On the Kalman-Yakubovich-Popov Lemma for Stabilizable Systems

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the case, problem (15) is equivalent to problem (17). We further note that the inequality in (29) is equivalent to

$$X \succeq X_{\rm opt} + (Z - Z_{\rm opt})\tilde{X}_{22}^{\dagger}(Z - Z_{\rm opt})^T,$$
$$(Z - Z_{\rm opt})\left(I - \tilde{X}_{22}\tilde{X}_{22}^{\dagger}\right) = 0.$$

Both in the case of trace and log-determinant, the function $f(X)$ is concave on the cone of positive–definite matrices. This implies that the optimal value of X, Z are $X = X_{\rm opt}$, $Z = Z_{\rm opt}$, as claimed. □

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On Kalman–Yakubovich–Popov Lemma for Stabilizable Systems

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***Abstract*—The Kalman–Yakubovich–Popov (KYP) Lemma has been a cornerstone in System Theory and Network Analysis and Synthesis. It relates an analytic property of a square transfer matrix in the frequency domain to a set of algebraic equations involving parameters of a minimal realization in time domain. This note proves that the KYP lemma is also valid for realizations which are stabilizable and observable.**

***Index Terms*—Nonminimal realization, positive-real functions.**

I. INTRODUCTION

Given a square transfer matrix $Z(s)$, the Kalman–Yakubovich–Popov (KYP) Lemma relates an analytic property of a square transfer matrix in the frequency domain to a set of algebraic equations involving parameters of a minimal realization in time domain. See the original references [7], [18], and [13], [20]. Further important developments were given in [3], [12]. The lemma was generalized to the multivariable case in [2]. Extensions and clarifications appeared on [5], [16], and [10]. Clear presentations and

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relationships with other related results appeared in [17] and [8]. A novel proof based on convexity properties and linear algebra appeared recently in [14]. Based on this classical result, the following question with respect to minimality arises: is the KYP lemma valid for nonminimal realizations? This note addresses this question and gives a positive answer, i.e., the KYP lemma is valid for realizations which are stabilizable and observable. This extension has important applications in control systems theory and in the stability analysis of adaptive output feedback systems [6]. Some comments have appeared in the literature with respect to this relaxation. Meyer [11] made early comments on the minimality issue. A method for construction of Lyapunov functions for a positive real nonminimal systems was proposed in [6]. In a recent survey paper, the authors stated that the KYP lemma is valid for stabilizable realizations. However, they did not provide details of the proof. The objective of this note is to clarify and establish that the KYP lemma holds also for stabilizable and observable realizations.

II. Preliminaries

Let us consider a linear time-invariant m-inputs m-outputs transfer matrix $Z(s)$ with a minimal realization given by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^n$; $u, y \in \mathbf{R}^m$, $m \leq n$, and A, B, C, D are matrices of the corresponding dimensions. Let us denote the realization of $Z(s)$ given in (1) by

$$\Sigma_{Z(s)} = (A, B, C, D)$$

or

$$\Sigma_{Z(s)} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

In order to avoid trivialities, let us make the following assumption.

General Assumption: The transfer matrix $Z(s) = C(sI - A)^{-1}B + D$ is such that $Z(s) + Z^T(-s)$ has normal rank m, i.e., its rank is m almost everywhere in the complex plane.

The following are standard definitions of positive-real (PR) and strictly positive-real (SPR) systems, see [3] and [12].

Definition 1: The transfer matrix $Z(s)$ is said to be PR if: i) All elements of $Z(s)$ are analytical in Re$[s] > 0$; and ii) $Z(s)+Z^T(-s) \geq 0$ for all Re$[s] > 0$; $Z(s)$ is said to be SPR if $Z(s - \varepsilon)$ is PR for some $\varepsilon > 0$.

The following lemma give us a general procedure to generate uncontrollable *equivalent* realizations from two minimal realizations of a given transfer matrix $Z(s)$. The uncontrollable modes should be similar and the augmented matrices should be related by a change of coordinates as explained next.

