Error Bounds for Balanced Truncation of Linear Time-Varying Systems

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Error bounds for balanced truncation of linear time-varying systems*

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Abstract

In this paper error bounds for truncated balanced linear time-varying systems in discrete time are obtained. The analysis is based on direct calculations with the time-varying observability and controllability Lyapunov inequalities. The obtained bounds in the induced ℓ_2-norm generalize well-known error-bound formulas for time-invariant systems. The case of time-varying state-space dimension is considered, and this proves to be valuable both for technical and practical reasons. Input-output stability of truncated models is shown to be guaranteed.

1. Introduction

This paper will treat model reduction of time-varying linear systems. Time-varying linear systems are of interest not only for modeling of time-varying physical processes, but also because of the fact that *time-invariant* nonlinear systems can be well approximated by *time-varying* linear systems about nominal trajectories.

To reduce the order of linear time-invariant systems balanced truncation is often used. Balanced realizations were introduced for this purpose in [9]. Since then an error bound has been proven, [3, 4, 1], which gives a bound on the worst-case error between the original and reduced model and justifies the approximation. This result is now considered to be standard. Balanced realization for time-varying linear systems have also gotten attention, see for example [11, 13] for some early references. However, until recently no error bound has been given for the time-variable case. To obtain bounds, methods for uncertain systems could be utilized, see for example [2]. However, these bounds would be conservative as the known time-variance is encapsulated in an uncertainty ball.

The first explicit error bound for balanced time-varying models, to the authors' best knowledge, was given in [7] and later in [6]. There, an operator-theoretic framework was used to give bounds similar to those that apply to time-invariant models. For time-periodic linear systems bounds have been proven in [8, 12]. There, a special form of lifting isomorphism was used.

In this paper we will work directly with the observability and controllability Lyapunov inequalities (LMIs). It will be seen that it is natural to allow the state-space dimension to vary in size over time. The approach will give stronger error bounds and the method will give stability results on the reduced models. As special cases we will recover the known results for time-invariant and periodic systems. Similar results for systems in continuous time are given in [10].

The ability to vary the state-space dimension over time is not only of interest for technical reasons. In for example stiff problems, such as chemical reactions, it is frequent that in the initial phase, many complex reactions take place and that the dynamics then slows down. It is then reasonable to have a model with many states in the initial phase and then switch to a low-order model after some time. The analysis presented will help to decide when to switch the number of states and also how much loss in accuracy a particular choice might give.

2. Preliminaries and notation

It will be useful to utilize time-varying state-space dimension as commented in the introduction. It is known that minimal realization of linear systems in general have this property, see [5]. However, it will also be a useful technical tool for reducing the order of systems where the state-space dimension originally is constant. Let the state-space dimension at time k be n_k.

In the following the weighted Euclidean norm will be used for $x_k \in R^{n_k}$: $|x_k|^2_{P_k} = x_k^T P_k x_k$, $P_k > 0$.

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Discrete-time signals x over a time interval $[0,T]$, with $x_k \in R^{n_k}$, belong to $\ell_2^n[0,T]$ iff the norm

$$||x||_{2,P} = \left(\sum_{k=0}^{T} |x_k|_{P_k}^2\right)^{1/2}$$

is finite for $P_k = I$. Linear systems $G : \ell_2^m[0,T] \rightarrow \ell_2^p[0,T]$ are used with the induced norm

$$||G|| = \sup_{||u||_2 \neq 0} \frac{||Gu||_2}{||u||_2}.$$

The linear systems $y = Gu$ that we consider are assumed to have finite dimensional state-space realizations:

$$G : \begin{cases} x_{k+1} = A_k x_k + B_k u_k, & x_0 = 0 \\ y_k = C_k x_k + D_k u_k, & x_k \in R^{n_k} \end{cases} \quad (1)$$

with m inputs and p outputs. As the model order may vary with k, A_k is not necessarily a square matrix but rather rectangular. The matrices have the following structure:

$$\begin{aligned} &A_k \in R^{n_{k+1}\times n_k}, \quad B_k \in R^{n_{k+1}\times m}, \\ &C_k \in R^{p\times n_k}, \qquad D_k \in R^{p\times m}. \end{aligned}$$

