Robust Control under Parametric Uncertainty via Primal-Dual Convex Analysis

Ghulchak, Andrey; Rantzer, Anders

Published in:
IEEE Transactions on Automatic Control

DOI:
10.1109/9.995040

2002

Robust Control Under Parametric Uncertainty Via Primal-Dual Convex Analysis

Andrey Ghulchak and Anders Rantzer

***Abstract*—A numerical method is proposed for optimal robust control synthesis. The method applies to the case when the coefficients of the characteristic polynomial depend *linearly* on the uncertain parameters. A primal/dual pair of infinite-dimensional convex problems is solved by successive finite-dimensional approximations. The primal/dual pair has no duality gap, and both upper and lower bounds produced by the approximations converge monotonically to the optimal value.**

***Index Terms*—Convex optimization, duality, finite-approximation, parametric uncertainty, robust stabilization.**

I. Notation

The following notation will be used throughout the note. By $\mathbb{T}$, the unit circle in the complex plane is denoted. The notation $|\cdot|_p$ stands for a vector norm in $\mathbb{R}^n$ (p for *primary*), and the dual norm is denoted by $|\cdot|_d$, i.e., $|x|_d = \sup\{\delta^T x: |\delta|_p \le 1\}$. The notation $\mathbf{L}^q$ denotes the standard Lebesgue space on $\mathbb{T}$, with the $\mathbf{L}^q$ norm referred to as $\|\cdot\|_q$. The Hardy space $\mathbf{H}^q$ consists of functions that are analytical in the open unit disc and belong to $\mathbf{L}^q$ on $\mathbb{T}$. The space $\mathbf{RH}^q$ contains all real-rational functions of $\mathbf{H}^q$, and $\mathbf{H}_0^q = z\mathbf{H}^q = \{f \in \mathbf{H}^q: f(0) = 0\}$ is the shifted $\mathbf{H}^q$ space. The prefix $\mathcal{B}$ refers to the unit ball.

II. Introduction

During the last decade, much progress has been made in robustness analysis of linear time-invariant systems with uncertainties [1], [3], [14], [18], [19]. In contrast, few results have addressed *synthesis* of the control systems with parametric uncertainties. The classical methods of controller design, such as the root locus and the frequency response methods, have been extended to uncertain linear systems in a number of papers [2], [4], [6], [13]. In more general situations, different heuristic methods like "D-K iteration" [7] or "QFT" [12] have been proposed.

However, there is still a lack of a nonconservative and easy-to-handle design procedure for systems with real parametric uncertainty. The synthesis problem has turned out to be very hard. In general, a real-valued uncertainty is harder to deal with than a complex one [9], [15].

Recently, a large number of analysis and synthesis problems in robust control have been stated in terms of convex optimization. This gives great benefits both for theoretical analysis and for practical computations. In particular, it has been shown in [17] that the robust stabilization problem under parametric uncertainties has a convex formulation if the characteristic polynomial depends linearly on the uncertain parameters (so called *rank one* problem). The authors consider the uncertainty as an artificial feedback loop

$$G_\delta: \begin{cases} \begin{pmatrix} y \\ z \end{pmatrix} = G(s) \begin{pmatrix} u \\ w \end{pmatrix} \\ w = \delta^T z \end{cases} \tag{1}$$

where $G(s)$ is the nominal plant, w is the *scalar* input and δ is the uncertain vector in $\mathbb{R}^m$. The objective is to robustly stabilize the plant (1) for all real $\delta \in \mathbb{R}^m$ satisfying the norm bound $|\delta|_p \le \nu$. As pointed out in [17] one can also add a complex uncertainty to treat performance specifications in the same framework.

All closed-loop transfer functions from w to z attainable by nominally stabilizing controllers are of the form $T_{zw} = T_1 + T_2 Q$ where Q is stable and T_1 and T_2 are determined by G. (Note that there is no T_3 term in rank-one case). The condition for robust stability becomes

$$[1 + \delta^T(T_1 + T_2 Q)]^{-1} \in \mathbf{H}^\infty, \quad \forall\, \delta: |\delta|_p \le \nu.$$

A convex parameterization of all robustly stabilizing controllers was found in [17] as follows.

Proposition 1: Suppose $T_1 \in \mathbf{RH}^\infty_{m\times 1}, T_2 \in \mathbf{RH}^\infty_{m\times n}$. Then, the following two conditions on $Q \in \mathbf{RH}^\infty_{n\times 1}$ are equivalent:

1) $[1 + \delta^T(T_1 + T_2 Q)]^{-1} \in \mathbf{RH}^\infty$ for all $\delta \in \mathbb{R}^m$ with $|\delta|_p \le \nu$;
2) there exist $\alpha \in \mathbf{RH}^\infty$ and $\beta \in \mathbf{RH}^\infty_{n\times 1}$ such that $Q = \beta/\alpha$ and

$$\nu |\mathrm{Re}\,[T_1\alpha + T_2\beta](z)|_d < \mathrm{Re}\,\alpha(z), \quad \forall\, z \in \mathbb{T}. \tag{2}$$

Remark: In the following, we will omit indexes denoting the size of matrix functions, which is usually clear from context.

