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LECTURE NOTES ON A GEOMETRIC THEORY FOR LINEAR DYNAMICAL SYSTEMS

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LECTURE NOTES ON A GEOMETRIC THEORY FOR LINEAR DYNAMICAL SYSTEMS

G. Bengtsson

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1. INTRODUCTION

Historics.

The geometric concepts which are described in this report were introduced by Wonham and Morse [1], [3] and to some extent by Basile and Marro [2]. In their original paper [1], Wonham and Morse intended to find a more general solution to the decoupling problem, i.e. the problem of finding a suitable control which allows different subsets of output signals to be manipulated independently. For this purpose, a set of algebraic concepts, the geometric state space theory, were introduced by which it was possible to describe "input-output" properties of a linear system without using transfer functions or other input-output operators. These concepts can be regarded as extensions of the concepts "eigenspace" and "controllable subspace".

The geometric state space theory has later been used within other areas of control theory, e.g.

- the algebraic regulator problem [6], [9],
- invariant properties [7], [8]
- minimal observers for a linear functional of the state [5], [7]
- invertibility and minimal system inverses [4], [11], [12]
- system zeros [8], [12]
- model following [10]

This report is a summary of a series of lectures given at the Division of Automatic Control in Lund. It intends to survey and interpret the basic results within the geometric state space theory. The more applied part of the theory is only briefly squetched and is used rather as illustrations of the algebraic concepts. The proofs in the original papers are sometimes slightly changed and additional explanations and comments are provided. References for the theorems will not be given in the sequel. The reader can find most of the results in [1], [2], [3], [4].

Notations and Mathematical Background.

The notations that are used in the report are listed separately in the Appendix, where also rules for evaluation of algebraic expressions and properties of invariant subspaces are summarized.

Model.

Only time invariant linear systems will be considered. The system is assumed to be described by a triple of linear maps A: $\mathcal{X} \to \mathcal{X}$, B: $\mathcal{U} \to \mathcal{X}$, C: $\mathcal{X} \to \mathcal{Y}$ and an initial condition $x_0 \in \mathcal{X}$, where $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are linear vector spaces of finite dimension. By this we mean either a continuous time system of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \qquad\qquad x(0) = x_0 \\ y &= Cx \end{aligned} \tag{1}$$

or a discrete time system

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t \qquad\qquad x_0 = x_0 \\ y_t &= Cx_t \end{aligned} \tag{2}$$

In most cases it is immaterial which system is considered even if the details in the proofs sometimes may be slightly different.

2. A-INVARIANT SUBSPACES.

Definition.

Let X be a vector space and A: $X \rightarrow X$ a linear map.

Definition 2.1. A subspace $V \subset X$ is A-invariant if $AV \subset V$.

An A-invariant subspace has thus the property that all elements in the subspace are mapped into the subspace itself under A.

Interpretation.

What is the meaning of an A-invariant subspace V for the systems (1) and (2)? Consider e.g. (2) and let $x_0 \in V$ and $u = 0$, i.e.

$$x_{t+1} = Ax_t \qquad x_0 \in V$$

Since V is A-invariant, it follows directly that $x_t \in V$ for $t \geq 0$, i.e. the trajectory remains within the subspace V. The interpretation for the continuous time system (1) is identical. Assume for instance that the A-invariant subspace V is such that

$$V \subset \ker(C)$$

Since the trajectory is completely within V, this means that the output signal y_t, $t \geq 0$, is identical zero for all initial conditions $x_0 \in V$. The subspace V is thus an unobservable subspace to the systems (1) and (2).

Maximal A-invariant Subspace.

An A-invariant subspace contained in ker(C) can thus be interpreted as an unobservable subspace. We intend to develop this interpretation further and to give a characterization of the A-invariant subspaces which are contained in a given subspace $S \subset X$. We intend to show that there exists a unique maximal A-invariant subspace V^* contained in S, and how this subspace is constructed.

Introduce the class $\underline{V}(S)$ consisting of all A-invariant subspaces V contained in S, i.e.

$$\underline{V}(S) = \{V \mid V \subset S;\ AV \subset V\}$$

$$= \{V \mid V \subset S \cap A^{-1}V\} \qquad (3)$$

This class has the following properties:

(i) $\underline{V}(S)$ is closed under summation, since let V_1 , $V_2 \in \underline{V}(S)$. Then

$$V_1 + V_2 \subset S + S = S$$

$$A(V_1+V_2) = AV_1 + AV_2 \subset V_1 + V_2$$

Thus $V_1 + V_2 \in \underline{V}(S)$

(ii) $\underline{V}(S)$ is partially ordered by $\subset$.

(iii) S has finite dimension.

The properties (i) - (iii) means that $\underline{V}(S)$ has a maximal element which is denoted V^*. (Take e.g. the sum of all elements in $\underline{V}(S)$.)

The maximal element V^* can be constructed by means of a sequence. Besides being maximal, V^* shall also be an element of $\underline{V}(S)$, i.e.

$$V^* \subset S \cap A^{-1}V^* \qquad (4)$$

To get a suitable sequence, make the following attempt. Let

$$V_0 = S$$

If V_0 is A-invariant, i.e. satisfies (4), then by necessity $V^* = V_0$, since all elements in $\underline{V}(S)$ satisfy $V \subset S = V_0$. If this is not the case, the dimension must be decreased. Take instead

$$V_1 = S \cap A^{-1}V_0$$

etc. This leads to the following sequence

$$\begin{aligned} V_0 &= S \\ V_{i+1} &= S \cap A^{-1}V_i \end{aligned} \qquad (5)$$

<u>Theorem 2.1</u>. The maximal element V^* is given by $V^* = V_\sigma$ where σ is the least integer such that $V_\sigma = V_{\sigma+1}$ in the sequence (5). Moreover, the sequence (5) converges in at most $\dim(S)$ steps.

<u>Proof</u>. Let us first show that V_σ is an element of $\underline{V}(S)$. By construction (5) it follows that

$$V_\sigma = V_{\sigma+1} = S \cap A^{-1}V_\sigma$$

i.e. $V_\sigma \in \underline{V}(S)$ according to (4). It then remains to show that V_σ is maximal. The sequence (5) has the following properties:

(a) V_i is monotonously decreasing, i.e. $S \supset V_0 \supset V_1 \supset \ldots$

(b) V^* is the lower bound, i.e. $V_i \supset V^*$ for all i.

These properties are easily shown by induction. Property (a) is true for i = 1 since

$$V_1 = S \cap A^{-1}V_0 \subset S = V_0$$

Assume that (a) holds for $i \leq q$. Then for $i = q + 1$

$$V_{q+1} = S \cap A^{-1}V_q \subset S \cap A^{-1}V_{q-1} = V_q$$

Property (b) is shown in a similar way. It is true for i = 0 since

$$V^* \subset S = V_0$$

Assume (b) holds for $i \leq q$. Then for $i = q+1$

$$V_{q+1} = S \cap A^{-1}V_q \supset S \cap A^{-1}V^* \supset V^* \qquad (6)$$

From (a) and (b) it follows immediately that the sequence V_i converges towards the maximal element V^*. From (a) we also see that as long as the sequence does not converge, the dimension of V_i must be decreased by at least one in each step. Since $V_i \subset S$, it thus follows that the sequence converges in at most dim(S) steps.