The system we would like to obtain, $\hat{y} = \hat{G}u$, will be called a reduced-order system. It will have the state-space dimension $\hat{n}_k$, where $\hat{n}_k \leq n_k$ for all k. The set

$$\mathcal{T} = \{k : \hat{n}_k < n_k\}$$

contains the time points where the state-space dimension differs. We will construct $\hat{G}$ from G based on the matrix partitions

$$\begin{aligned} A_k &= \begin{bmatrix} A_{k,11} & A_{k,12} \\ A_{k,21} & A_{k,22} \end{bmatrix}, \qquad && A_{k,11} \in R^{\hat{n}_{k+1}\times \hat{n}_k} \\ B_k &= \begin{bmatrix} B_{k,1} \\ B_{k,2} \end{bmatrix}, && B_{k,1} \in R^{\hat{n}_{k+1}\times m} \\ C_k &= \begin{bmatrix} C_{k,1} & C_{k,2} \end{bmatrix}, && C_{k,1} \in R^{p\times \hat{n}_k}. \end{aligned}$$

If the realization (1) is chosen such that the states in the lower part of x_k are "small" in some sense, a reasonable reduced-order candidate is obtained by truncating those states:

$$\hat{G} : \begin{cases} \hat{x}_{k+1} = A_{k,11}\hat{x}_k + B_{k,1}u_k, & \hat{x}_0 = 0 \\ \hat{y}_k = C_{k,1}\hat{x}_k + D_k u_k, & \hat{x}_k \in R^{\hat{n}_k}. \end{cases} \quad (2)$$

The auxiliary signal

$$\hat{z}_{k+1} = A_{k,21}\hat{x}_k + B_{k,2}u_k \quad (3)$$

will be useful. $\hat{z}_k \in R^{n_k - \hat{n}_k}$ and is only defined if $k \in \mathcal{T}$. $\hat{z}$ is not needed to evaluate the map $\hat{G}$. It will be used to evaluate the difference between the outputs of G and $\hat{G}$.

If the systems G and $\hat{G}$ shall have a similar input-output behavior when the above truncation scheme is used, it is important that the coordinate system in the realization of G is well chosen. As we will see, such coordinate systems exist in many cases. A change in coordinate system, $x_k = T_k\tilde{x}_k$, for invertible T_k, will transform the realization according to

$$\begin{gathered} \{A_k, B_k, C_k, D_k\} \xrightarrow{T_k} \\ \{\tilde{A}_k, \tilde{B}_k, \tilde{C}_k, \tilde{D}_k\} = \{T_{k+1}^{-1}A_k T_k,\ T_{k+1}^{-1}B_k,\ C_k T_k,\ D_k\}. \end{gathered} \quad (4)$$

The topic of this paper is to answer the question of how to choose the numbers $\hat{n}_k$, and then to give a bound on the error $||G - \hat{G}||$ that results from this choice.

We will bound $||G - \hat{G}||$ by finding numbers $C > 0$ such that $||y - \hat{y}||_2 \leq C||u||_2$.

3. Observability results

Consider the *Lyapunov observability inequality*:

$$A_k^T Q_{k+1} A_k + C_k^T C_k \leq Q_k. \quad (5)$$

It determines how much energy there will be in the output for a given initial state of the system G with zero input. It can, however, also be used to determine the difference in energy between the outputs from G and $\hat{G}$ when both systems are driven by the same input signal. To see this, assume there is a positive semidefinite solution Q_k with a block-diagonal structure

$$Q_k = \begin{bmatrix} Q_{k,1} & 0 \\ 0 & q_k \cdot I_{n_k - \hat{n}_k} \end{bmatrix} \in R^{n_k \times n_k} \quad (6)$$

and q_k scalar. Then rewrite (5) in the following way:

$$\begin{bmatrix} A_k \\ I \end{bmatrix}^T \begin{bmatrix} Q_{k+1} & 0 \\ 0 & -Q_k \end{bmatrix} \begin{bmatrix} A_k \\ I \end{bmatrix} + C_k^T C_k \leq 0. \quad (7)$$