Lemma 2: Let $\Sigma_i(A_i, B_i, C_i, D_i)$ for $i = 1, 2$ be two minimal realizations of $Z(s)$, i.e., $Z(s) = C_i(sI - A_i)^{-1}B_i + D_i$ for $i = 1, 2$. Now, define the augmented systems

$$\begin{aligned} \overline{A}_i &:= \begin{bmatrix} A_i & 0 \\ 0 & A_{0i} \end{bmatrix} \quad \overline{B}_i := \begin{bmatrix} B_i \\ 0 \end{bmatrix} \\ \overline{C}_i &:= \begin{bmatrix} C_i & C_{0i} \end{bmatrix} \quad \overline{D}_i := [D_i] \end{aligned} \tag{2}$$

where the dimensions of A_{01} and A_{02} are the same, moreover, there exist a nonsingular matrix T_0 such that $A_{01} = T_0 A_{02} T_0^{-1}$ and $C_{01} = C_{02} T_0^{-1}$. Then, $\overline{\Sigma}_i(\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ for $i = 1, 2$ are two *equivalent* realizations of $Z(s)$.

Proof: Simple algebraic manipulations. ■

As a dual result, we can generate unobservable augmented realizations of $Z(s)$ as established in the following corollary.

Corollary 3: Let $\Sigma_i(A_i, B_i, C_i, D_i)$ for $i = 1, 2$ be two minimal realizations of $Z(s)$, i.e., $Z(s) = C_i(sI - A_i)^{-1}B_i + D_i$ for $i = 1, 2$. Now, define the augmented systems

$$\begin{aligned} \overline{A}_i &= \begin{bmatrix} A_i & 0 \\ 0 & A_{0i} \end{bmatrix} \quad \overline{B}_i = \begin{bmatrix} B_i \\ B_{i0} \end{bmatrix} \\ \overline{C}_i &= \begin{bmatrix} C_i & 0 \end{bmatrix} \quad \overline{D}_i = [D_i] \end{aligned} \tag{3}$$

where the dimensions of A_{01} and A_{02} are the same, moreover, there exist a nonsingular matrix T_0 such that $A_{01} = T_0 A_{02} T_0^{-1}$ and $B_{01} = T_0 B_{02}$. Then $\overline{\Sigma}_i(\overline{A}_i, \overline{B}_i, \overline{C}_i, \overline{D}_i)$ for $i = 1, 2$ are two *equivalent* realizations of $Z(s)$. ■

Remark 1: Note also that if the eigenvalues of A_i and A_{0i} are different then the pair $(\overline{C}_i, \overline{A}_i)$ is observable if and only if the pair (C_{0i}, A_{0i}) is observable; and under the same conditions, the pair $(\overline{A}_i, \overline{B}_i)$ is controllable if and only if the pair (A_{0i}, B_{0i}) is controllable. The proof can be obtained by using the Popov–Belevitch–Hautus test [15].

III. Relaxed KYP Lemma

Following the nomenclature of Khalil [8], we may postulate our main result as follows.

Theorem 4: Let $Z(s) = \overline{C}(sI - \overline{A})^{-1}\overline{B} + \overline{D}$ be a $m \times m$ transfer matrix is such that $Z(s) + Z^T(-s)$ has normal rank m, where $\overline{A}$ is Hurwitz, $(\overline{A}, \overline{B})$ is stabilizable, and $(\overline{C}, \overline{A})$ is observable. Assume that if there are multiple eigenvalues, then all of them are controllable modes or all of them are uncontrollable modes. Then, $Z(s)$ is SPR if and only if there exist a positive definite symmetric matrix P, matrices W and L, and a positive constant ϵ such that

$$\begin{aligned} P\overline{A} + \overline{A}^T P &= -L^T L - \epsilon P \\ P\overline{B} &= \overline{C}^T - L^T W \\ W^T W &= \overline{D} + \overline{D}^T. \end{aligned} \tag{4}$$

Remark 2: The assumption that $Z(s) + Z^T(-s)$ has normal rank m is in order to avoid redundances in inputs and/or outputs. The assumption that the intersection of the set of controllable modes with the set of uncontrollable modes is empty, is used only in the *necessary* part of the proof given below.

