Concentration Bounds for Single Parameter Adaptive Control

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Concentration Bounds for Single Parameter Adaptive Control

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***Abstract*— The purpose of this paper is to analyse transient dynamics in adaptive control using statistical concentration bounds. For maximal clarity, the study is limited to a linear first order system with a single uncertain parameter. Two types of bounds are given: First we prove probabilistic bounds on the parameter estimation error as a function of time. In particular, we prove that the estimation error has finite variance after three time steps and finite fourth moments after five time steps. These bounds are independent of how the parameter estimates are used for feedback. Secondly, we bound the "regret" as a function of time, i.e. the difference in control performance between a self-tuning adaptive controller and the best controller given full knowledge of the plant. The conservatism of the bounds is investigated through simulation.**

I. INTRODUCTION

The history of adaptive control dates back at least to aircraft autopilot development in the 1950s. Later on, computer control and system identification lead to a surge of research activity during the 1970s. Following the landmark paper [2], a long sequence of contributions to adaptive control theory derived conditions for convergence, stability, robustness and performance under various assumptions. For example, [12] analysed adaptive algorithms using averaging, [7] derived an algorithm that gives mean square stability with probability one, while [9] gave conditions for the optimal asymptotic rate of convergence. On the other hand, conditions that may cause instability were studied in [6], [11] and [15]. Altogether, hundreds (maybe thousands) of papers have been written on adaptive control, followed by numerous textbooks, such as [3], [8], [13], [16] and [1].

The research activity in adaptive control was declining in the 1990s, but has recently started to grow again, for reasons similar to the growth of machine leaning; abundance of data and computing resources creates an ever-growing stream of engineering opportunities for adaptation. Also from the theoretical perspective there are new opportunities. While the literature during the 1970-90s was focused on stability and asymptotic performance, new tools for analysis of the transient behavior have recently emerged. In particular, statistical theory for tail and concentration bounds has seen a dramatic development during the last twenty years, with numerous applications in fields like machine learning, compressed sensing, network routing and pattern recognition and is now taught in regular university curriculum. See for example [14] and [19]. Moreover, the attractivity of the theory from a controls perspective has grown with recent extensions to random matrices, as summarized in [18] and [17].

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The purpose of this paper is to demonstrate how transient analysis of adaptive controllers can be carried out using statistical concentration bounds. The analysis is limited to a scalar system with only one estimated parameter, but, as explained later, the main arguments have natural generalizations to vectors and matrices.

After some notation and preliminaries in section II, the main bounds on parameter estimation errors are proved in section III. These bounds are then used in section IV to bound the "regret", i.e. the difference in control performance between a self-tuning adaptive controller and the best controller given full knowledge of the plant. We conclude the paper with some comments and ideas for multi-variable extensions.

II. NOTATION AND PRELIMINARIES

Let $\mathbb{P}$ denote probability and $\mathbb{E}$ expected value. For a random variable $x > 0$, the moments can be computed as

$$\mathbb{E}x^p = \int_0^\infty \mathbb{P}[x^p \geq t]dt.$$

The probability on the right hand side will in this paper be estimated using the *Chernoff bound*

$$\mathbb{P}[x \geq 0] \leq \inf_{\theta>0} \mathbb{E}e^{\theta x}.$$

A random variable $x \in \mathbb{R}$ with mean $\bar{x}$ is said to be *sub-Gaussian* with *variance proxy* ν^2 if

$$\mathbb{E}[e^{\theta(x-\bar{x})}] \leq e^{\theta^2\nu^2/2}$$

for all $\theta \in \mathbb{R}$. Some basic properties of sub-Gaussian variables are:

Proposition 1: Let x and y be independent and sub-Gaussian with variance proxy ν_x^2 and ν_y^2 respectively. Then $x + y$ is sub-Gaussian with variance proxy $\nu_x^2 + \nu_y^2$.

Proposition 2: Let $x \in \mathbb{R}$ be a zero mean sub-Gaussian random variable with unit variance proxy. Let $y \in \mathbb{R}$ be fixed. Then

$$\mathbb{E}\exp\left[\theta(x+y)^2\right] \leq \frac{1}{\sqrt{1-2\theta}}\exp\left(\frac{\theta y^2}{1-2\theta}\right)$$

for $\theta < 1/2$.