If we apply the same input signal u to (1) and (2) we obtain the trajectories x and $\hat{x}$. Introduce the partition

$$x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \end{bmatrix}$$

and multiply (7) from the right with

$$\begin{bmatrix} x_{k,1} - \hat{x}_k \\ x_{k,2} \end{bmatrix} \in R^{n_k}$$

and with the transpose from the left. We then obtain

$$\begin{bmatrix} x_{k+1,1}-\hat{x}_{k+1} \\ x_{k+1,2}-\hat{z}_{k+1} \\ x_{k,1}-\hat{x}_k \\ x_{k,2} \end{bmatrix}^T \begin{bmatrix} Q_{k+1} & 0 \\ 0 & -Q_k \end{bmatrix} \begin{bmatrix} x_{k+1,1}-\hat{x}_{k+1} \\ x_{k+1,2}-\hat{z}_{k+1} \\ x_{k,1}-\hat{x}_k \\ x_{k,2} \end{bmatrix} + \\ +|y_k-\hat{y}_k|^2 \leq 0$$

which is equivalent to

$$\begin{aligned} &\Delta|x_{k,1}-\hat{x}_k|^2_{Q_{k,1}} + \Delta|x_{k,2}|^2_{q_k} - \\ &2q_{k+1}\hat{z}^T_{k+1}x_{k+1,2} + |\hat{z}_{k+1}|^2_{q_{k+1}} + |y_k-\hat{y}_k|^2 \leq 0 \end{aligned} \tag{8}$$

using the structure (6) of Q_k. The forward difference operator Δ is defined as

$$\Delta a_k = a_{k+1} - a_k.$$

on a scalar sequence $\{a_k\}$. Now we can state the following lemma:

LEMMA 1—OBSERVABILITY
If there is a solution Q_k with the structure (6) to the Lyapunov inequality (5) on the interval $[0, T+1]$, then the solutions of (1) and (2) satisfy

(i)

$$\begin{aligned} &|x_{T+1,1}-\hat{x}_{T+1}|^2_{Q_{T+1,1}} + |x_{T+1,2}|^2_{q_{T+1}} + \\ &||y-\hat{y}||^2_2 + \sum_{k\in\mathcal{T}} \left(|\hat{z}_k|^2_{q_k} - 2q_k\hat{z}^T_k x_{k,2}\right) \leq 0 \end{aligned} \tag{9}$$

where equality holds if (5) was solved with equality.

(ii) For every non-increasing positive scalar sequence $\{a_k\}^T_{k=0}$ we have

$$||y-\hat{y}||^2_{2,a} - \sum_{k\in\mathcal{T}} a_{k-1}2q_k\hat{z}^T_k x_{k,2} \leq 0. \tag{10}$$

Proof. (i): Sum the inequalities (8) over $k = 0,\dots,T$ and notice the canceling terms.

(ii): Multiply (8) with a_k for each k, and sum for $k = 0,\dots,T$. For non-increasing a_k the partially canceling terms are necessarily non-negative. The sum over $\mathcal{T}$ is the only sign-indefinite term, which leads to the inequality (10). □

As seen if $\mathcal{T} = \varnothing$ the difference in output is zero, as $G = \hat{G}$. Also notice that all terms in the lemma are necessarily non-negative except the terms with $\hat{z}^T_k x_{k,2}$. These terms are the price we pay for removing states. It is also seen that if the numbers q_k are small for $k \in \mathcal{T}$, we might expect $||y-\hat{y}||_2$ to be small. This is only true if $\hat{z}^T_k x_{k,2}$ is simultaneously small. We will bound these terms by an analysis of controllability next.