Proof: Sufficiency:

Let $\mu \in (0, \epsilon/2)$ then from (4)

$$P(\overline{A} + \mu I) + (\overline{A} + \mu I)^T P = -L^T L - (\epsilon - 2\mu)P \tag{5}$$

which implies that $(\overline{A} + \mu I)$ is Hurwitz and, thus, $Z(s - \mu)$ is analytic in Re$[s] \geq 0$. Define now for simplicity

$$\overline{\Phi}(s) := (sI - \overline{A})^{-1}.$$

Therefore

$$\begin{aligned} &Z(s-\mu) + Z^T(-s-\mu) \\ &\quad = \overline{D} + \overline{D}^T + \overline{C}\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)\overline{C}^T \\ &\quad = W^T W + \left[\overline{B}^T P + W^T L\right]\overline{\Phi}(s-\mu)\overline{B} \\ &\qquad + \overline{B}^T\overline{\Phi}^T(-s-\mu)\left[P\overline{B} + L^T W\right] \\ &\quad = W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W \\ &\qquad + \overline{B}^T P\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)P\overline{B} \\ &\quad = W^T W + W^T L\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T W \end{aligned}$$

$$
\begin{aligned}
&+ \overline{B}^T\overline{\Phi}^T(-s-\mu)\left[\overline{\Phi}^{-T}(-s-\mu)P + P\overline{\Phi}^{-1}(s-\mu)\right]\\
&\cdot \overline{\Phi}(s-\mu)\overline{B}\\
&= W^TW + W^TL\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^TW\\
&+ \overline{B}^T\overline{\Phi}^T(-s-\mu)\\
&\cdot\left\{\left[-(s+\mu)I - \overline{A}^T\right]P + P\left[(s-\mu)I - \overline{A}\right]\right\}\\
&\cdot \overline{\Phi}(s-\mu)\overline{B}\\
&= W^TW + W^TL\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^TW\\
&+ \overline{B}^T\overline{\Phi}^T(-s-\mu)\left\{-2\mu P - \overline{A}^TP - P\overline{A}\right\}\overline{\Phi}(s-\mu)\overline{B}\\
&= W^TW + W^TL\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^TW\\
&+ \overline{B}^T\overline{\Phi}^T(-s-\mu)\left\{L^TL + (\epsilon-2\mu)P\right\}\overline{\Phi}(s-\mu)\overline{B}\\
&= W^TW + W^TL\overline{\Phi}(s-\mu)\overline{B} + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^TW\\
&+ \overline{B}^T\overline{\Phi}^T(-s-\mu)L^TL\overline{\Phi}(s-\mu)\overline{B} + (\epsilon-2\mu)\overline{B}^T\overline{\Phi}^T\\
&\cdot(-s-\mu)P\overline{\Phi}(s-\mu)\overline{B}\\
&= \left[W^T + \overline{B}^T\overline{\Phi}^T(-s-\mu)L^T\right]\left[W + L\overline{\Phi}(s-\mu)\overline{B}\right]\\
&+ (\epsilon-2\mu)\overline{B}^T\overline{\Phi}^T(-s-\mu)P\overline{\Phi}(s-\mu)\overline{B}.
\end{aligned}
$$

From the above, it follows that $Z(j\omega-\mu)+Z^T(-j\omega-\mu) \geq 0\ \forall\omega$ and $Z(s)$ is SPR.

Necessity:

Assume that $Z(s) \in$ SPR. Let $\overline{\Sigma}(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ be a stabilizable and observable realization of $Z(s)$ and $\Sigma(A, B, C, D)$ a minimal realization of $Z(s)$. Given that the controllable and uncontrollable modes are different we can consider that the matrix $\overline{A}$ is block diagonal and, therefore, $Z(s)$ can be written as

$$
Z(s) = \underbrace{[C \quad C_0]}_{\overline{C}}\underbrace{\begin{bmatrix} sI - A & 0\\ 0 & sI - A_0\end{bmatrix}^{-1}}_{[sI-\overline{A}]}\underbrace{\begin{bmatrix} B\\ 0\end{bmatrix}}_{\overline{B}} + \underbrace{D}_{\overline{D}} \tag{6}
$$

where the eigenvalues of A_0 correspond to the uncontrollable modes. As stated in the preliminaries, the condition $\sigma(A)\cap\sigma(A_0) = \emptyset$ [where $\sigma(T)$ means the spectrum of the square matrix T] means that the pairs (C, A) and (C_0, A_0) are observable if and only if $(\overline{C}, \overline{A}) = ([C \quad C_0], [\begin{smallmatrix} A & 0\\ 0 & A_0\end{smallmatrix}])$ is observable.