□

Remark. Since $V_i \subset V_{i-1} \subset S$ according to the proof of Theorem 1.1, the sequence (5) can be replaced by the following equivalent sequence

$$V_0 = S$$

$$V_{i+1} = V_i \cap A^{-1}V_i \tag{7}$$

Example 2.1. Consider the discrete time system

$$\begin{aligned} x_{t+1} &= Ax_t \qquad x_0 \text{ given} \\ y_t &= Cx_t \end{aligned} \tag{8}$$

Find the set of states x_0 for which the output y_t is zero in the first k step, i.e. $y_0 = y_1 = \ldots = y_k = 0$. From (8) we have directly

$$y_j = CA^j x_0 = 0 \qquad j = 0, 1, \ldots, k$$

The initial states which produce zero outputs in k steps are thus given by

$$x_0 \in \bigcap_{j=0}^{k} \ker(CA^j) \triangleq Q$$

Introduce the subspaces V_i, i = 0, 1, ..., k, by

$$V_i = \bigcap_{j=0}^{i} \ker(CA^j) \tag{9}$$

It is immediately seen that $V_0 = \ker(C)$ and $V_k = Q$. By rewriting (9) we have

$$\mathcal{V}_{i+1} = \bigcap_{j=0}^{i+1} \ker(CA^j) = \ker(C) \cap \left(\bigcap_{j=1}^{i+1} \ker(CA^j) \right) \tag{10}$$

Observe that

$$\begin{aligned}
\bigcap_{j=1}^{i+1} \ker(CA^j) &= \left\{ x \mid CA^j x = 0,\ j = 1, 2, \ldots, i+1 \right\} \\
&= \left\{ x \mid CA^j (Ax) = 0,\ j = 0, 1, \ldots, i \right\} \\
&= \left\{ x \mid Ax \in \bigcap_{j=0}^{i} \ker(CA^j) \right\} \\
&= A^{-1} \left(\bigcap_{j=0}^{i} \ker(CA^j) \right) \\
&= A^{-1} \mathcal{V}_i
\end{aligned}$$

A substitution into (10) gives a recursion for $\mathcal{V}_i$ which is identical to the sequence (5) i.e.

$$\mathcal{V}_0 = \ker(C)$$

$$\mathcal{V}_{i+1} = \ker(C) \cap A^{-1} \mathcal{V}_i$$

$$\mathcal{Q} = \mathcal{V}_k$$

This result gives further interpretation to the sequence (5). The subspaces $\mathcal{V}_k$ produced by this sequence is thus the set of initial states x_0 which produces zero outputs in the first k step if $\mathcal{S} = \ker(C)$.

Example 2.2. The following numerical example shows the construction of V^*. Let

$$A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad S = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

where $\{\cdot\}$ denotes the subspace spanned by the column vectors. Application of the sequence (5) gives

$$V_0 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

$$V_1 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}^{-1} \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{vmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{Bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$V_2 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Since $V_1 = V_2$, the sequence has converged in one step and

$$\mathcal{V}^* = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Note that the inverse of A exists in this case, which simplifies the computation of the inverse image.

Exercise. Show that $\mathcal{V}^*$ produced by the sequence (5) with $\mathcal{S} = \ker(C)$ equals the unobservable subspace for the pair (A,C), i.e. the null space of the observability matrix

$$Q = \begin{vmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{vmatrix}$$

Cf. Example 2.1.

3. (A,B)-INVARIANT SUBSPACES.

Definition.

Let $\mathcal{X}$ and $\mathcal{U}$ be vector spaces and A: $\mathcal{X} \to \mathcal{X}$ and B: $\mathcal{U} \to \mathcal{X}$ a pair of linear maps.

Definition 3.1. The subspace $\mathcal{V} \subset \mathcal{X}$ is (A,B)-invariant if there is a linear map F: $\mathcal{X} \to \mathcal{U}$ such that $(A+BF)\mathcal{V} \subset \mathcal{V}$.

This means that $\mathcal{V}$ can be made (A+BF)-invariant by a suitable choice of F.

Interpretation.

The map F in Definition 3.1 means a state feedback for the systems (1) and (2), e.g. for (2)

$$x_{t+1} = (A+BF)x_t \qquad x_0 \text{ given}$$

Analogously with the interpretation of A-invariant subspaces, this means that a trajectory can be forced to remain within the subspace $\mathcal{V}$ by application of a suitable feedback $u_t = Fx_t$.

Characterization.

The following theorem gives necessary and sufficient conditions for a given subspace to be (A,B)-invariant.

Theorem 3.1. The subspace V is (A,B)-invariant if and only if $AV \subset V + B$.

Proof. (only if) If V is (A,B)-invariant there is a linear map F such that $(A+BF)V \subset V$. Let $v \in V$ be arbitrary. Then

$$(A+BF)v = w \in V$$

Thus

$$Av = w - BFv \in V + B$$

(if) Let $v_1, v_2 \ldots, v_q$ be a basis for V. There exist $w_i \in V$ and $z_i \in U$ such that

$$Av_i = w_i + Bz_i \qquad i = 1, 2, \ldots, q$$

Let $F: X \to U$ be a map such that $Fv_i = - z_i$. This map exists since $v_1, v_2, \ldots, v_q$ is a basis. Thus

$$(A+BF)v_i = w_i \in V \qquad i = 1, 2, \ldots, q$$

□

Remark. Note the similarity between the condition in Theorem 3.1 and the definition of A-invariant subspaces.

Remark. The proof of Theorem 3.1 shows how a linear map F can be constructed if the subspace is (A,B)-invariant.

Associated with an (A,B)-invariant subspace V there are many linear maps F such that $(A+BF)V \subset V$. Introduce the class $\underline{F}(V)$ consisting of all such F:

$$\underline{F}(V) \triangleq \{F \mid (A+BF)V \subset V\}$$

It is possible to characterize all elements in this feedback class by means of a single element F_0:

<u>Theorem 3.2</u>. Assume that $F_0 \in \underline{F}(V)$. Then $F \in \underline{F}(V)$ if and only if $B(F-F_0)V \subset V$.

<u>Proof</u>. (if) Follows directly by

$$(A+BF)V = \big(A + BF_0 + B(F-F_0)\big)V$$

$$\subset (A+BF_0)V + B(F-F_0)V$$

$$\subset V + V = V$$

(only if) Let $v \in V$ be arbitrary. Then

$$(A+BF_0)v = w \in V$$

$$(A+BF)v = z \in V$$

Subtract

$$B(F-F_0)v = z - w \in V$$

□

<u>Remark</u>. Note that the condition of the theorem is equivalent to either of the following two conditions

(a) $B(F-F_0)V \subset V \cap B$

(b) $(F-F_0)V \subset B^{-1}(V)$

□

Sometimes it is desirable to find a common F such that a set of (A,B)-invariant subspaces V_i, i = 1, ..., q are (A+BF)-invariant, i.e.

$$F \in \bigcap_{i=1}^{q} \underset{\sim}{F}(V_i)$$

If there exists such a common F, the (A,B)-invariant subspaces V_i are said to be compatible. General necessary and sufficient conditions for a given set of subspace to be compatible are not known. It is, however, possible to give such conditions in a special case:

Theorem 3.3. The (A,B)-invariant subspaces V_1 and V_2 are compatible if and only if $V_1 \cap V_2$ is (A,B)-invariant.

Proof. (only if) Let F be such that V_1 and V_2 are (A+BF)-invariant. It follows directly that $(A+BF)(V_1 \cap V_2) \subset \subset V_1 \cap V_2$.

(if) Let

$$F_1 \in \underset{\sim}{F}(V_1) \quad F_2 \in \underset{\sim}{F}(V_2) \quad F_{12} \in \underset{\sim}{F}(V_1 \cap V_2)$$

Moreover, let $\hat{V}_1$ and $\hat{V}_2$ be any subspaces such that

$$V_1 = \hat{V}_1 \oplus V_1 \cap V_2 \qquad V_2 = \hat{V}_2 \oplus V_1 \cap V_2$$

Then also

$$V_1 + V_2 = \hat{V}_1 \oplus V_1 \cap V_2 \oplus \hat{V}_2$$

It follows from the independence of the subspaces that there are projections P_1, P_2 and P_{12} such that

$P_1 = $ proj. onto $\hat{V}_1$ along V_2

$P_2 = $ proj. onto $\hat{V}_2$ along V_1

$P_{12} = $ proj. onto $V_1 \cap V_2$ along $\hat{V}_1 \oplus \hat{V}_2$

Set

$$F = F_1P_1 + F_{12}P_{12} + F_2P_2$$

Then $F \in \underline{F}(V_1)$, since

$$(A+BF)V_1 = (A+BF_1P_1+BF_{12}P_{12}+BF_2P_2)(\hat{V}_1 \oplus V_1 \cap V_2)$$

$$= (A+BF_1)\hat{V}_1 + (A+BF_{12})(V_1 \cap V_2)$$

$$\subset V_1 + V_1 \cap V_2 = V_1$$

where properties of the projections have been used. By symmetry, it also follows that $F \in \underline{F}(V_2)$.

