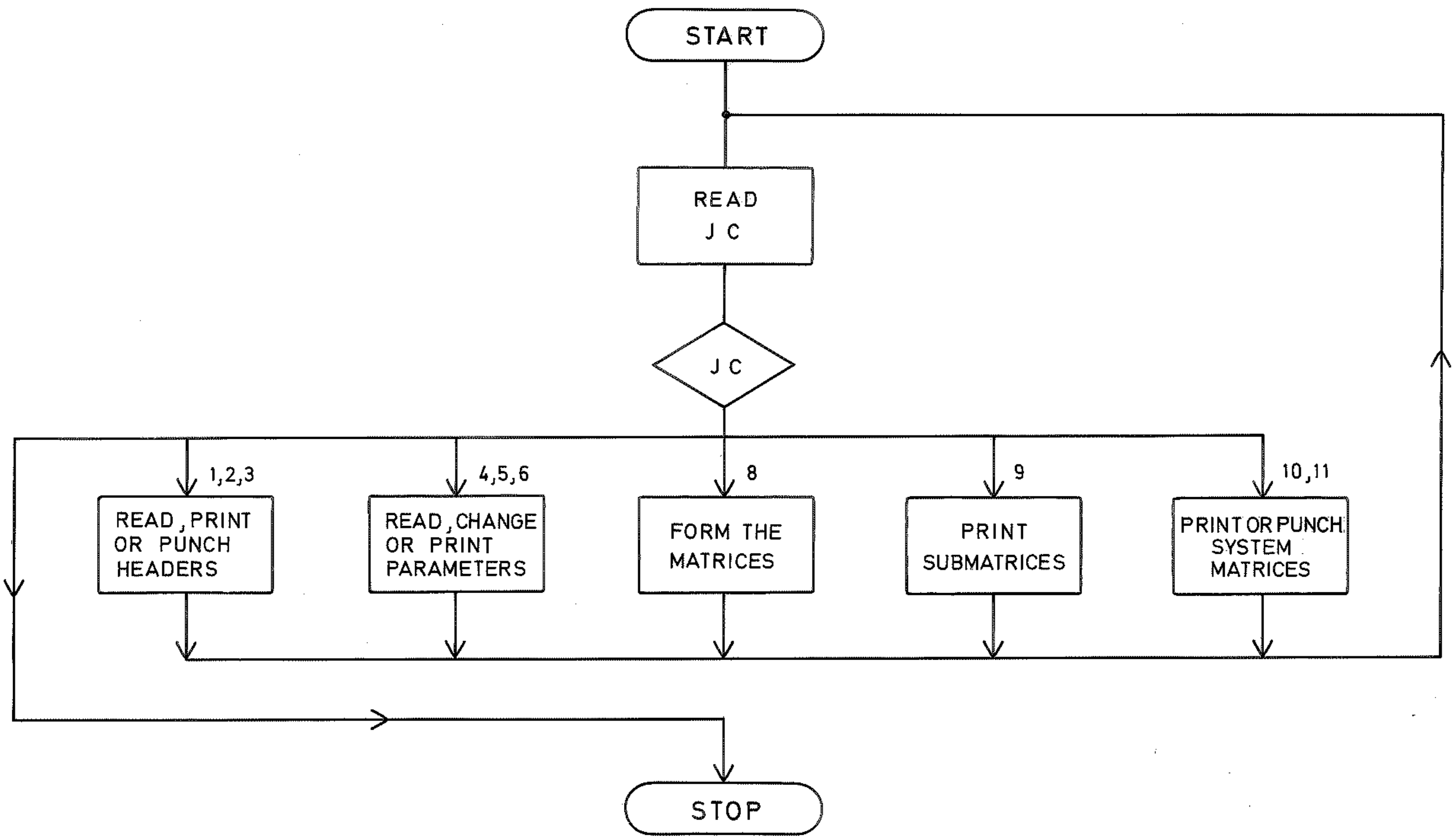

Fig. 3.1 - Flow chart of the main program LIPS (LInear Power System)

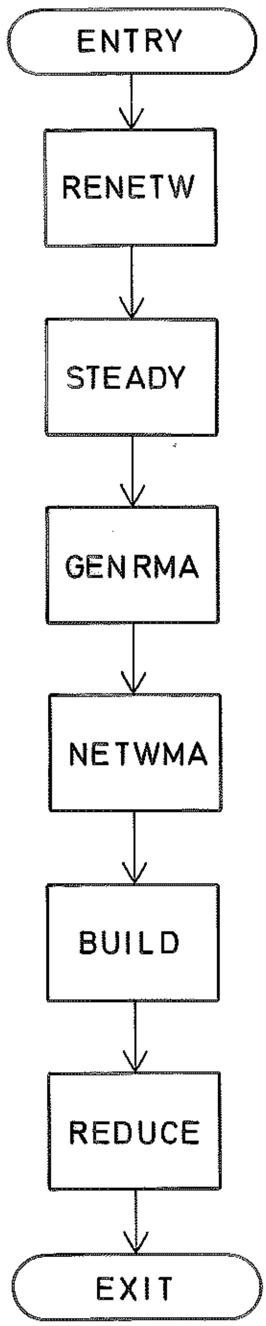

Fig. 3.2 - The main steps in forming of the coefficient matrices.

We now give a short description of the blocks in Fig. 3.2.

RENETW
The admittance matrix for the whole system is formed. The admittances representing the loads are computed. The non-generator node voltages are eliminated and the impedance matrix is computed.

STEADY
The steady state values are computed starting with the result of a load flow calculation.

GENRMA
The matrices relating machine current deviations to flux linkage deviations are computed.

NETWMA
The matrices relating machine voltage deviations to machine current deviations and rotor angle deviations are computed.

BUILD
The submatrices A_{12}, A_{21}, ..., C_{47} are computed.

REDUCE
The submatrices A_{12}, A_{21}, ..., C_{47} are placed into the system matrices A, B, C, and D. The states, inputs, and outputs are renumbered so that all states (inputs, outputs) associated with one plant are assigned consecutive indices in the state (input, output) vector. Finally all algebraic equations ($d\psi_d/dt = 0$, $d\psi_q/dt = 0$) are eliminated.

The program is written completely in FORTRAN and consists of a large number of subroutines altogether 2000 statements. The code occupies 9k of core memory. During the debugging the program was requested to print results before and after each step. The results were checked manually. Some linearizations were compared with results of numerical differentiations.

4. A REDUCED MODEL OF THE SCANDINAVIAN NETWORK.

The numerical example considered in this report is a reduced model of the Scandinavian network, which is shown in Fig. 4.1. A detailed model of the Swedish part of the network consists of 150 nodes and 250 lines. The reduction of the model has been performed in two steps. First the Swedish network has been reduced to a network with ten generators and the Norwegian network has been reduced to a network with three generators. This reduction has been performed by Mr. K. Walve at the Swedish State Power Board. The resulting network with thirteen generators has been reduced to a network with three generators. This second reduction has been performed by Mr. B. Hallberg, also at the Swedish State Power Board. The three machine power system is shown in Fig. 4.2.

By comparison with field tests it has been shown [13] that the detailed model well describes the Scandinavian network, but it can be questioned if the reduced model does. In any case the reduced model has many of the characteristics of the Scandinavian network. On the other hand it may lead to erroneous results if we predict stability in the Scandinavian network based on the reduced model.

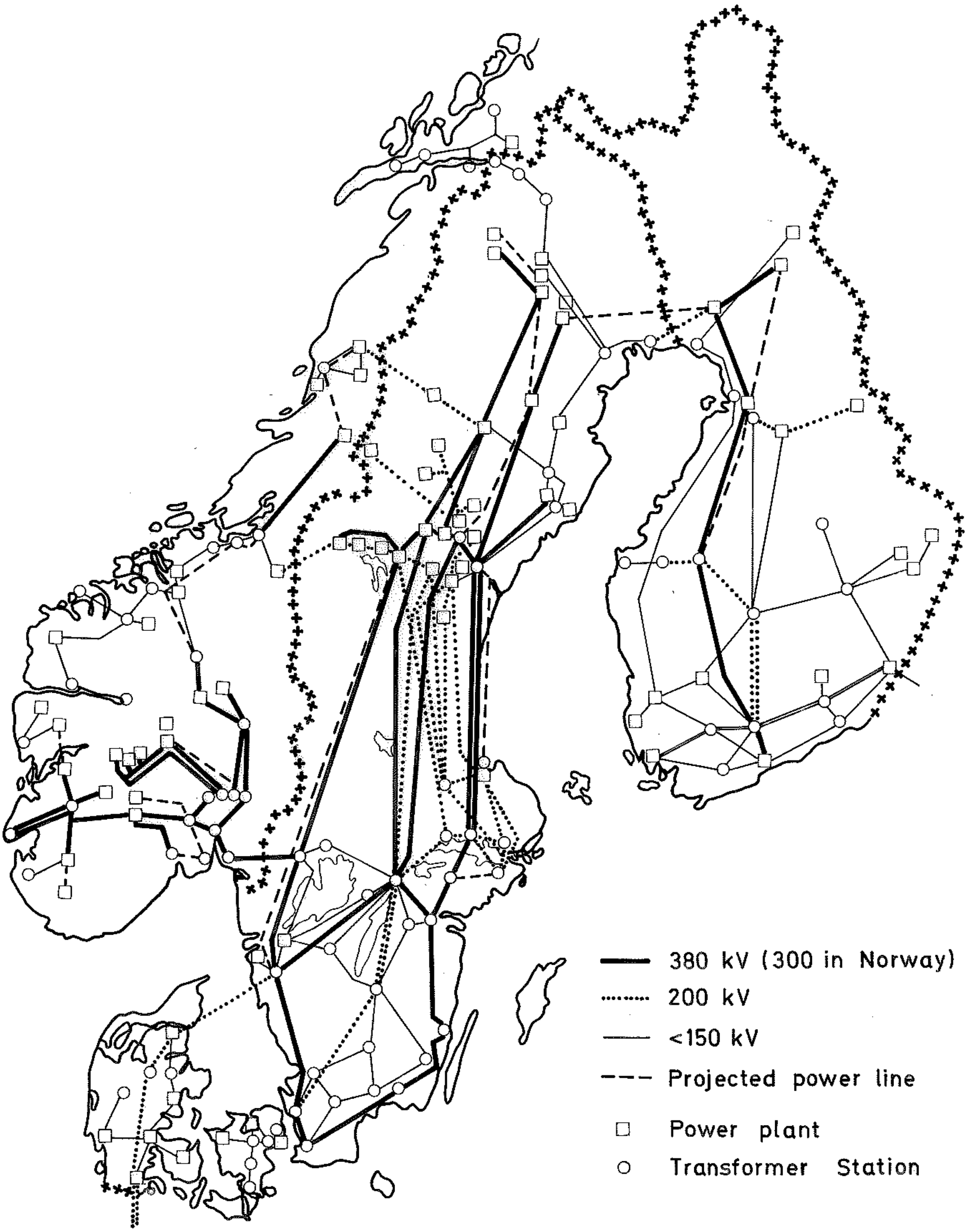

Fig. 4.1 - The Scandinavian network.

