Structure of H∞-optimal controllers : the golden section example

Hagander, Per; Bernhardsson, Bo

1990

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name
INTERNAL REPORT

Date of issue
October 1990

Document Number
CODEN: LUTFD2/(TFRT-7468)/1–12/(1990)

Author(s)
Per Hagander
Bo Bernhardsson

Supervisor

Sponsoring organisation

Title and subtitle
Structure of H_∞-optimal Controllers – The Golden Section Example

Abstract

The structure of H_∞-optimal controllers is investigated. A simple second order feed-forward problem is analyzed which has one free parameter, the control weight ρ in the loss function. The minimal H_∞-norm is calculated as a function of ρ and it is found that regulator structure and uniqueness changes when ρ is varied.

For small ρ the central controller is the only optimal controller. It is first order and 'equalizing'. The optimal γ_o solves the equation $\gamma^4 + 2\gamma^3 = \rho^{-2}$. The γ-iteration will give this controller as a limit case of second order controllers having parameters going off to infinity.

Above a critical value $\rho_c^2 = \frac{1}{2}(\sqrt{5}+1)$, the golden section ratio, the optimal central controller changes structure and becomes second order. The optimal loss is given simply by $\gamma_o^{-2} = 1 + \rho^2$. The central controller is no longer equalizing, although there exists a second order equalizing controller with the same optimal γ-value. Optimality could now also be achieved by zero order or first order controllers.

As a side-effect the note also describes properties of some suggested methods to obtain H_∞-optimal controllers. Although not explicitly discussed there are strong implications for the parametrization of the numerical methods involved in the γ-iteration and in the search for optimal equalizing controllers.

Key words

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title

ISBN

Language
English

Number of pages
14

Recipient's notes

Security classification

1. Introduction

H_∞-design is a maturing field. Software packages are marketed and applications are emerging. Still very few people seem to have an intuitive feeling for the properties of the resulting controllers and how they depend on design parameters. It easily happens that the obtained H_∞-solutions to a problem show obvious disadvantages, and you have to know how to modify your problem formulation. In that respect LQG-control, supplemented with LTR, seems to be easier to understand so far.

In this note we give a stripped down example still showing much of the structure of H_∞-controllers and of the properties of the solution methods.

2. Problem formulation

Regard the feed-forward problem

$$\begin{aligned}(s+1)x_1 &= u\\ (s+1)x_2 &= x_1 + d\\ z &= \begin{pmatrix} x_2/\rho \\ -u \end{pmatrix}\end{aligned} \tag{1}$$

Use causal feed-forward $u = K(s)d$ from the disturbance d.

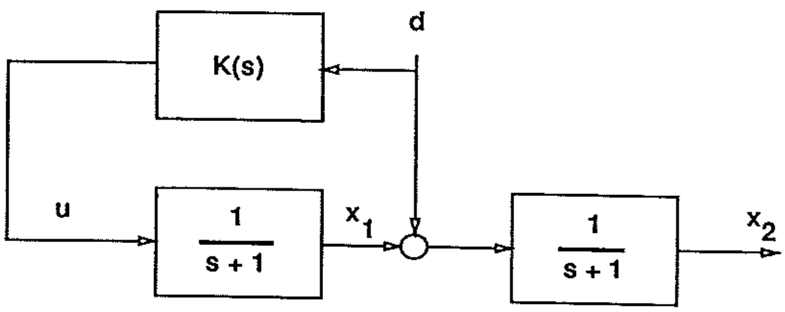

Figure 1. Block diagram of the problem studied in this paper

Find γ_o and corresponding controllers that solve the optimization problem

$$\gamma_o = \min_{K(s)} \max_{\|d\|_2=1} \|z\|_2 \tag{2}$$

for $\rho \in (0, \infty)$. The parameter ρ could be seen as a weight multiplying the control, but it is introduced in this way to avoid a normalization below.

3. Solution

First notice that the feed-forward $K(s) = -(s+1)$ totally eliminates the disturbance, but the input contribution to the loss would be very large for a high-frequency d. Thus we would expect that an optimal $K(s)$ approaches $K(s) = -(s+1)$ as the penalty ρ on the input goes to zero. Similarly $K(s) = 0$ would be optimal as $\rho \to \infty$.