□

A sufficient condition for compatibility can, however, be given in a more general case:

Theorem 3.4. If the (A,B)-invariant subspaces V_i, $i = 1, 2, \ldots, q$, are independent then they are compatible.

Proof. Since the subspaces V_i are independent we have

$$V_1 + V_2 + \ldots + V_q = V_1 \oplus V_2 \oplus \ldots \oplus V_q$$

Then there exist projections P_i onto V_i along $\sum_{j \neq i} V_j$.

Set

$$F = \Sigma F_j P_j$$

Then $F \in \underline{F}(V_i)$ for all i since

$$(A+B\Sigma F_j P_j)V_i = (A+BF_i)V_i \subset V_i$$

□

Maximal (A,B)-invariant Subspaces.

Let $S \subset X$ be a given subspace and consider all (A,B)-invariant subspaces contained in S. Introduce the class $\underline{W}(S)$ of all such subspaces:

$$\begin{aligned}\underline{W}(S) &\triangleq \{V \mid AV \subset V + B,\ V \subset S\} \\ &= \{V \mid V \subset S \cap A^{-1}(V+B)\}\end{aligned} \qquad (11)$$

As was the case for A-invariant subspaces, there is a unique maximal element V^* in this class since

(i) $\underline{W}(S)$ is closed under summation. Let V_1 and V_2 be elements in $\underline{W}(S)$. Then $V_1 + V_2 \in \underline{W}(S)$ since

$$\begin{aligned}A(V_1+V_2) &= AV_1 + AV_2 \subset V_1 + B + V_2 + B \\ &= V_1 + V_2 + B\end{aligned}$$

$$V_1 + V_2 \subset S + S = S$$

(ii) $\underline{W}(S)$ is partially ordered by $\subset$.

(iii) S has finite dimension.

Based upon the sequence (5), we construct a similar sequence for (A,B)-invariant subspaces:

$$\mathcal{V}_0 = \mathcal{S}$$
$$\mathcal{V}_{i+1} = \mathcal{S} \cap A^{-1}(\mathcal{V}_i + \mathcal{B}) \tag{12}$$

<u>Theorem 3.5</u>. The maximal element $\mathcal{V}^*$ is given by $\mathcal{V}^* = \mathcal{V}_\sigma$ where σ is the least integer such that $\mathcal{V}_\sigma = \mathcal{V}_{\sigma+1}$. Moreover, the sequence (12) converges after at most $\dim(\mathcal{S})$ steps.

<u>Proof</u>. The proof of Theorem 3.5 is only a simple generalization of the proof of Theorem 2.1. It is left as an exercise for the reader.

□

<u>Remark</u>. Note that Theorem 2.1 is a special case of Theorem 3.5 with $\mathcal{B} = 0$.

The section is concluded by some examples.

<u>Example 3.1</u>. Consider the system

$$x_{t+1} = Ax_t + Bu_t + Gv_t \qquad x_0 = 0$$
$$y_t = Cx_t$$

where v_t is a disturbance which is measured. The following feedforward problem shall be solved. Find a control law of the form

$$u_t = Fx_t + Hv_t$$

such that the output signal y_t is uneffected by the disturbance v_t. Under what conditions can this problem be solved?

The closed-loop system becomes

$$x_{t+1} = (A+BF)x_t + (G+BH)v_t \qquad x_0 = 0$$

$$y_t = Cx_t$$

Consider the impulse response. The impulse response shall be zero since v_t is not permitted to influence y_t. Thus

$$x_{t+1} = (A+BF)x_t \qquad \forall x_1 \in \mathrm{Im}(G+BH)$$

$$y_t = Cx_t = 0$$

This means that an (A+BF)-invariant subspace V and a map H shall be found so that

$$\ker(C) \supset V \supset \mathrm{Im}(G+BH)$$

Take V as the maximal (A,B)-invariant subspace contained in ker(C). The feedforward problem can thus be solved if and only if there is a map H such that

$$\mathrm{Im}(G+BH) \subset V^* .$$

It can be shown that this condition is equivalent to the following

$$\mathrm{Im}(G) \subset V^* + B \qquad (13)$$

Hence, there is a control law with the desired properties if and only if (13) holds.

Example 3.2. The following example illustrates the construction of $\mathcal{V}^*$ from the sequence (12):

$$A = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad B = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \qquad \mathcal{S} = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

Cf. Example 3.2. The sequence (12) becomes

$$\mathcal{V}_0 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

$$\mathcal{V}_1 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}^{-1} \left(\begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \right)$$

$$= \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{vmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \begin{Bmatrix} 0 & 1 & \vdots & 1 \\ 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{Bmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

Since $\mathcal{V}_0 = \mathcal{V}_1$, the sequence has converged and the maximal (A,B)-invariant subspace contained in $\mathcal{S}$ is given by

$$\mathcal{V}^* = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix}$$

Example 3.3. Consider the same maps and subspace as in Example 3.2. Characterize the feedback class $\underline{F}(\mathcal{V}^*)$. The map

$$F_0 = [0 \quad 0 \quad -1]$$

belongs to $\underline{F}(\mathcal{V}^*)$ since

$$(A+BF_0)\mathcal{V}^* = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} = \mathcal{V}^*$$

Moreover, in this case

$$\mathcal{V}^* \cap \mathcal{B} = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} \cap \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \{0\}$$

According to the remark of Theorem 3.2, it follows that every $F = [f_1 \ f_2 \ f_3] \in \underline{F}(\mathcal{V}^*)$ is characterized by

$$\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} (f_1 \ f_2 \ f_3{+}1) \begin{Bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{Bmatrix} = \begin{Bmatrix} f_2 & f_1+f_3+1 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix} = \{0\}$$

i.e.

$$f_2 = 0$$

$$f_1 + f_3 = -1$$

<u>Exercise</u>. Formulate and solve the correspondence of Example 2.1 for $(\dot{A},B)$-invariant subspaces.

4. CONTROLLABILITY SUBSPACES.

Definition.

Let A: $X \to X$ and B: $U \to X$ be a pair of linear maps, where U and X are vector spaces. Let us first give a definition which directly relates to the systems (1) and (2).

Definition 4.1. A subspace $R \subset X$ is a controllability subspace if there are linear maps F and G such that

$$R = \{A + BF \mid \mathrm{Im}(BG)\} \qquad \square$$

Definition 4.1 depends on two unknown parameters, the maps F and G. An equivalent definition which only depends on one parameter is the following.

Definition 4.2. A subspace $R \subset X$ is a controllability subspace if there is a linear map F such that

$$R = \{A + BF \mid B \cap R\} \qquad \square$$

The equivalence between the definitions follows by the following lemma and the fact that $\mathrm{Im}(BG) \subset B$.

Lemma 4.1. If $\hat{B} \subset B$ and $\{A|\hat{B}\} = R$, then $\{A|B \cap R\} = R$. Conversely, if $\{A|B \cap R\} = R$ then there is a linear map G such that $\{A|\mathrm{Im}(BG)\} = R$.