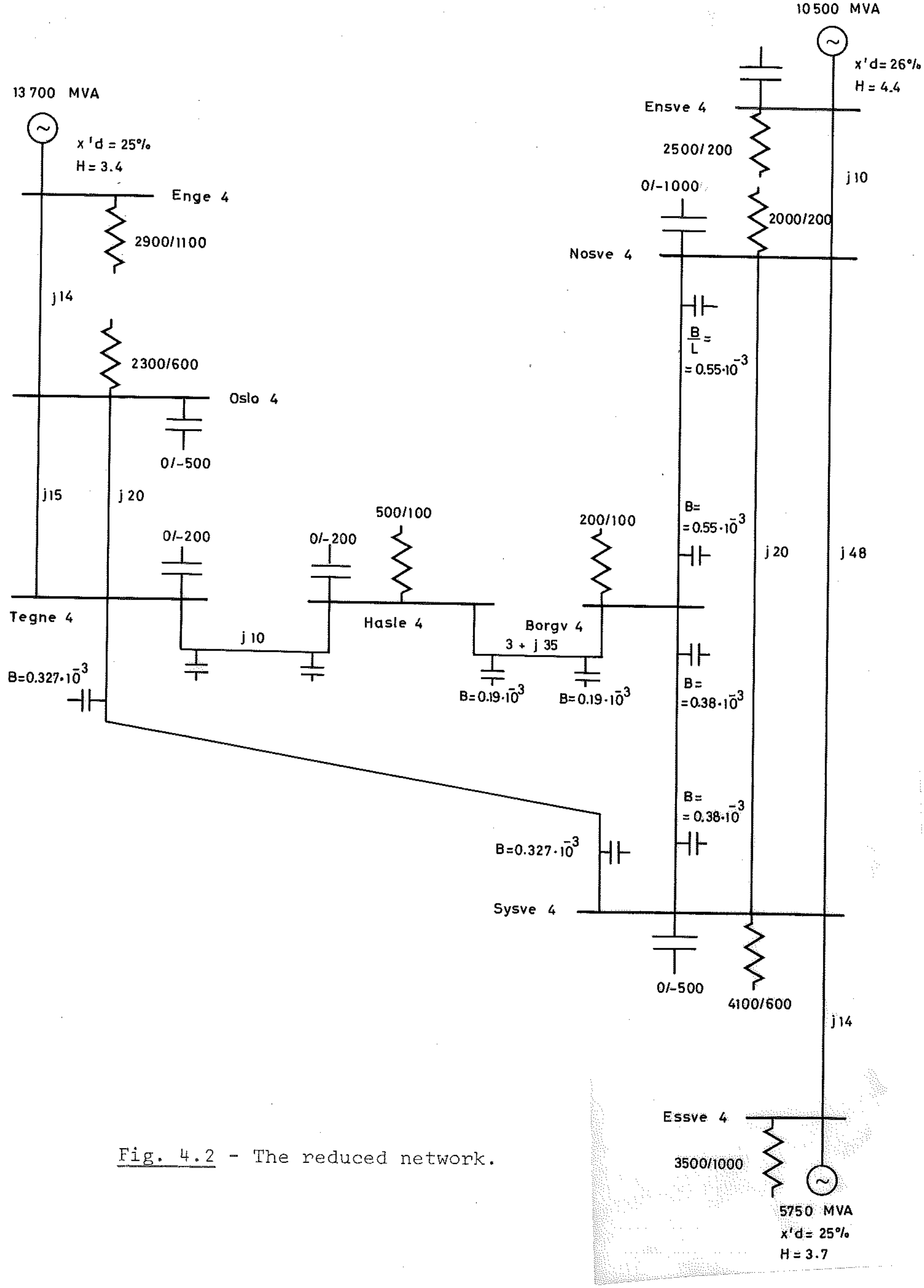

Fig. 4.2 - The reduced network.

5. ELIMINATION OF ψ_d AND ψ_q TERMS FROM THE STATE VECTOR.

A straightforward use of the modelling program results in a model with 21 states, given in Appendix 1. The flux linkages of d-axis armature and q-axis armature windings are included in the state vector. These states are associated with very short time constants. The possibility of reducing the order of the system by assuming $d\psi_{di}/dt = d\psi_{qi}/dt = 0$ is included in the program. To perform this reduction the state vector is partitioned into two components x_1 and x_2 where $x_2^T = (\psi_{d1}, \psi_{q1}, \ldots, \psi_{qn})$ while the other states are collected in the vector x_1. The system matrices are partitioned in a similar way and we have the equations

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 \\ A_3 & A_4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} B_1 \\ B_2 \end{vmatrix} u \qquad (5.1a)$$

$$y = \begin{vmatrix} C_1 & C_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + Du \qquad (5.1b)$$

If we put $\dot{x}_2 = 0$ in (5.1) the system equations become

$$\dot{x}_1 = (A_1 - A_2A_4^{-1}A_3)x + (B_1 - A_2A_4^{-1}B_2)u \qquad (5.2a)$$

$$y = (C_1 - C_2A_4^{-1}A_3)x_1 + (D - C_2A_4^{-1}B_2)u \qquad (5.2b)$$

To check if the reduction is reasonable the eigenvalues of the both models (5.1) and (5.2) are computed and shown in Table 5.1. It is observed that reduction causes only a minor change in the eigenvalues, that are present in both models. This observation strengthens us in the belief that

i	λ_i		μ_i	
1	$6.5382\cdot10^{-5}$		$9.4265\cdot10^{-8}$	
2	$-7.3225\cdot10^{-3}$		$-7.3224\cdot10^{-3}$	
3	$-7.6923\cdot10^{-2}$		$-7.6923\cdot10^{-2}$	
4	$-7.6924\cdot10^{-2}$		$-7.6923\cdot10^{-2}$	
5	$-1.0000\cdot10^{-1}$		$-1.0000\cdot10^{-1}$	
6	$-3.3205\cdot10^{-1}$		$-3.3207\cdot10^{-1}$	
7	$-3.6987\cdot10^{-1}$		$-3.6986\cdot10^{-1}$	
8	$-4.2225\cdot10^{-1}$		$-4.2229\cdot10^{-1}$	
9	$-7.5387\cdot10^{-1}$	$+i2.9711$	$-7.5467\cdot10^{-1}$	$+i2.9706$
10	$-7.5387\cdot10^{-1}$	$-i2.9711$	$-7.5467\cdot10^{-1}$	$-i2.9706$
11	-1.2434	+i4.1314	-1.2406	+i4.1326
12	-1.2434	-i4.1314	-1.2406	-i4.1326
13	-1.4086		-1.4086	
14	-1.8356		-1.8356	
15	-2.3828		-2.3832	
16	-78.667	+i415.53		
17	-78.667	-i415.53		
18	-1169.9	+i1677.1		
19	-1169.9	-i1677.1		
20	-1471.3	+i1111.9		
21	-1471.3	-i1111.9		

Table 5.1 - The eigenvalues (λ_i) of the original model (5.1) and the eigenvalues (μ_i) of the reduced model (5.2).

the reduction is reasonable. The system matrices for the reduced model is given in Appendix 2.

The system matrices of the three machine model is given in Appendix 2. The states are:

x_1 rotor angle, GNOSVE
x_2 rotor angular velocity, GNOSVE
x_3 flux linkage of field winding, GNOSVE
x_4 excitation voltage, GNOSVE
x_5 velocity of water, GNOSVE
x_6 rotor angle, GSYSVE
x_7 rotor angular velocity, GSYSVE
x_8 flux linkage of field winding, GSYSVE
x_9 excitation voltage, GSYSVE
x_{10} steam pressure, GSYSVE
x_{11} rotor angle, GNGE
x_{12} rotor angular velocity, GNGE
x_{13} flux linkage of field winding, GNGE
x_{14} excitation voltage, GNGE
x_{15} velocity of water, GNGE

The input variables are:

u_1 excitation input, GNOSVE
u_2 gate opening, GNOSVE
u_3 excitation input, GSYSVE
u_4 steam valve setting, GSYSVE
u_5 fuel flow, GSYSVE
u_6 excitation input, GNGE
u_7 gate opening, GNGE

The output variables are:

y_1 rotor angular velocity, GNOSVE
y_2 terminal voltage, GNOSVE
y_3 excitation voltage, GNOSVE

y_4 rotor angular velocity, GSYSVE
y_5 terminal voltage, GSYSVE
y_6 excitation voltage, GSYSVE
y_7 steam pressure, GSYSVE
y_8 rotor angular velocity, GNGE
y_9 terminal voltage, GNGE
y_{10} excitation voltage, GNGE

where GNOSVE, GSYSVE, and GNGE denote the generators in North Sweden, South Sweden, and Norway respectively.