Proposition 3: Let $w_1, \ldots, w_t$ be i.i.d. sub-Gaussian random variables with unit variance proxy and symmetric distribution around zero. Then

$$\mathbb{E}\big[(w_1^2 + \cdots + w_t^2)^p\big] \leq t^p(2p-1)!!$$

where $n!!$ denotes the product of all odd numbers up to n. See the Appendix for references and proofs.

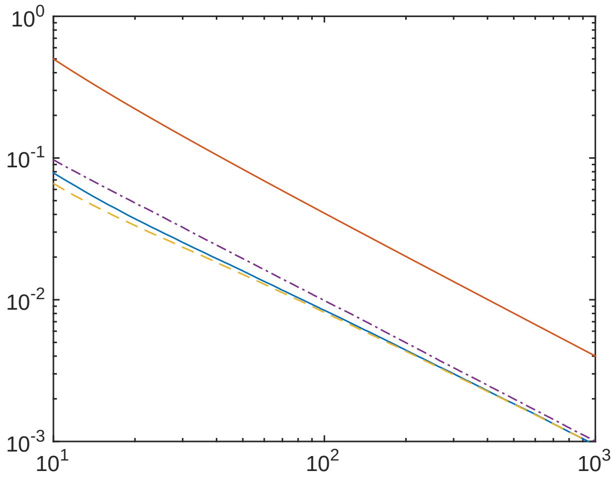

Fig. 1. The upper solid plot is the upper bound $4/(t-2)$ for the variance $\mathbb{E}(\widehat{a}_t - a)^2$ according to Theorem 4. This bound is valid for all values of a and all control laws. The second plot (dash-dotted) is the average of $(\widehat{a}_t - a)^2$ over 10^5 simulations with $a = 0$ and $u_k \equiv 0$. The third and fourth plots are averages of $(\widehat{a}_t - a)^2$ over 10^5 simulations with $u_k = -\hat{a}_k x_k$ and with $a = 1$ and $a = 2$ respectively.

III. Finite Time Estimation Error Bounds

Consider a scalar linear system

$$x_{k+1} = ax_k + u_k + w_k, \qquad x_0 = 0 \tag{1}$$

where the parameter $a \in \mathbb{R}$ and $w_0, \ldots, w_t$ are independent zero mean sub-Gaussian random variables with variance proxy σ^2. The least squares estimate of a given $u_0, x_0, \ldots, u_t, x_t$ is then

$$\widehat{a}_t = \frac{\sum_{k=1}^{t-1}(x_{k+1} - u_k)x_k}{\sum_{k=1}^{t-1} x_k^2}. \tag{2}$$

Our first result, giving probabilistic bounds on the estimation error $\widehat{a}_t - a$, can then be stated as follows:

Theorem 4: Define x_k and $\widehat{a}_t$ according to (1) and (2). Suppose for every k that u_k is a measurable function of $x_0, \ldots, x_k$. Define $\rho > 0$. Then

$$\mathbb{P}\big[|\widehat{a}_t - a| > \rho\big] \leq 2\left(1 + \rho^2\right)^{-t/2} \tag{3}$$

$$\mathbb{E}(\widehat{a}_t - a)^{2p} \leq \tfrac{2^{p+1}p!}{(t-2)(t-4)\cdots(t-2p)} \tag{4}$$

for $t \geq 2p + 1$.

Remark 1. Notice that for any given error level $\rho > 0$, the probability that the estimation error exceeds ρ will decay at least as fast as the exponential rate (3), which is *independent of the control law* determining u_k. As an illustration, we have in Figure 1 plotted the bound on $\mathbb{E}(\widehat{a}_t - a)^2$ together with simulated values of $\sum_{t=1}^{T}(\hat{a}_t - a)^2/T$.