Proof. It follows that

$$R = \{A|\hat{B}\} = \hat{B} + A\hat{B} + \dots + A^{n-1}\hat{B} \supset \hat{B}$$

Since we also have $\hat{B} \subset B$, it follows that $\hat{B} \subset B \cap R$. Thus

$$\mathcal{R} = \{A|\hat{\mathcal{B}}\} \subset \{A|\mathcal{B} \cap \mathcal{R}\} \tag{14 a}$$

The subspace $\mathcal{R}$ is A-invariant, since by the Cayleigh-Hamilton theorem

$$A\mathcal{R} = A\mathcal{B} + A^2\mathcal{B} + \ldots + A^n\mathcal{B}$$

$$= A\mathcal{B} + A^2\mathcal{B} + \ldots + \left(\sum_{i=0}^{n-1} \alpha_i A^i\right)\mathcal{B}$$

$$\subset \mathcal{R}$$

where α_i are the coefficients in the ch.p. of A. Thus

$$A(\mathcal{B} \cap \mathcal{R}) \subset A\mathcal{R} \subset \mathcal{R}$$

and by induction

$$A^j(\mathcal{B} \cap \mathcal{R}) \subset \mathcal{R}$$

Hence

$$\{A|\mathcal{B} \cap \mathcal{R}\} = \mathcal{B} \cap \mathcal{R} + A(\mathcal{B} \cap \mathcal{R}) + \ldots + A^{n-1}(\mathcal{B} \cap \mathcal{R})$$

$$\subset \mathcal{R} + \mathcal{R} + \ldots + \mathcal{R} = \mathcal{R} \tag{14 b}$$

By (14 a) and (14 b) it follows that $\mathcal{R} = \{A|\mathcal{B} \cap \mathcal{R}\}$ and the first part in the theorem is proven. To prove the second part, let b_i be the i:th column in B and let $r_1 \ldots r_q$ be a basis for $\mathcal{B} \cap \mathcal{R}$. Then

$$r_j = \sum_{i=1}^{m} g_{ij} b_i$$

for suitable g_{ij}. Let $G = \{g_{ij}\}$. Then $\mathcal{B} \cap \mathcal{R} = \mathrm{Im}(BG)$. □

Interpretation.

Definition 4.1 gives a direct interpretation of a controllability subspace $\mathcal{R}$ for the systems (1) and (2). The maps F and G are parts of a control of the form

$$u = Fx + Gv \qquad (15)$$

where v is some external input. The closed-loop system becomes

$$\dot{x} = (A+BF)x + BGv$$

The controllable subspace from the input v shall be exactly $\mathcal{R}$. If this can be achieved by a suitable control of the form (15), $\mathcal{R}$ is thus a controllability subspace. The use of controllability subspaces will be further illustrated in connection with invertibility and decoupling below.

Characterization.

A controllability subspace $\mathcal{R}$ can be characterized in the following way.

Theorem 4.1. A subspace $\mathcal{R} \subset \mathcal{X}$ is a controllability subspace if and only if

(i) $\mathcal{R}$ is (A,B)-invariant

(ii) $\mathcal{R} = \{A + BF \mid \mathcal{B} \cap \mathcal{R}\}$ where $F \in \underline{F}(\mathcal{R})$

Remark. Note that condition (i) in Theorem 4.1 is equivalent to the condition that $\underline{F}(\mathcal{R})$ is nonempty.

Proof. (only if) There exists a map F such that $\{A + BF \mid \mathcal{B} \cap \mathcal{R}\} = \mathcal{R}$. Then

$$(A+BF)\mathcal{R} \subset \mathcal{R}$$

cf. the proof of Lemma 4.1. Thus $\mathcal{R}$ is (A,B)-invariant and $F \in \underline{F}(\mathcal{R})$. It remains to show that F can be chosen arbitrary in $\underline{F}(\mathcal{R})$. Set

$$\mathcal{R}^{(0)} = 0$$

$$\mathcal{R}^{(i)} = \mathcal{B} \cap \mathcal{R} + (A+BF)(\mathcal{B} \cap \mathcal{R}) + \dots + (A+BF)^{i-1}(\mathcal{B} \cap \mathcal{R})$$

The following recursion is obtained.

$$\mathcal{R}^{(0)} = 0$$

$$\mathcal{R}^{(i)} = (A+BF)\mathcal{R}^{(i-1)} + \mathcal{B} \cap \mathcal{R} \qquad (16)$$

$$\mathcal{R}^{(n)} = \mathcal{R} = \{A + BF \mid \mathcal{B} \cap \mathcal{R}\}$$

Consider the same recursion for an arbitrary $F_1 \in \underline{F}(\mathcal{R})$

$$\hat{\mathcal{R}}^{(0)} = 0$$

$$\hat{\mathcal{R}}^{(i)} = (A+BF_1)\hat{\mathcal{R}}^{(i-1)} + \mathcal{B} \cap \mathcal{R}$$

$$\hat{\mathcal{R}}^{(n)} = \hat{\mathcal{R}} = \{A + BF_1 \mid \mathcal{B} \cap \mathcal{R}\}$$

It is shown by induction that $\mathcal{R}^{(i)} = \hat{\mathcal{R}}^{(i)}$ for all i. The equality holds for i = 0. Assume it holds for $i \leq q$. For $i = q + 1$ we have

$$\hat{R}^{(q+1)} = (A+BF_1)\hat{R}^{(q)} + \mathcal{B} \cap R$$

$$= (A+BF_1)R^{(q)} + \mathcal{B} \cap R$$

$$= \big(A + BF + B(F_1-F)\big)R^{(q)} + \mathcal{B} \cap R$$

$$\subset (A+BF)R^{(q)} + B(F_1-F)R^{(q)} + \mathcal{B} \cap R$$

$$\subset (A+BF)R^{(q)} + \mathcal{B} \cap R = R^{(q+1)}$$

where the remark of Theorem 3.2 and the fact that $R^{(q)} \subset R$ have been used in the last step. In a similar way

$$R^{(q+1)} = (A+BF)R^{(q)} + \mathcal{B} \cap R$$

$$= \big(A + BF_1 + B(F_1-F)\big)R^{(q)} + \mathcal{B} \cap R$$

$$\subset (A+BF_1)R^{(q)} + \mathcal{B} \cap R$$

$$= (A+BF_1)\hat{R}^{(q)} + \mathcal{B} \cap R = \hat{R}^{(q+1)}$$

showing that $R^{(q+1)} = \hat{R}^{(q+1)}$. The equality holds then for all i and especially for $i = n$. Thus $R = \hat{R}$.

(ii) Follows directly from (ii) and Definition 4.2). □

A controllability subspace can be characterized in an alternative way which does not require the computation of F. For this we need the sequence (16) and the following lemma. The proof of the lemma is omitted here and can be found in [1].

Lemma 4.2. Let $\mathcal{R}$ be (A,B)-invariant and $\tilde{\mathcal{R}} \subset \mathcal{R}$. Then the following equality holds for an arbitrary $F \in \underline{F}(\mathcal{R})$

$$\mathcal{B} \cap \mathcal{R} + (A+BF)\tilde{\mathcal{R}} = \mathcal{R} \cap (A\tilde{\mathcal{R}}+\mathcal{B})$$

□

If Lemma 4.2 is applied to the sequence (16) we get

$$\mathcal{R}^{(0)} = 0$$

$$\mathcal{R}^{(i)} = \mathcal{B} \cap \mathcal{R} + (A+BF)\mathcal{R}^{(i-1)} = \mathcal{R} \cap (A\mathcal{R}^{(i-1)}+\mathcal{B})$$

$$\mathcal{R}^{(n)} = \mathcal{R}$$

since $\mathcal{R}^{(i-1)} \subset \mathcal{R}$ and $\mathcal{R}$ is (A,B)-invariant. The following criterion for a subspace to be a controllability subspace is thus obtained.

Theorem 4.2. The subspace $\mathcal{R}$ is a controllability subspace if and only if

(i) $\mathcal{R}$ is (A,B)-invariant

(ii) $\mathcal{R} = \mathcal{R}^{(n)}$ where

$$\mathcal{R}^{(0)} = 0$$

$$\mathcal{R}^{(i)} = \mathcal{R} \cap (A\mathcal{R}^{(i-1)}+\mathcal{B}), \; i = 1, 2, \ldots, n$$

Maximal Controllability Subspaces.