6. NUMERICAL SOLUTION OF THE RICCATI EQUATION.

To find the solution of the algebraic equation

$$A^T S + SA - SBQ^{-1}B^T S + Q_1 = 0 \qquad (6.1)$$

the matrix differential equation

$$-\frac{d\Pi}{dt} = A^T\Pi + \Pi A - \Pi BQ_2^{-1}B^T\Pi + Q_1 \qquad (6.2)$$

is integrated until a stationary solution is reached. The integration procedure used is a fourth order Runge-Kutta method with fixed step size. This method is chosen because of its numerical simplicity. The drawback of the chosen method is the computation time required. To reduce the computation time it is desirable to increase the step size as much as possible. The maximum step size is determined by numerical stability of the Runge-Kutta integration procedure. When integrating linear differential equations $dz/dt = Fz$ it is well-known [14] that the fourth order Runge-Kutta method is numerical stable if $\lambda_i h \in D$ where λ_i are the eigenvalues of F, h is the step size and

$$D = \{\mu \,|\, |1 + \mu + \mu^2/2 + \mu^3/3 + \mu^4/4| < 1\}$$

Numerical experiments with the Riccati equation indicates that this criterion can be applied to the Jaccobian $\partial f(z,t)/\partial z$ of the nonlinear differential equation $dz/dt = f(z,t)$.

It is possible to give an explicit expression of λ_i, the eigenvalues of the Jaccobian, in A, B, and $\Pi(t;Q_0,t_f)$ where $\Pi(t;Q_0,t_f)$ is the solution of (6.2) with $\Pi(t_f;Q_0,t_f) = Q_0$. Consider two solutions $\Pi_1(t;Q_{01},t_f)$ and $\Pi_2(t;Q_{02},t_f)$ where $||Q_{01}-Q_0|| \leq \delta/2$ and $||Q_{02}-Q_0|| \leq \delta/2$.

Let $\Delta\Pi = \Pi_1 - \Pi_2$. The differential equation for $\Delta\Pi$ now becomes

$$- \frac{d(\Delta\Pi)}{dt} = A^T\Pi_1 + \Pi_1 A - \Pi_1 BQ_2^{-1}B^T\Pi_1 + Q_1 -$$

$$- A^T\Pi_2 - \Pi_2 A + \Pi_2 BQ_2^{-1}B^T\Pi_2 - Q_1 =$$

$$= \Delta\Pi\left(A-BQ_2^{-1}B^T\Pi_1\right) + \left(A-BQ_2^{-1}B^T\Pi_2\right)^T\Delta\Pi$$

or

$$- \frac{d(\Delta\Pi)}{dt} = \Delta\Pi\{A - BL(t;Q_{01},t_f)\} + \{A - BL(t;Q_{02},t_f)\}^T\Delta\Pi \qquad (6.3)$$

$$||\Delta\Pi(t_f)|| \leq \delta$$

equation (6.3) can be written

$$dy/dt = F(t;Q_0,t_f)y$$

The eigenvalues λ_k of F are [15] given by

$$\lambda_k = \mu_i + \mu_j \qquad i,j = 1, 2, \ldots, n$$

where μ_k are the eigenvalues of

$$\{A - BL(t;Q_0,t_f)\}$$

As a rule of thumb we require that $h(\mu_i+\mu_j) \in D$.

Observe that we have to choose the step size so that the integration procedure is stable when we reach the stationary solution. The matrix Q_0 can be used to increase the step size at the beginning of the integration.

7. CHOICE OF LOSS FUNCTION AND SIMULATION.

The loss function J in (1.2) depends on the two matrices Q_1 and Q_2. The model has fifteen states and seven inputs, which means that the loss function J depends on 15·16/2 + + 7·8/2 = 148 parameters. It is neither possible nor desirable to investigate all possible parameter combinations. On the contrary it is desirable to reduce the number of parameters but to retain the freedom to affect the time responses. To do so we have to affect the contribution to the loss function J from

1) the states (the inputs, the outputs),
2) the states (the inputs, the outputs) of a certain type.

The objective of the control is to

1) keep the rotor angle differences small,
2) keep the frequency constant,
3) keep the terminal voltage constant.

The matrices Q_1 and Q_2 was formed in the following way

$$Q_1 = P + w_x^2 Q + w_y^2 C^T R C$$

$$Q_2 = w_u^2 T$$

where

$$x^T P x = \sum_i \sum_j w_{\Delta\theta}^2 (\theta_i - \theta_j)^2$$

$$Q = \text{diag}(Q_x, Q_x, \ldots, Q_x)$$

$$R = \mathrm{diag}(R_y, R_y, \ldots, R_y)$$

$$T = \mathrm{diag}(T_u, T_u, \ldots, T_u)$$

and

$$Q_x = \mathrm{diag}(w_\theta^2, w_\omega^2, w_{\psi f}^2, w_{vf}^2, w_{z_p}^2)$$

$$R_y = \mathrm{diag}(w_f^2, w_{vt}^2, w_{y_e}^2, w_{y_p}^2)$$

$$T_u = \mathrm{diag}(w_{ef}^2, w_{ut}^2, w_{uf}^2)$$

The number of parameters, determining the loss function J, is reduced to 16 by this approach.

The value of the parameter vector was changed until reasonable time responses were obtained when the system was subjected to an initial value disturbance. It is particularly easy to find the time responses for this type of disturbance. Assume we are interested in $x(\tau)$, $u(\tau)$ and $y(\tau)$ for $\tau = 0, T, \ldots, NT$, and that $x(0)$ is given. We then have $y(\tau) = Hx(\tau)$, $u(\tau) = -Lx(\tau)$, $x(\tau+T) = Fx(\tau)$ where $H = C-DL$, $L = Q_2^{-1}B^TS$, and $F = \exp(A-BL)$ are constant matrices.

In general it is difficult to give rules for the choice of Q_1 and Q_2 but as rules of thumb we can use:

1) The response time is decreased by an increase in w_x.
2) The inputs are reduced by an increase in w_u.

After a couple of iterations the following values of the parameters gave rise to reasonable time responses:

$w_x = 1$	$w_u = 20$	$w_y = 1$
$w_{\Delta\theta} = 5$	$w_\Theta = 1$	$w_\omega = 5$
$w_{\psi_f} = 1/5$	$w_{v_f} = 1/10$	$w_{z_p} = 5$
$w_{e_f} = 2$	$w_{u_t} = 5$	$w_{u_f} = 5$
$w_f = 2$	$w_{v_t} = 1/5$	$w_{y_e} = 1$
$w_{y_p} = 1/5$		

The numerical values of Q_1 and Q_2 are given in Appendix 3.

The solution of the Riccati equation (6.1) is given in Appendix 4 and the corresponding feedback matrix $L = Q_2^{-1}B^TS$ is given in Appendix 5.

In Table 7.1 we show the eigenvalues of the optimal system $(A-BQ_2^{-1}B^TS)$. We observe that the system is asymptotically stable as stated in Section 1. It is also observed that the complex eigenvalues are better damped than the corresponding eigenvalues in the open loop system (Table 5.1).

In Fig. 7.1 we show the time responses of the states, inputs and outputs when θ_1 is given an initial value disturbance of 0.3 rad.

Earlier in this section we stated that the objective of the control is to keep the rotor angle differences small. To take care of the angular differences we formed the term

$$\sum_i \sum_j w_{\Delta\theta}^2(\theta_i - \theta_j)^2$$

in the loss function.

From Fig. 7.1 we can see that the rotor angle differences decay very rapidly. The rotor angles on the other hand tend to zero more sluggish but it is no need for a rapid return.

i	λ_i	
1	$-7.3346\cdot10^{-3}$	
2	$-2.0932\cdot10^{-1}$	
3	$-2.7667\cdot10^{-1}$	$+i3.5530\cdot10^{-1}$
4	$-2.7667\cdot10^{-1}$	$-i3.5530\cdot10^{-1}$
5	$-3.1735\cdot10^{-1}$	
6	$-3.8322\cdot10^{-1}$	$+i2.5255\cdot10^{-1}$
7	$-3.8322\cdot10^{-1}$	$-i2.5255\cdot10^{-1}$
8	$-5.1386\cdot10^{-1}$	
9	-1.3595	$+i3.1247$
10	-1.3595	$-i3.1247$
11	-1.3721	$+i4.1815$
12	-1.3721	$-i4.1815$
13	-1.4856	$+i3.7911\cdot10^{-2}$
14	-1.4856	$-i3.7911\cdot10^{-2}$
15	-2.4613	

Table 7.1 - Eigenvalues of the optimal system.

It was also stated that the objective of the control is to keep the frequency constant and to keep the terminal voltage constant. From Fig. 7.1 we can see that the frequency and terminal voltage deviations are fairly small and well damped.

Turning to the inputs we observe that the prime mover inputs are small. The deviations in steam pressure and fuel flow are particularly small. The inputs to the exciters on the other hand are strongly affected by the control law.

To implement the optimal control law it is necessary to measure or reconstruct the whole state vector. This may be a formidable task in a large power system. Therefore, it is desirable to use only locally available signals and a small number of transmitted signals. It is still an open question how to choose these signals.

In this report it has been demonstrated how one powerful tool from control theory can be applied to power systems. It has also been demonstrated how system specifications can be formulated in terms of quadratic loss function. Finally it has been demonstrated that the required time responses can be obtained in a couple of iterations.