The main issue of this note is to develop a convex programming algorithm that solves the problem (2) for the maximum possible ν. The algorithm is a combination of two finite-dimensional approximations of the primal and dual infinite-dimensional problems. It produces lower and upper bounds on the optimal uncertainty norm bound ν_{opt} and gives a robustly stabilizing controller with any prespecified level of suboptimality.

The note is organized as follows. In Section III, we derive the primal convex programming algorithm in case the uncertainty norm bound ν is given. The dual convex programming algorithm is proposed in Section IV. Section V refers to the important case when the uncertainty set is a polytope. The numerical example is considered in Section VI.

III. Convex Programming Algorithm for a Given Uncertainty Bound

A. The Primal Problem

The following problem is of our main interest in the note.

Primal Problem: Given $\nu > 0$ and $F,\, G \in \mathbf{RH}^\infty$, find a function $h \in \mathbf{RH}^\infty$ such that

$$\mathcal{J}(h, z) = \mathrm{Re}\,(F(z)h(z)) - \nu|\mathrm{Re}\,(G(z)h(z))|_d > 0, \quad \forall\, z \in \mathbb{T}. \tag{3}$$

The problem (2) takes this form if we define $F = (\,1 \quad 0 \quad \ldots \quad 0\,) \in \mathbb{R}^{n+1}$, $G = (\,T_1 \quad T_2\,) \in \mathbf{RH}^\infty$ and $h = [\alpha; \beta]$. If the set of solutions is nonempty, ν is a lower bound for the optimal norm bound

$$\nu_{\mathrm{opt}} = \sup\{\nu | \exists\, h \in \mathbf{RH}^\infty: \mathcal{J}(h, z) > 0, \forall\, z \in \mathbb{T}\}. \tag{4}$$

We can construct a finite-dimensional approximation by solving the problem on a finite-dimensional subspace of $\mathbf{RH}^\infty$ and on a finite grid of points $z \in \mathbb{T}$.

Let $\{\phi_i\}_{i=0}^{+\infty}$ be a (Schauder) basis of the space $\mathbf{RH}^\infty$ of scalar functions (say, $\phi_i(z) = z^i$). Then, the real span of the first N functions $\{\phi_i\}_{i=0}^{N-1}$

$$\mathcal{H}_N = \left\{ h \mid h(z) = \sum_{i=0}^{N-1} h_i \phi_i(z),\ h_i \in \mathbb{R}^{n+1} \right\} \tag{5}$$

Manuscript received October 1, 2000; revised December 11, 2001. Recommended by Associate Editor V. Balakrishnan.

The authors are with the Department of Automatic Control, Lund Institute of Technology (LTH), 22100 Lund, Sweden (e-mail: ghulchak@control.lth.se; rantzer@control.lth.se).

Publisher Item Identifier S 0018-9286(02)03741-8.

forms an $N(n+1)$th-dimensional subspace of $\mathbf{RH}^\infty$. Consider a finite grid of points $\mathcal{Z}_K = \{z_k\}_{k=1}^K$ of the upper half of the unit circle. The condition (3) for a function $h \in \mathcal{H}_N$ over the grid $\mathcal{Z}_K$ takes the form

$$\mathcal{J}\left(\{h_i\}_{i=0}^{N-1}, z_k\right) > 0 \quad \forall\, z_k \in \mathcal{Z}_K. \tag{6}$$

We suggest the following scheme.

1) Take $N = 1$, $K \geq 2$ and $\{0, \pi\} \subset \mathcal{Z}_K$.
2) Find a function h for given N, K as a solution to (6). If the problem is infeasible then $N := N + 1$ and repeat.
3) Check the condition (3) for *all* z in the upper half of $\mathbb{T}$. If it does not hold, increase K by adding some of "bad" points to the set $\mathcal{Z}_K$ and go to Step 2), otherwise STOP.

The main numerical questions here are: a) how to check if (3) holds for all z, and b) how to refine the frequency grid (increase K) at Step 3)? The questions will be treated Section III-B.