Let $S \subset X$ be a given subspace. We shall show that there is a unique maximal controllability subspace R^* contained in S.

Theorem 4.3. Let $S \subset X$ be a given subspace. There is a unique maximal controllability subspace R^* contained in S. This subspace is given by

$$R^* = \{A + BF \mid B \cap V^*\} \tag{17}$$

where V^* is the maximal (A,B)-invariant subspace contained in S and $F \in \underline{F}(V^*)$.

Remark. To construct R^* we must thus first apply the sequence (5) to obtain V^*. After that an $F \in \underline{F}(V^*)$ is computed. The subspace R^* is then obtained by (17).

Proof. According to Lemma 4.1, R^* given by (17) is a controllability subspace since $\hat{B} = B \cap V^* \subset B$. Moreover, by Theorem 4.1, R^* is independent by the choice of $F \in \underline{F}(V^*)$ and is thus unique. It then remains to show that R^* is maximal.

Let $\hat{R}$ be an arbitrary controllability subspace contained in S, i.e.

$$S \supset \hat{R} = \{A + B\hat{F} \mid B \cap \hat{R}\} \qquad \hat{F} \in \underline{F}(\hat{R})$$

Since $\hat{R}$ is (A,B)-invariant according to Theorem 4.1, it follows that $\hat{R} \subset V^*$ by the maximality of V^*. Hence, there is a subspace $\hat{V}$ so that

$$\mathcal{V}^* = \hat{\mathcal{R}} \oplus \hat{\mathcal{V}}$$

There exist projections with the following properties:

P_1 = proj. onto $\hat{\mathcal{V}}$ along $\hat{\mathcal{R}}$

P_2 = proj. onto $\hat{\mathcal{R}}$ along $\hat{\mathcal{V}}$

Moreover, let $F \in F(\mathcal{V}^*)$ and set

$$F_1 = FP_1 + \hat{F}P_2$$

Then $F_1 \in \underline{F}(\mathcal{V}^*)$ since

$$(A+BF_1)\mathcal{V}^* = (A+BFP_1+B\hat{F}P_2)(\hat{\mathcal{V}}\oplus\mathcal{R})$$

$$= (A+BF)\hat{\mathcal{V}} + (A+B\hat{F})\hat{\mathcal{R}}$$

$$\subset \mathcal{V}^* + \hat{\mathcal{R}} = \mathcal{V}^*$$

where the properties of the projections P_1 and P_2 have been used. We also have $F_1 \in \underline{F}(\hat{\mathcal{R}})$ since

$$(A+BF_1)\hat{\mathcal{R}} = (A+BFP_1+B\hat{F}P_2)\hat{\mathcal{R}}$$

$$= (A+B\hat{F})\hat{\mathcal{R}} \subset \hat{\mathcal{R}}.$$

Since F_1 belongs to both the classes $\underline{F}(\mathcal{V}^*)$ and $F(\hat{\mathcal{R}})$, it follows that

$$\hat{\mathcal{R}} = \{A + BF_1 \mid \mathcal{B} \cap \hat{\mathcal{R}}\} \subset \{A + BF_1 \mid \mathcal{B} \cap \mathcal{V}^*\} = \mathcal{R}^*$$

since $\hat{\mathcal{R}} \subset \mathcal{V}^*$. The first equality follows from Theorem 4.1. Since $\hat{\mathcal{R}}$ is arbitrary, $\mathcal{R}^*$ must be maximal.

□

Remark. Theorems 3.5 and 4.3 show that there exist a maximal (A,B)-invariant subspace V^* and a maximal controllability subspace R^* contained in a given subspace S. Since R^* is also (A,B)-invariant, cf. Theorem 4.1, we thus have the ordering $S \supset V^* \supset R^*$.

Properties.

Let $R \subset X$ be a given controllability subspace. Since R is also (A,B)-invariant, there is a linear map F such that

$$(A+BF)R \subset R$$

Associated with the (A+BF)-invariant subspace R is some of the eigenvalues of A+BF, namely the eigenvalues of

$$(A+BF) \mid R$$

Let us first examine how the eigenvalues of $(A+BF) \mid R$ change for different choices of $F \in \underline{F}(R)$.

Theorem 4.4. Let R be a controllability subspace and let $\alpha(s)$ be an arbitrary monic polynomial of degree $\dim(R)$. Then there exists $F \in \underline{F}(R)$ so that the ch.p. for $(A+BF) \mid R$ is $\alpha(s)$. Moreover, if $0 \neq b \in B \cap R$ is arbitrary, then F can be chosen so that b in addition generates R under (A+BF).

Remark. b generates R under (A+BF) if the vectors b, (A+BF)b, ..., $(A+BF)^{n-1}b$ span R.

Proof. Let $F_1 \in \underline{F}(R)$ be arbitrary. By the definition of a controllability subspace:

$$R = \{A + BF_1 \mid B \cap R\} \tag{19}$$

Let $A_1 \triangleq A + BF_1$. Let $b_1\ b_2\ \ldots\ b_q$ be a basis for $B \cap R$ and let ρ_1 be the largest integer so that the vectors

$$b_1, A_1b_1, \ldots, A_1^{\rho_1-1} b_1$$

are linearly independent. Set

$$r_1 = b_1$$

$$r_j = A_1r_{j-1} + b_1 \qquad j = 2, \ldots, \rho_1$$

Then $r_j \in R$ and the vectors $r_1, r_2, \ldots, r_{\rho_1}$ are linearly independent. If $\rho_1 < \dim(R)$ let ρ_2 be the largest integer so that the vectors

$$b_1, A_1b_1 \ldots A_1^{\rho_1-1} b_1; b_2, A_1b_2, \ldots, A_1^{\rho_2-1} b_2$$

are linearly independent. Set

$$r_j = A_1r_{j-1} + b_2 \qquad j = \rho_1 + 1, \ldots, \rho_2 \tag{20}$$

Then $r_j \in R$ and the vectors $r_1, r_2, \ldots, r_{\rho_1+\rho_2}$ are linearly independent. Proceed recursively in this way until

$$R = \mathrm{span}(r_1, r_2, \ldots, r_s) \qquad s = \dim(R)$$

According to (19) this will happen after a finite number of steps.

For the vectors r_i we have

$$r_1 = b_1$$

$$r_{i+1} = A_1 r_i + \tilde{b}_i \qquad i = 1, \ldots, s-1 \tag{21}$$

where $\tilde{b}_i \in \mathcal{B} \cap \mathcal{R}$ and $\tilde{b}_s$ is arbitrary in $\mathcal{B} \cap \mathcal{R}$. Let $z_i \in U$ be such that

$$Bz_i = \tilde{b}_i \qquad i = 1, 2, \ldots, s$$

Since the vectors r_i are linearly independent there is a map F_2 such that

$$F_2 r_i = z_i$$

A substitution into (21) yields

$$r_1 = b_1$$

$$r_{i+1} = (A_1 + BF_2) r_i = \big(A + B(F_1 + F_2)\big) r_i \tag{22}$$

Thus, $\mathcal{R}$ is cyclic with generator b_1, i.e.

$$\mathcal{R} = \{A + BF_0 \mid \mathrm{Im}(b_1)\} \qquad F_0 = F_1 + F_2$$

Well-known theorems for single-input systems can now be applied to show that the ch.p. for $(A+BF_0+b_1k^T) \mid \mathcal{R}$ can be made equal to $\alpha(s)$ by a suitable k. The choice $F = F_0 + z_1 k^T$, where $Bz_1 = b_1$, proves the theorem.