The current standard method for a general H_∞-problem is the γ-iteration approach, e.g. Doyle et. al. (1989). Introduce their notation

$$G(s) := \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & I & 0 \end{array}\right) \tag{3}$$

Notice that we have a pure disturbance feed-forward problem. The solution only requires one Riccati-equation based on

$$\begin{aligned} H_\infty &= \begin{pmatrix} A & P \\ Q & -A^T \end{pmatrix} \\ P &= -B_2 B_2^T + B_1 \gamma^{-2} B_1^T \\ Q &= -C_1^T C_1 \end{aligned} \tag{4}$$

with the solution $X_2 X_1^{-1} = X_\infty \geq 0$ satisfying

$$H_\infty \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} A_r \tag{5}$$

and A_r stable. Thus for this example

$$G(s) := \left(\begin{array}{cc|cc} -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ \hline 0 & \rho^{-1} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad H_\infty = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & \gamma^{-2} \\ 0 & 0 & 1 & -1 \\ 0 & -\rho^{-2} & 0 & 1 \end{pmatrix} \tag{6}$$

Notice that we fulfill the scalings required in Doyle et al. (1989), by

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}^T \begin{pmatrix} 0 & \rho^{-1} & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$$

The central controller and ρ_c

The central controller for $\gamma > \gamma_o$ is given by

$$u = F_\infty \hat{x}, \qquad [sI - A]\,\hat{x} = B_2 u + B_1 d \tag{7}$$

or

$$K(s) = F_\infty \left[sI - (A + B_2 F_\infty)\right]^{-1} B_1 \tag{8}$$

where $F_\infty = -B_2^T X_\infty$. Now γ is limited from below by γ_o by

(A) $X_\infty \to \infty$, or really X_1 becomes singular, so that $X_\infty \geq 0$ no longer holds.

(B) the eigenvalues of $A + PX_\infty$ reach the imaginary axis.

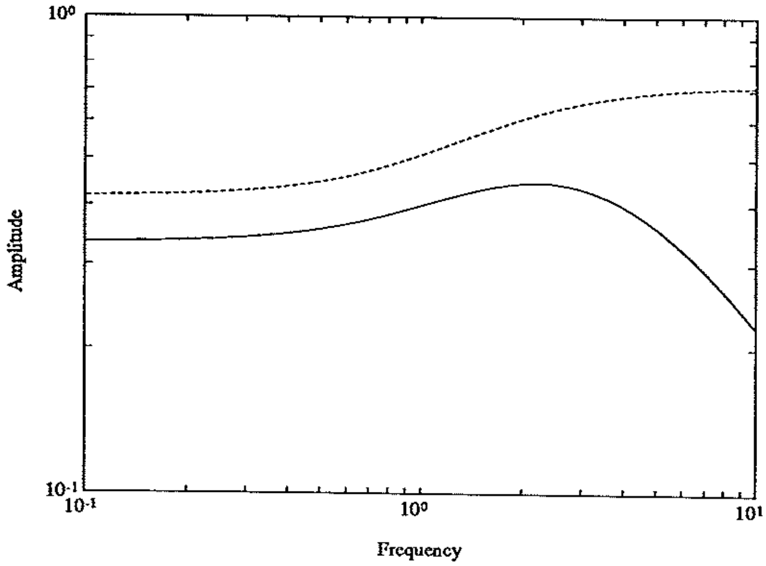

Figure 2. Amplitude of the central controller with $\rho^2 = 2$ (—) and $\rho^2 = 1$ (- - -). For $\rho > \rho_c$ the optimal central controller is second order while for $\rho < \rho_c$ it is first order.

Numerical results

Numerical calculations show that in this example (A) occurs first for $\rho < \rho_c$ and (B) first for $\rho > \rho_c$, and detailed search gives the well-known number $\rho_c^2 = 1.6180339887\ldots$.

With $F_\infty = -\begin{pmatrix} l_1 & l_2 \end{pmatrix}$ (8) gives

$$z = G_c(s)d = \begin{pmatrix} \frac{s+1+l_1}{(s+1)^2+l_1(s+1)+l_2}/\rho \\ \frac{l_2(s+1)}{(s+1)^2+l_1(s+1)+l_2} \end{pmatrix} d \tag{9}$$

The central controller is thus strictly proper of second order. It is a lead-network with steady-state gain equal to $\frac{l_2}{1+l_1+l_2}$ and high-frequency roll-off (-1). For $F_\infty \to \infty$ in the low-ρ case (A) the numerical results suggest that we get a finite $k = \lim l_1/l_2$, so that

$$z = G_c(s)d = \begin{pmatrix} \frac{k}{k(s+1)+1}/\rho \\ \frac{(s+1)}{k(s+1)+1} \end{pmatrix} d \tag{10}$$

and the limiting controller has steady-state gain $1/(k+1)$ and high-frequency gain $1/k$, see figure 2. Thus a low weight on the control signal gives a feed-forward with a direct feed-through, while the high frequency gain rolls off, if the input is penalized harder.