The following lemma will be useful for the proof of Theorem 4:

Lemma 5: For $t \geq 1$, let $w_0, \ldots, w_t$ be independent zero mean sub-Gaussian random variables with variance proxy σ^2. Define $x_{k+1} = z_k + w_k$, where $z_0 = 0$ and for $k \geq 1$ the variable z_k is a measurable function of $w_0, \ldots, w_{k-1}$. Then

$$\mathbb{E}\exp\left[\frac{\rho}{\sigma^2}\sum_{k=1}^{t}\left(w_k x_k - \rho x_k^2\right)\right] \leq \frac{1}{(1+\rho^2)^{t/2}}$$

Proof. For $k \geq 1$, define $\mathcal{F}_k$ to be the σ-algebra generated by $w_0, \ldots, w_k$. Then

$$\begin{aligned}
&\mathbb{E}\exp\left[\frac{\rho}{\sigma^2}\sum_{k=1}^{t}\left(w_k x_k - \rho x_k^2\right) \,\middle|\, \mathcal{F}_{t-1}\right] \\
&\quad\leq \exp\left[\frac{\rho}{\sigma^2}\left(\sum_{k=1}^{t-1}\left(w_k x_k - \rho x_k^2\right) - \frac{\rho}{2}x_t^2\right)\right]
\end{aligned} \tag{5}$$

by the definition of w_t being sub-Gaussian. Only two terms in the bracket depend on w_{t-1}, namely $w_{t-1}x_{t-1}$ and $\frac{\rho}{2}x_t^2$. Isolating w_{t-1} by completion of squares

$$\begin{aligned}
&\frac{\rho}{\sigma^2}\left(w_{t-1}x_{t-1} - \frac{\rho}{2}x_t^2\right) \\
&= \frac{\rho}{\sigma^2}w_{t-1}x_{t-1} - \frac{\rho^2}{2\sigma^2}(z_{t-1} + w_{t-1})^2 \\
&= \frac{x_{t-1}^2}{2\sigma^2} - \frac{\rho z_{t-1}x_{t-1}}{\sigma^2} - \frac{\rho^2}{2}\left(\frac{z_{t-1}}{\sigma} - \frac{x_{t-1}}{\rho\sigma} + \frac{w_{t-1}}{\sigma}\right)^2
\end{aligned}$$

and applying Proposition 2 gives

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{\rho}{\sigma^2}\left(w_{t-1}x_{t-1} - \frac{\rho}{2}x_t^2\right)\right) \,\middle|\, \mathcal{F}_{t-2}\right] \\
&\leq \frac{1}{\sqrt{1+\rho^2}}\exp\left(\frac{x_{t-1}^2}{2\sigma^2} - \frac{\rho z_{t-1}x_{t-1}}{\sigma^2} - \frac{\rho^2/2}{1+\rho^2}\left(\frac{z_{t-1}}{\sigma} - \frac{x_{t-1}}{\rho\sigma}\right)^2\right) \\
&= \frac{1}{\sqrt{1+\rho^2}}\exp\left(\frac{\rho^2 x_{t-1}^2}{2\sigma^2} - \frac{\rho^2}{2\sigma^2(1+\rho^2)}\left(\rho x_{t-1} + z_{t-1}\right)^2\right) \\
&\leq \frac{1}{\sqrt{1+\rho^2}}\exp\left(\frac{\rho^2 x_{t-1}^2}{2\sigma^2}\right).
\end{aligned}$$

Repeated use of this argument implies

$$\begin{aligned}
&\mathbb{E}\exp\left[\frac{\rho}{\sigma^2}\sum_{k=1}^{t}\left(w_k x_k - \rho x_k^2\right)\right] \\
&\leq \mathbb{E}\exp\left[\frac{\rho}{\sigma^2}\left(\sum_{k=1}^{t-1}\left(w_k x_k - \rho x_k^2\right) - \frac{\rho}{2}x_t^2\right)\right] \\
&\leq \frac{1}{\sqrt{1+\rho^2}}\mathbb{E}\exp\left[\frac{\rho}{\sigma^2}\left(\sum_{k=1}^{t-2}\left(w_k x_k - \rho x_k^2\right) - \frac{\rho}{2}x_{t-1}^2\right)\right] \\
&\vdots \\
&\leq \frac{1}{(1+\rho^2)^{(t-1)/2}}\mathbb{E}\exp\left[-\frac{\rho^2}{2\sigma^2}x_1^2\right] \\
&\leq (1+\rho^2)^{-t/2}
\end{aligned}$$