□

Another interpretation of controllability subspaces is the following. Consider the system

$\dot{x} = Ax + Bu \qquad\qquad x(0) = 0$

Set

$$x(u,t) = \int_0^t e^{A(t-s)}Bu(s)ds$$

Let $S \subset X$ be a given subspace. To which state $x(u,t_1) \in S$ can we control the system if we require that the trajectory $x(u,t)$, $t \in [0,t_1]$, remain within S? This question is answered by the following theorem.

Theorem 4.5. Let Z be the set of states $x \in S$ such that for some u

(i) $\quad x(u,t) \in S \qquad\qquad \forall t \in [0,t_1]$

(ii) $\quad x(u,t_1) = x$

Then $Z = R^*$, where R^* is the maximal controllability subspace contained in S. Moreover, for all trajectories with the properties (i) and (ii) we also have $x(u,t) \in R^*$, $t \in [0,t_1]$.

Proof. Take $u = Fx + Gv$ so that

$R^* = \{A + BF \mid Im(BG)\}$

Linear maps F and G with this property can be found according to Lemma 4.1. Set

$$\hat{A} \triangleq A + BF \qquad\qquad \hat{B} \triangleq BG \tag{23}$$

It is well known that the controllable subspace for the pair $(\hat{A},\hat{B})$ equals the image of the controllability gramian R, i.e.

$$R^* = \mathrm{Im}(R) \tag{24}$$

$$R = \int_0^{t_1} e^{\hat{A}(t_1-s)}\hat{B}\hat{B}^T e^{\hat{A}^T(t_1-s)}\,ds \tag{25}$$

Let us first show that $R^* \subset Z$. Let $x \in R^*$. Then $x = Rw$ for some w by (24). Choose

$$v(t) = \hat{B}^T e^{\hat{A}^T(t_1-t)} w$$

A substitution into (25) yields

$$x = Rw = \int_0^{t_1} e^{\hat{A}(t_1-s)}\hat{B}v(s)\,ds$$

This means that the equation

$$\dot{x} = \hat{A}x + \hat{B}v \qquad x(0) = 0$$

has the solution $x(t_1) = x$. Moreover, since the controllable subspace for the pair $(\hat{A},\hat{B})$ equals R^*, we also have $x(t) \in R^*$, $t \in [0,t_1]$. Set

$$u = Fx + Gv$$

and compare with (23). Then $x(u,t) \in R^* \subset S$, $t \in [0,t_1]$, and $x(u,t_1) = x$. Thus $x \in Z$ and $R^* \subset Z$.

We shall also show that $R^* \supset Z$. Let V^* be the maximal (A,B)-invariant subspace contained in S. According to Theorem 4.1

$$V_0 = S$$

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A^{-1}(\mathcal{V}_i+\mathcal{B})$$

$$\mathcal{V}_\sigma = \mathcal{V}^*$$

where $\mathcal{V}_\sigma = \mathcal{V}_{\sigma+1}$. Assume $x \in \mathcal{Z}$. Then there exists an input u so that

$$x(u,t) \in \mathcal{S} \qquad t \in [0,t_1]$$

$$x(u,t_1) = x$$

and $x(u,t) \in \mathcal{V}_0$. If $x(u,t) \in \mathcal{V}_i$, then $\dot{x}(u,t) \in \mathcal{V}_i$. Thus

$$Ax(u,t) = \dot{x}(u,t) - Bu(t) \in \mathcal{V}_i + \mathcal{B}$$

This means that

$$x(u,t) \in \mathcal{V}_i \cap A^{-1}(\mathcal{V}_i+\mathcal{B}) = \mathcal{V}_{i+1}$$

By induction it follows that $x(u,t) \in \mathcal{V}_\sigma = \mathcal{V}^*$. Let $F \in$ $\in \underline{F}(\mathcal{V}^*)$ and define v(t) by

$$v(t) = Fx(u,t) - u(t)$$

Then

$$\dot{x}(u,t) = (A+BF)x(u,t) + Bv(t)$$

and thus

$$Bv(t) = \dot{x}(u,t) - (A+BF)x(u,t) \in \mathcal{V}^* + \mathcal{V}^* = \mathcal{V}^*$$

since $\dot{x}(u,t)$ and $x(u,t)$ belong to $\mathcal{V}^*$. We have then shown that

$$\mathrm{Im}\big(Bv(t)\big) \subset \mathcal{B} \cap \mathcal{V}^* \qquad \forall t \in [0,t_1]$$

Then

$$x = x(u,t_1) = \int_0^{t_1} e^{(A+BF)(t_1-s)} Bv(s)ds$$

$$\in \{A + BF \mid \mathcal{B} \cap \mathcal{V}^*\} = \mathcal{R}^*$$

and thus $\mathcal{Z} \supset \mathcal{R}^*$. Since we have also shown that $\mathcal{Z} \subset \mathcal{R}^*$, it follows that $\mathcal{Z} = \mathcal{R}^*$. The last statement in the theorem follows directly by varying the final point of time t_1.

□

Remark. The maximal controllability subspace $\mathcal{R}^*$ contained in $\mathcal{S}$, can thus be interpreted as the set of states in $\mathcal{S}$ which can be reached from the origin with a trajectory which does not leave $\mathcal{S}$. If $\mathcal{S} = \ker(C)$, where C is the output matrix in (1), we then have a clear interpretation of the set of inputs which leaves the output uneffected, i.e. the kernel of the input-output operator. This interpretation is essential in the system inversion problem treated below.

System Inversion.

Consider the system (1) with $x(0) = 0$. The system (2) defines a map θ from the input space $\underline{U}$ to the output space $\underline{Y}$ by

$$y(t) = (\theta u)(t) = C \int_0^t e^{A(t-s)} Bu(s)ds \qquad t \geq 0 \qquad (27)$$

We will give conditions when the map θ is injective, i.e. when the system (1) is left invertible. Since θ is a linear map, θ is injective if and only if

$$\theta(u) = 0 \Rightarrow u = 0 \tag{28}$$

The following theorem gives necessary and sufficient conditions for left invertibility.

Theorem 4.6. The system (1) is left invertible if and only if

(i) $\ker(B) = 0$

(ii) $V^* \cap \mathcal{B} = 0$

Remark. Note that (ii) is equivalent to $R^* = 0$ by Theorem 4.3. Moreover, it is easily seen that (i) and (ii) is equivalent to the condition $B^{-1}(V^*) = 0$.

Proof. (only if) Let us first show (i). Let $w \in \ker(B)$ and take the input as

$$u(t) = w \qquad t \geq 0$$

Then

$$(\theta u)(t) = \int_0^t ce^{A(t-s)}Bwds = 0$$

and according to (28), $u(t) = w = 0$, i.e. $\ker(B) = 0$. To show (ii) let $x \in V^* \cap \mathcal{B}$. Then we also have $x \in R^*$ according to Theorem 4.3. By Theorem 4.5, there exists an input $\tilde{u}$ so that the state of (1) satisfies $x(t_1) = x$

and $x(t) \in \ker(C)$ for $t \in [0,t_1]$. Take

$$u = \begin{cases} \tilde{u} & 0 \leq t \leq t_1 \\ Fx(t) & t > t_1 \end{cases}$$

where $F \in \underline{F}(V^*)$. Then

$$(\theta u)(t) = Cx(t) = 0 \qquad t \geq 0$$

and thus $u = 0$ by (28). Hence

$$x = x(t_1) = \int_0^{t_1} e^{A(t_1-s)} Bu(s)ds = 0$$

which shows that $x = 0$, i.e. $V^* \cap B = 0$.

(if) Assume that $y(t) = (\theta u)(t) = Cx(t) = 0$ for $t \geq 0$. Then $x(t) \in \ker(C)$ $\forall t \geq 0$. According to Theorem 4.5, it follows that $x(t) \in R^*$. But $R^* = 0$ by (ii) and thus $x(t) = 0$ $\forall t \geq 0$. Hence

$$0 = x(t) = \int_0^t e^{A(t-s)} Bu(s)ds \qquad t \geq 0$$

which implies that $Bu(t) = 0$. Since $\ker(B) = 0$ by (i), it follows that $u(t) = 0$ $\forall t \geq 0$. Left invertibility then follows by (28).