4. Controllability results

Here it will be seen how far away the states in G and $\hat{G}$ can be forced with the input signal u. The following inequality will be called the *Lyapunov controllability inequality*:

$$A_k P_k A^T_k + B_k B^T_k \leq P_{k+1}. \tag{11}$$

Assume again there is a block diagonal solution

$$P_k = \begin{bmatrix} P_{k,1} & 0 \\ 0 & p_k \cdot I_{n_k-\hat{n}_k} \end{bmatrix} \in \mathbf{R}^{n_k\times n_k} \tag{12}$$

and that it is positive definite for all k. Notice that (11) can be rewritten as

$$\begin{bmatrix} P^{-1/2}_{k+1}A_kP^{1/2}_k & P^{-1/2}_{k+1}B_k \end{bmatrix} \begin{bmatrix} P^{1/2}_k A^T_k P^{-1/2}_{k+1} \\ B^T_k P^{-1/2}_{k+1} \end{bmatrix} \leq I$$

which is equivalent to

$$\begin{bmatrix} A_k & B_k \\ I & 0 \end{bmatrix}^T \begin{bmatrix} P^{-1}_{k+1} & 0 \\ 0 & -P^{-1}_k \end{bmatrix} \begin{bmatrix} A_k & B_k \\ I & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{13}$$

Now, assume we again apply the same input signal u to G and $\hat{G}$. We then obtain the system trajectories x and $\hat{x}$. Multiply (13) with

$$\begin{bmatrix} x_{k,1}+\hat{x}_k \\ x_{k,2} \\ 2u_k \end{bmatrix} \in \mathbf{R}^{n_k+m}$$

from the right and with the transpose from the left. This gives

$$\begin{aligned} &\Delta|x_{k,1}+\hat{x}_k|^2_{P^{-1}_{k,1}} + \Delta|x_{k,2}|_{p^{-1}_k} + \\ &2p^{-1}_{k+1}\hat{z}^T_{k+1}x_{k+1,2} + |\hat{z}_{k+1}|^2_{p^{-1}_{k+1}} \leq 4|u_k|^2 \end{aligned} \tag{14}$$

if the structure of P_k is used. Now the following lemma can be stated:

LEMMA 2—CONTROLLABILITY
If there is a solution P_k to the inequality (11) with the structure (12) on the interval $[0, T+1]$ then the solutions to (1) and (2) satisfy

(i)

$$\begin{aligned} &|x_{T+1,1}+\hat{x}_{T+1}|^2_{P^{-1}_{T+1,1}} + |x_{T+1,2}|^2_{p^{-1}_{T+1}} + \\ &\sum_{k\in\mathcal{T}} \left(|\hat{z}_k|^2_{p^{-1}_k} + 2p^{-1}_k\hat{z}^T_k x_{k,2}\right) \leq 4||u||^2_2. \end{aligned} \tag{15}$$

(ii) For every positive non-increasing scalar sequence $\{b_k\}^T_{k=0}$ we have

$$\sum_{k\in\mathcal{T}} b_{k-1}2p^{-1}_k\hat{z}^T_k x_{k,2} \leq 4||u||^2_{2,b}. \tag{16}$$

Proof. As in Lemma 1. Use (14) instead of (8). □

The lemma gives boundaries on the reachable set in the state-space for fixed amounts of input energy. Notice that when $\mathcal{T} = \varnothing$ equation (15) reduces to the well-known result

$$x_{T+1}^T P_{T+1}^{-1} x_{T+1} \leq ||u||_2^2$$

as $x_k = \hat{x}_k$ for all k. Also notice that the sum in (16) potentially can cancel the sum in (10), namely if

$$a_{k-1} q_k = b_{k-1} p_k^{-1} \tag{17}$$

for all $k \in \mathcal{T}$. This will be utilized in the following.