B. Modification of the Primal Algorithm and Related Numerical Issues

Consider the following modification of the algorithm. Let $H_N \subset \mathbb{R}^{N(n+1)}$ be a convex bounded set containing a neighborhood of the origin. Then, $\{h_i\}_{i=0}^{N-1} \in H_N$ implies

$$\left\|\sum_{i=0}^{N-1} h_i z^i\right\|_\infty \leq C_N \qquad \left\|\sum_{i=1}^{N-1} i h_i z^i\right\|_\infty \leq C'_N$$

where the constants C_N and C'_N do not depend on $\{h_i\}_{i=0}^{N-1}$. For example, the convex set

$$\left\{\{h_i\}_{i=0}^{N-1} \mid \sum_{i=0}^{N-1} |h_i| \leq C_N\right\}$$

can be chosen as H_N. Let us fix a tolerance $\varepsilon_0 > 0$ and replace Step 2) with 2$'$. Find a function h for given N, K as a solution to

$$\varepsilon_{\max} = \max\left\{\varepsilon \mid \{h_i\}_{i=0}^{N-1} \in H_N, \mathcal{J}\left(\{h_i\}_{i=0}^{N-1}, z_k\right) \geq \varepsilon, \forall\, z_k \in \mathcal{Z}_K\right\}. \tag{7}$$

If $\varepsilon_{\max} \leq \varepsilon_0$ then $N := N + 1$ and repeat.

The idea to introduce the set H_N is to obtain the uniform boundedness of the solution $h(z)$ and its derivative $h'(z)$ independently on the coefficients $\{h_i\}_{i=0}^{N-1}$. It guarantees that the function $J(t) = \mathcal{J}(h(e^{jt}), e^{jt})$ does not vary very fast, which makes it possible to conclude its global positiveness from values at a grid.

Theorem 1: Let $\nu < \nu_{\text{opt}}$. Then

1) there exists an $N < +\infty$ such that a solution to the primal problem can be found as $\sum_{i=0}^{N-1} h_i z^i$ with $\{h_i\}_{i=0}^{N-1} \in H_N$;
2) for each N, (3) holds for all $z \in \mathbb{T}$ if (7) holds for the grid $\mathcal{Z}_K = \{e^{jt_k}\}_{k=1}^K$ that satisfies

$$0 = t_1 \leq t_2 \leq \cdots \leq t_K = \pi, \quad |t_{k+1} - t_k| \leq \frac{\varepsilon_{\max}}{M_N} \tag{8}$$

where

$$M_N = \|F'\|_\infty C_N + \|F\|_\infty C'_N + \sup_{|\delta|_p \leq \nu} |\delta|(\|G'\|_\infty C_N + \|G\|_\infty C'_N).$$

Proof:

1) The claim follows easily from the fact that the polynomials are uniformly dense in $\mathbf{RH}^\infty$ and the set of all solutions is a cone.
2) Denote $J_\delta(t) = \operatorname{Re} F(e^{jt})h(e^{jt}) + \delta^T \operatorname{Re} G(e^{jt})h(e^{jt})$. Then

$$\left|\frac{dJ_\delta}{dt}\right| = \left|\operatorname{Im}[F'h + Fh']\, e^{jt} + \delta^T \operatorname{Im}\, [G'h + Gh']\, e^{jt}\right| \leq M_N$$

for all $|\delta|_p \leq \nu$ and for all $h \in \mathcal{H}_N$ with $\{h_i\}_{i=0}^{N-1} \in H_N$. Hence, for a grid that satisfies (8), we have

$$J_\delta(t) = J_\delta(t_k) + \int_{t_k}^t \frac{dJ_\delta}{ds}(s)ds \geq J_\delta(t_k) - M_N(t - t_k) \geq J_\delta(t_k) - \varepsilon_0$$

for all $t \in [0, \pi]$. Finally, $J(t) \geq J(t_k) - \varepsilon_0 > 0$. □

Thus for every N, the modified algorithm takes at most $K_{\max}(N) = [\pi M_N/\varepsilon_0] + 1$ points. However, the actual number K depends on a grid refinement strategy and usually is much less than $K_{\max}$ in practice. A rather obvious idea of a good refinement is not to add new points where the function $J(t)$ is already large. One possible choice of "bad" points to add at Step 3) is the local negative minima of $J(t)$ calculated, for instance, with the (low) accuracy ε_0/M_N. Another reasonable solution is to use the function $\|J\|_\infty - J$ (properly scaled) as a distribution density for the new grid. So we add more points where the function J is small.

IV. Duality. Optimization of the Uncertainty Bound Via Primal and Dual Problems

A. The Dual Problem

A feasible solution to the primal problem gives a lower bound ν to the optimal value ν_{opt}. Conversely, if a given ν is a lower bound of ν_{opt}, the proposed algorithm finds a feasible solution in finite number of steps. However, the algorithm is unable to determine if $\nu > \nu_{\text{opt}}$ since at each step we solve a finite-dimensional approximation, and the finite-dimensional infeasibility does not imply that of the original problem. In this section, we use the duality result to obtain an upper bound. The next theorem is extracted from [10] and [11].