□

Decoupling.

The geometric concepts introduced above can be used to formulate and solve the so called decoupling problem, cf. Wonham and Morse [1], [3], [4].

Consider the system

$$\dot{x} = Ax + Bu \tag{30}$$

$$y_i = C_i x \qquad i = 1, 2, \ldots, r$$

where y_i denotes different output vectors. Apply a control of the form

$$u = Fx + \sum_{i=1}^{r} G_i v_i$$

i.e.

$$\dot{x} = (A+BF)x + \sum_{i=1}^{r} BG_i v_i \tag{31}$$

$$y_i = C_i x \qquad i = 1, 2, \ldots, r$$

To have the system decoupled, the maps F and G_i shall be chosen so that the input v_i in (31) only influences the output y_i and no other output. Let R_i denote the controllable subspace from v_i in (31), i.e.

$$R_i = \{A + BF \mid \mathrm{Im}(BG_i)\} \tag{32}$$

We note that R_i is a controllability subspace. The condition that v_i does not influence y_j if $i \neq j$, can now be formulated in the following way

$$C_j \mathcal{R}_i = 0 \qquad i \neq j \tag{33}$$

Moreover, the input v_i shall influence the output y_i. We require pointwise controllability in the output space, i.e.

$$C_i \mathcal{R}_i = \mathrm{Im}(C_i) \tag{34}$$

By a simple rewriting of the conditions (32), (33) and (34), we are now able to formulate the restricted decoupling problem (i.e. no dynamic compensation is allowed).

Find controllability subspaces $\mathcal{R}_i$, $i = 1, 2, \ldots, r$, and a map F such that

(i) $\mathcal{R}_i = \{A + BF \mid \mathcal{B} \cap \mathcal{R}_i\}$

(ii) $\mathcal{R}_i \subset \bigcap_{j \neq i} \ker(C_j)$

(iii) $\mathcal{R}_i + \ker(C_i) = \mathcal{X}$ (35)

Note that (i) is equivalent to (32) by Lemma 4.1, (ii) is equivalent to (33) and (iii) is equivalent to (34).

The condition which is most difficult to handle is (i), i.e. to find a common F in the classes $\underline{F}(\mathcal{R}_i)$. If only the conditions (ii) and (iii) are considered it is temptating to take $\mathcal{R}_i$ maximal in

$$\bigcap_{j \neq i} \ker(C_j)$$

i.e. $\mathcal{R}_i = \mathcal{R}_i^*$. This subspace can be constructed according to Theorem 4.3. It is, however, not certain that the subspaces $\mathcal{R}_i^*$ are compatible, i.e. condition (i) may fail.

This is the reason why, the restricted decoupling problem is yet unsolved in its most general form. Different conditions can, however, be introduced which guarantees that R_i^* are compatible. If dynamic compensation is permitted, the general problem is solved, see Wonham and Morse [3].

Here, the decoupling problem will be solved in a special case, namely if

$$\text{rank}(C) = n \qquad C = \begin{vmatrix} c_1 \\ c_2 \\ \vdots \\ c_r \end{vmatrix} \tag{36}$$

This condition means that the state vector has been partitioned into r subvectors which are to be controlled independently by the inputs v_i in (30). The condition (36) is equivalent to the following condition

$$\bigcap_{i=1}^{r} \ker(C_i) = 0 \tag{37}$$

Necessary and sufficient conditions for the existence of a decoupling control law of the form (30) in this special case are given in the following theorem.

<u>Theorem 4.7</u>. Assume that the outputs y_i in (30) have been chosen so that (37) is satisfied. Then, the restricted decoupling problem has a solution if and only if

$$R_i^* + \ker(C_i) = X \qquad i = 1, 2, \ldots, r$$

where R_i^* is the maximal controllability subspace contained in

$$\bigcap_{j \neq i} \ker(C_j)$$

Proof. (if) It is immediately seen that (ii) and (iii) in (35) are satisfied. It then remains to verify (i). The subspaces R_i^* are independent since

$$R_i^* \cap \left(\sum_{j \neq i} R_j\right) \subset \left(\bigcap_{j \neq i} \ker(C_j)\right) \cap \ker(C_i)$$

$$= \bigcap_{j=1}^{r} \ker(C_j) = 0$$

according to the assumption (37). It then follows by Theorem 3.4 that $\{R_i^*\}$ are compatible, i.e. there exists an

$$F \in \bigcap_{i=1}^{r} \underline{F}(R_i^*)$$

Then from Theorem 4.1

$$R_i^* = \{A + BF \mid \mathcal{B} \cap R_i\}$$

i.e. (i) is satisfied.

(only if) Trivial.

□

The section is concluded by some examples.

Example 4.1. Consider

$$A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad B = \begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{vmatrix}$$

$$\mathcal{R} = \begin{Bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{Bmatrix}$$

Is $\mathcal{R}$ a controllability subspace?

Use Theorem 4.1. Let us first examine if condition (i) of the theorem is satisfied. Compute

$$A\mathcal{R} = \begin{Bmatrix} 1 & 1 \\ -1 & -2 \\ 1 & 0 \end{Bmatrix}$$

$$\mathcal{R} + \mathcal{B} = \begin{Bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{Bmatrix} + \begin{Bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

It is seen that $A\mathcal{R} \subset \mathcal{R} + \mathcal{B}$, i.e. $\mathcal{R}$ is (A,B)-invariant by Theorem 3.1. We have then verified condition (i) in Theorem 4.1. To verify (ii) of Theorem 4.1 an $F \in \underline{F}(\mathcal{R})$ must be calculated. Such an F is given by

$$F = \begin{vmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

Evaluate $\{A + BF \mid \mathcal{B} \cap \mathcal{R}\}$. In this case

$$\mathcal{B} \cap \mathcal{R} = \begin{Bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{Bmatrix} \cap \begin{Bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$A + BF = \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Since n = 3:

$$\{A + BF \mid \mathcal{B} \cap \mathcal{R}\} = \mathcal{B} \cap \mathcal{R} + (A+BF)(\mathcal{B} \cap \mathcal{R}) + (A+BF)^2(\mathcal{B} \cap \mathcal{R})$$

$$= \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{Bmatrix} = \mathcal{R}$$

It then follows that $\mathcal{R}$ is a controllability subspace.

Example 4.2. Let

$$A = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \qquad B = \begin{vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{vmatrix} \qquad \mathcal{S} = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{Bmatrix}$$

Compute the maximal controllability subspace $\mathcal{R}^*$ contained in $\mathcal{S}$. Use Theorem 4.3. Compute first the maximal (A,B)-invariant subspace $\mathcal{V}^*$ contained in $\mathcal{S}$ by the sequence (5), i.e.

$$\mathcal{V}_0 = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{Bmatrix}$$

$$V_1 = \left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\} \cap \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \left[\left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\} + \left\{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\right\} \right]$$

$$= \left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\} \cap \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \left\{\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{matrix}\right\}$$

$$= \left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\} \cap \left\{\begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{matrix}\right\} = \left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\}$$

The sequence has converged in one step and

$$V^* = \left\{\begin{matrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{matrix}\right\}$$

Compute $F \in \underline{F}(V^*)$:

$$F = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

Moreover,

$$B \cap V^* = \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\}$$

R^* is then computed by Theorem 4.3 as

$$R = \{A + BF \mid B \cap V^*\}$$

$$= \left\{\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right\} + \left\{\begin{matrix} 0 \\ 1 \\ 1 \end{matrix}\right\} = \left\{\begin{matrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \end{matrix}\right\} = \left\{\begin{matrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{matrix}\right\}$$

Example 4.3. Can the following system be decoupled by a control law of the form (31)?