□

Proof of Theorem 4. Note that

$$\widehat{a}_t - a = \frac{\sum_{k=1}^{t-1} w_k x_k}{\sum_{k=1}^{t-1} x_k^2}.$$

Define $z_k = ax_k + u_k$. Then

$$\begin{aligned}\mathbb{P}\big[\widehat{a}_t - a > \rho\big] &= \mathbb{P}\big[\textstyle\sum_{k=1}^t (w_k x_k - \rho x_k^2) > 0\big] \\ &\leq \mathbb{E}\exp\left[\rho\sigma^{-2}\textstyle\sum_{k=1}^t \big(w_k x_k - \rho x_k^2\big)\right] \\ &\leq \big(1+\rho^2\big)^{-t/2}.\end{aligned}$$

The first inequality is the Chernoff bound and the second is given by Lemma 5. An identical argument gives the same upper bound for $\mathbb{P}\big[a - \widehat{a}_t > \rho\big]$. Adding the two probabilities gives (3).

For positive integers p, we get (4) as follows:

$$\begin{aligned}&\mathbb{E}(\widehat{a}_t - a)^{2p} = \int_0^\infty \mathbb{P}\big[|\widehat{a}_t - a|^{2p} > y^p\big] d(y^p) \\ &\leq \int_0^\infty \frac{2}{(1+y)^{t/2}} d(y^p) \\ &= \int_0^\infty \frac{2p(p-1)y^{p-2}}{(t/2-1)(1+y)^{t/2-1}} dy \\ &= \int_0^\infty \frac{2p(p-1)(p-2)y^{p-3}}{(t/2-1)(t/2-2)(1+y)^{t/2-2}} dy \\ &\;\;\vdots \\ &= \int_0^\infty \frac{2p(p-1)(p-2)\cdots 2\cdot 1}{(t/2-1)\cdots(t/2-p+1)(1+y)^{t/2-p+1}} dy \\ &= \frac{2p!}{(t/2-1)(t/2-2)\cdots(t/2-p)}\end{aligned}$$

□

IV. A Regret Bound for Self-Tuning Control

Combining (1) with the feedback law $u_k = -\widehat{a}_k x_k$ for $k \geq 2$ and $v_k = 0$ gives the closed loop system

$$\begin{cases} x_1 = w_0, \quad x_2 = ax_1 + w_1 \\ x_{t+1} = \frac{\sum_{k=1}^{t-1} w_k x_k}{\sum_{k=1}^{t-1} x_k^2} x_t + w_t \qquad t \geq 2 \end{cases} \tag{6}$$

or alternatively

$$\begin{cases} x_1 = w_0, \quad x_2 = ax_1 + w_1, & X_0 = Y_0 = 0 \\ X_t = X_{t-1} + x_t^2 & t \geq 1 \\ Y_t = Y_{t-1} + x_t w_t & t \geq 1 \\ W_t = W_{t-1} + w_t^2 & t \geq 1 \\ x_{t+1} = \frac{Y_{t-1}}{X_{t-1}} x_t + w_t & t \geq 2 \end{cases} \tag{7}$$

By extending arguments of the previous section, we will now prove the following decay bound on the state variance.

Theorem 6: Suppose that $w_0, \ldots, w_t$ are independent sub-Gaussian random variables with unit variance proxy. Let x_t be defined by (6). Then

$$\mathbb{E}x_t^2 \leq \frac{704(t-2)^2}{(t-9)^3} + 1$$

for $t \geq 2p + 8 \geq 10$.

Remark 2. Notice that when the parameter a is known, the optimal controller is $u_k = -ax_k$, with cost $\mathbb{E}x_t^2 = 1$. Hence the term $\frac{704(t-2)^2}{(t-9)^3}$ gives a bound on the "regret". (Strictly speaking, the regret is the accumulated cost degradation up to time t, but this number would be infinite already at $t = 1$.)