Analysis of the γ-iteration

Case (B) The condition (B) means that the eigenvalues of the Hamiltonian matrix H_∞, i.e. the roots of

$$\det(sI - H_\infty) = (s^2-1)^2 + \rho^{-2}\gamma^{-2}(s^2-1) + \rho^{-2} = 0 \tag{11}$$

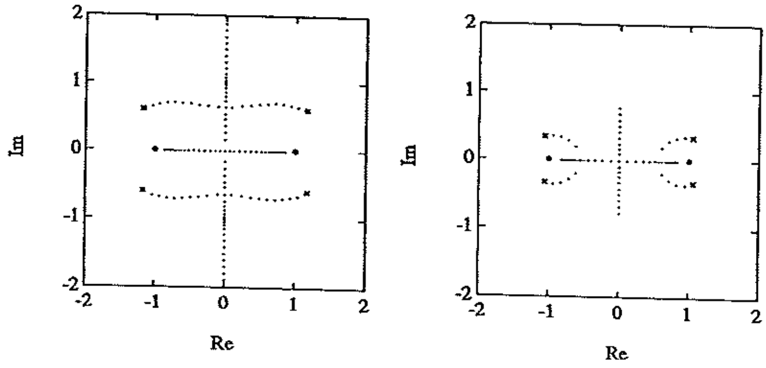

Figure 3. Root-loci of (11), i.e. the eigenvalues of H_∞, with respect to γ^{-2} for $\rho^2 = 1/2$ (left) and $\rho^2 = 2$ (right).

reach the imaginary axis as γ^{-2} increases from zero. For $\rho = 1$ we get four roots at the origin for $\gamma^{-2} = 2$. For smaller ρ the four roots reach the axis at imaginary values for

$$(B1) \qquad \gamma^{-2} = 2\rho,$$

while for large ρ the roots first reach and split at the real axis. In the latter case two roots then meet at the origin for

$$(B2) \qquad \gamma^{-2} = 1 + \rho^2 \tag{12}$$

Case (A) In order to describe case (A) introduce s_1 and s_2 for the stable roots of (11) i.e. the stable eigenvalues of H_∞. The corresponding eigenvectors $q = [q_1\, q_2\, q_3\, q_4]^T$ fulfill

$$(sI - H_\infty)q = 0$$

giving

$$(s-1)(s^2-1)q_1 = -\rho^{-2}q_2$$

Thus the upper part $[q_1\, q_2]^T$ of the "stable eigenvectors" q would be linearly dependent, and we would have condition (A) above, in case

$$(s_1 - 1)(s_1^2 - 1) = (s_2 - 1)(s_2^2 - 1) \tag{13}$$

From the full expression for the two stable eigenvectors q it then also follows that the limit of l_1/l_2 is finite.

$$k = \lim \frac{l_1}{l_2} = -\frac{q_2}{q_1} = -\frac{1}{s_1 + s_2} \tag{14}$$

Now rewrite (11) as

$$s^4 - (2 - \rho^{-2}\gamma^{-2})s^2 + (1 + \rho^{-2} - \rho^{-2}\gamma^{-2}) = 0$$

which implies

$$\begin{cases} s_1^2 + s_2^2 = 2 - \rho^{-2}\gamma^{-2} \\ s_1^2 s_2^2 = 1 + \rho^{-2} - \rho^{-2}\gamma^{-2} \end{cases}$$

and rewrite (13) as

$$(s_1 - s_2)\left\{(s_1^2 + s_2^2 + s_1 s_2) - (s_1 + s_2 + 1)\right\} = 0$$

Introduce

$$\begin{aligned} x &= s_1 + s_2 \\ y &= s_1 s_2 \\ z &= \rho^{-2}\gamma^{-2} \end{aligned}$$

Since $s_1 \neq s_2$ we get the three equations

$$\begin{cases} x^2 - 2y = 2 - z \\ y^2 = 1 + \rho^{-2} - z \\ x^2 - y = x + 1 \end{cases}$$

Now solve x and y from the first and the last equations, remembering that s_1 and s_2 stable means $x < 0$ and $y > 0$,

$$\begin{aligned} x &= 1 - \sqrt{1+z} \\ y &= z - \sqrt{1+z} \end{aligned}$$

so that the second equation gives

$$(z - \sqrt{1+z})^2 = 1 + \rho^{-2} - z$$

Expanding the square results in

$$(z^2 - \rho^{-2}) + 2z = 2z\sqrt{1+z}$$

which after squaring simplifies to

$$(z^2 - \rho^{-2})^2 = 4z\rho^{-2}$$

Now introduce the expression for z, and we finally see that (11) and (13) give $\gamma^4 \pm 2\gamma^3 = \rho^{-2}$ Both these equations have a unique positive solution γ. The minus sign gives a false root that corresponds to $(z^2 - \rho^{-2}) + 2z < 0$, so the resulting equation is

$$(A) \qquad \gamma^4 + 2\gamma^3 = \rho^{-2} \tag{15}$$

The stability requirement $y > 0$ is fulfilled for γ from (15) only in case $\rho^2 < \rho_c^2$, so unstable roots of (11) are required to solve (13) in case of large ρ-values.

