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# An converse theorem for density functions

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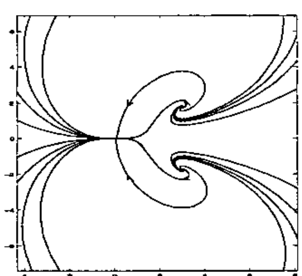
***Abstract*** It is proved that existence of a density function is both necessary and sufficient for almost global stability in a nonlinear system.

***Keywords*** Density function, converse theorem, stability

## 1. Introduction

A new convergence criterion for nonlinear systems was recently derived by the author [Rantzer, 2001]. From existence of a scalar "density" function satisfying certain inequalities, it was proved that for "almost all initial states" the system trajectory tends to zero. The new criterion is similar to Lyapunov's second theorem but differs in several respects. An important difference is that the statement leaves room for exceptional sets, such as unstable equilibria, while still yielding a global conclusion. For example, the criterion can be applied to a system with the following phase plot, to verify that almost all trajectories tend to the origin.



The new criterion also has a remarkable convexity property in the context of control synthesis. While the set of control Lyapunov functions for a given system may not even be connected, the corresponding set of density functions is always convex. This property has has been explored for numerical computations in [Rantzer and Parrilo, 2000] and for smooth transitions beteen different nonlinear controllers in [Rantzer and Ceragioli, 2001].

The present paper is concerned with the converse implication, to prove that under appropriate assumptions, a continuously differentiable density function must exist. An elegant result of this type has recently been independently reported by [Monzon, 2002].

### Notation

Given any $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ for $t \geq 0$ be the solution $x(t)$ of $\dot{x}(t) = f(x(t))$, $x(0) = x_0$.

## 2. Main result

THEOREM 1
Given $f \in \mathbf{C}^2(\mathbf{R}^n, \mathbf{R}^n)$, suppose that the system $\dot{x} = f(x)$ has a stable equilibrium in $x = 0$ and no solutions with finite escape time. Then, the following two conditions are equivalent.

(*i*) For almost all initial states $x(0)$ the solution $x(t)$ tends to zero as $t \to \infty$.

(*ii*) There exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ which is integrable outside a neighborhood of zero and such that

$$[\nabla \cdot (f\rho)](x) > 0 \qquad \text{for almost all } x$$

□

The proof will be based on the following lemma:

LEMMA 1
Let $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and $f(x)/|x|$ bounded. Let $h, h_j \in \mathbf{C}^1(\mathbf{R}^n, [0,1])$ with $h(x) = 0$ for $|x| < 0.5$ and $h(x) = 1$ for $|x| \geq 1$ while $0 \leq h_j \leq h$. Define

$$\psi_j(x) = \exp\left(-\int_0^\infty h(\phi_s(x))ds\right) h_j(x)$$
$$\rho_j(x) = \int_0^\infty \psi_j(\phi_{-t}(x)) \left|\frac{\partial \phi_{-t}(x)}{\partial x}\right| dt$$

Then $\int_{|x|\geq 1} \rho_j(x)dx \leq \int_{\mathbf{R}^n} h_j(x)dx$ and for all $Z \subset \mathbf{R}^n$

$$\int_{\phi_t(Z)} \rho_j(x)dx - \int_Z \rho_j(x)dx = \int_0^t \int_{\phi_\tau(Z)} \psi_j(x)dxd\tau$$

□

*Proof of Lemma 1* For every $Y \subset \mathbf{R}^n$, the change of variables $z = \phi_{-t}(x)$ gives

$$\int_Y \rho_j(x)dx = \int_0^\infty \int_{\phi_{-t}(Y)} \psi_j(z)dzdt$$

In particular, with $P = \{x \ : \ |x| \geq 1\}$

$$\begin{aligned}\int_P \rho_j(x)dx &= \int_0^\infty \int_{\phi_{-t}(P)} \psi_j(z)dzdt \\ &\leq \int_0^\infty \int_{\mathbf{R}^n} \psi_j(z)h(\phi_t(z))dzdt \\ &= \int_{\mathbf{R}^n} \left(\int_0^\infty h(\phi_t(z))dt\right)\psi_j(z)dz \\ &\leq \int_{\mathbf{R}^n} h_j(z)dz\end{aligned}$$