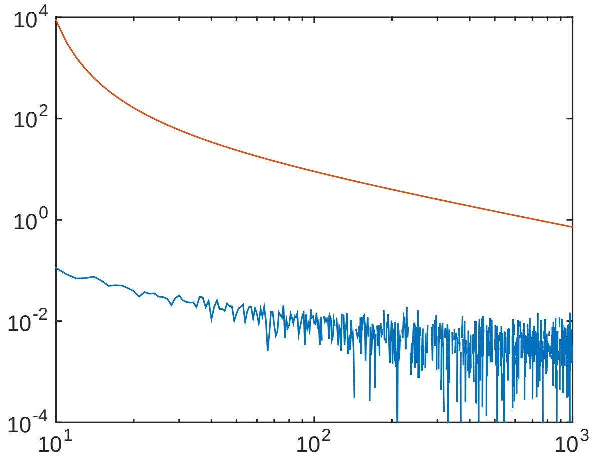

Fig. 2. The upper solid plot is the upper bound $\frac{704(t-2)^2}{(t-10)^3}$ for $\mathbb{E}x_t^2 - 1$ versus t. This bound is valid for all values of a and t. The lower plot is the average of $x_t^2 - 1$ over 10^5 simulations with $a = 1$ and $u_k = -\hat{a}_k x_k$.

To illustrate the result, Figure 2 plots the upper bound together with simulated values of $\sum_{t=1}^T x_t^2/T$.

Lemma 7: Let $\{x_k\}_{k=1}^\infty$ be defined as in Lemma 5. Then $\mathbb{E}\big[(x_1^2 + \cdots + x_t^2)^{-p}\big] \leq 2t^{-p}$ for $t = 1, 2, 3\ldots$

Proof. Notice that

$$\begin{aligned}&\mathbb{P}\left[\big(x_1^2 + \cdots + x_t^2\big)^{-p} > (y/t)^p\right] \\ &= \mathbb{P}\left[t/y > x_1^2 + \cdots + x_t^2\right] \\ &\leq \min_{\theta\geq 0} \mathbb{E}\exp\left[\theta(t/y - x_1^2 - \cdots - x_t^2)\right] \\ &\leq \min_{\theta\geq 0} \frac{\mathbb{E}\exp\left[\theta(t/y - x_1^2 - \cdots - x_{t-1}^2)\right]}{\sqrt{1+2\theta}} \\ &\;\;\vdots \\ &\leq \min_{\theta\geq 0} \frac{e^{\theta t/y}}{(1+2\theta)^{t/2}} \qquad (8) \\ &= \left[y^{-1}\exp(1-y^{-1})\right]^{t/2} \qquad \text{for } y \geq 1\end{aligned}$$

where the minimum in (8) is attained for $\theta = (y-1)/2$. Hence

$$\begin{aligned}&\mathbb{E}\big[(x_1^2 + \cdots + x_t^2)^{-p}\big] \cdot t^p \\ &= \int_0^\infty \mathbb{P}\left[\big(x_1^2 + \cdots + x_t^2\big)^{-p} > (y/t)^p\right] d(y^p) \\ &\leq 1 + \int_1^\infty \left[y^{-1}\exp(1-y^{-1})\right]^{t/2} d(y^p) \\ &\leq 1 + \int_1^\infty \left[y^{-1}\exp(1-y^{-1})\right]^{p+2} d(y^p) \\ &= 1 + pe^{p+2}\int_0^\infty y^{-3}e^{-(p+2)/y} dy \\ &= 1 + p\left(\frac{e}{p+2}\right)^{p+2} \leq 2\end{aligned}$$

for $t \geq 2p + 4 \geq 6$. □

Proof of Theorem 6. We will first prove that

$$\left|\frac{x_{t+1}}{\sqrt{X_t}}\right| \leq \left|\frac{Y_{t-1}}{X_{t-1}}\right| + \left|\frac{w_t}{\sqrt{X_{t-1}}}\right| \tag{9}$$

$$|x_{t+2}| \leq \left|\frac{Y_{t-1}^2}{X_{t-1}^2}x_t\right| + \left|\frac{Y_{t-1}}{X_{t-1}}w_t\right| + \left|\frac{Y_t}{X_t}w_t\right| + |w_{t+1}| \tag{10}$$