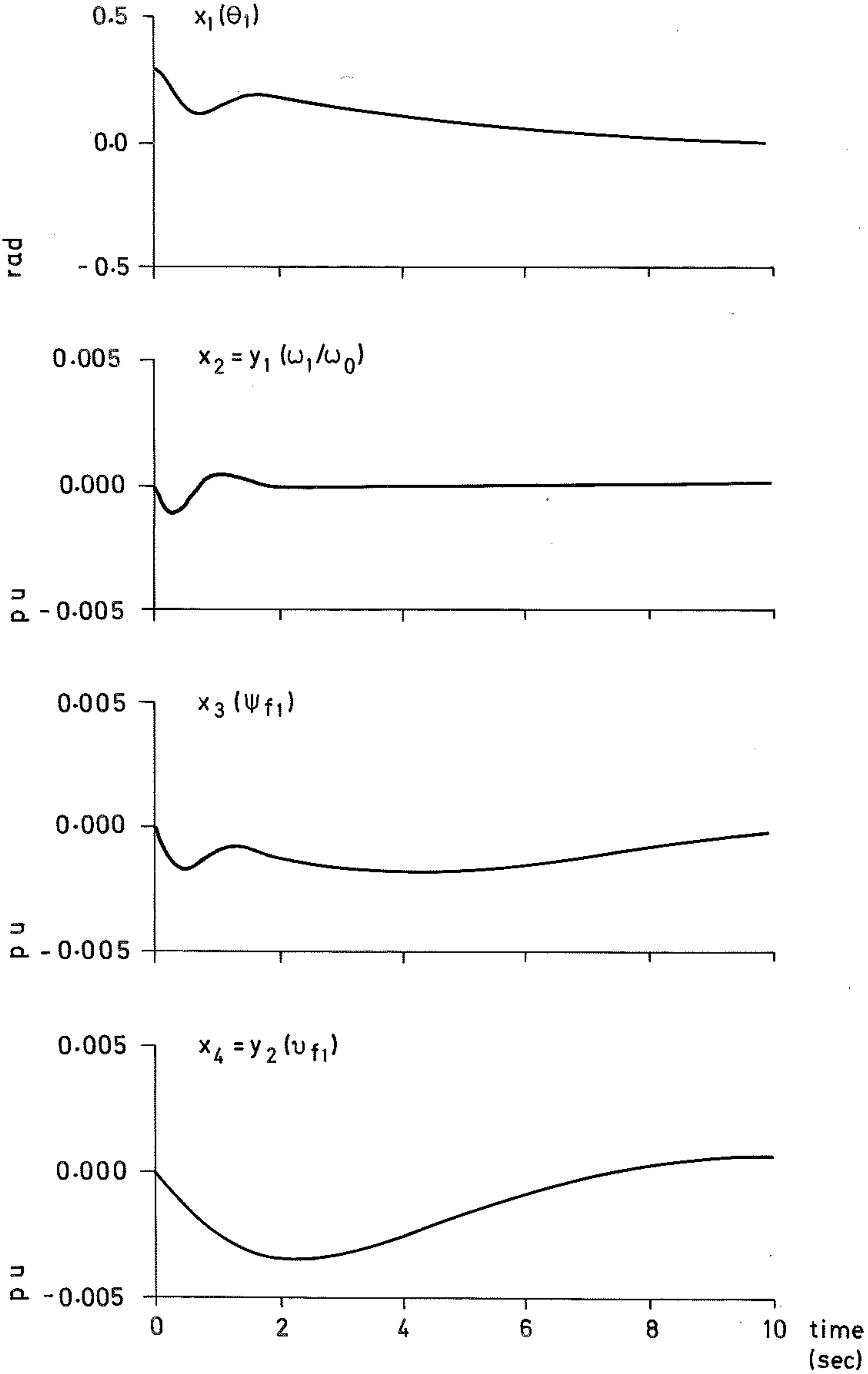

Fig. 7.1 - Time responses for the states, inputs, and outputs.

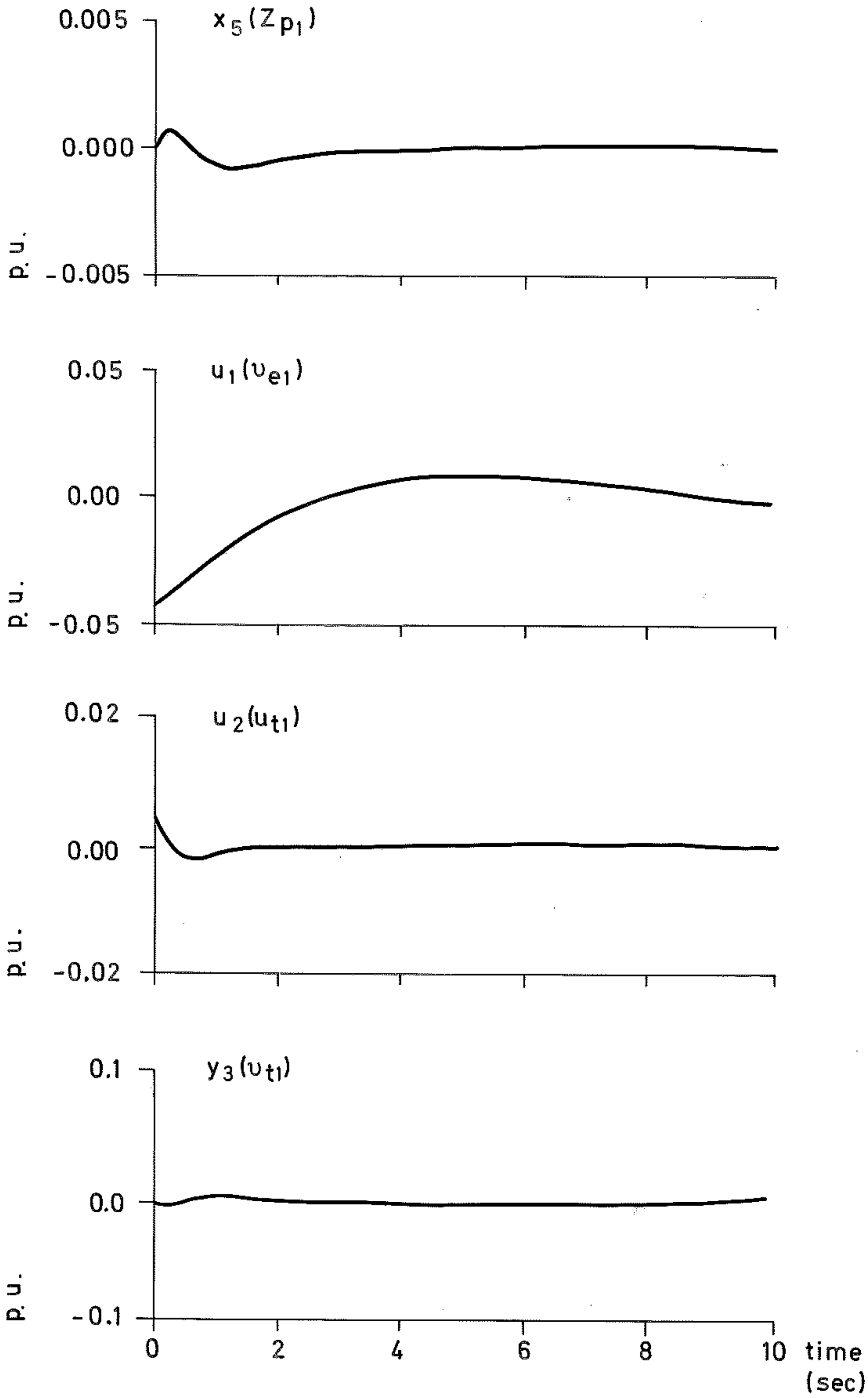
0.005
$x_5(Zp_1)$
0.000
p.u.
-0.005
0.05
$u_1(\upsilon_{e1})$
0.00
p.u.
-0.05
0.02
$u_2(u_{t1})$
0.00
p.u.
-0.02
0.1
$y_3(\upsilon_{t1})$
0.0
p.u.
-0.1
0
2
4
6
8
10
time
(sec)

Fig. 7.1 - (Contd.)

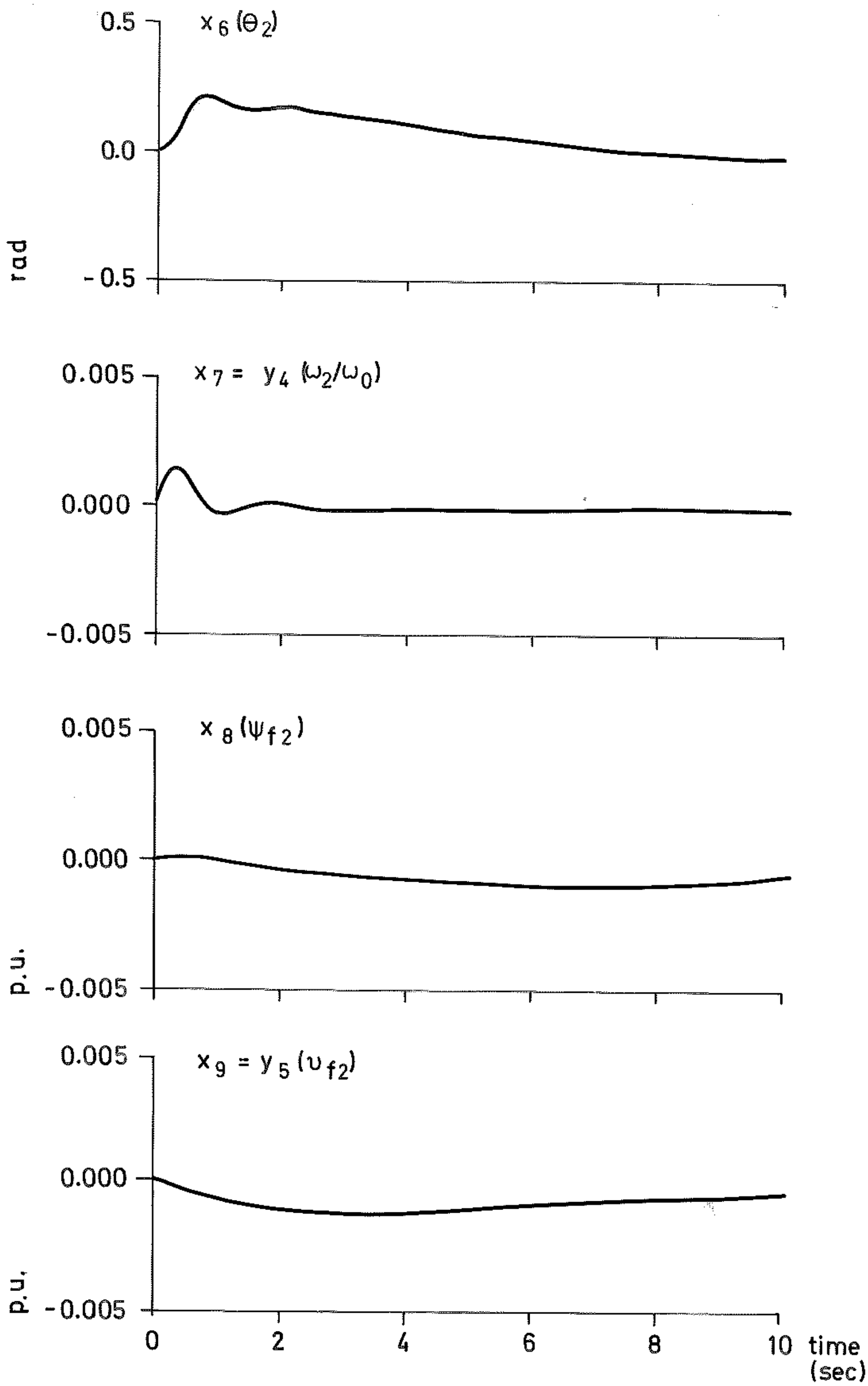
0.5
0.0
-0.5
rad
x6 (Θ2)
0.005
0.000
-0.005
x7 = y4 (ω2/ω0)
0.005
0.000
-0.005
p.u.
x8 (ψf2)
0.005
0.000
-0.005
p.u.
x9 = y5 (υf2)
0
2
4
6
8
10
time
(sec)