5. Balanced realizations and error bounds

The two previous sections rely heavily on the ability to obtain block-diagonal solutions to the inequalities (5) and (11). Luckily, there are such solutions if the system is completely controllable and observable, that is, $P_k > 0$ and $Q_k > 0$ for all k. These are called *balanced solutions* and are diagonal and equal:

$$P_k = Q_k = \Sigma_k = \text{diag}\{\sigma_{k,1}, \sigma_{k,2}, \ldots, \sigma_{k,n_k}\},$$

with each element positive. This normally requires a coordinate transformation (4), of the original realization of G. It is always possible to permute the elements in Σ_k. The elements will be called the *singular values* of G. The singular value $\sigma_{k,l}$ quantifies how observable and controllable the state $x_{k,l}$ is, and thereby how important it is, as will be shown. How to obtain balanced solutions and other properties has been throughly studied, see for example [12] and references therein.

We would like to get an upper error bound between G and $\hat{G}$, i.e. $||G - \hat{G}||$, for different choices of $\hat{n}_k$. This can be done rather easily with Lemma 1 and 2, if there is a balanced realization of G at hand. It is useful to group states that share a common singular value. Therefore introduce the notation

$$\Sigma_k = \text{diag}\{\sigma_{k,1} \cdot I_{s_{k,1}}, \ldots, \sigma_{k,N_k} \cdot I_{s_{k,N_k}}\}.$$

Thus there are N_k unique singular values at time k and $s_{k,1} + \cdots + s_{k,N_k} = n_k$. As there are monotonicity conditions on the weight sequences $\{a_k\}$ and $\{b_k\}$ in Lemma 1 and 2 we expect this to show up somehow in the error bound. The following types of subsequences will be used.

Non-increasing subsequence: Assume there is a non-increasing subsequence in time among the elements in Σ. Collect them in a set S_i:

$$S_i = \{\sigma_{d,N_d}, \ldots, \sigma_{e,N_e}, \ldots\}$$
$$d < \ldots < e < \ldots \in \mathcal{T}_i$$
$$\sigma_{d,N_d} \geq \ldots \geq \sigma_{e,N_e} \geq \ldots.$$

The corresponding states in G can be truncated to obtain $\hat{G}$ with an error bound by choosing $a_k = 1$ for all k, and $b_{k-1} = \sigma_{k,N_k}^2$ if $\sigma_{k,N_k} \in S_i$, in order to fulfill (17). As b_k needs to defined for all $k \in [0, T]$, we need to assign values to the possible gaps. Actually, we can assign any values to the gaps as long as the full sequence $\{b_k\}_0^T$ is kept non-increasing. Next, from Lemma 1 and 2, add (10) and (16):

$$||y - \hat{y}||_2^2 \leq 4||u||_{2,b}^2 \leq 4 \sup_k b_k \cdot ||u||_2^2$$
$$\Rightarrow ||G - \hat{G}|| \leq 2 \sup_{k \in \mathcal{T}_i} \sigma_{k,N_k} = 2 \sup S_i. \tag{18}$$

Non-decreasing subsequence: A corresponding result exists for non-decreasing subsequences from Σ:

$$S_i = \{\sigma_{d,N_d}, \ldots, \sigma_{e,N_e}, \ldots\}$$
$$d < \ldots < e < \ldots \in \mathcal{T}_i$$
$$\sigma_{d,N_d} \leq \ldots \leq \sigma_{e,N_e} \leq \ldots.$$

Then choose $b_k = 1$ for all k and $a_{k-1} = \sigma_{k,N_k}^{-2}$ if $k \in \mathcal{T}_i$, to fulfill (17). For the possible gaps in $\{a_k\}$, any assigned numbers that keeps the complete sequence $\{a_k\}_0^T$ non-increasing will do. If we add (10) and (16) we again obtain (18).

5.1 Error bounds

If we have several different sequences S_i we can remove them iteratively, as a truncated system is still balanced with the remaining singular values. This is seen from straightforward calculations. We can now formulate the main result:

THEOREM 1
Assume G in (1) is balanced with Σ_k. If there are L sequences S_i, defined as above, then

$$||G - \hat{G}|| \leq 2 \sum_{i=1}^{L} \sup S_i. \tag{19}$$

$\hat{G}$ is balanced with the singular values in the set $\Sigma - \bigcup_{i=1}^{L} S_i$.