Theorem 2: The optimal value ν_{opt} from (4) has the following dual representation $\nu_{\text{opt}} = \min\{\nu_{\text{opt}|c}, \nu_{\text{opt}|s}\}$ where:

$$\begin{aligned}\nu_{\text{opt}|c} &= \inf\{\nu \mid w \in \mathbf{L}^1 \setminus \{0\}, w(z) \geq 0, \delta(z) \in \mathbb{R}^m, |\delta(z)|_p \\ &\qquad \leq \nu\colon \left(F + \delta^T G\right) w \in \mathbf{H}_0^1\} \\ \nu_{\text{opt}|s} &= \inf\{|\delta|_p \mid \exists\, z \in \mathbb{T}, \delta \in \mathbb{R}^m\colon F(z) + \delta^T G(z) = 0\}.\end{aligned}$$

Calculation of an upper bound for $\nu_{\text{opt}|s}$ can be organized relatively easy as a finite-dimensional convex programming at each z, followed by sweeping out the unit half circle. The upper bound on $\nu_{\text{opt}|c}$ is given by the following corollary (the proof is trivial by putting $x = w\delta$).

Corollary 1: The number $\nu \geq 0$ is the upper bound of $\nu_{\text{opt}|c}$ iff there exist real functions x, $w \in \mathbf{L}^1$ and a complex function $h \in \mathbf{H}_0^1$ that satisfy the condition

$$\begin{aligned}|x(z)|_p &\leq \nu w(z) \\ w(z)F(z) + x(z)^T G(z) &= h(z) \\ \|w\|_1 &> 0.\end{aligned} \tag{9}$$

The condition (9) is linear in w and h and convex in x. Applying ideas similar to those of Section III-A we can obtain a finite-dimensional approximation of this problem in terms of convex programming. Let $\{\phi_i\}_{i=1}^{+\infty}$ be a (Schauder) basis of the space $\mathbf{H}_0^1$ (say $\phi_i(z) = z^i$). Then, we approximate h by the series expansion $h(z) = \sum_{i=1}^N h_i\phi_i(z)$ and consider the condition (9) over a finite grid $\mathcal{Z}_K = \{z_k\}_{k=1}^K$ of the upper half of $\mathbb{T}$ to get

$$\begin{aligned}\mathcal{J}_{\rm ineq}(x_k, w_k) &\le 0, \quad \forall\, k\\ \mathcal{J}_{\rm eq}(x_k, w_k, \{h_i\}_{i=1}^N, z_k) &= 0, \quad \forall\, k\\ \frac{1}{K}\sum_{k=1}^K w_k &> 0.\end{aligned} \tag{10}$$

Our optimization variable contains all the coefficients h_i as well as w_k and x_k—the pointwise values of w, x on $\mathcal{Z}_K$. An implementation of the dual algorithm to estimate $\nu_{{\rm opt}|c}$ may be as follows.

1) For given N and $K > N(n+1)$ solve the convex problem (10). If it is infeasible, then $N := N+1$ and repeat.
2) Check the condition (9) for all $z \in \mathbb{T}$. If it does not hold, increase K by refining the grid $\mathcal{Z}_K$ and go to Step 1), otherwise STOP.

Summing up, the problem of stability radius optimization may be solved by the finite-dimensional approximations to primal and dual problems in parallel. Both approximations can be implemented as the standard convex programming. For sufficiently big N and K, either the primal or the dual algorithm finds a solution, and we can obtain an arbitrarily good approximation of $\nu_{\rm opt}$ by decreasing the gap between the lower and upper bounds.

B. Numerical Issues for the Dual Algorithm

First, let us briefly outline numerical difficulties related to calculation of $\nu_{s|{\rm opt}} = \min_{z\in\mathbb{T}} \nu_s(z)$ where

$$\nu_s(z) = \inf\{|\delta|_p \colon A(z)\delta = b(z)\} \tag{11}$$

with $A = [{\rm Re}\,(G)\,{\rm Im}\,(G)]^T$ and $b = [{\rm Re}\,(F)\,{\rm Im}\,(F)]^T$. The problem (11) is similar to that considered in [16]. The difference is that in [16] the matrix A has only two rows whereas in our case $A \in \mathbb{R}^{2(n+1)\times m}$. The main numerical problem here is that the function $\nu_s(z)$ is not continuous when z goes along $\mathbb{T}$, and a search for the global optimum over a grid can easily miss it. The following result similar to [16, Lemma 1] shows that $\nu_s(z)$ is piecewise continuous, with the points of possible discontinuity being described explicitly.

Lemma 1: Let A be a continuous $n_A \times m_A$ matrix function on $\mathbb{T}$ and $0 \le r \le \min(n_A, m_A)$. Then ν_s in (11) is continuous on $\mathbb{T}_r = \{z \in \mathbb{T} \colon {\rm rank}\,(A(z)) = {\rm rank}\,([A(z)\,b(z)]) = r\}$.