$$\dot{x} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} x + \begin{vmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{vmatrix} u$$

$$y_1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} x$$

$$y_2 = [0 \quad 0 \quad 1]x$$

First we see that

$$\ker(C) = \ker\begin{vmatrix} C_1 \\ C_2 \end{vmatrix} = \ker\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hdashline 0 & 0 & 1 \end{vmatrix} = 0$$

and condition (37) is satisfied. This means that Theorem 4.7 can be applied. We have

$$\ker(C_1) = \ker\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \left\{\begin{matrix} 0 \\ 0 \\ 1 \end{matrix}\right\}$$

$$\ker(C_2) = \ker[0 \quad 0 \quad 1] = \left\{\begin{matrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{matrix}\right\}.$$

Compute the maximal controllability subspaces R_1^* and R_2^* contained in $\ker(C_2)$ and $\ker(C_1)$ respectively. This is done by Theorem 4.3. The maximal (A,B)-invariant subspaces V_1^* and V_2^* contained in $\ker(C_2)$ and $\ker(C_1)$ respectively are given by:

$$V_1^* = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{Bmatrix} \qquad V_2^* = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

A common F in the classes $F(V_1^*)$ and $F(V_2^*)$ are given by

$$F = \begin{vmatrix} 0 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} \tag{38}$$

Moreover,

$$B \cap V_1^* = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$B \cap V_2^* = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$$A + BF = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

By application of Theorem 4.3:

$$R_1^* = \{A + BF \mid B \cap V_1^*\}$$

$$= \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{Bmatrix}$$

$$R_2^* = \{A + BF \mid B \cap V_2^*\}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

It is easily verified that

$$\mathcal{R}_1^* + \ker(C_1) = \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} = \mathbb{R}^3$$

$$\mathcal{R}_2^* + \ker(C_2) = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{Bmatrix} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} = \mathbb{R}^3$$

By Theorem 4.3 it follows that the system can be decoupled. The feedback matrix in (31) is given by (38). To compute the feedforward matrices G_1 and G_2 in (31), solve $BG_i = S_i$, $i = 1, 2$, where S_i is a basis matrix for $\mathcal{B} \cap \mathcal{V}_i^*$. Hence

$$\begin{vmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{vmatrix} G_1 = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{vmatrix} G_2 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

The solutions become

$$G_1 = \begin{vmatrix} 1 \\ -1 \end{vmatrix} \qquad\qquad G_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

A decoupling control law is thus given by

$$u = \begin{vmatrix} 0 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} x + \begin{vmatrix} 1 \\ -1 \end{vmatrix} v_1 + \begin{vmatrix} 0 \\ 1 \end{vmatrix} v_2$$

5. REFERENCES.

[1] Wonham - Morse: "Decoupling and pole assignement in linear multivariable systems: A geometric approach", Siam. J. Control, 8, 1 - 18, Feb. 1970.

[2] Basile - Marro: "Controlled and conditioned invariants subspaces in linear system theory", J. Optimiz. Theory Appl., 3, 306 - 315, 1969.

[3] Morse - Wonham: "Decoupling and pole assignement by dynamic compensation", Siam. J. Control, 8, 317 - 337, Aug. 1970.

[4] Morse - Wonham: "Status of noninteracting control", IEEE Trans. Aut. Contr., AC-16, 568 - 580, Dec. 1971.

[5] Wonham: "Dynamic observers - geometric theory", IEEE Trans. Aut. Contr., AC-15, 258 - 259, 1970.

[6] Wonham: "Tracking and regulation in linear multivariable systems", Siam J. Control, 11, 424 - 437, 1973.

[7] Wonham - Morse: "Feedback invariants of linear multivariable systems", Automatica, 8, 93 - 100, 1972.

[8] Morse: "Structural invariants of linear multivariable systems", Siam J. Contr., 11, 446 - 465, Aug. 1973.

[9] Wonham - Pearson: "Regulation and internal stabilization in linear multivariable systems", SIAM J. Control, 12, 5-18, Febr. 1974

[10] Morse: "Structure and design of linear model following systems" IEEE Trans. Automatic Control, AC-18, 346 - 354, Aug. 1973

[11] Bengtsson: "Minimal system inverses for linear multivariable systems", J. Math. An. Appl., 46, 261 - 274, May 1974

[12] Bengtsson: "A theory for control of linear multivariable systems", Ph.D. Thesis, Lund Inst. of Techn., Div. of Automatic Control, Jan. 1974

APPENDIX

Notations.

$\mathcal{V}, \mathcal{R}, \mathcal{B}$	- linear subspaces
A, B, C	- linear maps
0	- zero space
ϕ	- empty space
Im(A) or $\mathcal{A}$	- the range space of A
ker(A)	- the null space of A
$A\mathcal{V}$	- $\{x \mid x = Av, v \in \mathcal{V}\}$ the image of $\mathcal{V}$ under A
$A^{-1}\mathcal{V}$	- $\{x \mid Ax \in \mathcal{V}\}$ the inverse image of $\mathcal{V}$ under A
$\mathcal{V}_1+\mathcal{V}_2$	- $\{x \mid x = x_1 + x_2, x_1 \in \mathcal{V}_1, x_2 \in \mathcal{V}_2\}$ the sum of $\mathcal{V}_1$ and $\mathcal{V}_2$
$\mathcal{V}_1 \oplus \mathcal{V}_2$	- Direct sum of $\mathcal{V}_1$ and $\mathcal{V}_2$. The sum of $\mathcal{V}_1$ and $\mathcal{V}_2$ is direct if every element x in $\mathcal{V}_1+\mathcal{V}_2$ can be written $x = x_1 + x_2$ with unique $x_1 \in \mathcal{V}_1$, $x_2 \in \mathcal{V}_2$.
$\mathcal{V}_1 \cap \mathcal{V}_2$	- Intersection of $\mathcal{V}_1$ and $\mathcal{V}_2$.
$\mathcal{V}^{\perp}$	- The orthogonal complement of $\mathcal{V}$
Projection	- P is a projection onto $\mathcal{V}_1$ along $\mathcal{V}_2$ if $Px_1=x_1$ and $Px_2=0$ for all $x_1 \in \mathcal{V}_1$ and all $x_2 \in \mathcal{V}_2$. The projection exists if $\mathcal{V}_1 \cap$ $\cap \mathcal{V}_2 = 0$.
Basis matrix	- V is a basis matrix for $\mathcal{V}$ if the columns of V are linearly independent and span $\mathcal{V}$.
Span $(x_1, \ldots, x_k)$	- The subspace spanned by the vectors $x_1, x_2, \ldots, x_k$
$A \mid \mathcal{V}$	- The restriction of A to $\mathcal{V}$.

Algebraic Expressions.

(A1) $(V_1+V_2)^\perp = V_1^\perp \cap V_2^\perp$

(A2) $(A^{-1}V)^\perp = A^T V^\perp$

(A3) $A(V_1+V_2) = AV_1 + AV_2$

(A4) $A(V_1 \cap V_2) \subset AV_1 \cap AV_2$

(A5) $(A+B)V_1 \subset AV_1 + BV_1$

(A6) $V_1 \cap (V_2+V_3) \supset V_1 \cap V_2 + V_1 \cap V_3$ (= if $V_1 \supset V_2$)

(A7) $(\ker(A))^\perp = \mathrm{Im}(A^T)$

Properties of Invariant Subspaces.

Let V_i, i=1,2, be two A-invariant subspaces and let $d_i(s)$ be the ch.p. for $A|V_i$, i=1,2.

(A9) $A|V_i$ is a map $V_i \to V_i$.

(A10) If $V_1 \subset V_2$ then $d_1(s)$ divides $d_2(s)$.

(A11) $V_i^\perp$ is A^T-invariant and the ch.p. $d(s)$ for A can be written $d(s) = d_i(s)\hat{d}_i(s)$, where $\hat{d}_i(s)$ is the ch.p. for $A^T|V_i^\perp$.