for $t \geq 1$. Using that $\hat{a} - a = Y_{t-1}/X_{t-1}$, we will now combine (9) and (10) with Theorem 4 to get the desired result. The triangle inequality gives

$$\left|\frac{x_{t+1}}{\sqrt{X_t}}\right| = \frac{\left|\frac{Y_{t-1}}{X_{t-1}}x_t + w_t\right|}{\sqrt{X_{t-1}+x_t^2}} \leq \left|\frac{Y_{t-1}}{X_{t-1}}\right| + \left|\frac{w_t}{\sqrt{X_{t-1}}}\right|$$

which proves (9). Similarly

$$\begin{aligned} |x_{t+2}| &= \left|\frac{Y_t}{X_t}\left(\frac{Y_{t-1}}{X_{t-1}}x_t + w_t\right) + w_{t+1}\right| \\ &= \left|\frac{(Y_{t-1}+x_t w_t)Y_{t-1}}{(X_{t-1}+x_t^2)X_{t-1}}x_t + \frac{Y_t}{X_t}w_t + w_{t+1}\right| \\ &\leq \left|\frac{Y_{t-1}^2}{X_{t-1}^2}x_t\right| + \left|\frac{Y_{t-1}}{X_{t-1}}w_t\right| + \left|\frac{Y_t}{X_t}w_t\right| + |w_{t+1}| \end{aligned}$$

which proves (10). Using the Cauchy-Schwarz inequality

$$Y_t^2 \leq X_t W_t,$$

together with (9), the first term of (10) can be bounded for $t \geq 2$ as

$$\begin{aligned} \left|\frac{Y_{t-1}^2}{X_{t-1}^2}x_t\right| &\leq \left|\frac{Y_{t-1}^2}{X_{t-1}^{3/2}}\right| \cdot \left|\frac{x_t}{\sqrt{X_{t-1}}}\right| \\ &\leq \left|\frac{Y_{t-1}}{X_{t-1}}\right|\sqrt{W_t}\left|\frac{x_t}{\sqrt{X_{t-1}}}\right| \\ &\leq \left|\frac{Y_{t-1}}{X_{t-1}}\right|\sqrt{W_t}\left(\left|\frac{Y_{t-2}}{X_{t-2}}\right| + \left|\frac{w_{t-1}}{\sqrt{X_{t-2}}}\right|\right) \end{aligned}$$

The inequality of arithmetic and geometric means then gives

$$\left|\frac{Y_{t-1}^2}{X_{t-1}^2}x_t\right| \leq \frac{2t}{3}\left|\frac{Y_{t-1}^3}{X_{t-1}^3}\right| + \frac{2W_t^{3/2}}{3t^2} + \frac{t}{3}\left|\frac{Y_{t-2}^3}{X_{t-2}^3}\right| + \frac{t}{3}\frac{|w_{t-1}|^3}{X_{t-2}^{3/2}}.$$

Injecting this inequality in to (10), squaring, taking expectation, and applying the bounds of Theorem 4 gives

$$\begin{aligned} &\mathbb{E}x_{t+2}^2 \\ &\leq \mathbb{E}\left(4\frac{t^2Y_{t-1}^6}{X_{t-1}^6} + \frac{4W_t^3}{t^4} + 2\frac{t^2Y_{t-2}^6}{X_{t-2}^6} + 2\frac{t^2w_{t-1}^6}{X_{t-2}^3}\right. \\ &\qquad\left. + \frac{Y_{t-1}^2}{X_{t-1}^2}w_t^2 + \frac{Y_t^2}{X_t^2}w_t^2\right) + \mathbb{E}w_{t+1}^2 \\ &\leq \frac{4t^2\cdot 2^{3+1}3!}{(t-2)(t-4)(t-6)} + \frac{4\cdot 15}{t} + \frac{2t^2\cdot 2^{3+1}3!}{(t-3)(t-5)(t-7)} \\ &\qquad + \frac{2\cdot 15\cdot 2}{t} + \frac{4}{t-2} + \frac{4}{t-3} + 1 \\ &\leq \frac{704t^2}{(t-7)^3} + 1 \end{aligned}$$

which completes the proof. □

V. Conclusions and Extensions

The main conclusion of the paper is that the theory of statistical concentration bounds is a powerful tool for analysis of adaptive feedback systems. Unlike previous analysis methods, it makes it possible to prove rigorous bounds on the behavior of an adaptive system, even after a just small number of adaptation steps. As expected the bounds come with a degree of conservatism. However, the qualitative behavior of the upper bounds is very similar to the simulations, which is useful in the context of controller design.