Comparison of (A) and (B) It is now easy to see that it is (A) that gives the optimal γ-value if $\rho < \rho_c$ and (B2) if $\rho > \rho_c$, see figure 4. If we require both (A) and (B2), i.e. insert $\gamma^{-2} = (\rho^2 + 1)$ from (12) into (15), we get the condition $\rho^4 = \rho^2 + 1$ giving $\rho_c^2 = (\sqrt{5} + 1)/2$.

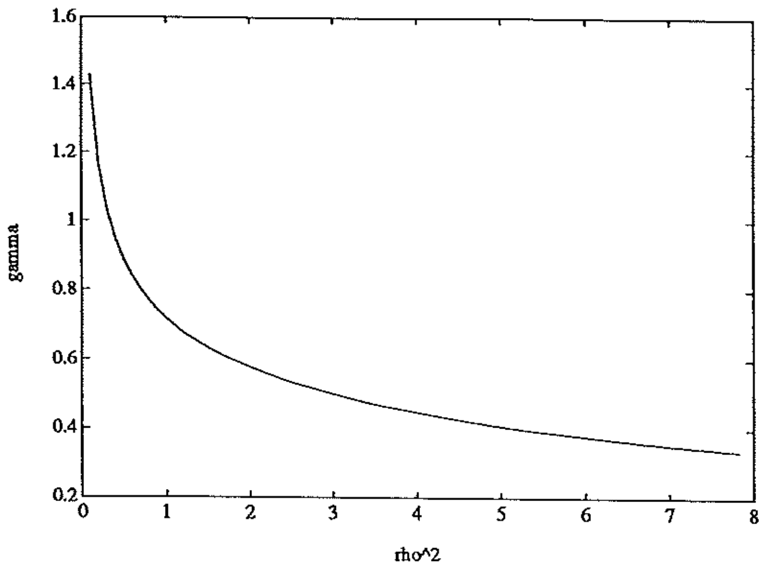

Figure 4. Optimal γ_o as a function of ρ^2. For $\rho^2 < \rho_c^2 = \frac{\sqrt{5}+1}{2}$ eq. (15) is used while (12) is used for larger ρ.

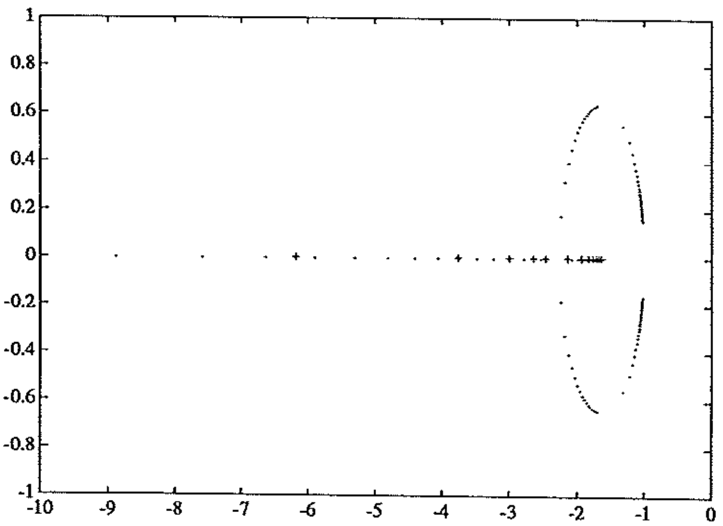

Figure 5. Root-loci of the optimal closed-loop poles for the central controller, i.e. the eigenvalues of $A + B_2F_\infty$, with respect to ρ^2. For $\rho^2 < \rho_c^2$ (+) there is one pole at $-\infty$ and one coming in from $-\infty$ going to -1.618... . For $\rho^2 > \rho_c^2$ (.) one pole is coming in from $-\infty$ meeting the one coming out from -1.618... .