Proof. Use (18) iteratively and the triangular inequality. Denote the mth truncation G_m.

$$||G - \hat{G}|| = ||G - G_1 + G_1 + \ldots + G_{L-1} - G_L||$$
$$\leq ||G - G_1|| + \ldots + ||G_{L-1} - G_L|| \leq 2 \sum_{i=1}^{L} \sup S_i.$$

□

In several special cases we can simplify Theorem 1. Let us consider two cases: monotonous and periodic systems.

COROLLARY 1—MONOTONOUS SYSTEMS

Assume G is balanced with $\Sigma_k = \text{diag}\{\Sigma_{k,1}, \Sigma_{k,2}\}$,

$$\Sigma_{k,1} = \text{diag}\{\sigma_{k,1} \cdot I_{s_1}, \ldots, \sigma_{k,r} \cdot I_{s_r}\},$$
$$\Sigma_{k,2} = \text{diag}\{\sigma_{k,r+1} \cdot I_{s_{r+1}}, \ldots, \sigma_{k,N} \cdot I_{s_N}\}$$

and that every singular value $\sigma_{k,i}$, $i = r+1 \ldots N$, is monotonous in time, then the truncated $(s_1+\ldots+s_r)$-order system $\hat{G}$ is balanced with $\Sigma_{k,1}$ and

$$||G - \hat{G}|| \leq 2 \sum_{i=r+1}^{N} \sup_k \sigma_{k,i}. \tag{20}$$

Proof. Use Theorem 1 with the sequences $S_i = \{\sigma_{k,i}\}_{k=1}^{T+1}$ for $i = r+1 \ldots N$. □

Here we have assumed $s_k = s$ and $N_k = N$ are constant for simplicity. The corollary is essentially Theorem 1 in [10], but there it was formulated for continuous-time systems. These results generalize the well-known error bound for time-invariant systems derived in [3, 4, 1]: for constant Σ, the error bound is recovered.

For ω-periodic systems G there is a realization $A_{k+\omega} = A_k$, $B_{k+\omega} = B_k$, $C_{k+\omega} = C_k$, and $D_{k+\omega} = D_k$ for all k. If the system is completely controllable and observable there is a ω-periodic balanced solution $\Sigma_k = \Sigma_{k+\omega}$, see [8].

COROLLARY 2—PERIODIC SYSTEMS

If G is ω-periodic and balanced with $\Sigma_{k+\omega} = \Sigma_k$ for all k then the truncation $\hat{G}$ is balanced with the remaining singular values and

$$||G - \hat{G}|| \leq 2 \sum_{k=1}^{\omega} \sum_{i=r_k+1}^{N_k} \sigma_{k,i}$$

over an infinite time horizon.

Proof. Use Theorem 1. Choose the first infinite sequence S_1 as

$$S_1 = \{\sigma_{1,N_1}, \sigma_{1+\omega,N_1}, \sigma_{1+2\omega,N_1}, \ldots\}$$
$$\mathcal{T}_1 = \{1, 1+\omega, 1+2\omega, \ldots\}$$

and observe that all elements in S_1 are equal and thus fulfill the monotonicity condition. After we have removed the states in S_1, we continue iteratively as before and remove the wanted amount of states. The result follows. □

Results similar to Corollary 2 were also obtained in [7, 8, 12] with other techniques. One should however notice that this bound easily gets conservative. For a system with a large period ω (from fast sampling of a continuous-time system for instance) the bound gets large very quickly if states over the whole period are removed. A better bound is obtained if we search for *constant* balanced solutions $\Sigma_k = \Sigma$ to the Lyapunov *inequalities*, as the sets S_i get larger and fewer[1]. Such solutions always exist. Constant Σ were also used in [6].