We have to prove that $\overline{\Sigma}(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ satisfies the KYP equations (4).

Note that A, A_0 are both Hurwitz. Indeed, A is stable because $\Sigma(A, B, C, D)$ is a minimal realization of $Z(s)$ which is SPR. A_0 is stable because the system is stabilizable. Thus $\exists\delta > 0$: $Z(s-\delta) \in$ PR and $Z(s-\mu) \in$ PR $\forall\mu \in [0, \delta]$. Choose now ϵ sufficiently small such that $U(s) := Z(s-(\epsilon/2)) \in$ SPR, then the following matrices are Hurwitz:

$$
\begin{aligned}
\overline{A}_\epsilon &= \overline{A} + \frac{\epsilon}{2}I \in \mathbf{R}^{(n+n_0)\times(n+n_0)}\\
A_\epsilon &= A + \frac{\epsilon}{2}I \in \mathbf{R}^{n\times n}\\
A_{0\epsilon} &= A_0 + \frac{\epsilon}{2}I \in \mathbf{R}^{n_0\times n_0}.
\end{aligned} \tag{7}
$$

Note that $\overline{A}_\epsilon$ is also block diagonal having block elements A_ϵ and $A_{0\epsilon}$ and the eigenvalues of A_ϵ and $A_{0\epsilon}$ are different. Let $\Sigma_\epsilon(A_\epsilon, B, C, D)$ be a minimal realization of $U(s)$ and $\overline{\Sigma}_\epsilon(\overline{A}_\epsilon, \overline{B}, \overline{C}, \overline{D})$ an observable and stabilizable realization of $U(s)$. Therefore

$$
U(s) = C(sI - A_\epsilon)^{-1}B + D = \overline{C}(sI - \overline{A}_\epsilon)^{-1}\overline{B} + \overline{D}. \tag{8}
$$

Note that the controllability of the pair (A_ϵ, B) follows from the controllability of (A, B). Since $A_{0\epsilon}$ is Hurwitz, it follows that $(\overline{A}_\epsilon, \overline{B})$ is stabilizable.

From the spectral factorization lemma for SPR transfer matrices [19], [8, Lemma A.11, p. 691], or [2], there exists an $m\times m$ stable transfer matrix $V(s)$ such that

$$
U(s) + U^T(-s) = V^T(-s)V(s). \tag{9}
$$

Remark 3: Here, the assumption that $Z(s) + Z^T(-s)$ has normal rank m is used implicitly, otherwise the matrix $V(s)$ would be of dimensions $(r\times m)$, where r is the normal rank of $Z(s) + Z^T(-s)$.

Let $\Sigma_V(F, G, H, J)$ be a minimal realization of $V(s)$, F is Hurwitz because $V(s)$ is stable; a minimal realization of $V^T(-s)$ is $\Sigma_{V^T}(-F^T, H^T, -G^T, J^T)$. Now, the series connection $V^T(-s)V(s)$ has realization (see [9, p. 15] for the formula of a cascade interconnection)

$$
\Sigma_{V^T(-s)V(s)} \cdot\left(\begin{bmatrix} F & 0\\ H^TH & -F^T\end{bmatrix}, \begin{bmatrix} G\\ H^TJ\end{bmatrix}, [J^TH \quad -G^T], [J^TJ]\right). \tag{10}
$$

Although we will not require the minimality of $\Sigma_{V^T(-s)V(s)}$ in the sequel, it can be proved to follow from the minimality of $\Sigma_V(F, G, H, J)$, see [8] or [1].