Proof: The statement is rather obvious by the following geometrical interpretation. The function $\nu_s(z)$ is the $|\cdot|_p$-distance in $\mathbb{R}^{m_A}$ from the origin to the affine subspace $M_z = \{\delta \in \mathbb{R}^{m_A} \colon A(z)\delta = b(z)\}$. If $z \in \mathbb{T}_r$ then $\dim(M_z) = m_A - r$. Since the function A is continuous, the subspace M_z moves continuously, so the distance is a continuous function. □

The rest of the section is devoted to the second step of the dual algorithm for $\nu_{{\rm opt}|c}$. Since we find the functions x and w only at finite grid $\mathcal{Z}_K$, we should extrapolate their values to all other points subject to the condition (10). This is the main difficulty since the equality is not likely to hold at other points for any choice of real vectors. So we are not able to find proper candidates for the pointwise values $x(z)$ and $w(z)$ between the grid points to satisfy the equality exactly. Let us introduce the pointwise approximation error

$$E(z) = \min_{x,w}\{|wF(z) + x^T G(z) - h(z)| \colon |x|_p \le \nu w,\ x \in \mathbb{R}^m,\ w \in \mathbb{R}\}. \tag{12}$$

Since we have found h on Step 1), the calculation of E at each z becomes a low-dimensional convex programming. We know that $E(z)$ is zero at $z \in \mathcal{Z}_K$ and should be zero for all $z \in \mathbb{T}$ for h to be a solution to the dual problem. To estimate E, the same ideas from Section III-B can be used. First, because the set of all solutions is a cone, we can impose the constraint $\{h_i\}_{i=1}^N \in H_N$ at Step 1) (in order to guarantee boundedness of the derivative of $E(e^{jt})$) and maximize ε subject to $\nu w - |x|_p \ge \varepsilon$. Second, a similar grid refinement strategy of adding those points where $E(z)$ is large can be used. Finally, we can decide that Step 2) is successfully done if $E(z)$ is around zero within a small tolerance.

V. Linear Optimization if the Uncertainty Set Is a Polytope

An important case arises when the uncertainty set is a polytope. In this case, both the primal and dual conditions are linear.

Suppose that the unit ball $\{|x|_p \le 1\}$ is a polytope. Then the polar unit ball $\{|y|_d \le 1\}$ is also a polytope, and for any $a, b \ge 0$, both conditions $|x|_p \le a$ and $|y|_d \le b$ have the form of linear inequalities.

Hence, the primal finite-dimensional approximation problem (6) can be reduced to a system of linear inequalities in the form

$$A_{KN} X_N \succ 0 \tag{13}$$

where $X_N = \{h_i\}_{i=0}^{N-1} \in \mathbb{R}^{N(n+1)}$ and A_{KN} is a real matrix. Thus, finding a function $h \in \mathcal{H}_N$ which satisfies (3) at the grid $\mathcal{Z}_K$ is the standard LP feasibility problem.

For the dual condition (10) the situation is the same. It can be reduced to

$$\begin{aligned}A_{KN}^{(1)} X_{KN} &\preceq 0\\ A_{KN}^{(2)} X_{KN} &= 0.\end{aligned} \tag{14}$$

Here, the vector $X_{KN} \in \mathbb{R}^{(m+1)K+(n+1)N}$ contains $\{x(z_k)\}_{k=1}^K$, $\{w(z_k)\}_{k=1}^K$, and $\{h_i\}_{i=1}^N$.

VI. Example: Robust Performance Problem for a Mechanical System With Resonance

A. The Problem Statement

Consider a system of two masses connected by a spring (see Fig. 1). A simple mathematical model of the system is

$$\begin{aligned}m_1\ddot{y}_1 + c_1\dot{y}_1 + k(y_1 - y_2) &= u,\\ m_2\ddot{y}_2 + c_2\dot{y}_2 + k(y_2 - y_1) &= 0\end{aligned}$$

where m_i is the i-th mass, c_i is the damping coefficient for the i-th mass, y_i is the position of the i-th mass, k is the spring constant and u is the control force. The position of the second mass y_2 is assumed to be measurable.