Direct minimization of the frequency function

An alternative method to determine γ_o and optimal controllers $K(s)$ is to analytically solve

$$\begin{aligned} \gamma_o^2 &= \min_{K(s)} \|G_c\|_\infty^2 = \min_K J^{max} = \min_K \max_\omega J(\omega) \\ J(\omega) &= G_c^T(-i\omega)G_c(i\omega) \end{aligned} \tag{16}$$

under stability requirement. If for example the controller structure is restricted

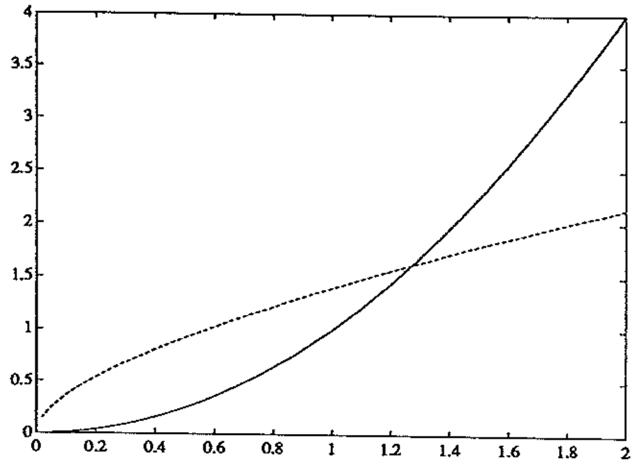

Figure 6. ρ^2 (—) and k_2 (- - -) as functions of ρ

to proportional control $u(t) = Kd(t)$, it follows that

$$\min_K \max_\omega J(\omega) = (\rho^2 + 1)^{-1} \tag{17}$$

It also follows that (17) is the lowest possible $\omega = 0$ value for any controller. For the first order central controller (10) it is straight-forward to evaluate

$$J_1(\omega) = \frac{(k/\rho)^2 + \omega^2 + 1}{(k\omega)^2 + (k+1)^2} = \frac{1}{k^2}\frac{\omega^2 + a(k)}{\omega^2 + b(k)} \tag{18}$$

where $a(k) = (k/\rho)^2 + 1$ and $b(k) = (1/k + 1)^2$. Thus

$$J_1^{max}(k) = \max_{\omega^2 \in [0,\infty)} J_1(\omega) = \begin{cases} 1/k^2, & \text{for } \omega = \infty, \quad \text{if} \quad a(k) < b(k) \\ \dfrac{a(k)}{b(k)k^2}, & \text{for } \omega = 0, \quad \text{otherwise} \end{cases} \tag{19}$$

The condition $a(k) = b(k)$ is equivalent to

$$k^4 - \rho^2(2k+1) = 0 \tag{20}$$

which has two real solutions $k_1 \in (-0.5, 0)$ and $k_2 \in (0, \infty)$, with $|k_2| > |k_1|$. Thus $a(k) < b(k)$ for $k \in (k_1, k_2)$. This means that

$$\min_{k \in [k_1, k_2]} J_1^{max}(k) = J_1^{max}(k_2) = k_2^{-2}$$

Further

$$\frac{d}{dk}\frac{a(k)}{b(k)k^2} = 2\frac{k/\rho^2 - 1}{(k+1)^3} \le 0, \quad \text{for } k \in (-1, \rho^2]$$

with equality for $k = \rho^2$. Therefore we finally get

$$\min_k J_1^{max} = \begin{cases} J_1^{max}(k_2) = k_2^{-2}, & k_2 > \rho^2 \\ J_1^{max}(\rho^2) = (\rho^2+1)^{-1}, & \text{otherwise} \end{cases} \tag{21}$$

Notice that it for $a(k) = b(k)$ follows that $J_1(\omega) = k^{-2}$ for all ω, and we have a flat (all pass) frequency response. Since k_2 grows with ρ, but slower than ρ^2,

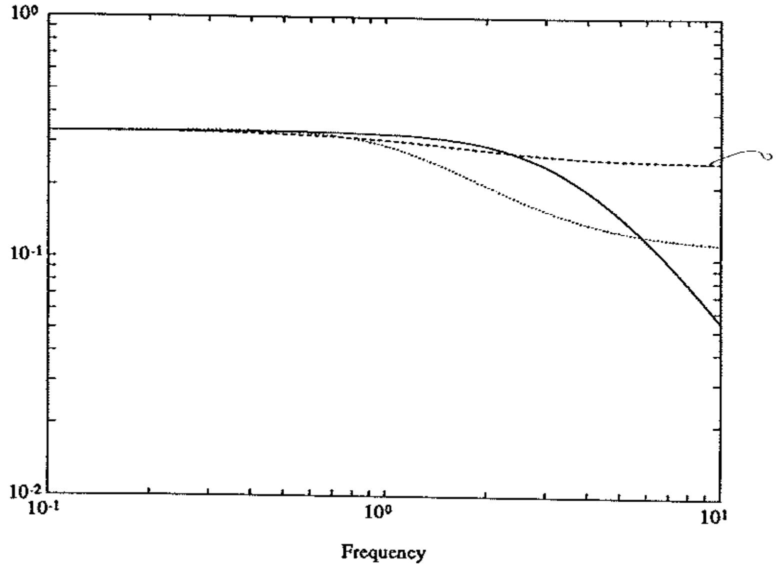

Figure 7. Amplitude plots for different H_∞-optimal controllers for $\rho^2 = 2$. Second order central controller (—); First order controller given by (10) and $k = \rho^2$ (- - -); Optimal P-controller (···). Notice that the central controller is not uniformly better than other controllers.