Fig. 7.1 - (Contd.)

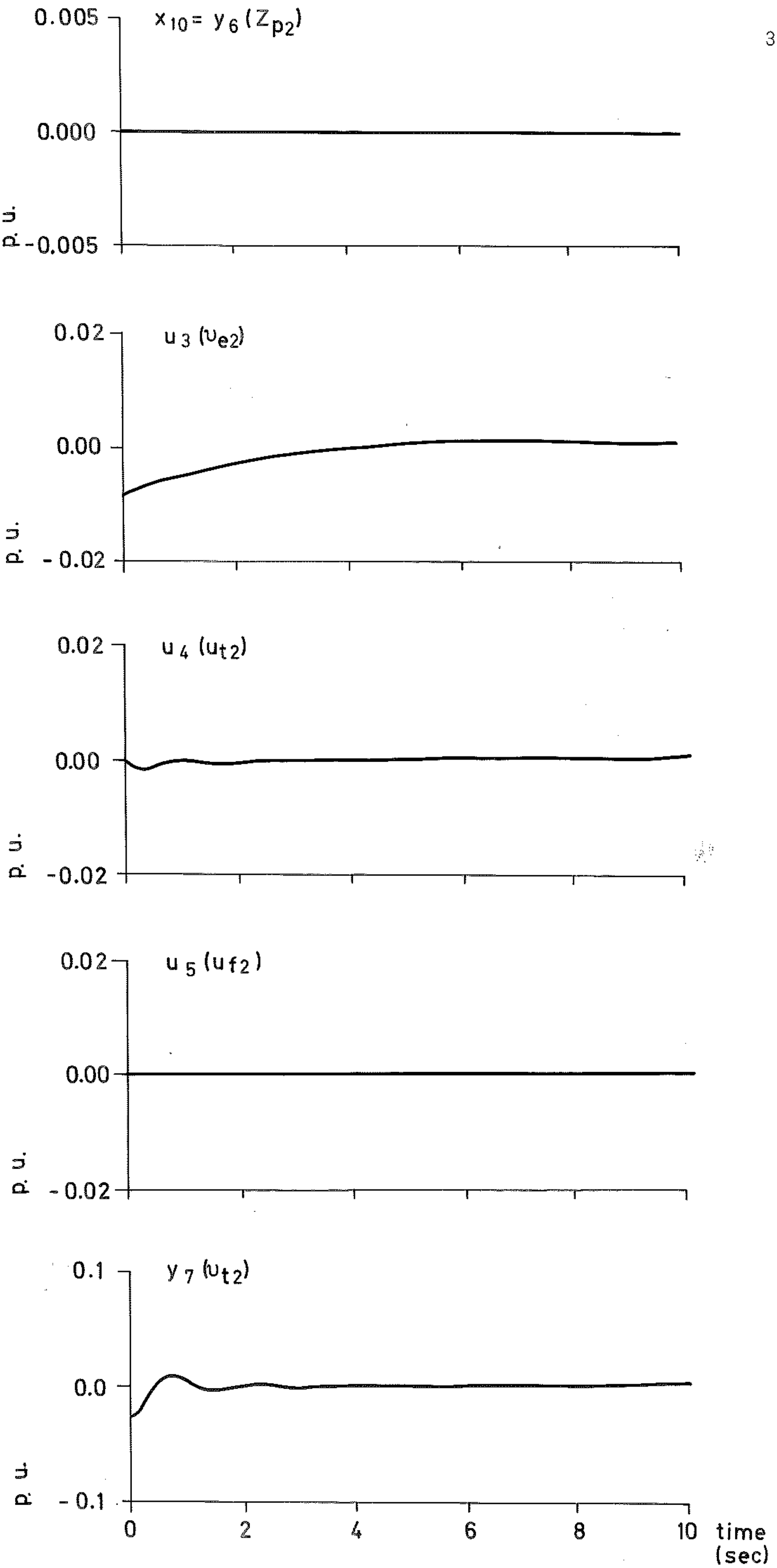
0.005
$x_{10} = y_6 (Z_{p2})$
0.000
p. u. -0.005
0.02
$u_3 (\upsilon_{e2})$
0.00
p. u. - 0.02
0.02
$u_4 (u_{t2})$
0.00
p. u. -0.02
0.02
$u_5 (u_{f2})$
0.00
p. u. - 0.02
0.1
$y_7 (\upsilon_{t2})$
0.0
p. u. - 0.1
0
2
4
6
8
10
time (sec)

Fig. 7.1 - (Contd.)

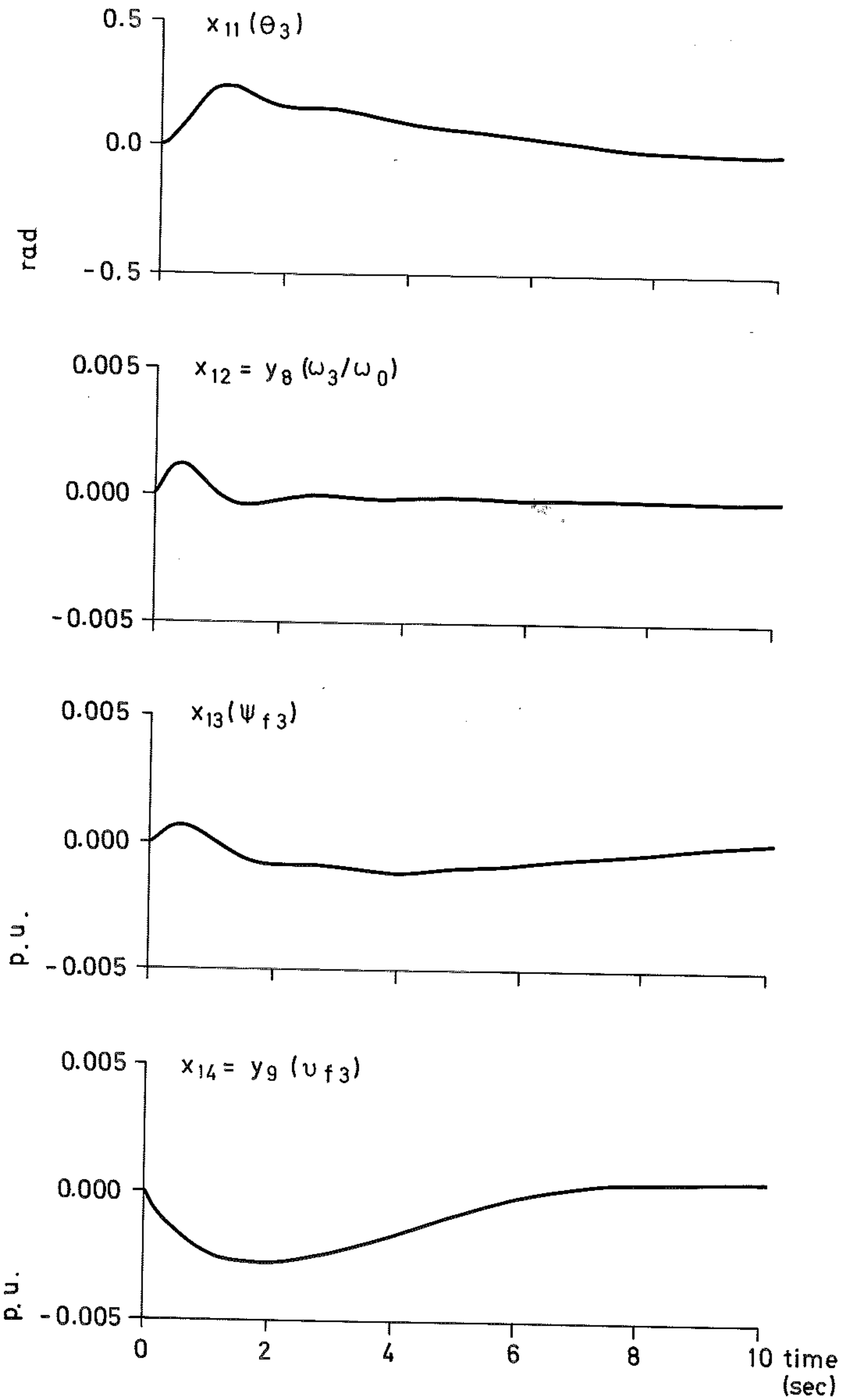
0.5
0.0
-0.5
rad
$x_{11} (\theta_3)$
0.005
0.000
-0.005
$x_{12} = y_8 (\omega_3/\omega_0)$
0.005
0.000
-0.005
p.u.
$x_{13} (\psi_{f3})$
0.005
0.000
-0.005
p.u.
$x_{14} = y_9 (\upsilon_{f3})$
0
2
4
6
8
10
time
(sec)

Fig. 7.1 - (Contd.)