Let us now define a nonminimal realization of $V(s)$ obtained from $\Sigma_V(F, G, H, J)$ as follows:

$$
\overline{F} = \begin{bmatrix} F & 0\\ 0 & F_0\end{bmatrix}; \quad \overline{G} = \begin{bmatrix} G\\ 0\end{bmatrix}; \quad \overline{H} = [H \quad H_0]; \quad \overline{J} = J \tag{11}
$$

and such that F_0 is similar to $A_{0\epsilon}$ and the pair (H_0, F_0) is observable, i.e., $\exists\ T_0$ nonsingular such that

$$
F_0 = T_0A_{0\epsilon}T_0^{-1}. \tag{12}
$$

This constraint will be clarified later on. Since $\sigma(F_0)\cap\sigma(F) = \emptyset$, then the pair

$$
(\overline{H}, \overline{F}) = \left([H \quad H_0], \begin{bmatrix} F & 0\\ 0 & F_0\end{bmatrix}\right) \tag{13}
$$

is observable. Thus, the nonminimal realization $\overline{\Sigma}_V(\overline{F}, \overline{G}, \overline{H}, \overline{J})$ of $V(s)$ is observable and stabilizable.

Now, a nonminimal realization of $V^T(-s)V(s)$ based on $\overline{\Sigma}_V(\overline{F}, \overline{G}, \overline{H}, \overline{J})$

$$
\overline{\Sigma}_{V^T(-s)V(s)} \cdot\left(\begin{bmatrix} \overline{F} & 0\\ \overline{H}^T\overline{H} & -\overline{F}^T\end{bmatrix}, \begin{bmatrix} \overline{G}\\ \overline{H}^T\overline{J}\end{bmatrix}, [\overline{J}^T\overline{H} \quad -\overline{G}^T], [\overline{J}^T\overline{J}]\right) \tag{14}
$$

is (see [9, p. 15])

$$
\overline{\Sigma}_{V^T(-s)V(s)} = \left[\begin{array}{cccc|c}
F & 0 & 0 & 0 & G\\
0 & F_0 & 0 & 0 & 0\\
H^TH & H^TH_0 & -F^T & 0 & H^TJ\\
H_0^TH & H_0^TH_0 & 0 & -F_0^T & H_0^TJ\\
\hline
J^TH & J^TH_0 & -G^T & 0 & J^TJ
\end{array}\right]. \tag{15}
$$

From the diagonal structure of the above realization, it could be concluded that the eigenvalues of F_0 correspond to uncontrollable modes and the eigenvalues of $(-F_0{}^T)$ correspond to a unobservable modes. A constructive proof is given below.

Since the pair $(\overline{H}, \overline{F})$ is observable and $\overline{F}$ is stable, there exists a positive–definite matrix

$$\overline{K} = \overline{K}^T = \begin{bmatrix} K & r \\ r^T & K_0 \end{bmatrix} > 0 \tag{16}$$

solution of the Lyapunov equation

$$\overline{KF} + \overline{F}^T\overline{K} = -\overline{H}^T\overline{H}. \tag{17}$$

This explains why we imposed the constraint that (H_0, F_0) should be observable. Otherwise, there will not exist a positive definite solution for (17).

Define

$$\overline{T} := \begin{bmatrix} I & 0 \\ \overline{K} & I \end{bmatrix}; \qquad \overline{T}^{-1} = \begin{bmatrix} I & 0 \\ -\overline{K} & I \end{bmatrix}$$

and use it as a change of coordinates for the nonminimal realization $\overline{\Sigma}_{V^T(-s)V(s)}$ above to obtain

$$\overline{\Sigma}_{V^T(-s)V(s)} = \left[\begin{array}{cccc|c} F & 0 & 0 & 0 & G \\ 0 & F_0 & 0 & 0 & 0 \\ 0 & 0 & -F^T & 0 & (J\overline{H} + \overline{G}^T\overline{K})^T \\ 0 & 0 & 0 & -F_0^T & \\ \hline J\overline{H}+\overline{G}^T\overline{K} & -G^T & 0 & & J^TJ \end{array}\right]. \tag{18}$$

Now, it is clear that the eigenvalues of F_0 correspond to uncontrollable modes and the eigenvalues of $(-F_0{}^T)$ correspond to unobservable modes.