see fig 6, the first order controller would give a flat response for small ρ up to $k_2 = \rho^2$, i.e. $\rho^4 = \rho^2 + 1$ and $\rho^2 = \rho_c^2 = (\sqrt{5}+1)/2$. If we use $k = k_2$, the "flattening" controller, also in case $\rho > \rho_c$ i.e. $k_2 < \rho^2$, we would get the flat response $J_1(\omega) = k_2^{-2}$, but that is then worse than the best first order controller $k = \rho^2$ with

$$J_1(\omega) = \rho^{-4}\frac{\omega^2 + a(\rho^2)}{\omega^2 + b(\rho^2)} \tag{22}$$

The low frequency value of this function $J_1(0) = (\rho^2+1)^{-1}$ is its largest value, while the high frequency value is $J_1(\infty) = \rho^{-4}$. Notice that this is the same worst value as for the proportional controller (17). On the other hand, if we use $k = \rho^2$ also for $\rho < \rho_c$, then the high frequency value ρ^{-4} would be worse than the flat response k_2^{-2} obtained with $k = k_2$ from (20), although the low frequency value $(\rho^2+1)^{-1}$ still would be smaller.

If we now compare with the results from the γ-iteration, we see that we get the first order controller as long as the control is cheap enough to allow the flat all-pass response to be optimal. Notice that (20) with $\gamma = 1/k$ is the same as (15). For larger ρ the γ-iteration results in a second order controller with loss $\gamma^{-2} = 1 + \rho^2$. We have now found that the same loss could be obtained with a first order controller or with a proportional controller.

Uniqueness of the optimal controller

We will now see that the controller is unique in the low ρ-case. As we have seen above the controller is not unique for $\rho > \rho_c$. From Doyle et al (1989) it follows that all controllers giving $\|z\|_2 < \gamma\|d\|_2$ can be found by

$$\begin{cases} u = F_\infty \Phi B_1 d + (F_\infty \Phi B_2 + I)v \\ r = (-\gamma^{-2} B_1^T X_\infty \Phi B_1 + I)d - \gamma^{-2} B_1^T X_\infty \Phi B_2 v \end{cases} \tag{23}$$

where $v = Qr$ and $\|Q\|_\infty < \gamma$, and where $\Phi^{-1}(s) = sI - (A + B_2 F_\infty)$. Notice that the γ^{-1} in Doyle et al (1989) has been corrected to γ^{-2}. Further with

$X_\infty = \begin{pmatrix} l_1 & l_2 \\ l_2 & l_3 \end{pmatrix}$ we have $\Phi(s) = \frac{1}{P(s)} \begin{pmatrix} s+1 & -l_2 \\ 1 & s+1+l_1 \end{pmatrix}$ and the characteristic polynomial $P(s) = (s+1)^2 + l_1(s+1) + l_2$. Thus we can rewrite (23) as

$$\begin{cases} P(s)u = -l_2(s+1)d + (s+1)^2 v \\ P(s)r = \left[P(s) - \gamma^{-2}(l_3(s+1) + l_1 l_3 - l_2^2)\right] d - \gamma^{-2}(l_2(s+1) + l_3)v \end{cases}$$

and

$$u = \frac{s+1}{P(s)} \left\{ -l_2 + (s+1)Q(s) \frac{P(s) - \gamma^{-2}(l_3(s+1) + l_1 l_3 - l_2^2)}{P(s) + \gamma^{-2}(l_2(s+1) + l_3)Q(s)} \right\} d \tag{24}$$

Now regard case A, i.e. $\rho < \rho_c$, where $X_\infty \to \infty$, and introduce apart from $k = l_1/l_2$ also $l = l_3/l_2$ and $m = l_1 l_3/l_2 - l_2$. Similarly to (14) it is straightforward to see that l and m are finite. We thus get from (24)

$$u = \frac{s+1}{k(s+1)+1} \left\{ -1 + \frac{1}{l_2}(s+1)Q(s) \frac{k(s+1) + 1 - \gamma^{-2}(l(s+1)+m)}{k(s+1) + 1 + \gamma^{-2}(s+1+l)Q(s)} \right\} d$$

and the controller is unique for this case, since the second term within the parenthesis becomes zero.