Given, the restrictive nature of the system studied in this paper, a natural next step is to extend the results to higher order systems with several parameters. A first minor step would be to consider systems where the effect of control is uncertain as in [10], [4]. Analysis of the system

$$x_{k+1} = x_k + bu_k, \qquad x_0 = 0$$

with the least squares parameter estimate

$$\widehat{b}_t = \frac{\sum_{k=1}^{t-1}(x_{k+1}-x_k)u_k}{\sum_{k=1}^{t-1}u_k^2}.$$

becomes very similar to Theorem 4 provided that the control signal has the form $u_k = z_k + w_k$, where z_k is determined from past data and w_k (e.g. the effect of measurement noise) is independent of the past.

As pointed out earlier, most of the concepts used above also have matrix counterparts. Hence consider the state equation $x_{k+1} = Ax_k + Bu_k + w_k$, where $w_k \in \mathbb{R}^n$ is white noise. Least squares estimation of A and B then gives the estimation error

$$[\tilde{A} \quad \tilde{B}] = \sum_{k=1}^{t-1} w_k \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \left(\sum_{k=1}^{t-1}\begin{bmatrix} x_k \\ u_k \end{bmatrix}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T\right)^{-1}$$

To generalize (3), we need a bound on the probability that $\|\,[\tilde{A} \quad \tilde{B}]\,\| > \rho$. This inequality can hold only if one of the two matrices

$$\sum_{k=1}^{t-1}\begin{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T & \begin{bmatrix} x_k \\ u_k \end{bmatrix} w_k^T \\ w_k \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T & \frac{\sqrt{\rho}}{t-1}I \end{bmatrix} \qquad \sum_{k=1}^{t-1}\begin{bmatrix} x_k \\ u_k \end{bmatrix}\begin{bmatrix} x_k \\ u_k \end{bmatrix}^T - \frac{t-1}{\sqrt{\rho}}$$

has a negative eigenvalue. The probability for this can be bounded using matrix Chernoff bounds [17] in analogy with the scalar theory of section III.

VI. Acknowledgement

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Appendix: Proofs and references

An excellent tutorial on probabilistic tail and concentration bounds is [19], where section 2.1.3 explains Proposition 1 and many related results.

Proof of Proposition 2 The statement is a generalization of Lemma 1.6 in [5], but the proof is analogous. The identity

$$\frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}\exp\left(\lambda z-\frac{\lambda^2}{2\theta}\right)d\lambda=\exp(\theta z^2)$$

is used twice:

$$\begin{aligned}&\mathbb{E}\exp\left[\theta(x+y)^2\right]\\&=\mathbb{E}\frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}\exp\left(\lambda(x+y)-\frac{\lambda^2}{2\theta}\right)d\lambda\\&\le\frac{1}{\sqrt{2\pi\theta}}\int_{-\infty}^{\infty}\exp\left(\frac{\lambda^2}{2}+\lambda y-\frac{\lambda^2}{2\theta}\right)d\lambda\\&=\frac{1}{\sqrt{1-2\theta}}\exp\left(\frac{\theta y^2}{1-2\theta}\right)\end{aligned}$$

□

Proof of Proposition 3 The variable $w=(w_1+\cdots+w_t)/\sqrt{t}$ is sub-Gaussian with unit variance. Hence

$$\begin{aligned}&t^{-p}\mathbb{E}(w_1^2+\cdots+w_t^2)^p\\&=\mathbb{E}w^{2p}\\&=\int_0^{\infty}\mathbb{P}\left[w^2>y\right]d(y^p)\\&=\int_0^{\infty}\mathbb{P}\left[|w|>r\right]d(r^{2p})\\&\le 2\int_0^{\infty}\exp(-r^2/2)d(r^{2p})=(2p-1)!!\end{aligned}$$

□