Denoting $g_i(s) = m_i s^2 + c_i s + k$, $i = 1, 2$, the system can be rewritten as

$$y_2 = \frac{k}{g_1 g_2 - k^2} u = Gu. \tag{15}$$

Suppose that our plant G contains a real parametric uncertainty δ_0 in the second mass $m_2 + \delta_{01}$ and in the second damping coefficient $c_2 + \delta_{02}$ as well as a complex additive uncertainty due to neglected nonlinear dynamics

$$G_{\delta_0,\Delta_u}(s) = \frac{k}{g_1(s)g_2(s) - k^2 + (s\delta_{01} + \delta_{02})sg_1(s)} + W_u\Delta_u,$$
$$|\delta_{01}| + |\delta_{02}| \le \nu_\delta, |\Delta_u| \le 1$$

and our problem is to find a stabilizing controller with integral action $u = K_{\rm opt}(s)y_2$ that solves the robust performance $\mathbf{H}^\infty$ optimization problem

$$\gamma_{\rm opt} = \inf_K \sup_{|\delta_0|_1 \le \nu_\delta, |\Delta_u| \le 1} \|W_y S_{\delta_0,\Delta_u}\|_\infty \tag{16}$$

for the standard input sensitivity function $S_{\delta_0,\Delta_u} = 1/(1 - G_{\delta_0,\Delta_u} K)$.

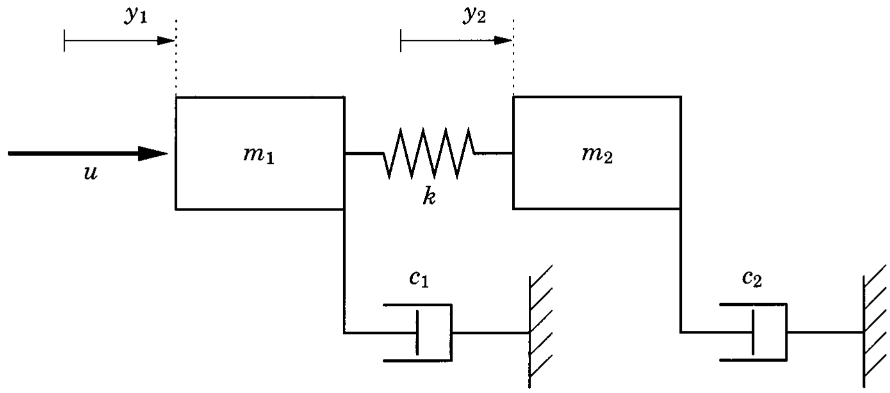

Fig. 1. A schematic representation of the system in the example.

The problem is very difficult and does not fit the method of this note directly. However, a reasonable "convexification" can be performed to obtain a closely related problem that has the necessary quasiconvex form.

B. The Convexification

The problem (16) is equivalent to [8]

$$\nu_{\rm opt} = \sup_K \{\nu_y : \| \nu_y |W_y S_{\delta_0,0}| + |W_u K S_{\delta_0,0}| \|_\infty < 1, \quad \forall\, |\delta_0|_1 \le \nu_\delta\}. \quad (17)$$

Consider a plant representation

$$G_{\delta_0,\Delta_u} = \frac{N + \delta_0^T N_\delta}{M + \delta_0^T M_\delta} + \Delta_u W_u \qquad |\delta_0|_1 \le \nu_\delta, \ \ |\Delta_u| \le 1$$

where N and $M \in \mathbf{H}^\infty$ are the nominal coprime factors of the plant $G_{0,0}$, and W_u, N_δ and $M_\delta \in \mathbf{H}^\infty$ are the perturbation functions. Consider also a controller in the form $K = \beta/(\alpha_0 \alpha)$ where α_0 is the fixed part of the controller (integrator in our case) and $\alpha, \beta \in \mathbf{H}^\infty$. The condition in (17) can be replaced by stability of $\alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha - \left(N + \delta_0^T N_\delta\right) \beta$ plus

$$\nu_y |W_y \alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha| + |W_u \left(M + \delta_0^T M_\delta\right) \beta| < |\alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha - \left(N + \delta_0^T N_\delta\right) \beta|. \quad (18)$$

Since common factors of α and β do not change the controller K, we can use this freedom to remove the imaginary part of right-hand side in (18) and replace (17) with

$$\hat{\nu}_{\rm opt} = \sup_{\alpha,\beta} \Big\{ \nu_y : \nu_y |W_y \alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha| + |W_u \left(M + \delta_0^T M_\delta\right) \beta| < \mathrm{Re}\, \Big(\alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha - \left(N + \delta_0^T N_\delta\right) \beta\Big) \Big\}. \quad (19)$$

Note that the closed-loop stability is included into the condition since the real part of the characteristic polynomial is strictly positive. The problem (19) is conservative in the sense that in general it gives only a lower bound $\hat{\nu}_{\rm opt} \le \nu_{\rm opt}$. However, the gap is very often small in practice and depends on the degree of "nonconvexity on δ_0" of the relation (18). An accurate derivation of this fact is similar to [17] and is omitted for the sake of brevity.