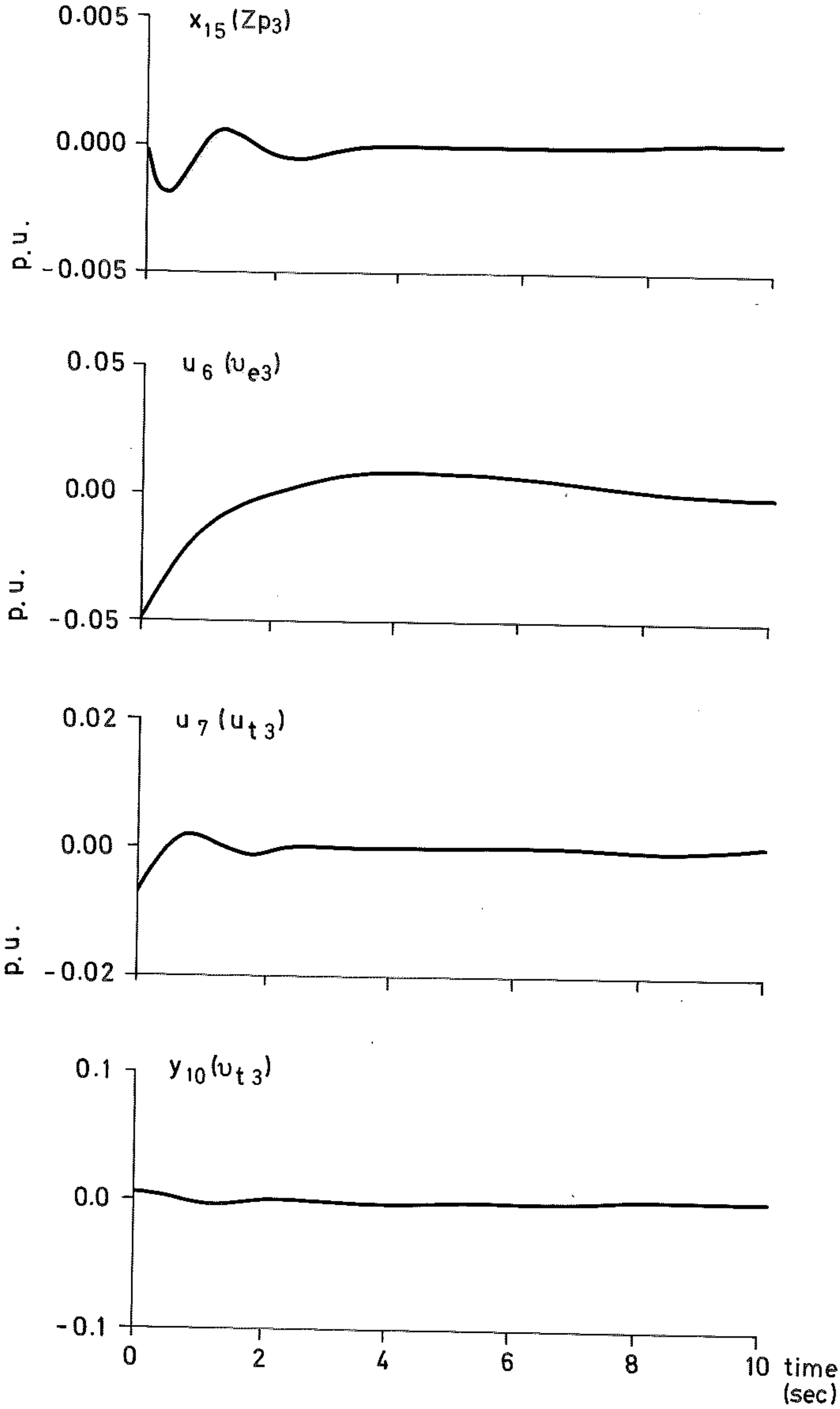
0.005
$x_{15}(Zp_3)$
0.000
p.u.
-0.005
0.05
$u_6\ (\upsilon_{e3})$
0.00
p.u.
-0.05
0.02
$u_7\ (u_{t\,3})$
0.00
p.u.
-0.02
0.1
$y_{10}(\upsilon_{t\,3})$
0.0
-0.1
0
2
4
6
8
10
time
(sec)

Fig. 7.1 - (Contd.)

8. REFERENCES

[1] deMello, F.P., and Concordie, C.: "Concepts of Synchronous Machine Stability as Affected by Excitation Control", IEEE Trans. on Power Apparatus and Systems, Vol. PAS-88, pp. 316-329, April, 1969.

[2] Ölwegård, Å.: "Pendlingar i det svenska kraftsystemet och i samkörningsförbindelserna med grannländerna. Pendlingarnas uppkomst och möjligheterna till förbättrad utdämpning av dem.", PM SK-4325, Swedish State Power Board, 1968.

[3] Yu, Y.N., Vongsuiya, K., and Wedman, L.N.: "Application of an Optimal Control Theory to a Power System", IEEE Trans. Power Apparatus and Systems, Vol. PAS-89, No. 1, January, 1970, pp. 55 - 62.

[4] Anderson, J.H.: "The Control of a Synchronous Machine Using Optimal Control Theory", Proc. IEEE, Vol. 59, No. 1, January, 1971, pp. 25-35.

[5] Elgerd, O.I., and Fosha, C.E.: "Optimum Megawatt-Frequency Control of Multiarea Electric Energy Systems", IEEE Trans. on Power Apparatus and Systems, Vol. PAS-89, No. 4, April, 1970, pp. 556--563.

[6] Fosha, C.E., and Elgerd, O.I.: "The Megawatt-Frequency Problem: A New Approach Via Optimal Control Theory", IEEE Trans. on Power Apparatus and Systems, Vol. PAS-89, No. 4, April, 1970, pp. 563-577.

[7] Kalman, R.: "Contributions to the Theory of Optimal Control", Bol. Soc. Mat. Mexicana, 5 (1960), 102-119.

[8] Bucy: "Global Theory of the Riccati Equation", J. Comput. System Sci., 1 (1967), pp. 349-361.

[9] Mårtensson, K.: "On the Matrix Riccati Equation", Information Sciences, 3 (1971), 17-49.

[10] Lindahl, S.: "A State Space Model of a Multimachine Power System", Research Report 7118, November, 1971, Lund Institute of Technology, Division of Automatic Control.

[11] Taylor, D.G.: "Analysis of Synchronous Machines Connected to Power System Networks", Proc. IEE (London), Vol. 109, pt.C., pp. 606-610, July, 1962.

[12] Åström, K.J., and Eklund, K.: "A Simplified Non-Linear Model of a Drum Boiler-Turbine Unit", Int. J. Control 16, 145-169, 1972.

[13] Ölwegård, Å., and Langer, P.: "Determination of Power Transfer Capability of Transmission Systems and Tie-Lines on the Occurrence of Severe Disturbances", CIGRE 1968, Report 32.04.

[14] Gear, C.W.: "Numerical Initial Value Problems in Ordinary Differential Equations", Prentice-Hall, Englewood, Cliffs, 1971.

[15] Gantmacher: "The Theory of Matrices", Chelsea, New York, 1959.

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A-MATRIX, PAGE 1

A(1, 2)= .314159+03	A(2, 2)=-.319444+01	A(2, 3)= .143268+00
A(2, 4)= .672515-01	A(2, 5)= .251738+00	A(2, 7)= .340985+00
A(3, 3)=-.770647+00	A(3, 4)= .520194+00	A(3, 6)= .250453+00
A(4, 1)= .468926+02	A(4, 2)=-.102954+03	A(4, 3)= .152614+04
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A(4,15)=-.414687+02	A(4,17)= .383212+03	A(4,18)=-.383212+03
A(4,19)=-.212512+03	A(5, 1)=-.125570+03	A(5, 2)=-.309397+03
A(5, 3)= .125328+04	A(5, 4)=-.156744+04	A(5, 5)=-.809753+03
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A(5,12)=-.306221+02	A(5,15)= .639610+02	A(5,17)=-.400500+03
A(5,18)= .400500+03	A(5,19)=-.203338+03	A(6, 6)=-.769231-01
A(7, 7)=-.140858+01	A(8, 9)= .314159+03	A(9, 9)=-.250580+01
A(9,10)= .652683-01	A(9,11)=-.107332-01	A(9,12)= .115498+00
A(9,14)= .159024+00	A(10,10)=-.369241+00	A(10,11)= .327499-01
A(10,13)= .336492+00	A(11, 1)=-.118987+03	A(11, 3)= .660972+03
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A(11, 9)=-.317946+03	A(11,10)= .752605+02	A(11,11)=-.752605+02
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A(12, 3)= .761389+02	A(12, 4)=-.761389+02	A(12, 5)=-.350704+03
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A(18, 8)= .528687+01	A(18,10)= .118602+02	A(18,11)=-.118602+02
A(18,12)=-.322485+02	A(18,15)= .304515+02	A(18,16)=-.799841+02
A(18,17)= .192078+04	A(18,18)=-.192078+04	A(18,19)= .132015+04
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A(19,19)=-.101919+04	A(20,20)=-.769231-01	A(21,21)=-.183560+01