From (8), a nonminimal realization of $U(s)$ is $\overline{\Sigma}_\epsilon(\overline{A}_\epsilon, \overline{B}, \overline{C}, \overline{D})$. Thus, a nonminimal realization for $U^T(-s)$ is $\overline{\Sigma}_\epsilon$ $(-\overline{A}_\epsilon^T, \overline{C}^T, -\overline{B}^T, \overline{D}^T)$. Using the results in the preliminaries, a nonminimal realization of $U(s) + U^T(-s)$ is

$$\Sigma_{U(s)+U^T(-s)} \cdot \left(\begin{bmatrix} \overline{A}_\epsilon & 0 \\ 0 & -\overline{A}_\epsilon^T \end{bmatrix}, \begin{bmatrix} \overline{B} \\ \overline{C}^T \end{bmatrix}, \begin{bmatrix} \overline{C} & -\overline{B}^T \end{bmatrix}, \begin{bmatrix} \overline{D} + \overline{D}^T \end{bmatrix} \right). \tag{19}$$

Using (9) we conclude that the stable (unstable) parts of the realizations of $U(s) + U^T(-s)$ and $V^T(-s)V(s)$ are identical. Therefore, in view of the block diagonal structure of the system and considering only the stable part, we have

$$\begin{aligned} \overline{F} = \begin{bmatrix} F & 0 \\ 0 & F_0 \end{bmatrix} &= R\overline{A}_\epsilon R^{-1} = R \begin{bmatrix} A_\epsilon & 0 \\ 0 & A_{0\epsilon} \end{bmatrix} R^{-1} \\ \overline{G} = \begin{bmatrix} G \\ 0 \end{bmatrix} &= R\overline{B} = R\begin{bmatrix} B \\ 0 \end{bmatrix} \\ J\overline{H} + \overline{G}^T\overline{K} &= \overline{C}R^{-1} = [C \quad C_0]R^{-1} \\ J^TJ &= \overline{D} + \overline{D}^T. \end{aligned} \tag{20}$$

The above relationships imposes that the uncontrollable parts of the realizations of $U(s)$ and $V(s)$ should be similar. This is why we imposed that F_0 is similar to $A_{0\epsilon}$ in the construction of the nonminimal realization of $V(s)$.

From the Lyapunov equation (17), and using $\overline{F} = R\overline{A}_\epsilon R^{-1}$ in (20), we get

$$\begin{aligned} \overline{KF} + \overline{F}^T\overline{K} &= -\overline{H}^T\overline{H} \\ \overline{K}R\overline{A}_\epsilon R^{-1} + R^{-T}\overline{A}_\epsilon^T R^T\overline{K} &= -\overline{H}^T\overline{H} \\ R^T\overline{K}R\overline{A}_\epsilon + \overline{A}_\epsilon^T R^T\overline{K}R &= -R^T\overline{H}^T\overline{H}R \\ P\overline{A}_\epsilon + \overline{A}_\epsilon^T P &= -L^TL \end{aligned} \tag{21}$$

where we have used the definitions $P := R^T\overline{K}R$; $L := \overline{H}R$. Introducing (7), we get the first equation of (4).

From the second equation of (20), we have $\overline{G} = R\overline{B}$. From the third equation in (20) and using $W = J$, we get

$$\begin{aligned} J\overline{H} + \overline{G}^T\overline{K} &= \overline{C}R^{-1} \\ J^T\overline{H}R + \overline{G}^TR^{-T}R^T\overline{K}R &= \overline{C} \\ W^TL + \overline{B}^TP &= \overline{C} \\ P\overline{B} &= \overline{C}^T - L^TW \end{aligned} \tag{22}$$

which is the second equation of (4).

Finally, from the last equation of (20), we get the last equation of (4) since $W = J$. ■

IV. Examples

Next, we will consider two examples to illustrate the result.

1) Let a nonminimal realization of $Z(s) = (1/(s+1)) + ((s+2)/(s+2))$ be

$$\Sigma \begin{cases} \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ \alpha \\ 0 \end{bmatrix} u & \alpha \neq 0 \\ y = [\alpha \quad \beta]x + [1]u & \beta \neq 0. \end{cases} \tag{23}$$