The problem (19) is already quasiconvex and can be solved by a primal-dual convex algorithm followed by a line search. However, we have to simplify it even further for the technical reason that the software we use cannot yet handle the setting (19) for now. So, we have to remove the term $\delta_0^T M_\delta$ from left-hand side of the inequality in (19) and to consider the following problem instead:

$$\sup_{\alpha,\beta \in \mathbf{H}^\infty} \Big\{ \nu_y : \nu_y |W_y \alpha_0 M \alpha| + |W_u M \beta| < \mathrm{Re}\, \Big(\alpha_0 \left(M + \delta_0^T M_\delta\right) \alpha - \left(N + \delta_0^T N_\delta\right) \beta\Big) \Big\}. \quad (20)$$

This corresponds to changing $M + \delta_0^T M_\delta$ to M in the numerator of S_{δ_0,Δ_u}. Intuitively it is clear that it does not affect much the value $\|W_y S_{\delta_0,\Delta_u}\|_\infty$ since cardinal changes of the value are due to the closed-loop poles, *ie* due to the denominator. The problem (20) differs slightly from (3) due to the fact that only the first term of left-hand side of the inequality is scaled by ν_y. However the primal-dual method can be adapted easily (see [11] for a general case) since the primal and the dual problems deal with a *fixed* ν and can handle unequal scaling. Finally, the optimal controller is given by $K = \beta/(\alpha_0 \alpha)$.

C. The Numerical Result

In the numerical example, we take $m_1 = 2.25$ kg, $m_2 = 2.07$ kg, $c_1 = 3.25$ Ns/m, $c_2 = 8.18$ Ns/m and $k = 423$ N/m. The functions N and M are chosen as the normalized coprime factors of the nominal plant, followed by a close zero-pole cancellation in M with the tolerance 0.001 (to reduce numerical errors in the algorithm), that is:

$$\begin{aligned} N &= \frac{90.82}{s^4 + 5.484 s^3 + 398.5 s^2 + 1073 s + 90.87} \\ M &= \frac{s}{s + 0.087\,53} \\ M_\delta &= \begin{pmatrix} s^2 \\ s \end{pmatrix} \frac{0.4831 s^2 + 0.6978 s + 90.82}{s^4 + 5.484 s^3 + 398.5 s^2 + 1073 s + 90.87} \\ N_\delta &= 0. \end{aligned}$$

The parametric uncertainty level ν_δ is chosen to be 0.5. The weighting function W_u is chosen as $(s + 10)/(s + 1000)$ to capture larger uncertainty at high frequencies. The sensitivity weight W_y is chosen as $(s + 1.4)^2/s^2$ to penalize low frequencies up to the sensitivity function peak which happens to be around 2 rad/s. Finally, a fixed factor $\alpha_0 = s/(s + 1)$ is added to α to obtain an integral action in the resulting controller.

Let us make one more minor modification of the problem, namely, in (20) we replace ℓ_2 norm to ℓ_∞ norm (which is maximum of real and imaginary parts) in order to use linear programming as explained in Section V. Again, it does not change the problem much since these two norms are topologically equivalent.

For $\nu_y = 0.424$ the primal algorithm finds a solution (α, β) of order 80. The final grid consists of 225 points. After pole-zero cancellation

$$K = \frac{-346.2777(s+25.55)(s+3.656)(s+0.5069)(s^2+4.028s+494.2)}{s(s+28.6)(s^2+14.1s+75.06)(s^2+3.574s+397.9)}. \tag{21}$$

with the tolerance 0.01 the controller becomes (21), as shown at the top of the page. For $\nu_y = 0.4372$ the dual algorithm finds a solution. Hence, the controller K has a sufficiently good level of suboptimality (around 3%).

Thus we have found the controller which provides us with the value $1/0.424 = 2.3585$ as an approximation of the robust performance bound

$$\gamma = \sup_{|\delta_0|_1 \le 0.5, |\Delta_u| \le 1} \|W_y S_{\delta_0, \Delta_u}\|_\infty .$$

Of course, after several simplifications being made, we must expect that the actual bound is larger. The straightforward calculation of γ for the controller K gives 3.3415 which is not that far away from our result. This is another confirmation that all the simplifications were quite reasonable.

VII. Conclusion

In this note, we have presented a convex primal-dual technique for optimal robust control design in the particular case when uncertain parameters appear linearly in the closed-loop characteristic polynomial (rank-one problem). Both the primal and dual algorithms are based on finite-dimensional convex optimization. Running both algorithms simultaneously, it is possible to find the largest uncertainty bound, that is the maximum allowable perturbation of parameters without losing stability, as well as to design the optimal robust controller. With the uncertainty set chosen as a polytope (approximating the original uncertainty set if it is not), linear optimization can be used to solve the problem by efficient LP solvers.

References

[1] V. Balakrishnan and F. Wang, "Efficient computation of a guaranteed lower bound on the robust stability margin for a class of uncertain systems," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 2185–2190, 1999.

[2] R. B. Barmish and R. Tempo, "The robust root locus," *Automatica*, vol. 26, pp. 283–292, 1990.