111 NONZERO ELEMENTS
330 ZERO ELEMENTS

B-MATRIX, PAGE 1

B(2, 2)=-.227323+00	B(6, 1)= .769231-01	B(7, 2)= .140858+01
B(9, 4)= .162229+00	B(13, 3)= .100000+00	B(14, 4)=-.783733-02
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B(21, 7)= .183560+01		

10 NONZERO ELEMENTS
137 ZERO ELEMENTS

C-MATRIX, PAGE 1

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C( 2, 4)=-.520055+01  C( 2, 5)=-.178025+01  C( 2, 8)= .182443+00
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C( 2,15)= .155987+00  C( 2,17)=-.861515+00  C( 2,18)= .861515+00
C( 2,19)=-.817029+00  C( 3, 6)= .100000+01  C( 4, 9)= .100000+01
C( 5, 1)=-.132271+00  C( 5, 3)= .127815+01  C( 5, 4)=-.127815+01
C( 5, 5)=-.895992+00  C( 5, 8)= .189646+00  C( 5,10)= .316653+00
C( 5,11)=-.316653+00  C( 5,12)=-.103969+00  C( 5,15)=-.573755-01
C( 5,17)= .745393+00  C( 5,18)=-.745393+00  C( 5,19)=-.893768+00
C( 6,13)= .100000+01  C( 7,14)= .100000+01  C( 8,16)= .100000+01
C( 9, 1)= .192920+00  C( 9, 3)=-.768531+00  C( 9, 4)= .768531+00
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   43 NONZERO ELEMENTS
  167 ZERO ELEMENTS
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D-MATRIX, PAGE 1

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    0 NONZERO ELEMENTS
   70 ZERO ELEMENTS
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A-MATRIX, PAGE 1

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A(13,14)= .247081+00	A(14,14)=-.769231-01	A(15,15)=-.183560+01

69 NONZERO ELEMENTS
156 ZERO ELEMENTS

B-MATRIX, PAGE 1

B(2, 2)=-.227323+00	B(4, 1)= .769231-01	B(5, 2)= .140858+01
B(7, 4)= .162229+00	B(9, 3)= .100000+00	B(10, 4)=-.783733-02
B(10, 5)= .730000-02	B(12, 7)=-.294089+00	B(14, 6)= .769231-01
B(15, 7)= .183560+01		

10 NONZERO ELEMENTS
95 ZERO ELEMENTS

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C-MATRIX, PAGE 1

C( 1, 2)= .100000+01  C( 2, 1)=-.212729-01  C( 2, 2)= .932214+00
C( 2, 3)= .889955+00  C( 2, 6)= .194903-01  C( 2, 7)=-.100683-01
C( 2, 8)= .734920-02  C( 2,11)= .178260-02  C( 2,12)= .227022-01
C( 2,13)= .232592-01  C( 3, 4)= .100000+01  C( 4, 7)= .100000+01
C( 5, 1)=-.913649-01  C( 5, 2)= .240934+00  C( 5, 3)= .233823+00
C( 5, 6)= .162358+00  C( 5, 7)= .277790+00  C( 5, 8)= .286660+00
C( 5,11)=-.709951-01  C( 5,12)= .159072+00  C( 5,13)= .143982+00
C( 6, 9)= .100000+01  C( 7,10)= .100000+01  C( 8,12)= .100000+01
C( 9, 1)= .108115-01  C( 9, 2)= .114114-01  C( 9, 3)= .151457-01
C( 9, 6)= .126540-01  C( 9, 7)=-.743007-02  C( 9, 8)= .117922-02
C( 9,11)=-.234655-01  C( 9,12)= .102604+01  C( 9,13)= .912437+00
C(10,14)= .100000+01