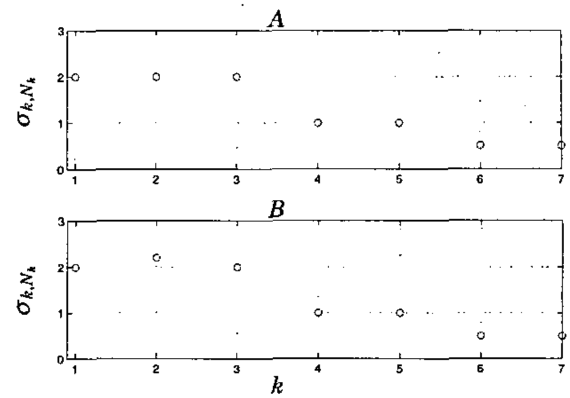

Figure 1: Two sequences $\{\sigma_{k,N_k}\}$ of singular values of a system G. In A the singular values are all non-increasing so that we can put them all in a sequence S_1. In B the value at $k = 2$ destroys this property. But we can still use the result by introducing a second sequence S_2.

Let us look at an example of how Theorem 1 works:

EXAMPLE 1

In Figure 1 two sequences $\{\sigma_{k,N_k}\}$ are shown. Sequence A is non-increasing so all the corresponding states can be removed with

$$||G - \hat{G}|| \leq 2 \max_k \sigma_{k,N_k} = 4.$$

In sequence B the value at $k = 2$ destroys the monotonicity property. However, two separate monotonic subsequences can be found:

$$S_1 = \{\sigma_{1,N_1}, \sigma_{3,N_3}, \ldots, \sigma_{7,N_7}\}$$
$$S_2 = \{\sigma_{2,N_2}\}$$

for example. If all the states are removed as in A then

$$||G - \hat{G}|| \leq 2 \cdot \max S_1 + 2 \cdot \max S_2 = 2 \cdot 2 + 2 \cdot 2.2 = 8.4.$$

As seen it is advisable to look for solutions where many of the singular values $\sigma_{k,l}$ are equal or change monotonically as this leads to larger and fewer sequences S_i. To find such solutions one should use the freedom the Lyapunov inequalities and the boundary conditions give. □

[1] A solution with many, but necessarily not all, constant singular values over the period also lowers the bound.

6. Input-output stability

As was noted in [10], where similar results were developed for continuous-time systems, the previous results have implications on the stability of the truncated system $\hat{G}$. This will be made plausible by an informal discussion.

If the original system G is input-output stable (finite ℓ_2-gain) and subsequences of states S_i are removed to obtain $\hat{G}$, as in Theorem 1, bounds of the type

$$||y - \hat{y}||_2 \leq C\,||u||_2, \quad C > 0 \tag{21}$$

are obtained over an infinite time-horizon. If finite-time input signals u are applied, we know from the input-output stability of G that $y \to 0$ as $t \to \infty$. From (21) $||y - \hat{y}||_2$ is bounded. This implies $\hat{y} \to y \to 0$. Therefore the system $\hat{G}$ will also be input-output stable.

One of the advantages of the approach in this paper is that we have not been so concerned with stability, as we have worked over finite time horizons. This allows us even to work with unstable systems. However, that the original system and its truncation automatically have the same stability property is important as it allows us to let $T \to \infty$ in the results. It should be noted that truncated models might become non-minimal, but they will behave nicely, as they are input-output stable and all states will be bounded.

7. Conclusion

Error bounds (in induced ℓ_2-norm) for truncated linear time-varying systems were presented. The method requires block-diagonal solutions to Lyapunov inequalities. Systems with balanced realizations fit well to this requirement. The presented error bounds generalize known time-invariant results into the time-varying setting. Each state at each time instant is associated with a singular value. If the singular values fulfill certain monotonicity conditions over time and are small, it is possible to truncate many states over long, possibly infinite, time horizons with a small error. Time-varying state-space dimension is considered. Input-output stability of truncated models is guaranteed if the original system is input-output stable.

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