Moreover

$$\begin{aligned}&\int_{\phi_t(Z)} \rho_j(x)dx - \int_Z \rho_j(z)dz \\ &= \int_0^\infty \int_{\phi_{t-\tau}(Z)} \psi_j(z)dzd\tau - \int_0^\infty \int_{\phi_{-\tau}(Z)} \psi_j(z)dzd\tau \\ &= \int_0^t \int_{\phi_\tau(Z)} \psi_j(z)dzd\tau\end{aligned}$$

□

The main difficulty in applying Lemma 1 for the proof of Theorem 1 is that $\rho_j$ need not necessarily be differentiable. This is where the assumption about stability of $x = 0$ becomes useful:

*Proof of Theorem 1* Suppose $(i)$ holds. Stability of the equilibrium $x = 0$ guarantees existence of $\varepsilon > 0$ such that

$$\sup_{t\geq 0} |\phi_t(x)| < 1 \qquad \text{for all } x \text{ with } |x| \leq \varepsilon$$

Define $g \in \mathbf{C}^1(\mathbf{R}^n, [0,1])$ with $g(x) > 0$ for $1 < |x| < 2$ and $g(x) = 0$ otherwise. Define $\rho_j$ according to Lemma 1 with

$$h_j(x) = g(x)g(2\phi_{j\nu}(x)/\varepsilon)$$

where the number $\nu > 0$ is chosen so small that $h_j(x) > 0$ for some integer $j$ whenever $1 < |x| < 2$ and $\lim_{t\to\infty} |\phi_t(x)| = 0$.

Given any $x_0$, either $\rho_j(x)$ is identically zero near $x_0$, or there exists $T > 0$ such that neither $\psi_j(\phi_{-T}(\cdot))$ nor $h_j(\phi_{-T}(\cdot))$ is identically zero near $x_0$. It follows that $|\phi_{-T}(x_0)| \geq 1$, so $|\phi_{-T-t}(x_0)| > \varepsilon$ for $t \geq 0$ and therefore $h_j(\phi_{-t}(\cdot))$ is identically zero near $x_0$ for every $t > j\nu + T$. In particular

$$\rho_j(x) = \int_0^{j\nu+T} \psi_j(\phi_{-t}(x)) \left|\frac{\partial \phi_{-t}(x)}{\partial x}\right| dt$$

for $x$ in a neighborhood of $x_0$. Note that $\psi_j(\phi_t(z))$ is positive and differentiable at $z$ for all $t$ whenever $h_j(z) > 0$. In fact, all the expressions under the integral sign have continuous derivatives on the compact interval $[0, j\nu + T]$, so $\rho_j \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\})$. Furthermore, comparing the last equality of Lemma 1 to Liouville's theorem (Lemma 1 in [Rantzer, 2001]) shows that

$$[\nabla \cdot (f\rho_j)](x) = \psi_j(x) \qquad \text{almost everywhere}$$

Note that, according to $(i)$, for almost all $x$ with $1 < |x| < 2$ there is a number $j$ such that $\psi_j(x) > 0$. The desired $\rho$ will be constructed using a linear combination such $\rho_j$:s.

Define $\bar{\rho}(x) = \sum_{n=1}^\infty c_j\rho_j(x)$ with

$$c_j = 2^{-j}\left[\sup_{-j\leq \log|x|\leq j} (1 + \rho_j + |\partial\rho_j/\partial x|)(x)\right]^{-1}$$

Then $\bar{\rho} \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\})$ because of uniform convergence on the compact sets $\{x \ : \ -j \leq \log|x| \leq j\}$. Moreover $\int_{|x|\geq 1} \bar{\rho}(x)dx \leq \sum_j c_j \int_{\mathbf{R}^n} h_j(x)dx < \infty$ and

$$[\nabla \cdot (f\bar{\rho})](x) = \sum_j c_j\psi_j(z)$$

is positive for almost all $x$ with $1 < |x| < 2$.

The same construction can be used to define $\bar{\rho}_m \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\})$ for $m = \pm 1, \pm 2, \ldots$ to get positive divergence on the intervals $2^m < |x| < 2^{m+1}$. Then, define $\rho = \sum_{m=1}^\infty d_m\bar{\rho}_m$ with proper choice of $d_m$ to get $[\nabla \cdot (f\rho)](x) > 0$ almost everywhere in $\mathbf{R}^n$. This proves the implication from $(i)$ to $(ii)$.

The opposite implication was in proved in Theorem 1 of [Rantzer, 2001] □

### 3. References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