    34 NONZERO ELEMENTS
   116 ZERO ELEMENTS


D-MATRIX, PAGE 1

     0 NONZERO ELEMENTS
    70 ZERO ELEMENTS
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Q1-MATRIX, PAGE 1

```
Q1( 1, 1)= .510004+02 Q1( 1, 2)=-.166882-02 Q1( 1, 3)=-.160526-02
Q1( 1, 6)=-.250006+02 Q1( 1, 7)=-.100986-02 Q1( 1, 8)=-.105337-02
Q1( 1,11)=-.249998+02 Q1( 1,12)=-.156940-03 Q1( 1,13)=-.151397-03
Q1( 2, 1)=-.166882-02 Q1( 2, 2)= .290371+02 Q1( 2, 3)= .354455-01
Q1( 2, 6)= .229724-02 Q1( 2, 7)= .229834-02 Q1( 2, 8)= .303722-02
Q1( 2,11)=-.628426-03 Q1( 2,12)= .284791-02 Q1( 2,13)= .267140-02
Q1( 3, 1)=-.160526-02 Q1( 3, 2)= .354455-01 Q1( 3, 3)= .738769-01
Q1( 3, 6)= .222001-02 Q1( 3, 7)= .223524-02 Q1( 3, 8)= .294344-02
Q1( 3,11)=-.614753-03 Q1( 3,12)= .291756-02 Q1( 3,13)= .272743-02
Q1( 4, 4)= .101000+01 Q1( 5, 5)= .250000+02 Q1( 6, 1)=-.250006+02
Q1( 6, 2)= .229724-02 Q1( 6, 3)= .222001-02 Q1( 6, 6)= .510011+02
Q1( 6, 7)= .179245-02 Q1( 6, 8)= .186799-02 Q1( 6,11)=-.250005+02
Q1( 6,12)= .157011-02 Q1( 6,13)= .141504-02 Q1( 7, 1)=-.100986-02
Q1( 7, 2)= .229834-02 Q1( 7, 3)= .223524-02 Q1( 7, 6)= .179245-02
Q1( 7, 7)= .290031+02 Q1( 7, 8)= .318194-02 Q1( 7,11)=-.782592-03
Q1( 7,12)= .145346-02 Q1( 7,13)= .131933-02 Q1( 8, 1)=-.105337-02
Q1( 8, 2)= .303722-02 Q1( 8, 3)= .294344-02 Q1( 8, 6)= .186799-02
Q1( 8, 7)= .318194-02 Q1( 8, 8)= .432892-01 Q1( 8,11)=-.814618-03
Q1( 8,12)= .187906-02 Q1( 8,13)= .170083-02 Q1( 9, 9)= .101000+01
Q1(10,10)= .260000+02 Q1(11, 1)=-.249998+02 Q1(11, 2)=-.628426-03
Q1(11, 3)=-.614753-03 Q1(11, 6)=-.250005+02 Q1(11, 7)=-.782592-03
Q1(11, 8)=-.814618-03 Q1(11,11)= .510002+02 Q1(11,12)=-.141317-02
Q1(11,13)=-.126364-02 Q1(12, 1)=-.156940-03 Q1(12, 2)= .284791-02
Q1(12, 3)= .291756-02 Q1(12, 6)= .157011-02 Q1(12, 7)= .145346-02
Q1(12, 8)= .187906-02 Q1(12,11)=-.141317-02 Q1(12,12)= .290431+02
Q1(12,13)= .383853-01 Q1(13, 1)=-.151397-03 Q1(13, 2)= .267140-02
Q1(13, 3)= .272743-02 Q1(13, 6)= .141504-02 Q1(13, 7)= .131933-02
Q1(13, 8)= .170083-02 Q1(13,11)=-.126364-02 Q1(13,12)= .383853-01
Q1(13,13)= .741525-01 Q1(14,14)= .101000+01 Q1(15,15)= .250000+02

  87 NONZERO ELEMENTS
 138 ZERO ELEMENTS
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Q2-MATRIX, PAGE 1

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Q2( 1, 1)= .160000+02 Q2( 2, 2)= .100000+05 Q2( 3, 3)= .160000+02
Q2( 4, 4)= .100000+05 Q2( 5, 5)= .100000+05 Q2( 6, 6)= .160000+02
Q2( 7, 7)= .100000+05

   7 NONZERO ELEMENTS
  42 ZERO ELEMENTS
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S-MATRIX, PAGE 1

S(1, 1)= .203860+02	S(1, 2)= .101585+04	S(1, 3)= .743783+02
S(1, 4)= .302601+02	S(1, 5)= .715187+02	S(1, 6)=-.384797+01
S(1, 7)= .766209+02	S(1, 8)= .755451+01	S(1, 9)= .446757+01
S(1,10)= .807461+02	S(1,11)=-.865898+01	S(1,12)= .960728+02
S(1,13)= .143991+03	S(1,14)= .360370+02	S(1,15)= .142612+03
S(2, 1)= .101585+04	S(2, 2)= .935765+05	S(2, 3)= .101740+05
S(2, 4)= .303696+04	S(2, 5)= .111942+05	S(2, 6)=-.418587+02
S(2, 7)=-.147338+04	S(2, 8)= .461763+03	S(2, 9)= .419858+03
S(2,10)= .678545+04	S(2,11)=-.276443+03	S(2,12)=-.120050+05
S(2,13)= .949794+04	S(2,14)= .318795+04	S(2,15)= .627374+04
S(3, 1)= .743785+02	S(3, 2)= .101740+05	S(3, 3)= .325859+04
S(3, 4)= .135880+04	S(3, 5)= .260531+04	S(3, 6)= .400279+02
S(3, 7)= .322374+04	S(3, 8)= .106886+03	S(3, 9)= .109112+03
S(3,10)= .185506+04	S(3,11)= .443203+02	S(3,12)= .620256+04
S(3,13)= .188243+04	S(3,14)= .814177+03	S(3,15)= .126421+04
S(4, 1)= .302601+02	S(4, 2)= .303696+04	S(4, 3)= .135880+04
S(4, 4)= .106988+04	S(4, 5)= .806525+03	S(4, 6)= .149514+02
S(4, 7)= .156623+04	S(4, 8)= .647538+02	S(4, 9)= .670827+02
S(4,10)= .108110+04	S(4,11)= .128997+02	S(4,12)= .378373+04
S(4,13)= .912253+03	S(4,14)= .392927+03	S(4,15)= .759418+03
S(5, 1)= .715188+02	S(5, 2)= .111942+05	S(5, 3)= .260531+04
S(5, 4)= .806525+03	S(5, 5)= .251845+04	S(5, 6)= .341937+02
S(5, 7)= .205315+04	S(5, 8)= .777058+02	S(5, 9)= .875639+02
S(5,10)= .140309+04	S(5,11)= .348995+02	S(5,12)= .219265+04
S(5,13)= .149463+04	S(5,14)= .647988+03	S(5,15)= .746650+03
S(6, 1)=-.384796+01	S(6, 2)=-.418586+02	S(6, 3)= .400279+02
S(6, 4)= .149514+02	S(6, 5)= .341937+02	S(6, 6)= .144613+02
S(6, 7)= .504360+03	S(6, 8)=-.337133+01	S(6, 9)= .954338+00
S(6,10)= .101177+02	S(6,11)=-.753854+01	S(6,12)= .668095+02
S(6,13)= .690354+02	S(6,14)= .151486+02	S(6,15)= .795090+02
S(7, 1)= .766208+02	S(7, 2)=-.147338+04	S(7, 3)= .322374+04
S(7, 4)= .156623+04	S(7, 5)= .205315+04	S(7, 6)= .504360+03
S(7, 7)= .605327+05	S(7, 8)= .622454+03	S(7, 9)= .154657+03
S(7,10)= .422973+04	S(7,11)=-.266101+03	S(7,12)=-.171835+05
S(7,13)= .439026+04	S(7,14)= .148996+04	S(7,15)= .289523+04
S(8, 1)= .755449+01	S(8, 2)= .461765+03	S(8, 3)= .106886+03
S(8, 4)= .647538+02	S(8, 5)= .777060+02	S(8, 6)=-.337130+01
S(8, 7)= .622454+03	S(8, 8)= .652021+02	S(8, 9)= .418016+02
S(8,10)= .440490+03	S(8,11)= .623551+01	S(8,12)= .467503+02
S(8,13)= .243401+02	S(8,14)= .350393+02	S(8,15)=-.190228+02
S(9, 1)= .446757+01	S(9, 2)= .419858+03	S(9, 3)= .109112+03
S(9, 4)= .670827+02	S(9, 5)= .875639+02	S(9, 6)= .954339+00
S(9, 7)= .154657+03	S(9, 8)= .418016+02	S(9, 9)= .102356+03
S(9,10)= .784217+03	S(9,11)= .200922+01	S(9,12)= .491126+03
S(9,13)= .820370+02	S(9,14)= .291089+02	S(9,15)= .878113+02
S(10, 1)= .807461+02	S(10, 2)= .678545+04	S(10, 3)= .185506+04
S(10, 4)= .108110+04	S(10, 5)= .140309+04	S(10, 6)= .101177+02
S(10, 7)= .422973+04	S(10, 8)= .440491+03	S(10, 9)= .784217+03
S(10,10)= .164577+05	S(10,11)= .338548+02	S(10,12)= .616678+04
S(10,13)= .167598+04	S(10,14)= .781880+03	S(10,15)= .128280+04
S(11, 1)=-.865900+01	S(11, 2)=-.276444+03	S(11, 3)= .443203+02
S(11, 4)= .128997+02	S(11, 5)= .348993+02	S(11, 6)=-.753853+01
S(11, 7)=-.266101+03	S(11, 8)= .623556+01	S(11, 9)= .200922+01
S(11,10)= .338549+02	S(11,11)= .188228+02	S(11,12)= .650976+03
S(11,13)=-.444694+02	S(11,14)=-.155470+01	S(11,15)=-.573370+02
S(12, 1)= .960717+02	S(12, 2)=-.120051+05	S(12, 3)= .620255+04
S(12, 4)= .378373+04	S(12, 5)= .219264+04	S(12, 6)= .668082+02
S(12, 7)=-.171835+05	S(12, 8)= .467518+02	S(12, 9)= .491126+03
S(12,10)= .616679+04	S(12,11)= .650982+03	S(12,12)= .216110+06
S(12,13)= .192296+05	S(12,14)= .312826+04	S(12,15)= .288912+05

S-MATRIX, PAGE 2

S(13, 1)= .143991+03	S(13, 2)= .949794+04	S(13, 3)= .188243+04
S(13, 4)= .912253+03	S(13, 5)= .149463+04	S(13, 6)= .690351+02
S(13, 7)= .439026+04	S(13, 8)= .243399+02	S(13, 9)= .820370+02
S(13,10)= .167598+04	S(13,11)=-.444685+02	S(13,12)= .192298+05
S(13,13)= .559696+04	S(13,14)= .172711+04	S(13,15)= .517351+04
S(14, 1)= .360370+02	S(14, 2)= .318795+04	S(14, 3)= .814177+03
S(14, 4)= .392927+03	S(14, 5)= .647988+03	S(14, 6)= .151486+02
S(14, 7)= .148996+04	S(14, 8)= .350393+02	S(14, 9)= .291089+02
S(14,10)= .781880+03	S(14,11)=-.155470+01	S(14,12)= .312826+04
S(14,13)= .172711+04	S(14,14)= .120020+04	S(14,15)= .958286+03
S(15, 1)= .142612+03	S(15, 2)= .627374+04	S(15, 3)= .126421+04
S(15, 4)= .759418+03	S(15, 5)= .746649+03	S(15, 6)= .795088+02
S(15, 7)= .289523+04	S(15, 8)=-.190225+02	S(15, 9)= .878113+02
S(15,10)= .128280+04	S(15,11)=-.573360+02	S(15,12)= .288912+05
S(15,13)= .517351+04	S(15,14)= .958286+03	S(15,15)= .647548+04

225 NONZERO ELEMENTS
0 ZERO ELEMENTS

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L-MATRIX, PAGE 1

L( 1, 1)= .145481+00  L( 1, 2)= .146007+02  L( 1, 3)= .653267+01
L( 1, 4)= .514367+01  L( 1, 5)= .387752+01  L( 1, 6)= .718818-01
L( 1, 7)= .752995+01  L( 1, 8)= .311316+00  L( 1, 9)= .322513+00
L( 1,10)= .519761+01  L( 1,11)= .620178-01  L( 1,12)= .181910+02
L( 1,13)= .438583+01  L( 1,14)= .188907+01  L( 1,15)= .365105+01
L( 2, 1)=-.130186-01  L( 2, 2)=-.550412+00  L( 2, 3)= .135702+00
L( 2, 4)= .445685-01  L( 2, 5)= .100274+00  L( 2, 6)= .576800-02
L( 2, 7)= .322695+00  L( 2, 8)= .448535-03  L( 2, 9)= .278974-02
L( 2,10)= .433868-01  L( 2,11)= .112001-01  L( 2,12)= .581754+00
L( 2,13)=-.537885-02  L( 2,14)= .188049-01  L( 2,15)=-.374450-01
L( 3, 1)= .279223-01  L( 3, 2)= .262411+01  L( 3, 3)= .681950+00
L( 3, 4)= .419267+00  L( 3, 5)= .547274+00  L( 3, 6)= .596462-02
L( 3, 7)= .966605+00  L( 3, 8)= .261260+00  L( 3, 9)= .639725+00
L( 3,10)= .490135+01  L( 3,11)= .125576-01  L( 3,12)= .306954+01
L( 3,13)= .512731+00  L( 3,14)= .181930+00  L( 3,15)= .548821+00
L( 4, 1)= .117973-02  L( 4, 2)=-.292205-01  L( 4, 3)= .508446-01
L( 4, 4)= .245616-01  L( 4, 5)= .322084-01  L( 4, 6)= .817428-02
L( 4, 7)= .978703+00  L( 4, 8)= .975280-02  L( 4, 9)= .189437-02
L( 4,10)= .557202-01  L( 4,11)=-.434347-02  L( 4,12)=-.283600+00
L( 4,13)= .699094-01  L( 4,14)= .235587-01  L( 4,15)= .459637-01
L( 5, 1)= .589446-04  L( 5, 2)= .495338-02  L( 5, 3)= .135419-02
L( 5, 4)= .789205-03  L( 5, 5)= .102425-02  L( 5, 6)= .738593-05
L( 5, 7)= .308770-02  L( 5, 8)= .321558-03  L( 5, 9)= .572478-03
L( 5,10)= .120141-01  L( 5,11)= .247140-04  L( 5,12)= .450175-02
L( 5,13)= .122347-02  L( 5,14)= .570772-03  L( 5,15)= .936447-03
L( 6, 1)= .173255+00  L( 6, 2)= .153267+02  L( 6, 3)= .391431+01
L( 6, 4)= .188907+01  L( 6, 5)= .311533+01  L( 6, 6)= .728300-01
L( 6, 7)= .716327+01  L( 6, 8)= .168458+00  L( 6, 9)= .139947+00
L( 6,10)= .375904+01  L( 6,11)=-.747451-02  L( 6,12)= .150397+02
L( 6,13)= .830344+01  L( 6,14)= .577019+01  L( 6,15)= .460714+01
L( 7, 1)= .233525-01  L( 7, 2)= .150466+01  L( 7, 3)= .496477-01
L( 7, 4)= .281233-01  L( 7, 5)= .725717-01  L( 7, 6)= .126299-01
L( 7, 7)= .103680+01  L( 7, 8)=-.486669-02  L( 7, 9)= .167516-02
L( 7,10)= .541131-01  L( 7,11)=-.296693-01  L( 7,12)=-.105230+01
L( 7,13)= .384120+00  L( 7,14)= .839044-01  L( 7,15)= .338981+00

 105 NONZERO ELEMENTS
   0 ZERO ELEMENTS
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