[3] C. Boussios and E. Feron, "Estimating the conservatism of Popov's criterion for real parametric uncertainties," *Syst. Control Lett.*, vol. 31, pp. 173–183, 1997.

[4] M. Bozorg and E. M. Nebot, "l^p parameter perturbation and design of robust controllers for linear systems," *Int. J. Control*, vol. 72, no. 3, pp. 267–275, 1999.

[5] M. A. Dahleh and I. J. Diaz-Bobillo, *Control of Uncertain Systems: A Linear Programming Approach*. Upper Saddle River, NJ: Prentice-Hall, 1995.

[6] K. B. Datta and V. V. Patel, "H_∞-based synthesis for a robust controller of interval plants," *Automatica*, vol. 32, no. 11, pp. 1575–1579, 1996.

[7] J. C. Doyle, "Structured uncertainty in control system design," in *Proc. IEEE Conf. Decision Control*, 1985, pp. 260–265.

[8] J. Doyle, B. Francis, and A. Tannenbaum, *Feedback Control Theory*. New York: Macmillan, 1992.

[9] M. Fu, "The real structured singular value is hardly approximable," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 1286–1288, 1997.

[10] A. Ghulchak and A. Rantzer, "Robust controller design via linear programming," in Proc. 38th IEEE Conf. Decision Control, Phoenix, AZ, 1999.

[11] ——, "Duality in $\mathbf{H}^\infty$ cone optimization," SIAM J. Control Optim., submitted for publication.

[12] I. Horowitz, "Survey of quantitative feedback theory," *Int. J. Control*, vol. 53, no. 2, pp. 255–291, 1991.

[13] L. H. Keel and S. P. Bhattacharyya, "Robust parametric classical control," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1524–1530, 1994.

[14] C. Marsh and H. Wei, "Robustness bound for systems with parametric uncertainty," *Automatica*, vol. 32, no. 10, pp. 1447–1453, 1996.

[15] L. Qiu, B. Bernhardsson, A. Rantzer, E. J. Davison, P. M. Young, and J. C. Doyle, "A formula for computation of the real stability radius," *Automatica*, vol. 31, no. 6, pp. 879–890, 1995.

[16] L. Qiu and E. J. Davison, "A unified approach for the stability robustness of polynomials in a convex set," *Automatica*, vol. 28, no. 5, pp. 945–959, 1992.

[17] A. Rantzer and A. Megretski, "A convex parameterization of robustly stabilizing controllers," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 1802–1808, 1994.

[18] M. Teboulle and J. Kogan, "Application of optimization methods to robust stability of linear systems," *J. Optim. Theory Appl.*, vol. 81, pp. 169–192, 1994.

[19] A. Tesi and A. Vichino, "Robustness analysis for linear dynamic systems with linearly correlated parametric uncertainties," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 321–329, 1990.

Invariance Control for a Class of Cascade Nonlinear Systems

Jörg Mareczek, Martin Buss, and Mark W. Spong

***Abstract*—We consider the control of partially linear cascade systems using switching control of the states of the linear subsystem. We give sufficient conditions under which feedback of the linear states with switching gains guarantees both exponential stability of the linear subsystem and positive invariance of a prespecified region in state space. We refer to a control scheme incorporating these two objectives as *invariance control*. Semiglobal asymptotic stabilization follows under some additional conditions. The key idea of our design is to keep a given state space region positively invariant by switching on the boundary of the region. Thus, the transient response of the system can be kept within prescribed bounds which is important in many practical applications. Our approach can also be viewed as an alternative to high gain designs. The results in this note can be extended to nonlinear cascades and even to noncascaded systems.**

***Index Terms*—Cascade nonlinear systems, internal dynamics, invariance control, switching gains.**

I. Introduction

In this note, we consider the partially linear cascade nonlinear system

$$\dot{z} = f(z, x) \tag{1}$$

$$\dot{x} = Ax + bu \tag{2}$$

Manuscript received June 21, 2000; revised January 17, 2001 and November 12, 2001. Recommended by Associate Editor W. Lin. This work was done at the Institute of Automatic Control Engineering, Technische Universität München, 80290 München, Germany, and was supported in part by the National Science Foundation under Grant ECS-9812591, and by the Alexander von Humboldt Foundation, Bonn, Germany.

J. Mareczek is with Diehl GmbH Co KG, 90552 Roethenbach a. d. Pegnitz, Germany (e-mail: joerg.mareczek@ieee.org).

M. Buss is with the Control Systems Group, Faculty of Electrical Engineering and Computer Science, Technical University Berlin, D-10587 Berlin, Germany (e-mail:m.buss@ieee.org).

M. W. Spong is with the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA (e-mail: spong@control.csl.uiuc.edu).

Publisher Item Identifier S 0018-9286(02)03742-X.