For the large-ρ case (B) the optimal controller is not unique. It is possible to use (24) to determine a $Q(s)$ for all the different optimal controllers discussed for instance in fig 7.

Conditions for Equalizing Control

A natural question arising from fig 7 is whether there is a controller giving a flat response $\gamma^2 = J(\omega) = (\rho^2+1)^{-1}$ for $\rho > \rho_c$. Of course that would mean the worst among all the H_∞-optimal controllers. Such equalizing controllers have been investigated, also in the multivariable case, e.g. in Kwakernaak (1990). Using the control $u = \frac{T}{R}d$ it follows that

$$\begin{aligned} x_2 &= \frac{M}{(s+1)^2 R} d \\ M &= T + (s+1)R \end{aligned} \tag{25}$$

so that we require

$$\gamma^2 = \frac{T(-s)T(s)}{R(-s)R(s)} + \rho^{-2} \frac{M(-s)M(s)}{(-s^2+1)^2 R(-s)R(s)} \tag{26}$$

for all s. From (26) it follows that $(s+1)^2$ should be a factor of $M(s)$ and then from (25) that $(s+1)$ is factor of $T(s)$.

First order controller. First assume that

$$\deg T(s) \le \deg R(s) = 1$$

so that

$$\begin{aligned} M(s) &= (s+1)^2 \\ T(s) &= t(s+1) \\ R(s) &= s + r = s + 1 - t \end{aligned}$$

and

$$\gamma^2 = \frac{t^2(-s^2+1)}{(-s^2+r^2)} + \rho^{-2}\frac{1}{(-s^2+r^2)}$$

Equating for different powers of $(-s^2)$ gives

$$\gamma^2 = t^2$$
$$\gamma^2 r^2 = t^2 + \rho^{-2}$$

Using R stable this now gives $t = -\gamma$ and an equation to determine γ

$$r^2 = 1 + \rho^{-2}\gamma^{-2} = (1+\gamma)^2 \tag{27}$$

Rewriting (27) we get the same equation as in (15):

$$\gamma^4 + 2\gamma^3 = \rho^{-2}$$

This equation has a unique positive solution. For $\rho > \rho_c$ this γ is larger than that obtained by all the controllers in fig 7, while the controller is the same as the central controller for smaller ρ.

Second order controller. If we assume

$$\deg T(s) \le \deg R(s) = 2$$

and introduce

$$T(s) = t(s+t_1)(s+1)$$
$$R(s) = (s+r_1)(s+r_2)$$
$$M(s) = (s+m)(s+1)^2$$

then (25) gives

$$\begin{cases} m+1 = t + r_1 + r_2 \\ m = tt_1 + r_1 r_2 \end{cases}$$

while (26) gives

$$\gamma^2(-s^2+r_1^2)(-s^2+r_2^2) = t^2(-s^2+t_1^2)(-s^2+1) + \rho^{-2}(-s^2+m^2)$$

and

$$\begin{cases} \gamma^2 = t^2 \\ \gamma^2(r_1^2+r_2^2) = t^2(t_1^2+1) + \rho^{-2} \\ \gamma^2 r_1^2 r_2^2 = t^2 t_1^2 + \rho^{-2} m^2 \end{cases}$$

Eliminating r_1 and r_2 we get

$$\begin{cases} ((1-\gamma^2)t_1 - \gamma m)^2 = \left((\rho^2+1) - \gamma^{-2}\right)\rho^{-2} m^2 & = a(\gamma) m^2 \\ (m+\gamma)^2 - (t_1+\gamma)^2 = 1 + \rho^{-2}\gamma^{-2} - (1+\gamma)^2 & = b(\gamma) \end{cases} \tag{28}$$

and the resulting controller

$$K(s) = -\gamma \frac{(s+t_1)(s+1)}{s^2 + (m+1+\gamma)s + (m+\gamma t_1)} \tag{29}$$

The equations (28) could be interpreted as the intersection of two lines through the origin with a hyperbola, provided that $a(\gamma) \geq 0$, while there is no solution for $a(\gamma) < 0$. Equality $a(\gamma) = 0$ is obtained for $\gamma^{-2} = 1 + \rho^2$, which also gives the lowest possible γ-value, as could be expected from (12).

However, for $\rho < \rho_c$, the intersection giving stability goes to $m = t_1 = 0$ as $b(\gamma) \to 0$, i.e. γ approaches the solution to (27), the condition for first order controller. A pole at zero cancels in the controller. The optimal controller becomes first order. This fact was discussed e.g. in Kwakernaak (1990). For lower γ, between $b(\gamma) = 0$ and $a(\gamma) = 0$, all intersections give unstable systems.