Note that the system realization is stabilizable and observable for all $\beta \neq 0$. The KYP equations (4) for $\epsilon = 0.2$ give us

$$A^TP + PA = -L^TL - 0.2P \tag{24}$$

give us

$$\begin{bmatrix} -1.8P_1 & -2.8P_2 \\ -2.8P_2 & -3.8P_3 \end{bmatrix} = -\begin{bmatrix} l_1^2 & l_1l_2 \\ l_1l_2 & l_2^2 \end{bmatrix} \tag{25}$$

$$\overline{B}^TP + W^TL = \overline{C} = \begin{bmatrix} \frac{1}{\alpha} & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} + \sqrt{2}[l_1 \quad l_2] = [\alpha \quad \beta]$$

with solutions $L \simeq [0.5765\alpha \quad 0.6172\beta]$ and

$$P \simeq \begin{bmatrix} 0.1847\alpha^2 & 0.1271\alpha\beta \\ 0.1271\alpha\beta & 0.1003\beta^2 \end{bmatrix} > 0$$

for all α, β different from zero.

2) Let the nonminimal realization of

$$Z(s) = \frac{(s+a)}{(s+a)(s+b)}$$

for some $a > 0$, $b > 0$ and $b \neq a$ be

$$\Sigma \begin{cases} \dot{x} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ \alpha \end{bmatrix} u & \alpha \neq 0 \\ y = [\beta \quad \alpha]x + [0]u & \beta \neq 0 \end{cases} \tag{26}$$

it is easy to verify that for all $\epsilon < \min(a, b)$

$$P = \begin{bmatrix} \frac{(a+b-\epsilon)^2\beta^2}{(2b-\epsilon)(2a-\epsilon)} & \alpha\beta \\ \alpha\beta & \alpha^2 \end{bmatrix} > 0$$

for all $a > 0$, $b > 0$, $\alpha \neq 0$, $\beta \neq 0$

$$L = \begin{bmatrix} \frac{(a+b-\epsilon)}{\sqrt{2b-\epsilon}}\beta & \sqrt{2b-\epsilon}\alpha \end{bmatrix}$$

satisfy the equations of the KYP Lemma.

V. Conclusion

We have removed the minimality assumption in the Kalman–Yakubovich–Popov lemma, and proven that the lemma is still valid for stabilizable and observable realizations provided that the set of controllable modes and the set of uncontrollable modes do not intersect. Some examples illustrate the result.

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Asymptotic Behavior of Nonlinear Networked Control Systems

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***Abstract*—The defining characteristic of a networked control system (NCS) is having a feedback loop that passes through a local area computer network. Our two-step design approach includes using standard control methodologies and choosing the network protocol and bandwidth in order to ensure important closed-loop properties are preserved when a computer network is inserted into the feedback loop. For sufficiently high data rates, global exponential stability is preserved. Simulations are included to demonstrate the theoretical result.**

***Index Terms*—Asynchronous packets, networked control systems.**

I. INTRODUCTION

Using a (local area) networked control architecture has many advantages over a traditional point-to-point design including low cost of installation, ease of maintenance, lower cost, and greater flexibility [3], [4]. For these reasons the networked control architecture is already used in many applications, particularly where weight and volume are of consideration, for example in automobiles [2] and aircraft [5], [6]. The introduction of a computer network in the feedback loop unfortunately invalidates the traditional analytic stability and performance guarantees that control design typically produces. In this note, we reconnect the analysis of the control design to the networked control context, and provide guarantees of stability and certain levels of asymptotic performance to the control systems employing networked feedback loops.

We focus on a multiple-input–multiple-output (MIMO) nonlinear plant with a nonlinear controller connected by a communication network. A block diagram of this system is presented in Fig. 1.

We assume that the controller is designed without regard to the network, meaning that if the input to the controller is connected directly to the output of the plant the system would be globally (or locally) exponentially stable. We provide conditions under which these stability properties are preserved when the communication network is inserted into the loop between the outputs of the plant and the controller input. Each output, or group of outputs, is assumed to be monitored by a smart sensor with a network interface. Specifically, in the laboratory we us a Controller Area Network (CAN-II) operating at 1 Mb/sec because CAN-II is commonly used in automobiles and manufacturing plants. Each smart sensor must compete with the others for access to the network. The resulting communication constraint is the primary focus of this note, hence propagation delays, communication errors and observation noise will not be treated.

The general system consists of the time-varying plant, the time-varying controller, and the network. We denote the plant dynamics by $\dot{x}_p(t) = f_p(t, x_p(t), u_p(t))$, $y(t) = g_p(t, x_p(t))$,

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