Also for $\rho > \rho_c$, $b(\gamma) \to 0$ occurs before $a(\gamma) \to 0$, when we decrease γ. In this interval we now obtain two possible equalizing controllers of second order. In one of the controllers a stable pole is canceled as $b(\gamma) \to 0$, while a pole at the origin is cancelled in the other one. After cancellation they actually become equal. The two controllers approach each other without any cancellation at optimality, ie as $a(\gamma) \to 0$. This proves the fact that there exists one optimal equalizing controller of second order for $\rho > \rho_c$.

Robustness to nonzero initial conditions

All the H_∞ controllers are optimal under the assumption of zero initial conditions of the process. With nonzero initial conditions the central controller (7) really gives

$$z = \begin{pmatrix} 0 & \rho^{-1} \\ l_1 & l_2 \end{pmatrix} \hat{x} + \begin{pmatrix} 0 & \rho^{-1} \\ 0 & 0 \end{pmatrix} \tilde{x}$$

where

$$\hat{x} = \Phi B_1 d + \Phi \hat{x}(0)$$

and

$$\tilde{x}(t) = e^{At} x(0)$$

With no information on $x(0)$ it is natural to choose $\hat{x}(0) = 0$ giving

$$z(t) = \begin{pmatrix} \rho^{-1}(\hat{x}_2(t) + \tilde{x}_2(t)) \\ l_1 \hat{x}_1(t) + l_2 \hat{x}_2(t) \end{pmatrix} = (G_c d)(t) + \begin{pmatrix} \rho^{-1} \tilde{x}_2(t) \\ 0 \end{pmatrix}$$

It is obvious that small initial conditions $x(0)$ will only marginally influence the norm $\|z\|_2$.

If on the other hand we regard the full information problem, i.e. we allow both feed-forward from d and feedback from x, then the central controller, would be $u = F_\infty x$ and would not contain any feed-forward. We would obtain the same ρ_c etc and for the low ρ-case any initial condition $x(0)$ would in the limit as $\gamma \to \gamma_{opt}$ give infinite control action and an infinite norm of z. The optimality with respect to only disturbances d leads to impractical control. Similarly we would get infinite loss also from (7) if we try to use any information on $x(0)$ as nonzero starting values $\hat{x}(0)$ for the observer in (7).

4. Conclusions

The main purpose of this investigation has been to find and analyze H_∞-optimal controllers. Special attention was given to how the structure of the H_∞-optimal controllers change as you modify some parameter of the problem formulation, as when the controller becomes non-unique for increasing input penalty ρ.

As a side-effect the note also contains properties of two suggested methods to obtain H_∞-optimal controllers, and although not explicitly commented upon there are strong implications for the parametrization of the numerical methods involved in the γ-iteration (Doyle et. al. 1989) and the search for optimal equalizing controllers (Kwakernaak 1990). No attempt was made to use any of the other methods suggested in the literature, like the descriptor formulation (eg. Safonov et. al. 1989) or the general four block distance problem in Glover et. al. (1991).

It might also be interesting to compare with other solution techniques like those based on Nevanlinna-Pick algorithms, (eg. Chang and Pearson 1984). The relation between the nonuniqueness for $\rho > \rho_c$ and the Parrot-bound (eg. Young 1988) could also be elaborated. Another area that might be fruitful to expand is the relation to zero-sum differential games (eg. Bryson and Ho 1969).

Similarly it would be nice to rank the nonunique optimal controllers by some other criteria. Intuitively the equalizing controller would be worse than all the other controllers in Figure 7. Some frequency weighted square integral could be one such measure.

5. References

Bryson, A. E., and Y. C. Ho (1969): *Applied optimal control*, Blaisdell Publishing Company, Waltham, Mass..

Chang, B.-C., and J. B. Pearson (1984): "Optimal disturbance reduction in linear multivariable systems," *IEEE Trans. Aut. Control*, **29**, 880–887.

Doyle, J. C., K. Glover, P. P. Khargonekar, B. A. Francis (1989): "State space solution to standard H_2 and H_∞ control problems," *IEEE Trans. Aut. Control*, **34**, 831–847.

Glover, K., D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov (1991): "A characterization of all solutions to the four block general distance problem," *SIAM J. Control Optimization*, To appear.

Kwakernaak, H. (1990): "Progress in the polynomial solution of the standard H_∞ optimal control problem," *11th IFAC world congress Tallinn, Estonia, USSR*, pp. 122–129.

Safonov, M. G., D. J. N. Limebeer, and R. Y. Chiang (1989): "Simplifying the H_∞-theory via loop-shifting, matrix pencil and descriptor concepts," *Int. J. Control*, **50**, 2467–2488.

Young, N. (1988): *An introduction to Hilbert space*, Cambridge University Press.